MATH4045, Mathematics Group Project, Project 2, Group D, Thursday 7th December 2023

MODELLING FOR THEME PARK RIDES: DODGEM CARS

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ABSTRACT. We propose mathematical models of the forces involved in theme park dodgem cars to produce realistic predictions of the forces exerted on the car during collisions and their trajectories. This report explores three mathematical models, all developing from the previous, demonstrating different possible collision scenarios that can occur. Throughout, we model the rubber bumper as a Maxwell Spring Dashpot to represent its elastic and energy dissipation properties during collisions. These bumpers are vital to shock absorption during collisions, resulting in a safer ride. Through this investigation, we show that the trajectory angle before colliding affects the experienced force by the rider, thus making a head-on collision the most dangerous as direct force is exerted on the rider and no dampening due to the friction of the bumpers sliding together. This experienced force will also vary with the masses of the riders, as more force will be experienced when colliding with a heavier opposition.

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1. Introduction

Dodgem Cars are a classic amusement park ride, allowing collaborative enjoyment with friends and the sharp thrills of low-speed collisions. While they may be a staple to theme parks, they can be dangerous with sudden forces being exerted onto the rider, often leading to injuries [1].

Throughout this report, we have derived three models that have allowed us to evaluate three possible collision scenarios, including the collision between a car and a wall and collisions between two moving cars . The principle idea of these models stems from Newton's 2nd Law, the net forces on an object are equivalent to the mass of that object multiplied by its acceleration, $F = ma$ [2]. The layout of this report explores collision, firstly in 1 dimension and then into a more inclusive 2-dimensional situation. In all models, energy lost through air resistance, heat and sound have been assumed to be negligible.

Firstly, to mimic a realistic collision of dodgem cars we had to explore ways of modeling the rubber bumpers. These bumpers absorb the majority of the shock during collisions, thus making them a vital aspect of all subsequent models. Rubber has viscoelastic properties, meaning it exhibits both elastic and viscous properties [3], making it an appropriate approach to model the bumper as a Maxwell Spring Dashpot, $F = kx + b\frac{dx}{dt}$ [4]. The model consists of a spring to represent the elastic property of rubber and a dashpot to mimic the energy dissipation during the collisions. Through a thorough investigation by plotting the amount of damping occurring in collisions and the values of the Spring Dashpot Model, b and k, it was identified as the most fitting and realistic method to model the deformation mechanics and energy dissipation of rubber (see Figure 3).

Our first model explores the concept of a dodgem car colliding with the metal walls of the arena at varying angles. We first investigated this in a 1D evaluation, the car hitting a wall perpendicularly, and the forces present during this. Forming a second-order ODE using Newton's second law [2] we were able to find a relationship between b and k values to ensure dampened harmonic motion, which we then assumed for all subsequent models. This was then translated into a 2D model of the dodgem car colliding into a wall at varying acute angles. By predicting the resulting trajectory after colliding, we found that the car bounces off at a rate related to the angle of approach and initial velocity and then continues traveling parallel to the wall.

In the next two models, we evaluated the concept of two dodgem cars colliding together. For this, we build on the concepts of Model 1 but also incorporating Newton's 3rd Law, an object exerting a force on a second object will experience an equal and opposite force [2]. This allowed us to simultaneously evaluate the forces experienced by both cars. Model 2 focuses on the 1D scenario of a head-on collision using Lagrange equations and demonstrates the effect of different masses of riders. This is important for the analysis of safety, as it allows a broader evaluation of the maximum force that can be exerted on a rider. As shown, the lighter rider will be exerted much more force if they were both of equal mass. In Model 3, we analysis two dodgem cars colliding into the body of each other at varying acute angles. This is shown as a safer type of collision than Model 2, as the friction between the rubber bumpers when colliding and the wheels sliding in an opposing direction decreases the velocity significantly, resulting in a safer experience for both riders. Model 2, a head-on collision, is now concluded as the most dangerous type of collision due to the direct force being exerted on the rider and no dampening due to friction occurring.

Our ultimate aim throughout our models was to create a more inclusive and broader analysis of forces experienced by the dodgem cars during collisions and the resulting trajectories. Hence, we purposely chose for our equations to contain multiple unknowns. This allows us to evaluate multiple situations, including the possibility of various velocities, masses for both cars and riders and angle of approach.

2. Maxwell model for modelling the bumper

It was not realistic for the dodgem car to be modelled as a rigid body, as this does not capture the elasticity and energy dissipation properties of rubber. In the investigation of deformation mechanics, we found the spring dashpot system [4]. The study "Introduction to Viscoelasticity in Polymers and its Impact on Rolling Resistance in Pneumatic Tyres" [3] informed us of the viscoelastic properties of rubber and made it feasible to use a spring dashpot system. The Maxwell model uses a linear spring and a dashpot in series which models a rubber bumper well, the spring represents the elastic properties of rubber and the dashpot represents the energy dissipation through the bumper during the collision.

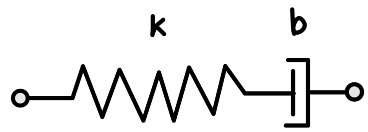


Figure 1. Maxwell Spring Dashpot Model

Thus, the total force exerted can be expressed with k as the spring constant and b is the dashpot constant;

$$F = kx + b\frac{dx}{dt} \tag{2.1}$$

3. Model 1: Collision with a rigid side wall

3.1. 1-Dimensional Model.

The 1-dimensional model considers the dodgem car hitting the wall perpendicularly. There is a constant driving force F_m and we assume that the dodgem car hits the wall at its maximum velocity. We decided to neglect friction because the motion is in the same direction as the wheel orientation and the bumper is modelled as the Maxwell spring dashpot system as stated above.

An equation of forces during the collision can be equated from Newton's 2nd Law, then used to evaluate the impulse at the point of collision over the time colliding. The impulse of Car 1 is equivalent to the impulse of -Car 2, due to Newton's 3rd Law [5].

$$\int_{t_0}^{t} F dt = \int_{t_0}^{t} madt. \tag{3.1}$$

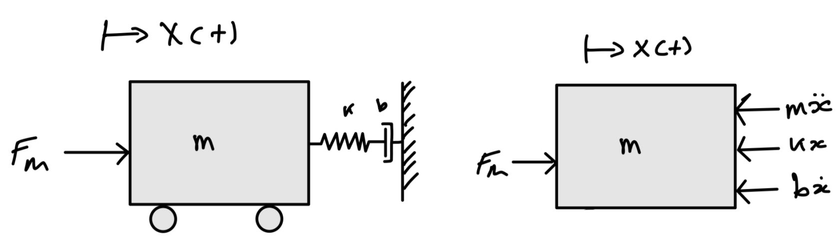


Figure 2. 1D model (left), forces acting on dodgem car (right)

1-Dimensional Model

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_m \tag{3.2}$$

with $x(0) = 0, x'(0) = v_0, m, b, k > 0$

where m is the mass of the dodgem car and driver combined, b is the dashpot constant, k is the spring constant and F_m is the driving force from the motor.

Referring to Figure 2, $b\frac{dx}{dt}$ is the force that the dashpot exerts on the system and kx is the force of the spring. The model is based on Newtonian Mechanics using Newton's Second Law $F = ma$.

This is a non-homogeneous, second order ordinary differential equation with constant coefficients. The complementary function is solvable with a characteristic equation and the particular solution is solved using a constant trial function.

$$m\lambda^2 + b\lambda + k = 0 \implies \lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{3.3}$$

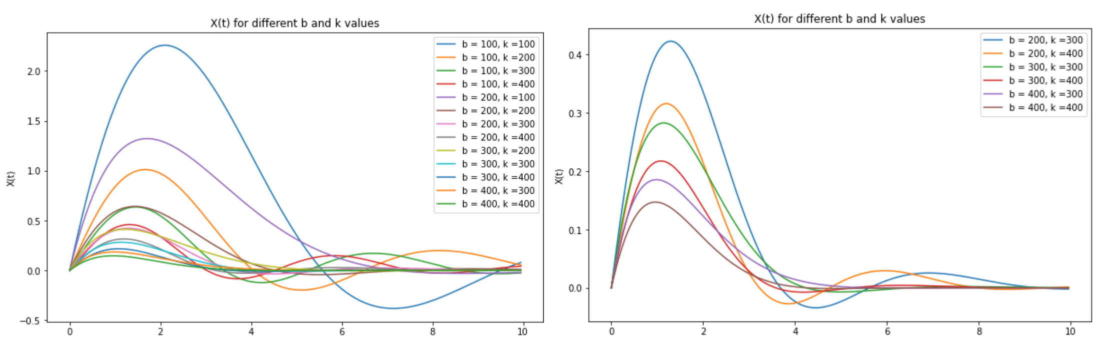
Now we need to model a dampened harmonic motion, since we have a dashpot and a spring in the model. This is only possible if $b^2 - 4mk < 0 \iff b^2 < 4mk$, creating a relationship on the values that b and k can take for our solution to be valid. Thus,

$$\lambda = \frac{-b}{2m} \pm \frac{i\sqrt{4mk - b^2}}{2m} \tag{3.4}$$

and our solution is,

$$x(t) = \frac{F_m}{k} e^{-\frac{b}{2m}t} \left[\left(\frac{v_0 - \frac{b}{2m}}{w} \right) \sin(wt) - \cos(wt) + 1 \right] \tag{3.5}$$

where $w = \frac{\sqrt{4mk-b^2}}{2m}$



(A) A range of b,k values (B) b,k values causing damping

FIGURE 3. Motion for wall collision with varying b,k values

Figure 3 shows the varying levels of harmonic motion with different values of b and k. We see that for small b and k values, the motion is more oscillatory and the dodgem car is pushed back over 2m. This is unrealistic suggesting the use of larger b and k values. Looking at Graph (B) we can see that larger b and k values cause a sufficient amount of damping. The dodgem car is now pushed back a smaller distance resulting in a more realistic model.

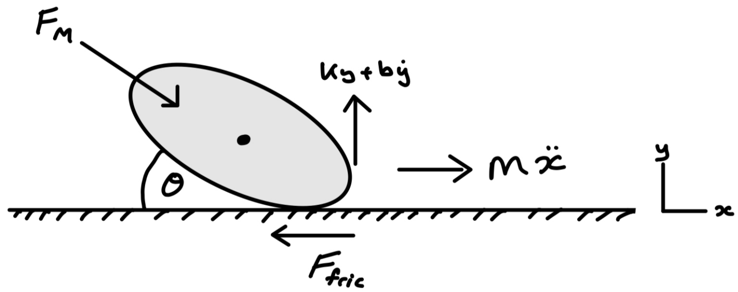


FIGURE 4. 2D force diagram

3.2. **2-Dimensional Model.** The 2D model considers the collisions of a dodgem car with a metal wall at various acute angles. The car initially hits the wall at angle θ, then gets pushed away from the wall continuing to move parallel to the wall due to the constant motor force. This model is for angles less than 45º. The constant μ_0 of rubber against stainless steel is 0.64. [6]. This is a sufficient value of coefficient of friction between the rubber bumper of the dodgem car and the arena walls.

$$m\frac{d^2x}{dt^2} = F_m(cos(\theta) - \mu_0 sin(\theta)) \tag{3.6}$$

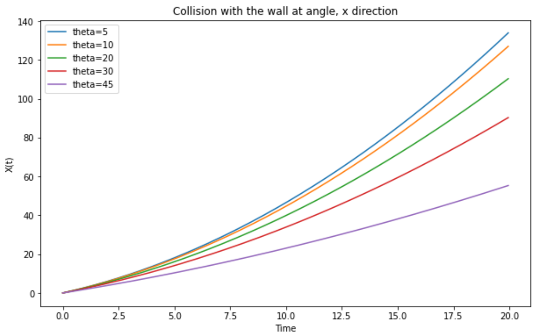
$$b\frac{dy}{dt} + ky = F_m sin(\theta) \tag{3.7}$$

with initial conditions, $x(0) = 0, x'(0) = v_0cos(\theta), y(0) = 0, y'(0) = v_0sin(\theta)$
Solving yields,

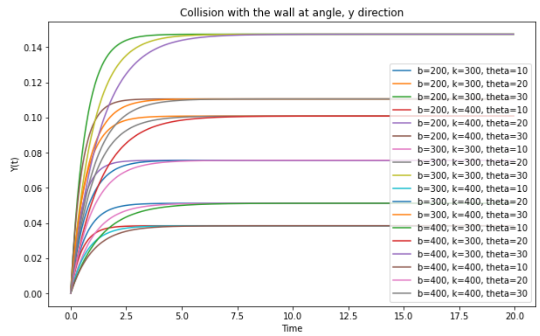
$$x(t) = \frac{F_m}{k}(cos(\theta) - \mu_0 sin(\theta))t^2 + v_0cos(\theta)t \tag{3.8}$$

$$y(t) = \frac{F_m}{k}sin(\theta)(1 - e^{-\frac{k}{b}t}) \tag{3.9}$$

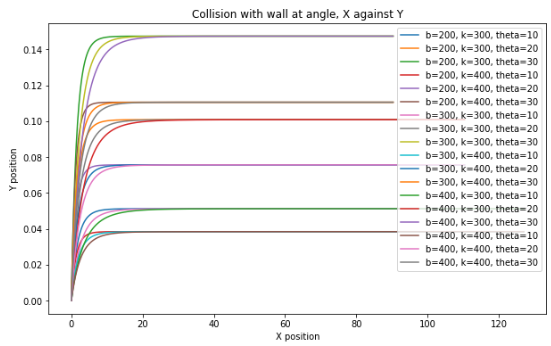
Figure 3 showed us that larger values of b and k give dampened harmonic motion, using the relationship $b^2 < 4mk$ and varying the angle θ we can see the motion.



(A) X position against time



(B) Y position against time



(C) X against Y

FIGURE 5. Displacement graphs for varying values of b, k and θ

Since the $x(t)$ equation depends only on θ, there is no need to vary b and k.

On impact the dodgem car will lose some velocity since kinetic energy is transferred to potential energy and some is dissipated by the dashpot. This will create a linear velocity graph, and so a constant acceleration, until it reaches its maximum velocity and will level off. Thus, the dodgem car will travel constantly away from the collision point. Smaller θ values show that the dodgem car travels away faster since less kinetic energy is dissipated by the dashpot.

The $Y(t)$ graph shows that upon collision the dodgem car is pushed away from the collision point quickly and will remain there once all of the potential energy is transferred to kinetic energy. We assume that once the dodgem car stops touching the wall it will never touch it again because the driving force in the x direction is greater than the bounce in the y. Larger values of θ show that the dodgem car bounces away faster since the bumper compresses more and the spring can store more potential energy.

Ideally we would want the 1D model to be a specific case in the 2D model in terms of the maths. If we had assumed that the angle the dodgem hit the wall was equal to the angle it exits this would have been possible however this only works if momentum is conserved. Thus we decided that angles less than 45º would cause it to travel parallel to the wall and angles greater than this would cause the bumper car hit the wall and stop.

In conclusion, a dodgem car colliding with a wall in 1D exhibits dampened harmonic motion because of our spring dashpot model of the bumper. We have found a relationship between b and k values to ensure a realistic motion is exhibited. Our 2D model shows that the constant driving force of the car when it collides with the wall causes the car to then bounce off in the y direction and continue to travel parallel to the wall. Being unable to find data about spring and dashpot constants we theorised through the Figures 3 and 5 that larger values for b and k yield the most realistic motion, the car bounces away a short distance. Commenting on the rotation of the car during the collision, the rotation would be induced by the normal force against the bumper and the wall, and the spring would exert a force at the front of the car. We decided that this was too complex to model and decided to omit it.

4. Model 2: Two dodgem cars collide head-on

It is evident that this situation cannot be treated as a perfectly elastic collision as in the case where it would be considered elastic, there would be no net loss of kinetic energy due to the collision [7]. Various forms of energy transfer are important to consider, however, friction between the wheels and the floor is considered negligible due to its minimal impact on the collision. The main transfers of energy in the head-on collision of two bumper cars occur in the forms of kinetic energy, elastic potential energy and energy dissipated by the bumper. During a collision, initial kinetic energy is converted into these forms of energy. After the collision collectively both cars have less kinetic energy due to the energy dissipated by the bumper through damping in the Maxwell Model (Figure 1). As the total mechanical energy of the system is not conserved and the cars have a lower total momentum after the collision, the laws of conservation of energy and momentum cannot be applied [8].

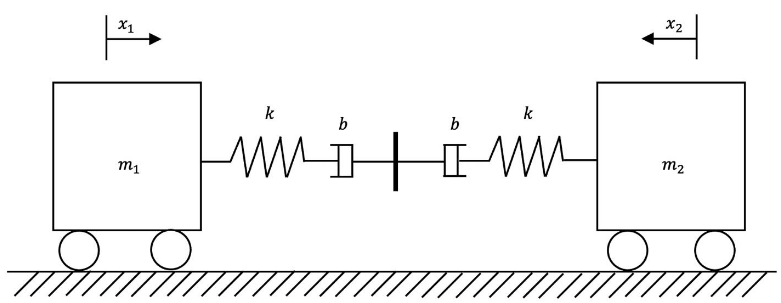


Figure 6. Spring Dashpot Model for Head on Collision

In this situation, the head-on collision of two dodgem cars is modelled and analysed in 1 dimension (See Figure 6 above), as no angular forces are acting on the system. Each (empty) car is assumed to have the same mass of however the two cars have masses m_1 and m_2 in the system due to the varying masses of the riders. In this model, the loss of energy and momentum is assumed to be entirely as a result of the dampening effect of the bumper, as explained below

4.1. Lagrange equations for modelling the motions.

Lagrange equation is a fundamental equation describing the movement of a system. The general form of Lagrange equation is [9]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = Q_s \tag{4.1}$$

where L is the Lagrangian of the system, the difference between the kinetic energy (E_c) and potential energy (E_p), q_i is the generalised coordinates, $\dot{q}_i$ is the generalised velocities, and Q_s represents the generalised forces acting on the system.

Replacing equation (4.1) with $L = E_c - E_p$, $q_s = x_i$, $\dot{q}_s = \dot{x}_i$, $Q_s = -\frac{\partial E_d}{\partial \dot{x}_i}$, where $i = 1, 2$ and E_d is the energy dissipation, two Lagrange equations can be obtained [10]:

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{x}_1}\right) - \frac{\partial E_c}{\partial x_1} = -\frac{\partial E_d}{\partial \dot{x}_1} - \frac{\partial E_p}{\partial x_1} \tag{4.2}$$

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{x}_2}\right) - \frac{\partial E_c}{\partial x_2} = -\frac{\partial E_d}{\partial \dot{x}_2} - \frac{\partial E_p}{\partial x_2} \tag{4.3}$$

4.2. **Model: Head-On Collision.** When two dodgem cars collide head-on, both cars are assumed to have the same spring constant and damping coefficient. The force generated by spring and dashpot can be expressed as (2.1). In this model, x represents the position of the car (along x axis). The kinetic energy of the system is:

$$E_k = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} \tag{4.4}$$

the potential energy can be expressed by:

$$E_p = \frac{k(x_1 - x_2)^2}{2} \tag{4.5}$$

and the expression of energy dissipation is:

$$E_d = \frac{b(\dot{x}_1 - \dot{x}_2)^2}{2} \tag{4.6}$$

Replacing the Lagrange equations (4.2) and (4.3) with expression of kinetic energy, potential energy and energy dissipation, and using Newton's Second Law ($F = ma$), the equations describing the movement of two cars can be expressed as:

$$F_1 = m_1 \ddot{x}_1 = -k(x_1 - x_2) - b(\dot{x_1} - \dot{x_2}) \tag{4.7}$$

$$F_2 = m_2 \ddot{x}_2 = k(x_1 - x_2) + b(\dot{x_1} - \dot{x_2}) \tag{4.8}$$

F_1 and F_2 are the forces acting on each car respectively, which are dependent on each other. As a result of this dependence, this situation must be modelled as a coupled second order differential equations. Looking at the (4.7) and (4.8), the significance of the spring and dashpot terms k and b are both determined by their relative speeds and masses, so the elastic potential energy generated by the collision is 'shared' between the two bumpers. This gives a relative elastic potential force in the equation. The same logic is applied in the damping term, as the damping effect again is 'shared' between the bumpers of each car.

A coupled ordinary differential equation is necessary here to represent the effect of Newton's Third Law resulting in the interdependence of each car on one another.

4.3. **Analysis Part 1.** At the origin (time=0s, displacement=0m), the two dodgem cars have just collided. The three graphs below represent three different head-on collision situations 1 second after the collision has taken place. All situations have both cars having the same mass, however, different speeds at the moment just before colliding (v_1 and v_2). Each dodgem car has a maximum speed of 7mph (3.129m/s) [4]. It is also assumed that both cars have a constant motor force up until the point of collision.

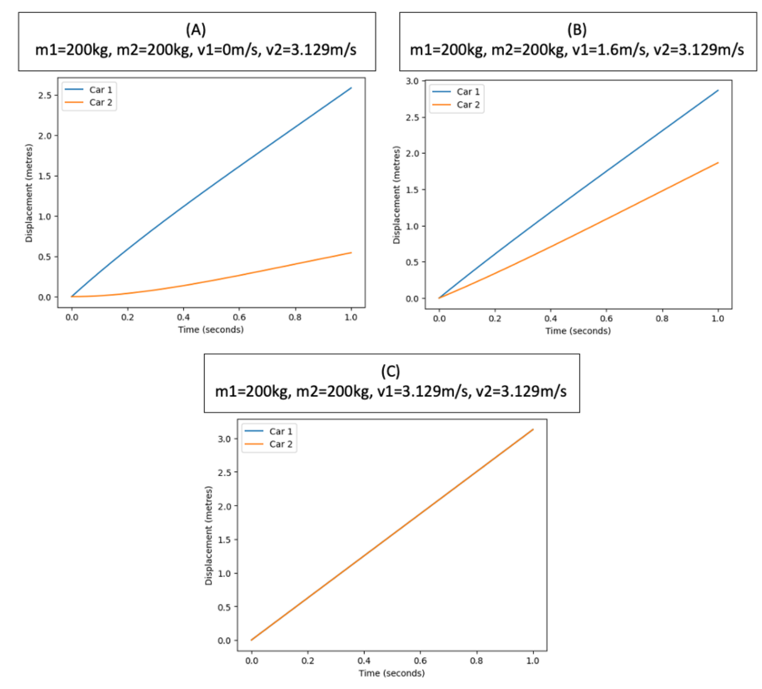


FIGURE 7. Trajectories of two dodgems after colliding (same masses)

Graph (A) shows that two cars of the same mass travelling at different speeds in the first second after collision have different displacements. Car 2 collides with stationary Car 1 with an initial maximum speed, exhibiting the maximum amount of displacement for Car 1 compared to the other graphs. As expected given that the majority of the momentum is passed from Car 2 to Car 1, due to the conservation of momentum mechanics and the remaining energy is absorbed by the bumper. This explains the steep slope of Car 1 in the first second after the collision, as more energy transferred means that Car 1 has more momentum than Car 2 after this has been transferred during the collision.

Graph (B) has Car 1 at a different speed compared to Graph (A); roughly half of Car 2. As Car 2 still has double the momentum of Car 1, the graph of our model again shows that Car 1 has much of the transferred energy. This is, however, less than in Graph (A) due to the opposing forces causing the damping effect to be much larger in the spring dashpot model. Much more of the energy is dissipated by the bumper in Car 1, but also transferred via momentum. Therefore, the displacement of Car 1 is much larger.

Graph (C) shows the head-on collision where both cars have collided at the same maximum speed. Here after the collision, both cars will have the same momentum, due to them having the same weight and same bumpers. All of the energy lost in this system comes down to the damping effect of the bumpers, which explains why both lines for Car 1 and Car 2 overlap. As a result, their respective displacements will be the same. However, there are obvious limitations to this model due to the linearity of these analytical results.

4.4. **Analysis Part 2.** To analyse a head-on collision, we need to consider the different masses of the riders, in turn affecting the momentum of the system. A heavier car acting on a lighter car is much more likely to impact the force exerted and therefore the safety of the system. This emphasises the importance of the bumpers in the system.

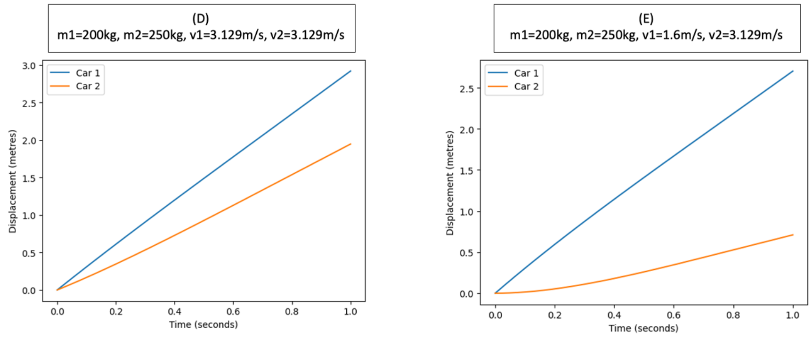


FIGURE 8. Trajectories of two dodgems after colliding (different masses)

Graph (D) shows a mostly linear relationship between displacement and time for both cars after the collision. Both cars are travelling at the slightly different speeds just before the point of colliding, however after colliding car 1 is travelling at a higher speed just after, due to it being the lighter car. Graph (D) has a very similar distribution to that of Graph (B) (Figure 7), largely because the difference in mass has accounted for the difference in speed before the collision. Despite initially travelling at the same speed, Car 1 has a higher velocity after the collision as a result of the transfer in momentum from Car 2 to Car 1.

Graph (E) represents two cars of different speeds and masses colliding. Before the collision Car 1 is moving at roughly half the speed of car 2. As we can see Car 1 (lighter car) and travelling at a greater speed than Car 2. Once again, this results in Car 1 having a much larger displacement after the collision due to this transfer in momentum.

4.5. **Review of Model 2.** Overall, the model of a head-on collision is the most important regarding safety as there is a higher risk of injury due to the largest changes in momentum over a short period of time. Unlike Model 1, this system has two bumpers of which the impact of a collision can be absorbed, which is vital for keeping the activity both thrilling and safe. Through the spring dashpot model, large amounts of energy are dissipated through damping, which has a significant effect on the trajectory of both cars in equations (4.7) and (4.8). The nature of the coupled equations demonstrates the interdependence of all the variables, particularly how changing mass and speeds impact the damping effect and final force felt by the rider. This is reinforced by the viscoelastic properties of rubber in the Maxwell Model. Much of the elastic potential energy generated during the collision (from the initial kinetic energy) is, in quick succession, put back into the system, as shown by the linearity of our results. As seen above, different situations in both Figures 7 and 8 with different masses and speeds for both cars affect the speed of separation post-collision, especially the force exerted on the rider. Different from the other models, there is a larger force felt by the rider, especially if a rider is hit by a heavier rider (see Figure 8 (D)), emphasizing the importance of the bumper. The closer the speeds and masses are, the more linear the transfer of momentum in the system, as expressed by Figure 7(C).

5. MODEL 3: TWO DODGEM CARS COLLIDING AT AN ANGLE

In this model, one dodgem car is colliding into the body of a second at varying acute angles. These cars are moving at initial velocities v_1 and v_2 for Car 1 and Car 2 respectively. This model follows from Model 1 and Model 2 in combination to demonstrate a greater range of collision possibilities, as this is the most common collision that occurs and can be thought as a 2D version of Model 2. In this situation we cannot assume the loss of energy is only as a result of dampening from the bumper, like we did in Model 2, but also the high level of friction between the two rubber bumpers and friction due to wheels sliding in an opposing direction during the collision.

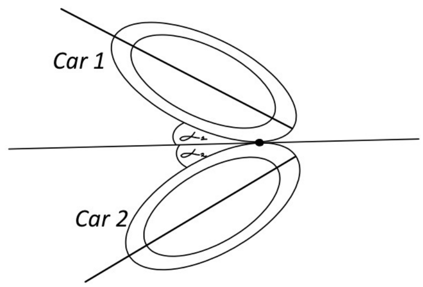


FIGURE 9. 2D collision between two dodgem cars

5.1. **Model.** Similarly to (3.1), an equation of forces during the collision can be equated from Newton's second Law, then used to evaluate the impulse at the point of collision over the time colliding. The impulse of Car 1 is equivalent to the impulse of the negative of Car 2, due to Newton's 3rd Law. From Newton's second Law we can equate the forces as;

$$ma = F_m - (F_{friction} + F_{springdashpot}) \tag{5.1}$$

With constant driving force F_m assumed the same for both cars, acting in the direction that the cars' velocities. The spring dashpot model, $F_{springdashpot}$ (2.1), and friction, $F_{friction}$, between the two rubber bumpers and sliding wheels are included as opposing forces during the collision.

F_1, F_2 is representative of the forces from Car 1 and Car 2 respectively. m_1, m_2 is represents the combined weight of mass of the driver and the dodgem's masses of Car 1, Car 2, respectively .

$$F_1 = m_1 \frac{d^2\mathbf{x}_1}{dt^2} = F_m - b|\frac{d\mathbf{x}_1}{dt} - \frac{d\mathbf{x}_2}{dt}| - k|\mathbf{x}_1 - \mathbf{x}_2| - \mu_1 N\mathbf{x}_1 - \mu_2 m_1 g\frac{d\mathbf{x}_1}{dt} \tag{5.2}$$

$$F_2 = m_2 \frac{d^2\mathbf{x}_2}{dt^2} = F_m - b|\frac{d\mathbf{x}_1}{dt} - \frac{d\mathbf{x}_2}{dt}| - k|\mathbf{x}_1 - \mathbf{x}_2| - \mu_1 N\mathbf{x}_2 - \mu_2 m_2 g\frac{d\mathbf{x}_2}{dt} \tag{5.3}$$

Due to Newton's 3rd Law, $F_1 = -F_2$ [2], where μ_1 is the coefficient of friction between rubber and rubber and μ_2 the coefficient of friction between plastic wheels and the metal arena floor [11] travelling in the opposite direction, with N as the normal to the point of collision. The coefficient of μ_1 can be approximated as 1.15 and μ_2 as 0.2 [6]. These overall forces, equations (5.3) and (5.4), can now be split into x and y components. This results in a second order differential equation that models the x direction and simultaneous first order differential equations for the y direction.

$$\Rightarrow m_1 \frac{d^2x_1}{dt^2} = F_m(cos(\alpha_1) - \mu_1 sin(\alpha_1)x_1) \tag{5.4}$$

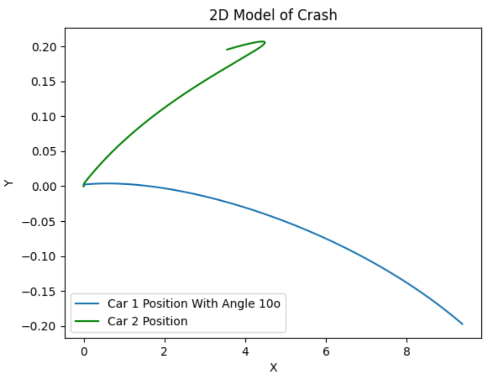
$$m_2 \frac{d^2x_2}{dt^2} = F_m(cos(\alpha_2) - \mu_1 sin(\alpha_2)x_2) \tag{5.5}$$

$$\Rightarrow b|\frac{dy_1}{dt} - \frac{dy_2}{dt}| + k|y_1 - y_2| = F_m sin(\alpha_1) - \mu_2 m_1 g\frac{dy_1}{dt} \tag{5.6}$$

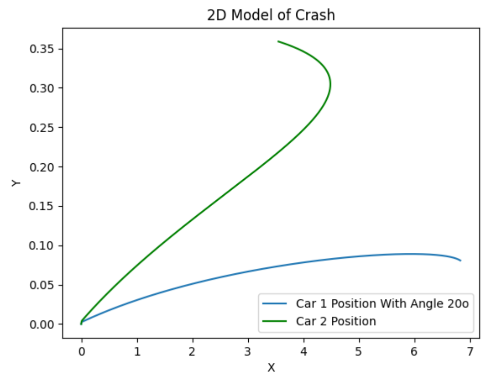
$$b|\frac{dy_1}{dt} - \frac{dy_2}{dt}| + k|y_1 - y_2| = F_m sin(\alpha_2) - \mu_2 m_2 g\frac{dy_2}{dt} \tag{5.7}$$

In this model the friction between the two rubber bumpers ,$\mu_1 N$, is modelled as a direct correlation to the distance travelled during the collision, as the more contact between the two bumpers, the more friction there will be. The friction of the wheels sliding in the opposing direction, $\mu_2 m_2 g$, correlates to the velocity the car during the collision. Friction between the wheels rolling in the direction the car is travelling at is neglected.

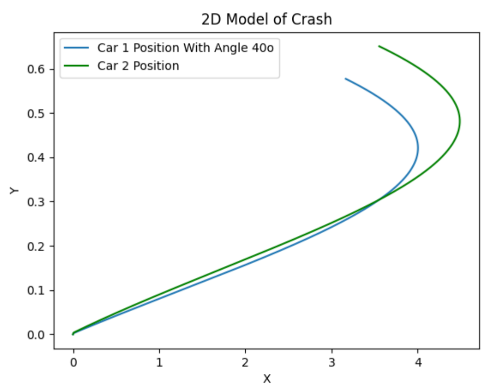
With initial conditions, $x(0) = 0, x'(0) = v_i cos(\alpha_i), y(0) = 0, y'(0) = v_i sin(\alpha_i)$ for $i = 1, 2$, similar to (3.8) and (3.9). These equations are then solved in Python to produce these results, showing Car 1 in blue and Car 2 in green;



(A) Car 1 colliding at a 10 degree angle



(B) Car 1 colliding at a 20 degree angle



(C) Car 1 colliding at a 40 degree angle

FIGURE 10. Trajectories of two dodgem cars colliding at varying angles

In Figure 10, the point of collision occurs at the origin, seen on the graph as the point both lines meet, and the subsequent separation represents their resulting trajectories. Car 1 is modelled at 50% heavier than Car 2.

Graph (A) shows Car 1 colliding with Car 2 at an angle of 10 degrees to the line of collision. It can be seen that after colliding, Car 1 gets deflected and continues travelling at a decreasing velocity. While Car 2 after the collision gets deflected more and then comes to a stop. This aligns with our first predictions as Car 1 is heavier so able to displace Car 2 but with a much lower velocity after colliding. This is due to the dissipation of the kinetic energy and acceleration being inversely proportional to mass. Car 2 stops quickly after colliding due to the large amount of contact between the rubber bumpers during the collision, thus a large amount of kinetic energy is lost to friction between the bumpers.

Graph (B) has Car 1 colliding with Car 2 at an angle of 20 degrees. This now shows a more even displacement of the cars, than Graph (A), due to the increased angle of approach, accounting for Car 1 being heavier. Both cars get forced into the line of approach of Car 1, resulting in less spin of Car 2 than in Graph (A).

Graph (C) shows Car 1 colliding with Car 2 at a 40 degree angle. These results follow directly from Graph (B), by reinforcing the concept of Car 2 getting forced into the direction of Car 1's approach after the collision. Due to this resulting redirection, there is an extended period of contact between the bumpers causing a high level of friction between both cars, thus forcing both cars to stop shortly after colliding.

5.2. **Analysis.** This model appropriately represents the subsequent trajectories of both cars after colliding at varying acute angles and follows an intuitive impression of the situation [12]. This model reinforces the concepts mentioned in Model 2, where heavier masses displace lighter masses. This also explores the concept of increasing the angle of approach will result in the colliding car being deflected less and forcing the car being collided into to follow the direction of the colliding car. In smaller angles of collision both cars will be displaced and stop shortly after the collision due high levels of friction between the rubber bumpers on both cars. In all these situations each car's velocity decreases significantly after the collision and in many cases one of both cars are forced to stop. After this stopping, shown on the graphs at the point of change in direct of the trajectory, the subsequent motion can be neglected as realistically at this point the rider will then redirect themselves and continue accelerating.

6. Conclusions and future works

There are many directions for further evaluation of the forces and trajectories of dodgem cars during collisions. These collisions lack a lot of mathematical research, while 20% of injuries sustained at theme parks are related to dodgem cars [13]. Thus, an extension of our models could be utilized to calculate the inertia and forces experienced directly by the rider to evaluate the rider's safety further.

Our research demonstrates the vitality of the rubber bumpers to the safety of low-speed collision throughout each model (seen in Figure 3), with the use of the Maxwell Spring Dashpot Model (2.1). This appropriately models the viscoelastic properties of rubber and is responsible for absorbing the majority of the exerted forces during a collision. This results in the car being forced backward less and a safer experience for the rider. The most dangerous type of collision in terms of the most force exerted on the rider, as we have shown, is a head-on collision as the force is directly exerted on the rider and there is no sliding friction from the rubber bumpers that significantly decrease the speed during collisions. An additional factor that could be taken into consideration in future models is the oval shape of dodgem cars and how the geometry could affect the contact friction between bumpers and the trajectory after colliding.

7. References

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8. Appendix

Seven meetings, outside of the scheduled Friday seminars, were held.

Temple, abstract, contents and intro written by Eleanor Cole.

Chapter 2 (Maxwell Model) and Chapter 3 (Model 1) written and researched by Thomas Sharples.

Figures 1,2,3,4 and 5 created by Thomas Sharples.

Chapter 4, Model 2, formulated, written and researched by Robert Snell. Ruian Li helped with first paragraph and provided literature. Figure 6, Lagrange equations section written and researched by Ruian Li.

Chapter 5 (Model 3) written and researched by Eleanor Cole. Some analysis of graphs done by Hongyi Gu and written and contextualised by Eleanor Cole.

Conclusions and future works written by Eleanor Cole.

Coding for head on car collision and collision between two cars at an angle by Alex Jednorowicz.

Presentation slides created by Ruian Li and Alex Jednorowicz.

Presented by Alex Jednorowicz.

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