Deriving equations of motions of a double pendulum on a cart using the Lagrangian

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1 Introduction

This file is made to document the simulation of a double pendulum on a cart using lagrangian mechanics. In this file a double pendulum on a cart will be discussed.

2 The system

Where in figure 1:

M:	Mass of the cart [kg]
m_1:	Mass of the first pendulum [kg]
m_2:	Mass of the second pendulum [kg]
θ_1:	Degrees first pendulum in reference to the cart [rad]
θ_2:	Degrees second pendulum in reference to the cart [rad]
ℓ_1:	Length of the first pendulum [m]
ℓ_2:	Length of the second pendulum [m]
x_c:	Position of the cart [m]

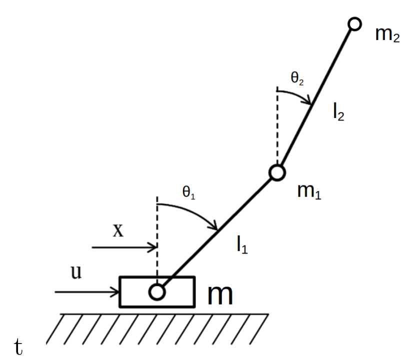

Figure 1: The pendulum

3 Lagrangian

Let the position and velocity of the pendulum be

$$
\begin{aligned}
x_1 &= \ell_1 \sin(\theta_1) + x_c & \dot{x}_1 &= \ell_1 \dot{\theta}_1 \cos(\theta_1) + \dot{x} \\
y_1 &= -\ell_1 \cos(\theta_1) & \dot{y}_1 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) \\
x_2 &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) + x_c & \dot{x}_2 &= \ell_1 \theta_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2) + \dot{x} \\
y_2 &= -\ell_1 \cos(\theta_1) - \ell_2 \cos(\theta_2) & \dot{y}_2 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) + \ell_2 \dot{\theta}_2 \sin(\theta_2)
\end{aligned}
$$

To derive the equations of motion, the Lagrangian (1) will be used. First, T, the kinetic energy of the system will be calculated.

$$
\mathcal{L} = T - V \tag{1}
$$

$$
\begin{aligned}
T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_s^2 + \dot{y}_s^2) \\
T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m ((\dot{x} + \ell \dot{\theta} cos(\theta))^2 + (\ell \dot{\theta} sin(\theta))^2) \\
T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 + 2\dot{x}\ell\dot{\theta} cos(\theta) + \ell^2 \dot{\theta}^2 cos(\theta)^2 + \ell^2 \dot{\theta}^2 sin(\theta)^2) \\
T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 + 2\dot{x}\ell\dot{\theta} cos(\theta) + \ell^2 \dot{\theta}^2) \\
T &= \frac{1}{2} (M + m) \dot{x}^2 + \frac{1}{2} m (2\dot{x}\ell\dot{\theta} cos(\theta) + \ell^2 \dot{\theta}^2) \\
T &= \frac{1}{2} (M + m) \dot{x}^2 + m\dot{x}\ell\dot{\theta} cos(\theta) + \frac{1}{2} m \ell^2 \dot{\theta}^2
\end{aligned} \tag{2}
$$

Then, for V, the potential energy of the system

$$
\begin{aligned}
V &= m_1 g (y_1) + m_2 g (y_2) \\
V &= -m_1 g \ell_1 \cos(\theta_1) - m_2 g (\ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2)) \\
V &= -(m_1 + m_2) g \ell_1 \cos(\theta_1) - m_2 g \ell_2 \cos(\theta_2)
\end{aligned} \tag{3}
$$

Lastly, using (2) and (3) the Lagrangian, $\mathcal{L}$, can be formulated

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2} (M + m_1) \dot{x}_c^2 \\
&+ \frac{1}{2} (m_1 + m_2) \ell_1^2 \dot{\theta}_1^2 \\
&+ \frac{1}{2} m_2 \left(2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \ell_2^2 \dot{\theta}_2^2 + 2\dot{x}_c (\ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2)) \right) \\
&+ (m_1 + m_2) g \ell_1 \cos(\theta_1) + m_2 g \ell_2 \cos(\theta_2)
\end{aligned} \tag{4}
$$

4 Solve for $\ddot{\theta}_1$

Using the Lagrangian equation (12)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{5}$$

Substituting (4) in equation (12) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\dot{x}_c\ell_1\cos(\theta_1) \tag{6}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) =& (m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ &+ m_2\ddot{x}_c\ell_1\cos(\theta_1) - m_2\dot{x}_c\ell_1\sin(\theta_1)\dot{\theta}_1 \end{aligned} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\ell_1\sin(\theta_1) - m_2\dot{x}_c\ell_1\dot{\theta}_1\sin(\theta_1) \tag{8}$$

Which give the following differential equation:

$$\begin{aligned} &(m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ &+ m_2\ddot{x}_c\ell_1\cos(\theta_1) - m_2\dot{x}_c\ell_1\sin(\theta_1)\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) \\ &+ (m_1 + m_2)g\ell_1\sin(\theta_1) + m_2\dot{x}_c\ell_1\dot{\theta}_1\sin(\theta_1) = 0 \end{aligned} \tag{9}$$

Which simplifies to:

$$\begin{aligned} &(m_1 + m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) \\ &+ m_2\ddot{x}_c\cos(\theta_1) + (m_1 + m_2)g\sin(\theta_1) = 0 \end{aligned} \tag{10}$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + m_2\ddot{x}_c\cos(\theta_1) + (m_1 + m_2)g\sin(\theta_1)}{-(m_1 + m_2)\ell_1} = \ddot{\theta}_1 \tag{11}$$

5 Solve for $\ddot{\theta}_2$

Using the Lagrangian equation (12)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \tag{12}$$

Substituting (4) in equation (12) and solving for θ_1 gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_2 \ell_1 \ell_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 \ell_2^2 \dot{\theta}_2 + m_2 \dot{x}_c \ell_2 \cos(\theta_2) \tag{13}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = {} & m_2 \ell_1 \ell_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ & + m_2 \ell_2^2 \ddot{\theta}_2 + m_2 \ddot{x}_c \ell_2 \cos(\theta_2) - m_2 \dot{x}_c \ell_2 \sin(\theta_2) \dot{\theta}_2 \end{aligned} \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 \dot{x}_c \dot{\theta}_2 \ell_2 \sin(\theta_2) - m_2 g \ell_2 \sin(\theta_2) \tag{15}$$

Which give the following differential equation:

$$\begin{aligned} & m_2 \ell_1 \ell_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \ell_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ & + m_2 \ell_2^2 \ddot{\theta}_2 + m_2 \ddot{x}_c \ell_2 \cos(\theta_2) - m_2 \dot{x}_c \ell_2 \sin(\theta_2) \dot{\theta}_2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ & + m_2 \dot{x}_c \dot{\theta}_2 \ell_2 \sin(\theta_2) + m_2 g \ell_2 \sin(\theta_2) = 0 \end{aligned} \tag{16}$$

Which simplifies to:

$$m_2 \ell_2 \ddot{\theta}_2 + m_2 \ell_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 \ddot{x}_c \cos(\theta_2) + m_2 g \sin(\theta_2) = 0 \tag{17}$$

With the differential equation solved for $\ddot{\theta}_1$:

$$\frac{m_2 \ell_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \ell_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 \ddot{x}_c \cos(\theta_2) + m_2 g \sin(\theta_2)}{-m_2 \ell_2} = \ddot{\theta}_2 \tag{18}$$

6 Solve for $\ddot{x}_c$

Using the Lagrangian equation (12)