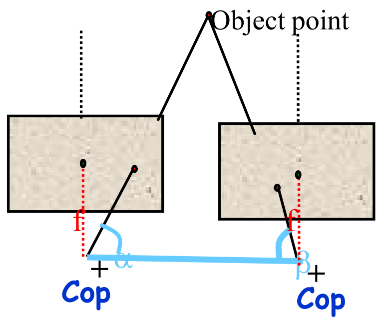
Geometry

Class 5

Last Class

- Traingulation in a rectified image
- Correspondence
- From 3D to 2D: projection matrix



Stereo Main Issues

- Correspondence
  - Geometric constraints
  - Local information
  - Global information
- Calibration
- Reconstruction (triangulation)

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3D to 2D: $\tilde{p} = M\tilde{P}$

- Object + image are in the camera's coordinate system: $M = M_{int}$

$$M_{int} = \begin{pmatrix} s_x f & 0 & o_x & 0 \\ 0 & s_y f & o_y & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Object – general coordinate system:

$$M = M_{int} M_{ext}$$

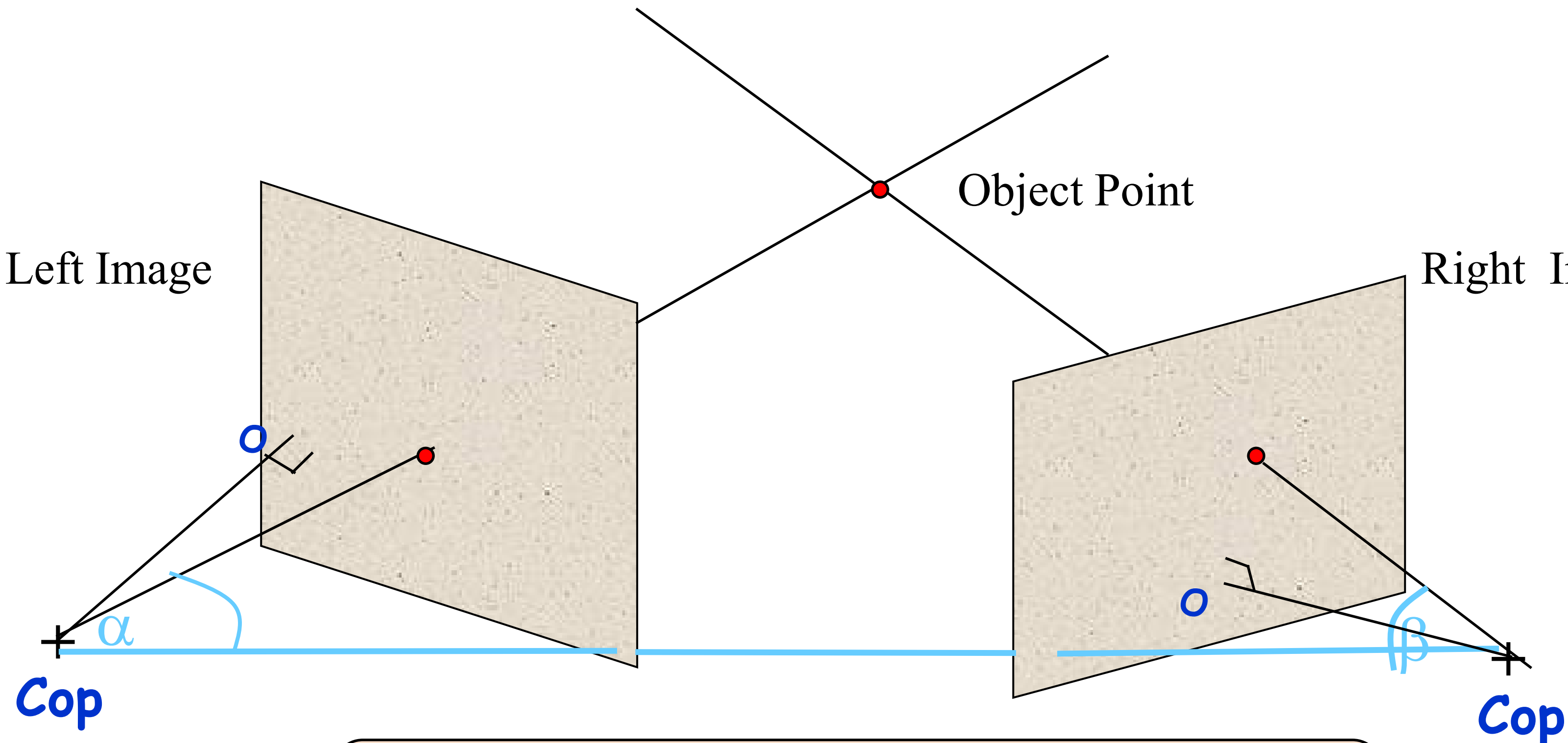
- $M_{ext} = \begin{pmatrix} & R & & -RT \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Next

- Strereo – general case
- Epipolar geometry
- Rectification
- Other type of stereo images
- Homography
- RANSAC

Back to Stereo

General Case Triangulation



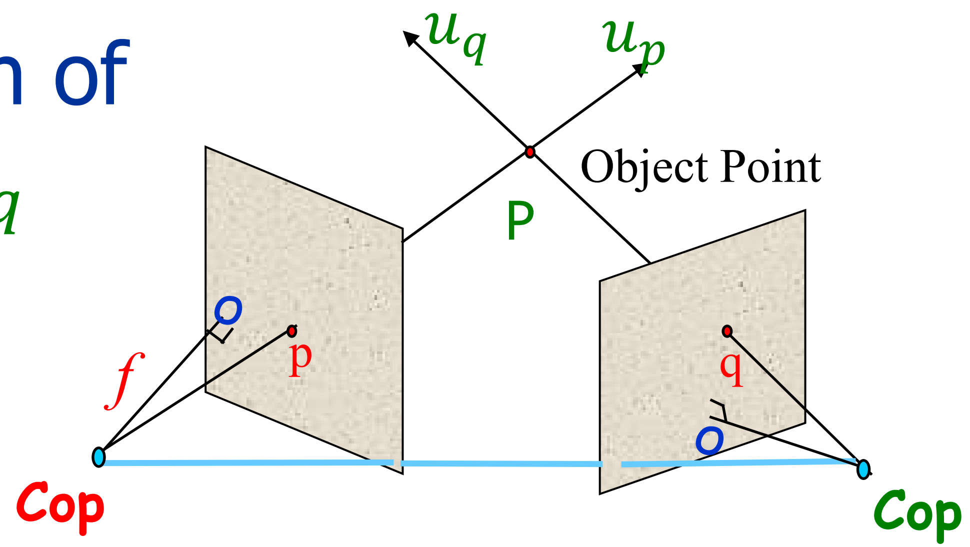
Possible to use trigonometry but projective geometry is nicer & easier

Triangulation

- We are looking for P such that:
  $\tilde{p} = M_L \tilde{P}$ and $\tilde{q} = M_R \tilde{P}$
- P is the intersection of two lines u_p and u_q

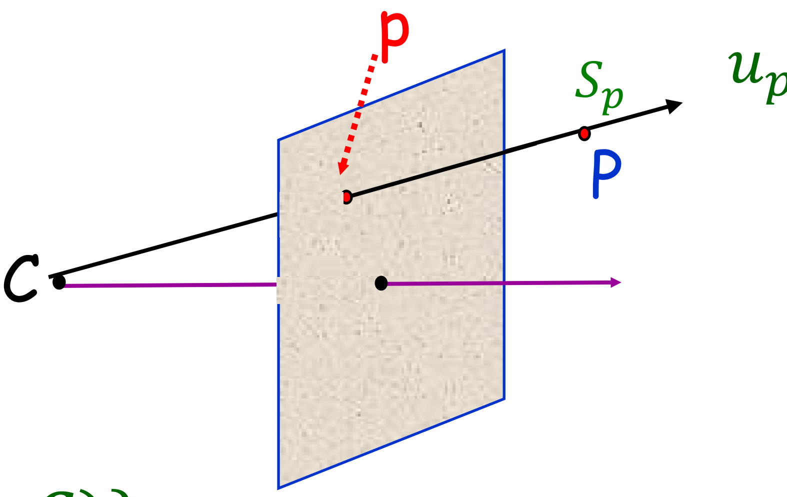
How to compute?

How to compute?



Back Projection Ray

- Given M and p
- Let S_p be a point in 3D s.t. $\tilde{p} = M\,\tilde{S}_p$
- The ray u_p is given by:
  - $u_p = \{P_\lambda \mid P_\lambda = C + \lambda(S_p - C)\}$
  - Denote $\mathrm{u_p}(\lambda) = C + \lambda\overrightarrow{CS_p}$ a point on u_p where $\overrightarrow{CS_p} = S_p - C$

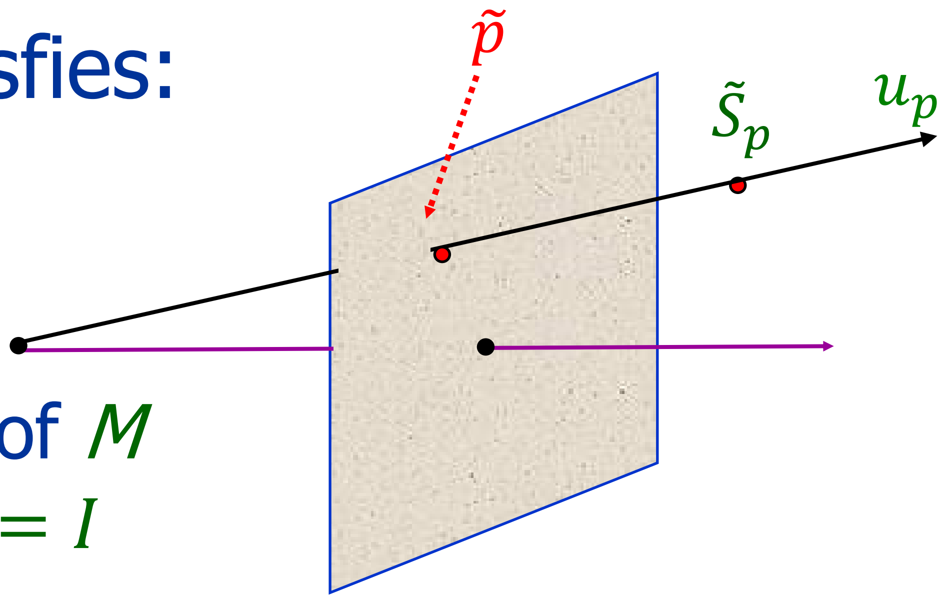


Note: S_p and C are in Euclidean coordinates

Back Projection Ray

- Given M and p
- A point, $S_p \in u_p$ satisfies:
  - $\tilde{p} = M\,\tilde{S}_p$
  - $\tilde{S}_p = M^{+}\tilde{p}$
  - M^{+}: pseudo inverse of M

    $$MM^{+} = I$$
  - $M^{+} = M^{T}(MM^{T})^{-1}$
  - $MM^{T}(MM^{T})^{-1}\,\tilde{p} = \tilde{p}$



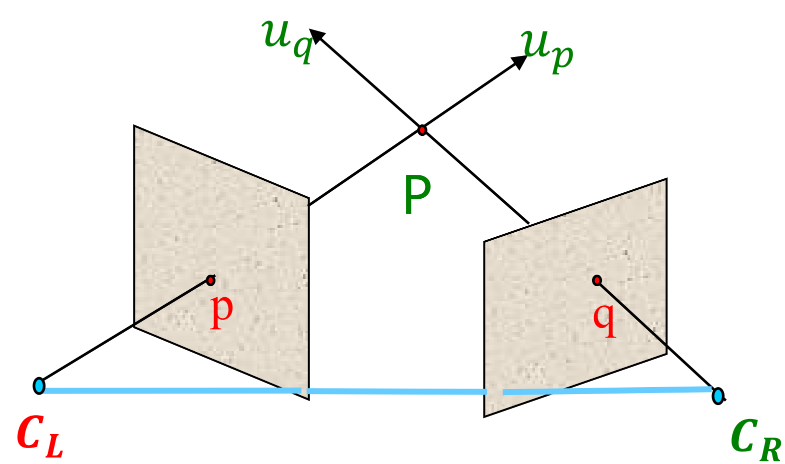
Pseudoinverse

- Let A be a $n \times m$ matrix where $n<m$
- Define $A^{+}=A^T(A\,A^T)^{-1}$

- $AA^{+}= I$

Triangulation

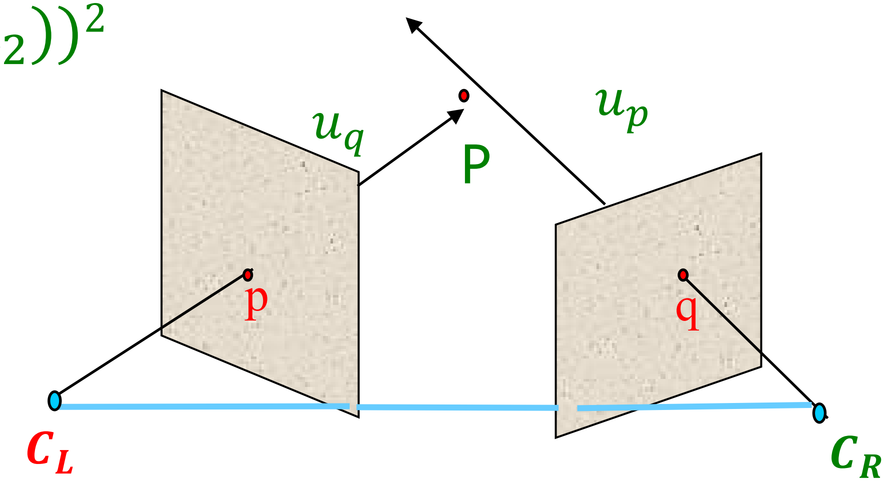
- We are looking for P such that: $\tilde{p} = M_L \tilde{P}$ and $\tilde{q} = M_R \tilde{P}$
- P is the intersection of the lines u_p and u_q
- $P = u_p(\lambda_1) = u_q(\lambda_2)$



Triangulation

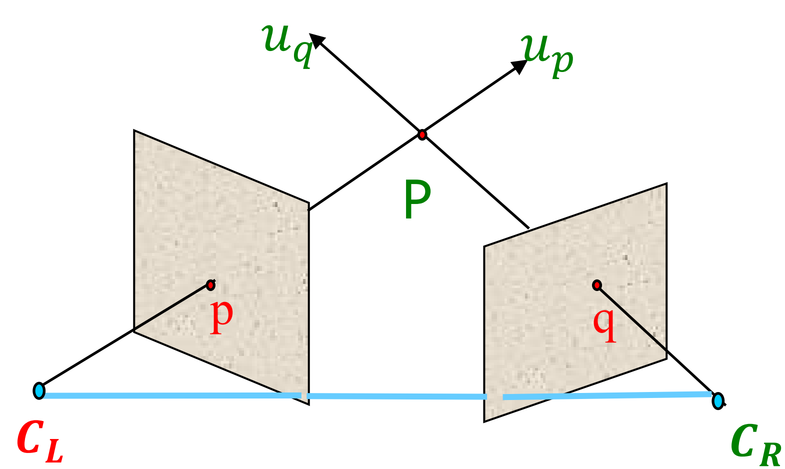
When u_p and u_q do not intersect,
we are looking for P that minimizes:

- $d\left(P, u_p(\lambda_1)\right)^2 + d(P, u_q(\lambda_2))^2$
- Other possibilities…



Triangulation

- We look for λ_1 and λ_2 such that
  - $u_p(\lambda_1) = u_q(\lambda_2)$
  - It follows that $P = u_p(\lambda_1)$
- Some algebraic manipulation:
  - $u_p(\lambda_1) = C_L + \lambda_1 \overrightarrow{C_L S_p}$
    $= C_L + \lambda_1 \vec{v}_p$
  - $u_q(\lambda_2) = C_R + \lambda_2 \overrightarrow{C_R S_q}$
    $= C_R + \lambda_1 \vec{v}_q$
  - $C_L + \lambda_1 \vec{v}_p = C_R + \lambda_2 \vec{v}_q$



Denote $u_p(\lambda) = C + \lambda \overrightarrow{CS}$ a point on u_p where $\overrightarrow{CS} = S_p - C$

Triangulation

- $C_L - C_R = \lambda_2 \vec{v}_r - \lambda_1 \vec{v}_L$
- In matrix notation: $(-v_L, v_R)\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = (C_L - C_R)$
- We search for (λ_1, λ_2) that minimizes:

$$\left\| (-v_p, v_q)\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} - (C_L - C_R) \right\|$$

- In python use: (np.linalg.lstsq)
- $P_L = u_p(\lambda_1)$, $P_R = u_q(\lambda_2)$
- $\boxed{P = \frac{P_L + P_R}{2}}$

Triangulation: Another Minimization

- Minimize the distance of the projected point to the two image.
- Choose P that minimizes:

$$\min_{\tilde{P}}(\left|\tilde{p} - M_L\tilde{P}\right|| + |\left|\tilde{p} - M_R\tilde{P}\right||)$$

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The COP

- Can we compute C from M?
- Claim: The camera center, C is the null vector of M
- Proof idea:
  - Note that the degree of M is 3
  - Hence there exists a C s.t. $M\tilde{C} = (0,0,0)^T$
  - We will show: that C is the center of projection

Cont.

- We have to show that for any given 3D point, all points on the ray that connects it to the COP project to the same image point:
  - Consider a 3D point A and
  - Let $\tilde{a} = M\,\tilde{A}$
  - C is the center of projection of M if all points on the ray $P(\lambda) = C + \lambda(A - C)$ projects to $\tilde{a}$

Algebra ...

- $P(\lambda) = C + \lambda(A - C)$ Cartesian coordinates
- $P(\lambda) = (1-\lambda)C + \lambda A$
- $\tilde{P}(\lambda) \cong \begin{pmatrix} (1-\lambda)C_x + \lambda A_x \\ (1-\lambda)C_y + \lambda A_y \\ (1-\lambda)C_z + \lambda A_z \\ 1 \end{pmatrix}$ Homogenous coordinates

$$\cong \begin{pmatrix} (1-\lambda)C_x + \lambda A_x \\ (1-\lambda)C_y + \lambda A_y \\ (1-\lambda)C_z + \lambda A_z \\ 1-\lambda+\lambda \end{pmatrix} = (1-\lambda)\tilde{C} + \lambda\tilde{A}$$

Cont.

It follows that:

- $M\ \tilde{P}(\lambda) = M(1-\lambda)\tilde{C} + \lambda\tilde{A} = M\tilde{A} = \tilde{a}$

Q.E.D.

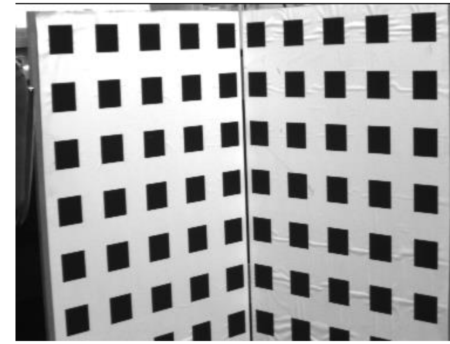
Camera Calibration

Camera Calibration

Find the camera matrices:

- Intrinsic parameters
- Extrinsic parameters

General Idea



Assumption:

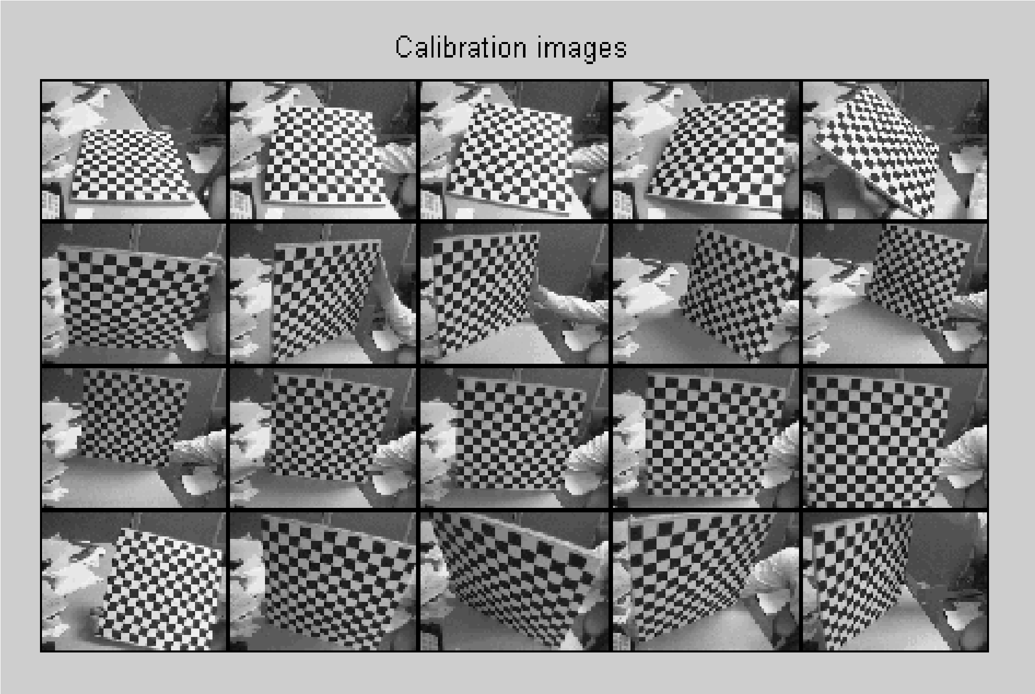
- Known 3D pattern $\{P_i\}$
- Known correspondence between image and object points $\{p_i\}$

Solution:

- Solve the linear equations: $\tilde{p}_i = M\tilde{P}_i$
  - Note: $\tilde{u} \cong \tilde{v}$ in the projective space: $\tilde{u} \times \tilde{v} = 0$

Calibration based on a planar surface

- Using several images allow to compute *M* and their location with respect to each other

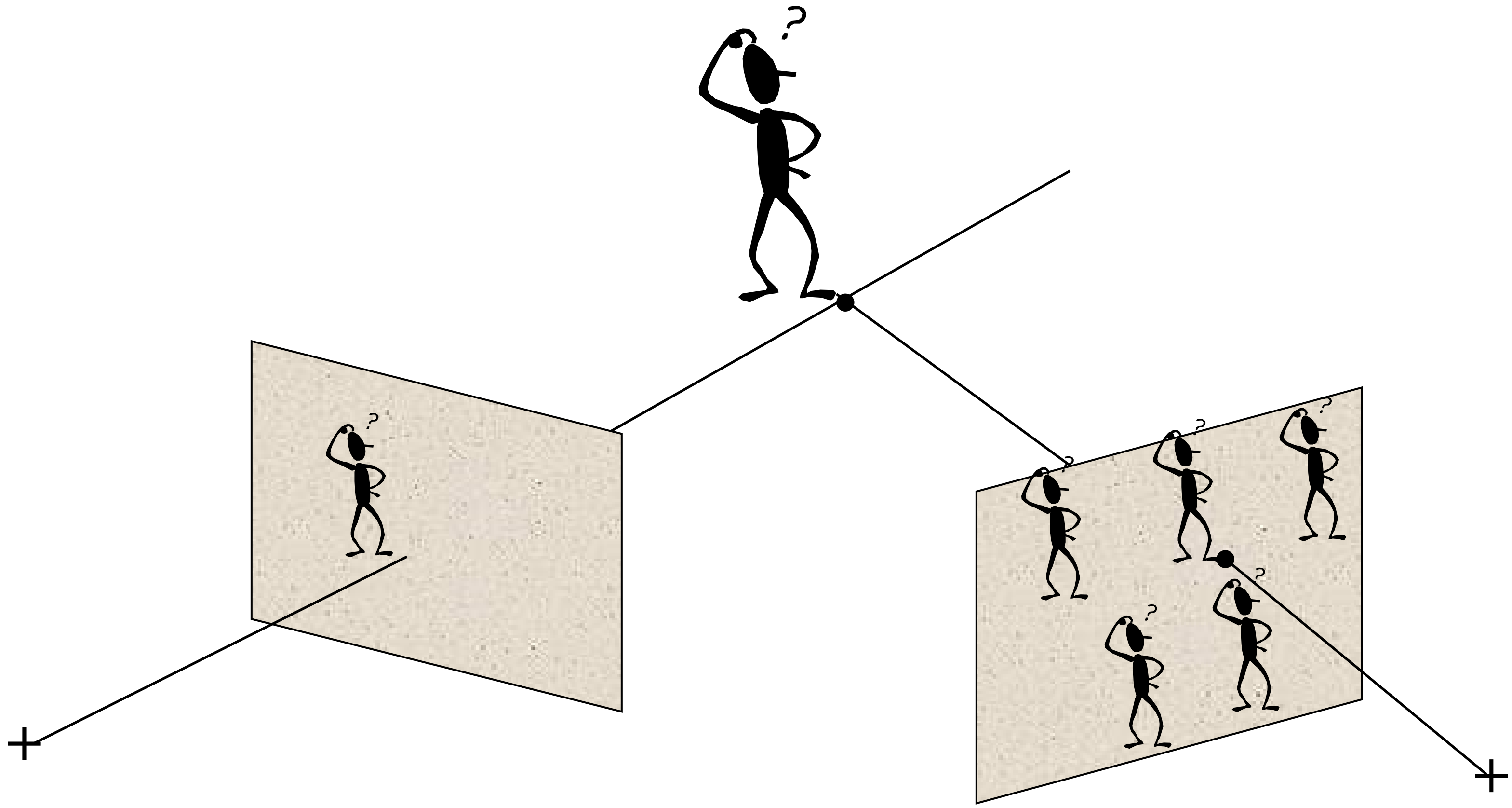


So far

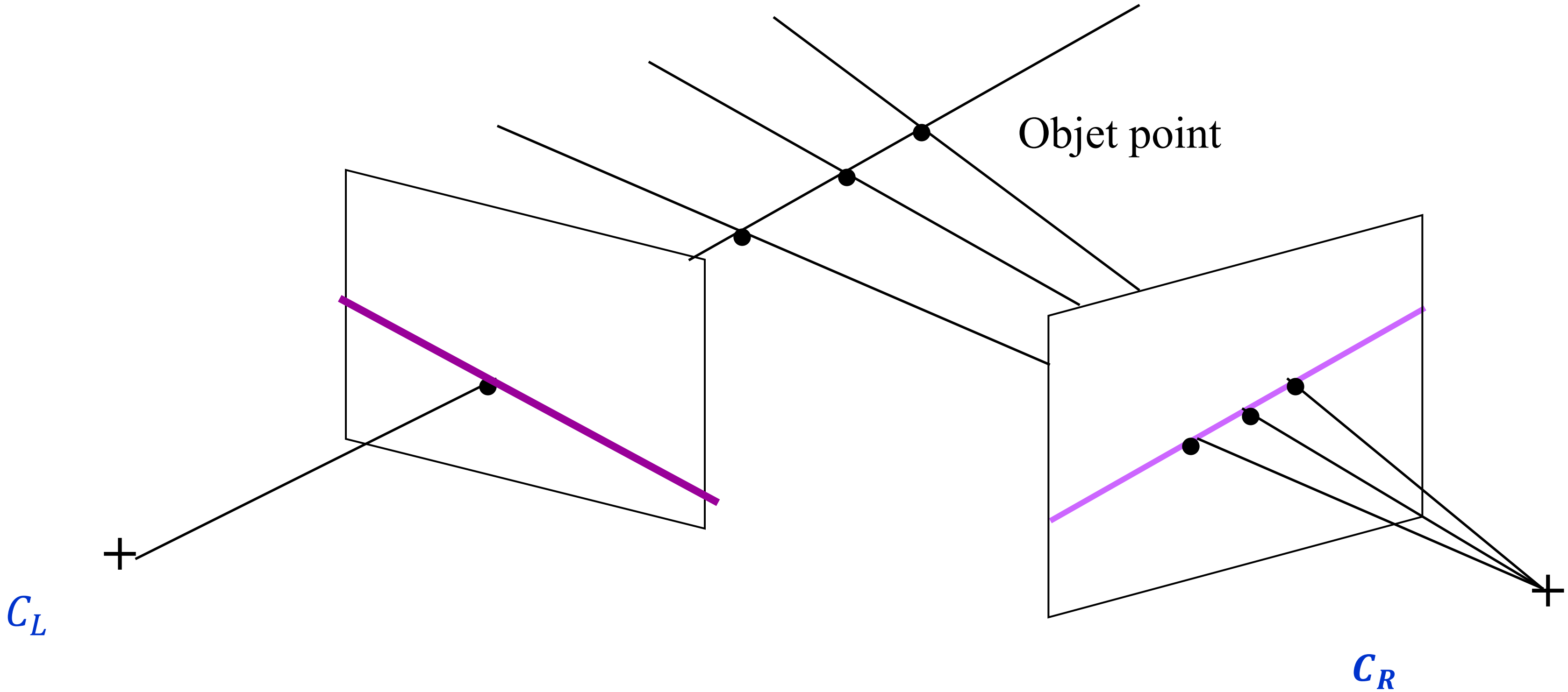
- Using geometry & algebra to compute:
  - Projection matrix
  - Traingulation
  - Calibration
- Next: correspondence

Epipolar Geometry

Correspondence: Geometry



Epipolar Lines



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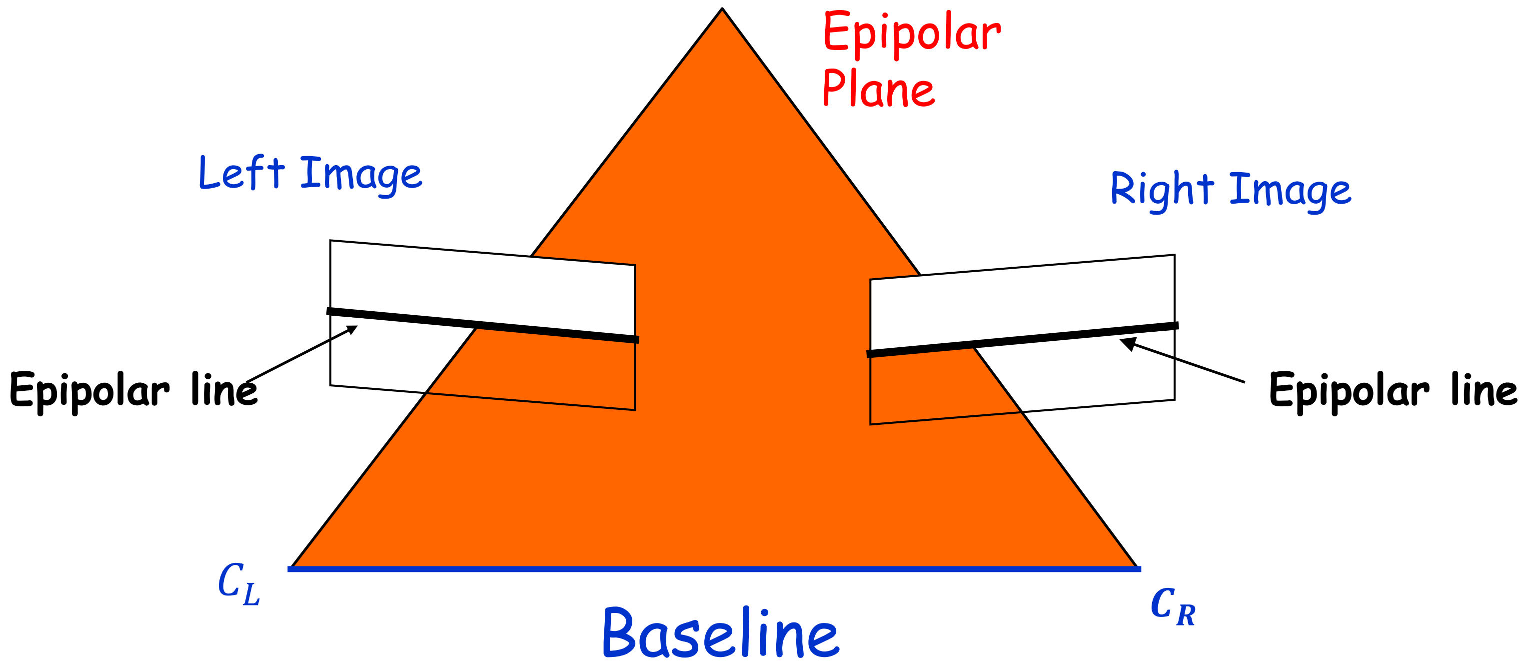
Epipolar Geometry

A pair of corresponding points must lay on corresponding epipolar lines



Epipolar Geometry

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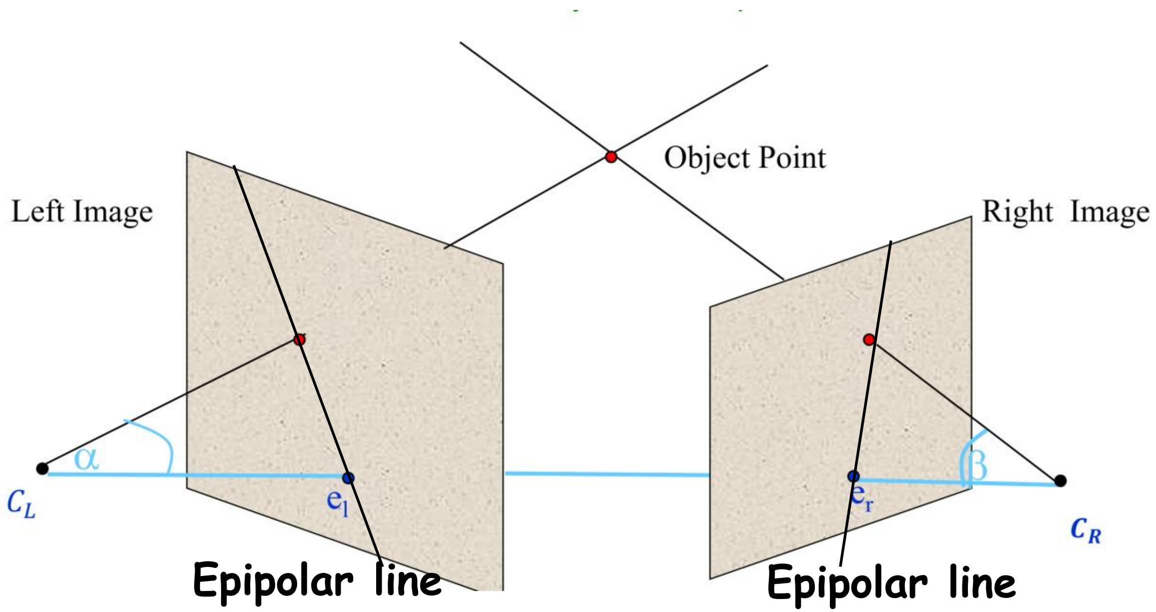
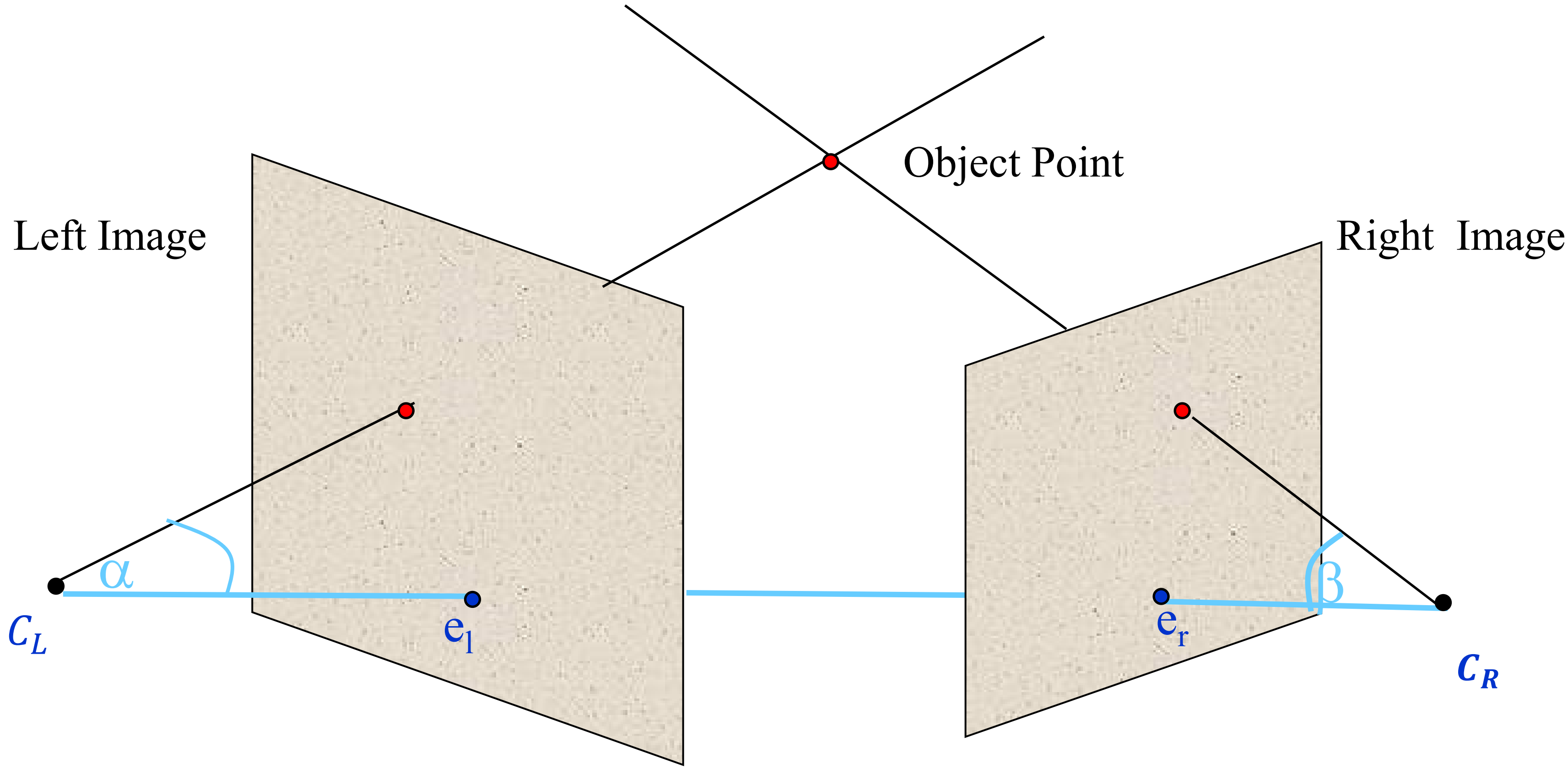
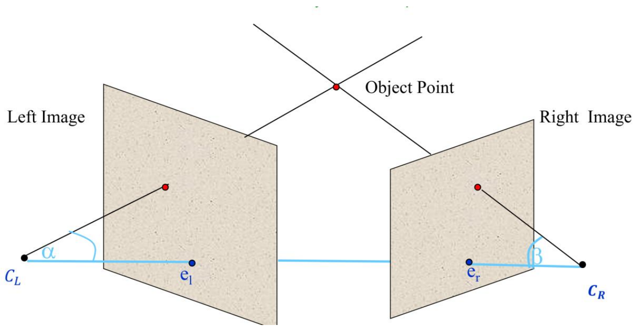


The epipole points are e_l and e_r



Epipole

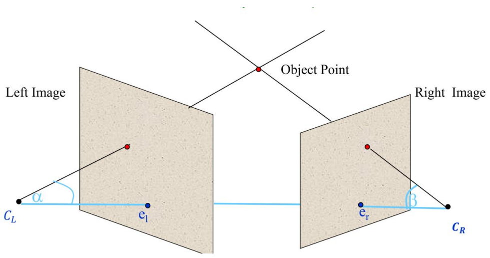
- The projection of the *cop* of the other camera
- Except for the epipole, only one epipolar line goes through any image point
- All the epipolar lines go through the epipole



The Fundamental Matrix

That's the way to learn it

Fundamental Matrix



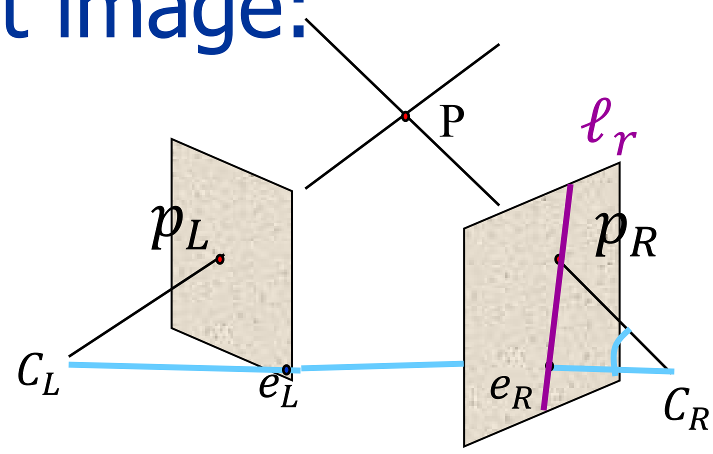
- F is a 3×3 matrix of rank 2
- Let p_L and p_R be corresponding image points: $\tilde{p}_R^T F \tilde{p}_L = 0$
- A point in one image defines the epipolar line in the other image:
  - $\tilde{\ell}_R = F\tilde{p}_L$ and $\tilde{\ell}_L = \tilde{p}_R^T F$
- F is based on intrinsic and extrinsic parameters

Determining the Epipolar Lines

- Depends on the intrinsic and extrinsic parameters
- Algebraic description homogenous coordinates:
  - A line is given by $\tilde{\ell}$
  - A point $\tilde{p}$ lay on a line $\tilde{\ell}$ if $\tilde{p} \cdot \tilde{\ell} = 0$

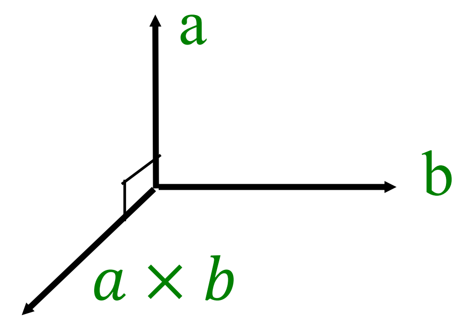
Computing F

- Let M_L and M_R be the projection matrices
- A point in 3D that projects to p_L:
  - $\tilde{P} = M_L^{+}\tilde{p}_l$
- Its projection to the right image:
  - $\tilde{p}_r = M_R \tilde{P}$
- A right image line:
  - $\ell_r = e_r \times (M_R M_L^{+}\tilde{p}_l)$



Vector Product

- Let $a = (a_x, a_y, a_z)$ and b $= (b_x, b_y, b_z)$
- Vector (Cross product):
  - $a \times b = (a_y b_z - b_y a_z, -a_x b_z + b_x a_z, a_x b_y - b_x a_y)$
  - $a \times b = \boxed{\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}} b$
  - $a \times b = [a]_\times b$

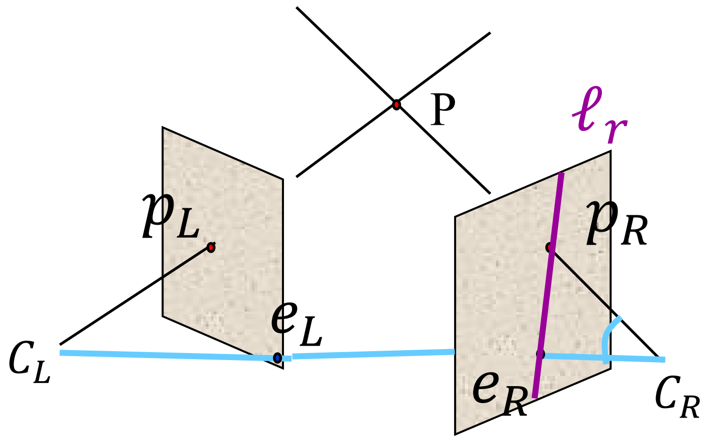


$\boxed{[a]_\times}$

Computing F

All points, q, on the line defined by e_r and p_R satisfy:

- $\tilde{p}_R^T(\mathrm{e_r} \times (M_R M_L^+ \tilde{p}_L)) = 0$
- $\tilde{p}_R^T([\mathrm{e_r}]_\times M_R M_L^+ \tilde{p}_L)) = 0$
- $\ell_R = [\mathrm{e_r}]_\times M_R M_L^+ \tilde{p}_L$
- $\boxed{F = [\mathrm{e_r}]_\times M_R M_L^+}$



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Summary Fundamental Matrix

- If p_L and p_R are corresponding image points then:
  - $\tilde{p}_R^T F \tilde{p}_L = 0$
- Epipolar lines:
  - $\tilde{\ell}_R = F\tilde{p}_L$ and $\tilde{\ell}_L = \tilde{p}_R^T F$
- The epipoles:
  - $\tilde{e}_R^T F = 0$ and $F\tilde{e}_L = 0$

A Question

- What is kF?

Next

- More on Epipolar Geometry
- Unclaibrated pairs
- Other stereo pairs
- Special cases - Homography
  - planar surfaces, camera rotation
- More than 2 images:
  - Structure from motion