MATH2400 Notes

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Contents

1 Foundations

What are the real numbers - $\mathbb{R}$?
Which *basic properties* characterise it?
Doing mathematics is like building: foundations, or *axioms*, are the basic, essential properties that are necessary to build upon. Axioms are basic, essential properties that we assume to be true. The bricks are *depositions* and the glue is *theorms*, *proposistions*.

Example 1.1

Euclidean Geometry has five axioms. One is that, given 2 distinct points in the plane, there is a distinct line joining them.

Definition 1.1

A triangle is a collection of 3 different points joined by 3 line segments.

Definition 1.2

A right triangle is a triangle with one angle at 90°.

Theorem 1.1

Pythagorean theorm - the lengths a, b, and c of the sides of a right triangle satisfy

$$a^2 + b^2 = c^2$$

Let's go back to real numbers.

1.1 Basic Properties of $\mathbb{R}$

There is an order relation $<$ in $\mathbb{Q}$, satisfying

- (O_1) for any x, $y \in \mathbb{Q}$, exactly one of the following holds

$$x < y, \text{ or } x = y, \text{ or } x > y$$

- (O_2) (transitivity) if $x < y$ and $y < z$, $x < z$.

There are operations

$$\begin{aligned} + : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad & \text{addition} \\ \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad & \text{multiplication} \end{aligned}$$

The rest of the axioms are

$$(A_1)\ (x+y)+z = x+(y+z)$$
$$(A_2)\ x+y = y+x$$
$$(A_3)\ \exists\,\text{element}\, 0\in\mathbb{R}\,\text{s.t.}\, x+0 = x$$
$$(A_4)\ \forall x\in\mathbb{R},\ \exists(-x)\in\mathbb{R}\text{s.t.}x+(-x)=0$$
$$(M_1)\ (x\cdot y)\cdot z = x\cdot(y\cdot z)$$
$$(M_2)\ x\cdot y = y\cdot x$$
$$(M_3)\ \exists\,\text{element}\,1\in\mathbb{R}\,\text{s.t.}\,x\cdot 1 = x$$
$$(M_4)\ \forall x\in\mathbb{R},\ x\neq 0,\ \exists x^{-1}\in\mathbb{R}\text{s.t.}x\cdot x^{-1}=1$$
$$(D)\ x\cdot(y+z) = xy+xz$$
$$(O_1)\ \text{stated above}$$
$$(O_2)\ \text{stated above}$$
$$(O_3)\ \text{if}\, x>0\,\text{and}\, y>0 \implies xy>0$$
$$(O_4)\ \text{similarly} \implies x+y>0$$

Rmk: If we replace $\mathbb{R}$ with arbitrary set F, then $(F,+,\cdot)$ is called a *field* if it satisfies $(A_1)\ldots(A_4)$, $(M_1)\ldots(M_4)$, (D). It is called an *ordered field* if it also has a relation $<$ satisfying $(O_1)\ldots(O_4)$

With these axioms one can show many basic properties of $\mathbb{R}$, for example:
For all $x\in\mathbb{R}$ it holds that $0\cdot x = 0$.

Proof.

$$0 = 0+0 \qquad (A_3) \qquad (1)$$
$$0\cdot x = (0+0)\cdot x = 0\cdot x + 0\cdot x \qquad (D) \qquad (2)$$

We add $-0\cdot x$ to both sides.

$$\begin{aligned} 0 = (0\cdot x) + (-\cdot x) &= (0\cdot x + 0\cdot x) + (-0\cdot x) && (A_4 \text{ and } 2) && (3)\\ &= 0\cdot x + (0\cdot x + (-0\cdot x)) && (A_1) && (4)\\ &= 0\cdot x + 0 && (A_4) && (5)\\ &= 0\cdot x && (A_3) && (6)\\ \therefore 0 &= 0\cdot x && && (7)\end{aligned}$$

□

Q: Do the properties (axioms) characterise the real numbers? No!

$$\mathbb{N} = \{1,\,2,\,3,\,\ldots\} \qquad (8)$$
$$\mathbb{Z} = \{\ldots,\,-3,\,-2,\,-1,\,0,\,1,\,2,\,3,\,\ldots\}\ \text{Integers} \quad (A_1,\,\ldots,\,A_3) \qquad (9)$$
$$\mathbb{Q} = \left\{\frac{a}{b} : a,\, b\in\mathbb{Z},\, b\neq 0\right\}\ \text{Rational numbers}\ \left(\text{where}\ \frac{1}{b} := b^{-1}\right) \qquad (10)$$

$\mathbb{Q}$ satisfies all the properties mentioned above. What's better about $\mathbb{R}$?

Definition 1.3

Let $A\subseteq\mathbb{R}$. We say:

i A is *bounded above* if $\exists b\in\mathbb{R}$ s.t. $a\leq b$ $\forall a\in A$. We call such b and *upper bound.*

ii A is *bounded below* if $\exists b\in\mathbb{R}$ s.t. $b\leq a$ $\forall a\in A$. We call such b a *lower bound.*

iii If there is an upper bound $b_0\in\mathbb{R}$ for A s.t. for all other upper bounds we have $b_0\leq b$, then b_0 is called the *least upper bound* (or *supremum*) of A, and is denoted by

$$\sup A := b_0$$

iv Similarly, if there is a lower bound b_0 for A s.t. $b_0 \geq b$ for all other lower bounds, we call b_0 the *greatest lower bound* (or *infimum*) of A, denoted

$$\inf A := b_0$$

Example 1.2

Some supremums,

1. $A = [0, 1] \implies \sup A = 1$ $A = (0, 1) \implies \sup A = 1$
2. $A = \left\{\frac{1}{n} : n \in \mathbb{N} = \{1, 2, 3, \ldots\}\right\}$
3. Consider $A = \left\{x \in \mathbb{Q} : x^2 < 2\right\}$. Then clearly $\sup A = \sqrt{2}$. But $\sqrt{2} \notin \mathbb{Q}$, so in $\mathbb{Q}$ we don't always have least upper bounds!

Why is it true that $\sqrt{2} \notin \mathbb{Q}$? Proof by contradiction:

Proof. Suppose that $\sqrt{2} \in \mathbb{Q}$. Then $\exists a, b \in \mathbb{Z}$, $b \neq 0$, s.t.

$$\sqrt{2} = \frac{a}{b} \implies a = \sqrt{2} \cdot b \implies a^2 = 2 \cdot b^2$$

Now, using the fundamental theorem of arithmetics, we know that every $n \in \mathbb{N}$ can be expressed conversely as

$$n = p_1^{\alpha_1} \cdot \cdots \cdot p_k^{\alpha_k}$$

such that p_i are difference primes, $\alpha_i \in \mathbb{N}$.
In particular, we may assume $a, b > 0$, and write

$$a = 2^k \cdot r, \qquad b = 2^l \cdot s$$

where $r, s \in \mathbb{N}$ are odd numbers.

$$a^2 = 2 \cdot b^2 \implies \left(2^k \cdot r\right)^2 = 2 \cdot \left(2^l \cdot s\right)^2 \tag{11}$$

$$2^{2k} \cdot r^2 = 2^{2l+1} \cdot s^2 \qquad \text{r and s are odd, so the square must also be odd} \tag{12}$$

$$2k = 2l + 1 \tag{13}$$

by converse. A contradiction! □

The last basic property/axiom characterising $\mathbb{R}$ is

Definition 1.4

Least upper bound (LUB). For any set $A \subseteq \mathbb{R}$ bounded above, we have that $\sup A \in \mathbb{R}$

Theorem 1.2

The properties $(O_1) \ldots (O_4)$, $(A_1) \ldots (A_4)$, $(M_1) \ldots (M_4)$, (LUB), (D) completely characterise the real numbers $\mathbb{R}$. In other words, there exists a unique order field with the least upper bound property (LUB).

> **Rmk**: In practice, unless it is explicity stated that you can only use axioms, you are of course free to use all the well known properties of the real numbers, e.g. $1 > 0$, $(-x)(-y) = xy$, etc.

1.2 The Archimedes Principle

The following is a key fact about $\mathbb{R}$, which we prove using our basic properties.

Theorem 1.3: Archimedes property

In two parts,

i let $x, y \in \mathbb{R}$, $x > 0$. Then $\exists n \in \mathbb{N}$ s.t. $nx > y$

ii $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, for any $x, y \in \mathbb{R}$ and $x < y$, $\exists r \in \mathbb{Q}$ s.t. $x < r < y$

Proof. (i) diving by x, the statement becomes equivalent to showing that

$$\exists n \in \mathbb{N} \text{ s.t. } n > \frac{y}{x} =: z \in \mathbb{R}$$

In other words, we want to show that $\forall z \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $n > z$.
i.e. z is not an upper bound for $\mathbb{N}$. $\therefore$ let us show $\mathbb{N}$ is not bounded above.
Proof by contradiction.
Suppose $\mathbb{N}$ is bounded above by LUB $\exists S \in \mathbb{R}$ s.t. $S = \sup \mathbb{N}$
In particular, $S \geq \mathbb{N} \forall n \in \mathbb{N}$.
If $n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$

$$\begin{aligned} &\implies S \geq n + 1 \\ &\implies S - 1 \geq n \\ &\implies S - 1 \text{ is an upper bound, but this contradicts the assumption } S = \sup \mathbb{N} \end{aligned}$$

$\therefore \mathbb{N}$ is not bounded above. □

Proof. (ii) first assume $x > 0$. Since $y > x \implies y - x > 0$
Apply (i) to $y - x$ and 1:

$$\exists n \in \mathbb{N} \text{ s.t. } n\,(y - x) > 1 \qquad (1)$$

Now consider the set

$$A := \{k \in \mathbb{N} : k > n \cdot x\} \geq \mathbb{N}$$

By (i), $A \neq \emptyset$. Let $m \in \mathbb{A}$ be the least element in A.
Since $m \in A$, we have

$$m > n \cdot x \implies \frac{m}{n} > x \qquad (2)$$

Since m is the least element in A,

$$m - 1 \leq n \cdot x \qquad (3)$$

$$m \leq n \cdot x + 1$$

Use (1) and (3) to get

$$\begin{aligned} ny > 1 + nx &\geq m \\ \implies ny &> m \\ \implies y &> \frac{m}{n} \end{aligned}$$

Calling $r := \frac{m}{n} \in \mathbb{Q}$, we get

$$x < r < y$$

as desired.
If $x < 0$, $y > 0$, take $r = 0 \in \mathbb{Q}$.
If $x < 0$, $y \geq 0$, notice $x < y \implies 0 \leq (-y) < (-x)$
By first case, $\exists r \in \mathbb{Q}$ s.t.

$$-y < r < -x$$

Take $-r \in \mathbb{Q}$ then

$$x < -r < y$$

□

Corollary:

$$\inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = 0$$

Proof. 1. 0 is a lower bound

2. 0 is the greatest lower bound

1 is clear, since $\frac{1}{n} > 0 \forall n \in \mathbb{N}$

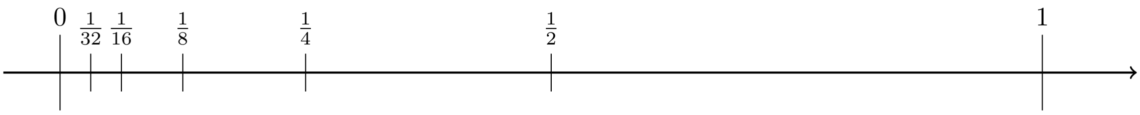


Figure 1.1: Elements in $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

2: Suppose $\exists a > 0$ which is also a lower bound for the set. By Archimedes property, $\exists n \in \mathbb{N}$ s.t. $n \cdot a > 1$

$$\implies a > \frac{1}{n}$$

a contradiction since a was a lower bound (i.e. $a \leq \frac{1}{n} \forall n \in \mathbb{N}$)
$\therefore$ 0 is the greatest lower bound. $\square$

1.3 Absolute Value

Definition 1.5: Absolute Value

The function $|\cdot| : \mathbb{R} \to [0, \infty)$. In otherwords, $x \to |x|$ given by

$$|x| := \begin{cases} x, & \text{if}\, x \geq 0 \\ -x, & \text{if}\, x < 0 \end{cases}$$

is called the absolute value.

Idea: $|x|$ is the "size" of the real number, or the distance from x to 0 on the real line.

Proposition 1.1

Some results of the absolute value,

i $|x| \geq 0$, $|x| = 0$ iff $x = 0$

ii $|-x| = x, \qquad \forall x \in \mathbb{R}$

iii $|x \cdot y| = |x| \cdot |y|, \qquad \forall x, y \in \mathbb{R}$

iv $|x|^2 = x^2, \qquad \forall x \in \mathbb{R}$

v $|x| \leq y$ iff $-y \leq x \leq y$

vi $-|x| \leq x \leq |x|, \qquad \forall x \in \mathbb{R}$

Proposition 1.2

(triangle inequality) $\forall x, y \in \mathbb{R}$, we have that

$$|x + y| \leq |x| + |y|$$

Proof. from (v), we have that

$$\begin{aligned} &-|x| \leq x \leq |x| \\ &-|y| \leq y \leq |y| \\ &\implies -(|x| + |y|) \leq x + y \leq (|x| + |y|) \end{aligned}$$

Recall (v) as

$$|a| \leq b \text{ iff } -b \leq a \leq b$$

By (v), the above inequality implies

$$|x+y| \leq |x| + |y|$$

$\square$

2 Sequences

What's the precise definition of limits for a sequence of real numbers?

Definition 2.1

a sequence of real numbers is a function $x : \mathbb{R} \to \mathbb{R}$. Each term is usually denoted x_n. The sequence is denoted by, for example,

$$\{x_n\}_{n=1}^{\infty}, \{x_n\}_{n\in\mathbb{N}}, (x_n)_{n\geq 1}$$

Rmk: Do not confuse sequences with sets. Even though we may think of the set of elements in a sequence,

$$A = \{x_n : n \in \mathbb{N}\}$$

they are not the same.

Example 2.1

Consider the expression

$$x_n = (-1)^n, n \in \mathbb{N}$$

As a sequence, $1, -1, 1, -1, \ldots$
A a set, $A = \{x_n : n \in \mathbb{N}\} = \{-1, 1\} \subseteq \mathbb{R}$
We lose ordering!

Definition 2.2

We say a sequence is

- bounded above, if $\exists B \in \mathbb{R}$ s.t. $B \geq x_n$, $\forall x \in \mathbb{N}$
- bounded below, if $\exists B \in \mathbb{R}$ s.t. $B \leq x_n$, $\forall x \in \mathbb{N}$
- bounded, if above and below

Example 2.2

The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is bounded below by 0, above by 1.
$\{n\}_{n\in\mathbb{N}}$ is bounded below by 1. By Archimedes property, not bounded above.

The following is one of the most important concepts in this course.

Definition 2.3

A sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to $L \in \mathbb{R}$ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|x_n - L| < \epsilon, \forall x \geq N$$

We call such an L the limit, and we denote

$$L := \lim_{n\to\infty} x_n, \text{ or } x \to_{n\to\infty} L$$

We say the sequence is convergent s.t. $\exists L \in \mathbb{R}$ s.t. the above definition holds. Otherwise, we say it is divergent.

Rmk: In general, N depends on ϵ.

Example 2.3

Convergence of a few sequences.

1. constant sequence $x_n \equiv C \in \mathbb{R}$. $\{C\}_{n=1}^{\infty}$ converges to C as $n \to \infty$. Indeed, given $\epsilon > 0$, take $N = 1$.

$$|x_n - C| = |c - c| = 0 < \epsilon \, \forall n \geq 1$$

2. the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

3. the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

Proof. of 2: let $\epsilon > 0$. By Archimedes property, $\exists N$ s.t.

$$\begin{aligned} N \cdot \epsilon &> 1 \\ \epsilon &> \frac{1}{N} \\ \epsilon &> \frac{1}{N} \geq \frac{1}{n} \, \forall n \geq N \\ \epsilon &> |\frac{1}{N} - 0| \, \forall n \geq N \end{aligned}$$

By def of limits, this implies $\lim_{n\to\infty} \frac{1}{n} = 0$ ☐

Proof. of 3: Suppose the contrary, that is converges to some $L \in \mathbb{R}$. By definition, this means for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|(-1)^n - L| < \epsilon, \, \forall n \geq N$$

Take $\epsilon = \frac{1}{2}$. Then, a contradiction. Therefore, diverges. ☐

Proposition 2.1

A convergent sequence has a unique limit.

Proof. Suppose that

$$\lim_{n\to\infty} x_n = L_1 \in \mathbb{R} \tag{14}$$

$$\lim_{n\to\infty} x_n = L_2 \in \mathbb{R} \tag{15}$$

By definition of a limit, given $\epsilon > 0$, $\exists N_1, \, N_2 \in \mathbb{N}$ s.t.

$$\begin{aligned} |x_n - L_1| &< \frac{\epsilon}{2} \qquad \forall n \geq N \\ |x_n - L_2| &< \frac{\epsilon}{2} \qquad \forall n \geq N \end{aligned}$$

For $n \geq \max(N_1, \, N_2) \in \mathbb{N}$, both statements above hold.

$$|L_2 - L_1| = |L_2 - x_n + x_n - L_1| \qquad \leq |L_2 - x_n| + |x_n - L_1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since $|L_2 - L_1| < \epsilon$ for all $\epsilon > 0$, we must have $|L_1 - L_2| = 0 \implies L_1 = L_2$ ☐

Proposition 2.2

A convergent sequence is bounded.

Proof. Let $L := \lim_{n\to\infty} x_n$. Choose $\epsilon = 1 > 0$. Then, $\exists N \in \mathbb{N}$ s.t.

$$|x_n - L| < 1,\ \forall n \geq N$$

By the reverse triangle inequality,

$$|a| - |b| \leq |a - b|$$

we have

$$|x_n| - |L| \leq |x_n - L| < 1,\ \forall n \geq N$$
$$|x_n| \leq 1 + |L| = c_1$$

Since the first terms (up to N) are only finitely many, they are bounded by

$$c_2 = \max\{|x_1|,\ \dots,\ |x_{n-1}|\}$$

We claim $|x_n| \leq \max\{c_1,\ c_2\}$
Indeed, if $n < N$ then $|x_n| \leq c_2 \leq \max\{c_1,\ c_2\}$
And if $n \geq N$ then $|x_n| \leq c_1 \leq \max\{c_1,\ c_2\}$ □

2.1 Monotone sequences

Definition 2.4

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called

i Monotone increasing if $x_n \leq x_{n+1}$

ii Strictly monotone increasing if $x_n < x_{n+1}$

iii monotone decreasing if $x_n \geq x_{n+1}$

iv Strictly monotone decreasing if $x_n > x_{n+1}$

In any of the above cases, $\{x_n\}$ is called *monotone*.

Example 2.4

The sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ is strictly monotone decreasing.

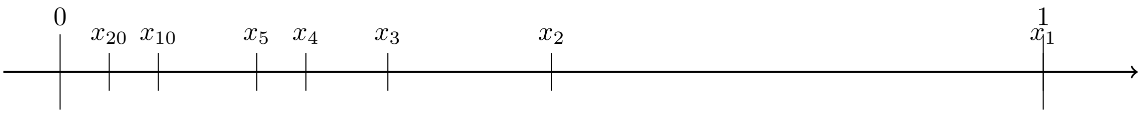


Figure 2.1: Elements in $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Example 2.5

The sequence $\left\{\frac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$ is not monotone.

Theorem 2.1: Convergence of Monotone Bounded Sequences

For some sequence,

i if it is monotone increasing and bounded above, then it converges

ii if it is monotone decreasing sequence and bounded below, then it converges

Guess: $L = \sup\{x_n : n \in \mathbb{N}\}$. Want to show for $\epsilon > 0$,

$$|x_n - L| < \epsilon \quad \forall n \geq N$$

and

$$x_n - L < 0$$

Proof. i Let $L = \sup\{x_n : n \in \mathbb{N}\} \in \mathbb{R}$. Let $\epsilon > 0$. We claim that $\exists N \in \mathbb{N}$ s.t. $L - \epsilon < x_N$.

Suppose on the contrary that no such N exists. Then $x_n \leq L - \epsilon \quad \forall n \in \mathbb{N}$

This implies $L - \epsilon$ is an upper bound for the set

$$\{x_n : n \in \mathbb{N}\}$$

But $L - \epsilon < L$ and L was the least upper bound - a contradiction. $\therefore \exists N \in \mathbb{N}$ s.t. $x_N > L - \epsilon$

ii Since $\{x_n\}$ is monotone increasing, the above implies

$$x_n \geq x_N > L - \epsilon \quad \forall n \geq N$$

On the other hand, we know

$$\begin{aligned} x_n &\leq L < L + \epsilon \\ &\implies L - \epsilon < x_n < L + \epsilon \\ &\implies -\epsilon < x_n - L < \epsilon \\ &\implies |x_n - L| < \epsilon \end{aligned}$$

as we wanted to show.

For ii, apply i to the sequence $\{-x_n\}_{n\in\mathbb{N}}$

$\square$

Example 2.6

Define x_n recursively by

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

Does $\{x_n\}$ converge?

Definition 2.5

Let $\{x_n\}$ be a sequence, and let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers:

$$n_1 < n_2 < \dots$$

The sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a *subsequence* of $\{x_n\}$.

Clearly, we have that $n_k \geq k \, \forall k \in \mathbb{N}$

Example 2.7

For the sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$, the sequence $\left\{\frac{1}{k^2}\right\}_{k\in\mathbb{N}}$ is a subsequence: simply set $n_k = k^2$.

Proposition 2.3

If $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence, then so is any subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, and $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$.

Proof. Set $L := \lim_{n\to\infty} x_n$. Given $\epsilon > 0$, by def. of limit we have $\exists N$ s.t. $|x_n - L| < \epsilon \forall n \geq N$. Since $n_k \geq k \forall k \in \mathbb{N}$, if $k \geq N$ then $n_k \geq N$.

$$\implies |x_{n_k} - L| < \epsilon \forall k \geq N$$

By def. of limit,

$$x_{n_k} \to_{k\to\infty} L$$

□

Limits behave well with respect to algebraic operations.

Proposition 2.4

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences.

i The sequence $\{x_n + y_n\}$ is convergent, and $\lim x_n \pm y_n = \lim x_n \pm \lim y_n$

ii The sequence $\{x_n \cdot y_n\}$ is convergent and $\lim x_n \cdot y_n = \lim x_n \cdot \lim y_n$

iii If $\lim y_n \neq 0$ and $y_n \neq 0 \forall n \in \mathbb{N}$, then $\left\{\frac{x_n}{y_n}\right\}$ is convergent, and $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$

iv If $x_n \geq 0$ then $\{x_n\}$ is convergent and $\lim \sqrt{x_n} = \sqrt{\lim x_n}$

Definition 2.6

Let $\{x_n\}$ be a bounded sequence. Set

$$a_n := \sup\{x_k : k \geq n\}, \quad b_n := \inf\{x_n : k \geq n\}$$

and define

$$\limsup_{n\to\infty} x_n := \lim_n \to \infty a_n \qquad \text{"limit superior"}$$
$$\liminf_{n\to\infty} x_n := \lim_n \to \infty b_n \qquad \text{"limit inferior"}$$

Even though a bounded sequence might not converge, we always have

Proposition 2.5

For a bounded sequence $\{x_n\}$, $\limsup x_n$ and $\liminf x_n$ always exist.

Proof. Observe that $\{a_n\}$ is bounded below (since $\{x_n\}$ is), and monotone decreasing.

$$a_{n+1} = \sup\{x_k : k \geq n+1\} \leq \sup\{x_k : k \geq n\}$$

(taking sup of a subseq is always $\leq$ that sup of the whole set.) Hence, $\{a_k\}$ converges. □

Example 2.8

Given the function $x_n = \begin{cases} \frac{n+1}{n} & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$

$$x_1 = 2,\ x_n = -1,\ \frac{4}{3},\ -1,\ \frac{6}{5},\ \ldots$$

Thus, bounded!

$$\limsup_{n\to\infty} x_n = 1, \qquad \liminf_{n\to\infty} x_n = -1$$

fix this

Theorem 2.2

A bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent if and only iff $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$

Theorem 2.3

(*Bolzano - Weistrass*): any bounded sequence of real numbers has a convergent subsequence.

Proof. Given $\{x_n\}$, we call one of the terms x_m a peak, if $x_m \geq x_n$ for all n. Either,

i there are infinitely many peaks, or

ii there are finitely many peaks

Case (i): let's call the peaks x_{n_1} etc, with $n_1 < n_2 < \dots$. Then, $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_n\}$ which is monotone decreasing (why?). It is also bounded, since $\{x_n\}$ is bounded. Therefore, it converges by previous theorem.
Case (ii): There is some $N \in \mathbb{N}$ s.t. no x_n is a peak for all $n \geq N$.
Take $n_1 := N$. Since x_{n_1} is not a peak, $\exists n_1 > n_2$ with $x_{n_1} < x_{n_2}$. Since x_{n_2} is not a peak, $\exists n_3 > n_2$ with $x_{n_2} < x_{n_3}$. In this way, we construct a monotone increasing subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and since it is bounded, it converges by the theorem quoted above. □

2.2 Cauchy Sequences

Definition 2.7

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called *Cauchy sequence*, if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \epsilon \qquad \forall m, n \geq N$$

Example 2.9

The sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is Cauchy.

Proof. Let $\epsilon > 0$. Choose N so that $N > \frac{2}{\epsilon}$ (Archimedean property). Then,
Thus, if $n, m \geq N$,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq |\frac{1}{n}| + |\frac{-1}{m}| = \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Theorem 2.4

A sequence is Cauchy is and only if it is convergent.

Proof. Suppose $\{x_n\}_{n\in\mathbb{N}}$ converges to $L \in \mathbb{R}$. Want to prove: $\{x_n\}$ is cauchy.
By definition of convergence, given $\epsilon > 0\, \exists N \in \mathbb{N}$ s.t.

$$|x_n - L| < \frac{\epsilon}{2}, \qquad \forall n \geq N$$

Hence, for any $n, m \geq N$ we have

$$\begin{aligned}|x_n - x_m| &= |(x_n - L) + (L - x_m)| \\ &\leq |x_n - L| + |x_m - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

By definition, this implies $\{x_n\}$ is cauchy. □

Idea: Start with a cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$.

- Cauchy $\implies$ bounded
  Let's call $L_S := \limsup x_n$, $L_I := \liminf x_n$

Fact: There exist the subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_{m_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} x_{n_k} = L_S, \; \lim_{k\to\infty} x_{m_k} = L_I$$

Goal: $L_S = L_I$
How: use what we know:

Proof. By definition, $\forall \epsilon > 0 \, \exists N_1, N_2 \in \mathbb{N}$ s.t.

$$\begin{aligned} |x_{n_k} - L_S| &< \frac{\epsilon}{3} \qquad \forall k \geq N_1 \\ |x_{m_k} - L_I| &< \frac{\epsilon}{3} \qquad \forall k \geq N_2 \end{aligned}$$

We also know $\{x_n\}_{n\in\mathbb{N}}$ is cauchy, i.e.

$$\forall \epsilon > 0, \quad \exists N_3 \in \mathbb{N}$$

s.t.

$$|x_n - x_m| < \frac{\epsilon}{3}, \qquad \forall n, m \geq N_3$$

Finally,

$$\begin{aligned} |L_S - L_I| &= |(L_S - x_{n_k}) + (x_{n_k} - x_{m_k}) + (x_{m_k} - L_1)| \\ &\leq |L_S - x_{n_k}| + |x_{n_k} - x_{m_k}| + |x_{m_k} - L_I| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}, \qquad \forall k \text{ "large enough"} \end{aligned}$$

$$\implies |L_S - L_I| = 0 \implies L_S = L_I$$

By a previous theorem, this implies that $\{x_n\}$ converges. □

Rmk: (not assessed) The theorem above relies on the LUB axiom.
In fact, many textbooks use that theorem as an axiom! and deduce LUB from it.

2.3 Summary

$\{x_n\}_{n\in\mathbb{N}}$ sequence of real numbers.

- Convergence: $\lim_{n\to\infty} x_n = L$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.s $|x_n - L < \epsilon|$, $\forall n \geq N$
- For a given sequence, such L is unique (if it exists)
- Convergent $\implies$ bounded ($|x_n| \leq B \, \forall n \in \mathbb{N}$)
- Monotone bounded sequences are convergent
- Basic properties of limits
- Subsequences, lim sup, lim inf
- Bolzano-Weistrass: Bounded $\implies$ $\exists$ convergent subsequences
- Cauchy Sequences
- Cauchy $\Leftrightarrow$ convergent

3 Series

Definition 3.1

Given a sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$, a *series* is the formal sum of all its terms

$$\sum_{n=1}^{\infty} x_n$$

We say a series is *convergent* if the sequence of partial sums $\{s_k\}$ defined by

$$s_k := \sum_{n=1}^{k} x_n = x_1 + \cdots + x_k$$

is convergent. In that case, we call $L = \sim_{k\to\infty} s_k$ the *value* of the series $\sum_{n=1}^{\infty} x_n$. If it is not convergent, we say the series diverges.

Obs:

1. Given $\sum_{n=1}^{\infty} x_n$, the partial sums $\{s_k\}_{k\in\mathbb{N}}$ form a sequence of real numbers; we say $\sum_{n=1}^{\infty} x_n$ converges if $\{s_k\}$ converges in the sense of sequences.
2. Sometimes we start summing from $n = 0$ for $n = 2$, or any fixed natural number. This is fine.

Example 3.1

Given the series $\sum_{n=1}^{\infty} = \frac{1}{2^n}$, does it converge? If so, to which value? We need to study the sequence of partial sums,

$$s_k := \sum_{n=1}^{k} \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$$

Which finds
We claim $s_k = 1 - \frac{1}{2^k}$

Proof. Induction of k: $k = 1$, $s_1 = \frac{1}{2} = 1 - \frac{1}{2}$ so holds.
Suppose $s_k = 1 - \frac{1}{2^k}$
The claim is proved. □

Now the sequence $\{1 - \frac{1}{2^k}\}_{k\in\mathbb{N}}$ converges to 1 as $k \to \infty$. Therefore, by def. of convergence of a series, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent and its value is 1.

fix

Theorem 3.1

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence with $a_n \geq 0 \quad \forall n \in \mathbb{N}$. Then, $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $\left\{s_k = \sum_{n=1}^{k} a_n\right\}_{k\in\mathbb{N}}$ is bounded above.

Proof. By definition, $\sum_{n=1}^{\infty} a_n$ converges as a series iff $\{s_k\}_{k\in\mathbb{N}}$ converges as a sequence.
$\implies$ if $\{s_k\}$ converges, then it is bounded (by an earlier proposition).
$\Leftarrow$ notice that $\{s_k\}$ is monotone increasing:

$$s_{k+1} = \underbrace{a_1 + \cdots + a_k}_{} s_k + a_{k+1} = s_k + \underbrace{a_{k+1}}_{} \geq 0 \geq s_k$$

$$\implies s_{k+1} \geq s_k \quad \forall k \in \mathbb{N}$$

Thus, if $\{s_K\}_{k\in\mathbb{N}}$ is bounded above, then by the theorem on monotone bounded sequences, it converges. □

Corollary: Suppose $\{a_n\}$, $\{b_n\}$ satisfy $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
(Contrapositive: if $\sum a_n$ diverges, then $\sum b_n$ diverges as well)

Proof. Let $s_k := \sum_{n=1}^k a_n$, $t_k := \sum_{n=1}^k b_n$. The assumption implies that

$$s_k \leq t_k \qquad \forall k \in \mathbb{N}$$

If $\sum b_n$ converges, then by the previous theorem we know $\{t_k\}$ is bounded above $\implies \exists B \in \mathbb{R}$ s.t.

$$t_k \leq B, \qquad \forall k \in \mathbb{N}$$

From the above 2 eqs., we deduce $\{s_n\}$ is also bounded above. Hence, again by previous theorem, $\sum a_n$ converges. $\square$

Example 3.2

The expression $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

Proof. Notice that $\frac{1}{n} \geq 0 \, \forall n \in \mathbb{N}$. Also,

$$s_1 = 1, \qquad s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \frac{3}{4} + \frac{1}{4} \geq 2$$

And so forth. In general, $S_{2^m} \geq 1 + \frac{m}{2}$, which is unbounded $\implies$ $\{s_k\}$ is unbounded $\implies$ $\sum \frac{1}{n}$ diverges. $\square$

Theorem 3.2

If $\sum_{n=1}^\infty a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Rmk: This is very useful as a criterion for deducing convergence: if $a_n \nrightarrow 0$, then $\sum a_n$ diverges!

Example 3.3

The expression $\sum_{n=1}^\infty \log(n)$ diverges, since $\log(n) \to +\infty$.

Proof. (of theorem) If the series converges then by definition

$$\lim_{k\to\infty} s_k = L \in \mathbb{R}$$

where $s_k = \sum_{k=1}^\infty a_n$. Given $\epsilon > 0$, by above $\exists N \in \mathbb{N}$ s.t.

$$|s_k - L| < \frac{\epsilon}{2}, \qquad \forall k \geq N$$

$$\begin{aligned} \implies |a_k| &= |s_k - s_{k-1}| = |(s_k - L) + (L - s_{k-1})| \\ &\leq |s_k - L| + |s_{k-1} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall k \geq N+1 \end{aligned}$$

We have shown that given $\epsilon > 0$, $\exists N$ s.t.

$$|a_k| < \epsilon, \quad \forall k \geq N$$

By def. of convergence (for sequences), this says $\lim a_n = 0$. $\square$

We now collect some basic properties of convergent series and algebraic operations between them:

Proposition 3.1

Some basic properties of convergent series,

(i) if $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ are two series which converge to A, $B \in \mathbb{R}$, respectively, then the series $\sum_{n=1}^\infty (a_n + b_n)$ is convergent, and converges to $A + B$. That is,

$$\sum_{n=1}^\infty (a_n + b_n) = \sum_{n=1}^\infty a_n + \sum_{n=1}^\infty b_n$$

(ii) if $\sum_{n=1}^{\infty} a_n$ converges to $A \in \mathbb{R}$ and $c \in \mathbb{R}$ is arbitrary, then $\sum (c \cdot a_n)$ converges to $c \cdot A$.

(iii) Let $\sum_{n=1}^{\infty} a_n$ be a series, and $m \in \mathbb{N}$. If one of the two series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=m}^{\infty} a_n$ is convergent, then so is the other one, and we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n$$

Proof. For (i) and (ii), consider the partial sums and apply the corresponding proposition for sequences. For (iii), suppose $\sum_{n=m}^{\infty}$ is convergent (the other case is similar). Let us call

$$s_k := \sum_{n=1}^{k} a_n,\ T_k := \sum_{n=m}^{k} a_n, \quad k \geq m$$

the partial sums for the series $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=m}^{\infty} a_n$, respectively. By assumption, $\{T_k\}_{k \in \mathbb{N}}$ converges to some limit $L \in \mathbb{R}$. But notice that for each $k \geq m$,

$$T_k = s_k - \left(\sum_{n=1}^{m-1} a_n \right) = s_k - c$$

where $c := \sum_{n=1}^{m-1} a_n \in \mathbb{R}$ is a fixed real number (independent of k or n). Then, by earlier proposition, the sequence $\{s_k\}_{k \in \mathbb{N}}$ also converges, and

$$\lim_{k \to \infty} s_k = c + \lim_{k \to \infty} T_k$$

$\implies$ by definition of convergence of series, $\sum_{n=1}^{\infty} a_n$ converges, and $\sum_{n=1}^{\infty} a_n = c + \sum_{n=m}^{\infty} a_n$ $\square$

Definition 3.2

Geometric Series

$$\sum_{n=0}^{\infty} a^n$$

where $a \in \mathbb{R}$ is fixed.

Lemma 3.1

If $|a| \geq 1$ then the series $\sum_{n=0}^{\infty} a^n$ diverges. If $|a| < 1$, then $\sum_{n=0}^{\infty} a^n$ converges to $\frac{1}{1-a}$

Pf/ Let $s_k := \sum_{n=0}^{k} a^n$. Then,

$$(1-a) \cdot s_k = s_k - a s_k$$

$$= 1 + \cancel{a} + \dots + \cancel{a^k} - \cancel{a} - \cancel{a^2} - \dots \cancel{} - a^{k+1}$$

$$= 1 - a^{k+1}$$

Assume $a \neq 1$. Then, the previous computation shows

$$s_k = \frac{1-a^{k+1}}{1-a} \qquad (1)$$

If $|a| < 1 \Rightarrow |a|^k \xrightarrow[k \to \infty]{} 0$.

(Proof of Claim: For $a = 0$, trivial. If $|a| > 0$ then

$$|a|^k = e^{\log |a|^k} = e^{k \cdot \log |a|},$$

but $k \cdot \log |a| < 0$ since $|a| < 1 \Rightarrow$

$$k \log |a| \xrightarrow[k \to \infty]{} -\infty \Rightarrow e^{k \log |a|} \longrightarrow 0.)$$

(48

Putting this in (1) yields

$$s_k \xrightarrow[k \to \infty]{} \frac{1}{1-a}, \quad \text{as long as } |a| < 1.$$

If $|a| \geq 1$ then $a^n \underset{n \to \infty}{\not\to} 0 \Rightarrow$ by the test seen before, $\sum_{n=0}^{\infty} a^n$ cannot converge. $\square$

Proof. $\square$

Proposition 3.2

The series $\sum_{n=1} \frac{1}{n^p}$ converges iff $p > 1$.

Proof. Exercise. Idea: as in the $p = 1$ case, compare (above or below, depending on whether $p > 1$ or $p < 1$) the partial sums S_{2^k} with easier-to-compute sequence. $\square$

Rmk: The function $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$ is the famous Riemann zeta function. One of the most important open problems in mathematics, the Riemann hypothesis, is concerned with the zeroes of $\zeta(p)$, when $p \in \mathbb{C}$. It It encodes fundamental information on the distribution of prime numbers.

Example 3.4

Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge?
Notice that $\frac{1}{n(n+1)} \leq \frac{1}{n^2}$, $\forall n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), thus by comparison $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ also converges.

3.1 Absolute Convergence

Definition 3.3

We say a series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Rmk: In order to distinguish this from the usual notion of convergence, we say a series is *conditionally convergent* if it is convergent but not absolutely convergent.

Proposition 3.3

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent. Moreover, we have the "triangle inequality":

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof. Consider the partial sums:

$$s_k := \sum_{n=1}^{k} a_n, \quad T_k := \sum_{n=1}^{k} |a_n|$$

By assumption, $\{T_k\}$ is convergent. In particular, it is Cauchy: given $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|T_k - T_j| < \epsilon, \qquad \forall k,\, j \geq N$$

Let us show that $\{s_k\}$ is Cauchy. By the triangle inequality, (say $k > j$)

$$\begin{aligned} |s_k - s_j| &= |(a_1 + \cdots + a_k) - (a_1 + \cdots + a_j)| \\ &= |a_{j+1} + \cdots + a_k| \\ &\leq |a_{j+1}| + \cdots + |a_k| \\ &= |T_k - T_j| < \epsilon \qquad \forall j,\, k \geq N \end{aligned}$$

Therefore, $\{s_k\}$ is Cauchy $\implies \sum_{n=1}^{\infty} a_n$ converges.
Regarding the last claim, for each k we have

$$|s_k| = \left| \sum_{n=1}^{k} a_n \right| \leq \sum_{n=1}^{k} |a_n| = T_k$$

Similarly, taking limits, which in turn implies

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

$\square$

Example 3.5

Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges:
It converges absolutely, because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by above proposition ($p = 2 > 1$). Then, it converges by previous proposition.

3.2 Rearrangements

Definition 3.4

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Given a series $\sum_{n=1}^{\infty} a_n$, we set $b_n := a_{\sigma(n)}$, and we call $\sum_{n=1}^{\infty} b_n$ a *rearrangement* of $\sum_{n=1}^{\infty} a_n$. That is, we sum the "same numbers in a different order". Unlike with finite sums, with series one has to be careful.

Example 3.6

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ does not converge absolutely. However, it converges conditionally.
But! One can rearrange the terms so that it diverges, e.g. instead of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$$

we sum

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{999} - \frac{1}{2} + \\ + \frac{1}{1001} + \frac{1}{1003} + \dots + \frac{1}{1999} - \frac{1}{4} + \\ + \dots \end{aligned}$$

For absolutely convergent series we have

Proposition 3.4

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, with $S = \sum_{n=1}^{\infty} a_n$. Then, for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ we have that $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is also absolutely convergent, and $\sum_{n=1}^{\infty} a_{\sigma(n)} = S$.

Theorem 3.3: Riemann

If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series, then for any $x \in \mathbb{R}$ there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ s.t. $\sum_{n=1}^{\infty} a_{\sigma(n)} = x$.

4 Continuity

In this section, we will study limits of functions, continuity, the intermediate value theorem, and uniform continuity.
Before discussing limits, we need

Definition 4.1

Let $S \subseteq \mathbb{R}$ be a set, we call $x \in \mathbb{R}$ an *accumulation point* (also called *limit point* or *cluster points*), if for every $\epsilon > 0$,

$$(x-\epsilon,\, x+\epsilon) \cap (S \setminus \{x\}) \neq 0$$

Intuitively: $x \in S$ is an acc. point if it can be approximated "as closely as we want" by using points in S, different from x itself.

Example 4.1

Some examples of acc. points,

1. $[0,\, 1] = S$. All points in $[0,\, 1]$ are acc. points.
2. $\{3\} = S$. No acc. points.
3. $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Claim: S has exactly one acc. pt. $= 0$.

Proof. Suppose 0 is an acc. pt: given $\epsilon > 0$, $(-\epsilon,\, \epsilon) \cap S \setminus \{0\} \neq 0$ because by the Archimedean property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \epsilon$.

$$\implies \frac{1}{n} \in (-\epsilon,\, \epsilon) \quad \text{and} \quad \frac{1}{n} \in S$$

Suppose there are no other acc. pts.: let $a \in \mathbb{R}$ be an acc. pt. $a \neq 0$.
If $a < 0$, choose $\epsilon = \frac{1}{2} \cdot |a| \implies (a-\epsilon,\, a+\epsilon)$ consists entirely of negative numbers $\implies$ since $S \subseteq R_{>0}$,

$$(a-\epsilon,\, a+\epsilon) \cap S = 0$$

This contradicts the def. of acc. points.
$a > 0$. Say $a \in S$, $a = \frac{1}{n}$, $n \in \mathbb{N}$.
$\epsilon_1 = \frac{1}{n} - \frac{1}{n+1} > 0$ $\epsilon_2 = \frac{1}{n-1} - \frac{1}{n} > 0$
Take $\epsilon := \frac{1}{n} \min\{\epsilon_1,\, \epsilon_2\} \implies (a-\epsilon,\, a+\epsilon)$ contains only one point of S, namely $\frac{1}{n}$.
$\implies (a-\epsilon,\, a+\epsilon) \cap \left(S \setminus \left\{\frac{1}{n}\right\}\right) = 0$
$\implies a = \frac{1}{n}$ is not an acc. point.
If $a \notin S$, $\frac{1}{a} > 0$. Claim $m \in \mathbb{N}$ s.t.

$$m < \frac{1}{a} < m+1$$

$$\implies \frac{1}{m+1} < a < \frac{1}{m}$$

$\epsilon = \frac{1}{2} \min\{\epsilon_3,\, \epsilon_4\} \therefore (a-\epsilon,\, a+\epsilon) \cap S \neq 0$. □

Example 4.2

More acc. points,

4. The acc. pts. of $(0, 1)$ are all points in $[0, 1]$.
5. Acc. pts. of $(0, 1) \cup (1, 2)$ are $[0, 2]$.
6. For $S = \mathbb{Q}$, acc. pts. are $\mathbb{R}$.
7. $\mathbb{Z}$ has no acc. pts.

4.1 Limits of Functions

Definition 4.2: $\epsilon - \delta$ limit of a function

Let $f : S \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be an acc. pt. We say that $f(x)$ *converges* to L as x goes to a, denoted by

$$\lim_{x \to a} f(x) = L, \quad \text{or} \quad f(x) \to_{x \to a} L$$

if for every $\epsilon > 0$ there exists some $\delta > 0$ s.t.

$$|f(x) - L| < \epsilon, \quad \forall x \in S \setminus \{a\}, |x - a| < \delta$$

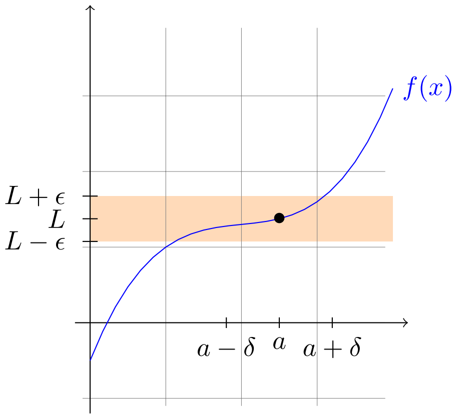


Figure 4.1: Limit of a function

Observe, δ will depend on ϵ (and on a, f).

Example 4.3

Consider

$$\lim_{x \to 2} 3x + 1 = 7$$

Proof. Let $\epsilon > 0$. We want to find $\delta > 0$ s.t. if $|x - 2| < \delta \implies |3x + 1 - 7| < \epsilon$.
Start from what you want to estimate:

$$\begin{aligned} |3x + 1 - 7| &= |3(x - 2 + 2) + 1 - 7| \\ &= |3 \cdot (x - 2)| = 3 \cdot |x - 2| \\ &< 3 \cdot \delta \leq \epsilon \end{aligned}$$

For the last inequality to hold, it is enough to chose $\delta = \frac{\epsilon}{3}$.
The above computation (estimate) shows that if $|x - 2| < \frac{\epsilon}{\epsilon}$ then $|3x + 1 - 7| < \epsilon$.
$\therefore$ by def., $\lim_{x \to 2} 3x + 1 = 7$. $\square$

Example 4.4

Consider $f : (0, \infty) \to \mathbb{R}$, $f(x) = \begin{cases} x, & x > 0 \\ 1, & x = 0 \end{cases}$ Claim: $\lim_{x \to 0} f(x) = 0$. Observe: the value $f(0)$ is not relevant for $\lim_{x \to 0} f(x)$.

Proof. Given $\epsilon > 0$, choose $\delta = \epsilon$. Then, for any $x \in [0, \infty]$, $x \neq 0$ we have that if $|x - 0| < \delta = \epsilon \implies$

$$|f(x) - 0| = |x| < \delta = \epsilon$$

$\square$

Example 4.5

Consider

$$\lim_{x \to 1} x^2 + 3 = 4$$

Proof. Let $\epsilon > 0$. Want to find $\delta > 0$ s.t. if $|x-1| < \delta \implies \left|\left(x^2+3\right) - 4\right| < \epsilon$.

$$\begin{aligned} |x^2+3-4| = |x^2-1| &= |(x-1)\cdot(x+1)| \\ &= \underbrace{|x-1|}_{<\delta} \cdot |x+1| < \\ &< \delta|(x-1)+2| \leq \\ &\leq \delta\left(|x-1|+2\right) < \\ &< \delta(\delta+2) = * \end{aligned}$$

We would need to find $\delta = \delta(\epsilon)$ s.t. $\delta^2 + 2\delta \leq \epsilon$. More elegant solution: make an "a priori" assumption: assume also that $\delta < 1$. Why? Because we can!
Then,

$$* = \delta(\delta+2) < \delta(1+2) = 3\delta \underbrace{\leq}_{\text{want}} \epsilon$$

Choose $\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{6}\right\} \implies \delta \leq \frac{1}{2} < 1$ and $\implies \delta \leq \frac{\epsilon}{6} < \frac{\epsilon}{3}$
$\implies$ if $|x-1| < \delta \implies |x^2+3-4| < \epsilon$.
Then, by def., $\lim_{x \to 1} x^2 + 3 = 4$ □

4.2 Limits at Infinity and Infinite Limits

Sometime we are interested in the behavior of $f(x)$ for very large x. For that purpose, we define

Definition 4.3: Limits at ∞

Given a function $f : (c, \infty) \to \mathbb{R}$, we say that

$$\lim_{x \to \infty} f(x) = L \in \mathbb{R}$$

if for every $\varepsilon > 0$, there exists some $\Pi > 0$ ($\Pi \in \mathbb{R}$) s.t. if $x > \Pi \implies |f(x) - L| < \varepsilon$.

Observe, we have of course the analogous definition for $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$, for an $f : (-\infty, c) \to \mathbb{R}$, simply replace "$x > \Pi$" with "$x < -\Pi$" above.

Example 4.6

Consider

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Proof. Given $\varepsilon > 0$, we look for $\Pi > 0$ s.t. if $x > \Pi$ then $\left|\frac{1}{x}\right| < \varepsilon$.

$$\left|\frac{1}{x}\right| = \frac{1}{|x|} = \frac{1}{x} = \frac{1}{\Pi} \underbrace{\leq}_{)} \text{want?}\varepsilon$$

In order for $\frac{1}{n} \leq \epsilon$ to hold, it's enough to take $\Pi = \frac{1}{\epsilon} > 0$. For that Π, the above computation shows that if $x > \Pi$ then $|\frac{1}{x}| < \frac{1}{n} = \varepsilon$. Therefore, by definition, $\lim_{x\to\infty} \frac{1}{x} = 0$. □

To could happen that the values of a function do *not* approach any particular real number, as $x \to a$, but that they become arbitrarily large. We call this an *infinite limit*.

Definition 4.4

Let $f : S \to \mathbb{R}$ be a function, $a \in \mathbb{R}$ an acc. pt. for S. We say $\lim_{x\to a} f(x) = \infty$, if for every $B > 0$ there exists $\delta > 0$ s.t. if $x \in S$, $0 < |x - a| < \delta \implies f(x) > B$

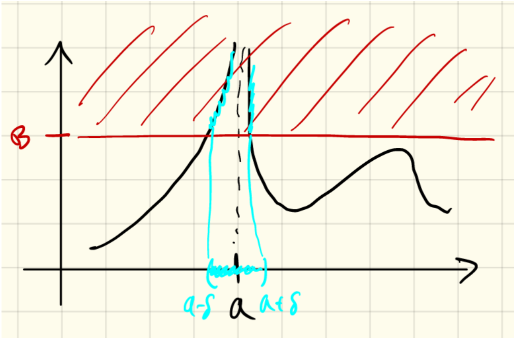


Figure 4.2: Infinite Limits

Observe: we can define $\lim_{x\to a} f(x) = -\infty$ analogously, by simply replacing "$f(x) > B$" by "$f(x) < -B$" above.
What's the connection between these notions of limits of functions and limits of sequences?

Theorem 4.1: $\epsilon - \delta$ limit equivalent to sequential limits

Let $f : S \to \mathbb{R}$ be a function, $a \in \mathbb{R}$ be an acc. pt. for S. Then, $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}$ with $x_n \in S \setminus \{a\} \quad \forall n$ and $x_n \to_{n\to\infty} a$, we have that $f(x_n) \to_{x\to\infty} L$.

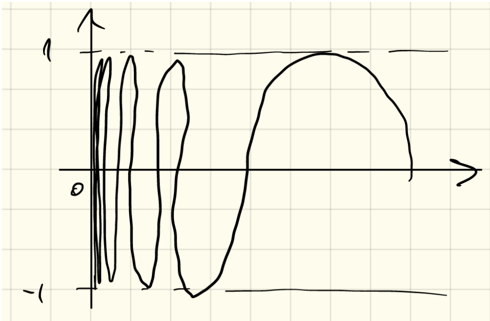
Proof. ($\implies$) Suppose that $\lim_{x\to a} f(x) = L$. Then we know that

1. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$
Let $\{x_n\}$ be a sequence w/ $x_n \in S \setminus \{a\}, \quad \forall n \in \mathbb{N}$, and $x_n \to_{n\to\infty} a$. WTS that $f(x) \to_{n\to\infty} L$. To do that, we use the def. of sequential limits. Let $\epsilon > 0$. By (1), $\exists \delta > 0$ s.t.
2. if $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$.
Now, can we ensure that $0 < |x_n - a| < \delta$ for all n "large enough"? Yes!
Since we know that $x_n \to_{n\to\infty} a$, we have
3. $\forall \varepsilon_1 > 0$, $\exists N$ s.t. if $n \geq N \implies |x_n - a| < \varepsilon_1$. Take $\varepsilon_1 = \delta$ in (3). Then, $\exists N \in \mathbb{N}$ s.t.
4. if $n \geq N \implies |x_n - a| < \delta$.
(2) and (4) yeild that if $n \geq N$, then $0 < |x_n - a| < \delta \implies |f(x_n) - L| < \varepsilon$. By definition of sequential limits, $f(x_n) \to_{x\to\infty} L$.

($\Leftarrow$) We prove the contrapositive statement:
$$(p \to g, \text{ contrapositive} L \sim g \to \sim p)$$
If $\lim_{x\to a} f(x) \neq L$, then there exists some sequence $\{x_n\}$, $x_n \in S \setminus \{a\}$, $x_n \to a$, with $f(x_n) \nrightarrow L$.
If $\lim_{x\to a} f(x) \neq L$, then the following fails:
$$(\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (\text{if } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon))$$
That is, we know that
$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \text{ but } |f(x) - L| \geq \varepsilon$$
In other words, for each $\delta > 0$ there must be some x_δ with $0 < |x_\delta - a| < \delta$ but $|f(x_\delta) - L| \geq \varepsilon$. For $\delta = \frac{1}{n}$, $\exists x_n \in S$ s.t.
$$0 < |x_n - a| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon > 0$$
We have thus constructed a sequence $\{x_n\}$ with $x_n \in S \setminus \{a\} \quad \forall n \in \mathbb{N}$, $x_n \to_{n\to\infty} a$, but $f(x_n) \nrightarrow L$. ∎

Example 4.7

Does $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ exist?



No. We use the above theorem to justify.
Take $x_n = \frac{1}{2\pi n}$, $n \in \mathbb{N}$. Clearly, $x_n \to_{n\to\infty} 0$.

$$f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin(2\pi n) = 0, \quad \forall n \in \mathbb{N} \implies f(x_n) \to_{n\to\infty} 0$$

So if there is a limit for $\sin\left(\frac{1}{x}\right)$, it must be 0. But: consider $y_n := \frac{1}{2\pi n + \frac{\pi}{2}} \implies y_n \to_{n\to\infty} 0$,

$$f(y_n) = \sin\left(2\pi n + \frac{\pi}{2}\right) \equiv 1 \to_{n\to\infty} 1$$

Since $1 \neq 0$, the limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does *not* exist (by the above theorem).

4.3 Squeeze Theorems

For sequences we have the following

Theorem 4.2

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of real numbers, with $a_n \le b_n \le c_n$, $\forall n \in \mathbb{N}$. If $a_n \to_{n\to\infty} L$ and $c_n \to_{n\to\infty} L$, then $b_n \to_{n\to\infty} L$.

Proof. Let $\varepsilon > 0$. By assumption, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$|L - a_n| < \frac{\varepsilon}{3}, \quad \forall n \ge N_1$$
$$|L - c_n| < \frac{\varepsilon}{3}, \quad \forall n \ge N_2$$

Then,

$$\begin{aligned}|a_n - c_n| &= |a_n - L + L - c_n| \le \\ &= |L - a_n| + |L + c_n| < \frac{2\varepsilon}{3}\end{aligned}$$

$\forall n \ge \max\{N_1, N_2\} =: N$. Thus,

$$\begin{aligned}|b_n - L| &= |b_n - a_n + a_n - L| \le \\ &\le |b_n - a_n| + |L - a_n| = \\ &= b_n - a_n + |L - a_n| \\ &\le c_n - a_n + |L - a_n| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

$\forall n \ge N$. $\therefore b_n \to L$. $\square$

From that and the equivalent between $\varepsilon - \delta$ limits and sequential limits, we deduce

Theorem 4.3: Squeeze theorem for Functions

Let f, g, $h : S \to \mathbb{R}$, $a \in \mathbb{R}$ an acc. pt. for S. Assume that

$$f(x) \leq g(x) \leq h(x), \quad \forall x \in S$$

If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) =: L \in \mathbb{R}$, then the limit of $g(x)$ as x goes to a exists, and equals L.

Proof. By the equivalence, it is enough to show that $g(x_n) \to_{n\to\infty} L$, for every sequence $\{x_n\}$, $x_n \in S \setminus \{a\}$, $x_n \to_{n\to\infty} a$. But for each such sequence, we apply the squeeze theorem for sequences. □

Example 4.8

Study $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$.
Since $-1 \leq \sin(t) \leq 1 \quad \forall t \in \mathbb{R}$, then $-x \leq x\sin\left(\frac{1}{x}\right) \leq x \quad \forall x \in \mathbb{R} \setminus \{0\}$.
But $\lim_{x\to 0} -x = 0$, $\lim_{x\to 0} x = 0$, thus $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ by squeeze theorem.

4.4 Algebraic Properties of Limits

From the algebraic properties of limits for sequences, and the equivalent between $\varepsilon - \delta$ limits and sequential limits, we deduce the following

Proposition 4.1

Let f, $g : S \to \mathbb{R}$ be functions, $a \in \mathbb{R}$ be an acc. pt. for S, and suppose that $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist. Then,

i) $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

ii) $\lim_{x\to a} (f(x) \cdot g(x)) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x))$

iii) If $\lim_{x\to a} g(x) \neq 0$ then for the function $f/g : S \to \mathbb{R}, \quad x \to \frac{f(x)}{g(x)}$, where $S = \{x \in S : g(x) \neq 0\}$, we have $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

4.5 Continuous Functions

These are those functions whose graph can be drawn "without lifting the pen from the paper".

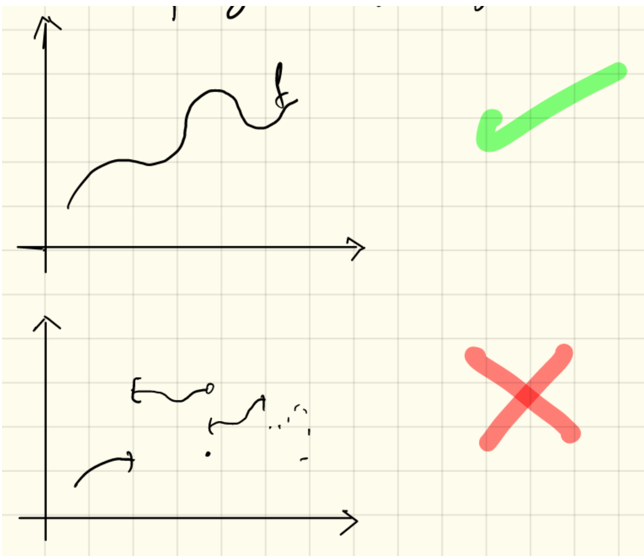


Figure 4.3: Example of continuous vs discontinuous functions

What's the precise definition?

Definition 4.5: Continuous functions

A function $f : S \to \mathbb{R}$ is said to be continuous at a, for $a \in S$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(x \in S,\, |x - a| < \delta) \implies |f(x) - f(a)| < \varepsilon$$

We say f is *continuous*, if f is continuous at every $a \in S$.

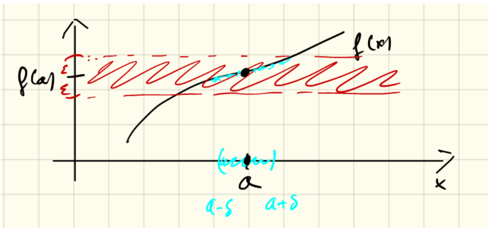


Figure 4.4: Showing that f is continuous at a

Rmk: If $a \in S$ is an acc. pt., then the def. is equivalent to saying that

$$\lim_{x \to a} f(x) = f(a)$$

If however a is *not* an acc. pt. of S, then f is conditionally continuous at a.

Rmk: Notice that, as opposed to the def. of limits of functions, here we do require that $a \in S$ (because we need to evaluate $f(a)$). Thus for continuity, the value of f at a is extremely important.

Rmk: By the equiv. between $\varepsilon - \delta$ limits and sequential limits, f is cont. at $a \in S$ iff for all sequences $\{x_n\}$, $x_n \in S\ \forall n \in \mathbb{N}$, $x_n \to a$, we have $\lim_{n\to\infty} f(x_n) = f(a)$.

Example 4.9

Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$, is a cont. func.

Proof. Let $a \in \mathbb{R}$, we show that f is cont. at a.
Given $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then, wherever $|x - a| < \delta$, we have $|f(x) - f(a)| = |x - a| < \delta = \varepsilon$.
Hence by def., f is cont. at a. □

Example 4.10

Let

$$f(x) = \begin{cases} x, & x \neq 0 \\ \pi, & x = 0 \end{cases}$$

Then f is *not* cont. at 0.
Indeed, we have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0 \neq f(0)$$

Let us mention the following

Proposition 4.2

Let f, $g : S \to \mathbb{R}$ be functions, which are continuous at $a \in S$. Then,

i) $f(x) \pm g(x)$ is cont. at a

ii) $f(x) \cdot g(x)$ is cont. at a

iii) If $g(a) \neq 0$ then $\frac{f(x)}{g(x)}$ is cont. at a.

Proof. Exercise. Follows from def. of continuity, and the algebraic properties of limits. ☐

Corollary: Polynomial functions $p : \mathbb{R} \to \mathbb{R}$

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

are continuous on all of $\mathbb{R}$. Moreover, rational functions ($f(x) = \frac{p(x)}{q(x)}$, p, q polynomials) are continuous on their domain ($q \neq 0$).

Proof. Constant functions are cont., and $f(x) = x$ as well. Then applying the previous proposition, we obtain what is stated.
e.g. $f(x) \equiv a_1$ constant, $g(x) = x$, cont. $\implies a_1 \cdot x$ cont.
$h(x) \equiv a_0$ cont. $\implies a_1 \cdot x + a_0$ cont. ☐

Facts: the trigonometric functions $\sin x$, $\cos x$, etc are continuous in their entire domain. Similarly for e^x, $\ln x$.

Example 4.11

Let

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$$

Then f is discontinuous at every $a \in \mathbb{R}$

Theorem 4.4: Composition of Continuous Functions

Let A, $B \subseteq \mathbb{R}$, $f : A \to B$, $g : B \to \mathbb{R}$ be functions with f is continuous at $a \in A$, and g is continuous at $b := f(a) \in B$. Then $g \circ f : A \to \mathbb{R}$ is continuous at a.

Proof. Let $\{x_n\}$ be a sequence in A with $\lim_{n\to\infty} x_n = a$.
f cont. at $a \implies f(x_n) \to_{n\to\infty} f(a) = b$.
g cont. at $b \implies g(f(x_n)) \to_{n\to\infty} g(b) = g(f(a)) \implies (g \circ f)(x_n) \to_{n\to\infty} (g \circ f)(a)$
Since $\{x_n\}$ was arbitrary, by eariler theorem, we conclude that $g \circ f$ is cont. at a. ☐

Corollary: Let $d \in \mathbb{R}$. Then, the function $h : (0, \infty) \to \mathbb{R}$, $\ln(x) = x^d$, is continuous.

Proof. By definition, $x^d = e^{d \cdot \ln x} = g \circ f(x)$, $g(x) = e^x$, $f(x) = d \cdot \ln x$, both continuous. By above theorem, $g \circ f = \ln$ is cont. ☐

4.6 Uniform Continuity

Example 4.12

Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is continuous (using only the definition of continuity).
We need to prove that for all $a \in \mathbb{R}$, we have that for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $|x - a| < \delta$, $\implies |x^2 - a^2| < \varepsilon$.
Fix a; given $\varepsilon > 0$, we look for $\delta > 0$ for which $|x - a| < \delta$ implies $|x^2 - a^2| < \epsilon$.
Thinking:

$$\begin{aligned} |x^2 - a^2| &= |(x-a)(x+a)| = |x-a| \cdot |x+a| \\ &= |x-a| \cdot |(x-a) + 2a| \\ &\leq |x-a| \cdot (|x-a| + 2 \cdot |a|) \\ &< \delta \cdot (\delta + 2 \cdot |a|) \\ &\leq \delta \cdot (1 + 2 \cdot |a|) \leq \epsilon \end{aligned}$$

Need: $\delta \leq \frac{\epsilon}{1+2|a|}$.
Choose $\delta = \min\left\{1, \frac{\epsilon}{1+2 \cdot |a|}\right\}$. Then, the above shows that for $|x - a| < \delta$, we have that $|x^2 - a^2| < \cdots <$

$\delta\,(\delta + 2\cdot|a|) \le \delta\,(1 + 2|a|) \le \epsilon \implies |x^2 - a^2| < \epsilon$
$\therefore x^2$ is continuous at a.

Obs: δ depends on ϵ and on a! The larger $|a|$ gets, the smaller we need to take δ.

Sometimes it is extremely important to have δ independant of the point a. For that, we define

Definition 4.6: Uniformly Continuous Functions

We call a function $f : S \to \mathbb{R}$ *uniformly continuous*, if for ever $\varepsilon > 0$, there exists $\delta > 0$ s.t. whenever $x,\, y \in S$, $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Rmk: $\delta > 0$ is indep. of the points x, y (but it still depends on $\varepsilon > 0$).

Rmk: f uniformly continuous $\implies$ f continuous.

Rmk: This notion depends strongly on the domain.

Example 4.13

The function $f(x) = x^2$ is *not* unif. cont. on $\mathbb{R}$, but it *is* unif. cont. of $[0,\,1]$.

Claim: $f : [0,\,1] \to \mathbb{R}$, $f(x) = x^2$ is unif. cont.

Proof. From previous example, we know that for proving continuity at a it suffices to choose $\delta = \min\left\{1,\, \frac{\varepsilon}{1+2\cdot|a|}\right\}$. If we know that $|a| \le 1$, then

$$\frac{\varepsilon}{3} \le \frac{\varepsilon}{1 + 2\cdot|a|}$$

Thus, if we choose $\delta = \min\left\{1,\, \frac{\varepsilon}{3}\right\}$ then $\delta \le 1$, $\delta \le \frac{\varepsilon}{1+2|a|}$ for all $a \in [0,\,1]$. The computation from the example shows that $|x - a| < \delta \implies |x^2 - a^2| < \varepsilon$, for that choice of $\delta > 0$. Change a by y, this is nothing but the definition of uniform continuity. $\square$

Example 4.14

The function $f : (0,\,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is not unif. cont.

Proof. By contradiction, suppose f were unif. cont.. Then, given any $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x - y| < \delta$, $x,\, y \in (0,1) \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$.
We took a particular $y = \delta$, $x \in (0, \delta)$. Then $|x - y| = y - x = \delta - x < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$.
$\implies \varepsilon > \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{x} - \frac{1}{\delta}$
$\implies \varepsilon + \frac{1}{\delta} > \frac{1}{x}$, a contradiction.
$\therefore f$ is *not* uniformly continuous. $\square$

Theorem 4.5

Let $f : [a, b] \to \mathbb{R}$ be a continuous functions. Then, f is uniformly continuous.

Proof. Suppose that f is not unif. cont. THen, by negating the def. of unif. cont., we get: $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, there are points $x, y \in [a, b]$ with $|x - y| < \delta$, but $|f(x) - f(y)| \ge \varepsilon$.
Take $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. Then, $\exists \{x_n\}_n, \{y_n\}_n$, s.t. $|x_n - y_n| < \frac{1}{n}$, $|f(x_n) f(y_n)| \ge \varepsilon$, $\forall n \in \mathbb{N}$. Since $\{x_n\}_n$ is in $[a, b]$, it is bounded. Thus, by the Theorem of Bolzano-Weirstrass, $\exists$ convergent subseq. $\{x_{n_k}\}_k$, converging to some $c \in [a, b]$, as $k \to \infty$.
Indeed, $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, we have

$$-\frac{1}{n_k} < y_{n_k} - x_{n_k} < \frac{1}{n_k}$$

$$\implies x_{n_k} - \frac{1}{n_k} < y_{n_k} < x_{n_k} + \frac{1}{n_k}$$

So the claim is a consequence of the squeeze theorem. By continuity, $\lim_{k\to\infty} f(x_{n_k}) = f(c)$. However, $|f(x_{n_k} - f(y_{n_k}| \geq \varepsilon > 0$ says that $\{f(y_{n_k})\}$ cannot converge to $f(c)$ as $k \to \infty$.
But this contradicts the fact that f is cont. at c, and $\{y_{n_k}\}_n$ is a seq. with $y_{n_k} \to c$. $\square$

Example 4.15

The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is not unif. cont. But, it is unif. cont. when restricted to a closed interval, say $f : (-1, 1) \to \mathbb{R}$.

4.7 Properties of Continuous Functions

Definition 4.7: Boundedness of Functions

We say a function $f : S \to \mathbb{R}$ is bounded if $\exists B \in \mathbb{R}$ s.t. $|f(x)| \leq B$, $\forall x \in S$.

Example 4.16

The function $\frac{1}{x}$ is unbounded on $(0, \infty)$ ($\frac{1}{x} \to_{x\to\infty} +\infty$), but it is bounded if restricted to $[1, 2]$.

Lemma 4.1

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is bounded.

Proof. Suppose f is not bounded. Then, $\forall B \in \mathbb{R} \exists x \in [a, b]$ with $|f(x)| > B$. Choose $B = n \in \mathbb{N}$, and we get a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in [a, b]$, $|f(x_n)| \geq n$, $\forall n \in \mathbb{N}$.
By the Theorem of Bolzano-Weirstrass, $\exists$ convergent subsequence $\{x_{n_k}\}_k$, $x_{n_k} \to_{k\to\infty} c \in [a, b]$. f is continuous $\implies f(x_{n_k}) \to_{k\to\infty} f(c)$, $\implies \{f(x_{n_k})\}_k$ is bounded. $\implies \exists \Pi \in \mathbb{R}$ s.t. $|f(x_{n_k})| \leq \Pi$, $\forall k \implies \Pi \geq |f(x_{n_k}| \geq n_k \geq k \quad \forall k \in \mathbb{N}$, a contradiction. (Archimedean property). $\therefore f$ is bounded. $\square$

Theorem 4.6: Min and Max for Continuous Functions

A cont. function $f : [a, b] \to \mathbb{R}$ attains both its minimum and maximum values. That is,

$$\exists c_{\min} \in [a, b] \text{ s.t. } f(c_{\min}) \leq f(x), \forall x \in [a, b]$$
$$\exists c_{\max} \in [a, b] \text{ s.t. } f(c_{\max}) \geq f(x), \forall x \in [a, b]$$

Proof. By the lemma, $f([a, b]) := \{f(x) : x \in [a, b]\}$ is a bounded set $\implies$ it has an infimum and a supremum. Let $\{f(x_n)\}_n$, $\{f(y_n)\}_n$ be sequences on $f([a, b])$ s.t.

$$\lim_{n\to\infty} f(x_n) = \inf f([a, b])$$
$$\lim_{n\to\infty} f(y_n) = \sup f([a, b])$$

By Bolzano-Weirstrass, we may extract convergent subsequences $\{x_{n_k}\}_k$, $\{y_{n_g}\}_g$, s.t.

$$x_{n_k} \to_{k\to\infty} c_{\min} \in [a, b] \implies f(x_{n_k}) \to_{k\to\infty} f(C_{\min})$$
$$y_{n_g} \to_{g\to\infty} c_{\max} \in [a, b] \implies f(y_{n_g}) \to_{g\to\infty} f(C_{\max})$$

From the above equations,

$$f(C_{\min}) = \inf \{f(x) : x \in [a, b]\}$$
$$f(C_{\max}) = \sup \{f(x) : x \in [a, b]\}$$

$\square$

Example 4.17

Consider $f : [-1, 2] \to \mathbb{R}$, $f(x) = x^2$ $c_{\text{min}} = 0$, $c_{\text{max}} = 2$.
These depend on the domain; e.g. for $[1, 3]$, $c_{\text{min}} = 1$, c_{max}

Example 4.18

The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$, doesn't achieve its min + max values.

Example 4.19

Continuity is crucial: $f(x) = \begin{cases} \frac{1}{x}, & x \in (0,\ 1] \\ 0, & x = 0 \end{cases}$
Def on $[0,\ 1]$ but unbounded. Problem: f is not cont. at $x = 0$.

Theorem 4.7: Intermediate Value Theorem

Let $f : [a,\ b] \to \mathbb{R}$ be continuous, and suppose that $\exists y$ "in between" $f(a)$ and $f(b)$ (ie either $f(a) < y < f(b)$, or $f(a) > y > f(b)$). Then, $\exists c \in [a, b]$ s.t. $f(c) = y$.

Proof. Assume that $f(a) < y < f(b)$, the other case being analogous. Consider $g(x) := f(x) - y$, also continuous on $[a, b]$, with $g(a) < 0$, $g(b) > 0$. Then the theorem follows from applying the result to $g(x)$. $\square$

Lemma 4.2

Let $f : [a, b] \to \mathbb{R}$ be continuous, with $f(a) < 0 < f(b)$. Then, $\exists c \in [a, b]$ with $f(c) = 0$.

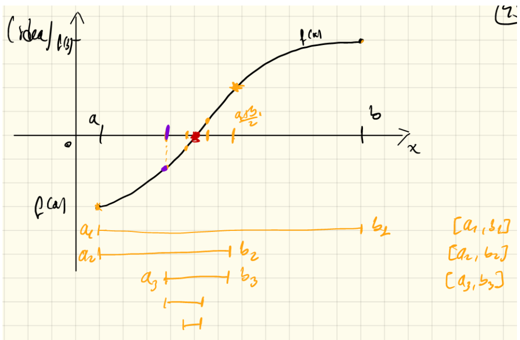


Figure 4.5: Sketch of the Intermediate Value Theorem

Proof. Define $\{a_n\}$, $\{b_n\}$ recursively:

1. $a_1 = a$, $b_1 = b$
2. if $f\left(\frac{a_n+b_n}{2}\right) \geq 0$, set $a_{n+1} = a_n$, $b_{n+1} = \frac{a_n+b_n}{2}$
3. if $f\left(\frac{a_n+b_n}{2}\right) < 0$, set $a_{n+1} = \frac{a_n+b_n}{2}$, $b_{n+1} = b_n$

It is clear that (can be proved by incduction):

1. $a_n < b_n, \quad \forall n \in \mathbb{N}$
2. $|b_n - a_n| = \frac{|a-b|}{2^n} \to_{n\to\infty} 0$

3. $\{a_n\}$ monotone increasing and bdd above (by b)
4. $\{b_n\}_n$ is monotone decreasing, bdd below $\implies a_n \to_{n\to\infty} c, \quad b_n \to_{n\to\infty} d$
   $|a_n - b_n| \to 0$, we must take $c = d$.
   $f(c) = \lim_{n\to\infty} f(a_n) \leq 0$
   $f(c) = \lim_{n\to\infty} f(b_n) \geq 0$

□

Corollary: Fixed point theorem. Let $f : [0,1] \to [0,1]$ be cont. Then, f has a fixed point, i.e. $\exists c \in [0,1]$ for which $f(c) = c$.

Proof. Consider $g(x) = x - f(x)$, cont.
$g(0) - f(0) \leq 0$, $g(1) = 1 - f(1) \geq 0$
Then, by I.V.T., $\exists c \in [0,1]$ with $g(c) = 0 \implies c - f(0) = 0$. □

5 The Derivative

Intuition: the functions that are easiest to understand are the *linear functions*:

$$\ell : \mathbb{R} \to \mathbb{R}, \quad \ell(x) = c \cdot x + d, \quad c, d \in \mathbb{R}$$

Given an arbitrary function $f : (a, b) \to \mathbb{R}$, we would like to approximate it by a linear function "in the best possible way". That is, given $x_0 \in (a, b)$, we look for $\ell(x) = cx + d$ such that $\ell(x)$ is the best linear approx for $f(x)$ near x_0. How?

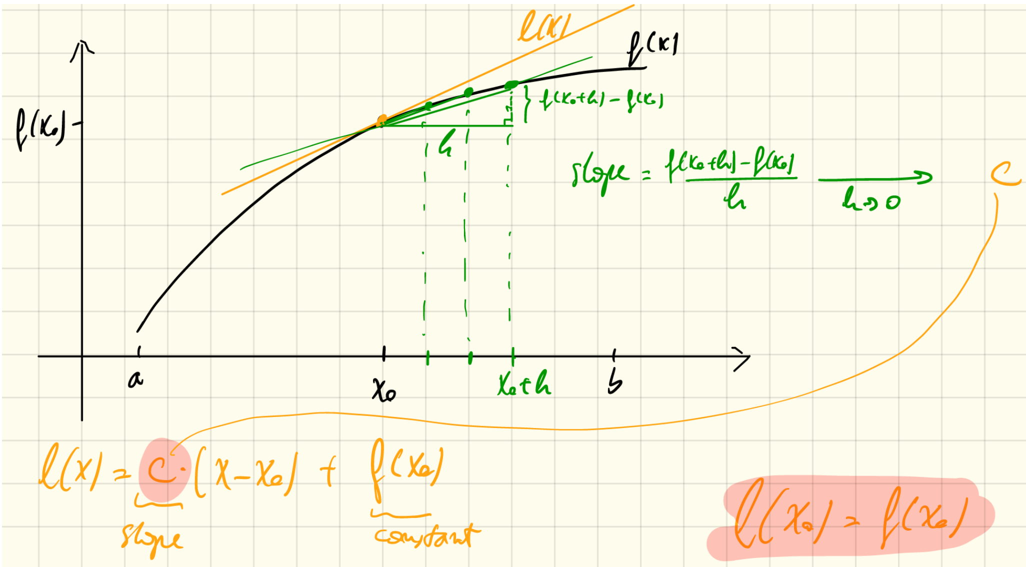


Figure 5.1: Approximating some function, f, with a linear equation, ℓ

Definition 5.1: Derivative of a Function

Let $f : (a, b) \to \mathbb{R}$ be a function, $x_0 \in (a, b)$. We say f is *differentiable* at x_0 if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} := L$$

exists. In that case, we call L the *derivative* of f at x_0, and we write $f'(x_0) = L$.
We say f is *differentiable* if it is differentiable at all $x_0 \in (a, b)$.

The expression $\frac{f(x_0+h)-f(x_0)}{h}$ is called Newton's difference quotient. It can also be written as

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(x) - f(x_0)}{x - x_0} \qquad x := x_0 + h$$

Thus, f is differentiable at x_0 iff

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

Example 5.1

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
$x_0 \in \mathbb{R}$ is arbitrary.

$$\begin{aligned}
\lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} &= \lim_{h \to 0} \frac{x_0^2 + 2h \cdot x_0 + h^2 - x_0^2}{h} \\
&= \lim_{h \to 0} \frac{2h \cdot x_0 + h^2}{h} \\
&= \lim_{h \to 0} 2x_0 + h = 2 \cdot x_0
\end{aligned}$$

Therefore, x^2 is differentiable at x_0, and $f'(x_0) = 2x_0$.

Example 5.2

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$, is *not* differentiable at $x_0 = 0$. We must show that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

does not exist.
Take $h = \frac{1}{n} \to_{n\to\infty} 0, \quad n \in \mathbb{N}$,

$$\frac{|\frac{1}{n}|}{\frac{1}{n}} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$

Take $h = \frac{-1}{n} \to_{n\to\infty} 0, \quad n \in \mathbb{N}$,

$$\frac{|\frac{-1}{n}|}{|\frac{-1}{n}|} = \frac{\frac{1}{n}}{\frac{-1}{n}} = -1 \neq 1$$

$\therefore$ the limit doesn't exist. By def., $|x|$ is not diff. at 0.

Proposition 5.1

If $f : (a, b) \to \mathbb{R}$ is differentiable at x_0, then it is continuous at x_0

Proof. We know that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

exists.

$$\begin{aligned}\lim_{x\to x_0} f(x) &= \lim_{h\to 0} f(x_0 + h)\\ &= f(x_0) + \lim_{h\to 0} f(x_0 + h) - f(x_0)\\ &= f(x_0) + \lim_{h\to 0} h(\frac{f(x_0+h) - f(x_0)}{h})\\ &= f(x_0)\end{aligned}$$

By def., f is continuous at x_0. □

The derivative is, in some sense, a "linear operation".

Proposition 5.2

Let $f, g : (a, b) \to \mathbb{R}$ be functions which are differentiable at $x_0 \in (a, b)$, and let $a \in \mathbb{R}$. Then,

(i) $h : (a, b) \to \mathbb{R}$, $h(x) = a \cdot f(x)$, is also diff. at x_0, adn $h'(x_0) = a \cdot f'(x_0)$.

(ii) $h : (a, b) \to \mathbb{R}$, $h(x) = f(x) + g(x)$, is diff. at x_0, and $h'(x_0) = f'(x_0) + g'(x_0)$.

For products or quotients, the formula is not so straightforward.

Proposition 5.3

Let $f, g : (a, b) \to \mathbb{R}$ be two functions which are differentiable at $x_0 \in (a, b)$. Then,

(i) $h(x) := f(x)g(x)$ is diff. at x_0, and

$$h'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \qquad \text{product rule}$$

(product rule)

(ii) If $g(x_0) \neq 0$, then $h(x) = f(x)/g(x)$ is well def. in some interval $(x_0 - \delta,\, x_0 + \delta)$, $\delta > 0$, it is diff. at

x_0, and

$$h'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2} \qquad \text{quotient rule}$$

Rmk: In (ii), g diff. at x_0 implies g cont. at x_0. Thus, $g(x_0) \neq 0 \implies$ by continuity, $\exists \delta > 0$ s.t.

$$g(x) \neq 0 \quad \forall x : |x - x_0| < \delta$$

Proof. We prove only (i): we need to study the limit

$$\lim_{h \to 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0)}{h}$$

knowing that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$

$$\lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = g'(x_0)$$

In the first expression, we would like the next two expressions to appear.

$$\begin{aligned}
&\frac{1}{h}\left(f(x_0 + h(g(x_0 + h)) - f(x_0)g(x_0)\right) \\
&= \frac{1}{h}\left(\left(f(x_0 + h) - f(x_0)\right) g(x_0 + h) + f(x_0)g(x_0 + h)\right) - f(x_0)g(x_0) \\
&= g(x_0 + h)\left(\frac{f(x_0 + h) - f(x_0)}{h}\right) + f(x_0)\frac{g(x_0 + h) - g(x_0)}{h} \\
&\to_{h \to 0} g(x_0)f'(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

$\therefore h(x) = f(x)g(x)$ is diff. at x_0. □

5.1 The Chain Rule

The composition of two differentiable functions is again differentiable. The formula for obtaining the derivative of the composition is called the "chain rule".

Proposition 5.4

Let $g : (a, b) \to (c, d)$, $f : (c, d) \to \mathbb{R}$ be functions, assume that g is differentiable at $x_0 \in (a, b)$ and that f is differentiable at $g(x_0)$. Then, the composition $f \circ g : (a, b) \to \mathbb{R}$, $(f \circ g)(x) = f(g(x))$, is differentiable at x_0 and

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$$

Proof. By definition, we need to show that

$$\lim_{h \to 0} \frac{f(g(x_0 + h)) - f(g(x_0))}{h} = f'(g(x_0)) \cdot g'(x_0)$$

Take a sequence $\{h_n\}_{n \in \mathbb{N}}$, $h_n \xrightarrow[n \to \infty]{} 0$, arbitrary. Let us show that

$$\frac{f(g(x_0 + h)) - f(g(x_0))}{h_n} \xrightarrow[n \to \infty]{} f'(g(x_0)) \cdot g'(x_0)$$

Call $y_0 := g(x_0)$, $y_n := g(x_0 + h_n)$. g is differentiable at x_0 $|implies$ g is continuous at x_0. $\implies y_n \xrightarrow[n \to \infty]{} y_0$.

- f is differentiable at $g(x_0) = y_0 \implies$

$$\lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0} = f'(y_0)$$

$$\frac{f(y_n) - f(y_0)}{y_n - y_0} \xrightarrow[n \to \infty]{} f'(y_0)$$

- g is differentiable at $x_0 \implies$

$$\lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = g'(x_0)$$

$$\frac{y_n - y_0}{h_n} \xrightarrow[n \to \infty]{} g'(x_0)$$

Multiply the two above results, and the result is evident. $\square$

Example 5.3

Let $h(x) := \log(1 + x^2)$. Then $h'(x) = \frac{2x}{1+x^2}$. Indeed, $h = f \circ g$, $f(x) = \log x$ differentiable on $(0, \infty)$. $g(x) = 1+x^2$, differentiable on $\mathbb{R}$. $f'(x) = \frac{1}{x}$, $g'(x) = 2x$. By chain rule, $h'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{x^2+1} \cdot 2x$.

Rmk: It can be shown that all trigonometric functions are differentiable on their entire domain. The same holds for e^x, $\log x$.

By algebraic properties, all rational functions are also differentiable (wherever they are defined).

5.2 Local Extrema

Another idea behind the concept of derivative is that it measure how the functions changes, at least infinitesimally. If $f'(x) > 0$ is "large" then f grows fast. If $f'(x) < 0$ "very negative", then f decreases rapidly. If $f'(x)$ is "small" then f doesn't change much near x.

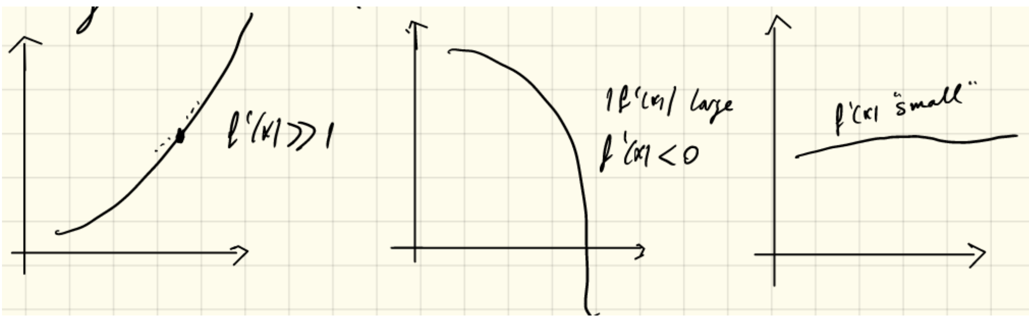


Figure 5.2: Classifying local extrema with derivatives

Proposition 5.5

Let $f : (a, b) \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$. If $f'(x_0) > 0$ (respectively < 0), there exists $\delta > 0$ such that $f(x_0 - u) < f(x_0) < f(x_0 + u) \quad \forall u \in (0, \delta)$ (respectively, $f(x_0 - u) > f(x_0) > f(x_0 + u) \quad \forall u \in (0, \delta)$). In particular, if $f'(x) > 0$ (respectively < 0) $\forall x \in (a, b)$, then f is strictly increasing (respectively decreasing).

Proof. Assume $f'(x_0) > 0$.
Know: $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) > 0$
Take $\varepsilon = \frac{f'(x_0)}{2} > 0$ in the definition of limits, then $\exists \delta > 0$ s.t. if $|h| < \delta$ then

$$\left| f'(x_0) - \frac{f(x_0 + h) - f(x_0)}{h} \right| < \frac{f'(x_0)}{2}$$

$$\leq f'(x_0) - \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\implies \frac{f'(x_0)}{2} > f'(x_0) - \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\implies \frac{f(x_0 + h) - f(x_0)}{h} > \frac{f'(x_0)}{2} > 0$$

If $h = u \geq 0$, then $f(x_0 + u) - f(x_0) > 0$.

If $h = -u < 0$, $u > 0$, then $f(x_0 - u) - f(x_0) < 0$.

$$\implies f(x_0 - u) < f(x_0) < f(x_0 + u), \quad \forall u \in (0, \delta)$$

□

Definition 5.2: Local Extrema

Let $f : (a, b) \to \mathbb{R}$ be a function. We say f has a *local maximum* at $x_0 \in (a, b)$, if $\exists \delta > 0$ s.t. $f(x) \leq f(x_0)$ $\forall x$ with $|x - x_0| < \delta$.
We say f has a *local minimum* at $x_0 \in (a, b)$, if $\exists \delta > 0$ s.t. $f(x) \geq f(x_0)$ $\forall x$, $|x - x_0| < \delta$.
In either case, we say that x_0 is a *local extremum* for f.

Corollary: If a differentiable function $f : (a, b) \to \mathbb{R}$ has a local extremum at $x_0 \in (a, b)$, then $f'(x_0) = 0$.

Proof. This is an immediate consequence of the definition and the previous proposition.

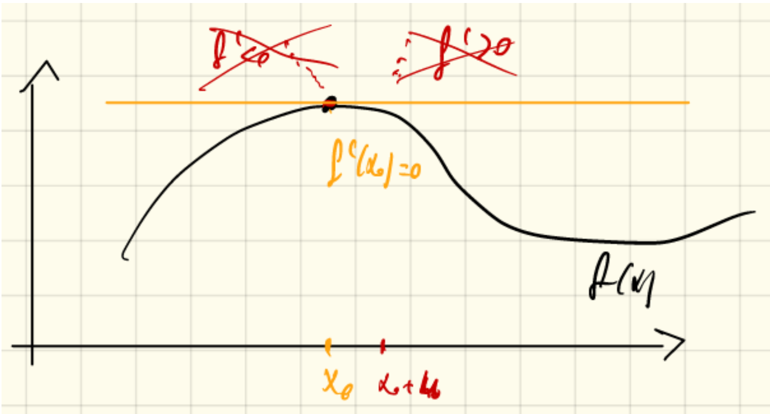


Figure 5.3: Proof that derivative is 0 at local extremum.

□

Example 5.4

- The function $f : (-3, 3) \to \mathbb{R}$, $f(x) = x^2 - 2x + 3$ has a local extremum at $x_0 = 1$.

  $$f'(x) = 2x - 2 = 0 \text{ iff } x = 1$$

  We see that 1 is a local min:

  $$f'(1 + u) = 2 \cdot u \geq 0$$
  $$f'(1 - u) = -2u < 0$$

- $f'(x_0) = 0$ is not enough for concluding that x_0 is a local extremum: take $f(x) = x^3$.

  $$f'(x) = 3x^2 \implies f'(0) = 0$$

  However, 0 is not an extremum: the function is strictly increasing (if $x < y \implies x^3 < y^3$).

5.3 Mean Value Theorem

Theorem 5.1: Rolle

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b). Assume that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Recall that by an earlier theorem theorem, f attains its maximum and minimum on $[a, b]$. Let $x_{\min}, x_{\max} \in [a, b]$ be such that

$$f(x_{\min}) = \min_{x \in [a,b]} f(x)$$
$$f(x_{\max}) = \max_{x \in [a,b]} f(x)$$

Notice that if $x_{\min} = x_{\max} \implies f$ is constant $\implies f' = 0$. So assume $x_{\min} \neq x_{\max}$.
If $x_{\min} = a \implies f(x_{\min}) = f(a) = f(b)$ is the minimum value for $f \implies c := x_{\max} \in (a, b)$.
If $x_{\max} = a \implies f(x_{\max}) = f(a) = f(b) \implies x_{\min} \neq a, b$ (f is not constant), set $c := x_{\min} \in (a, b)$.
If $x_{\min} \neq a$ and $x_{\max} \neq a$, $c := x_{\min} \in (a, b)$.
By the corollary, $f'(c) = 0$ (min & max are of course local extremum). □

Theorem 5.2: Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b). Then, there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

Proof. Define $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - f(b) + (f(b) - f(a)) \cdot \frac{b-x}{b-a}$
g continuous on $[a, b]$. g differentiable on $[a, b]$. $g(a) = 0$, $g(b) = 0 \implies$ by Rolle's theorem (see theorem 5.1), $\exists c$ s.t. $g'(c) = 0 \implies 0 = g'(c) = f'(0) - \frac{f(b)-f(a)}{b-a}$. □

5.4 Applications of the Mean Value Theorem

Corollary: (to the MVT) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function, with $f'(x) = 0$ $\forall x \in (a, b)$. Then, f is constant.

Proof. Take $x, y \in (a, b)$, $x < y$. Want to show that $f(x) = f(y)$.
We apply mean value theorem to $f : [x, y] \to \mathbb{R}$: observe that f is differentiable on $(a, b) \implies f$ is continuous on $(a, b) \implies f$ is continuous on $[x, y] \subseteq (a, b)$.
f differentiable on (a, b) implies f differentiable on (x, y).
Thus, the assumptions are satisfied. Then, $\exists c \in (x, y)$ s.t.

$$f(y) - f(x) = f'(c) \cdot (y - x)$$

However $f'(c) = 0$ which implies

$$f(y) - f(x) = 0 \implies f(y) = f(x)$$

□

Corollary: Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then, f is increasing (respectively decreasing) if and only if $f'(x) \geq 0$ (respectively $f'(x) \leq 0$), $\forall x \in (a, b)$.

Proof. Let us prove the increasing part.
Suppose f is increasing. Then, for all $x_0 \in (a, b)$, all $h \neq 0$ small enough (such that $x_0 + h \in (a, b)$), we have

$$\frac{f(x_0 + h) - f(x_0)}{h} > 0$$

(Indeed, if $h > 0$, then $x_0 + h > x_0 \implies f(x_0 + h) \geq f(x_0)$), and if $h < 0$ then $x_0 + h < x_0 \implies f(x_0 + h) \leq f(x_0)$). Thus, $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \geq 0$
Conversely, if $f'(c) \geq 0$ $\forall c \in (a, b)$. Let $x, y \in (a, b)$, $x < y$. Wish to show $f(x) \leq f(y)$. Apply MVT $\implies \exists c \in (x, y)$ s.t.

$$f(y) - f(x) = \underbrace{f'(c)}_{\geq 0} \cdot \underbrace{(y - x)}_{>0} \geq 0$$

$$\implies f(y) \geq f(x)$$

□

5.5 Taylor's Approximation

Mean Value Theorem tells us that given a differentiable function $f : (a, b) \to \mathbb{R}$, $x_0 \in (a, b)$ fixed, we may write

$$f(x) = \underbrace{f(x_0)}_{\text{constant}} + \underbrace{f'(\underbrace{c_x}_{\text{dep. } x}) \cdot (\underbrace{x - x_0}_{\text{linear}})}_{\text{small if } x \text{ close to } x_0}, \qquad c_x \text{ in between } x \ \& \ x_0$$

Taylor's theorem generalises this idea to higher orders,

> **Q**: Whi is the intuitive notion of derivative (i.e. $f'(x_0)$ is the slope of the best linear approximation of f around x_0) correct?

By Taylor's Approximation, we can find a polynomial of degree n whose values best approximate f near x_0 among all polynomials of degree n (provided f is n times differentiable).

> **Notation**: If $f : (a, b) \to \mathbb{R}$ is a differentiable function, then $f' : (a, b) \to \mathbb{R}$ is another function (it need not be continuous!). If f' is again differentiable, we call $f^{(2)} = f''$ it's derivative.
>
> Analogously, we set $f^{(k)} = \left(f^{k-1}\right)'$.

Definition 5.3: Taylor Polynomials

If $f : (a, b) \to \mathbb{R}$ is n-times differentiable, and $x_0 \in (a, b)$, we call

$$\begin{aligned} P_n^{x_0}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x - x_0)^k \\ &= f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{f''(x_0)}{2} \cdot (x - x_0)^2 + \frac{f^{(3)}(x_0)}{6} \cdot (x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n \end{aligned}$$

the n^{th} *Taylor polynomial* of f at x_0.

> **Rmk**: The derivatives of $P_n^{x_0}(x)$ at x_0 are
>
> $$P_n^{x_0}(x_0) = f(x_0), \quad \left(P_n^{x_0}\right)'(x_0) = f'(x_0), \quad \left(P_n^{x_0}\right)^{(k)}(x_0) = f^{(k)}(x_0)$$

Theorem 5.3: Taylor

Suppose $f : (a, b) \to \mathbb{R}$ has $n + 1$ continuous derivatives. Then, one can write

$$f(x) = P_n^{x_0}(x) = R_n^{x_0}(x)$$

where $R_n^{x_0}(x)$ is defined by (1) is called the *remainder* (or error) term, satisfies

$$\lim_{x \to x_0} \frac{R_n^{x_0}(x)}{(x - x_0)^n} = 0$$

> **Rmk**: In fact, it can be shown that for each x, $\exists c_{n,x}$ between x & x_0, s.t.
>
> $$R_n^{x+0}(x) = \frac{f^{(n+1)(c_{n,x})}}{(n + 1)!} \cdot (x - x_0)^{n+1}$$

Proof. See Lebl's book, 4.3.2. □

Example 5.5

1. $f(x) = e^x$ is ∞-times diff. It's Taylor approximation at $x_0 = 0$ is given by

$$P_n^0(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

2. $f(x) = \sin x$, $x_0 = 0$.

$$f(0) = 0, \quad f' = \cos x, \quad f^{(2)} = -\sin x, \quad f^{(3)} = -\cos x, \quad f^{(4)} = \sin x, \dots$$

$$\text{at } x_0 = 0 : f'(0) = 1,\ 0,\ -1,\ 0,\ 1 \implies P_3^0(x) = 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2!} - 1 \cdot \frac{x^3}{3!} + 0 \cdot \frac{x^4}{4!} + 1 \cdot \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$\sin(0,1) = 0.09983341664$$
$$P_5^0(0,1) = 0.09983341666$$

5.6 Inverse Function Theorem

Recall that given $f : A \to B$, $A, B \subset \mathbb{R}$, a function $g : B \to A$ is an *inverse* for f if

$$f \circ g = id_B, \quad \text{and} \quad g \circ f = id_A$$

i.e.

$$f(g(y)) = y, \quad \forall y \in B$$
$$g(f(x)) = x, \quad \forall x \in A$$

When such a g exists, it can be shown that it is unique, and we denote it by f^{-1}[1]; we say f is invertible. A necessary and sufficient condition for f to be invertible is that f is bijective, that it, f is injective and surjective.

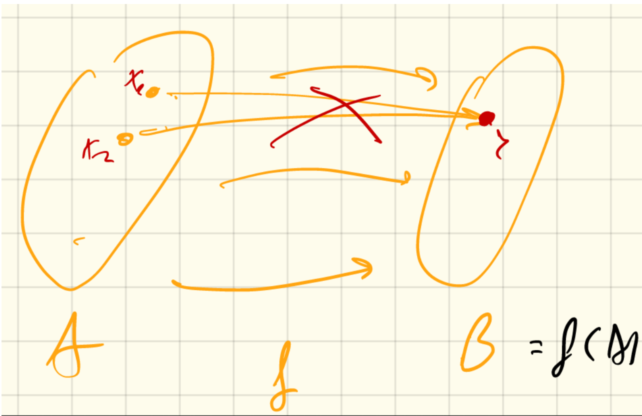


Figure 5.4: f is bijective

Example 5.6

Let $f : (a, b) \to \mathbb{R}$ be strictly monotone. Then, there is an inverse $f^{-1} : f((a, b)) \to (a, b)$ $(f(s) = \{f(s) : s \in S\})$. Because strict monotonicity implies injectivity, and surjectivity is ensured by co-restricting the f:

$$f : (a, b) \to f((a, b))$$

Q: Suppose $f : (a, b) \to (c, d)$ is cont./diff. and invertible, What can we say about f^{-1}?

Lemma 5.1

Let $f : (a, b) \to \mathbb{R}$ be a strictly monotone function (not necessarily continuous!). Then, $f^{-1} : f((a, b)) \to (a, b)$ is continuous.

[1] not to be confused with $\frac{1}{f}$

Definition 5.4

We say a function $f : (a, b) \to \mathbb{R}$ is of class $C^\perp$ (denoted by $f \in C'((a, b))$), if it is differentiable on (a, b) and $f'(x)$ is a continuous function.

Lemma 5.2

Let $f : (a, b) \to (c, d)$ be a strictly monotone ($\implies$ injective) surjective function. If f is differentiable at $x_0 \in (a, b)$, and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0) =: y_0$, and

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0)} = \frac{1}{f'(x_0)}$$

Rmk: If I knew that f^{-1} is differentiable at y, then by chain rule we would have

$$\begin{aligned} &f^{-1}(f(x)) = x \implies \\ &(f^{-1})'(f(x)) \cdot f'(x) = 1 \end{aligned}$$

Proof. WTS; $\lim\limits_{y \to y_0} \dfrac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \dfrac{1}{f'(x_0)}$.
We know: $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \neq 0 \implies$ by continuity of the function $t \to \frac{1}{t}$, $\lim\limits_{x \to x_0} \dfrac{x - x_0}{f(x) - f(x_0)} = \dfrac{1}{f'(x_0)}$.
Let's prove the first statement using sequences: let $\{y_n\}$ a sequence with $y_n \xrightarrow[n \to \infty]{} y_0$, $y_n \in (c, d) = f((a, b))$ $\implies y_n = f(x_n)$, $y_0 = f(x_0)$, $x_n \in (a, b)$. By previous lemma, f^{-1} is continuous $\implies$

$$\begin{aligned} f^{-1}(y_n) \xrightarrow[n \to \infty]{} f^{-1}(y_0) \implies \\ x_n \xrightarrow[n \to \infty]{} x_0 \end{aligned}$$

Using this in the second statement, we deduce

$$\frac{x_n - x_0}{f(x_n) - f(x_0)} \xrightarrow[n \to \infty]{} \frac{1}{f'(x_0)}$$

Finally,

$$\begin{aligned} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} &= \frac{f^{-1}f((x_n)) - f^{-1}(f(x_0))}{f(x_n) - f(x_0)} \\ &= \frac{x_n - x_0}{f(x_n) - f(x_0)} \xrightarrow[n \to \infty]{} \frac{1}{f'(x_0)} \end{aligned}$$

□

Corollary: Under the assumptions of the previous lemma, if moreover $f \in C^\perp((a, b)) \implies f^{-1} \in C^\perp((c, d))$.

Example 5.7

$$\arctan = \tan^{-1}, \quad \tan(x) = \frac{\sin(x)}{\cos(x)} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

bijective,

$$\begin{aligned} \tan'(x) &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + (\tan x)^2 > 0 \end{aligned}$$

By lemma, the derivative of arctan is given by

$$\arctan'(y) = \frac{1}{\tan'(x)} = \frac{1}{1+(\tan x)^2}$$
$$= \arctan'(y) = \frac{1}{1+y^2}$$

Theorem 5.4: Inverse Function Theorem

Let $f:(a,b)\to\mathbb{R}$ be of class $C^\perp$, and let $x_0\in(a,b)$ be s.t. $f'(x_0)\neq 0$. Then, there exists some interval $I\subseteq(a,b)$, $x_0\in I$, s.t. $f|_I : I\to f(I) =: J$ is invertible, and then inverse $f^{-1}:J\to I$ is of class $C^\perp$ and satisfies
$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Rmk: The importance relies of the followign: from an infintesimal property at one point (+ some regularity), the theorem gives us the existence of a local inverse.

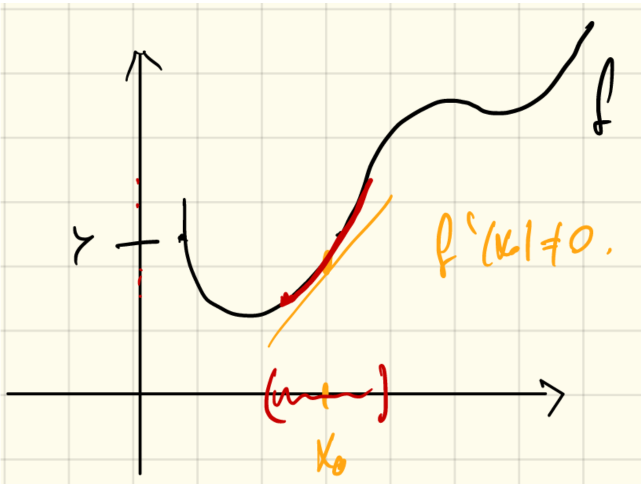


Figure 5.5: Sketch of inverse function theorem

Proof. Assume WLOG that $f'(x_0)>0$ (the other case is analogous). By continuity of $f'(x),\exists\delta>0$ s.t. $f'(x)>0\forall x\in(x_0-\delta,\,x_0+\delta)$. $I:=(x_0-\delta,\,x_0+\delta)$. $f|_I$ is thus strictly monotone increasing and continous. The imagine $f(I)$ is also an interval by the intermediate value theorem.
The, we can apply the previous corollary, and conclude that f^{-1} exists and is of class $C^\perp$. The formula for $(f^{-1})'$ follows from the previous lemma. □

Rmk: The fact that $f\in C^\perp$ is crucial. For instance, $f(x)=\begin{cases} x+2x^2\sin(1/x), & x\neq 0\\ 0 & x=0\end{cases}$ is differentuable on $\mathbb{R}$, $f'(0)>0$. But, f has no local inverse around 0.

6 Riemann Integral

Do not confused the *integral* and *antiderivative* - the integral is (informally) the area under the curve, and nothing else. The fact we may compute the antiderivative using the integral is a result we will prove later.

6.1 Partitions and lower and upper integrals

Definition 6.1: Partitions

A partition P of the interval $[a, b]$ si a finite set of numbers $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

We write

$$\Delta x_i := x_i - x_{i-1}$$

Definition 6.2: Darboux sums

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let P be a partition of $[a, b]$. Define

$$\begin{aligned} m_i &:= \inf \{f(x) : x_{i-1} \leq x \leq x_i\} \\ M_i &:= \sup \{f(x) : x_{i-1} \leq x \leq x_i\} \\ L(P, f) &:= \sum_{i=1}^{n} m_i \Delta x_i \\ U(P, f) &:= \sum_{i=1}^{n} M_i \Delta x_i \end{aligned}$$

We call $L(P, f)$ the *lower Darbouc sum* and $U(P, f)$ the *upper Darboux sum.*

Figure 6.1 shows an example of the Darboux sum. The width of the ith rectangle is Δx_i, the height of the shaded rectangle is m_i and the height of the entire rectangle is M_i.

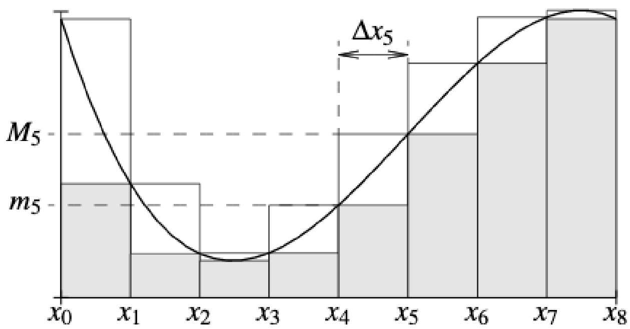


Figure 6.1: Sample Darboux sums. Shaded area is lower Darboux, unshaded is upper

Proposition 6.1

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all x we have $m \leq f(x) \leq M$. For any partition P of $[a, b]$, we have

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$$

See pg 164 of Lebl for proof.

Definition 6.3: Darboux Integrals

As the sets of lower and upper Darboux sums are bounded, we define

$$\underline{\int_a^b} f(x)\,\mathrm{d}x := \sup\{L(P,f) : P \text{ a partition of } [a,b]\}$$
$$\overline{\int_a^b} f(x)\,\mathrm{d}x := \inf\{U(P,f) : P \text{ a partition of } [a,b]\}$$

We call $\underline{\int}$ the *lower Darboux integral* and $\overline{\int}$ the *upper Darboux integral.*

For the *integral* to make sense, we will need the upper and lower Darboux integrals to be equal. This is not always true.

Example 6.1

Take the Dirichlet function $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$. Then,

$$\underline{\int_0^1} f = 0 \quad \text{and} \quad \overline{\int_0^1} f = 1$$

Occurs because that for every i we have $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0$ and $M_i = \sup\{f(x); x \in [x_{i-1}, x_i]\} = 1$. Thus, the lower sum $= 0$ and the upper sum $= 1$.

Definition 6.4: Refinement

Let $P := \{x_0, x_1, \ldots, x_n\}$ and