MATH1072 Summary - Ordinary Differential Equations

Dimensional Analysis

Base Dimensions

Majority of dimensions consist of the main base dimensions:

- Length (L)
- Time (T)
- Mass (M)

There are other non-mechanical base dimensions like temperature, electrical current etc.

Derived Dimensions

Some common derived dimensions include

- Speed - $\dfrac{L}{T}$
- Force - $N = M\dfrac{L}{T^2}$
- Energy - $J = M\dfrac{L^2}{T^2}$

It is important that the dimensions on both sides of the equation are **equal**. We then call the equation **dimensionally homogeneous**

ODEs

Equilibrium Solutions

An equilibrium solution is a constant solution such that $y(t) = c$ satisfies the ODE.

In other words, the solution to $f(t, y) = 0$ where $f(t, y) = y'$, representing the ODE.

Stability of Solutions

If the general solutions in a small neighbourhood converge towards an equilibrium solution $y = c$, then it is said to be the solution is *stable* at $y = c$.

Analytical Solutions

Linear First Order ODEs

The key thing is if the ODE is separable, the form of a linear (first order) ODE is

$$y'(t) = f(t)y(t) + g(t)$$

Example 1. The linear ODE $ty' + y = t\cos t$ is separable. Note that by the chain rule, $(ty)' = ty' + y$.

$$(ty)' = t\cos t$$

Integrate both sides

$$\int (ty)'\ dy = \int t\cos t\ dt$$

By integration by parts

$$ty = t\sin t - \int \sin t\ dt$$
$$ty = t\sin t + \cos t + c_1$$

Therefore our ODE solution is

$$y = \sin t + \frac{\cos t}{t} + \frac{c_1}{t}$$

Approximating Solutions

Euler's Method

Given $y' = f(t, y),\ y(0) = c$

$$y'(t) = \lim_{\Delta \to 0} \frac{y(t + \Delta) - y(t)}{\Delta}$$
$$\approx \frac{y(t + \Delta) - y(t)}{\Delta}$$

Euler's Method is iterative, so $y_{k+1} = y_k + f(t_k, y_k)$. So

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta} \approx f(t_k, y_k)$$

Heun's Method

Second Order Differential Equations

Say we have a linear second order ODE of form $f(y'', y', y, t) = 0$, assuming f is homogeneous and constant coefficient, we can denote the ODE as

$$y'' + ay' + by = 0$$

For any ODE of this form, we can represent its characteristic equation as

$$\lambda^2 + a\lambda + b = 0$$

(You should know where this comes from, but its ok if you dont). We can solve the characteristic equation with the quadratic formula

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Case 1: There are 2 real solutions for λ The general solution is then

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Case 2: There is only 1 real solution for λ The general solution is then

$$(c_1 t + c_2)e^{\lambda t}$$

Case 3: There is no real solutions (only complex) for λ As there is no real solutions, then $\lambda = \alpha \pm \beta i$, thus the general solution is

$$y(t) = e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$$

Generally analytically solvable 2nd order ODES may look like

$$y'' + p(t)y' + q(t)y = r(x)$$

If $r(x) = 0$, we say it is homogeneous. If it is homogeneous, we can use the **principle of superposition** for the general solution.

Method of Reduction of Order

The steps to solve with this method are

1. y" + p(t)y' + q(t)y = 0
2. Assume that $y_1(t)$ is a solution.
3. Look for solutions of the form $y = u(t)y_1(t)$
4. Substitute into the equation
5. $(uy_1)'' + p(t)(uy_1)' + q(t)uy_1 = 0$

MATH1072 Summary - Multivariate Calculus

Two Variable Limits

For a function $f(x,y)$, we can check if the limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

exists with the following method

$$\text{If } \begin{cases} f(x,y) \to L_1 \text{ as } (x,y) \to (a,b) \text{ along } C_1 \in D \\ f(x,y) \to L_2 \text{ as } (x,y) \to (a,b) \text{ along } C_2 \in D \end{cases}$$

Where C_1, C_2 are paths in the domain D.

We define multivariate limits similar to single variable limits,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If $\forall \varepsilon > 0, \exists \delta > 0$ such that if $(x,y) \in D$ then

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \epsilon$$

Partial Derivatives

Given a function $f(x,y)$, the partial derivatives of x and y respectively at a point $P = (a,b)$ are

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Generally, the partial derivative $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial f}{\partial y}$ can be thought of (and computed) as the derivative with respect to x or y respectively.

Example 2. Find the partial derivatives of $f(x,y) = x\sin y + y\cos x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\Big(x\sin y + y\cos x\Big)$$

Anything with a y is basically treated as a constant

$$= \sin y - y\cos x$$

The same applies to the partial derivative of y

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\Big(x\sin y + y\cos x\Big) \\ &= x\cos y + \cos x\end{aligned}$$

Higher Order Partial Derivatives

Higher order partials are usually notated as

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

We can also have higher orders of unique partials

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Chain Rule for Partials

For a function $f(x,y)$ where x, y are functions of t, the chain rule is defined as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

You can extrapolate this to as many dimensions as you want, Given a function $f(a_1(t), a_2(t), a_3(t)\ldots, a_n(t))$, you can represent its derivative as

$$\frac{df}{dt} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial a_i}\frac{da_i}{dt}\right) = \frac{\partial f}{\partial a_1}\frac{da_1}{dt} + \cdots + \frac{\partial f}{\partial a_n}\frac{da_n}{dt}$$

Where n is the highest dimension of the function f.

Gradient Vectors ∇f

The gradient vector of f is defined as

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = f_x\mathbf{i} + f_y\mathbf{j}$$

Directional Derivative

We can express the direction derivative with the gradient vector

$$f_u = \nabla f \cdot \frac{\mathbf{u}}{||\mathbf{u}||}$$

As this is the dot product, recall that

$$\underline{A} \cdot \underline{B} = ||A|| \;\; ||B|| \cos\theta$$

Tangent Planes

For $z = f(x,y)$ at $(a, b, f(a,b))$, the tangent plane is defined as

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Line Integrals

For higher dimensions, we can compute work done by a force field with a line integral. Given a function defined by

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$$

which moves along a curve C. Then we denote the line integral to be

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Where $\mathbf{r}$ is a parametrisation of the curve C.

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \quad a \le t \le b$$

To compute work is to compute the line integral given a domain $[a, b]$ on C.

$$W = \int_C F \cdot d\mathbf{r} = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

Conservative Fields

F is conservative if the line integral between A and B, $\int_A^B \mathbf{F} \cdot\, d\mathbf{r}$ will give the same result for **any** path you choose in between A and B

If a gradient field **F** is conservative, then there exists a function f such that $\nabla f = \mathbf{F}$, which you can use to compute line integrals with

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

Where f is called the **potential function**. It is expected to use this formula as apposed to the parametrisation of the path when asked to evaluate the work done on a conservative field. (Or other cases of computing a line integral over a path)

Checking if a field is conservative

An easy check to see if a force field is conservative is given $\mathbf{F} = (F_1, F_2) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y} \implies \text{Field is conservative}$$