CS 5830
Cryptography

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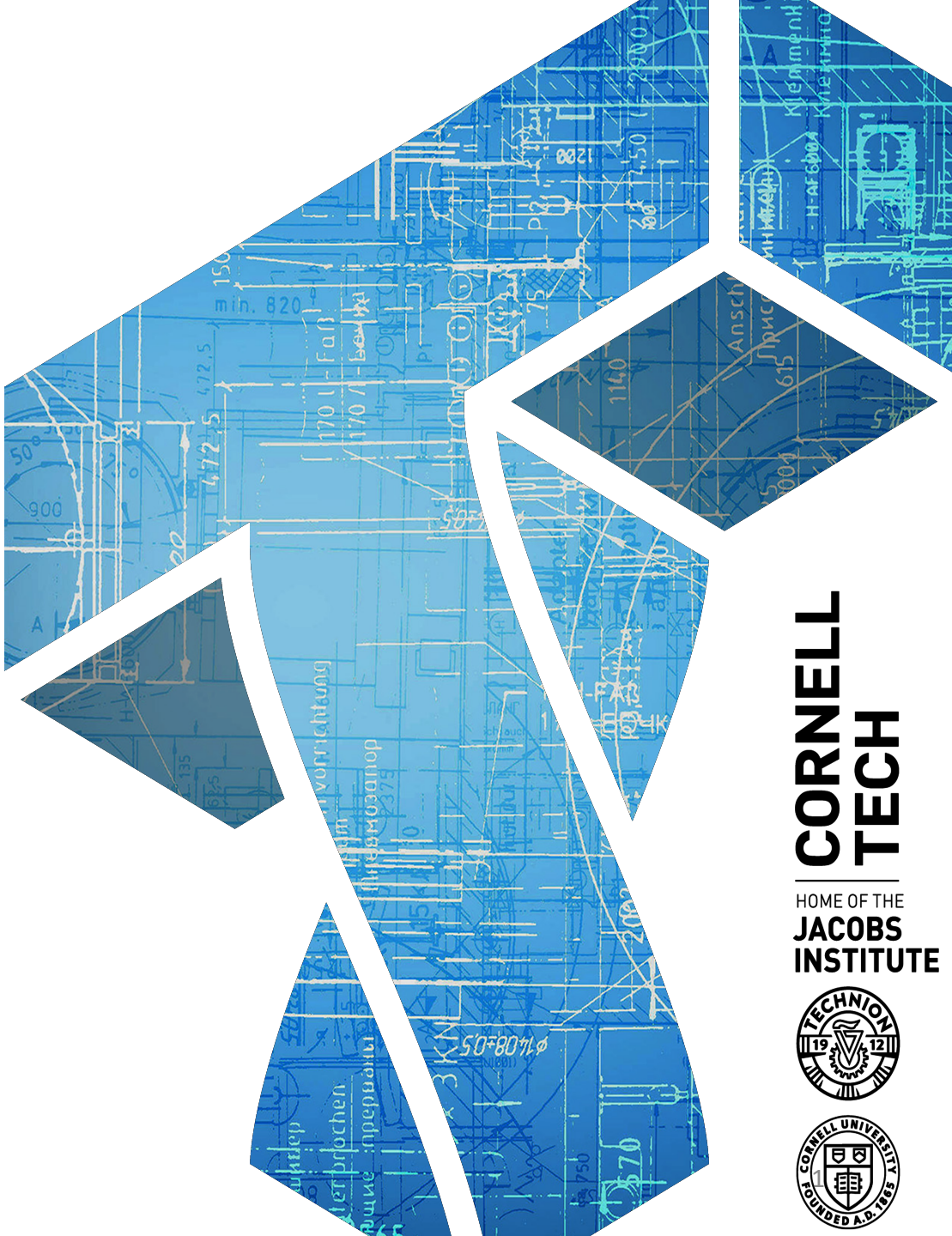

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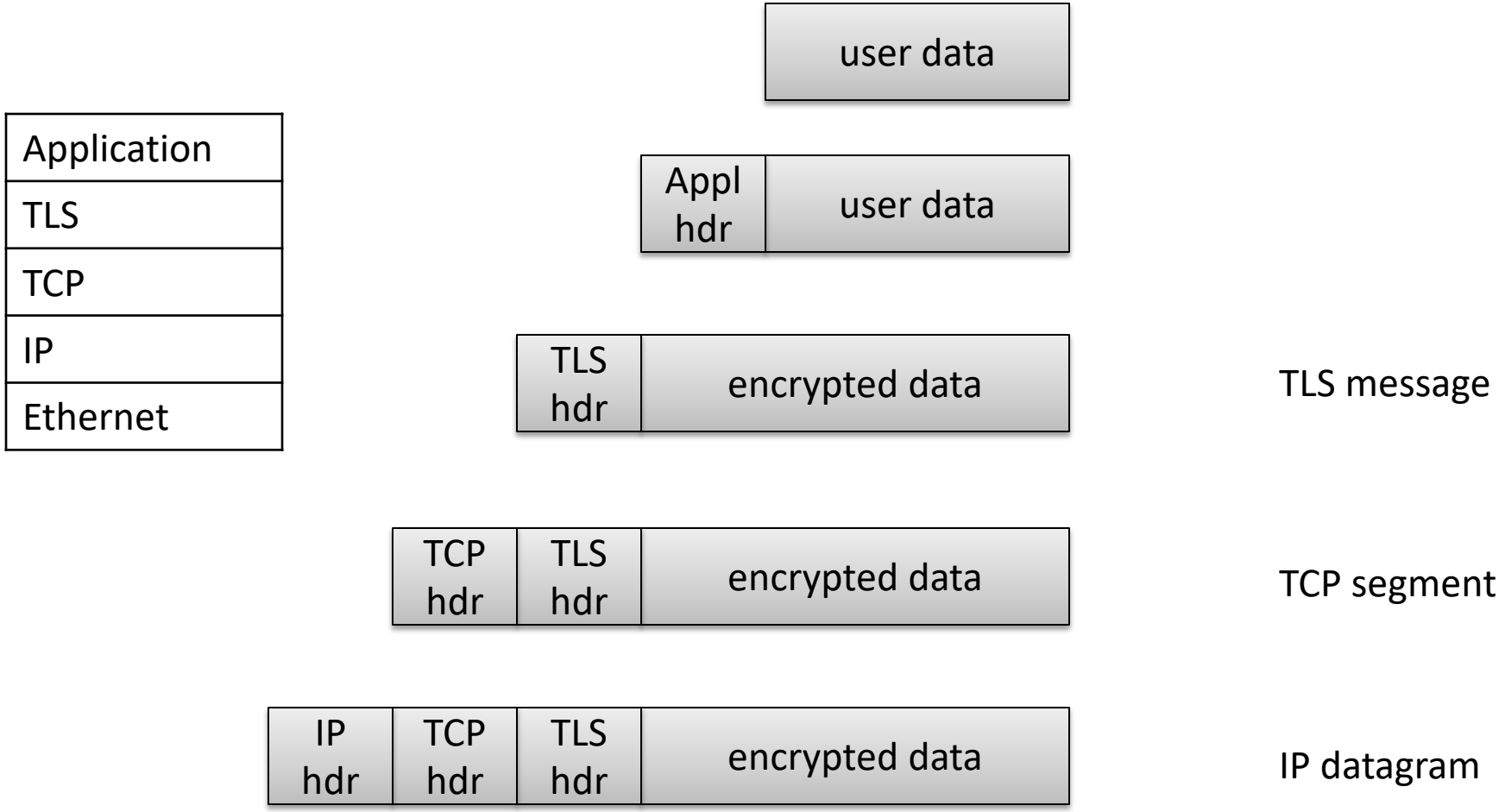
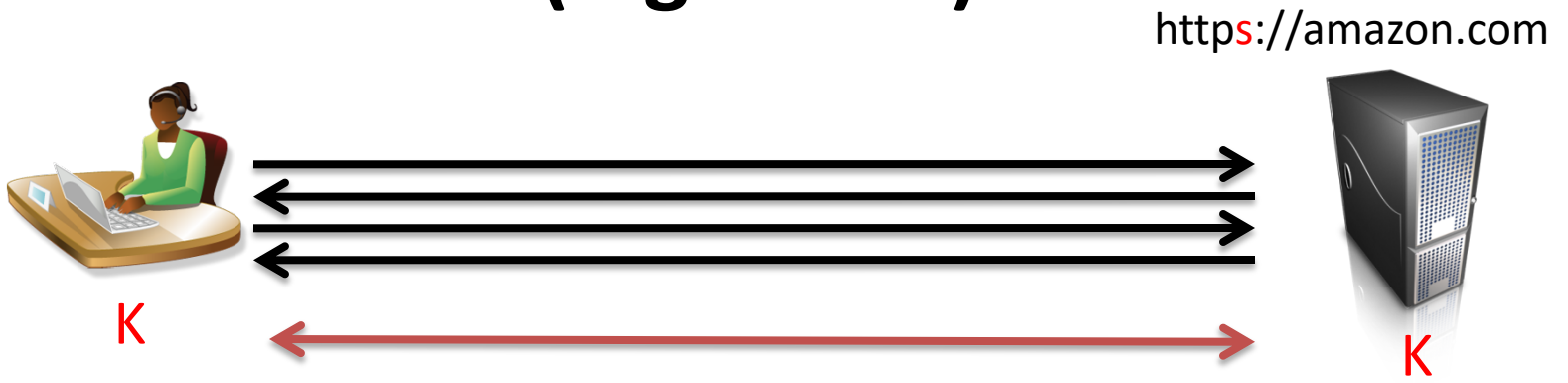
Background on TLS

- Secure Sockets Layer (SSL) designed at Mozilla in 1990s
  - SSL 1.0, 2.0, 3.0
  - Provide point-to-point encryption solution that can be used to protect sensitive data on the web (and elsewhere)
- Renamed to Transport Layer Security (TLS) in late 1990s
  - TLS 1.0, 1.1, 1.2, 1.3
  - 1.2 and before: multiple key exchange mechanisms including RSA key transport, Diffie-Hellman key exchange
  - 1.3: DH-only plus lower latency handshakes

TLS sits between application layer (HTTP) and TCP



How does TLS work (high level)?



Step 1:
Key exchange protocol to share secret K

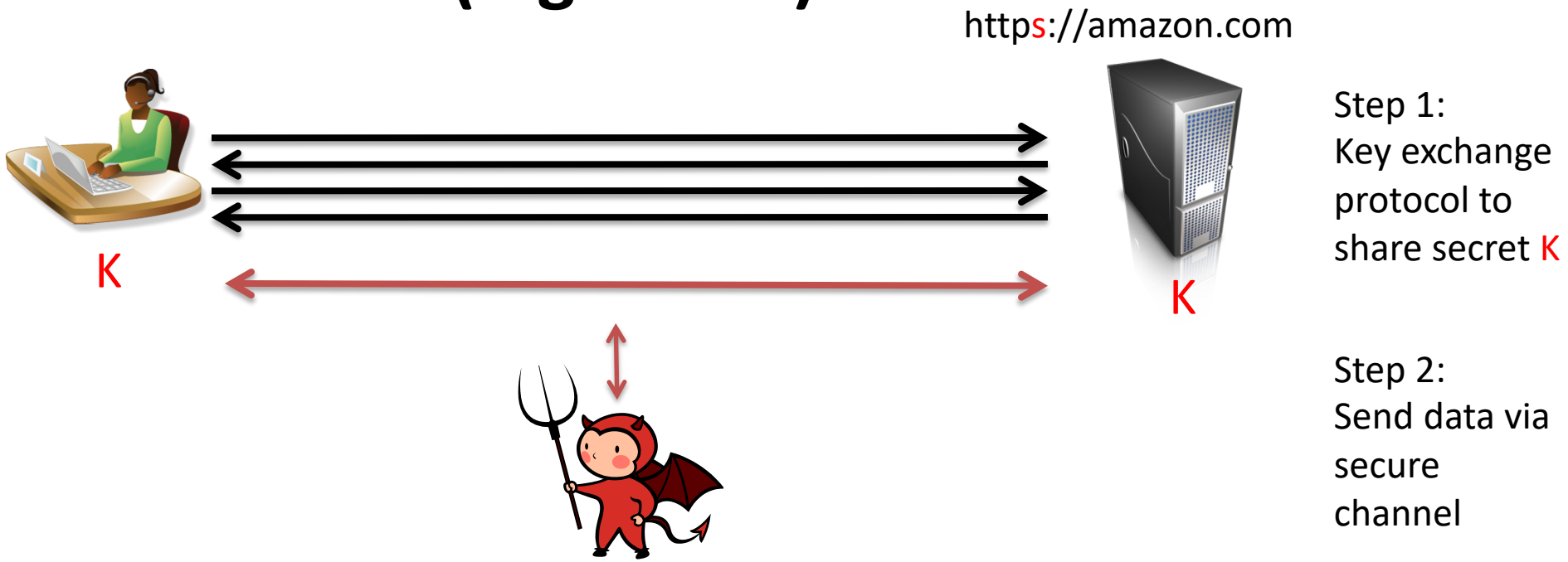
Step 2:
Send data via secure channel

The secure channel is implemented via our now familiar symmetric encryption primitives

Goals of handshake:

- Negotiate version
- Negotiate parameters (crypto algorithms to use)
- Authenticate server (Is server actually Amazon.com?)
  - Digital signatures and certificates
- Establish shared secret
  - Asymmetric cryptography

How does TLS work (high level)?



Security goals:

1. Resist passive adversaries
2. Resist active meddler-in-the-middle (often called man-in-the-middle)
3. Forward secrecy: old sessions not decryptable even if server later compromised

TLS 1.2 handshake for RSA transport

Client

Server

Pick random Nc

ClientHello, MaxVer, Nc, Ciphers/CompMethods →

Pick random Ns

← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod

Check CERT using CA public verification key

← CERT = (pk_s , signature over it)

Pick random PMS
C <- E(pk,PMS)

C →

PMS <- D(sk,C)

ChangeCipherSpec,
{ Finished, PRF(MS, “Client finished” || H(transcript)) } →

ChangeCipherSpec,
{ Finished, PRF(MS, “Server finished” || H(transcript’)) } ←

Bracket notation means contents encrypted

MS <- PRF(PMS, “master secret” || Nc || Ns)

TLS 1.2 key derivation & use

MS <- PRF(PMS, “master secret” || Nc || Ns)

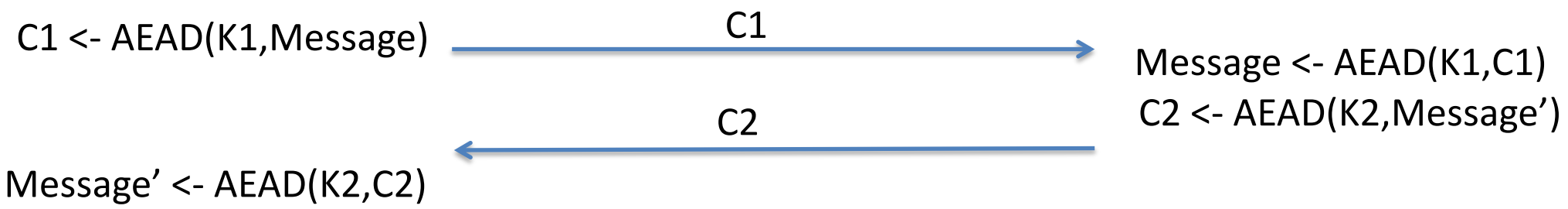
K1,K2 <- PRF(MS, “key expansion” || Ns || Nc)

concatenation

PRF(secret, seed) = HMAC-HASH(secret, A(1) + seed) +
HMAC-HASH(secret, A(2) + seed) +
HMAC-HASH(secret, A(3) + seed) + ...

Replaced with HKDF in TLS 1.3

Where A(0) = seed and A(i) = HMAC_hash(secret, A(i-1))



Record layer details

- Fragmentation
  - Maximum TLS ciphertext handles 2^{14} bytes of message data
  - Split longer message into multiple fragments
  - Encrypt each fragment separately
- Sequence numbers keep track of count of fragments sent in each direction
- Compression methods
- AEAD used in TLS 1.2 and before vulnerable to padding oracle attacks, or RC4-abuse attacks
  - HMAC-then-CBC
  - HMAC-then-RC4

TLS record layer attacks

Attack	Year	Vulnerability	Countermeasure
Vaudenay	2002	Padding oracle (theoretical)	---
Rogaway	2002	IV chaining (theoretical)	---
Kelsey	2002	Compression before encryption (theoretical)	---
Canval et al.	2003	Padding oracle via timing	Always compute HMAC
BEAST (Duong & Rizzo)	2011	IV chaining	Dedicated IVs
CRIME (Duong & Rizzo)	2012	TLS compression before encryption	Turn off TLS compression
Lucky13 (AlFardan & Paterson)	2013	Padding oracle via HMAC timing	Constant-time decryption attempted; move to RC4
RC4 attack (AlFardan et al.)	2013	RC4 cryptanalysis made practical	Move to CBC-based cipher suites
BREACH (Prado et al.)	2013	HTTP compression before encryption	Turn off HTTP compression (if possible)

TLS handshake for RSA transport

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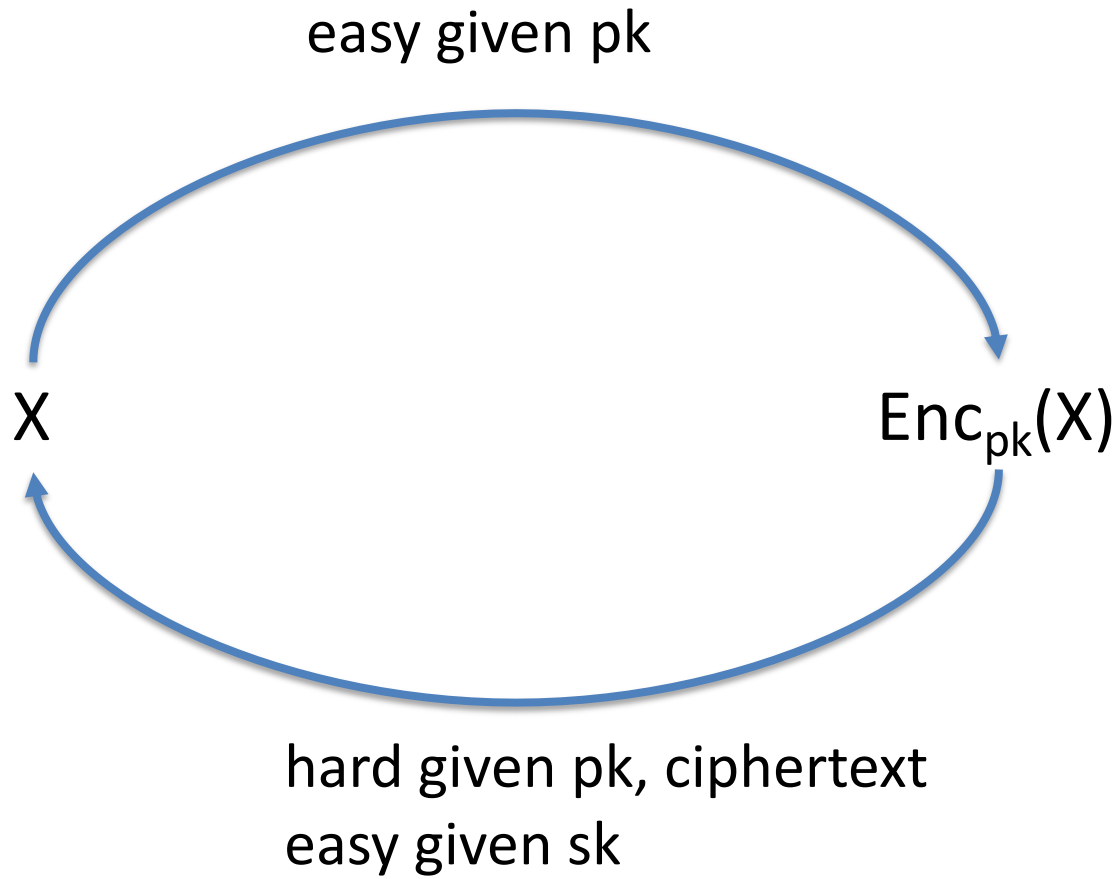
MS <- PRF(PMS, “master secret” || Nc || Ns)

Security against passive adversaries rely on inability to invert asymmetric encryption



The RSA trapdoor function

- Rivest, Shamir, Adleman 1978
- Garnered them a Turing award

RSA math

Let N be a positive number

Looking ahead: N = pq for large primes p,q

N will be called the modulus

p = 7, q = 13, gives N = 91

p = 17, q = 53, gives N = 901

RSA math

Let N be a positive number

Looking ahead: N = pq for large primes p,q

N will be called the modulus

$\mathbf{Z}_N = \{0,1,2,3,\ldots, N-1\}$

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \text{ and } i < N\}$

gcd(X,Y) = 1 if greatest common divisor of X,Y is 1

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

N = 13 $\quad \mathbf{Z}_{13}^* = \{ 1,2,3,4,5,6,7,8,9,10,11,12 \}$

N = 15 $\quad \mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$

The size of a set S is denoted by $|S|$

Def. $\phi(N) = |\mathbf{Z}_N^*|$ (This is Euler's totient function)

$\phi(13) = 12$

$\phi(15) = 8$

$\mathbf{Z}_{\phi(15)}^* = \mathbf{Z}_8^* = \{ 1,3,5,7 \}$

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

Fact. For any a,N with N > 0, there exists unique q,r such that

$$a = Nq + r \qquad \text{and} \qquad 0 \le r < N$$

$$17 \bmod 15 = 2 \qquad 105 \bmod 15 = 0$$

Def. $a \bmod N = r \in \mathbf{Z}_N$

Def. $a \equiv b \pmod N$ iff $(a \bmod N) = (b \bmod N)$

Operations work in natural way:

$$a \bullet b \bmod N \qquad a+b \bmod N$$

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

$(\mathbf{Z}_N^* , \bullet)$ is a **group** where $\bullet$ denotes multiplication mod N

Group $(G,\bullet)$ is a set G and operator $\bullet$ that satisfy:

1. *Closure*: for all $a,b \in G$ it holds that $a \bullet b \in G$
2. *Associativity*: for all $a,b,c \in G$ it holds that $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
3. *Identity*: Exists $I \in G$ s.t. for all $a \in G$ $a \bullet I = a$
4. *Inverses*: for $a \in G$ there exists $a^{-1} \in G$ s.t. $a \bullet a^{-1} = I$

Abelian group is additionally commutative:

for all $a,b \in G$ it holds that $a \bullet b = b \bullet a$

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

$(\mathbf{Z}_N^* , \bullet)$ is a **group**

$$\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$$

$2 \bullet 7 \equiv 14 \pmod{15}$

$4 \bullet 8 \equiv 2 \pmod{15}$

Closure: for any $a,b \in \mathbf{Z}_N^*$ $\quad a \bullet b \bmod N \in \mathbf{Z}_N^*$

Def. $a^i \bmod N = \underbrace{a \bullet a \bullet a \bullet \ldots \bullet a}_{i \text{ times}} \bmod N$

Some needed algorithms

Algorithm	**Running time (n = log N)**
Modular multiplication ab mod N	$O(n^2)$
Modular exponentation a^i mod N	$O(n^3)$
Modular inverse a^{-1} mod N	$O(n^2)$

Textbook exponentiation

Let G be a group.
How do we compute h^x for any $h \in G$?

```
Exp(h,x)
X' = h
For i = 2 to x do
        X' = X'•h mod N
Return X'
```

Requires time O(|G|) in worst case.

```
SqrAndMulExp(h,x)
b_k,…,b_0 = x
f = 1
For i = k down to 0 do
            f = f^2 mod N
            If b_i = 1 then
                        f = f•h mod N
Return f
```

Requires time O(k) multiplies and squares in worst case.

SqrAndMulExp(h,x)

$b_k,\ldots,b_0 = x$

f = 1

For i = k down to 0 do

    f = f^2 mod N

    If b_i = 1 then

        f = f•h mod N

Return f

$$x = \sum_{b_i \neq 0} 2^i$$

$$h^x = h^{\sum_{b_i \neq 0} 2^i} = \prod_{b_i \neq 0} h^{2^i}$$

$$h^{11} = h^{8+2+1} = h^8 \bullet h^2 \bullet h$$

$b_3 = 1$ $f_3 = 1 \bullet h$

$b_2 = 0$ $f_2 = h^2$

$b_1 = 1$ $f_1 = (h^2)^2 \bullet h$

$b_0 = 1$ $f_0 = (h^4 \bullet h)^2 \bullet h = h^8 \bullet h^2 \bullet h$

Don't implement this algorithm: side-channel attacks

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

Claim: Suppose $e,d \in \mathbf{Z}_{\phi(N)}^*$ satisfying $ed \bmod \phi(N) = 1$
then for any $x \in \mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$$\begin{aligned}(x^e)^d \bmod N &= x^{(ed \bmod \phi(N))} \bmod N \\ &= x^1 \bmod N \\ &= x \bmod N\end{aligned}$$

First equality is
by Euler's Theorem:
$x^{\phi(N)} \bmod N = x \bmod N$

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

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then for any $x \in \mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$ $\mathbf{Z}_{\phi(15)}^* = \{ 1,3,5,7 \}$

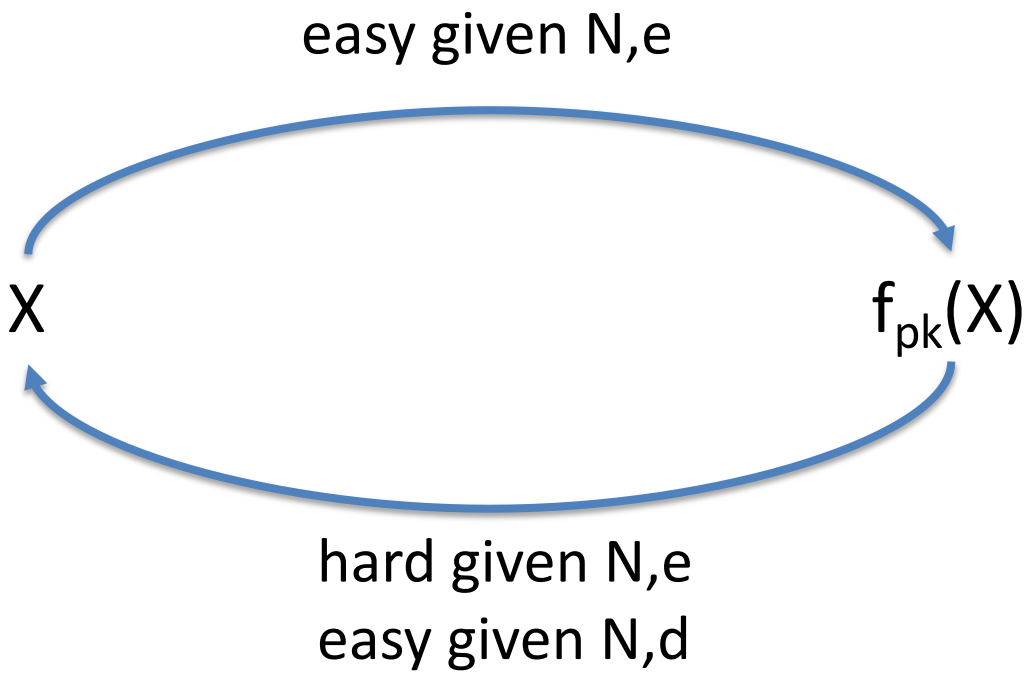
$e = 3$, $d = 3$ gives $ed \bmod 8 = 1$

x	1	2	4	7	8	11	13	14
$x^3 \bmod 15$	1	8	4	13	2	11	7	14
$y^3 \bmod 15$	1	2	4	7	8	11	13	14

The RSA trapdoor permutation

pk = (N,e) sk = (N,d) with ed mod $\phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$



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$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

But how do we find suitable N,e,d ?

If p,q distinct primes and N = pq then $\phi(N) = (p-1)(q-1)$

Why?

$\phi(N) = |\{1,\ldots,N-1\}| - |\{ip : 1 \le i \le q-1\}| - |\{iq : 1 \le i \le p-1\}|$

$= N-1 - (q-1) - (p-1)$

$= pq - p - q + 1$

$= (p-1)(q-1)$

The RSA trapdoor permutation

pk = (N,e) sk = (N,d) with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

But how do we find suitable N,e,d ?

If p,q distinct primes and N = pq then $\phi(N) = (p-1)(q-1)$

Given $\phi(N)$, choose $e \in \mathbf{Z}^*_{\phi(N)}$ and calculate

$d = e^{-1} \bmod \phi(N)$

How to find suitable p,q prime?

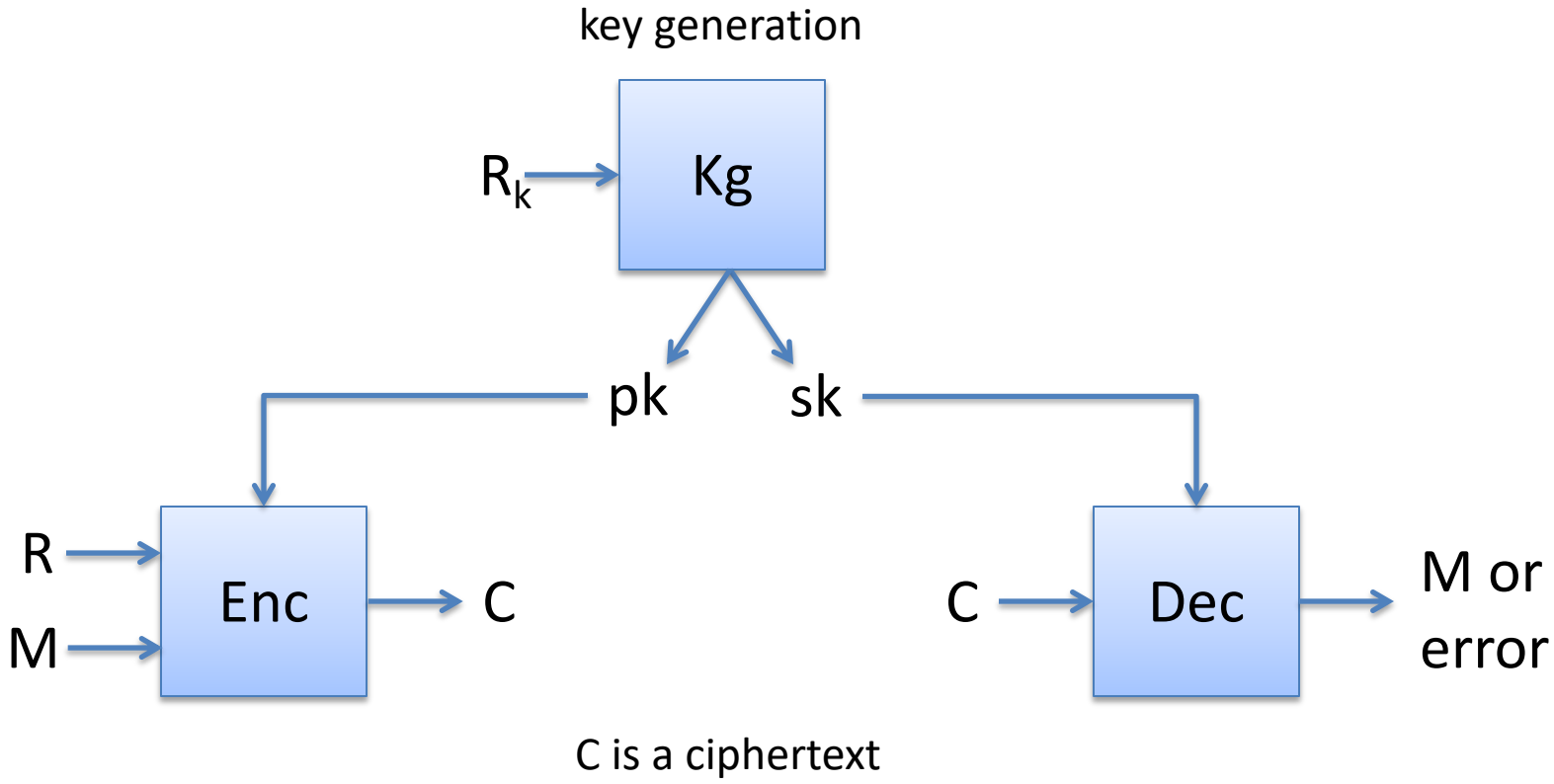
Choose random numbers and test primality (Miller-Rabin testing)

https://eprint.iacr.org/2018/749.pdf

Summary

- Find 2 large primes p, q . Let N = pq
  - random integers + primality testing
- Choose e (usually 65,537)
  - Compute d using $\phi(N) = (p-1)(q-1)$
- pk = (N,e) and sk = (N,d)
  - Often store p,q with sk to use Chinese Remainder Theorem

Public-key encryption



Correctness: D(sk , E(pk,M,R)) = M with probability 1 over randomness used

Textbook RSA

- Why not just use $M^e \bmod N$ directly to encrypt?
- Lots of reasons:
  - It is deterministic!
  - Ciphertext leaks Jacobi symbol of M (partial information revealed)
  - Meet-in-the-middle attack for short M that requires $2^{|M|/2}$ time and space (e.g., |M| = 64)
  - Malleable (we'll return to chosen-ciphertext attacks later on)
  - Small e attacks.
    - e = 3 and M smallish, $M^3 < N$, solve by taking 3rd root of M^3

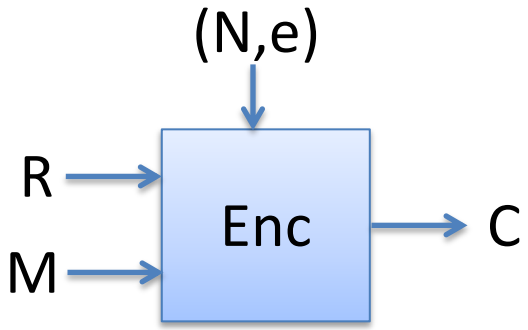
https://crypto.stanford.edu/~dabo/papers/RSA-survey.pdf

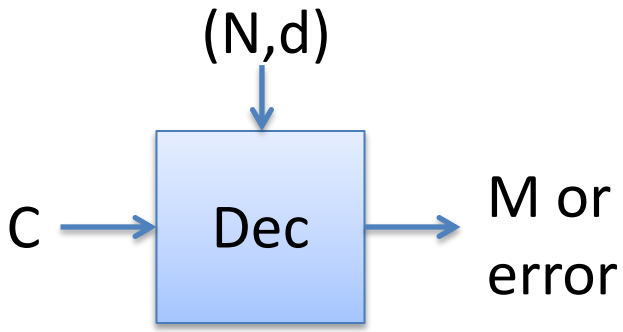
PKCS #1 RSA encryption

Kg outputs (N,e),(N,d) where $|N|_8 = n$

Let $B = \{0,1\}^8 / \{00\}$ be set of all bytes except 00

Want to encrypt messages of length $|M|_8 = m$





Enc((N,e), M, R)

pad = first n - m - 3 bytes from R that are in B

X = 00 || 02 || pad || 00 || M

Return X^e mod N

Dec((N,d), C)

$X = C^d \bmod N$; aa||bb||w = X

If (aa ≠ 00) or (bb ≠ 02) or ($00 \notin w$)

  Return error

pad || 00 || M = w

Return M

TLS handshake for RSA transport

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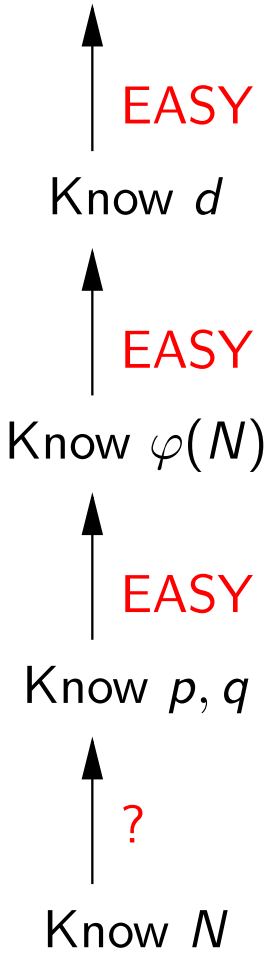
Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?

Inverting RSA : given N, e, y find x such that $x^e \equiv y \pmod{N}$



because $f^{-1}(y) = y^d \bmod N$

because $d = e^{-1} \bmod \varphi(N)$

because $\varphi(N) = (p-1)(q-1)$

Learning p,q from N is the factoring problem

We don't know if inverse is true, whether inverting RSA implies ability to factor

Factoring composites

- What is p,q for N = 901?

```
Factor(N):
for i = 2 , … ,  sqrt(N) do
   if N mod i = 0 then
          p = i
          q = N / p
          Return (p,q)
```

Woops… we can always factor

But not always efficiently:
Run time is sqrt(N)

$O(\text{sqrt}(N)) = O(e^{0.5 \ln(N)})$

Factoring composites

Algorithm	**Time to factor N**
Naïve	$O(e^{0.5 \ln(N)})$
Quadratic sieve (QS)	$O(e^{c})$ $c = d (\ln N)^{1/2} (\ln \ln N)^{1/2}$
Number Field Sieve (NFS)	$O(e^{c})$ $c = 1.92 (\ln N)^{1/3} (\ln \ln N)^{2/3}$

Factoring records

Challenge	Year	Algorithm	Time
RSA-400	1993	QS	830 MIPS years
RSA-478	1994	QS	5000 MIPS years
RSA-515	1999	NFS	8000 MIPS years
RSA-768	2009	NFS	~2.5 years
RSA-512	2015	NFS	$75 on EC2 / 4 hours

RSA-x is an RSA challenge modulus of size x bits

Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?
  - Pick |N| > 1024 and factoring will fail
  - Active attacks?

Bleichanbacher attack

C_1

I've just learned some information about $C_1^d \bmod N$

padding error?

C_2

padding error?

…

Dec((N,d), C)
X = $C^d \bmod N$; aa||bb||w = X
If (aa ≠ 00) or (bb ≠ 02) or (00 ∉ w)
Return error
pad || 00 || M = w
Return M

We can take a target C and decrypt it using a sequence of chosen ciphertexts $C_1, \ldots, C_q$ where q ≈ 1 million

[Bardou et al. 2012] q = 9400 ciphertexts on average

Response to this attack

- Ad-hoc fix: Don’t leak whether padding was wrong or not
  - This is harder than it looks (timing attacks, control-flow side channel attacks, etc.)
- Better:
  - use chosen-ciphertext secure encryption
  - OAEP is common choice

Summary

- RSA is example of trapdoor one-way function
  - Security conjectured. Relies on factoring being hard
- RSA security scales somewhat poorly with size of primes
- RSA PKCS#1 v1.5 is insecure due to padding oracle attacks. Don’t use it in new systems.
  - Use OAEP instead