Multivar Gaussians and GMMs

Statistics and data analysis

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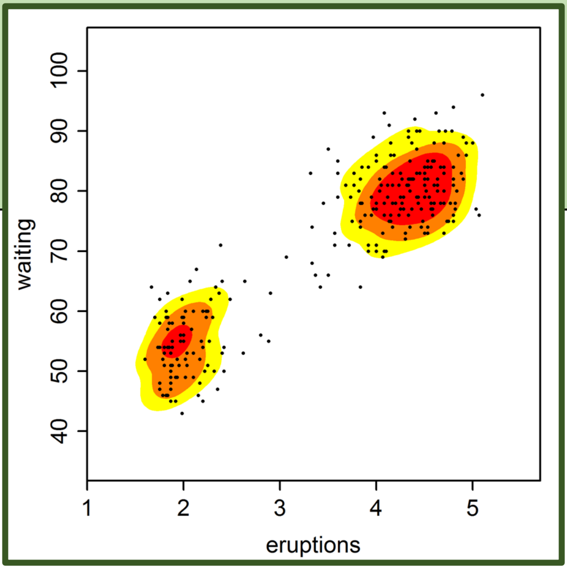

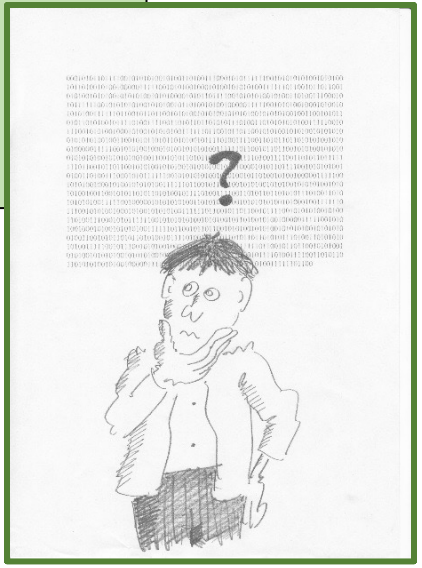

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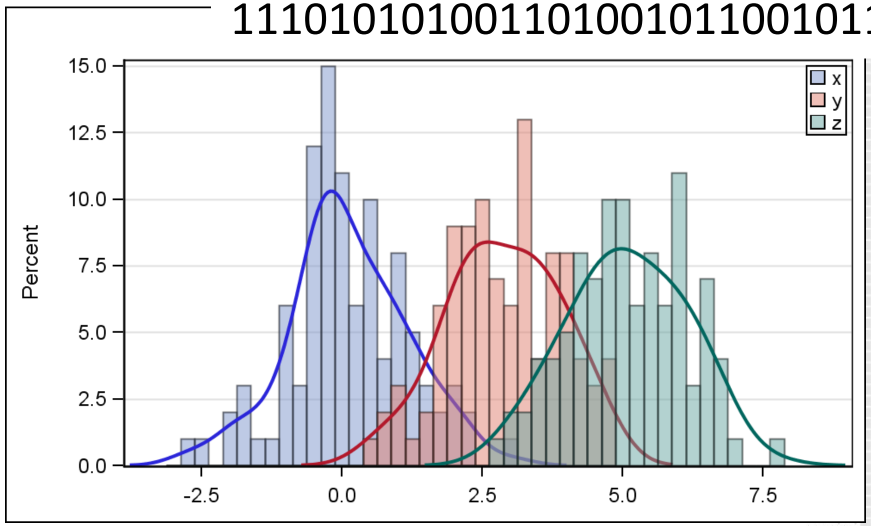
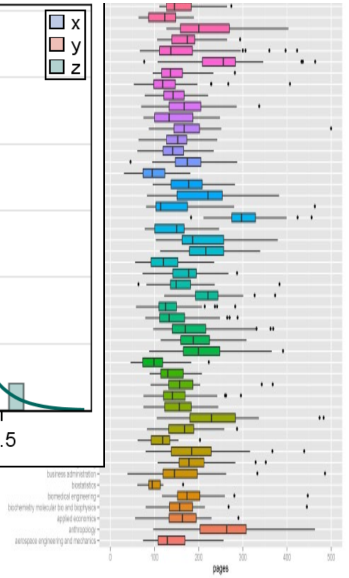




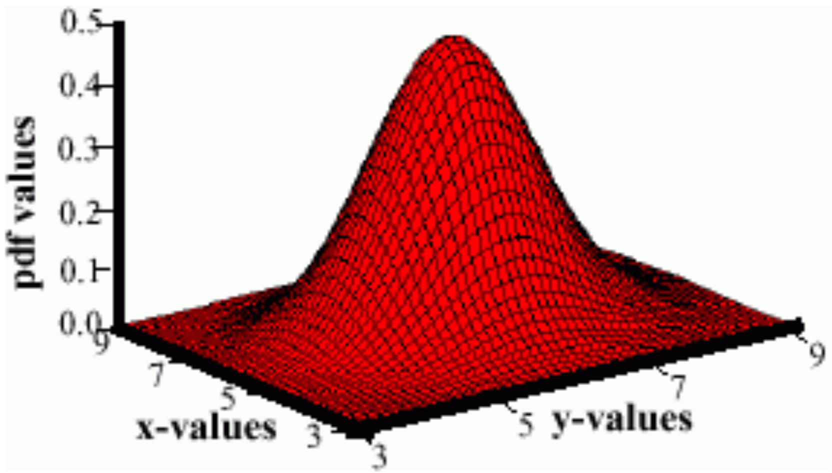
Multivariate Normal Distributions

- **A multivariate normal distribution is defined by its pdf:**

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^{d} \cdot Det\,\Sigma}}\, exp\left(-\frac{1}{2} \cdot \langle \vec{x} - \vec{\mu}, \Sigma^{-1}(\vec{x} - \vec{\mu}) \rangle \right)$$

where μ represents the mean (vector) and ∑ represents the covariance matrix.

- **The covariance is always symmetric and positive semidefinite. (invertible, Det > 0)**
- **Does the covariance matrix ∑ represent covariances?**
- **How does the shape vary as a function of the covariance? – will be discussed**



Simple case – diagonal Σ

$$p(x;\mu,\Sigma) = \frac{1}{2\pi\begin{vmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{vmatrix}^{1/2}} \exp\left(-\frac{1}{2}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}^T \begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}^{-1} \begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}\right)$$

$$= \frac{1}{2\pi(\sigma_1^2\cdot\sigma_2^2 - 0\cdot 0)^{1/2}} \exp\left(-\frac{1}{2}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}^T \begin{bmatrix}\frac{1}{\sigma_1^2} & 0\\ 0 & \frac{1}{\sigma_2^2}\end{bmatrix} \begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}\right)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2 - \frac{1}{2\sigma_2^2}(x_2-\mu_2)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2\right) \cdot \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2\right).$$

An alternative definition

Consider a standard bivariate normal random variable (Z_1, Z_2).
Note that we can directly define this by defining the joint pdf:

$$\varphi(x, y) = \frac{1}{2\pi} exp\left(-\frac{1}{2}(x^2 + y^2)\right)$$

Using a set of parameters $\alpha, \beta, \sigma, \tau$ and ρ we now define new random variables as follows:

$$A = \sigma Z_1 + \alpha \quad \text{and} \quad B = \tau\left(\rho Z_1 + \sqrt{1-\rho^2} \cdot Z_2\right) + \beta$$

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Bivar normal – cont …

We can show (change of variables and some heavy calculus) that the pdf of (A, B) will have the form:

$$p(x,y) = \frac{1}{2\pi\sqrt{Det\,\Sigma}}\, exp\left(-\frac{1}{2}\cdot\left\langle (x,y)-\vec{\mu}\ ,\ \Sigma^{-1}\left((x,y)-\vec{\mu}\right)\right\rangle\right)$$

What are $\vec{\mu}$ and Σ ?

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Bivar normal - cont

$$E(A) = E(\sigma Z_1 + \alpha) = \alpha$$

$$V(A) = V(\sigma Z_1 + \alpha) = \sigma^2$$

Similarly for B

$$E(B) = E\left(\tau\left(\rho Z_1 + \sqrt{1-\rho^2}\cdot Z_2\right) + \beta\right) = \beta$$

$$V(B) = V\left(\tau\left(\rho Z_1 + \sqrt{1-\rho^2}\cdot Z_2\right) + \beta\right)$$

$$= \tau^2\left(\rho^2 V(Z_1) + (1-\rho^2)V(Z_2)\right)$$

$$= \tau^2$$

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Bivar normal – cont …

We therefore have:

$$E(A) = \alpha \quad , \quad V(A) = \sigma^2 \quad ,$$

$$E(B) = \beta \quad \text{and} \quad V(B) = \tau^2$$

In fact, our construction implies that

$A \sim Nor(\alpha, \sigma^2)$ and $B \sim Nor(\beta, \tau^2)$

A useful fact

If $X \sim N(\mu, \sigma^2)$

then

$Z = (X - \mu)/\sigma$

is standard normal

If $Z \sim N(0, 1)$

then when defining

$X = \sigma Z + \mu$

we have

$X \sim N(\mu, \sigma^2)$

Covariance

$A = \sigma Z_1 + \alpha$ and $B = \tau\left(\rho Z_1 + \sqrt{1-\rho^2} \cdot Z_2\right) + \beta$

$$Cov(A, B) = E\big((A - EA) \cdot (B - EB)\big)$$

$$= E\left(\sigma Z_1 \cdot \tau\left(\rho Z_1 + \sqrt{1-\rho^2} \cdot Z_2\right)\right)$$

$$= \sigma\tau \cdot E\left(\rho Z_1^2 + \sqrt{1-\rho^2} Z_1 Z_2\right)$$

$$= \sigma\tau \cdot \rho$$

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Equivalence – the interpretation

Lets use an example to see how this construction reflects our first definition:

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^d \cdot Det\ \Sigma}}\ exp\left(-\frac{1}{2} \cdot \langle \vec{x} - \vec{\mu}, \Sigma^{-1}(\vec{x} - \vec{\mu})\rangle\right)$$

In the bivariate case, and assuming a mean at 0:

$$p(x, y) = \frac{1}{2\pi\sqrt{Det\ \Sigma}}\ exp\left(-\frac{1}{2} \cdot \langle (x, y), \Sigma^{-1}(x, y)\rangle\right)$$

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What is Σ ?

We constructed a bivariate random variable that has normal marginal by defining:

$$A = \sigma Z_1 \quad (+\alpha)$$

$$B = \tau\left(\rho Z_1 + \sqrt{1-\rho^2}\cdot Z_2\right) \quad (+\beta)$$

It can be shown, by a change of variables and setting α and β to 0, that the density function for our constructed bivariate normal is given by:

$$p(x,y) = \frac{1}{2\pi\sqrt{Det\,\Sigma}}\, exp\left(-\frac{1}{2}\cdot\langle(x,y),\Sigma^{-1}(x,y)\rangle\right)$$

With Σ being:

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma\tau\cdot\rho \\ \sigma\tau\cdot\rho & \tau^2 \end{pmatrix}$$

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A third alternative definition

- X is multivariate normal if for any constant vector v, the random variable $\mathrm{W} = (v\,, X)$ has a univariate normal distribution.
- X is multivariate normal with a diagonal Σ iff for any constant vector v, the random variable $\mathrm{W} = (v\,, X)$ has a univariate normal distribution with $\sigma^2 = (v.^2\,, \mathrm{diag}(\Sigma))$
- In general we have
  $E(W) = (v, E(X))$ and $Var(W) = (v, \Sigma(v))$

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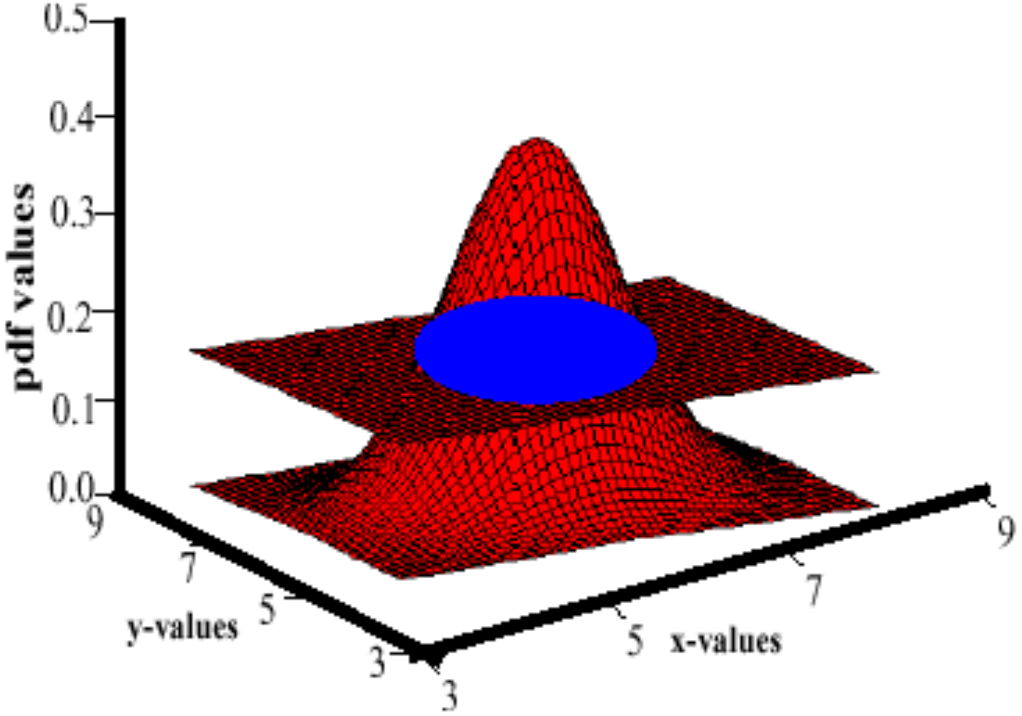
Support Regions or Isocontours

- Horizontal planes cut the pdf graph defining the shapes of ellipses of points that all have equal probability density.
- The shape of these support regions is determined by the covariance matrix.



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The diagonal case

What is the shape of isocontours for a diagonal covariance matrix?
Write:

$$
\begin{aligned}
c &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2-\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2\right)\\
2\pi c\sigma_1\sigma_2 &= \exp\left(-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2-\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2\right)\\
\log(2\pi c\sigma_1\sigma_2) &= -\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2-\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2\\
\log\left(\frac{1}{2\pi c\sigma_1\sigma_2}\right) &= \frac{1}{2\sigma_1^2}(x_1-\mu_1)^2+\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2\\
1 &= \frac{(x_1-\mu_1)^2}{2\sigma_1^2\log\left(\frac{1}{2\pi c\sigma_1\sigma_2}\right)}+\frac{(x_2-\mu_2)^2}{2\sigma_2^2\log\left(\frac{1}{2\pi c\sigma_1\sigma_2}\right)}.
\end{aligned}
$$

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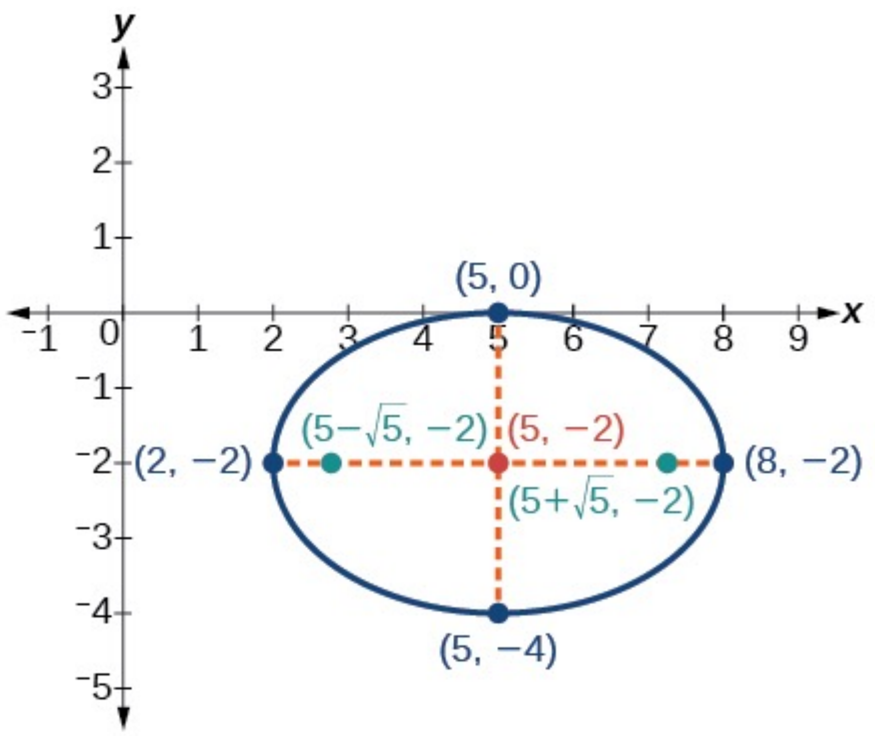
Isocontours - continued

Setting

$$r_1 = \sqrt{2\sigma_1^2 \ln\left(\frac{1}{2\pi c\sigma_1\sigma_2}\right)}$$

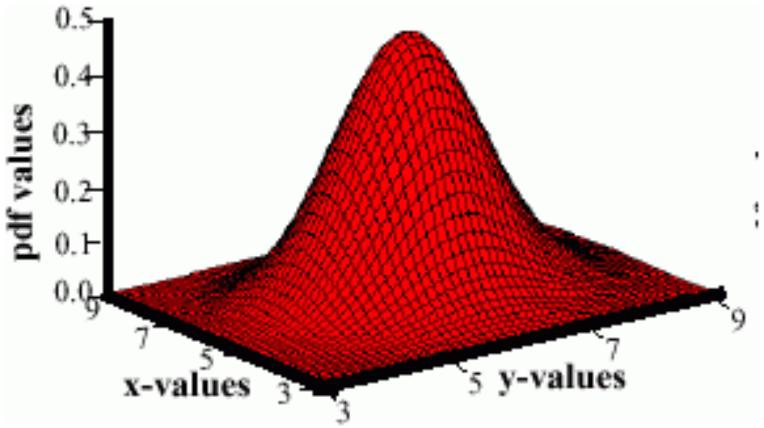
$$r_2 = \sqrt{2\sigma_2^2 \ln\left(\frac{1}{2\pi c\sigma_1\sigma_2}\right)}$$



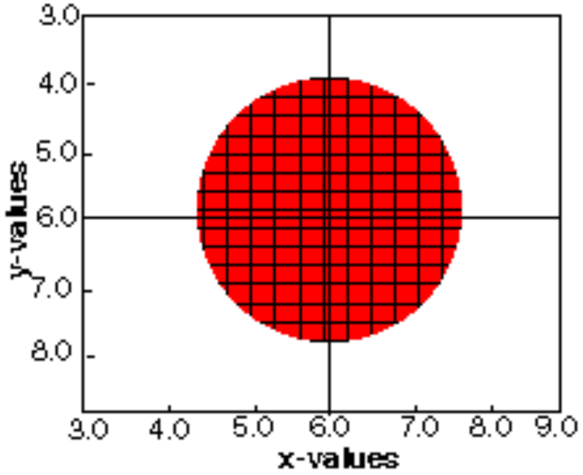
We get that (x_1, x_2) on the contour satisfies $$\left(\frac{x_1-\mu_1}{r_1}\right)^2 + \left(\frac{x_2-\mu_2}{r_2}\right)^2 = 1$$

Identity Covariance

Gaussian distribution



Support Region



- Gaussian distribution with identity covariance matrix has equal variances in all directions
- Support region for a Gaussian distribution with identity covariance matrix $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a circle

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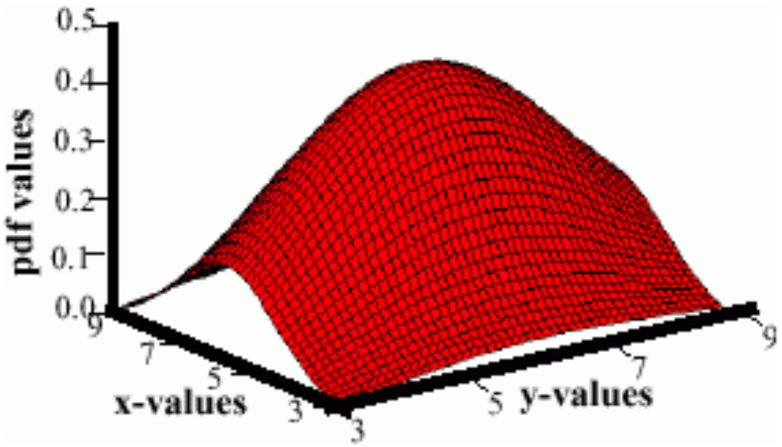
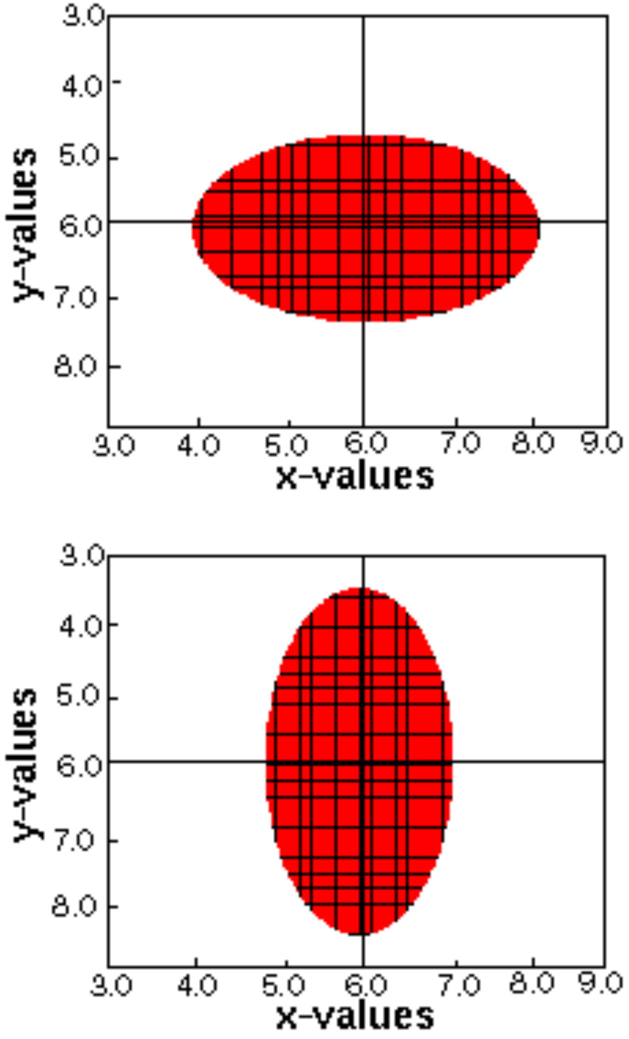
Unequal Variances

Gaussian distribution



Support Region



Support region for a Gaussian distribution with covariance matrix $C = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$ is an ellipse aligned with respect to the original axes

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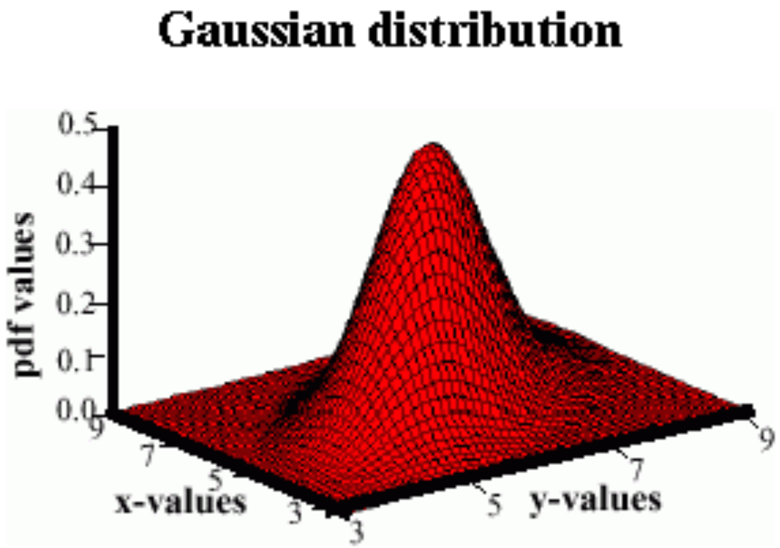
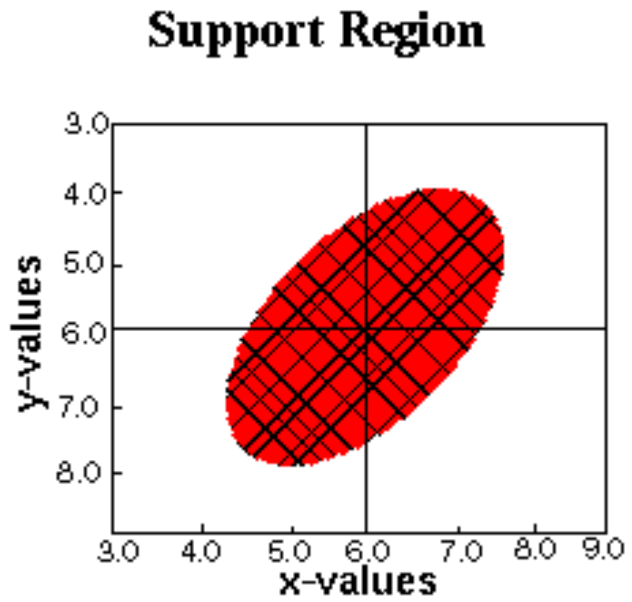
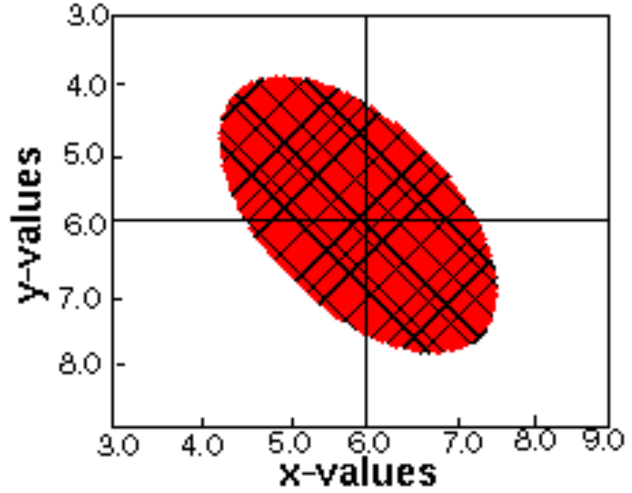
Nonzero Off-Diagonal Elements





Support region for a Gaussian distribution with covariance matrix $C = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ is an ellipse rotated at an angle of 45 degree with respect to the original axes



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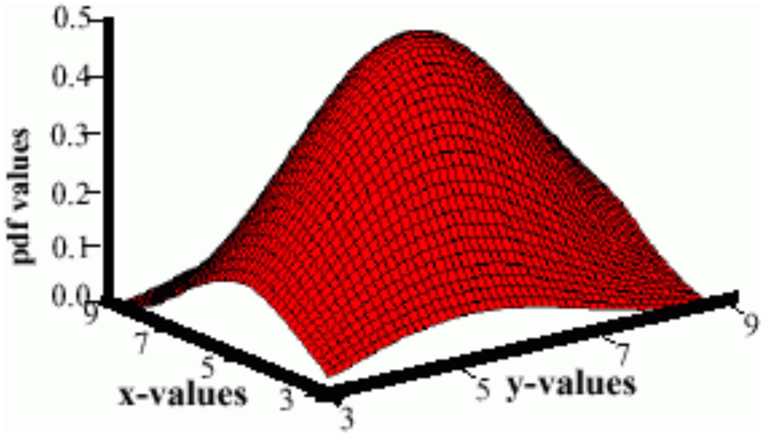
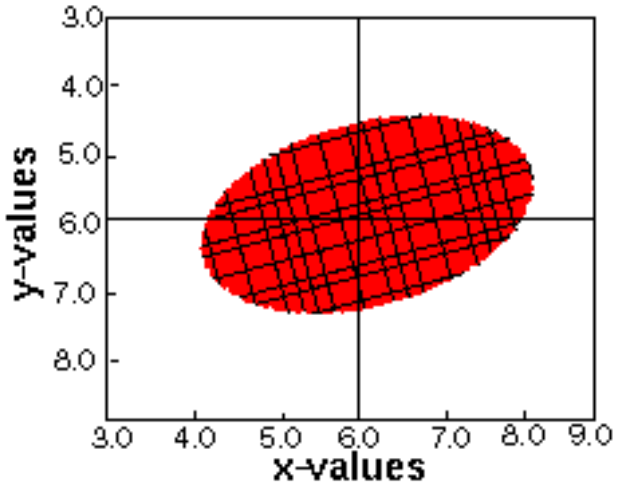
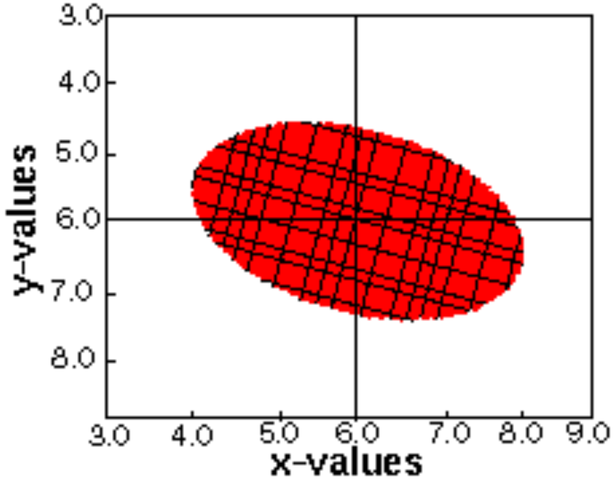
Unconstrained or “Full” Covariance

Gaussian distribution

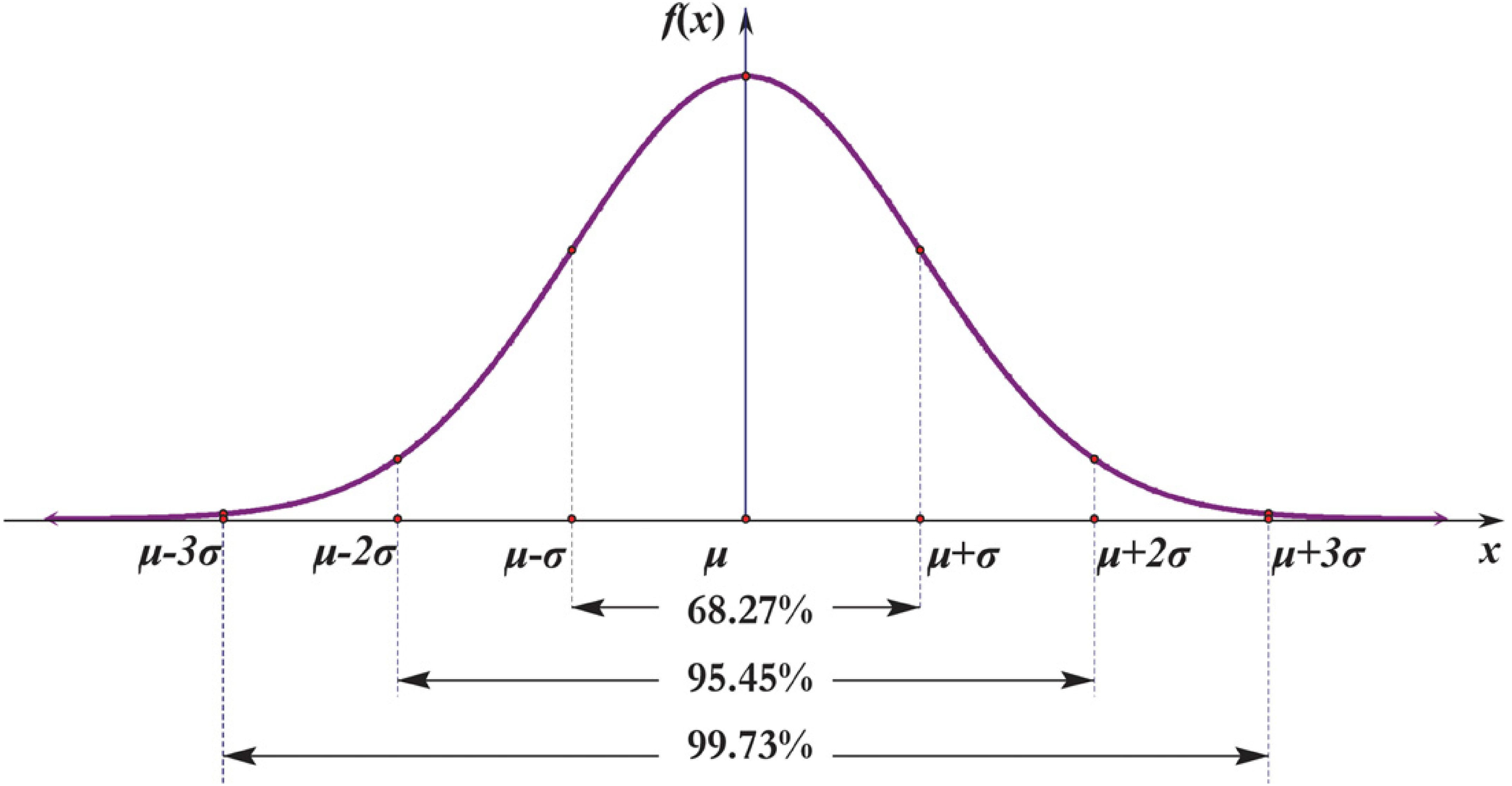


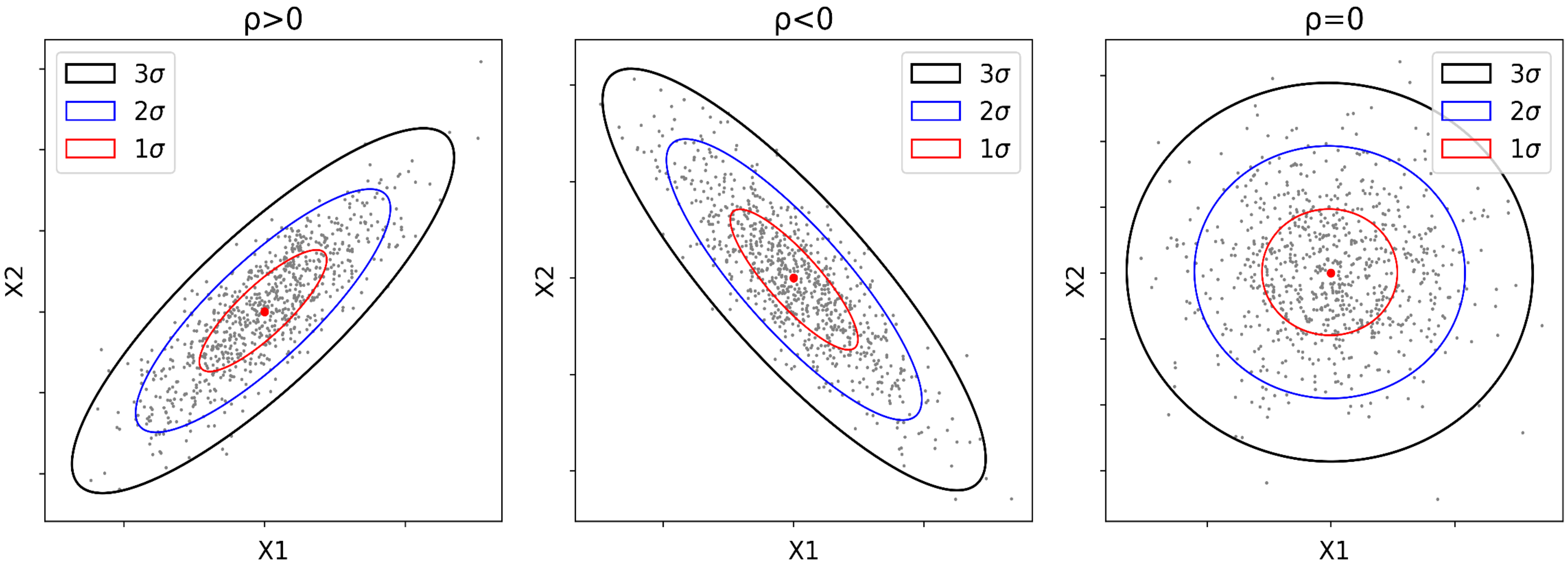
Support Region





Support region for a Gaussian distribution with covariance matrix $C = \begin{bmatrix} 5 & 0.5 \\ 0.5 & 2 \end{bmatrix}$ is an ellipse rotated at an angle of 9 degrees and -81 degrees with respect to the original axes


f(x)
x
μ-3σ
μ-2σ
μ-σ
μ
μ+σ
μ+2σ
μ+3σ
68.27%
95.45%
99.73%



ρ>0
ρ<0
ρ=0
3σ
2σ
1σ
X1
X2

Table 2. Magnification ratios of scaled SDHE corresponding to different specified confidence levels with space dimensionality not exceeding 10.

Dimensionality	Confidence Level (%)					
	80.0	85.0	90.0	95.0	99.0	99.9
1	1.2816	1.4395	1.6449	1.9600	2.5758	3.2905
2	1.7941	1.9479	2.1460	2.4477	3.0349	3.7169
3	2.1544	2.3059	2.5003	2.7955	3.3682	4.0331
4	2.4472	2.5971	2.7892	3.0802	3.6437	4.2973
5	2.6999	2.8487	3.0391	3.3272	3.8841	4.5293
6	2.9254	3.0735	3.2626	3.5485	4.1002	4.7390
7	3.1310	3.2784	3.4666	3.7506	4.2983	4.9317
8	3.3212	3.4680	3.6553	3.9379	4.4822	5.1112
9	3.4989	3.6453	3.8319	4.1133	4.6547	5.2799
10	3.6663	3.8123	3.9984	4.2787	4.8176	5.4395

doi:10.1371/journal.pone.0118537.t002

Wang B, Shi W, Miao Z (2015) Confidence Analysis of Standard Deviational Ellipse and Its Extension into Higher Dimensional Euclidean Space. PLOS ONE 10(3): e0118537. https://doi.org/10.1371/journal.pone.0118537
https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0118537

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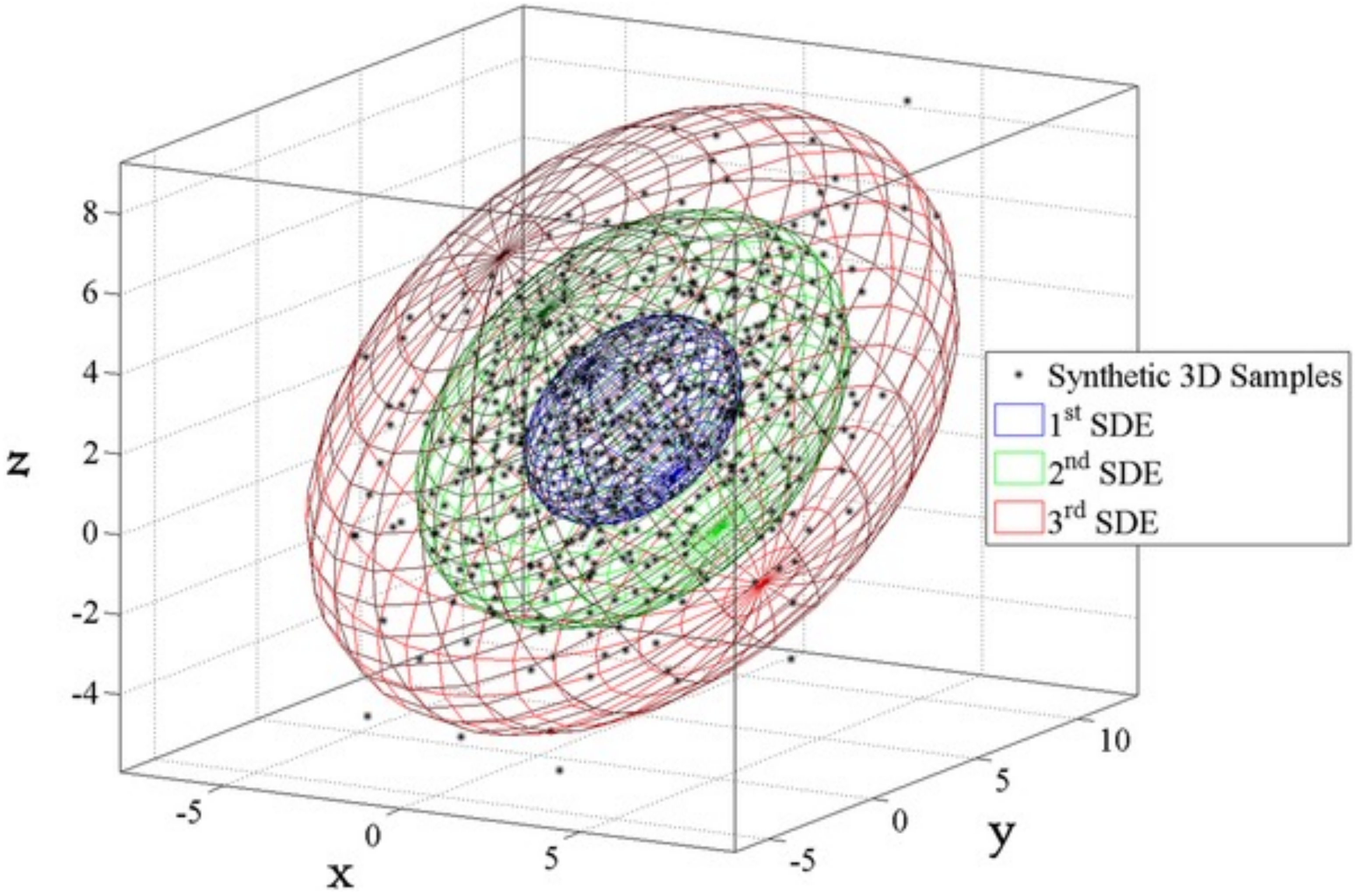
Wang et al 2015 –
Std ellipses in higher dimension



Wang B, Shi W, Miao Z (2015) Confidence Analysis of Standard Deviational Ellipse and Its Extension into Higher Dimensional Euclidean Space. PLOS ONE 10(3): e0118537. https://doi.org/10.1371/journal.pone.0118537
https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0118537

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Summary of Gauss related topics

- The normal distribution
- Central limit theorem – characterizing the distribution of averages.
- Approximately normal natural phenomena
- Multi D Gaussians and its defining characteristics and quantities.
- Log Normal distribution – next time

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