Assuaging Fears about Mathematical Diagrams

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Abstract

In her recent paper "Who's Afraid of Mathematical Diagrams?" Silvia De Toffoli argues that certain types of mathematical diagrams "form mathematical notation systems and therefore play a non-redundant role in proofs". Furthermore, she claims that "there is a plausible conception of proof according to which diagrams are ... in fact essential to those proofs." This paper expounds the latter claim, addressing some natural objections, clarifying the stakes, and establishing the plausibility of this conception beyond proofs containing mathematical diagrams. I will also discuss context- and agent-dependence in evaluating proofs by exploring the relationship of mathematical diagrams with aesthetics and explanation.

Introduction

Can mathematical diagrams play an essential justificatory role in mathematical proof? In a recent paper[1], Silvia De Toffoli suggests the affirmative, diverging from a commonly held view that diagrams play at most a heuristic or illustrative role in mathematical justification[2]. She provides two important examples to demonstrate that mathematical diagrams, subject to certain constraints, form genuine notational systems: fundamental polygons in topology and commutative diagrams in algebra. The basic argument is that, in either case, it is possible to equip these diagrams with a syntax capable of supporting logical inferences unambiguously corresponding to mathematical objects. Furthermore, they satisfy desirable qualities of notation such as reproduciblity, stability, accessibility, and so on.

For example, consider the polygon diagram in Figure 1. The constitutive features of this diagram – labeled edges and vertices – are "clearly identified, persistent, and stable." Furthermore, it is straightforward (and, indeed, common practice) for mathematical practitioners to reproduce

[1] De Toffoli (2023)
[2] Burgess(2015)

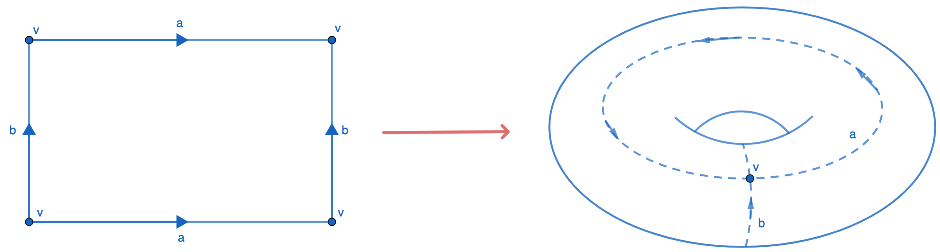

Figure 1: A polygon diagram and the corresponding torus, obtained by forming the quotient space with respect to identified edges and vertices.

these diagrams up to non-constitutive features[3]. Finally, practitioners can not only perform (physical or mental) manipulations to these diagrams, but, crucially, these manipulations correspond to well-defined mathematical operations. In this case, "gluing" together matching edges of a polygon corresponds to forming a quotient space, and "cutting and pasting" a polygon corresponds to a homeomorphism of the surface. Therefore, polygon diagrams form a notation system for topological surfaces that is not only intuitive[4] and natural for the topological setting, but also that supports inferences with precise mathematical meaning.

Examples like this settle the issue of whether diagrams *can* be used for mathematical justification, in so far as they meet agreed upon standards of rigor and criteria for acceptability[5]; the natural question, however, is whether there is actually epistemic merit to their use in proofs, beyond the heuristic or illustrative value they might provide. De Toffoli's second thesis addresses this explicitly: she asserts the existence of a "plausible conception of proof in which diagrams are not just abbreviations for more cumbersome non-diagrammatic displays but are in fact essential to those proofs." De Toffoli's justification for this claim hinges on the nonexistence of a useful context-independent criterion for proof identity. If one wishes to claim that two proofs of the same proposition are "the same," he must specify a particular context of investigation. Some examples of contexts are *rigor* – two proofs of a proposition are the same if they are equally rigorous – and *purity* – two proofs of a proposition are the same if they use the same central idea.

The context of investigation relevant to De Toffoli's argument is that of *epistemic benefits and drawbacks*, particularly as they relate to cognitive efficiency and the practitioner's ability to grasp

[3]See Manders (2008) for discussion of exact and co-exact features of mathematical diagrams.
[4]in the sense of De Toffoli (2020) and the discussion of rigor and intuition in topological arguments.
[5]See De Toffoli (2020) for discussion of criteria of acceptability in mathematical practice

the overall structure of a proof. In this setting, the primary differentiators between proofs of the same proposition are factors contributing to cognitive efficiency for a practitioner recreating or comprehending the proof. De Toffoli's example of a proof involving diagrams lessening the cognitive load on the practitioner is the *snake lemma*. This fundamental result in homological algebra is somewhat tedious to prove in prose alone, requiring the reader to bear in mind the properties of several maps and the relationships between several abstract vector spaces. Because of this, the result is almost always introduced, taught, and remembered in terms of a commutative diagram – indeed, the name "snake lemma" refers to the suggestive shape of the corresponding commutative diagram. With the help of the diagram, the practitioner can both check each step of the proof more reliably, as well as more easily grasp the global structure of the problem. In this sense, the diagram bears much of the cognitive burden that the practitioner carries when using the proof to convince himself of the existence of a formal proof of the result.

The crux of De Toffoli's argument is that, in the context of epistemic benefit, the discrepancy in *cognitive efficiency* of proofs of the snake lemma with and without diagrams renders them *different proofs* of the same proposition, even when they might be the same proof in the contexts of rigor or purity. Therefore, the commutative diagram is an essential feature of *that* proof in this context of investigation, and, as such, the diagram cannot be faithfully "translated" into prose while retaining the same proof. But how should qualities like explanatory power, mathematical beauty, and purity fit affect the individuation of proofs in the context of epistemic benefits, particularly in cases not involving diagrams? This question casts doubt on whether this is even a plausible conception of proof in the first place, or if this way of individuating proofs might lead to undesirable results. This paper serves as an introduction to the "lengthy discussion" that De Toffoli mentions is necessary to understand this phenomenon. In particular, I argue for a refined understanding of proof individuation that retains the focus on epistemic benefits and drawbacks as the primary individuator of proof presentations, while balancing this with mathematical rigor and mitigating undesirable judgments from subjective, agent-dependent criteria.

In the following section, I will discuss whether epistemic benefit actually yields a plausible, well-behaved identity criterion for proofs. I will answer in the affirmative, with some key caveats. I will also give a definition of an epistemic benefit, as well as a description of the generic practitioner that might enjoy such benefits. There will be a discussion of the commitments of engaging in this type of individuation, with respect to mathematical rigor and acceptability. The final section is a discussion of mathematical aesthetics and purity, which will clarify the interplay between epistemic benefits/drawbacks and aesthetic qualities often attributed to mathematical proofs.

Individuating Proofs

The set of all proof presentations surjects onto the set of provable true propositions $P \to Q$. We can partition the set of proof presentations by identifying any two presentations that are mapped by this surjection to the same proposition. However, further refinement of this partition requires a choice of a context of investigation; as De Toffoli discusses, there is not a satisfactory context-independent criterion for proof identity.[6] However, there is concern as to whether epistemic benefit is even a sensible condition for proof identity in the first place as it relates to desirable features of individuation, such as appropriate levels of specification, subjectivity, universality, and so on.

But it is not immediately clear that the context of epistemic benefit is well-behaved as a means to individuate proofs. For one – when, if ever, does this criterion actually equate proof presentations? In what sense can "cognitive efficiency" actually be measured and/or effectively compared across proof presentations? Furthermore, does this metric behave well with respect to comparison across media types? That is, it is not clear *a priori* whether it is even a coherent practice to compare diagrammatic and non-diagrammatic proofs with respect to cognitive efficiency, in general. Unless one can produce an example of a proof involving diagrams and a proof not involving diagrams that are similar in terms of epistemic benefits and drawbacks, the thesis that this proof criterion renders diagrams essential to certain proofs is not reasonable.

The purpose of this section is to remedy some of these objections, as well as to shed light on the motivation and meaning of some of the underlying concepts. To begin with, we address the notion of *cognitive efficiency*, as it relates to the epistemic benefits and drawbacks of a proof. A boon for cognitive efficiency proffered by De Toffoli is if one need not hold in mind a "heavy load of formulas" in order to see why steps are valid[7]. This benefit is clearly seen, for example, in the case of the snake lemma; as many of the relevant maps and their relationships are semantically embedded in a commutative diagram, the reader is free to grasp the overall structure of the proof, while still being able to quickly refer to the diagram to verify inferences. In this sense, the diagram shoulders much of the cognitive burden of understanding a proof of the snake lemma, therefore distinguishing

[6] See Gowers (2024)
[7] De Toffoli (2023)

it from a diagram-free version of the proof, in which the diagrammatic moves have been faithfully encoded in prose.

Importantly, mathematical diagrams whose roles are solely heuristic or illustrative *do not* contribute to cognitive efficiency in the same sense. According to De Toffoli's description, diagrams constitute mathematical notation when they meet the following three criteria:

1. a notation should be cognitively accessible: its constitutive formal features should be clearly identified, persistent, and stable;
2. a notation should be reproducible: it should be possible for an average practitioner to copy its constitutive formal features with relative ease and reliability, possibly with the aid of different tools such as a straightedge and/or a computer;
3. a notation should support calculations and/or inferences: it should be possible for an average practitioner to perform reasonably simple manipulations corresponding to mathematical operations.

For a given proposition $P \to Q$, along with a proof presentation A using diagrams that constitute mathematical notation and a proof presentation B using diagrams that do not constitute mathematical notation, A and B differ as proofs with respect to the context of epistemic benefits. To see this, we can evaluate the criteria in turn. To begin with, suppose B uses diagrams that are not cognitively accessible. This clearly works against cognitive efficiency – unraveling ambiguous, unstable diagrams imposes a highly nontrivial cognitive load. Secondly, if the diagrams in B are not easily and reliably reproduced by a reader, then making manipulations or changes to the diagrams corresponding to logical inferences can only be done in the mind of the practitioner – again, this arrests significant mental power that would otherwise be left to grasping the large-scale structure of the proof and making high level inferences. Finally, if the diagrams in B do not support logical inferences in the first place, then they certainly provide no epistemic benefit with respect to justification. None of these epistemic drawbacks, however, are imposed by diagrams-as-notation. Thus, the secondary refinement of proof presentations induced by epistemic benefit is fine enough to distinguish between proofs like A and proofs like B.

Subjectivity and agent-dependence

One might object that individuating proofs according to epistemic benefit is not reasonable, due to possible discrepancies between practitioners. I agree that it is necessary to mind possibly confounding effects of being overly specific with respect to the cognitive strengths and weaknesses of any one practitioner. However, it is still useful and productive to make these judgments with an agent in mind; we need simply to be careful about the precise role and nature of this agent. In order to frame this issue more concretely, consider the following theorem.[8]

Theorem 1 *Whenever a rectangle is tiled by rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.*

This theorem[9] was presented, along with a proof using a complex double integral, at the 1985 Summer Meeting of the MAA, as a call to attendees to find more simple or natural proofs of the theorem. This call was successful: in the months that followed, several alternative proofs of this theorem were presented, varying widely in simplicity, techniques used, and strength (that is, the amount of generality in which the proof may be applied). Two of these proofs are clear leaders in terms of cognitive efficiency, though they employ strikingly distinct techniques. Let R be the specially tiled rectangle. Here I sketch these two proofs.

1. **Checkerboard. (R. Rochberg)** Let R be an $a \times b$ rectangle with a tiling by rectangles with at least one integer side, where a and b are real numbers. We may assume that R is embedded in the plane with its lower-left corner at the origin. Then, consider the lattice obtained by tiling the plane by $\frac{1}{2} \times \frac{1}{2}$ black and white tiles, arranged in checkerboard fashion. Then, each of the tiles in R contains an equal amount of black and white. Therefore R contains an equal amount of black and white, as well.

   Now suppose for contradiction that neither a nor b is an integer. Then we may tile R with four non-empty rectangles: $R_1 = [0, \lfloor a \rfloor] \times [0, \lfloor b \rfloor]$, $R_2 = [\lfloor a \rfloor, a] \times [0, \lfloor b \rfloor]$, $R_3 = [0, \lfloor a \rfloor] \times [\lfloor b \rfloor, b]$, $R_4 = [\lfloor a \rfloor, a] \times [\lfloor b \rfloor, b]$, as in Figure 2. Since each of R_1, R_2, and R_3 have at least one integer side, they all contain equal amounts of black and white. However, R_4 does not contain equal amounts of black and white; this can be seen by an easy check using the facts that the sides

[8] Wagon (1987)

[9] This is a special case of a theorem of De Bruijn concerning packing n-dimensional bricks in an n-dimensional box. See De Bruijn (1969) for more details.

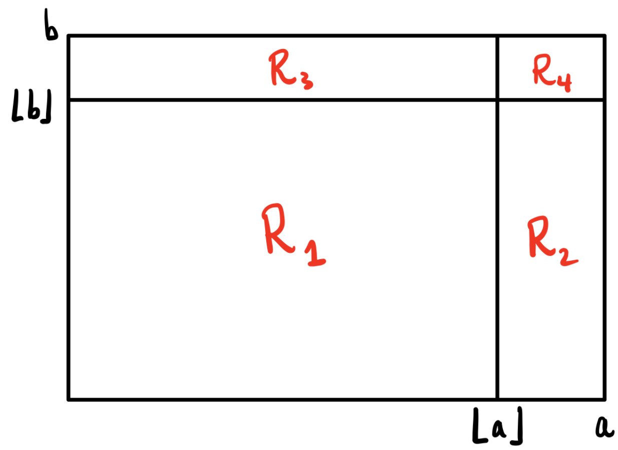

Figure 2: Subdividing the rectangle R for Rochberg's checkerboard proof.

of R_4 are all less than 1 and that its lower left corner lies on a lattice point. Thus, the union $R = R_1 \cup R_2 \cup R_3 \cup R_4$ does not contain an equal amount of black and white. This is a contradiction, so we may conclude that at least one of a or b is integral, as desired. ■

2. **Bipartite graph. (M. Paterson)** Again embed R in the plane with its lower left corner at the origin. Let S be the vertex set of corners of tiles with both coordinates integral, and let T be the set of tiles. We may form a bipartite graph G on the union $S \cup T$ by connecting with an edge each point in S with the elements in T of which it is a corner, as in Figure 3. By the hypothesis, each element in T has exactly 0, 2, or 4 corners in S. Therefore, G has an even number of edges. Now, note that each element of S which is not a corner of R must lie on either 2 or 4 tiles – hence it has even degree in G.

   But since the origin lies on only one element of T, there must be another point of S lying on an odd number of tiles. Thus, there must be another corner of R that lies in S. Therefore, either the width or the height of R is an integer. ■

From the point of view of a particular practitioner, these two proofs might present very different sets of epistemic benefits and drawbacks. To begin with, for a student with no knowledge of basic graph theory, the checkerboard proof is obviously more comprehensible. Therefore, with Burgess' conception of rigor in mind, it makes sense to evaluate these cognitive benefits in terms of a prac-

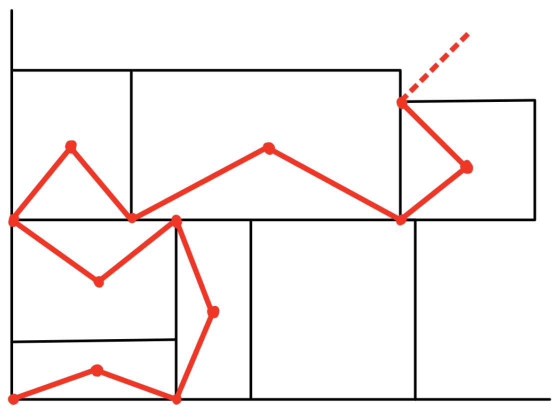

Figure 3: A portion of the bipartite graph formed by a special tiling of a rectangle.

titioner with sufficient background in relevant areas of mathematics. Even then, a mathematician with sufficient background for both proofs might feel a particular affinity, hence be more likely to recall (an epistemic benefit), for one or the other proof. For example, a graph theorist might be more drawn to the bipartite graph proof; similarly, a mathematician with a preference for constructive proof might prefer the bipartite graph proof.

Indeed, these proofs were posited by different mathematicians as being the most "natural" proofs of the fact, even though they are so different with respect to the techniques used; thus it is reasonable to assert that, to the mathematician that formulated each proof, his own proof provided the greatest epistemic benefit. Indeed, this is an *ipso facto* description of mathematical practice: a mathematician proves propositions in the way that makes the most sense to him. This principle contributes to an account for the common mathematical practice of publishing new proofs of previously established results. The rectangle-tiling theorem is a particularly concrete example of this phenomenon, but this practice in general speaks to the legitimacy of epistemic benefit (as well as mathematical purity) as a quality that is valued by mathematicians, even though it has no bearing on formal soundness.

To isolate what is happening here, it is helpful to actually name some potential epistemic benefits and drawbacks of mathematical proof, and to analyze them in the case of the two proofs of the rectangle tiling theorem. De Toffoli mentions cognitive efficiency as a major epistemic boon that is often bolstered by the inclusion of diagrams. Another benefit is the ease with which a practitioner can verify the claims made by the author of a proof. In this respect, the proofs of Theorem 1 have a similar structure: after relevant definitions and setup are given, each proof has one central inferential step which constitutes the crux of the argument. In the former, it is the verification that R_4 does not contain an equal amount of black and white; in the latter, it is the verification that each of the interior points of S have even degree in G, and that the origin has odd degree in G. Interestingly, the way in which one actually performs these verifications is also quite similar. In each case, these claims are quickly checked by means of a visualization of a general version of R, either in one's mind or drawn on paper. Suggestive heuristic diagrams were provided in the original versions of both proofs. In this sense, these proofs are similar with respect to the epistemic benefit of ease of verification.

Another epistemic benefit might be the ease with which one can communicate a proof in an informal setting – this also contributes to the ease with which a mathematician is able to recall and recreate a proof. Once again, these two proofs of the tiling theorem are quite similar in this respect. In presenting the checkerboard proof, one might draw (on a chalkboard, for example) a sketch of a tiled rectangle, along with a few suggestive squares of a checkerboard lattice. For the bipartite graph proof, the same sketch might be drawn, supplemented instead by a few edges of the bipartite graph, making sure to include the odd degree vertex at the origin. This also sheds light on a potential epistemic drawback shared by these two proofs: a dependence on a certain level of spatial intuition from the reader.

This analysis makes clear two things: (1) it is necessary to provide an explicit characterization of what is meant by an epistemic benefit or drawback. Here, we must tread carefully: if we are to redouble De Toffoli's defense of the context of epistemic benefits and drawbacks against Burgess' skepticism regarding heuristic or illustrative diagrams, it is desirable that this conception of proof identity does not distinguish along axes of explanatory power, methodological purity, mathematical beauty, or other features of proof presentations relating to mathematical aesthetics. If two proofs of $P \to Q$ differing only in explanatory power are individuated in this context, De Toffoli's argument meets a roadblock: then, a merely heuristic diagram (rather than a diagram comprising honest-to-goodness mathematical notation) could be considered essential to the identity of a proof.

And, (2), this characterization should be made in reference to a sufficiently generic mathematical practitioner; the cognitive benefit offered by a proof might vary significantly depending on the background and strengths of the reader.

In general, we may characterize an *epistemic benefit* as a feature of a proof presentation that contributes to a generic practitioner's accurate discovery, understanding, or communication of the proof. Similarly, an *epistemic drawback* is a feature that detracts from a generic practitioner's accurate discovery, understanding, or communication of the proof. Here, "accuracy" is in reference to consistency with the established and accepted body of mathematical knowledge. A generic practitioner is a typical working mathematician with sufficient background to understand and engage with the proof presentation in question.

Purity and Epistemic Benefit

De Toffoli describes the context of *mathematical purity* for individuating proofs. Roughly, two proofs may be identified in this setting if they use the same *central idea.* Furthermore, a proof of $P \to Q$ may or may not exhibit "methodological purity": in David Hilbert's formulation, whether it uses only the language, methods, and assumptions of that which lies in the presentation of the proposition[10]. But, in fact, this feature of a proof can grant significant epistemic benefit.

Consider, for example, the development of the modern field of p-adic Hodge theory. Inspired by the famous comparison theorem of de Rham, which establishes an isomorphism between the de Rham cohomology and the Betti cohomology of complex algebraic varieties, a deep and far-reaching link between geometry and topology. The classical proof of this result centers around an application of the Poincaré lemma, a statement about the same differential forms used to define de Rham cohomology in the first place.[11]

This field was revived in the 1970s, when, spurred by developments in algebraic geometry and Alexander Grothendieck's introduction of étale cohomology, mathematicians sought to devise an analog of the de Rham theorem for the étale cohomology of schemes over the p-adic numbers. In 1988, Gerd Faltings proved Jean-Marc Fontaine's 1981 conjecture of the existence of such an analogous isomorphism, but the structure of Faltings' proof bore almost no resemblance to the classical picture.[12]

This field was revived in the 2010s when Alexander Beilinson, in order to formulate a proof of Fontaine's conjecture using geometric techniques, formulated a p-adic analogue of the Poincaré

[10] Mancosu and Arana (2005)
[11] See, for example, Voisin and Shneps (2003) for more details.
[12] see Fontaine and Messings (1987) and Faltings (1988)

lemma that enabled a proof of the existence of the p-adic comparison isomorphism with the same overall structure as the classical proof. Beilinson's work was remarkable for several reasons. To begin with, it presented significant practical advantages for practitioners to understand and communicate the proof; it is much easier to grasp the overall structure of Beilinson's proof than Faltings', as the reader is able to compare it at each step with the well-established classical proof of de Rham's theorem. Indeed, the overarching structure of Beilinson's proof is remarkably similar to the picture for complex varieties, as shown in Figures 4 and 5. In each case, the desired map ρ factors through cleverly devised vector spaces, enabled in each case by an application of the appropriate version of the Poincaré lemma – note the similarity of the corresponding commutative diagrams. Though Beilinson's proof is not necessarily simpler or more accessible than Faltings' in its own right, it succeeds in placing the large-scale structure of the proof in a well-known existing mathematical framework.[13] This benefit demonstrates the epistemic power of methodological purity and aesthetic alignment.

$$\begin{tikzcd} & R\Gamma(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}) \arrow[dr] & \\ R\Gamma(X, \Omega^\bullet_X) \arrow[ur] \arrow[rr, dashed, "\rho"] & & R\Gamma(X^{\mathrm{an}}, \mathbb{C})) \end{tikzcd}$$

Figure 4: The comparison pattern for non-singular varieties over $\mathbb{C}$

$$\begin{tikzcd} & \mathcal{A}^{\natural}_{\mathrm{dR}} \arrow[dr] & \\ A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p \arrow[ur] \arrow[rr, dashed, "\rho"] & & A_{\mathrm{dR}} \end{tikzcd}$$

Figure 5: The p-adic comparison pattern.

Beilinson's proof led to an era of fruitful mathematical discovery, on the back of this new perspective in p-adic Hodge theory. The field of p-adic Hodge theory was revived, leading to several years of fruitful mathematical discovery using Beilinson's technique. Within the space of a few years after the publication of Beilinson's proof, more progress was made in the field than in the decade prior. Most notably, Bhargav Bhatt enjoyed great success extending the p-adic Poincaré lemma to

[13]See Beilinson's paper Beilinson (2011) for the original proof. Szamuely and Zabradi (2018) is a much more accessible version of Beilinson's argument, with helpful context and background.

much more general settings, and furthermore he adapted other classical results in Hodge theory to the p-adic setting in Beilinson's simpler framework, most notably the theory of spectral sequences.[14]

Note that Beilinson's proof should be distinguished from Faltings' in other respects as well. In the context of strength and generalizability, for example, it is much more far-reaching; in the context of purity, it is much more methodologically pure.[15] The techniques used, the definitions made, and the theory developed differ significantly for the two proofs. Even still, they are also individuated purely on the basis of epistemic benefit; this emphasizes the interplay between distinguishing proofs in different contexts of investigation. There is a case to be made that mathematical purity in most cases contributes positively to epistemic benefit. De Toffoli hints at this in the case of using polygon diagrams to prove statements about surfaces, mentioning that "if we prove things about geometric or topological objects, it is not surprising that proofs involving geometric or topological representations would stand out compared to those that do not involve them." It is thus reasonable to assert that using relevant diagrams as such, in contributing to methodological purity, both contribute to accurate understanding of a proof and to providing intuition about the proof technique that can stimulate the discovery of related propositions.

Here we must be careful. As mentioned before, it is not desirable for the criterion of proof identity induced by epistemic benefits and drawbacks to be so coarse as to allow merely heuristic mathematical diagrams to distinguish proof presentations from their non-diagrammatic counterparts. However, diagrams are a special case – in the purview of acceptability in mathematical practice,[16] it is paramount that diagrams meet some concrete standards for acceptability and mathematical rigor, like those laid out by De Toffoli. On the other hand, for a proof that does not contain notational diagrams as an essential component, explanatory arguments, can in many cases (as in the Beilinson example) contribute to accurate discovery, understanding, and communication in their own right. In either case we are requiring that the epistemic benefits are supported in a rigorous and mathematically acceptable way.

This discussion sheds light on the composite nature of proof identity, and consequently the difficulty of formulating a satisfactory set of criteria for individuating proofs. Though De Toffoli gave examples of differentiating proofs based on epistemic benefit in the cases of proofs containing notational diagrams, the aim of this section was to understand how we might differentiate proofs

[14] See Bhatt (2012)

[15] Luc Illusie compares the structure of Beilinson's proof in Illusie (2013)

[16] Acceptability in the sense of De Toffoli (2021), as distinct from mathematical rigor

along these lines in a more general setting, in order to suggest that this way of individuating proofs is well-behaved on a larger set of proof presentations.

Conclusion

Following De Toffoli's work justifying the essential role that certain diagrams can play in the identity of mathematical proofs, we have clarified the definition of epistemic benefits and drawbacks and addressed the nature of the generic mathematical practitioner on which this system relies. In short, an epistemic benefit (resp. drawback) is a feature of a proof presentation that enhances (resp. hinders) the discovery, understanding, and communication of the proof by a generic practitioner. Epistemic benefits manifest, among other ways, as features that reduce the cognitive load required to grasp, verify, or generalize a proof. Some ways that they are realized in practice might be by use of notational diagrams, as explored by De Toffoli, explanatory power, methodological purity, or by placing an argument in an existing mathematical framework.

The examples of the rectangle-tiling theorem and Beilinson's proof of the comparison isomorphism demonstrate that we can distinguish along lines of epistemic benefit in settings beyond proofs that contain notational diagrams versus those that do not. This suggests that this is a reasonable way to distinguish proofs in general, and indeed provides a more general structure in which De Toffoli's contentions might fit. We also addressed skepticism that might arise from the agent-dependence of this practice. We also discussed why one must be careful when evaluating epistemic benefits in the case of notational diagrams, as opposed to usage of non-notational diagrams in proofs, in order to maintain mathematical rigor and acceptability.

Additionally, we discussed the interplay between methodological purity, rigor, and epistemic benefit, as well as the relationship of these qualities with mathematical aesthetics. This is an area that merits more exploration in future work.

References

[1] Alexander Beilinson. *p-adic periods and derived de Rham cohomology*. Version Number: 5. 2011. DOI: 10.48550/ARXIV.1102.1294. URL: https://arxiv.org/abs/1102.1294 (visited on 06/04/2024).

[2] Bhargav Bhatt. *p-adic derived de Rham cohomology*. Version Number: 1. 2012. DOI: 10.48550/ARXIV.1204.6560. URL: https://arxiv.org/abs/1204.6560 (visited on 06/10/2024).

[3] Bhargav Bhatt. *p-divisibility for coherent cohomology*. Version Number: 1. 2012. DOI: 10.48550/ARXIV.1204.5831. URL: https://arxiv.org/abs/1204.5831 (visited on 06/10/2024).

[4] John P. Burgess. *Rigor and structure*. Oxford: Oxford university press, 2015. ISBN: 978-0-19-872222-9.

[5] N. G. De Bruijn. "Filling Boxes with Bricks". In: *The American Mathematical Monthly* 76.1 (Jan. 1969), pp. 37–40. ISSN: 0002-9890, 1930-0972. DOI: 10.1080/00029890.1969.12000125. URL: https://www.tandfonline.com/doi/full/10.1080/00029890.1969.12000125 (visited on 06/04/2024).

[6] Silvia De Toffoli. "Reconciling Rigor and Intuition". In: *Erkenntnis* 86.6 (Dec. 2021), pp. 1783–1802. ISSN: 0165-0106, 1572-8420. DOI: 10.1007/s10670-020-00280-x. URL: https://link.springer.com/10.1007/s10670-020-00280-x (visited on 06/03/2024).

[7] Silvia De Toffoli. "Who's Afraid of Mathematical Diagrams?" In: *Philosophers' Imprint* 23.1 (July 25, 2023). ISSN: 1533-628X. DOI: 10.3998/phimp.1348. URL: https://journals.publishing.umich.edu/phimp/article/id/1348/ (visited on 06/03/2024).

[8] Gerd Faltings. "-adic Hodge theory". In: *Journal of the American Mathematical Society* 1.1 (1988), pp. 255–299. ISSN: 0894-0347, 1088-6834. DOI: 10.1090/S0894-0347-1988-0924705-1. URL: https://www.ams.org/jams/1988-01-01/S0894-0347-1988-0924705-1/ (visited on 06/04/2024).

[9] Jean-Marc Fontaine and William Messing. "-adic periods and -adic étale cohomology". In: *Contemporary Mathematics*. Ed. by Kenneth A. Ribet. Vol. 67. Providence, Rhode Island: American Mathematical Society, 1987, pp. 179–207. ISBN: 978-0-8218-5074-9 978-0-8218-3385-8. DOI: 10.1090/conm/067/902593. URL: http://www.ams.org/conm/067/ (visited on 06/04/2024).

[10] Tim Gowers. *When are two proofs essentially the same?* Gowers's Weblog. 2007. URL: https://gowers.wordpress.com/2007/10/04/when-are-two-proofs-essentially-the-same/.

[11] Luc Illusie. *Around the Poincare lemma, after Beilinson*. 2013. URL: https://www.imo.universite-paris-saclay.fr/~luc.illusie/derived-deRham3a1.pdf.

[12] Paolo Mancosu and Andrew Arana. “Plane and Solid Geometry: A Note on Purity of Methods”. In: *From Logic to Practice*. Ed. by Gabriele Lolli, Marco Panza, and Giorgio Venturi. Vol. 308. Series Title: Boston Studies in the Philosophy and History of Science. Cham: Springer International Publishing, 2015, pp. 23–31. ISBN: 978-3-319-10433-1 978-3-319-10434-8. DOI: 10.1007/978-3-319-10434-8_2. URL: https://link.springer.com/10.1007/978-3-319-10434-8_2 (visited on 06/04/2024).

[13] Kenneth Manders. “The Euclidean Diagram (1995)”. In: *The Philosophy of Mathematical Practice*. Ed. by Paolo Mancosu. Oxford University Press, June 19, 2008, pp. 80–133. ISBN: 978-0-19-929645-3. DOI: 10.1093/acprof:oso/9780199296453.003.0005. URL: https://academic.oup.com/book/2780/chapter/143290265 (visited on 06/03/2024).

[14] Tamás Szamuely and Gergely Zábrádi. “The -adic Hodge decomposition according to Beilinson”. In: *Proceedings of Symposia in Pure Mathematics*. Ed. by Tomasso De Fernex et al. Vol. 97.2. Providence, Rhode Island: American Mathematical Society, June 1, 2018, pp. 495–572. ISBN: 978-1-4704-3578-3 978-1-4704-4680-2. DOI: 10.1090/pspum/097.2/01715. URL: http://www.ams.org/pspum/097.2 (visited on 06/04/2024).

[15] Claire Voisin and Leila Schneps. *Hodge Theory and Complex Algebraic Geometry I Volume 1*. OCLC: 668200573. Cambridge: Cambridge University Press, 2003. ISBN: 978-0-511-61534-4.

[16] Stan Wagon. “Fourteen Proofs of a Result About Tiling a Rectangle”. In: *The American Mathematical Monthly* 94.7 (Aug. 1987), pp. 601–617. ISSN: 0002-9890, 1930-0972. DOI: 10.1080/00029890.1987.12000693. URL: https://www.tandfonline.com/doi/full/10.1080/00029890.1987.12000693 (visited on 06/04/2024).