

Example of the mdput fonts.

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Abstract

The package mdput consists of a full set of mathematical fonts, designed to be combined with Adobe Utopia as the main text font.

This example is extracted from the excellent book *Mathematics for Physics and Physicists*, W. APPEL, Princeton University Press, 2007.

1 Conformal maps

1.1 Preliminaries

Consider a change of variable $(x, y) \mapsto (u, v) = (u(x, y), v(x, y))$ in the plane \mathbb{R}^2 , identified with \mathbb{R} . This change of variable really only deserves the name if f is locally bijective (i.e., one-to-one); this is the case if the jacobian of the map is nonzero (then so is the jacobian of the inverse map):

$$\left| \frac{D(u, v)}{D(x, y)} \right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0 \quad \text{and} \quad \left| \frac{D(x, y)}{D(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0.$$

Theorem 1.1. *In a complex change of variable*

$$z = x + iy \mapsto w = f(z) = u + iv,$$

and if f is holomorphic, then the jacobian of the map is equal to

$$J_f(z) = \left| \frac{D(u, v)}{D(x, y)} \right| = |f'(z)|^2.$$

Dem. Indeed, we have $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ and hence, by the Cauchy-Riemann relations,

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} = J_f(z).$$

□

Definition 1.1. A conformal map or conformal transformation of an open subset $\Omega \subset \mathbb{R}^2$ into another open subset $\Omega' \subset \mathbb{R}^2$ is any map $f : \Omega \mapsto \Omega'$, locally bijective, that preserves angles and orientation.

Theorem 1.2. Any conformal map is given by a holomorphic function f such that the derivative of f does not vanish.

This justifies the next definition:

Definition 1.2. A conformal transformation or conformal map of an open subset $\Omega \subset \mathbb{R}^2$ into another open subset $\Omega' \subset \mathbb{R}^2$ is any holomorphic function $f : \Omega \mapsto \Omega'$ such that $f'(z) \neq 0$ for all $z \in \Omega$.

Dem.[that the definitions are equivalent] We will denote in general $w = f(z)$. Consider, in the complex plane, two line segments γ_1 and γ_2 contained inside the set Ω where f is defined, and intersecting at a point z_0 in Ω . Denote by γ'_1 and γ'_2 their images by f .

We want to show that if the angle between γ_1 and γ_2 is equal to θ , then the same holds for their images, which means that the angle between the tangent lines to γ'_1 and γ'_2 at $w_0 = f(z_0)$ is also equal to θ .

Consider a point $z \in \gamma_1$ close to z_0 . Its image $w = f(z)$ satisfies

$$\lim_{z \rightarrow z_0} \frac{w - w_0}{z - z_0} = f'(z_0),$$

and hence

$$\lim_{z \rightarrow z_0} \text{Arg}(w - w_0) - \text{Arg}(z - z_0) = \text{Arg } f'(z_0),$$

which shows that the angle between the curve γ'_1 and the real axis is equal to the angle between the original segment γ_1 and the real axis, plus the angle $\alpha = \text{Arg } f'(z_0)$ (which is well defined because $f'(z) \neq 0$).

Similarly, the angle between the image curve γ'_2 and the real axis is equal to that between the segment γ_2 and the real axis, plus the same α .

Therefore, the angle between the two image curves is the same as that between the two line segments, namely, θ .

Another way to see this is as follows: the tangent vectors of the curves are transformed according to the rule $\vec{V}' = df_{z_0} \vec{V}$. But the differential of f (when f is seen as a map from \mathbb{R}^2 to \mathbb{R}^2) is of the form

$$df_{z_0} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = |f'(z_0)| \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad (1)$$

where α is the argument of $f'(z_0)$. This is the matrix of a rotation composed with a homothety, that is, a similitude.

Conversely, if f is a map which is \mathbb{R}^2 -differentiable and preserves angles, then at any point df is an endomorphism of \mathbb{R}^2 which preserves angles. Since f also preserves orientation, its determinant is positive, so df is a similitude, and its matrix is exactly as in equation (1). The Cauchy-Riemann equations are immediate consequences. \square

Rem. *An antiholomorphic map also preserves angles, but it reverses the orientation.*

Calcul différentiel

Pour obtenir la différentielle totale de cette expression, considérée comme fonction de x, y, \dots , donnons à x, y, \dots des accroissements dx, dy, \dots . Soient $\Delta u, \Delta v, \dots, \Delta f$ les accroissements correspondants de u, v, \dots, f . On aura

$$\Delta f = \frac{\partial f}{\partial u} \Delta u + \frac{\partial f}{\partial v} \Delta v + \dots + R \Delta u + R_1 \Delta v + \dots,$$

R, R_1, \dots tendant vers zéro avec $\Delta u, \Delta v, \dots$

Mais on a, d'autre part,

$$\begin{aligned} \Delta u &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \dots + S \Delta x + S_1 \Delta y + \dots \\ &= du + S dx + S_1 dy + \dots \\ \Delta v &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \dots + T \Delta x + T_1 \Delta y + \dots \\ &= dv + T dx + T_1 dy + \dots \\ &\dots \end{aligned}$$

$S, S_1, \dots, T, T_1, \dots$ tendant vers zéro avec dx, dy, \dots

Substituant ces valeurs dans l'expression de Δf , il vient

$$\begin{aligned} \Delta f &= \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \dots + \rho dx + \rho_1 dy + \dots \\ &= \left(\frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \dots \right) dx \\ &\quad + \left(\frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \dots \right) dy \\ &\quad + \dots + \rho dx + \rho_1 dy + \dots \end{aligned}$$

ρ, ρ_1, \dots tendant vers zéro avec dx, dy, \dots

On aura donc

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \dots, \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \dots, \\ &\dots\end{aligned}$$

et, d'autre part,

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \dots;$$

d'où les deux propositions suivantes :

La dérivée, par rapport à une variable indépendante x , d'une fonction composée $f(u, v, \dots)$ s'obtient en ajoutant ensemble les dérivées partielles $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \dots$, respectivement multipliées par les dérivées de u, v, \dots par rapport à x .

La différentielle totale df s'exprime au moyen de $u, v, \dots, du, dv, \dots$, de la même manière que si u, v, \dots étaient des variables indépendantes.

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