Deriving equations of motions of a double pendulum on a cart using the Lagrangian

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1 Introduction

This file is made to document the simulation of a double pendulum on a cart using lagrangian mechanics. In this file a double pendulum on a cart will be discussed.

2 The system

Where in figure 1:

M:	Mass of the cart [kg]
m_1:	Mass of the first pendulum [kg]
m_2:	Mass of the second pendulum [kg]
θ_1:	Degrees first pendulum in reference to the cart [rad]
θ_2:	Degrees second pendulum in reference to the cart [rad]
ℓ_1:	Length of the first pendulum [m]
ℓ_2:	Length of the second pendulum [m]
x_c:	Position of the cart [m]

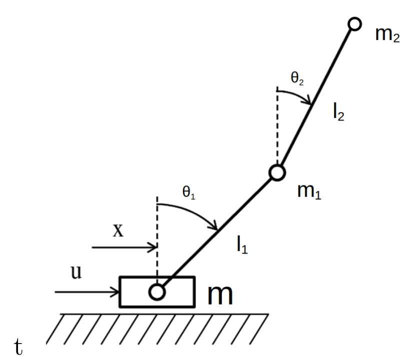

Figure 1: The pendulum

3 Lagrangian

Let the position and velocity of the pendulum be

$$
\begin{aligned}
x_1 &= \ell_1 \sin(\theta_1) + x_c & \dot{x}_1 &= \ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x} \\
y_1 &= -\ell_1 \cos(\theta_1) & \dot{y}_1 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) \\
x_2 &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) + x_c & \dot{x}_2 &= \ell_1 \theta_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2) + \dot{x} \\
y_2 &= -\ell_1 \cos(\theta_1) - \ell_2 \cos(\theta_2) & \dot{y}_2 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) + \ell_2 \dot{\theta}_2 \sin(\theta_2)
\end{aligned}
$$

To derive the equations of motion, the Lagrangian (1) will be used. First, T, the kinetic energy of the system will be calculated.

$$\mathcal{L} = T - V \tag{1}$$

$$
\begin{aligned}
T =& \frac{1}{2} M \dot{x}_c^2 + \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\
T =& \frac{1}{2} M \dot{x}_c^2 + \frac{1}{2} m_1 \left((\ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x}_c)^2 + (\ell_1 \dot{\theta}_1 \sin(\theta_1))^2 \right) \\
&+ \frac{1}{2} m_2 \left((\ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x}_c + \ell_2 \dot{\theta}_2 \cos(\theta_2))^2 + (\ell_1 \dot{\theta}_2 \sin(\theta_1) + \ell_2 \dot{\theta}_2 \sin(\theta_2))^2 \right) \\
T =& \frac{1}{2} M \dot{x}_c^2 + \frac{1}{2} m_1 (\dot{x}_c^2 + 2 \ell_1 \dot{x}_c \dot{\theta}_1 \cos(\theta_1) + \ell_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (...) \\
T =& ... + \frac{1}{2} m_2 (\ell_1^2 \dot{\theta}_1^2 \cos(\theta_1)^2 \\
&+ 2 \ell_1 \dot{\theta}_1 \cos(\theta_2) \dot{x}_c + 2 \ell_1 \dot{\theta}_1 \cos(\theta_1) \ell_2 \dot{\theta}_2 \cos(\theta_2) \\
&+ \dot{x}_c^2 + 2 \dot{x}_c \ell_2 \dot{\theta}_2 \cos(\theta_2) + \ell_2^2 \dot{\theta}_2^2 \cos(\theta_2)^2 \\
&+ \ell_1^2 \dot{\theta}_1^2 \sin(\theta_1)^2 + 2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1) \sin(\theta_2) + \ell_2^2 \dot{\theta}_2^2 \sin(\theta_2)^2) \\
T =& ... + \frac{1}{2} m_2 (\dot{x}_c^2 + \ell_1^2 \dot{\theta}_1^2 + 2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \ell_2^2 \dot{\theta}_2^2 + 2 \dot{x}_c (\ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2))) \\
T =& \frac{1}{2} (M + m_1 + m_2) \dot{x}_c^2 + \frac{1}{2} (m_1 + m_2) \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2 \\
&+ m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) \dot{x}_c \ell_1 \dot{\theta}_1 \cos(\theta_1) + m_2 \dot{x}_c \ell_2 \dot{\theta}_2 \cos(\theta_2)
\end{aligned}
\tag{2}
$$

Then, for V, the potential energy of the system

$$
\begin{aligned}
V &= m_1 g (y_1) + m_2 g (y_2) \\
V &= -m_1 g \ell_1 \cos(\theta_1) - m_2 g (\ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2)) \\
V &= -(m_1 + m_2) g \ell_1 \cos(\theta_1) - m_2 g \ell_2 \cos(\theta_2)
\end{aligned}
\tag{3}
$$

Lastly, using (2) and (3) the Lagrangian, $\mathcal{L}$, can be formulated

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2} (M + m_1 + m_2) \dot{x}_c^2 + \frac{1}{2} (m_1 + m_2) \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2 \\
&+ m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) \dot{x}_c \ell_1 \dot{\theta}_1 \cos(\theta_1) + m_2 \dot{x}_c \ell_2 \dot{\theta}_2 \cos(\theta_2) \\
&+ (m_1 + m_2) g \ell_1 \cos(\theta_1) + m_2 g \ell_2 \cos(\theta_2)
\end{aligned}
\tag{4}
$$

4 Solve for $\ddot{\theta}_1$

Using the Lagrangian equation (19)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{5}$$

Substituting (4) in equation (19) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1 - \theta_2) + (m_1 + m_2)\dot{x}_c\ell_1\cos(\theta_1) \tag{6}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) =&(m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\\ &+ (m_1 + m_2)\ddot{x}_c\ell_1\cos(\theta_1) - (m_1 + m_2)\dot{x}_c\ell_1\sin(\theta_1)\dot{\theta}_1\end{aligned} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\ell_1\sin(\theta_1) - (m_1 + m_2)\dot{x}_c\ell_1\dot{\theta}_1\sin(\theta_1) \tag{8}$$

Which give the following differential equation:

$$\begin{aligned}&(m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\\ &+ (m_1 + m_2)\ddot{x}_c\ell_1\cos(\theta_1) - (m_1 + m_2)\dot{x}_c\ell_1\sin(\theta_1)\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)\\ &+ (m_1 + m_2)g\ell_1\sin(\theta_1) + (m_1 + m_2)\dot{x}_c\ell_1\dot{\theta}_1\sin(\theta_1) = 0\end{aligned} \tag{9}$$

Which simplifies to:

$$\begin{aligned}&(m_1 + m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2)\\ &+ (m_1 + m_2)\ddot{x}_c\cos(\theta_1) + (m_1 + m_2)g\sin(\theta_1) = 0\end{aligned} \tag{10}$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)\ddot{x}_c\cos(\theta_1) + (m_1 + m_2)g\sin(\theta_1)}{-(m_1 + m_2)\ell_1} = \ddot{\theta}_1 \tag{11}$$

5 Solve for $\ddot{\theta}_2$

Using the Lagrangian equation (19)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{12}$$

Substituting (4) in equation (19) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_2\ell_1\ell_2\dot{\theta}_1\cos(\theta_1-\theta_2) + m_2\ell_2^2\dot{\theta}_2 + m_2\dot{x}_c\ell_2\cos(\theta_2) \tag{13}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) &= m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\ell_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\\ &+ m_2\ell_2^2\ddot{\theta}_2 + m_2\ddot{x}_c\ell_2\cos(\theta_2) - m_2\dot{x}_c\ell_2\sin(\theta_2)\dot{\theta}_2\end{aligned} \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2\dot{x}_c\dot{\theta}_2\ell_2\sin(\theta_2) - m_2 g\ell_2\sin(\theta_2) \tag{15}$$

Which give the following differential equation:

$$\begin{aligned}&m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\ell_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\\ &+ m_2\ell_2^2\ddot{\theta}_2 + m_2\ddot{x}_c\ell_2\cos(\theta_2) - m_2\dot{x}_c\ell_2\sin(\theta_2)\dot{\theta}_2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)\\ &+ m_2\dot{x}_c\dot{\theta}_2\ell_2\sin(\theta_2) + m_2 g\ell_2\sin(\theta_2) = 0\end{aligned} \tag{16}$$

Which simplifies to:

$$m_2\ell_2\ddot{\theta}_2 + m_2\ell_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2\ddot{x}_c\cos(\theta_2) + m_2 g\sin(\theta_2) = 0 \tag{17}$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2\ell_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2\ddot{x}_c\cos(\theta_2) + m_2 g\sin(\theta_2)}{-m_2\ell_2} = \ddot{\theta}_2 \tag{18}$$

6 Solve for $\ddot{x}_c$

Using the Lagrangian equation (19)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{19}$$

Substituting (4) in equation (19) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_c} = (M + m_1 + m_2)\dot{x}_c + (m_1 + m_2)\ell_1\dot{\theta}_1\cos(\theta_1) + m_2\ell_2\dot{\theta}_2\cos(\theta_2) \tag{20}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_c}\right) &= (M + m_1 + m_2)\ddot{x}_c + (m_1 + m_2)\ell_1\ddot{\theta}_1\cos(\theta_1) \\ &- (m_1 + m_2)\ell_1\dot{\theta}_1^2\sin(\theta_1) + m_2\ell_2\ddot{\theta}_2\cos(\theta_2) - m_2\ell_2\dot{\theta}_2^2\sin(\theta_2)\end{aligned} \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial x_c} = 0 \tag{22}$$

Which give the following differential equation:

$$\begin{aligned}&(M + m_1 + m_2)\ddot{x}_c + (m_1 + m_2)\ell_1\ddot{\theta}_1\cos(\theta_1) \\ &- (m_1 + m_2)\ell_1\dot{\theta}_1^2\sin(\theta_1) + m_2\ell_2\ddot{\theta}_2\cos(\theta_2) - m_2\ell_2\dot{\theta}_2^2\sin(\theta_2) = 0\end{aligned} \tag{23}$$

With the differential equation solved for $\ddot{x}_c$:

$$\frac{(m_1 + m_2)\ell_1\ddot{\theta}_1\cos(\theta_1) - (m_1 + m_2)\ell_1\dot{\theta}_1^2\sin(\theta_1) + m_2\ell_2\ddot{\theta}_2\cos(\theta_2) - m_2\ell_2\dot{\theta}_2^2\sin(\theta_2)}{-M + m_1 + m_2} = \ddot{x}_c \tag{24}$$

7 Liniearize using small angle approxiomation for: $\ddot{x}_c, \ddot{\theta}_1, \ddot{\theta}_2$

$$(M + m_1 + m_2)\ddot{x}_c + (m_1 + m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2 \;\; = F \tag{25}$$

$$(m_1 + m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2 + (m_1 + m_2)\ddot{x}_c - (m_1 + m_2)g\theta_1 = 0 \tag{26}$$

$$m_2\ell_2\ddot{\theta}_2 + m_2\ell_1\ddot{\theta}_1 + m_2\ddot{x}_c - m_2 g\theta_2 = 0 \tag{27}$$

Or in matrix form:

$$\begin{bmatrix} (M + m_1 + m_2) & (m_1 + m_2)\ell_1 & m_2\ell_2 \\ (m_1 + m_2) & (m_1 + m_2)\ell_1 & m_2\ell_2 \\ m_2 & m_2\ell_1 & m_2\ell_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(m_1 + m_2)g & 0 \\ 0 & 0 & -g \end{bmatrix} \begin{bmatrix} x_c \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

With state vector equalling:

$$X = \begin{bmatrix} x_c \\ \theta_1 \\ \theta_2 \\ \dot{x}_c \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \qquad \text{or} \qquad \dot{X} = \begin{bmatrix} \dot{x}_c \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{x}_c \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \tag{29}$$

Now write the second-order into the first-order form: $M\dot{X} + NX = F$:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (M + m_2 + m_1) & (m_1 + m_2)\ell_1 & m_2\ell_2 \\ 0 & 0 & 0 & (m_1 + m_2) & (m_1 + m_2)\ell_1 & m_2\ell_2 \\ 0 & 0 & 0 & m2 & m_2\ell_1 & m_2\ell_2 \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (m_1 + m_2)g & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

Using the following formula for calculating the A and B matrix for the state space:

$$A = M^{-1}N, B = M^{-1}F(\text{voeg bron in}) \tag{31}$$

Resulting in the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{-g(m_1+m_2)}{M} & 0 & 0 & 0 & 0 \\ 0 & \frac{g(M+m_1)(m_1+m_2)}{(Ml_1m_1)} & \frac{-gm_2}{(l_1m_1)} & 0 & 0 & 0 \\ 0 & \frac{-g(m_1+m_2)}{(l_2m_1)} & \frac{g(m_1+m_2)}{(l_2m_1)} & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M \\ -1/Ml_1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$