

Deriving equations of motions of a double pendulum on a cart using the Lagrangian

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1 Introduction

This file is made to document the simulation of a double pendulum on a cart using lagrangian mechanics. In this file a double pendulum on a cart will be discussed.

2 The system

Where in figure 1:

- M : Mass of the cart [kg]
- m_1 : Mass of the first pendulum [kg]
- m_2 : Mass of the second pendulum [kg]
- θ_1 : Degrees first pendulum in reference to the cart [rad]
- θ_2 : Degrees second pendulum in reference to the cart [rad]
- ℓ_1 : Length of the first pendulum [m]
- ℓ_2 : Length of the second pendulum [m]
- x_c : Position of the cart [m]

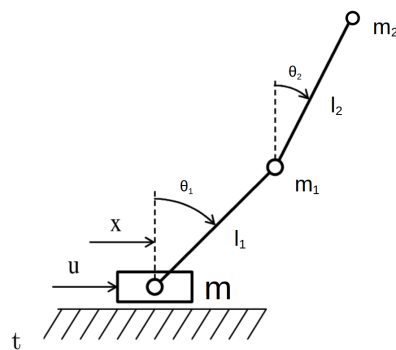


Figure 1: The pendulum

3 Lagrangian

Let the position and velocity of the pendulum be

$$\begin{aligned}
 x_1 &= \ell_1 \sin(\theta_1) + x_c & \dot{x}_1 &= \ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x} \\
 y_1 &= -\ell_1 \cos(\theta_1) & \dot{y}_1 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) \\
 x_2 &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) + x_c & \dot{x}_2 &= \ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2) + \dot{x} \\
 y_2 &= -\ell_1 \cos(\theta_1) - \ell_2 \cos(\theta_2) & \dot{y}_2 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) + \ell_2 \dot{\theta}_2 \sin(\theta_2)
 \end{aligned}$$

To derive the equations of motion, the Lagrangian (1) will be used. First, T , the kinetic energy of the system will be calculated.

$$\mathcal{L} = T - V \quad (1)$$

$$\begin{aligned}
 T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_s^2 + \dot{y}_s^2) \\
 T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \ell\dot{\theta}\cos(\theta))^2 + (\ell\dot{\theta}\sin(\theta))^2) \\
 T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\ell\dot{\theta}\cos(\theta) + \ell^2\dot{\theta}^2\cos^2(\theta) + \ell^2\dot{\theta}^2\sin^2(\theta)) \\
 T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\ell\dot{\theta}\cos(\theta) + \ell^2\dot{\theta}^2) \\
 T &= \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}m(2\dot{x}\ell\dot{\theta}\cos(\theta) + \ell^2\dot{\theta}^2) \\
 T &= \frac{1}{2}(M + m)\dot{x}^2 + m\dot{x}\ell\dot{\theta}\cos(\theta) + \frac{1}{2}m\ell^2\dot{\theta}^2
 \end{aligned} \quad (2)$$

Then, for V , the potential energy of the system

$$\begin{aligned}
 V &= m_1 g(y_1) + m_2 g(y_2) \\
 V &= -m_1 g \ell_1 \cos(\theta_1) - m_2 g(\ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2)) \\
 V &= -(m_1 + m_2)g\ell_1 \cos(\theta_1) - m_2 g \ell_2 \cos(\theta_2)
 \end{aligned} \quad (3)$$

Lastly, using (2) and (3) the Lagrangian, \mathcal{L} , can be formulated

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2}(M + m_1)\dot{x}_c^2 \\
 &+ \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 \\
 &+ \frac{1}{2}m_2 \left(2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + \ell_2^2\dot{\theta}_2^2 + 2\dot{x}_c(\ell_1\dot{\theta}_1 \cos(\theta_1) + \ell_2\dot{\theta}_2 \cos(\theta_2)) \right) \\
 &+ (m_1 + m_2)g\ell_1 \cos(\theta_1) + m_2 g \ell_2 \cos(\theta_2)
 \end{aligned} \quad (4)$$

4 Solve for $\ddot{\theta}_1$

Using the Lagrangian equation (12)

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{\partial \mathcal{L}}{\partial q} \quad (5)$$

Substituting (4) in equation (12) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2 \dot{\theta}_1 + m_2 \ell_1 \ell_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 \dot{x}_c \ell_1 \cos(\theta_1) \quad (6)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) = & (m_1 + m_2)\ell_1^2 \ddot{\theta}_1 + m_2 \ell_1 \ell_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ & + m_2 \ddot{x}_c \ell_1 \cos(\theta_1) - m_2 \dot{x}_c \ell_1 \sin(\theta_1) \dot{\theta}_1 \end{aligned} \quad (7)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2)g \ell_1 \sin(\theta_1) - m_2 \dot{x}_c \ell_1 \dot{\theta}_1 \sin(\theta_1) \quad (8)$$

Which give the following differential equation:

$$\begin{aligned} & (m_1 + m_2)\ell_1^2 \ddot{\theta}_1 + m_2 \ell_1 \ell_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ & + m_2 \ddot{x}_c \ell_1 \cos(\theta_1) - m_2 \dot{x}_c \ell_1 \sin(\theta_1) \dot{\theta}_1 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ & + (m_1 + m_2)g \ell_1 \sin(\theta_1) + m_2 \dot{x}_c \ell_1 \dot{\theta}_1 \sin(\theta_1) = 0 \end{aligned} \quad (9)$$

Which simplifies to:

$$\begin{aligned} & (m_1 + m_2)\ell_1 \ddot{\theta}_1 + m_2 \ell_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 \ell_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \\ & + m_2 \ddot{x}_c \cos(\theta_1) + (m_1 + m_2)g \sin(\theta_1) = 0 \end{aligned} \quad (10)$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2 \ell_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 \ell_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + m_2 \ddot{x}_c \cos(\theta_1) + (m_1 + m_2)g \sin(\theta_1)}{-(m_1 + m_2)\ell_1} = \ddot{\theta}_1 \quad (11)$$

5 Solve for $\ddot{\theta}_2$

Using the Lagrangian equation (12)

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{\partial \mathcal{L}}{\partial q} \quad (12)$$

Substituting (4) in equation (12) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_2 \ell_1 \ell_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 \ell_2^2 \dot{\theta}_2 + m_2 \dot{x}_c \ell_2 \cos(\theta_2) \quad (13)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) &= m_2 \ell_1 \ell_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \\ &\quad + m_2 \ell_2^2 \ddot{\theta}_2 + m_2 \ddot{x}_c \ell_2 \cos(\theta_2) - m_2 \dot{x}_c \ell_2 \sin(\theta_2) \dot{\theta}_2 \end{aligned} \quad (14)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 \dot{x}_c \dot{\theta}_2 \ell_2 \sin(\theta_2) - m_2 g \ell_2 \sin(\theta_2) \quad (15)$$

Which give the following differential equation:

$$\begin{aligned} &m_2 \ell_1 \ell_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \\ &+ m_2 \ell_2^2 \ddot{\theta}_2 + m_2 \ddot{x}_c \ell_2 \cos(\theta_2) - m_2 \dot{x}_c \ell_2 \sin(\theta_2) \dot{\theta}_2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ &+ m_2 \dot{x}_c \dot{\theta}_2 \ell_2 \sin(\theta_2) + m_2 g \ell_2 \sin(\theta_2) = 0 \end{aligned} \quad (16)$$

Which simplifies to:

$$m_2 \ell_2 \ddot{\theta}_2 + m_2 \ell_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 \ddot{x}_c \cos(\theta_2) + m_2 g \sin(\theta_2) = 0 \quad (17)$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2 \ell_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 \ddot{x}_c \cos(\theta_2) + m_2 g \sin(\theta_2)}{-m_2 \ell_2} = \ddot{\theta}_2 \quad (18)$$

6 Solve for \ddot{x}