Deriving equations of motions of a double pendulum on a cart using the Lagrangian

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1 Introduction

This file is made to document the simulation of a double pendulum on a cart using lagrangian mechanics. In this file a double pendulum on a cart will be discussed.

2 The system

Where in figure 1:

M:	Mass of the cart [kg]
m_1:	Mass of the first pendulum [kg]
m_2:	Mass of the second pendulum [kg]
θ_1:	Degrees first pendulum in reference to the cart [rad]
θ_2:	Degrees second pendulum in reference to the cart [rad]
ℓ_1:	Length of the first pendulum [m]
ℓ_2:	Length of the second pendulum [m]
x_c:	Position of the cart [m]

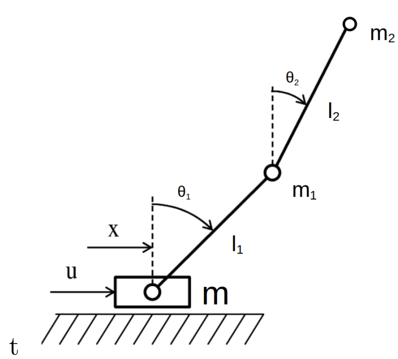

Figure 1: The pendulum

3 Lagrangian

Let the position and velocity of the pendulum be

$$
\begin{aligned}
x_1 &= \ell_1 \sin(\theta_1) + x_c & \dot{x}_1 &= \ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x} \\
y_1 &= -\ell_1 \cos(\theta_1) & \dot{y}_1 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) \\
x_2 &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) + x_c & \dot{x}_2 &= \ell_1 \theta_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2) + \dot{x} \\
y_2 &= -\ell_1 \cos(\theta_1) - \ell_2 \cos(\theta_2) & \dot{y}_2 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) + \ell_2 \dot{\theta}_2 \sin(\theta_2)
\end{aligned}
$$

To derive the equations of motion, the Lagrangian (1) will be used. First, T, the kinetic energy of the system will be calculated.

$$\mathcal{L} = T - V \tag{1}$$

$$
\begin{aligned}
T =& \frac{1}{2} M \dot{x}_c^2 + \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\
T =& \frac{1}{2} M \dot{x}_c^2 + \frac{1}{2} m_1 \left((\ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x}_c)^2 + (\ell_1 \dot{\theta}_1 \sin(\theta_1))^2 \right) \\
&+ \frac{1}{2} m_2 \left((\ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x}_c + \ell_2 \dot{\theta}_2 \cos(\theta_2))^2 + (\ell_1 \dot{\theta}_2 \sin(\theta_1) + \ell_2 \dot{\theta}_2 \sin(\theta_2))^2 \right) \\
T =& \frac{1}{2} M \dot{x}_c^2 + \frac{1}{2} m_1 (\ell_1^2 \dot{\theta}_1^2 + \dot{x}_c^2 + \dot{\theta}_1^2 \ell_1^2 \sin(\theta_1)) + \frac{1}{2} m_2 (...) \\
T =& ... + \frac{1}{2} m_2 (\ell_1^2 \dot{\theta}_1^2 \cos(\theta_1)^2 \\
&+ 2\ell_1 \dot{\theta}_1 \cos(\theta_2) \dot{x}_c + 2\ell_1 \dot{\theta}_1 \cos(\theta_1) \ell_2 \dot{\theta}_2 \cos(\theta_2) \\
&+ \dot{x}_c^2 + 2\dot{x}_c \ell_2 \dot{\theta}_2 \cos(\theta_2) + \ell_2^2 \dot{\theta}_2^2 \cos(\theta_2)^2 \\
&+ \ell_1^2 \dot{\theta}_1^2 \sin(\theta_1)^2 + 2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1) \sin(\theta_2) + \ell_2^2 \dot{\theta}_2^2 \sin(\theta_2)^2) \\
T =& ... + \frac{1}{2} m_2 (\ell_1^2 \dot{\theta}_1^2 + 2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \ell_2^2 \dot{\theta}_2^2 + 2\dot{x}_c (\ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2))) \\
T =& \frac{1}{2} (M + m_1) \dot{x}_c^2 + \frac{1}{2} (m_1 + m_2) \ell_1^2 \dot{\theta}_1^2 \\
&+ \frac{1}{2} m_2 (2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \ell_2^2 \dot{\theta}_2^2 + 2\dot{x}_c (\ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2)))
\end{aligned}
\tag{2}
$$

Then, for V, the potential energy of the system

$$
\begin{aligned}
V &= m_1 g(y_1) + m_2 g(y_2) \\
V &= -m_1 g \ell_1 \cos(\theta_1) - m_2 g(\ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2)) \\
V &= -(m_1 + m_2) g \ell_1 \cos(\theta_1) - m_2 g \ell_2 \cos(\theta_2)
\end{aligned}
\tag{3}
$$

Lastly, using (2) and (3) the Lagrangian, $\mathcal{L}$, can be formulated

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2} (M + m_1) \dot{x}_c^2 \\
&+ \frac{1}{2} (m_1 + m_2) \ell_1^2 \dot{\theta}_1^2 \\
&+ \frac{1}{2} m_2 \left(2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \ell_2^2 \dot{\theta}_2^2 + 2\dot{x}_c (\ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2)) \right) \\
&+ (m_1 + m_2) g \ell_1 \cos(\theta_1) + m_2 g \ell_2 \cos(\theta_2)
\end{aligned}
\tag{4}
$$

4 Solve for $\ddot{\theta}_1$

Using the Lagrangian equation (19)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{5}$$

Substituting (4) in equation (19) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\dot{x}_c\ell_1\cos(\theta_1) \tag{6}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) =&(m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\\ &+ m_2\ddot{x}_c\ell_1\cos(\theta_1) - m_2\dot{x}_c\ell_1\sin(\theta_1)\dot{\theta}_1\end{aligned} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\ell_1\sin(\theta_1) - m_2\dot{x}_c\ell_1\dot{\theta}_1\sin(\theta_1) \tag{8}$$

Which give the following differential equation:

$$\begin{aligned}&(m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\\ &+ m_2\ddot{x}_c\ell_1\cos(\theta_1) - m_2\dot{x}_c\ell_1\sin(\theta_1)\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)\\ &+ (m_1 + m_2)g\ell_1\sin(\theta_1) + m_2\dot{x}_c\ell_1\dot{\theta}_1\sin(\theta_1) = 0\end{aligned} \tag{9}$$

Which simplifies to:

$$\begin{aligned}&(m_1 + m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2)\\ &+ m_2\ddot{x}_c\cos(\theta_1) + (m_1 + m_2)g\sin(\theta_1) = 0\end{aligned} \tag{10}$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + m_2\ddot{x}_c\cos(\theta_1) + (m_1 + m_2)g\sin(\theta_1)}{-(m_1 + m_2)\ell_1} = \ddot{\theta}_1 \tag{11}$$

5 Solve for $\ddot{\theta}_2$

Using the Lagrangian equation (19)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{12}$$

Substituting (4) in equation (19) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_2 \ell_1 \ell_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 \ell_2^2 \dot{\theta}_2 + m_2 \dot{x}_c \ell_2 \cos(\theta_2) \tag{13}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) &= m_2 \ell_1 \ell_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ &+ m_2 \ell_2^2 \ddot{\theta}_2 + m_2 \ddot{x}_c \ell_2 \cos(\theta_2) - m_2 \dot{x}_c \ell_2 \sin(\theta_2) \dot{\theta}_2\end{aligned} \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 \dot{x}_c \dot{\theta}_2 \ell_2 \sin(\theta_2) - m_2 g \ell_2 \sin(\theta_2) \tag{15}$$

Which give the following differential equation:

$$\begin{aligned}&m_2 \ell_1 \ell_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ &+ m_2 \ell_2^2 \ddot{\theta}_2 + m_2 \ddot{x}_c \ell_2 \cos(\theta_2) - m_2 \dot{x}_c \ell_2 \sin(\theta_2) \dot{\theta}_2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ &+ m_2 \dot{x}_c \dot{\theta}_2 \ell_2 \sin(\theta_2) + m_2 g \ell_2 \sin(\theta_2) = 0\end{aligned} \tag{16}$$

Which simplifies to:

$$m_2 \ell_2 \ddot{\theta}_2 + m_2 \ell_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 \ddot{x}_c \cos(\theta_2) + m_2 g \sin(\theta_2) = 0 \tag{17}$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2 \ell_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 \ddot{x}_c \cos(\theta_2) + m_2 g \sin(\theta_2)}{-m_2 \ell_2} = \ddot{\theta}_2 \tag{18}$$

6 Solve for $\ddot{x}_c$

Using the Lagrangian equation (19)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{19}$$

Substituting (4) in equation (19) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (M + m_1)\dot{x}_c + m_2\ell_1\dot{\theta}_1\cos(\theta_1) + m_2\ell_2\dot{\theta}_2\cos(\theta_2) \tag{20}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = (M + m_1)\ddot{x}_c + m_2\ell_1\ddot{\theta}_1\cos(\theta_1) - m_2\ell_1\dot{\theta}_1^2\sin(\theta_1) + m_2\ell_2\ddot{\theta}_2\cos(\theta_2) - m_2\ell_2\dot{\theta}_2^2\cos(\theta_2) \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \tag{22}$$

Which give the following differential equation:

$$(M + m_1)\ddot{x}_c + m_2\ell_1\ddot{\theta}_1\cos(\theta_1) - m_2\ell_1\dot{\theta}_1^2\sin(\theta_1) + m_2\ell_2\ddot{\theta}_2\cos(\theta_2) - m_2\ell_2\dot{\theta}_2^2\cos(\theta_2) = 0 \tag{23}$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2\ell_1\ddot{\theta}_1\cos(\theta_1) - m_2\ell_1\dot{\theta}_1^2\sin(\theta_1) + m_2\ell_2\ddot{\theta}_2\cos(\theta_2) - m_2\ell_2\dot{\theta}_2^2\cos(\theta_2)}{-M + m_1} = \ddot{x}_c \tag{24}$$

7 Liniearize using small angle approxiomation for: $\ddot{x}_c, \ddot{\theta}_1, \ddot{\theta}_2$

$$(M+m_1)\ddot{x}_c + m_2\ell_1\ddot{\theta}_1 - m_2\ell_2\ddot{\theta}_2 \ = F \tag{25}$$

$$\ell_2\ddot{\theta}_2 + \ell_1\ddot{\theta}_1 - \ddot{x}_c + g\theta_2 = 0 \tag{26}$$

$$(m_1+m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2 + m_2\ddot{x}_c + (m_1+m_2)g\theta_1 = 0 \tag{27}$$

Or in matrix form:

$$\begin{bmatrix} (M+m_1) & m_2\ell_1 & -m_2\ell_2 \\ -1 & \ell_2 & \ell_2 \\ m_2 & (m_1+m_2)\ell_1 & m_2\ell_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (m_1+m_2)g & g \end{bmatrix} \begin{bmatrix} x_c \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

With state vector equalling:

$$X = \begin{bmatrix} x_c \\ \theta_1 \\ \theta_2 \\ \dot{x}_c \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \qquad \text{or} \qquad \dot{X} = \begin{bmatrix} \dot{x}_c \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{x}_c \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \tag{29}$$

Now write the second-order into the first-order form: $M\dot{X} + NX = F$:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (M+m_1) & m_2\ell_1 & -m_2\ell_2 \\ 0 & 0 & 0 & -1 & \ell_2 & \ell_2 \\ 0 & 0 & 0 & m_2 & (m_1+m_2)\ell_1 & m_2\ell_2 \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (m_1+m_2)g & g & 0 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

Using the following formula for calculating the A and B matrix for the state space:

$$A = M^{-1}N, B = M^{-1}F(\text{voeg bron in}) \tag{31}$$

Resulting in the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{g(m_1+m_2)(-l_1m_2-l_2m_2)}{Ml_1m_1+Ml_1m_2-Ml_2m_2+l_1m_1^2-3l_1m_2^2-l_2m_1m_2-l_2m_2^2} & \frac{g(-l_1m_2-l_2m_2)}{Ml_1m_1+Ml_1m_2-Ml_2m_2+l_1m_1^2-3l_1m_2^2-l_2m_1m_2-l_2m_2^2} & 0 & 0 & 0 \\ 0 & \frac{g(m_1+m_2)(M+m_1-m_2)}{Ml_1m_1+Ml_1m_2-Ml_2m_2+l_1m_1^2-3l_1m_2^2-l_2m_1m_2-l_2m_2^2} & \frac{g(M+m_1-m_2)}{Ml_1m_1+Ml_1m_2-Ml_2m_2+l_1m_1^2-3l_1m_2^2-l_2m_1m_2-l_2m_2^2} & 0 & 0 & 0 \\ 0 & \frac{g(m_1+m_2)(-Ml_2-l_1m_2-l_2m_1)}{Ml_1l_2m_1+Ml_1l_2m_2-Ml_2^2m_2+l_1l_2m_1^2-3l_1l_2m_2^2-l_2^2m_1m_2-l_2^2m_2^2} & \frac{g(-Ml_2-l_1m_2-l_2m_1)}{Ml_1l_2m_1+Ml_1l_2m_2-Ml_2^2m_2+l_1l_2m_1^2-3l_1l_2m_2^2-l_2^2m_1m_2-l_2^2m_2^2} & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{l_1m_1+l_1m_2-l_2m_2}{Ml_1m_1+Ml_1m_2-Ml_2m_2+l_1m_1^2-3l_1m_2^2-l_2m_1m_2-l_2m_2^2} \\ -\frac{2m_2}{Ml_1m_1+Ml_1m_2-Ml_2m_2+l_1m_1^2-3l_1m_2^2-l_2m_1m_2-l_2m_2^2} \\ \frac{l_1m_1+l_1m_2+l_2m_2}{Ml_1l_2m_1+Ml_1l_2m_2-Ml_2^2m_2+l_1l_2m_1^2-3l_1l_2m_2^2-l_2^2m_1m_2-l_2^2m_2^2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$