Exercise Set 2

1.1 Lotka-Volterra model

If the prey's reproductive term is substituted by a more realistic logistic term, the populations of prey $x(t)$ and predators $y(t)$ are described by the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \alpha(1-x)x - c\,x\,y = -\alpha x^2 + \alpha x - c\,x\,y$$
$$\frac{\mathrm{d}}{\mathrm{d}t}y = -b\,y + d\,x\,y = y(dx-b)$$

wherein $\alpha, b, c, d \in \mathbb{R}^+$.

a) Linear stability analysis

The first and trivial fix point is - as in the normal model - found to be the origin:

$$\vec{x}_1^* = \begin{pmatrix} x_1^* \\ y_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The second fix point is obtained by solving the bracket in $\dot{y} \overset{!}{=} 0$ for x^* and plugging into $\dot{x}$:

$$\vec{x}_2^* = \begin{pmatrix} \frac{b}{d} \\ \frac{\alpha}{c}\,(1-\frac{b}{d}) \end{pmatrix}$$

It is apparent, that the fix point can just exist in the physically senseful regime of positive populations $x, y \geq 0$, so only if $b/d < 1 \Leftrightarrow b < d$ in order to get a positive y_2^* . This also makes sense considering that $x(t) = N/K$ should be lower equal 1 in a logisitcal model with finite capacity K.
For stability analysis, we study a linearized system around the fix points

$$\vec{x}(t) = \vec{x}^* + \vec{u}(t) \qquad \Rightarrow \qquad \dot{\vec{u}} \approx \mathbf{A}\cdot\vec{u} \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} \frac{\delta\dot{x}}{\delta x} & \frac{\delta\dot{x}}{\delta y} \\ \frac{\delta\dot{y}}{\delta x} & \frac{\delta\dot{y}}{\delta y} \end{pmatrix}_{\vec{x}=\vec{x}^*}.$$

In this model we get

$$\mathbf{A} = \begin{pmatrix} -2\alpha\,x^* + \alpha - c\,y^* & -c\,x^* \\ d\,y^* & -b + d\,x^* \end{pmatrix}.$$

For the first fix point $\vec{x_1}^* = (0,0)$, the matrix becomes

$$\mathbf{A}_1 = \begin{pmatrix} \alpha & 0 \\ 0 & -b \end{pmatrix} \qquad \Rightarrow \qquad \lambda_1 = \alpha, \;\; \lambda_2 = -b,$$

which corresponds to a **saddle** with a stable manifold along the y-axis and an unstable manifold along the x-axis.

For the second fix point, we get

$$\mathbf{A}_2 = \begin{pmatrix} -2\alpha\,\frac{b}{d} + \alpha - \alpha\left(1-\frac{b}{d}\right) & -c\,\frac{b}{d} \\ \frac{d\alpha}{c}\left(1-\frac{b}{d}\right) & 0 \end{pmatrix} = \begin{pmatrix} -\alpha\,\frac{b}{d} & -c\,\frac{b}{d} \\ \frac{\alpha}{c}\,(d-b) & 0 \end{pmatrix}$$

Without explicitly calculating the eigenvalues the stability can directly be evaluated by a look on the trace and determinant of the matrix (see `Strogatz` p.138):

$$\tau = \mathrm{trace}(\mathbf{A}) = \frac{-\alpha\, b}{d} \qquad \Delta = \det(\mathbf{A}) = \frac{\alpha\, b}{d}\,(d-b)\,.$$

Since $\tau < 0$ and $\Delta > 0$ for $d > b$ (as required for the fix point anyway), the fix point is found to be **stable**. The borderline between a node (i.e. two negative real eigenvalues) and a spiral (i.e. two complex (conjugate) eigenvalues) is described by $\tau^2 - 4\Delta = 0$. Plugging in τ and Δ yields the following parameter conditions for the different kinds of fix points.

$$\begin{aligned} \text{stable node} &\ \text{ for } \alpha > \alpha_0 \\ \text{degenate node} &\ \text{ for } \alpha = \alpha_0 \qquad\qquad \text{with } \ \alpha_0 = 4\frac{d}{b}(d-b) \\ \text{stable spiral} &\ \text{ for } \alpha < \alpha_0 \end{aligned}$$

These cases are (maybe a bit more intuitively) reflected by a look on the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) \ = \ -\frac{\alpha\, b}{2d} \pm \frac{\alpha\, b}{2d}\sqrt{1 - \frac{4}{\alpha}\cdot\frac{d}{b}\,(d-b)}$$

For $\alpha > \alpha_0$, the square root is real, yielding two negative eigenvalues and thus a stable node. For $\alpha < \alpha_0$ the square root becomes imaginary, which leads to complex eigenvalues and thus a spiral. In the borderline case $\alpha = \alpha_0$, the square root becomes zero, giving just one eigenvalue and thus a degenerate node (due to just one eigenvector).

b) Computed vector fields and phase portraits

These analytical results are well confirmed for the different parameter regimes by the following vector field plots and phase portraits, generated with python's `matplotlib`.

$\underline{\alpha = \mathbf{16},\ \mathbf{b = 2},\ \mathbf{c = 16},\ \mathbf{d = 4}}$

degenerate.pdf