

# Example of the mdici fonts.

Paul Pichaureau

February 1, 2013

## Abstract

The package mdici consists of a full set of mathematical fonts, designed to be combined with ITC Charter ITC Std as the main text font.

This example is extracted from the excellent book *Mathematics for Physics and Physicists*, W. APPEL, Princeton University Press, 2007.

## 1 Conformal maps

### 1.1 Preliminaries

Consider a change of variable  $(x, y) \mapsto (u, v) = (u(x, y), v(x, y))$  in the plane  $\mathbb{R}^2$ , identified with  $\mathbb{R}$ . This change of variable really only deserves the name if  $f$  is locally bijective (i.e., one-to-one); this is the case if the jacobian of the map is nonzero (then so is the jacobian of the inverse map):

$$\left| \frac{D(u, v)}{D(x, y)} \right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0 \quad \text{and} \quad \left| \frac{D(x, y)}{D(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0.$$

**Theorem 1.1.** *In a complex change of variable*

$$z = x + iy \mapsto w = f(z) = u + iv,$$

*and if  $f$  is holomorphic, then the jacobian of the map is equal to*

$$J_f(z) = \left| \frac{D(u, v)}{D(x, y)} \right| = |f'(z)|^2.$$

**Dem.** Indeed, we have  $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$  and hence, by the Cauchy-Riemann relations,

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} = J_f(z).$$

□

**Definition 1.1.** A conformal map or conformal transformation of an open subset  $\Omega \subset \mathbb{R}^2$  into another open subset  $\Omega' \subset \mathbb{R}^2$  is any map  $f : \Omega \mapsto \Omega'$ , locally bijective, that preserves angles and orientation.

**Theorem 1.2.** Any conformal map is given by a holomorphic function  $f$  such that the derivative of  $f$  does not vanish.

This justifies the next definition:

**Definition 1.2.** A conformal transformation or conformal map of an open subset  $\Omega \subset \mathbb{R}$  into another open subset  $\Omega' \subset \mathbb{R}$  is any holomorphic function  $f : \Omega \mapsto \Omega'$  such that  $f'(z) \neq 0$  for all  $z \in \Omega$ .

**Dem.**[that the definitions are equivalent] We will denote in general  $w = f(z)$ . Consider, in the complex plane, two line segments  $\gamma_1$  and  $\gamma_2$  contained inside the set  $\Omega$  where  $f$  is defined, and intersecting at a point  $z_0$  in  $\Omega$ . Denote by  $\gamma'_1$  and  $\gamma'_2$  their images by  $f$ .

We want to show that if the angle between  $\gamma_1$  and  $\gamma_2$  is equal to  $\theta$ , then the same holds for their images, which means that the angle between the tangent lines to  $\gamma'_1$  and  $\gamma'_2$  at  $w_0 = f(z_0)$  is also equal to  $\theta$ .

Consider a point  $z \in \gamma_1$  close to  $z_0$ . Its image  $w = f(z)$  satisfies

$$\lim_{z \rightarrow z_0} \frac{w - w_0}{z - z_0} = f'(z_0),$$

and hence

$$\lim_{z \rightarrow z_0} \text{Arg}(w - w_0) - \text{Arg}(z - z_0) = \text{Arg } f'(z_0),$$

which shows that the angle between the curve  $\gamma'_1$  and the real axis is equal to the angle between the original segment  $\gamma_1$  and the real axis, plus the angle  $\alpha = \text{Arg } f'(z_0)$  (which is well defined because  $f'(z) \neq 0$ ).

Similarly, the angle between the image curve  $\gamma_2'$  and the real axis is equal to that between the segment  $\gamma_2$  and the real axis, plus the same  $\alpha$ .

Therefore, the angle between the two image curves is the same as that between the two line segments, namely,  $\theta$ .

Another way to see this is as follows: the tangent vectors of the curves are transformed according to the rule  $\vec{V}' = df_{z_0} \vec{V}$ . But the differential of  $f$  (when  $f$  is seen as a map from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ ) is of the form

$$df_{z_0} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = |f'(z_0)| \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad (1)$$

where  $\alpha$  is the argument of  $f'(z_0)$ . This is the matrix of a rotation composed with a homothety, that is, a similitude.

Conversely, if  $f$  is a map which is  $\mathbb{R}^2$ -differentiable and preserves angles, then at any point  $df$  is an endomorphism of  $\mathbb{R}^2$  which preserves angles. Since  $f$  also preserves orientation, its determinant is positive, so  $df$  is a similitude, and its matrix is exactly as in equation (1). The Cauchy-Riemann equations are immediate consequences.  $\square$

**Rem.** An antiholomorphic map also preserves angles, but it reverses the orientation.

## Calcul différentiel

Pour obtenir la différentielle totale de cette expression, considérée comme fonction de  $x, y, \dots$ , donnons à  $x, y, \dots$  des accroissements  $dx, dy, \dots$ . Soient  $\Delta u, \Delta v, \dots, \Delta f$  les accroissements correspondants de  $u, v, \dots, f$ . On aura

$$\Delta f = \frac{\partial f}{\partial u} \Delta u + \frac{\partial f}{\partial v} \Delta v + \dots + R \Delta u + R_1 \Delta v + \dots,$$

$R, R_1, \dots$  tendant vers zéro avec  $\Delta u, \Delta v, \dots$

Mais on a, d'autre part,

$$\Delta u = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \dots + S \Delta x + S_1 \Delta y + \dots$$

$$= du + S dx + S_1 dy + \dots$$

$$\Delta v = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \dots + T \Delta x + T_1 \Delta y + \dots$$

$$= dv + T dx + T_1 dy + \dots$$

...

$S, S_1, \dots, T, T_1, \dots$  tendant vers zéro avec  $dx, dy, \dots$

Substituant ces valeurs dans l'expression de  $\Delta f$ , il vient

$$\Delta f = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \dots + \rho dx + \rho_1 dy + \dots$$

$$= \left( \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \dots \right) dx$$

$$+ \left( \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \dots \right) dy$$

$$+ \dots + \rho dx + \rho_1 dy + \dots$$

$\rho, \rho_1, \dots$  tendant vers zéro avec  $dx, dy, \dots$

On aura donc

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \dots,$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \dots,$$

...

et, d'autre part,

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \dots;$$

d'où les deux propositions suivantes :

*La dérivée, par rapport à une variable indépendante  $x$ , d'une fonction composée  $f(u, v, \dots)$  s'obtient en ajoutant ensemble les dérivées partielles  $\frac{\partial f}{\partial u}$ ,  $\frac{\partial f}{\partial v}$ , ..., respectivement multipliées par les dérivées de  $u$ ,  $v$ , ... par rapport à  $x$ .*

*La différentielle totale  $df$  s'exprime au moyen de  $u$ ,  $v$ , ...,  $du$ ,  $dv$ , ..., de la même manière que si  $u$ ,  $v$ , ... étaient des variables indépendantes.*

CAMILLE JORDAN, *Cours d'analyse de l'École polytechnique*