The COS Method:

Defining a truncation range explicitly

Master of Quantitative Finance

Tom Haynes (14392144)

March 27, 2023

$$\int_{\mathbb{R}} v(x) f(x)\, dx$$

Table of Contents

The COS Method

- The COS method is a Fourier option pricing method.
- It is one of the fastest known methods for pricing options and calculating the greeks quickly and accurately.
- It is the fastest known fourier method.
- Calculating options quickly, often for a number of strikes at once, is a very important problem in computational finance

> *"...stock price models are typically calibrated to gives prices of liquid call and put options by minimizing the mean-square-error between model prices and given market prices. During the optimization routine, model prices of call and put options need to be evaluated very often for different model parameters."[1]*

The Basic Idea

The the problem the COS method solves is fairly straightforward given the following conditions:

Conditions

- **The stochastic component (underling price) is modelled by random variable X where in which the probability density function f is unknown.**
- **We do have a characteristic function of f explicitly, denoted φ.**
- **We have an option payoff $v(X)$. Then the integral we are trying to compute is the following as quickly as possible:**

$$\int_{\mathbb{R}} v(x) f(x) dx \tag{1}$$

Table of Contents

Fourier Cosine Expansion

2 Fang and Osterlee (2008)

Start with the density function $f(x)$ on truncated interval $[a, b]$, with $x \in [a, b] \subset \mathbb{R}$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right), \quad F_n = \frac{2}{b-a}\int_a^b f(x) \cos\left(n\pi\frac{x-a}{b-a}\right) dx$$

We have a direct relation of characteristic function to density function... where:

$$\varphi(u) := \int_{\mathbb{R}} f(x) e^{iux} dx, \quad \text{where} \quad [a, b] \quad \text{s.t.} \quad \int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0$$

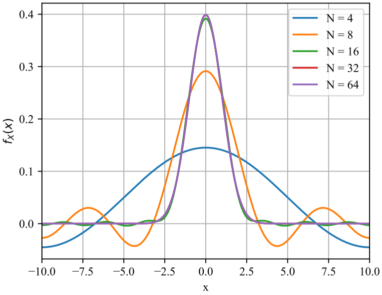
Then we can approximate F_n as the following:

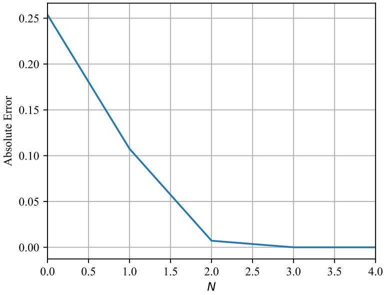
$$F_n \approx \frac{2}{b-a}\mathcal{R}\left(\left(\frac{n\pi}{b-a}\right) \exp\left(-i\frac{na\pi}{b-a}\right)\right)$$

Convergence Example

2 Fang and Osterlee (2008)

This method demonstrates exponential convergence in N. Below I demonstrate this with the normal probability distribution.





Pricing European Options

Starting with the risk neutral valuation formula - we derive the following:

$$V(x, t_0) = e^{-r\Delta t} \sum_{n=0}^{N-1}{}' \mathcal{R}\left(\varphi\left(\frac{n\pi}{b-a}; x\right) \cdot e^{\left(-in\pi\frac{a}{b-a}\right)}\right) H_n$$

These H_n are the payoff coefficients - these are discovered analytically. Fang and Osterlee define them for European options.There are three different directions for call and put coefficients for $a < 0 < b$.

$$H_k^{\text{call}} = \frac{2}{b-a} K(\chi_k(0, b) - \gamma_k(0, b)), \quad H_k^{\text{put}} = -\frac{2}{b-a} K(\chi_k(a, 0) - \gamma_k(a, 0))$$

where χ_k and γ_k are again complicated functions dependant entirely on the choice of the truncation range.

Table of Contents

3 Problems and Potential Solutions

Fang and Osterlee's Truncation Range

Fang and Osterlee suggest the use of cumulants - in Osterlee's book they define it as:

$$[a,b] = \left[(x+c_1) - L\cdot\sqrt{c_2+\sqrt{c_4}},\quad (x+c_1) + L\cdot\sqrt{c_2+\sqrt{c_4}}\right]$$

With $L \in [6,12]$, c_k being the cumulants of $\ln(S_0/K)$ in use, and $x = \ln(S_0)/K$

In this program, we are trying to rectify 2 main issues, those which arise from Fang and Osterlee's cumulants based definition of a truncation range.

1. Models where in which the COS method diverges (MJD).
2. Options with extreme moneyness.

The Common Thread

3 Problems and Potential Solutions

An underlying issue I have found in my research is the ambiguity surrounding the choice of truncation range of the COS expansion.

- *In Fang and Osterlee's seminal work they detail explicitly the use of the cumulants.*
- *In Mathematical Modeling and Computation in Finance, a book co-authored by Osterlee, he details this again, and offers the use of* $L \cdot \sqrt{T}$ *also.*

Yet among message boards and the like.. it still seems to be a point of contention, where often the solution is to divert away from the COS method. The benefit of doing this can be seen in Hirsa's results.

What I am looking to program here is a robust formula for the truncation ranges that can be easily implemented into python.

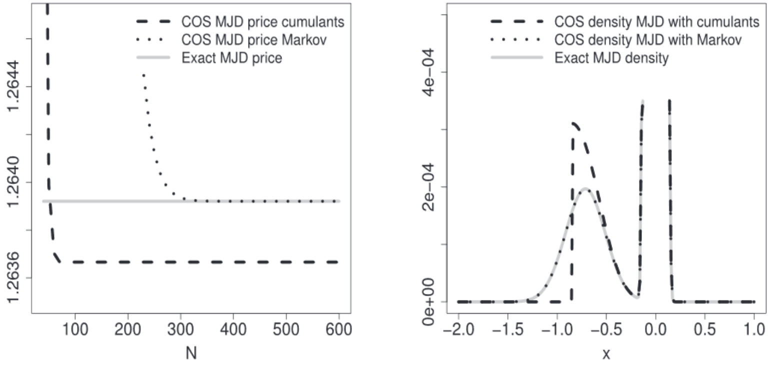
Divergence in the MJD Model - I

The Problem

- This is essentially because the cumulants range is not large enough to accommodate the jump size in the jump-diffusion model.
- i.e., under this model, if the jump size exceeds the bounds of the cumulants defined range (given there are 3 to select from)
- In Junike and Pankasharin's example using 6 cumulants, the method will fail to converge.

Divergence of MJD Model - II

$T = 0.1, \sigma = 0.1, \nu = 0.001, \kappa = -0.5, \delta = 0.2$ Jump Size $= -7$, $[a, b] = [-5.8, 5.8]$



Panel A Panel B

The Solution:

In Junike and Pankasharin's paper they discuss another proof to the COS method, that involves using Markov's inequality. This results in the following process (in summary):

- Choose a truncation range that is centred around 0. I.e. $a = -b$.
- Let μ_n be the n^{th} moment of the characteristic function in question.
- Use the following formula for M to define this truncation range $[-M, M]$.

$$M = \sqrt[n]{\frac{2K\mu_n}{\varepsilon}}, \quad \varepsilon \in [10^{-3}, 10^{-8}], \quad \{n \in \mathbb{Z} \mid 2 \leq n \leq 8\}$$

Extreme Moneyness - I

Problem:

Problems with extreme moneyness are well documented. Below I describe two instances of this:

- Divergence in GBM
- Divergence in Heston Stochastic Volatility Model

Le F'loch describes that the root of this innacuracy is that in Fang and Osterlee's calculation of cosine coefficients, described on slide 8, the coefficients are computed relative to the strike price K, but the truncation range is relative to the spot price.

Solution:

Fabien Le Floc'h derives a new option pricing formula that includes the forward prices - from here the recomendation is to change the truncation range as follows:

$$[a, b] \Rightarrow \left[a - \ln\frac{K}{F}, b - \ln\frac{K}{F}\right], \quad F(0, T) = S_0 e^{(e-q)\cdot T}$$

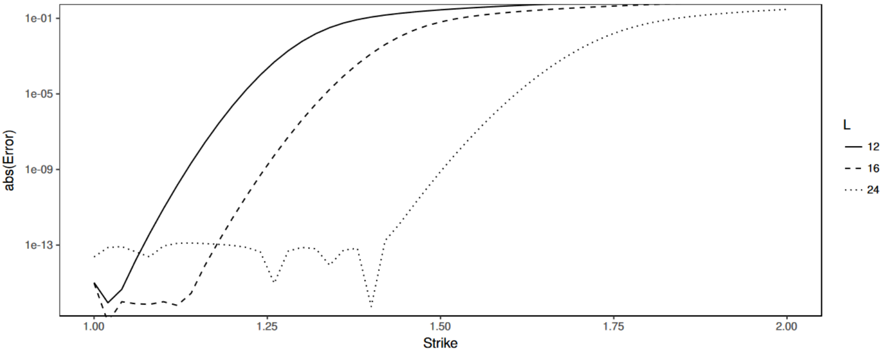
Extreme Moneyness - II - (Hirsa)

$$S_0 = 100, r = 0.05, T = 1/12, \sigma = 0.25$$

K	BS	COS	FrFFT	FFT	SP
10	90.0830	94.4112	90.0829	90.0830	90.0832
20	80.1660	82.5614	80.1660	80.1660	80.1483
30	70.2490	70.2490	70.2490	70.2490	69.0518
40	60.3319	60.3319	60.3319	60.3319	60.3529
50	50.4149	50.4149	50.4149	50.4149	50.4144
60	40.4979	40.4979	40.4979	40.4979	40.4977
70	30.5809	30.5809	30.5808	30.5809	30.5809
80	20.6651	20.6651	20.6650	20.6651	20.6650
90	10.9147	10.9147	10.9149	10.9147	10.9148
100	3.3006	3.3006	3.3004	3.3006	3.3004
110	0.4182	0.4182	0.4182	0.4182	0.4182
120	0.0207	0.0207	0.0207	0.0207	0.0207
130	4.42e-04	4.42e-04	4.52e-04	4.42e-04	4.42e-04
140	4.69e-06	4.42e-06	-2.05e-05	4.57e-06	4.69e-06
150	2.82e-08	-3.49e-05	2.85e-05	2.29e-07	2.82e-08
160	1.08e-10	-1.57e-03	-1.55e-05	-4.47e-08	1.08e-10
170	2.86e-13	-2.93e-02	-3.82e-06	-7.30e-08	2.89e-13
180	5.72e-16	-2.66e-01	1.18e-05	1.10e-07	2.57e-21
190	9.13e-19	-1.36e+00	-3.11e-06	-1.06e-07	-4.11e-24
200	1.22e-21	-4.45e+00	-6.58e-06	9.37e-08	2.39e-27

Extreme Moneyness - III - (Floc'h)

$$T = 2/365, \kappa = 1, \rho, \theta = 0.1, \sigma = 1, v_0 = 0.1, F = 1$$



Implementation - I

Start with GBM Stock Price Model and Hirsa's Results:

1. Start with an implementation of the COS method using the classical cumulants technique - compare results to Hirsa and BS model results (obviously using the same parameters).
2. Calculate first 8 moments of the characteristic function. Find where difference in M becomes negligible - that is the choice of M.
3. Perform step 1 with new cumulant ranges.

MJD Model:

1. Same as above, but with new characteristic function. Use classical cumulants based truncation range.
2. Alter this to the new Markov Inequality based values for $[a, b]$.
3. Compare results to Junike and Pankasharin.

Implementation - II