Example of the mdpus fonts.

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Abstract

The package mdpus consists of a full set of mathematical fonts, designed to be combined with Adobe Utopia Std as the main text font.

This example is extracted from the excellent book *Mathematics for Physics and Physicists*, W. APPEL, Princeton University Press, 2007.

1 Conformal maps

1.1 Preliminaries

Consider a change of variable $(x,y) \mapsto (u,v) = \big(u(x,y), v(x,y)\big)$ in the plane $\mathbb{R}^2$, identified with $\mathbb{R}$. This change of variable really only deserves the name if f is locally bijective (i.e., one-to-one); this is the case if the jacobian of the map is nonzero (then so is the jacobian of the inverse map):

$$\left|\frac{D(u,v)}{D(x,y)}\right| = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \neq 0 \qquad \text{and} \qquad \left|\frac{D(x,y)}{D(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0.$$

Theorem 1.1. *In a complex change of variable*

$$z = x + \mathrm{i}y \longmapsto w = f(z) = u + \mathrm{i}v,$$

and if f is holomorphic*, then the jacobian of the map is equal to*

$$J_f(z) = \left|\frac{D(u,v)}{D(x,y)}\right| = \left|f'(z)\right|^2.$$

Dem. Indeed, we have $f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}$ and hence, by the Cauchy-Riemann relations,

$$\left|f'(z)\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = J_f(z).$$

□

Definition 1.1. *A* conformal map *or* conformal transformation *of an open subset $\Omega \subset \mathbb{R}^2$ into another open subset $\Omega' \subset \mathbb{R}^2$ is any map $f : \Omega \mapsto \Omega'$, locally bijective, that preserves angles and orientation.*

Theorem 1.2. *Any conformal map is given by a holomorphic function f such that the derivative of f does not vanish.*

This justifies the next definition:

Definition 1.2. *A* conformal transformation *or* conformal map *of an open subset $\Omega \subset \mathbb{R}$ into another open subset $\Omega' \subset \mathbb{R}$ is any holomorphic function $f : \Omega \mapsto \Omega'$ such that $f'(z) \neq 0$ for all $z \in \Omega$.*

Dem.[that the definitions are equivalent] We will denote in general $w = f(z)$. Consider, in the complex plane, two line segments γ_1 and γ_2 contained inside the set Ω where f is defined, and intersecting at a point z_0 in Ω. Denote by γ_1' and γ_2' their images by f.

We want to show that if the angle between γ_1 and γ_2 is equal to θ, then the same holds for their images, which means that the angle between the tangent lines to γ_1' and γ_2' at $w_0 = f(z_0)$ is also equal to θ.

Consider a point $z \in \gamma_1$ close to z_0. Its image $w = f(z)$ satisfies

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = f'(z_0),$$

and hence

$$\lim_{z \longrightarrow z_0} \operatorname{Arg}(w - w_0) - \operatorname{Arg}(z - z_0) = \operatorname{Arg} f'(z_0),$$

which shows that the angle between the curve γ_1' and the real axis is equal to the angle between the original segment γ_1 and the real axis, plus the angle $\alpha = \operatorname{Arg} f'(z_0)$ (which is well defined because $f'(z) \neq 0$).

Similarly, the angle between the image curve γ_2' and the real axis is equal to that between the segment γ_2 and the real axis, plus the same α.

Therefore, the angle between the two image curves is the same as that between the two line segments, namely, θ.

Another way to see this is as follows: the tangent vectors of the curves are transformed according to the rule $\overrightarrow{V}' = \mathrm{d}f_{z_0}\overrightarrow{V}$. But the differential of f (when f is seen as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$) is of the form

$$\mathrm{d}f_{z_0} = \begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{pmatrix} = \left|f'(z_0)\right| \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{1}$$

where α is the argument of $f'(z_0)$. This is the matrix of a rotation composed with a homothety, that is, a similitude.

Conversely, if f is a map which is $\mathbb{R}^2$-differentiable and preserves angles, then at any point $\mathrm{d}f$ is an endomorphism of $\mathbb{R}^2$ which preserves angles. Since f also preserves orientation, its determinant is positive, so $\mathrm{d}f$ is a similitude, and its matrix is exactly as in equation (1). The Cauchy-Riemann equations are immediate consequences. □

Rem. *An* antiholomorphic *map also preserves angles, but it reverses the orientation.*

Calcul différentiel

Pour obtenir la différentielle totale de cette expression, considérée comme fonction de x, y, ..., donnons à x, y, ... des accroissements dx, dy, ... Soient Δu, Δv, ..., Δf les accroissements correspondants de u, v, ...,f. On aura

$$\Delta f = \frac{\partial f}{\partial u}\Delta u + \frac{\partial f}{\partial v}\Delta v + \ldots + R\Delta u + R_1 \Delta v + \ldots,$$

R, R_1, ... tendant vers zéro avec Δu, Δv, ...

Mais on a, d'autre part,

$$\begin{aligned}
\Delta u &= \frac{\partial u}{\partial x}dx + + \frac{\partial u}{\partial y}\Delta y + \ldots + S\Delta x + S_1 \Delta y + \ldots \\
&= du + S dx + S_1 dy + \ldots \\
\Delta v &= \frac{\partial v}{\partial x}dx + + \frac{\partial v}{\partial y}\Delta y + \ldots + T\Delta x + T_1 \Delta y + \ldots \\
&= dv + T dx + T_1 dy + \ldots \\
&\ldots
\end{aligned}$$

S, S_1, ..., T, T_1,... tendant vers zéro avec dx, dy, ...

Substituant ces valeurs dans l'expression de Δf, il vient

$$\begin{aligned}
\Delta f &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \ldots + \rho dx + \rho_1 dy + \ldots \\
&= \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \ldots\right)dx \\
&\quad + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \ldots\right)dy \\
&\quad + \ldots + \rho dx + \rho_1 dy + \ldots
\end{aligned}$$

ρ, ρ_1, ... tendant vers zéro avec dx, dy, ...

On aura donc

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \ldots, \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \ldots, \\
&\ldots
\end{aligned}$$

et, d'autre part,

$$df = \frac{\partial f}{\partial u}\mathrm{d}u + \frac{\partial f}{\partial v}\mathrm{d}v + \dots;$$

d'où les deux propositions suivantes :

La dérivée, par rapport à une variable indépendante x, d'une fonction composée $f(u,v,\dots)$ s'obtient en ajoutant ensemble les dérivées partielles $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, ..., respectivement multipliées par les dérivées de u, v, ... par rapport à x.

La différentielle totale df s'exprimer au moyen de u, v, ..., du, dv, ..., de la même manière que si u, v, ... étaient des variables indépendantes.

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