MATH1072 Notes

Ismael Khan

September 9, 2019

1 Lecture 1 - Introduction to Dimensional Analysis

To descrive real systems quantitatively, we use numbers and units of measurement. Eg. 3 meters, 5 years, 10 km/h. Each measurable quantity has a certain dimension.

1.1 Base dimensions

Length (L), time (T), mass (M), other non-mechanical base dimensions include temperature; electric charge/current etc.

1.2 Derived dimensions

Speed $\frac{L}{T}$, force ($N = M\frac{L}{T^2}$), energy ($J = M\frac{L^2}{T^2}$). The dimensions of the terms added on both sides must be equal. This is known as the equation being *dimensionally homogenous*. We can use the dimensional homogeneity to make dimensional estimateas for certain quantities (order of magnitude, not exact prediction)

2 Lecture 2 - Dimensional Analysis

3 Lecture 3 - Introduction to Differential Equations

Generally, an ordinary differential equation (ODE) is represented as:

$$F(t, y(t), y'(t), y''(t), ...) = 0$$

For Instance, Newton's Law

$$m\frac{d^2r}{dt^2} = F$$

Induction Law:

$$RI + L\frac{dI}{dt} + \frac{1}{c}$$

Population:

$$\frac{dP}{dt} = rP(1 - \frac{P}{k})$$

Maxwell's Equations:

$$\nabla \cdot \bar{E} = \frac{1}{\rho}\varepsilon_0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = \frac{\delta B}{\delta t}$$

$$\nabla \times \bar{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\delta E}{\delta t}$$

Navier-Stokes: (Modelling velocity of fluids in space)

$$\frac{\delta \overline{v}}{\delta t} = \overline{v}\nabla\overline{v} = ?$$

Schrodinger Wave Equation:

$$i\hbar\frac{\delta\psi}{\delta t} = \Big[\frac{-\hbar}{2m}\nabla^2 + V(r)\Big]\psi$$

$$\psi = ?$$

We generally have:

$$F(t, y(t), y'(t), ...) = 0,\ y(t) = ?$$

Where the order of ODE = order of the higest derivative. Take the equation $y' = y$, the solutions are $y = e^t$, $y = 0$ and $y = ce^t$. However the latter expression encapsulates the former, so $y(t) = ce^t$ is known as the general solution where $c \in \mathbb{R}$. The genreal solution is not unique.

If you take an ODE and add some initial condition, then the solution is a unique result. For Instance take the ODE $y' = y$ and say that $y(0) = 1$, then we achieve a unique solution of $c = 1$, $y = e^t$.

4 Lecture 4 - Ordinary Differential Equations

4.1 Equilibrium Solutions

An equilibrium solution (steady state solution) of an ODE (if such solution exists) is a constant solution $y(t) = c$ which satisfies the ODE (for any t).

Example 1. Take $y' = f(t, y)$, the equilibrium solution

$$f(t, y = c) = 0$$

This implies that the slope at $y = c$ must be 0 for a function defined on a (t, y) plane

Example 2. $y' = y$, $y = 0$ is an equilibrium solution.

Example 3. $y' = y(1 - y)$m $y = 0$ and $y = 1$ are equilibrium solutions

Slope fields can visualised with Mathematica using SteamPlot (or VectorPlot). The vector flow of a field corresponding to $f(t, y)$ is $\{1, f(t, y)\}$ since the flow in the horizontal direction (t) has a constant rate (the flow of time) which can be set to 1, the vertical flow is $f(t, y)$.

4.2 Stability of equilibrium solutions

If the solutions starting in condition a small neighbourhood of an equilibrium soltion $(y = c)$ converge towards the equilibrium solution for large t, then the equilibrium solution $y = c$ is stable.

Example 4. $y' = y$, the equilibrium solution $y = 0$ is unstable (for $y' = -y$, $y = 0$ is stable)

Example 5. $y' = y(1 - y)$, $y = 0$ is unstable, but $y = 1$ is stable.

5 Lecture 5 - Stability of ODE's

5.1 Condition for stability of equilibrium soltuions

Using a Taylor Series approximation of $f(t, y)$ near the equilibrium solution $y = c$, assume $f(t, y) = f(y)$ for simplicity

$$f(y) \approx f(c) + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c)^2 + ...$$

Where $y = c$ is the equilibrium solution $f(c) = 0$, thus

$$y'(t) = 0 + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c) + ...$$

Take $u = y - c$, then

$$u' = f'(c)u + ...$$

Implying

$$u(t) = Ae^{f'(c)t}$$

If $f'(c) > 0$, $y = c$ is unstable. If $f'(c) < 0$, $u(t) \to 0$, $y(t) \to c$ thus $y = c$ is stable.

5.2 Euler's Method

An iterative operation which models $y_k \approx y(k \cdot \Delta t)$. Given $y' = f(t, y)$, $y(0) = c$

$$\begin{aligned} y'(t) &\equiv \lim_{\Delta \to 0} \frac{y(t + \Delta) - y(t)}{\Delta} & (1) \\ &\approx \frac{y(t + \Delta) - y(t)}{\Delta} & (2) \end{aligned}$$

As it is an iterative method, $y_{k+1} = y_k + f(t_k, y_k)$

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta} \approx f(t_k, y_k)$$

Example 6. Given $y' = 2t$, $y(t) =?$, $y(0) = 0$. By Euler's

Method

$$N = 1,\ \Delta = 1 \tag{3}$$
$$N = 2,\ \Delta = \frac{1}{2} \tag{4}$$
$$N = 4,\ \Delta = \frac{1}{4} \tag{5}$$
$$N = 10\ \Delta = \frac{1}{10} \tag{6}$$

6 Lecture 6 - Continuation of Euler's Method

6.1 Error Generated in Euler's Method

Assume $y(t_k) = y_k$, the error is represented with the taylor series approximation

$$\Big| y_{k+1} - y(t_k + \Delta) \Big| = y(t_{k+1}) = y(t_k) + y'(t_k)\Delta + \frac{1}{2}y''(t_k)\Delta^2 + ...$$
$$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

Thus

$$\text{error} = |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2$$

However generalized for N steps,

$$N \cdot |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2 \cdot N$$
$$\propto \Delta^2 \cdot \frac{1}{\Delta} \propto \Delta$$

$$y_{k+1} = y_k + \frac{1}{2}\Big[f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \Big]\Delta$$

7 Lecture 7 - Solving Linear First Order ODEs

$$y' = f(t, y), y(0) = y_0$$
$$t \in [0, t_{max}]$$

1. Euler's Method

$$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

   Where total error $\propto \Delta$

2. Heun Method (Revised Euler's Method)

$$\begin{cases} y_{k+1} = y_k + f(t_k, y_k)\Delta \\ y_{k+1} = y_k + \frac{1}{2}[f(t_k, y_k) + f(t_k + \Delta, y_{k+1})] \end{cases}$$

   Where total error $\propto \Delta^2$. Note that Heun Method will possibly show in Assignment 3.

7.1 Runge Kutta Method

$$p_1 = f(t_k, y_k)\Delta$$
$$p_2 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_1}{2})\Delta$$
$$p_3 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_2}{2})\Delta$$
$$p_4 = f(t_k + \frac{\Delta}{2}, y_k + p_3)\Delta$$

for

$$y_{k+1} = y_k + \frac{1}{6}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 + \frac{1}{6}p_4 + ...$$

7.2 Adaptive Step Size

Fixed step size is mostly inefficient in most cases, so we use an adaptive step size for numerical methods to achieve better approximations.

7.3 Coupled Systems

$$y' = f(t, y_1, y_2)$$
$$x' = g(t, y_1, y_2)$$

$$x(t) =?\ y(t) =?$$

In the context of Euler's Method

$$\begin{cases} x_{k+1} = x_k + f(t_k, x_k, y_k)\Delta \\ y_{k+1} = y_k + g(t_k, x_k, y_k)\Delta \end{cases}$$

7.4 Analytical ODE Solutions

7.4.1 Linear First Order ODE's

By definition,

$$y(t) = f(t)y(t) + g(t)$$

is generally the standard form of a linear first order ODE. Note that

$$y'(t) = f(t)y(t)$$

is a special case of a linear first order ODE that is seperable.

Example 7.

$$ty' + y = t\cos t$$

Observe that, by the product rule for differentiation, $ty' + y = (ty)'$.

$$\begin{aligned}
(ty)' &= t\cos t \\
\int (ty)' dy &= \int t\cos t dt \\
ty &= t\sin t - \int \sin t dt \\
ty &= t\sin t + \cos t + c \\
\therefore y &= \frac{t\sin t + \cos t}{t} + \frac{c}{t}
\end{aligned}$$

8 Lecture 8 - Applications of First Order ODEs

Common applications of first order ODEs are

8.1 Radioactive Decay

The particles of a radioactive material decay spontaneously in a stochastic process. The total mass of the radioactive atoms decrease with time. We can represent this as

$$\frac{dM}{dt} = -kM$$

For $M(t)$ to represent the mass of the radioactive material over time t. Clearly the solution to this is

$$M(t) = M_0 e^{-kt}$$

Note that there is a limitation to this ODE Model. We assume that the mass changes continuously in time. Whereas in reality it changes in discrete steps following individual decay events. However for the macroscopic mass, the number of particles is much larger, so we can neglect the discrete jumps and assume a continuous deterministic model. Which works well for most cases.

The lifetime of particles varies, however we can categorize them with their average lifetime. We do this by asking how long it takes for a particle to reduce to half of the initial value. To obtain a more accurate value for average lifetime, consider grouping the lifetime of particles into discrete "bins". If the number of particles with lifetime in $[t_j, t_j + \Delta]$ is N_j, then the average lifetimes is represented as

$$\sum N_j = N_0$$

$$\tau \approx \frac{\sum t_j N_j}{\sum N_j}$$

If the number of particles decreases exponentially

$$N(t) = N_0 e^{-kt}$$

Then the number of particles lost in an interval $[t_j, t_j + \Delta]$ is

$$N_j = N[t_j] - N[t_j + \Delta]$$

Or

$$N_j = \frac{N(t_j) - N(t_j + \Delta)}{\Delta} \cdot \Delta \approx -N'(t)\Delta$$

Taking $\Delta \to 0$. The sum for calculating the average lifetime turns into an integral.

$$\tau = \frac{1}{N_0} \int_0^\infty t(-N'(t))\, dt = \int_0^\infty t e^{-kt}\, dt = \frac{1}{k}$$

8.2 Protein Synthesis and Degradation

Seriously who cares. I probably should write these notes though

9 Lecture 9 - Continuation of Protein Synthesis and Degradation

The concentration of a protein $C(t)$ that is synthesised at a constant rate S, k for the degradation rate.

$$\frac{dC}{dt} = S - kC$$

We can see that the equilibrium state is at $C = \frac{S}{k}$, the ODE is seperable, so the general solution is as follows

$$\int \frac{dC}{S - kC} = \int dt$$

And then... Poof! By the magic of Applied Mathematics, we achieve the general solution

$$C(t) = \frac{S}{k} + ce^{-kt}$$

Assume the initial value such that for $C(t = 0) = C_0$, then the general solution

$$C(t) = \frac{S}{k} + \left(C_0 - \frac{S}{k}\right)e^{-kt}$$

10 Lecture 10 - Heat Transfer

The heat flow (energy transferred per unit time) between two object of different temperature is proportional to the temperature difference between the two object (the heat flows from the higher to the low temperatures until it equilibrates). Thus the rate of change of the body temperature is proportional to the temperature difference between the body and the environment T_{env} (air).

$$\frac{dT}{dt} = -k(T - T_{env})$$

$$T(t) = T_{env} + (T_0 - T_{env})e^{-kt}$$

Setting $t = 0$ solves for T_0 and T_{env} should be predefined. The parameter k is a proportionality constant and is difficult to represent with a model (at this level)

10.1 Electric circuit with resistance and inductance

R resistance, L inductance in series, $E(t)$ external voltage source = the sum of the voltage drops over the resistance + the inductance. The ODE for the current $I(t)$,

$$E(t) = RI + L\frac{dI}{dt}$$

10.2 Population Dynamics

The rate of change of the population $P(t)$ is proportional with the actual current population

$$\frac{dP}{dt} = kP$$

,

$$P(t) = P_0 e^{kt}$$

The k parameter in this model is the cell division rate constant. Ideally, bacteria can divide every 20 minutes.

11 Lecture 11 - Continuation of Population Dynamics

To balance out the growth of population due to reproduction we can also include a loss term due to death. The number of individuals dying per unit time is also proportional to population size P

$$\frac{dP}{dt} = kp - dP = (k-d)P = \rho P$$

$$P(t) = P_0 e^{\rho t}$$

This modified population model still doesn't have a stable equilibrium state. The solution grows or decays exponentially depending on the sign of $\rho = k - d$ which is the **net reproduction rate constant**. It is expected that a population should stabilise after some time in a stable equilibrium; where repoduction and death balance out.

The thing missing from this model is that we assumed k and d are constant parameters, but the birth and death rates may be dependant on several external factors (e.g. food, habitat) and this may be dependent on the size of the population r, k (or ρ) are functions of P.

Thus a modified nonlinear population model

$$\frac{dP}{dt} = \rho(P)P$$

Where the exact form of the function $\rho(P)$ depends on the problem in question that we want to model. In general $\rho(P)$ is a decreasing function (less resources available when the population increases, which slows down reproduction and/or increases death rate).

The simplest functional form for a decreasing $\rho(P)$ is a linear function

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P$$

This is known as the **Logistic Equation**, where ρ is a constant (maximum net reproduction rate), K is the carrying capacity (the maximum sustainable population size). The net reproduction changes sign from positive to negative when $P = K$. The equilibrium solutions of this model are

- $P^* = 0$; unstable
- $P^* = K$; the derivative of RHS at $P = K$ is < 0 implying stable equilibrium state.

The exact solution to this ODE (seperable ODE) with initial condition $P(0) = P_0$ is

$$P(t) = K\frac{1}{1 + \Big(\frac{K}{P_0} - 1\Big)e^{-\rho t}}$$

For a long time $t \to \infty$, the solution $P(t)$ converges to the stable equilibrium. Assume that P describes a fish population and there is an additional loss rate due to fishing.

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P - f$$

f is a constant parameter; the amount of fish harvested per unit time.

Question:

How does the fishing rate f modify the state equilibrium of the population? How should the function $P^*(f)$ look graphed?.

Is there any qualitative (dramatic) change of the equilibrium as f is modified or only a smooth transition; are there any bifuractions?

12 Lecture 12 - ODE Models of Population Dynamics

12.1 Bifuraction

Consider the Differential equation $\frac{dy}{dt} = f(y;p)$ where p represents constant parameters. The equilibrium solutions are the roots of the equations $f(y) = 0 \implies \frac{dy}{dt}$ which will depend on the parameters. Bifuraction diagrams are qualitative changes of the solutions happening when a parameter p is varied, i.e the change in the number of stability type of the equilibria. A **bifuraction diagram** plots and shows the solutions of branches $y^*(p)$.

12.2 Logistic populations dynamics

Consider the equation

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P$$

Introduce non-dimensional variables for P and t in order to reduce the number of parameters in the problem. Choose $P' = \frac{P}{K}$ and $t' = \rho t$.

$$\frac{dP'}{dt'} = (1 - P')P'$$

The non-dimensional problem does not have any parameters. This shows that we can't have any bifuractiosn when we modify

K, and ρ as those are all mathematically equibva;lent problems they can only differ by strecthing or compeessing the axis P and t. Now consider the equation

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P - F$$

Introduce the same non-dimensional variables for P and t; ($P' = \frac{P}{K}$ and $t' = \rho t$).

$$\frac{dP'}{dt'} = (1 - P')P' - \frac{F}{\rho K}$$

Denoting $\dfrac{F}{\rho K} = F'$, F' cannot be elimated by using non-dimensional variable, so the problem has 1 real parameter $\Longrightarrow$ it may have qualitatively different solutions when F' is varied.

12.3 Harvesting at a constant rate

$$\frac{dP'}{dt'} = (1 - P')P' - F'$$

How does the equilibria change when F' is varied? ($P'^*(F') =?$). Solve it graphically by following the intersections of the two terms on the RHS as F' is varied?

13 Lecture 13 - Systems of Coupled 1st Order ODEs

$$y_1' = f(y_1, y_2),\ y_2' = g(y_1, y_2)$$

$y_1(t) =?$ and $y_2(t) =?$. There is no general method for finding solutions analytically for coupled systems of ODEs (except linear systems), but often the important information is related to the equilibrium states of the system. The equilbrium solutions (y_1^*, y_2^*) are the solutions of the algebraic system of simultaneous equations:

$$f(y_1, y_2) = 0,\ g(y_1, y_2) = 0$$

13.1 Dynamics of interacting populations

Example 8. Prey (P) and predator (R) population dynamics.

$$\frac{dP}{dt} = \rho P\Big(1 - \frac{P}{K}\Big) - aPR$$

$$\frac{dR}{dt} = bPR - dR$$

Given $\rho P\Big(1 - \dfrac{P}{K}\Big)$ is the prey without predator, aPR is the prey-predator interaction and $bPR - dR$ is the death of predator.

Non-dimensionalize the sytem by introducing new dimensionaless variables $P' = \dfrac{P}{K}$, $t' = \rho t$ (done similarly for the logistic equation) and choose $R' = \dfrac{a}{\rho}R$.

$$\frac{dP'}{dt'} = P'(1 - P') - P'R'$$

$$\frac{dR'}{dt'} = \frac{bK}{\rho}P'R' - \frac{d}{\rho}R'$$

$$\frac{dR'}{dt'} = \alpha P'R' - \beta R'$$

The non-dimensionalization shows that the behaviour of the solution can only depend on the two new parameters $\alpha = b\dfrac{K}{\rho}$ and $\beta = \dfrac{d}{\rho}$

14 Lecture 14 - Second Order Differential Equations

$$f(y'', y', y, t) = 0, \; y(t) = ?$$

Example 9. Newton's Law $F = ma$, where a is the acceleration. (second derivative of coordinate function $a = x''(t)$)

Many partial differentials contain second derivates over spatial coordinates. There is no general way to solve nonlinear second order ODEs analytically, however we can solve them numerically.

To solve numerically, we can rewrite it into the form of two coupled first order ODEs by introducing an unknown function defined as $y' = v$. Then $v' = F(v, y, t)$ and $y' = v$ forms a coupled system, needs to be complemented with initial conditions $y(t = 0)$ and $v(t = 0) = y'(t = 0)$

14.1 Solving linear 2nd order ODEs analytically

Some types of linear 2nd order ODEs may be solved analytically. A general form of a analytically solveable 2nd order ODE may be

$$y'' + p(t)y' + q(t)y = r(x)$$

Can be classified as homogenous if $r(x) = 0 \; \forall x$. Constant coefficients if $p(t)$ and $q(t)$ are constants. For homogenous ODEs ($r(x) = 0$), we can use the **Principle of superposition** for construction the general solution.

If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the homogenous linear 2nd order ODE then any linear combination $y(t) = c_1$. What?

14.2 Method of Reduction of Order

Assume that we have a solution $y_1(t)$ that satisfies a linear homogenous ODE $\implies$ then the reduction of order method leads to a problem of a lower order ODE for finding the other/general solution. The steps for solving with reduction of order method is as follows

1. $y'' + p(t)y' + q(t)y = 0$
2. Assume that $y_1(t)$ is a solution.
3. Look for solutions of the form $y = u(t)y_1(t)$
4. Substitute into the equation.
5. $(uy_1)'' + p(t)(uy_1)' + q(t)uy_1 = 0$

15 Lecture 15 -

16 Lecture 16 -

17 Lecture 17 -

18 Lecture 18 - Applications of 2nd order ODEs

18.1 Mechanical/Electrical Ossiclators

SMALL ANGLE TIME

$$\sin(\theta) = \theta$$

$$m\frac{dy}{dt} = -mg + kdy$$

Example 10. Given a pendulum,

$$m\frac{d^2y}{dt^2} = -ky$$

$$y(0) = y_0,\ y'(0) = 0$$

$$my'' + ky = 0$$

$$y'' + \frac{k}{m}y = 0$$

Denote $a = \dfrac{k}{m}$

$$y'' + ay = 0 \implies \lambda^2 + a = 0$$

$$\therefore y = e^{\lambda t}$$

For $\lambda = \pm i\sqrt{a}$

$$y(t) = A\cos(\sqrt{a}t) + B\sin(\sqrt{a}t)$$

However with this model there is no dampining over time.

$$m\frac{d^2y}{dt^2} = -ky - \gamma y'$$

$$y'' + \left(\frac{\gamma}{m}\right)y' + \left(\frac{k}{m}\right)y = 0$$

Denoting $\dfrac{\gamma}{m} = b$ and $\dfrac{m}{k} = a$

$$y'' + by' + ay = 0$$

If $b^2 > 4a$, $\lambda_1, \lambda_2 < 0$, if $b^2 < 4a$, $\lambda_{1,2} = \frac{-b}{2} \pm i\sqrt{4a - b^2}$

19 Lecture 19 - Introduction to Multivariate Calculus

19.1 Review of one-variable case

Let $f : D \to \mathbb{R}$ be a function with a domain D an open subset of $\mathbb{R}$. For $a \in D$ we say that the limit $\lim\limits_{x \to a} f(x)$ exists if and only if

1. The limit from the left exists
2. the limit from the right exists
3. these two limits coincide etc.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Furthermore if the limit exists and is equal to the actual value of f at a.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

We say that f is continuous at $x = a$. If f is continuous on all D, we say that f is continuous function on D

19.2 The two-variable case

When f is a function of more than one variable, the situation is more subtle. There are more than two ways to approach a given point of interest.

Example 11. Consider the function

$$f(x,y) = \frac{x^2}{x^2+y^2}$$

with domain given by $\mathbb{R}^2 \setminus (0,0)$

Approaching the origin along $y = 0$, if $x \neq 0$ $f(x,0) = \dfrac{x^2}{x^2+0} = 1$. Then

$$\lim_{x\to 0} f(x,0) = 1$$

If $y \neq 0$, $f(0,y) = \dfrac{0}{0+y^2} = 0$. Then,

$$\lim_{y\to 0} f(0,y) = 0$$

Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist, as $\lim\limits_{x\to 0} f(x,0) \neq \lim\limits_{y\to 0} f(0,y)$. In general, for the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ to exist, it is neccessary that every parth in D approaching (a,b) gives the same limiting value ((a,b) may not neccesarily be in D). This gives a method for finding if a limit does not exist for multivariate limits.

$$\text{If } \begin{cases} f(x,y) \to L_1 \text{ as } (x,y) \to (a,b) \text{ along the path } C_1 \in D \\ f(x,y) \to L_2 \text{ as } (x,y) \to (a,b) \text{ along the path } C_2 \in D \end{cases}$$

such that $L_1 \neq L_2$, then the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Remark. The above notation is somewhat deficient and perhaps one should write

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

to indicate that only paths D terminating in (a, b) (which itself may or may not be in D) are considered. For instance, $f(x, y) = x^2 + y^2$ with $D = \{(x, y) : x^2 + y^2 < 1\}$, then $\lim_{(x,y)\to(1,0)} f(x, y)$ exists and is 1. However if

$$\begin{cases} x^2 + y^2 & \text{for } D = \{(x, y) : x^2 + y^2 < 1\} \\ 0 & \text{for } D = \{(x, y) : x^2 + y^2 > 1\} \end{cases}$$

then $\lim_{(x,y)\to(1,0)} f(x, y)$ does not exist.

20 Lecture 20 - Continuation of Functions of Multiple Variables

Generally, we write $\lim_{(x,y)\to(a,b)} f(x, y) = L$ to mean the values of $f(x, y)$ approach L as the point (x, y) approaches (a, b) along any path in the domain f. That is, we can make the value of $f(x, y)$ as close to L as we like by taking (x, y) sufficiently close to (a, b). This is formalised through the following definition

Definition 1. Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b).

Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$, $\exists\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$. Where $|f(x, y) - L|$ can be described as the distance between $f(x, y)$ and L in $\mathbb{R}$. The $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between (x, y) and (a, b) in $\mathbb{R}^2$. The definition is essentially saying that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance between (x, y) and (a, b) sufficiently small, **but not 0**.