Deriving equations of motions of a cart pendulum using the Lagrangian

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1 Introduction

This is the introduction.

2 The system

Where in figure 1:
m: Mass of the pendulum [kg]
M: Mass of the cart [kg]
θ: Degrees pendulum in reference to the cart [rad]
ℓ: Length of the pendulum [m]
x: Position of the cart [m]

3 Lagrangian

Let the position and velocity of the pendulum be

$$x_s = x + \ell sin(\theta)$$
$$y_s = -\ell cos(\theta)$$
$$\dot{x}_s = \dot{x} + \ell\dot{\theta}cos(\theta)$$
$$\dot{y}_s = \ell\dot{\theta}sin(\theta)$$

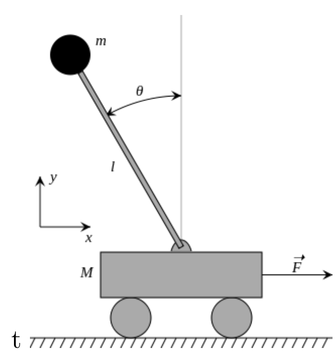

Figure 1: The pendulum

Then, to derive the equations of motion, the Lagrangion will be used. First, T, the kinetic energy of the system will be calculated.

$$\begin{aligned}
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_s^2 + \dot{y}_s^2) \\
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \ell\dot{\theta}cos(\theta))^2 + (\ell\dot{\theta}sin(\theta))^2) \\
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\ell\dot{\theta}cos(\theta) + \ell^2\dot{\theta}^2cos(\theta)^2 + \ell^2\dot{\theta}^2sin(\theta)^2) \\
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\ell\dot{\theta}cos(\theta) + \ell^2\dot{\theta}^2) \\
T &= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(2\dot{x}\ell\dot{\theta}cos(\theta) + \ell^2\dot{\theta}^2) \\
T &= \frac{1}{2}(M+m)\dot{x}^2 + m\dot{x}\ell\dot{\theta}cos(\theta) + \frac{1}{2}m\ell^2\dot{\theta}^2
\end{aligned} \tag{1}$$