MATH1071 - Advanced Calculus and Linear Algebra I

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Contents

Chapter 1

Set Theory

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1.1 Set Theory Notation:

A set is a 'collection' or group of things. Eg. a set of all integers $\mathbb{Z}$. A set is notated as:

$$\{a, b, c, d\}$$

In english said as "The set of elements a,b,c,d".

$$\{a \mid a \text{ satisfies } P\}$$

In english said as the set of all 'a' such that property P holds.
$a \in A$ means 'a' is an element of A. Suppose A, B are sets $A \subset B$ means A is a subset of B. $A \cup B$ implies a 'union', i.e:

$$A \cup B = \{c \mid c \in A \text{ or } c \in B\}$$

Similarly, you can imply a intersection

$$A \cap B = \{c \mid c \in A \text{ and } c \in B\}$$

$A \setminus B$ is A setminus B, where:

$$A \setminus B = \{c \mid c \in A \text{ but } c \notin B\}$$

The Cartesian Product of sets (denoted $A \times B$) is equivalent to:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Points can be 'mapped' between two sets, which is also called a function, Let A, B be two sets, then the function between A and B is denoted as:

$$f : A \to B$$

This function assigns a point in A, to a point in B. It takes a point from one set (called the Domain) and outputs a point from the other (the Range). which is a function (or mapping, map) from A to B, Assigning a point from the Set A to a point in Set B

1.1.1 Numbers

"You may think you know what numbers are, but you don't."

Natural Numbers: We often define natural numbers as the counting numbers. In set notation we can define it simply as.

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

Integer Numbers: Integers can be thought of as an extension of the naturals, including 0 and the countable negative numbers. This can be seen in our set notation.

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n \mid n \in \mathbb{N}\}$$

Rational Numbers: Rationals are numbers that can be expressed as a ratio between an integer number and a natural number.

$$\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$$

where we identify $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ if $p_1 = kp_2$ and $q_1 = kq_2$ for some $k \in \mathbb{Z}, k \neq 0$

Real Numbers: Reals cannot be defined rigorously like the $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ can be with this notation. For now, the real numbers are defined as $\mathbb{R}$, the real numbers are 'finite and infinite decimals' (not really)

Complex Numbers:

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

where i is such that $i^2 = -1$

1.2 Fields

Def: A field is a set F with two operations, $+ : F \times F \to F$ and $\cdot : F \times F \to F$ such that:

1. Associativity:
$$(a + b) + c = a + (b + c)$$
$$(ab) \times c = a \times (bc)$$
2. Communitivity:
$$a + b = b + a \; \forall a, b \in F$$
3. There exists $O \in F$ such that
$$0 + a = a + 0 = a$$
4. For each $a \in F$, there exists an $-a \in F$ such that
$$a + (-a) = (-a) + a = 0$$
5. There exists $1 \in F$ s.t $1 \cdot a = a \cdot 1 = a$
6. For every $a \neq 0$, there exists $a^{-1} \in F$ such that
$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$
7. Distrubutivity
$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Example 1. 1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields.

2. $\mathbb{F} = \{0, 1\}$, where we set $1 + 1 = 0$.

3. $\mathbb{Z}$ is NOT a field

4. $\mathbb{F} = \{0\}$ is a field where $0 = 1$

5. $\mathbb{F} = \{\Sigma, L\}$, Define $+, \cdot$ as follows, here $0 = \Sigma$, $1 = L$.

What if we define $\Sigma \cdot L = \Sigma \cdot \Sigma = L \cdot L = \Sigma$. Is it still a field? The answer is no.
Claim: there is no 1. Is Σ our 1? No, $\Sigma \cdot L = \Sigma$

Theorem 1.

$$a \cdot 0 = 0 \cdot a = 0 \; \forall \; a \in \mathbb{F}$$

Proof. By using the Distrubutivity axiom, the following can be implied:

$$a \cdot (0+0) = a \cdot 0 + a \cdot 0$$

Also, by Axiom 3 (Exsistance of Zero), $0+0=0$. Thus $a \cdot 0 = a \cdot 0 + a \cdot 0$. Now by adding $-a \cdot 0$ to both sides:

$$a \cdot 0 + -(a \cdot 0) = a \cdot 0 + a \cdot 0 - (a \cdot 0)$$

By Axiom 4, LHS = 0, For RHS, use Axiom 1 (Assoc.)

$$RHS = (a \cdot 0 + a \cdot 0) + -(a \cdot 0)$$

$$= \ a \cdot 0 + (a \cdot 0 - (a \cdot 0))$$

$$= \ a \cdot 0$$

Thus $0 = a \cdot 0$, and by Axiom 6 (Commutativity), $0 \cdot a = a \cdot 0 = 0$ □

Theorem 2. *0 is unique. If you have another element* $0 \in \mathbb{F} \mid \tilde{0} + a = a + \tilde{0}, \forall a \in \mathbb{F}$, *then* $0 = \tilde{0}$

Proof. Use Axiom 3 (There exists $0 \in \mathbb{F}$ such that $a + 0 = 0 + a = a$). Then $0 + \tilde{0} = \tilde{0} + 0 = \tilde{0}$. Now by assumption, for $a = 0$, $\tilde{0} + 0 = 0 + \tilde{0} = 0$ Thus, $0 = \tilde{0}$

□

Theorem 3. *If* $a \cdot b = 0 \ \forall \ a, b \in \mathbb{F}$. *Then* $a = 0$ *or* $b = 0$, *or both.*

Proof. If $a = 0$, then the statement holds true. If $a \neq 0$, then by Axiom 8 (Multiplicative Inverse) there exists an element a^{-1}. Multiplying both sides by a^{-1}

$$a \cdot b = 0$$

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$$

Now by Associativity, we can say that $a^{-1} \cdot (a \cdot b) = (a^{-1} \cdot a) \cdot b$. Thus a^{-1} and a cancel. For the right hand side, by the existence of a 0, the right hand side is thus 0. Therefore $b = 0$. Concluding that if $a \neq 0$, $b = 0$

□

1.3 Functions

Some common functions in this course include:

1. $|\cdot|$ = Absolute Value = $\mathbb{R} \to [0, \infty)$ for $x \in \mathbb{R}$. Where

$$|x| = \begin{cases} x, & x \geq 0 \\ x, & x < 0 \end{cases}$$

2. $[\cdot]$ = Integer Value = $\mathbb{R} \to \mathbb{Z}$, $[x]$ is the greatest integer such that $[x] \leq x$

3. $\sqrt{\cdot}$ = Square Root = $[0, \infty) \to [0, \infty)$. $\sqrt{x}$ is a unique non-negative number such that $(\sqrt{x})^2 = x$

1.4 Upper and Lower Bounds

Definition 1.4.1. Given a set $\Omega \subset \mathbb{R}$, the number $b \in \mathbb{R}$ is called an Upper Bound of Ω if $b \geq x \ \forall \ x \in \Omega$. It is a Lower Bound if $b \leq x \ \forall \ x \in \Omega$.

1.4.1 Supremum and Infimum

b is the 'least upper bound' (or Supremum) of Ω if:

1. b is an upper for Ω

2. $b \leq c$ for every upper bound c of Ω

The greatest lower bound (or Infimum) is defined analogously. The notation for Supremum and Infimum of Ω is $\sup \Omega$ and $inf\Omega$ respectively.

Note. Every subset of $\mathbb{R}$ that has an upper bound also has a supremum.

Remark.

1. The field $\mathbb{R}$ has the least upper bound property ($\mathbb{R}$ is the only field with this property (and/or subfields of R)).
2. $\mathbb{Q}$ does not have the least upper bound property??... Come back to this.
   Eg. Take $\mathbb{Q} \cap (0, \pi)$. It has an upper bound in $\mathbb{Q}$. But it does not have a Supremum.

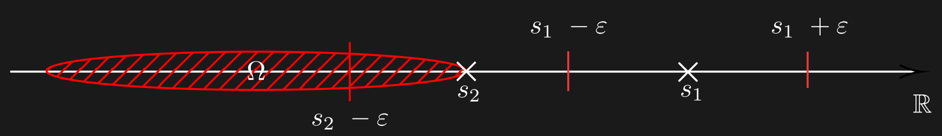
Example 2. The following are examples of the functionality of sup and inf.

1. $\sup[0, 1] = \sup(0, 1) = 1$
2. $\inf(0, 1) = 0$
3. $\sup[(0, 1) \cup 16] = 16$
4. $\sup(\mathbb{N}) =$ does not exist
5. $\inf(\mathbb{N}) = 1$

Proof. Suppose $s = \sup \Omega$, and $\tilde{s}$ also $= \sup \Omega$. Both s and $\tilde{s}$ are upper bounds of Ω. Since s is $\sup \Omega$, $s \leq \tilde{s}$. Since $\tilde{s}$ is also sup Ω, $\tilde{s} \leq s$. Thus you can infer that $s = \tilde{s}$. Shown the same for infimum. ☐

Theorem 4. *Suppose $\Omega \subset \mathbb{R}$ is not empty (written $\Omega \neq \varnothing$). Then some number $s = \sup \Omega$ if and only if (Vice-versa implication)*

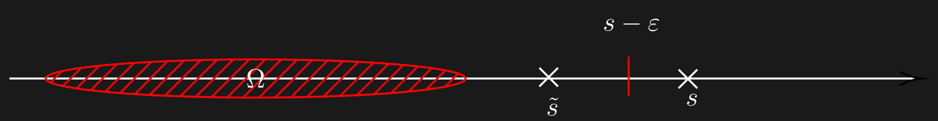
1. $\forall x \in \Omega, s \leq x$
2. *For every $\varepsilon < 0$, there exists $x \in \Omega \mid s - \varepsilon < x$*



Proof. As the if and only if statement implies a 'vice-versa implication'. The proof will be split into proving both the 'if part' and the 'only if part' individually.

Only If ($\Longleftarrow$) : Assume $s = \sup \Omega$. By default, s is an upper bound of Ω. Thus, 1. holds. Let us prove 2 using Proof by Contradiction. Assume that 2. is false. Then for some $\varepsilon < 0$. then there is no $x \in \Omega \mid s - \varepsilon < x$. $\forall x \in (s - \varepsilon, s]$, there holds $x \notin \Omega$. However, this would mean that $s - \varepsilon$ is also an Upper Bound for Ω. This is impossible since $s - \varepsilon < s$ and $s = \sup \Omega$. This is a contradiction. Hence proving statement 2.

If ($\Longrightarrow$) : Assume 1 and 2 both hold together. Prove that $s = \sup \Omega$ (1) implies s is an upper bound . It remains to show that any other Upper Bound $\tilde{s} \geq s$. By Contradiction, assume $\tilde{s} \leq s$.



Defining that $\varepsilon = \dfrac{s - \tilde{s}}{2} > 0$. We will show that there is no $x \in \Omega \mid x > s - \varepsilon$ and thus contradicts (2).

$$x > s - \frac{s - \tilde{s}}{2} = \frac{2s - s - \tilde{s}}{2}$$

$$\frac{s + \tilde{s}}{2} > \frac{\tilde{s} + \tilde{s}}{2} = \tilde{s}$$

Thus if $x > s - \varepsilon$, then $x > \tilde{s}$. This means $x \notin \Omega$ since $\tilde{s}$ is an Upper Bound for Ω. Thus 2 failed, showing that $s = \sup \Omega$. ☐

Chapter 2

Sequences

Definition 2.0.1. A sequence of elements of a set X is a map from $\mathbb{N}$ to X ($\mathbb{N} \to X$).

Example 3. For example,

1. $1, 2, 3, 4$
2. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

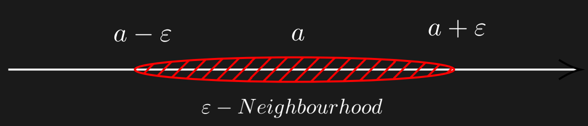
Definition 2.0.2 ($\varepsilon - N$)**.** Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. I.e,

$$(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, ... \subset \mathbb{R}$$

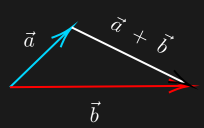
The limit of this sequence = 'a'. Written as:

$$\lim_{n \to \infty} a_n = a$$

If for every $\varepsilon > 0$, there exists $N \in \mathbb{N} \mid$ if $n \geq N$, then $|a_n - a| < \varepsilon$.



Theorem 5. *The Triangle Inequality: If you have $a, b \in \mathbb{R}$ then $|a + b| \leq |a| + |b|$. Analogous in a geometric perspective.*



$\therefore |a + b| \leq |a| + |b|$

Proof. Method 1: Consider the cases.

1. $a, b \geq 0$. Then as $a + b =$ positive $|a + b| = a + b = |a| + |b|$
2. $a, b \leq 0$. Then $|a + b| = |-(a + b)| = a + b = |a| + |b|$.
3. $a > 0, b < 0$. Then $|a + b| = |a - b|$, and as for the RHS, $|a| + |-b| = |a| + |b|$. Thus as the sum of two positives is greater than the difference... Get back to this definition... $|a + b| < |a| + |b|$.
4. $a < 0, b > 0$. Then the approach is analogous to case 3.

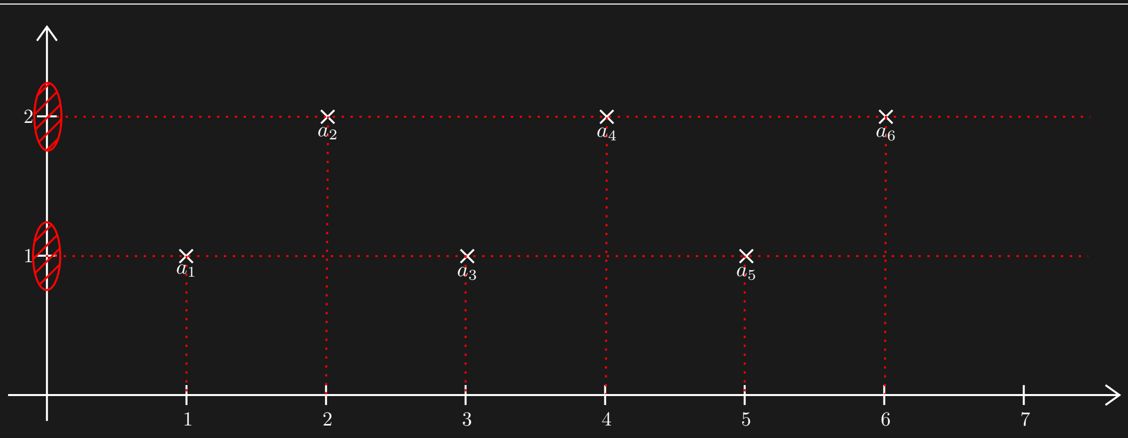
Method 2: Observe that $|a + b|^2 = (a + b)^2 = a^2 + b^2 + 2ab$. Therefore

$$a^2 + b^2 + 2ab \leq |a|^2 + |b|^2 + 2|a||b|$$

$$(a + b)^2 \leq (|a| + |b|)^2$$

Implying $|a + b| < |a| + |b|$ □

Theorem 6. *A limit is unique if* $\lim_{n\to\infty} a_n = a$ *and* $\lim\limits_{n\to\infty} a_n = b$. *Then* $a = b$.



Proof. Fix $\varepsilon > 0$. Since the $\lim\limits_{n\to\infty} a_n = a$, there exists $N_1 \in \mathbb{N}$ s.t if $n \geq N_1$ then $|a_n - a| < \frac{\varepsilon}{2}$. Since the limit $\lim\limits_{n\to\infty} a_n = b$, there exists $N_2 \in \mathbb{N}$ s.t if $n \geq N_2$, then $|a_n - b| < \frac{\varepsilon}{2}$. Now if $n \geq \max\{N_1, N_2\}$. Then

$$\begin{aligned} |a-b| &= |(a-a_n) + (b-a_n)| \\ &\leq |a-a_n| + |b-a_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This holds for any $\varepsilon > 0$ therefore $|a - b| = 0$ and thus $a = b$.

□

Theorem 7 (Squeeze Theorem)**.** *Suppose you have* $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}$ *are such that*

1. $a_n \leq b_n \leq c_n$, $\forall n \in \mathbb{N}$
2. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$.

Then $\lim\limits_{n\to\infty} b_n = L$.

Proof. Observe that

$$\begin{aligned} |b_n - L| &= |(b_n - a_n) + (a_n - L)| \\ &\leq |b_n - a_n| + |a_n - L| = (b_n - a_n) + |a_n - L| \\ &\leq (c_n - L) + |a_n - L| = |(c_n - L) + (L - a_n)| + |a_n - L| \end{aligned}$$

Fix $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$. Then there also exists some cutoff point $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|c_n - L| < \frac{\varepsilon}{3}$. Set $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$b_n - L \leq |c_n - L| + 2|a_n - L|$$

$$b_n - L \leq \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon$$

Thus $\lim\limits_{n\to\infty} b_n = L$ □

Example 4. Find

$$\lim_{n\to\infty} \frac{|\sin(n^2+1)|}{n^2}$$

Define that $a_n = 0 \forall n \in \mathbb{N}$. Define $c_n = \frac{1}{n} \forall n \in \mathbb{N}$. We know that $|\sin(x)| \leq 1 \forall x \in \mathbb{R}$.

$$0 = a_n \leq \frac{|\sin(n^2+a)|}{n^2} \leq \frac{1}{n^2} \leq \frac{1}{n} = c_n$$

Clearly, $\lim\limits_{n\to\infty} a_n = 0$. The claim is that $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$. Indeed, take $\varepsilon > 0$. Observe that $|c_n - 0| = c_n = \frac{1}{n}$. Then $\frac{1}{n} < \varepsilon$ if and only if $n > \frac{1}{\varepsilon}$ which happens if $n \geq [\frac{1}{\varepsilon}] + 100$. If we set $N = [\frac{1}{\varepsilon}] + 100$, then $n \geq N$, then $\frac{1}{n} < \varepsilon$. Thus $\lim\limits_{n\to\infty} c_n = 0$.

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = 0$$

$$\frac{|\sin(n^2+1)|}{n^2} = 0$$

Proof. Assume $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. Take $\varepsilon = 1$. There exists some $N \in \mathbb{N}$ s.t. if $n \geq N$, then $|a_n - L| < 1$. The second triangle inequality implies:

$$||x| - |y|| \leq ||x - y||$$

$|a_n| - |L| \leq 1$, which clearly implies that $|a_n| < |L| + 1$. Remember, $n \geq N$, however is not a problem. Define $M = max|L| + 1, |a_1|, |a_2|, ...|a_{n-1}|, |a_n|$. Obviously, $a_n \leq M \; \forall n \in \mathbb{N}$ □

Theorem 8. *Assume* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. *Fix* $\lambda \in \mathbb{R}$. *Then,*

1. $\lim_{n\to\infty} (a_n + b_n) = a + b$
2. $\lim_{n\to\infty} \lambda a_n = \lambda a$
3. $\lim_{n\to\infty} a_n b_n = ab$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ *if* $b_n \neq 0$ *for* $n \in \mathbb{N}$ *and* $b \neq 0$.

Corallary: $\lim_{n\to\infty} (a_n - b_n) = a - b$. This is proved by both 1 and 2, if you set $\lambda = -1$.

Proof. 1 and 2: Left as an exercise to the reader...

□

Proof. 3. $\lim_{n\to\infty} a_n b_n = ab \forall n \in \mathbb{N}$ and $b \neq 0$. Therefore $|a_n b_n - ab| < \varepsilon$.

$$\begin{aligned} &= |(a_n b_n - a_n b) + (a_n b - ab)| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

As $|a_n|$ converges, Eg. $|a_n| < M|b_n - b| + |b||a_n - a|$. Fix $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ s.t. if $n \geq N_1$. Thus, $|b_n - b| < \frac{\varepsilon}{2M}$. Also, there exists $N_2 \in \mathbb{N}$ s.t. if $n \geq N_2$ then $|a_n - a| \leq \frac{\varepsilon}{2|b|}$

$$|a_n b_n - ab| = M|b_n - b| + |b||a_n - a| < \varepsilon$$

$$M \cdot \frac{\varepsilon}{2M} + |b| \cdot \frac{\varepsilon}{2|b|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Proof. 4. Take 1 and 2 as □

2.0.1 Examples of Limits

Show that

$$\lim_{n\to\infty} \frac{n-1}{n^3+6} = 0$$

Observer that $\frac{n-1}{n^3+6} \geq 0$.

$$\frac{n-1}{n^3+6} \leq \frac{n}{n^3+6} \leq \frac{n}{n^3} = \frac{1}{n^2} \leq \frac{1}{n}$$

Thus, using the Squeeze Theorem.

$$0 \leq \frac{n-1}{n^3+6} \leq \frac{1}{n}$$

Taking limits we find:

$$\lim_{n\to\infty} \frac{n-1}{n^3+6} = 0$$

Theorem 9 (Convergent Sequences are bounded)**.** *More precisely, if* $(a_n)_{n=1}^{\infty}$ *converges, then*

$$\exists M > 0 : |a_n| \leq M \; \forall n \in \mathbb{N}$$

Proof. Let us first show that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$$

Fixing $\varepsilon > 0$ thus,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$$
$$\frac{b - b_n}{b \cdot b_n} < \varepsilon$$
$$\frac{|b_n - b|}{|b||b_n|} < \varepsilon$$

We need to show that $|b_n|$ does not get "too small". We know that $\lim_{n\to\infty} b_n = b \neq 0$. Use the def. of the limit with $\varepsilon = \frac{|b|}{2}$.

$$\exists\ N_1 \in \mathbb{N} : n > N_1, |b_n - b| < \frac{|b|}{2}$$

Now we need to show that $|b_n|$ does not get smaller. Using the Second Triangle Inequality:

$$||b_n| - |b|| \leq |b_n - b|$$

$$|b| - |b_n| \leq |b - b_n| < \frac{|b|}{2}$$

Which implies that,

$$|b| - |b_n| < \frac{|b|}{2}$$

Thus we can manipulate the inequality such that,

$$|b| \leq \frac{|b|}{2} + |b_n|$$

$$|b| - \frac{|b|}{2} \leq |b_n|$$

$$|b_n| \geq |b| - \frac{|b|}{2}$$

$$\therefore |b_n| \geq \frac{|b|}{2} > 0$$

Now we can show that,

$$|\frac{1}{b_n} - \frac{1}{b}| \leq \frac{|b - b_n|}{|b||b_n|} \leq \frac{|b - b_n|}{|b| \cdot \frac{|b|}{2}}$$
$$= \frac{2|b - b_n|}{|b|^2}$$

□

Remark. 1. If $\lambda \in (-1, 1)$ then,

$$\lim_{n\to\infty} \lambda^n = 0$$

2. Given $c > 0$,

$$\lim_{n\to\infty} c^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{c} = 1$$

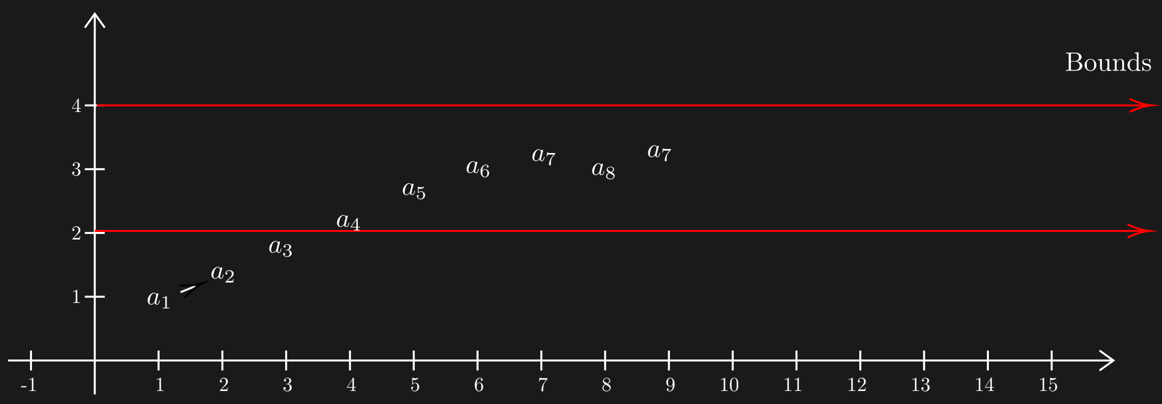
3.

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Definition 2.0.4. The sequence $(a_n)_{n=1}^{\infty}$ is 'monotone increasing' if every term 'a' is bigger than the next (if $a_{n+1} \geq a_n$). Monotone decreasing is defined analogously ($a_{n+1} < a_n$). Generally it is defined as monotone if it is either monotone decreasing or increasing.

Remark. Sometimes the terms non-increasing, non-decreasing, strictly increasing, stricly decreasing are used.

Theorem 10 (Convergence of a monotone sequence)**.** *A monotone sequence converges if and only if it is bounded.*



Proof. $\Rightarrow$ Already done in question.
$\Leftarrow$ Assume $(a_n)_{n=1}^{\infty}$ is increasing. Define

$$\alpha = \sup\{a_1, a_2, a_3 ...\}$$

Since $(a_n)_{n=1}^{\infty}$ is bounded, then the sup exist. By another theorem, given $\varepsilon > 0$, $\exists N \in \mathbb{N} : a_n \in (a - \varepsilon]$. $\square$

Definition 2.0.5. Given $(a_n)_{n=1}^{\infty}$, define

$$x_n = \sup\{a_n, a_{n+1}, a_{n+2}...\} \text{ and } y_n = \sup\{a_n, a_{n+1}, a_{n+2}...\}$$

The number $\lim\limits_{n\to\infty} x_n$ is called the upper limit of (ok boomer) Similarly, we define the no. $\lim\limits_{n\to\infty} y_n$ is the lower limit of $(a_n)_{n=1}^{\infty}$, written $\lim\limits_{n\to\infty} \inf a_n$

Note. $\lim\limits_{n\to\infty} \sup a_n$ can be ∞ and $\lim\limits_{n\to\infty} \inf a_n$ can be $-\infty$

Theorem 11. *Consider sequences* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *given by* $a_n = (a + \frac{1}{n})^n, b_n = (1 + \frac{1}{n})^n, n \in \mathbb{N}$. *Then* $(a_n)_{n=1}^{\infty}$ *is monotone increasing and* $(b_n)_{n=1}^{\infty}$ *is decreasing, and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

Chapter 3

Functions

3.1 Lemma: Bernoulli Inequality

If $x > -1$ and $n \in \mathbb{N} \cup \{0\}$,, then $(1+x)^n \geq 1 + nx$ We assume $x = -1$ and $n = 0$ do not hold simultaneously

Proof. Fix $\varepsilon > 0$. We need to find such that if $0 < |x-a| < \delta$, then $|f(x)g(x) - \ell m < \varepsilon$. Observe that

$$\begin{aligned} |f(x)g(x) - \ell m| &= |f(x)g(x) - f(x)m + f(x)m - \ell m| \\ &\leq |f(x)||g(x) - m| + |m||f(x) - \ell| \end{aligned}$$

As $f(x)$ converges to $\ell, \exists \delta_1 > 0 :$ if $0 < |x-a| < \delta_1$, then $|f(x) - \ell| < 1$ (definition of limit with $\varepsilon = 1$). In this case, $|(f(x) - \ell| < 1$ and $|f(x)| < 1 + |\ell|$. Thus, if $0 < |x-a| < \delta_1$ then $|f(x)g(x) - m\ell \leq (1+|\ell|)(|g(x) - m|) + |m||f(x) - \ell|$. $\exists \delta_2 > 0 :$ if $0 \leq |x-a| \leq \delta_2$, then

$$|g(x) - m| < \frac{\varepsilon}{2(1+|\ell|)}$$

Also $\exists \delta_3 > 0 :$ if $0 \leq |x-a| \leq \delta_3$, then

$$|f(x) - \ell| < \frac{\varepsilon}{2(|m|+1)}$$

Set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x-a| < \delta$, then

$$\begin{aligned} f(x)g(x) - \ell m| &\leq |1+|\ell||g(x) - m| + |m||f(x) - \ell| \\ &< \frac{\varepsilon}{2((1+|\ell|)} \cdot (1+|\ell|) + \frac{\varepsilon}{2(|m|+1)} \cdot |m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

□

Definition 3.1.1. $\lim_{x\to\infty} f(x) = \ell$ if $\forall \varepsilon > 0, \exists M > 0 :$ if $x > M$, then $|f(x) - \ell| < \varepsilon$. Similarly, you can define the limit $\lim_{x\to-\infty} f(x) = \ell$ (Reversed Definition)

Definition 3.1.2. $\lim_{x\to a} f(x) = \infty$ if $\forall M > 0 \exists \delta > 0 :$ if $0 < |x-a| < \delta$ then if $f(x) > M$. Analogously, define $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$ etc.

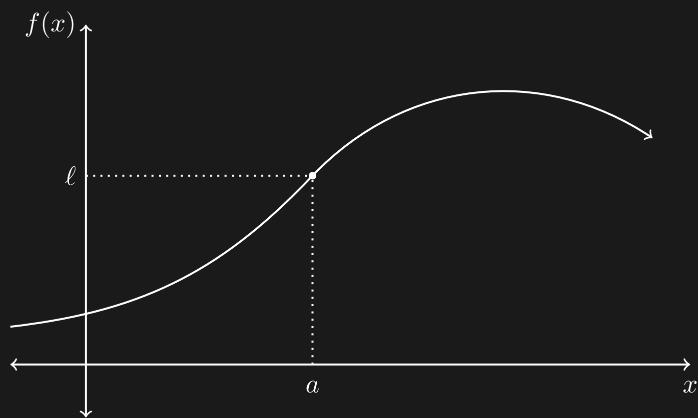
Theorem 12 (Squeeze Theorem)**.** *Given functions $f, gh : X \to \mathbb{R}$, if $f(x) \leq g(x) \leq h(x) \forall x \in X$, $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \ell$ then $\lim_{x\to a} h(x) = \ell$.*

Remark. We allow $a = \pm\infty$

Theorem 13 (Limit of Subsequences)**.** *The following are equivalent:*

1. $\lim_{x\to a} f(x) = \ell$
2. $\lim_{n\to\infty} f(x_n) = \ell$ *for every sequence* $(x_n)_{n=1}^{\infty}$ *s.t.* $\lim_{x\to\infty} x_n = a$

Proof. We need to prove from both sides of the equivalence, we begin by proving that (1) $\implies$ (2). Assume 2) holds, but 1) does not hold. Then $\lim_{x\to a} f(x)$ does not exist or it does not equal ℓ. Therefore $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, there is $x \in \{a-\delta, a+\delta\} \setminus \{a\}$

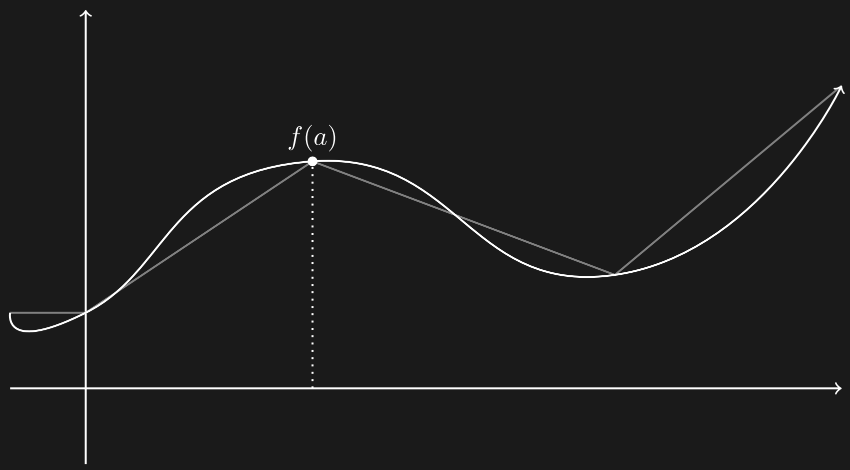


Take $\delta = 1$, this implies that $\exists x_1 \in (a-1, a+1) \setminus \{a\}$ such that $|f(x_1) - \ell| \geq \varepsilon_0$. Now take $\delta = \frac{1}{2} \implies \exists x_2 \in (a-\frac{1}{2}, a+\frac{1}{2})$ such that $|f(x_2) - \ell| \geq \varepsilon_0$. Setting $\delta = \frac{1}{3}, \frac{1}{4}$... so forth, allows us to obtain a sequence of $\mathbb{R}$, $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = a$ and $|f(x) - l| \geq \varepsilon_0$, $\forall n \in \mathbb{N}$. Which is a contraposition □

3.2 Continuity

Definition 3.2.1. If $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, for $a \in X$. We say that f is continuous at a if and only if

$$\lim_{x \to a} f(x) = f(a)$$



Alternatively, f is continuous at a if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x - a| < \delta$ then, $|f(x) - f(a)| < \varepsilon$.

Definition 3.2.2. We say $f : X \to \mathbb{R}$ is continuous on $X \subseteq \mathbb{R}$ if f is continuous $\forall x \in X$.

Example 5. Dumb spacing

1. $f(x) = x$, is continuous on $\mathbb{R}$.
2. $f(x) = a$, is continuous on $\mathbb{R}$.

Theorem 14. *Assume you have 2 functions, f and g, such that*

$$f, g : (a, b) \to \mathbb{R}$$

are continuous at $x_0 \in (a, b)$ then

1. *$(f + g)(x) = f(x) + g(x)$ is continuous at x_0.*
2. *f, g are continuous at x_0.*
3. *$\frac{f}{g}$ is continuous at x_0.*

Proof. This follows immediately from some previous theorem I can't read from my handwriting. $\square$

Collorary. Every polynomial function is contnuous on $\mathbb{R}$. moreover, every function of the form $\dfrac{P(x)}{Q(x)}$, P, Q, polynomial is continuous at every x such that $Q(x) \neq 0$. Whenever $P(x)$ and $Q(x)$ are rational functions.

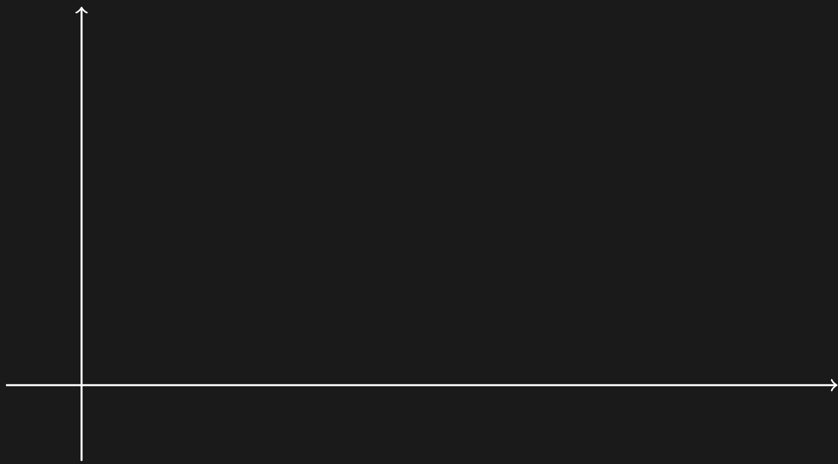
3.3 Character Building

Theorem 15 (Character Building Theorem)**.** *Suppose f is continuous on a closed bounced interval $[a, b]$. Then f is uniform continuous on $[a, b]$ on $[a, b]$ (if closed, bounded, uniform continuous $\implies$ continuous).*

Proof. By contradiction, assume that f is not uniform continuous. Then $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x, y \in [a, b]$ with

$$|x - y| < \delta \text{ but } |f(x) - f(y)| \geq \varepsilon_0$$

$\square$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