Deriving equations of motions of a cart pendulum using the Lagrangian

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1 Introduction

This is the introduction.

2 The system

Where in figure 1:
m: Mass of the pendulum [kg]
M: Mass of the cart [kg]
θ: Degrees pendulum in reference to the cart [rad]
ℓ: Length of the pendulum [m]
x: Position of the cart [m]

3 Lagrangian

Let the position and velocity of the pendulum be

$$
\begin{aligned}
x_s &= x + \ell sin(\theta) \\
y_s &= -\ell cos(\theta) \\
\dot{x}_s &= \dot{x} + \ell\dot{\theta} cos(\theta) \\
\dot{y}_s &= \ell\dot{\theta} sin(\theta)
\end{aligned}
$$

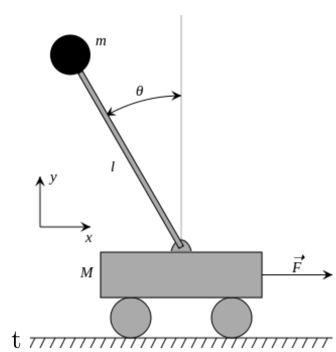

Figure 1: The pendulum

To derive the equations of motion, the Lagrangian (1) will be used. First, T, the kinetic energy of the system will be calculated.

$$\mathcal{L} = T - V \tag{1}$$

$$\begin{aligned}
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_s^2 + \dot{y}_s^2) \\
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \ell\dot{\theta}cos(\theta))^2 + (\ell\dot{\theta}sin(\theta))^2) \\
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\ell\dot{\theta}cos(\theta) + \ell^2\dot{\theta}^2cos(\theta)^2 + \ell^2\dot{\theta}^2sin(\theta)^2) \\
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\ell\dot{\theta}cos(\theta) + \ell^2\dot{\theta}^2) \\
T &= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(2\dot{x}\ell\dot{\theta}cos(\theta) + \ell^2\dot{\theta}^2) \\
T &= \frac{1}{2}(M+m)\dot{x}^2 + m\dot{x}\ell\dot{\theta}cos(\theta) + \frac{1}{2}m\ell^2\dot{\theta}^2
\end{aligned} \tag{2}$$

Then, for V, the potential energy of the system

$$V = -mg\ell cos(\theta) \tag{3}$$

Lastly, using (2) and (3) the Lagrangian, $\mathcal{L}$, can be formulated

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + m\dot{x}\ell\dot{\theta}cos(\theta) + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell cos(\theta) \tag{4}$$

4 Solve for $\ddot{\theta}$

Using the Lagrangian equation (5)

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) = \frac{\partial\mathcal{L}}{\partial q} \tag{5}$$

5 Solve for $\ddot{x}$