

# MATH1072 Notes

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## 1 Lecture 1 - Introduction to Dimensional Analysis

To describe real systems quantitatively, we use numbers and units of measurement. Eg. 3 meters, 5 years, 10  $km/h$ . Each measurable quantity has a certain dimension.

### 1.1 Base dimensions

Length ( $L$ ), time ( $T$ ), mass ( $M$ ), other non-mechanical base dimensions include temperature; electric charge/current etc.

### 1.2 Derived dimensions

Speed  $\frac{L}{T}$ , force ( $N = M \frac{L}{T^2}$ ), energy ( $J = M \frac{L^2}{T^2}$ ). The dimensions of the terms added on both sides must be equal. This is known as the equation being *dimensionally homogenous*. We can use the dimensional homogeneity to make dimensional estimates for certain quantities (order of magnitude, not exact prediction)

## 2 Lecture 2 - Dimensional Analysis

## 3 Lecture 3 - Introduction to Differential Equations

Generally, an ordinary differential equation (ODE) is represented as:

$$F(t, y(t), y'(t), y''(t), \dots) = 0$$

For Instance, Newton's Law

$$m \frac{d^2 r}{dt^2} = F$$

Induction Law:

$$RI + L \frac{dI}{dt} + \frac{1}{C}$$

Population:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{k}\right)$$

Maxwell's Equations:

$$\nabla \cdot \bar{E} = \frac{1}{\rho} \varepsilon_0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = \frac{\delta B}{\delta t}$$

$$\nabla \times \bar{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\delta E}{\delta t}$$

Navier-Stokes: (Modelling velocity of fluids in space)

$$\frac{\delta \bar{v}}{\delta t} = \bar{v} \nabla \bar{v} = ?$$

Schrodinger Wave Equation:

$$i\hbar \frac{\delta\psi}{\delta t} = \left[ \frac{-\hbar}{2m} \nabla^2 + V(r) \right] \psi$$
$$\psi = ?$$

We generally have:

$$F(t, y(t), y'(t), \dots) = 0, \quad y(t) = ?$$

Where the order of ODE = order of the highest derivative. Take the equation  $y' = y$ , the solutions are  $y = e^t$ ,  $y = 0$  and  $y = ce^t$ . However the latter expression encapsulates the former, so  $y(t) = ce^t$  is known as the general solution where  $c \in \mathbb{R}$ . The general solution is not unique.

If you take an ODE and add some initial condition, then the solution is a unique result. For Instance take the ODE  $y' = y$  and say that  $y(0) = 1$ , then we achieve a unique solution of  $c = 1$ ,  $y = e^t$ .

## 4 Lecture 4 - Ordinary Differential Equations

### 4.1 Equilibrium Solutions

An equilibrium solution (steady state solution) of an ODE (if such solution exists) is a constant solution  $y(t) = c$  which satisfies the ODE (for any  $t$ ).

**Example 1.** Take  $y' = f(t, y)$ , the equilibrium solution

$$f(t, y = c) = 0$$

This implies that the slope at  $y = c$  must be 0 for a function defined on a  $(t, y)$  plane

**Example 2.**  $y' = y$ ,  $y = 0$  is an equilibrium solution.

**Example 3.**  $y' = y(1 - y)$   $y = 0$  and  $y = 1$  are equilibrium solutions

Slope fields can visualised with Mathematica using `StreamPlot` (or `VectorPlot`). The vector flow of a field corresponding to  $f(t, y)$  is  $\{1, f(t, y)\}$  since the flow in the horizontal direction ( $t$ ) has a constant rate (the flow of time) which can be set to 1, the vertical flow is  $f(t, y)$ .

## 4.2 Stability of equilibrium solutions

If the solutions starting in condition a small neighbourhood of an equilibrium solution ( $y = c$ ) converge towards the equilibrium solution for large  $t$ , then the equilibrium solution  $y = c$  is stable.

**Example 4.**  $y' = y$ , the equilibrium solution  $y = 0$  is unstable (for  $y' = -y$ ,  $y = 0$  is stable)

**Example 5.**  $y' = y(1 - y)$ ,  $y = 0$  is unstable, but  $y = 1$  is stable.

# 5 Lecture 5 - Stability of ODE's

## 5.1 Condition for stability of equilibrium solutions

Using a Taylor Series approximation of  $f(t, y)$  near the equilibrium solution  $y = c$ , assume  $f(t, y) = f(y)$  for simplicity

$$f(y) \approx f(c) + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c)^2 + \dots$$

Where  $y = c$  is the equilibrium solution  $f(c) = 0$ , thus

$$y'(t) = 0 + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c) + \dots$$

Take  $u = y - c$ , then

$$u' = f'(c)u + \dots$$

Implied

$$u(t) = Ae^{f'(c)t}$$

If  $f'(c) > 0$ ,  $y = c$  is unstable. If  $f'(c) < 0$ ,  $u(t) \rightarrow 0$ ,  $y(t) \rightarrow c$  thus  $y = c$  is stable.

## 5.2 Euler's Method

An iterative operation which models  $y_k \approx y(k \cdot \Delta t)$ . Given  $y' = f(t, y)$ ,  $y(0) = c$

$$y'(t) \equiv \lim_{\Delta \rightarrow 0} \frac{y(t + \Delta) - y(t)}{\Delta} \quad (1)$$

$$\approx \frac{y(t + \Delta) - y(t)}{\Delta} \quad (2)$$

As it is an iterative method,  $y_{k+1} = y_k + f(t_k, y_k)$

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta} \approx f(t_k, y_k)$$

**Example 6.** Given  $y' = 2t$ ,  $y(t) = ?$ ,  $y(0) = 0$ . By Euler's

## Method

$$N = 1, \Delta = 1 \quad (3)$$

$$N = 2, \Delta = \frac{1}{2} \quad (4)$$

$$N = 4, \Delta = \frac{1}{4} \quad (5)$$

$$N = 10, \Delta = \frac{1}{10} \quad (6)$$

## 6 Lecture 6 - Continuation of Euler's Method

### 6.1 Error Generated in Euler's Method

Assume  $y(t_k) = y_k$ , the error is represented with the Taylor series approximation

$$\begin{aligned} \left| y_{k+1} - y(t_k + \Delta) \right| &= y(t_{k+1}) - y(t_k) - y'(t_k)\Delta - \frac{1}{2}y''(t_k)\Delta^2 + \dots \\ y_{k+1} &= y_k + f(t_k, y_k)\Delta \end{aligned}$$

Thus

$$\text{error} = |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2$$

However generalized for  $N$  steps,

$$\begin{aligned} N \cdot |y_{k+1} - y(t_k + \Delta)| &\propto \Delta^2 \cdot N \\ &\propto \Delta^2 \cdot \frac{1}{\Delta} \propto \Delta \end{aligned}$$

$$y_{k+1} = y_k + \frac{1}{2} \left[ f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \right] \Delta$$

# 7 Lecture 7 - Solving Linear First Order ODEs

$$y' = f(t, y), y(0) = y_0$$

$$t \in [0, t_{max}]$$

## 1. Euler's Method

$$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

Where total error  $\propto \Delta$

## 2. Heun Method (Revised Euler's Method)

$$\begin{cases} y_{k+1} = y_k + f(t_k, y_k)\Delta \\ y_{k+1} = y_k + \frac{1}{2}[f(t_k, y_k) + f(t_k + \Delta, y_{k+1})] \end{cases}$$

Where total error  $\propto \Delta^2$ . Note that Heun Method will possibly show in Assignment 3.

## 7.1 Runge Kutta Method

$$p_1 = f(t_k, y_k)\Delta$$

$$p_2 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_1}{2})\Delta$$

$$p_3 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_2}{2})\Delta$$

$$p_4 = f(t_k + \frac{\Delta}{2}, y_k + p_3)\Delta$$

for

$$y_{k+1} = y_k + \frac{1}{6}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 + \frac{1}{6}p_4 + \dots$$

## 7.2 Adaptive Step Size

Fixed step size is mostly inefficient in most cases, so we use an adaptive step size for numerical methods to achieve better approximations.

## 7.3 Coupled Systems

$$y' = f(t, y_1, y_2)$$

$$x' = g(t, y_1, y_2)$$

$$x(t) = ? \quad y(t) = ?$$

In the context of Euler's Method

$$\begin{cases} x_{k+1} = x_k + f(t_k, x_k, y_k)\Delta \\ y_{k+1} = y_k + g(t_k, x_k, y_k)\Delta \end{cases}$$

## 7.4 Analytical ODE Solutions

### 7.4.1 Linear First Order ODE's

By definition,

$$y'(t) = f(t)y(t) + g(t)$$

is generally the standard form of a linear first order ODE. Note that

$$y'(t) = f(t)y(t)$$

is a special case of a linear first order ODE that is separable.

**Example 7.**

$$ty' + y = t \cos t$$



Observe that, by the product rule for differentiation,  $ty' + y = (ty)'$ .

$$\begin{aligned}(ty)' &= t \cos t \\ \int (ty)' dy &= \int t \cos t dt \\ ty &= t \sin t - \int \sin t dt \\ ty &= t \sin t + \cos t + c \\ \therefore y &= \frac{t \sin t + \cos t}{t} + \frac{c}{t}\end{aligned}$$

## 8 Lecture 8 - Applications of First Order ODEs

Common applications of first order ODEs are

### 8.1 Radioactive Decay

The particles of a radioactive material decay spontaneously in a stochastic process. The total mass of the radioactive atoms decrease with time. We can represent this as

$$\frac{dM}{dt} = -kM$$

For  $M(t)$  to represent the mass of the radioactive material over time  $t$ . Clearly the solution to this is

$$M(t) = M_0 e^{-kt}$$

Note that there is a limitation to this ODE Model. We assume that the mass changes continuously in time. Whereas in reality

it changes in discrete steps following individual decay events. However for the macroscopic mass, the number of particles is much larger, so we can neglect the discrete jumps and assume a continuous deterministic model. Which works well for most cases.

The lifetime of particles varies, however we can categorize them with their average lifetime. We do this by asking how long it takes for a particle to reduce to half of the initial value. To obtain a more accurate value for average lifetime, consider grouping the lifetime of particles into discrete "bins". If the number of particles with lifetime in  $[t_j, t_j + \Delta]$  is  $N_j$ , then the average lifetimes is represented as

$$\sum N_j = N_0$$

$$\tau \approx \frac{\sum t_j N_j}{\sum N_j}$$

If the number of particles decreases exponentially

$$N(t) = N_0 e^{-kt}$$

Then the number of particles lost in an interval  $[t_j, t_j + \Delta]$  is

$$N_j = N[t_j] - N[t_j + \Delta]$$

Or

$$N_j = \frac{N(t_j) - N(t_j + \Delta)}{\Delta} \cdot \Delta \approx -N'(t) \Delta$$

Taking  $\Delta \rightarrow 0$ . The sum for calculating the average lifetime turns into an integral.

$$\tau = \frac{1}{N_0} \int_0^\infty t(-N'(t)) dt = \int_0^\infty t e^{-kt} dt = \frac{1}{k}$$

## 8.2 Protein Synthesis and Degradation

Seriously who cares. I probably should write these notes though

## 9 Lecture 9 - Continuation of Protein Synthesis and Degradation

The concentration of a protein  $C(t)$  that is synthesised at a constant rate  $S$ ,  $k$  for the degradation rate.

$$\frac{dC}{dt} = S - kC$$

We can see that the equilibrium state is at  $C = \frac{S}{k}$ , the ODE is seperable, so the general solution is as follows

$$\int \frac{dC}{S - kC} = \int dt$$

And then... Poof! By the magic of Applied Mathematics, we achieve the general solution

$$C(t) = \frac{S}{k} + ce^{-kt}$$

Assume the initial value such that for  $C(t = 0) = C_0$ , then the general solution

$$C(t) = \frac{S}{k} + \left(C_0 - \frac{S}{k}\right)e^{-kt}$$

## 10 Lecture 10 - Heat Transfer

The heat flow (energy transferred per unit time) between two object of different temperature is proportional to the temperature difference between the two object (the heat flows from the

higher to the low temperatures until it equilibrates). Thus the rate of change of the body temperature is proportional to the temperature difference between the body and the environment  $T_{env}$  (air).

$$\frac{dT}{dt} = -k(T - T_{env})$$

$$T(t) = T_{env} + (T_0 - T_{env})e^{-kt}$$

Setting  $t = 0$  solves for  $T_0$  and  $T_{env}$  should be predefined. The parameter  $k$  is a proportionality constant and is difficult to represent with a model (at this level)

## 10.1 Electric circuit with resistance and inductance

$R$  resistance,  $L$  inductance in series,  $E(t)$  external voltage source = the sum of the voltage drops over the resistance + the inductance. The ODE for the current  $I(t)$ ,

$$E(t) = RI + L\frac{dI}{dt}$$

## 10.2 Population Dynamics

The rate of change of the population  $P(t)$  is proportional with the actual current population

$$\frac{dP}{dt} = kP$$

,

$$P(t) = P_0 e^{kt}$$

The  $k$  parameter in this model is the cell division rate constant. Ideally, bacteria can divide every 20 minutes.

# 11 Lecture 11 - Continuation of Population Dynamics

To balance out the growth of population due to reproduction we can also include a loss term due to death. The number of individuals dying per unit time is also proportional to population size  $P$

$$\frac{dP}{dt} = kp - dP = (k - d)P = \rho P$$

$$P(t) = P_0 e^{\rho t}$$

This modified population model still doesn't have a stable equilibrium state. The solution grows or decays exponentially depending on the sign of  $\rho = k - d$  which is the **net reproduction rate constant**. It is expected that a population should stabilise after some time in a stable equilibrium; where reproduction and death balance out.

The thing missing from this model is that we assumed  $k$  and  $d$  are constant parameters, but the birth and death rates may be dependant on several external factors (e.g. food, habitat) and this may be dependent on the size of the population  $r$ ,  $k$  (or  $\rho$ ) are functions of  $P$ .

Thus a modified nonlinear population model

$$\frac{dP}{dt} = \rho(P)P$$

Where the exact form of the function  $\rho(P)$  depends on the problem in question that we want to model. In general  $\rho(P)$  is a decreasing function (less resources available when the population increases, which slows down reproduction and/or increases death rate).

The simplest functional form for a decreasing  $\rho(P)$  is a linear function

$$\frac{dP}{dt} = \rho \left(1 - \frac{P}{K}\right) P$$

This is known as the **Logistic Equation**, where  $\rho$  is a constant (maximum net reproduction rate),  $K$  is the carrying capacity (the maximum sustainable population size). The net reproduction changes sign from positive to negative when  $P = K$ . The equilibrium solutions of this model are

- $P^* = 0$ ; unstable
- $P^* = K$ ; the derivative of RHS at  $P = K$  is  $< 0$  implying stable equilibrium state.

The exact solution to this ODE (seperable ODE) with initial condition  $P(0) = P_0$  is

$$P(t) = K \frac{1}{1 + \left(\frac{K}{P_0} - 1\right) e^{-\rho t}}$$

For a long time  $t \rightarrow \infty$ , the solution  $P(t)$  converges to the stable equilibrium. Assume that  $P$  describes a fish population and there is an additional loss rate due to fishing.

$$\frac{dP}{dt} = \rho \left(1 - \frac{P}{K}\right) P - f$$

$f$  is a constant parameter; the amount of fish harvested per unit time.

### Question:

How does the fishing rate  $f$  modify the state equilibrium of the population? How should the function  $P^*(f)$  look graphed?.

Is there any qualitative (dramatic) change of the equilibrium as  $f$  is modified or only a smooth transition; are there any bifuractions?

## 12 Lecture 12 - ODE Models of Population Dynamics

### 12.1 Bifuraction

Consider the Differential equation  $\frac{dy}{dt} = f(y; p)$  where  $p$  represents constant parameters. The equilibrium solutions are the roots of the equations  $f(y) = 0 \implies \frac{dy}{dt}$  which will depend on the parameters. Bifuraction diagrams are qualitative changes of the solutions happening when a parameter  $p$  is varied, i.e the change in the number of stability type of the equilibria. A **bifuraction diagram** plots and shows the solutions of branches  $y^*(p)$ .

### 12.2 Logistic populations dynamics

Consider the equation

$$\frac{dP}{dt} = \rho \left(1 - \frac{P}{K}\right) P$$

Introduce non-dimensional variables for  $P$  and  $t$  in order to reduce the number of parameters in the problem. Choose  $P' = \frac{P}{K}$  and  $t' = \rho t$ .

$$\frac{dP'}{dt'} = (1 - P') P'$$

The non-dimensional problem does not have any parameters. This shows that we can't have any bifuractions when we modify

$K$ , and  $\rho$  as those are all mathematically equivalent problems they can only differ by stretching or compressing the axis  $P$  and  $t$ . Now consider the equation

$$\frac{dP}{dt} = \rho \left( 1 - \frac{P}{K} \right) P - F$$

Introduce the same non-dimensional variables for  $P$  and  $t$ ; ( $P' = \frac{P}{K}$  and  $t' = \rho t$ ).

$$\frac{dP'}{dt'} = (1 - P')P' - \frac{F}{\rho K}$$

Denoting  $\frac{F}{\rho K} = F'$ ,  $F'$  cannot be eliminated by using non-dimensional variable, so the problem has 1 real parameter  $\implies$  it may have qualitatively different solutions when  $F'$  is varied.

### 12.3 Harvesting at a constant rate

$$\frac{dP'}{dt'} = (1 - P')P' - F'$$

How does the equilibria change when  $F'$  is varied? ( $P'^*(F') = ?$ ). Solve it graphically by following the intersections of the two terms on the RHS as  $F'$  is varied?

## 13 Lecture 13 - Systems of Coupled 1st Order ODEs

$$y_1' = f(y_1, y_2), \quad y_2' = g(y_1, y_2)$$

$y_1(t) = ?$  and  $y_2(t) = ?$ . There is no general method for finding solutions analytically for coupled systems of ODEs (except linear systems), but often the important information is related to



the equilibrium states of the system. The equilibrium solutions  $(y_1^*, y_2^*)$  are the solutions of the algebraic system of simultaneous equations:

$$f(y_1, y_2) = 0, \quad g(y_1, y_2) = 0$$

## 13.1 Dynamics of interacting populations

**Example 8.** Prey ( $P$ ) and predator ( $R$ ) population dynamics.

$$\frac{dP}{dt} = \rho P \left(1 - \frac{P}{K}\right) - aPR$$

$$\frac{dR}{dt} = bPR - dR$$

Given  $\rho P \left(1 - \frac{P}{K}\right)$  is the prey without predator,  $aPR$  is the prey-predator interaction and  $bPR - dR$  is the death of predator.

Non-dimensionalize the system by introducing new dimensionless variables  $P' = \frac{P}{K}$ ,  $t' = \rho t$  (done similarly for the logistic equation) and choose  $R' = \frac{a}{\rho} R$ .

$$\frac{dP'}{dt'} = P'(1 - P') - P'R'$$

$$\frac{dR'}{dt'} = \frac{bK}{\rho} P'R' - \frac{d}{\rho} R'$$

$$\frac{dR'}{dt'} = \alpha P'R' - \beta R'$$

The non-dimensionalization shows that the behaviour of the solution can only depend on the two new parameters  $\alpha = b \frac{K}{\rho}$

and  $\beta = \frac{d}{\rho}$

# 14 Lecture 14 - Second Order Differential Equations

$$f(y'', y', y, t) = 0, \quad y(t) = ?$$

**Example 9.** Newton's Law  $F = ma$ , where  $a$  is the acceleration. (second derivative of coordinate function  $a = x''(t)$ )

Many partial differentials contain second derivatives over spatial coordinates. There is no general way to solve nonlinear second order ODEs analytically, however we can solve them numerically.

To solve numerically, we can rewrite it into the form of two coupled first order ODEs by introducing an unknown function defined as  $y' = v$ . Then  $v' = F(v, y, t)$  and  $y' = v$  forms a coupled system, needs to be complemented with initial conditions  $y(t = 0)$  and  $v(t = 0) = y'(t = 0)$

## 14.1 Solving linear 2nd order ODEs analytically

Some types of linear 2nd order ODEs may be solved analytically. A general form of a analytically solvable 2nd order ODE may be

$$y'' + p(t)y' + q(t)y = r(x)$$

Can be classified as homogenous if  $r(x) = 0 \forall x$ . Constant coefficients if  $p(t)$  and  $q(t)$  are constants. For homogenous ODEs ( $r(x) = 0$ ), we can use the **Principle of superposition** for construction the general solution.

If  $y_1(t)$  and  $y_2(t)$  are two linearly independent solutions of the homogenous linear 2nd order ODE then any linear combination  $y(t) = c_1$ . What?

## 14.2 Method of Reduction of Order

Assume that we have a solution  $y_1(t)$  that satisfies a linear homogenous ODE  $\implies$  then the reduction of order method leads to a problem of a lower order ODE for finding the other/general solution. The steps for solving with reduction of order method is as follows

1.  $y'' + p(t)y' + q(t)y = 0$
2. Assume that  $y_1(t)$  is a solution.
3. Look for solutions of the form  $y = u(t)y_1(t)$
4. Substitute into the equation.
5.  $(uy_1)'' + p(t)(uy_1)' + q(t)uy_1 = 0$

## 15 Lecture 15 - Solving 2nd Order ODES

Suppose we have a linear  $2^{nd}$  order ODE  $f(y'', y', y, t) = 0$ . Suppose we are solving for the general solution  $y(t)$ . If  $f$  is homogenous and constant-coefficient, the ODE is of the form  $y'' + ay' + by = 0$ . As a function that remains the same (same for a multiplicative constant) is  $e^t$ , we can guess  $y(t) = e^{\lambda t}$  is a solution for some  $\lambda$ .

$$\begin{aligned}y'' + ay' + b &= (e^{\lambda t})'' + a(e^{\lambda t})' + be^{\lambda t} \\0 &= \lambda^2 e^{\lambda t} + \lambda a e^{\lambda t} + b e^{\lambda t} \\0 &= e^{\lambda t}(\lambda^2 + a\lambda + b)\end{aligned}$$

Note that  $\lambda^2 + a\lambda + b = 0$  is the characteristic equation of an ODE

$$\implies \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Thus  $e^{\lambda_1 t}$  and  $e^{\lambda_2 t}$  are linearly independent solutions and the general solution is  $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$  for some  $c_1, c_2$ . However, this is only the case if  $\lambda_1 \neq \lambda_2$  ( $4b < a^2$ ).

If  $4b = a^2$ , then  $\lambda_1 = \lambda_2 = \lambda$  and we obtain only one solution  $e^{\lambda t}$ . With this, we can use reduction of order. As  $y_1(t) = e^{\lambda t}$  is a solution, let  $y(t) = u(t)e^{\lambda t}$ . Thus  $y'' + ay' + by = 0$

$$\implies (ue^{\lambda t})'' + a(ue^{\lambda t})' + bue^{\lambda t} = 0$$

$$(u'e^{\lambda t} + \lambda ue^{\lambda t})' + a(u'e^{\lambda t} + \lambda ue^{\lambda t}) + bue^{\lambda t} = 0$$

$$u''e^{\lambda t} + \lambda u'e^{\lambda t} + \lambda u'e^{\lambda t} + \lambda^2 ue^{\lambda t} + au'e^{\lambda t} + a\lambda ue^{\lambda t} + bue^{\lambda t} = 0$$

$$u'' + 2\lambda u' + \lambda^2 u + au' + a\lambda u + bu = 0$$

However note that  $\lambda = \frac{-a}{2}$ ,  $4b = a^2 \implies b = \frac{a^2}{4}$

$$u'' - au' + \frac{a^2}{4} + au' - \frac{a^2}{2}u + \frac{a^2}{4}u = 0$$

$$\implies u'' = 0$$

$$\implies u(t) = c_1 t + c_2$$

for some  $c_1, c_2$ . Thus the general solution is

$$y(t) = u(t)e^{\lambda t} = (c_1 t + c_2)e^{\lambda t}$$

If we obtain no solutions ( $4b > a^2$ ) in the real set, we get two complex conjugate roots.

$$\lambda_{1,2} = \frac{-a}{2} \pm \frac{\sqrt{4b - a^2}}{2}i = \alpha \pm \beta i$$

We get the general solution

$$\begin{aligned} y(t) &= c_1 e^{(\alpha + \beta i)t} + c_2 e^{(\alpha - \beta i)t} \\ &= c_1 e^{\alpha t} e^{\beta i t} + c_2 e^{\alpha t} e^{-\beta i t} \end{aligned}$$

However,  $e^{i\beta t} = \cos(\beta t) + i \sin(\beta t) \implies y(t) = e^{\alpha t} (c_1 \cos(\beta t) + c_1 i \sin(\beta t) + c_2 \cos(-\beta t) + i \sin(-\beta t))$

$$y(t) = (c_1 + c_2 e^{\alpha t} \cos(\beta t) + (c_1 - c_2) i e^{\alpha t} \sin(\beta t))$$

Let  $A = c_1 + c_2$  and  $B = (c_1 - c_2)i$

$$\begin{aligned} y(t) &= A e^{\alpha t} \cos(\beta t) + B e^{\alpha t} \sin(\beta t) \\ &= e^{\alpha t} (A \cos(\beta t) + B \sin(\beta t)) \end{aligned}$$

## 16 Lecture 16 - Constant-coefficient Non-Homogenous

Suppose we have an ODE  $y'' + p(t)y' + q(t)y = r(t)$  and we are attempting to solve for  $y(t)$ . This is a constant coefficient non homogenous linear order  $2^{nd}$  order ODE. (if  $p(t)$ ,  $q(t)$  are constant). Let  $L(x) = x'' + px' + qx$ . Suppose we have two linearly independent solutions  $y_1(t)$ ,  $y_2(t)$ . Thus  $L(y_1) = L(y_2) = r \implies L(y_1) - L(y_2) = 0$ . As the ODE is linear  $\implies L$  is linear,  $L(y_1) - L(y_2) = L(y_1 - y_2) = 0$ . This is a homogenous ODE and can be easily solved

### Steps:

1. Solve the corresponding homogenous ODE ( $r = 0$ )  $\rightarrow y_n = c_1 y_n + c_2 y_k$
2. Find solution  $y_p$  (reduction of order)  $\rightarrow y_p = u(t)y_{k1}$
3. General case is  $y(t) = y_p + y_n$

**Example 10.** Solve for  $y(t)$  given  $y'' + y' - 2y = t^2 - 2t + 3$

1. Solve  $y'' + y' - 2y = 0$ , characteristic equation is  $\lambda^2 + \lambda - 2 = 0$ . Implied

$$(\lambda + 2)(\lambda - 2) = 0$$

Thus  $\lambda_1 = -2$  and  $\lambda_2 = 1$ , therefore

$$y_n(t) = c_1 e^{-2t} + c_2 e^t$$

2. Reduction of Order: as  $y_1(t) = e^t$  is a solution we have a linearly independent solution  $y_p(t) = u(t)e^t$ . Subbing in

$$\begin{aligned}(ue^t)'' + (ue^t)' - 2ue^t &= t^2 - 2t + 3 \\ u''e^t + 2u'e^t t + ut^2 e^t + u'e^t + ute^t - 2ue^t &= t^2 - 2t + 3 \\ u'' + 2u't + ut^2 + u' + u5 - 2u &= (t^2 - 2t + 3)e^{-t} \\ \implies y(t) &= c_1 e^{-2t} + c_2 e^t - \frac{t^2}{2} + \frac{t}{2} - \frac{7}{4}\end{aligned}$$

## 17 Lecture 17 -

## 18 Lecture 18 - Applications of 2nd order ODEs

### 18.1 Mechanical/Electrical Oscillators

#### SMALL ANGLE TIME

$$\sin(\theta) = \theta$$

$$m \frac{dy}{dt} = -mg + kdy$$

**Example 11.** Given a pendulum,

$$m \frac{d^2 y}{dt^2} = -ky$$

$$y(0) = y_0, \quad y'(0) = 0$$

$$my'' + ky = 0$$

$$y'' + \frac{k}{m}y = 0$$

Denote  $a = \frac{k}{m}$

$$y'' + ay = 0 \implies \lambda^2 + a = 0$$

$$\therefore y = e^{\lambda t}$$

For  $\lambda = \pm i\sqrt{a}$

$$y(t) = A \cos(\sqrt{a}t) + B \sin(\sqrt{a}t)$$

However with this model there is no dampining over time.

$$m \frac{d^2 y}{dt^2} = -ky - \gamma y'$$

$$y'' + \left(\frac{\gamma}{m}\right)y' + \left(\frac{k}{m}\right)y = 0$$

Denoting  $\frac{\gamma}{m} = b$  and  $\frac{m}{k} = a$

$$y'' + by' + ay = 0$$

If  $b^2 > 4a$ ,  $\lambda_1, \lambda_2 < 0$ , if  $b^2 < 4a$ ,  $\lambda_{1,2} = \frac{-b}{2} \pm i\sqrt{4a - b^2}$

## 19 Lecture 19 - Introduction to Multivariate Calculus

### 19.1 Review of one-variable case

Let  $f : D \rightarrow \mathbb{R}$  be a function with a domain  $D$  an open subset of  $\mathbb{R}$ . For  $a \in D$  we say that the limit  $\lim_{x \rightarrow a} f(x)$  exists if and only if

1. The limit from the left exists
2. the limit from the right exists
3. these two limits coincide etc.



$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

Furthermore if the limit exists and is equal to the actual value of  $f$  at  $a$ .

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

We say that  $f$  is continuous at  $x = a$ . If  $f$  is continuous on all  $D$ , we say that  $f$  is continuous function on  $D$

## 19.2 The two-variable case

When  $f$  is a function of more than one variable, the situation is more subtle. There are more than two ways to approach a given point of interest.

**Example 12.** Consider the function

$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

with domain given by  $\mathbb{R}^2 \setminus (0, 0)$

Approaching the origin along  $y = 0$ , if  $x \neq 0$   $f(x, 0) = \frac{x^2}{x^2 + 0} = 1$ . Then

$$\lim_{x \rightarrow 0} f(x, 0) = 1$$

If  $y \neq 0$ ,  $f(0, y) = \frac{0}{0 + y^2} = 0$ . Then,

$$\lim_{y \rightarrow 0} f(0, y) = 0$$

Thus  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist, as  $\lim_{x \rightarrow 0} f(x, 0) \neq \lim_{y \rightarrow 0} f(0, y)$ .

In general, for the limit  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  to exist, it is necessary that every path in  $D$  approaching  $(a, b)$  gives the same

limiting value  $((a, b)$  may not necessarily be in  $D$ ). This gives a method for finding if a limit does not exist for multivariate limits.

$$\text{If } \begin{cases} f(x, y) \rightarrow L_1 \text{ as } (x, y) \rightarrow (a, b) \text{ along the path } C_1 \in D \\ f(x, y) \rightarrow L_2 \text{ as } (x, y) \rightarrow (a, b) \text{ along the path } C_2 \in D \end{cases}$$

*Remark.* The above notation is somewhat deficient and perhaps one should write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y)$$

to indicate that only paths  $D$  terminating in  $(a, b)$  (which itself may or may not be in  $D$ ) are considered. For instance,  $f(x, y) = x^2 + y^2$  with  $D = \{(x, y) : x^2 + y^2 < 1\}$ , then  $\lim_{(x,y) \rightarrow (1,0)} f(x, y)$  exists and is 1. However if

$$\begin{cases} x^2 + y^2 & \text{for } D = \{(x, y) : x^2 + y^2 < 1\} \\ 0 & \text{for } D = \{(x, y) : x^2 + y^2 > 1\} \end{cases}$$

then  $\lim_{(x,y) \rightarrow (1,0)} f(x, y)$  does not exist.

## 20 Lecture 20 - Continuation of Functions of Multiple Variables

Generally, we write  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$  to mean the values of  $f(x, y)$  approach  $L$  as the point  $(x, y)$  approaches  $(a, b)$  along any path in the domain  $f$ . That is, we can make the value of  $f(x, y)$  as close to  $L$  as we like by taking  $(x, y)$  sufficiently close to  $(a, b)$ . This is formalised through the following definition

**Definition 1.** Let  $f$  be a function of two variables whose domain  $D$  includes points arbitrarily close to  $(a, b)$ .

Then we say that the limit of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$  is  $L$  and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every number  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that if  $(x, y) \in D$  and  $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ , then  $|f(x, y) - L| < \varepsilon$ . Where  $|f(x, y) - L|$  can be described as the distance between  $f(x, y)$  and  $L$  in  $\mathbb{R}$ . The  $\sqrt{(x-a)^2 + (y-b)^2}$  is the distance between  $(x, y)$  and  $(a, b)$  in  $\mathbb{R}^2$ . The definition is essentially saying that the distance between  $f(x, y)$  and  $L$  can be made arbitrarily small by making the distance between  $(x, y)$  and  $(a, b)$  sufficiently small, **but not 0**.