Advanced Multivariate Calculus & Ordinary Differential Equations (MATH1072) Lecture Notes

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Chapter 1

Ordinary Differential Equations

1.1 Lecture 1 - Introduction to Dimensional Analysis

To descrive real systems quantitatively, we use numbers and units of measurement. Eg. 3 meters, 5 years, 10 km/h. Each measurable quantity has a certain dimension.

1.1.1 Base dimensions

Length (L), time (T), mass (M), other non-mechanical base dimensions include temperature; electric charge/current etc.

1.1.2 Derived dimensions

Speed $\frac{L}{T}$, force ($N = M\frac{L}{T^2}$), energy ($J = M\frac{L^2}{T^2}$). The dimensions of the terms added on both sides must be equal. This is known as the equation being *dimensionally homogenous*. We can use the dimensional homogeneity to make dimensional estimates for certain quantities (order of magnitude, not exact prediction)

1.2 Lecture 2 - Dimensional Analysis

1.3 Lecture 3 - Introduction to Differential Equations

Generally, an ordinary differential equation (ODE) is represented as:

$$F(t, y(t), y'(t), y''(t), ...) = 0$$

For Instance, Newton's Law

$$m\frac{d^2r}{dt^2} = F$$

Induction Law:

$$RI + L\frac{dI}{dt} + \frac{1}{c}$$

Population:

$$\frac{dP}{dt} = rP(1 - \frac{P}{k})$$

Maxwell's Equations:

$$\nabla \cdot \bar{E} = \frac{1}{\rho}\varepsilon_0$$
$$\nabla \cdot \bar{B} = 0$$
$$\nabla \times \bar{E} = \frac{\delta B}{\delta t}$$
$$\nabla \times \bar{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\delta E}{\delta t}$$

Navier-Stokes: (Modelling velocity of fluids in space)

$$\frac{\delta \overline{v}}{\delta t} = \overline{v} \nabla \overline{v} = ?$$

Schrodinger Wave Equation:

$$i\hbar \frac{\delta \psi}{\delta t} = \Big[\frac{-\hbar}{2m}\nabla^2 + V(r)\Big]\psi$$
$$\psi = ?$$

We generally have:

$$F(t, y(t), y'(t), ...) = 0, \ y(t) = ?$$

Where the order of ODE = order of the highest derivative. Take the equation $y' = y$, the solutions are $y = e^t$, $y = 0$ and $y = ce^t$. However the latter expression encapsulates the former, so $y(t) = ce^t$ is known as the general solution where $c \in \mathbb{R}$. The general solution is not unique.

If you take an ODE and add some initial condition, then the solution is a unique result. For Instance take the ODE $y' = y$ and say that $y(0) = 1$, then we achieve a unique solution of $c = 1$, $y = e^t$.

1.4 Lecture 4 - Ordinary Differential Equations

1.4.1 Equilibrium Solutions

An equilibrium solution (steady state solution) of an ODE (if such solution exists) is a constant solution $y(t) = c$ which satisfies the ODE (for any t).

Example 1. Take $y' = f(t, y)$, the equilibrium solution

$$f(t, y = c) = 0$$

This implies that the slope at $y = c$ must be 0 for a function defined on a (t, y) plane

Example 2. $y' = y$, $y = 0$ is an equilibrium solution.

Example 3. $y' = y(1 - y)$m $y = 0$ and $y = 1$ are equilibrium solutions

Slope fields can visualised with Mathematica using StreamPlot (or VectorPlot). The vector flow of a field corresponding to $f(t, y)$ is $\{1, f(t, y)\}$ since the flow in the horizontal direction (t) has a constant rate (the flow of time) which can be set to 1, the vertical flow is $f(t, y)$.

1.4.2 Stability of equilibrium solutions

If the solutions starting in condition a small neighbourhood of an equilibrium solution $(y = c)$ converge towards the equilibrium solution for large t, then the equilibrium solution $y = c$ is stable.

Example 4. $y' = y$, the equilibrium solution $y = 0$ is unstable (for $y' = -y$, $y = 0$ is stable)

Example 5. $y' = y(1 - y)$, $y = 0$ is unstable, but $y = 1$ is stable.

1.5 Lecture 5 - Stability of ODE's

1.5.1 Condition for stability of equilibrium soltuions

Using a Taylor Series approximation of $f(t, y)$ near the equilibrium solution $y = c$, assume $f(t, y) = f(y)$ for simplicity

$$f(y) \approx f(c) + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c)^2 + ...$$

Where $y = c$ is the equilibrium solution $f(c) = 0$, thus

$$y'(t) = 0 + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c) + ...$$

Take $u = y - c$, then

$$u' = f'(c)u + ...$$

Implying

$$u(t) = Ae^{f'(c)t}$$

If $f'(c) > 0$, $y = c$ is unstable. If $f'(c) < 0$, $u(t) \to 0$, $y(t) \to c$ thus $y = c$ is stable.

1.5.2 Euler's Method

An iterative operation which models $y_k \approx y(k \cdot \Delta t)$. Given $y' = f(t, y)$, $y(0) = c$

$$y'(t) \equiv \lim_{\Delta \to 0} \frac{y(t+\Delta) - y(t)}{\Delta} \tag{1.1}$$

$$\approx \frac{y(t+\Delta) - y(t)}{\Delta} \tag{1.2}$$

As it is an iterative method, $y_{k+1} = y_k + f(t_k, y_k)$

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta} \approx f(t_k, y_k)$$

Example 6. Given $y' = 2t$, $y(t) = ?$, $y(0) = 0$. By Euler's Method

$$N = 1,\ \Delta = 1 \tag{1.3}$$

$$N = 2,\ \Delta = \frac{1}{2} \tag{1.4}$$

$$N = 4,\ \Delta = \frac{1}{4} \tag{1.5}$$

$$N = 10\ \Delta = \frac{1}{10} \tag{1.6}$$

1.6 Lecture 6 - Continuation of Euler's Method

1.6.1 Error Generated in Euler's Method

Assume $y(t_k) = y_k$, the error is represented with the taylor series approximation

$$\Big| y_{k+1} - y(t_k + \Delta) \Big| = y(t_{k+1}) = y(t_k) + y'(t_k)\Delta + \frac{1}{2}y''(t_k)\Delta^2 + ...$$

$$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

Thus

$$\text{error} = |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2$$

However generalized for N steps,

$$N \cdot |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2 \cdot N$$

$$\propto \Delta^2 \cdot \frac{1}{\Delta} \propto \Delta$$

$$y_{k+1} = y_k + \frac{1}{2}\Big[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\Big]\Delta$$

1.7 Lecture 7 - Solving Linear First Order ODEs

$$y' = f(t, y), y(0) = y_0$$

$$t \in [0, t_{max}]$$

1. Euler's Method

   $$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

   Where total error $\propto \Delta$

2. Heun Method (Revised Euler's Method)

$$\begin{cases} y_{k+1} = y_k + f(t_k, y_k)\Delta \\ y_{k+1} = y_k + \frac{1}{2}[f(t_k, y_k) + f(t_k + \Delta, y_{k+1})] \end{cases}$$

Where total error $\propto \Delta^2$. Note that Heun Method will possibly show in Assignment 3.

1.7.1 Runge Kutta Method

$$p_1 = f(t_k, y_k)\Delta$$
$$p_2 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_1}{2})\Delta$$
$$p_3 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_2}{2})\Delta$$
$$p_4 = f(t_k + \frac{\Delta}{2}, y_k + p_3)\Delta$$

for

$$y_{k+1} = y_k + \frac{1}{6}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 + \frac{1}{6}p_4 + ...$$

1.7.2 Adaptive Step Size

Fixed step size is mostly inefficient in most cases, so we use an adaptive step size for numerical methods to achieve better approximations.

1.7.3 Coupled Systems

$$y' = f(t, y_1, y_2)$$
$$x' = g(t, y_1, y_2)$$
$$x(t) =?\ y(t) =?$$

In the context of Euler's Method

$$\begin{cases} x_{k+1} = x_k + f(t_k, x_k, y_k)\Delta \\ y_{k+1} = y_k + g(t_k, x_k, y_k)\Delta \end{cases}$$

1.7.4 Analytical ODE Solutions

Linear First Order ODE's

By definition,

$$y(t) = f(t)y(t) + g(t)$$

is generally the standard form of a linear first order ODE. Note that

$$y'(t) = f(t)y(t)$$

is a special case of a linear first order ODE that is separable.

Example 7.

$$ty' + y = t\cos t$$

Observe that, by the product rule for differentiation, $ty' + y = (ty)'$.

$$(ty)' = t\cos t$$
$$\int (ty)'dy = \int t\cos t dt$$
$$ty = t\sin t - \int \sin t dt$$
$$ty = t\sin t + \cos t + c$$
$$\therefore y = \frac{t\sin t + \cos t}{t} + \frac{c}{t}$$

1.8 Lecture 8 - Applications of First Order ODEs

Common applications of first order ODEs are

1.8.1 Radioactive Decay

The particles of a radioactive material decay spontaneously in a stochastic process. The total mass of the radioactive atoms decrease with time. We can represent this as

$$\frac{dM}{dt} = -kM$$

For $M(t)$ to represent the mass of the radioactive material over time t. Clearly the solution to this is

$$M(t) = M_0 e^{-kt}$$

Note that there is a limitation to this ODE Model. We assume that the mass changes continuously in time. Whereas in reality it changes in discrete steps following individual decay events. However for the macroscopic mass, the number of particles is much larger, so we can neglect the discrete jumps and assume a continuous deterministic model. Which works well for most cases.

The lifetime of particles varies, however we can categorize them with their average lifetime. We do this by asking how long it takes for a particle to reduce to half of the initial value. To obtain a more accurate value for average lifetime, consider grouping the lifetime of particles into discrete "bins". If the number of particles with lifetime in $[t_j, t_j + \Delta]$ is N_j, then the average lifetimes is represented as

$$\sum N_j = N_0$$

$$\tau \approx \frac{\sum t_j N_j}{\sum N_j}$$

If the number of particles decreases exponentially

$$N(t) = N_0 e^{-kt}$$

Then the number of particles lost in an interval $[t_j, t_j + \Delta]$ is

$$N_j = N[t_j] - N[t_j + \Delta]$$

Or

$$N_j = \frac{N(t_j) - N(t_j + \Delta)}{\Delta} \cdot \Delta \approx -N'(t)\Delta$$

Taking $\Delta \to 0$. The sum for calculating the average lifetime turns into an integral.

$$\tau = \frac{1}{N_0} \int_0^\infty t(-N'(t))\, dt = \int_0^\infty te^{-kt}\, dt = \frac{1}{k}$$

1.8.2 Protein Synthesis and Degradation

Seriously who cares. I probably should write these notes though

1.9 Lecture 9 - Continuation of Protein Synthesis and Degradation

The concentration of a protein $C(t)$ that is synthesised at a constant rate S, k for the degradation rate.

$$\frac{dC}{dt} = S - kC$$

We can see that the equilibrium state is at $C = \frac{S}{k}$, the ODE is seperable, so the general solution is as follows

$$\int \frac{dC}{S - kC} = \int dt$$

And then... Poof! By the magic of Applied Mathematics, we achieve the general solution

$$C(t) = \frac{S}{k} + ce^{-kt}$$

Assume the initial value such that for $C(t = 0) = C_0$, then the general solution

$$C(t) = \frac{S}{k} + \Big(C_0 - \frac{S}{k}\Big)e^{-kt}$$

1.10 Lecture 10 - Heat Transfer

The heat flow (energy transferred per unit time) between two object of different temperature is proportional to the temperature difference between the two object (the heat flows from the higher to the low temperatures until it equilibrates). Thus the rate of change of the body temperature is proportional to the temperature difference between the body and the environment T_{env} (air).

$$\frac{dT}{dt} = -k(T - T_{env})$$

$$T(t) = T_{env} + (T_0 - T_{env})e^{-kt}$$

Setting $t = 0$ solves for T_0 and T_{env} should be predefined. The parameter k is a proportionality constant and is difficult to represent with a model (at this level)

1.10.1 Electric circuit with resistance and inductance

R resistance, L inductance in series, $E(t)$ external voltage source = the sum of the voltage drops over the resistance + the inductance. The ODE for the current $I(t)$,

$$E(t) = RI + L\frac{dI}{dt}$$

1.10.2 Population Dynamics

The rate of change of the population $P(t)$ is proportional with the actual current population

$$\frac{dP}{dt} = kP$$

,

$$P(t) = P_0e^{kt}$$

The k parameter in this model is the cell division rate constant. Ideally, bacteria can divide every 20 minutes.

1.11 Lecture 11 - Continuation of Population Dynamics

To balance out the growth of population due to reproduction we can also include a loss term due to death. The number of individuals dying per unit time is also proportional to population size P

$$\frac{dP}{dt} = kp - dP = (k - d)P = \rho P$$

$$P(t) = P_0e^{\rho t}$$

This modified population model still doesn't have a stable equilibrium state. The solution grows or decays exponentially depending on the sign of $\rho = k - d$ which is the **net reproduction rate constant**. It is expected that a population should stabilise after some time in a stable equilibrium; where repoduction and death balance out.

The thing missing from this model is that we assumed k and d are constant parameters, but the birth and death rates may be dependant on several external factors (e.g. food, habitat) and this may be dependent on the size of the population r, k (or ρ) are functions of P.

Thus a modified nonlinear population model

$$\frac{dP}{dt} = \rho(P)P$$

Where the exact form of the function $\rho(P)$ depends on the problem in question that we want to model. In general $\rho(P)$ is a decreasing function (less resources available when the population increases, which slows down reproduction and/or increases death rate).

The simplest functional form for a decreasing $\rho(P)$ is a linear function

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P$$

This is known as the **Logistic Equation**, where ρ is a constant (maximum net reproduction rate), K is the carrying capacity (the maximum sustainable population size). The net reproduction changes sign from positive to negative when $P = K$. The equilibrium solutions of this model are

- $P^* = 0$; unstable
- $P^* = K$; the derivative of RHS at $P = K$ is < 0 implying stable equilibrium state.

The exact solution to this ODE (seperable ODE) with initial condition $P(0) = P_0$ is

$$P(t) = K\frac{1}{1 + \Big(\frac{K}{P_0} - 1\Big)e^{-\rho t}}$$

For a long time $t \to \infty$, the solution $P(t)$ converges to the stable equilibrium. Assume that P describes a fish population and there is an additional loss rate due to fishing.

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P - f$$

f is a constant parameter; the amount of fish harvested per unit time.

Question:

How does the fishing rate f modify the state equilibrium of the population? How should the function $P^*(f)$ look graphed?. Is there any qualitative (dramatic) change of the equilibrium as f is modified or only a smooth transition; are there any bifuractions?

1.12 Lecture 12 - ODE Models of Population Dynamics

1.12.1 Bifuraction

Consider the Differential equation $\frac{dy}{dt} = f(y;p)$ where p represents constant parameters. The equilibrium solutions are the roots of the equations $f(y) = 0 \implies \frac{dy}{dt}$ which will depend on the parameters. Bifuraction diagrams are qualitative changes of the solutions happening when a parameter p is varied, i.e the change in the number of stability type of the equilibria. A **bifuraction diagram** plots and shows the solutions of branches $y^*(p)$.

1.12.2 Logistic populations dynamics

Consider the equation

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P$$

Introduce non-dimensional variables for P and t in order to reduce the number of parameters in the problem. Choose $P' = \frac{P}{K}$ and $t' = \rho t$.

$$\frac{dP'}{dt'} = (1 - P')P'$$

The non-dimensional problem does not have any parameters. This shows that we can't have any bifuractiosn when we modify K, and ρ as those are all mathematically equibva;lent problems they can only differ by strecthing or compeessing the axis P and t. Now consider the equation

$$\frac{dP}{dt} = \rho\Big(1 - \frac{P}{K}\Big)P - F$$

Introduce the same non-dimensional variables for P and t; ($P' = \frac{P}{K}$ and $t' = \rho t$).

$$\frac{dP'}{dt'} = (1 - P')P' - \frac{F}{\rho K}$$

Denoting $\dfrac{F}{\rho K} = F'$, F' cannot be elimated by using non-dimensional variable, so the problem has 1 real parameter $\implies$ it may have qualitatively different solutions when F' is varied.

1.12.3 Harvesting at a constant rate

$$\frac{dP'}{dt'} = (1 - P')P' - F'$$

How does the equilibria change when F' is varied? ($P'^*(F') =?$). Solve it graphically by following the intersections of the two terms on the RHS as F' is varied?

1.13 Lecture 13 - Systems of Coupled 1st Order ODEs

$$y_1' = f(y_1, y_2),\ y_2' = g(y_1, y_2)$$

$y_1(t) =?$ and $y_2(t) =?$. There is no general method for finding solutions analytically for coupled systems of ODEs (except linear systems), but often the important information is related to the equilibrium states of the system. The equilbrium solutions (y_1^*, y_2^*) are the solutions of the algebraic system of simultaneous equations:

$$f(y_1, y_2) = 0,\ g(y_1, y_2) = 0$$

1.13.1 Dynamics of interacting populations

Example 8. Prey (P) and predator (R) population dynamics.

$$\frac{dP}{dt} = \rho P\Big(1 - \frac{P}{K}\Big) - aPR$$
$$\frac{dR}{dt} = bPR - dR$$

Given $\rho P\Big(1 - \dfrac{P}{K}\Big)$ is the prey without predator, aPR is the prey-predator interaction and $bPR - dR$ is the death of predator.

Non-dimensionalize the sytem by introducing new dimensionaless variables $P' = \dfrac{P}{K}$, $t' = \rho t$ (done similarly for the logistic equation) and choose $R' = \dfrac{a}{\rho}R$.

$$\frac{dP'}{dt'} = P'(1 - P') - P'R'$$
$$\frac{dR'}{dt'} = \frac{bK}{\rho}P'R' - \frac{d}{\rho}R'$$
$$\frac{dR'}{dt'} = \alpha P'R' - \beta R'$$

The non-dimensionalization shows that the behaviour of the solution can only depend on the two new parameters $\alpha = b\dfrac{K}{\rho}$ and $\beta = \dfrac{d}{\rho}$

1.14 Lecture 14 - Second Order Differential Equations

$$f(y'', y', y, t) = 0,\ y(t) =?$$

Example 9. Newton's Law $F = ma$, where a is the acceleration. (second derivative of coordinate function $a = x''(t)$)

Many partial differentials contain second derivates over spatial coordinates. There is no general way to solve nonlinear second order ODEs analytically, however we can solve them numerically.

To solve numerically, we can rewrite it into the form of two coupled first order ODEs by introducing an unknown function defined as $y' = v$. Then $v' = F(v, y, t)$ and $y' = v$ forms a coupled system, needs to be complemented with initial conditions $y(t = 0)$ and $v(t = 0) = y'(t = 0)$

1.14.1 Solving linear 2nd order ODEs analytically

Some types of linear 2nd order ODEs may be solved analytically. A general form of a analytically solveable 2nd order ODE may be

$$y'' + p(t)y' + q(t)y = r(x)$$

Can be classified as homogenous if $r(x) = 0\ \forall x$. Constant coefficients if $p(t)$ and $q(t)$ are constants. For homogenous ODEs ($r(x) = 0$), we can use the **Principle of superposition** for construction the general solution.

If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the homogenous linear 2nd order ODE then any linear combination $y(t) = c_1$. What?

1.14.2 Method of Reduction of Order

Assume that we have a solution $y_1(t)$ that satisfies a linear homogenous ODE $\implies$ then the reduction of order method leads to a problem of a lower order ODE for finding the other/general solution. The steps for solving with reduction of order method is as follows

1. $y'' + p(t)y' + q(t)y = 0$
2. Assume that $y_1(t)$ is a solution.
3. Look for solutions of the form $y = u(t)y_1(t)$
4. Substitute into the equation.
5. $(uy_1)'' + p(t)(uy_1)' + q(t)uy_1 = 0$

1.15 Lecture 15 - Solving 2nd Order ODES

Suppose we have a linear 2^{nd} order ODE $f(y'', y', y, t) = 0$. Suppose we are solving for the general solution $y(t)$. If f is homogenous and constant-coefficient, the ODE is of the form $y'' + ay' + by = 0$. As a function that remains the same (same for a multiplicative constant) is e^t, we can guess $y(t) = e^{\lambda t}$ is a solution for some λ.

$$\begin{aligned}
y'' + ay' + b &= (e^{\lambda t})'' + a(e^{\lambda t})' + be^{\lambda t} \\
0 &= \lambda^2 e^{\lambda} t + \lambda a e^{\lambda t} + be^{\lambda t} \\
0 &= e^{\lambda t}(\lambda^2 + a\lambda + b)
\end{aligned}$$

Note that $\lambda^2 + a\lambda + b = 0$ is the characteristic equation of an ODE

$$\implies \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Thus $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are linearly independent solutions and the general solution is $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ for some $c_1,\ c_2$. However, this is only the case if $\lambda_1 \neq$ something ($4b < a^2$).

If $4b = a^2$, then $\lambda_1 = \lambda_2 = \lambda$ and we obtain only one solution $\lambda^{\lambda t}$. With this, we can use reduction of order. As $y_1(t) = e^{\lambda t}$ is a solution, let $y(t) = u(t)e^{\lambda t}$. Thus $y'' + ay' + by = 0$

$$\implies (ue^{\lambda t})'' + a(ue^{\lambda t})' + bue^{\lambda t} = 0$$

$$\begin{aligned}
(u'e^{\lambda t} + \lambda ue^{\lambda t})' + a(u'e^{\lambda t} + \lambda ue^{\lambda t}) + bue^{\lambda t} &= 0 \\
u''e^{\lambda t} + \lambda u'e^{\lambda t} + \lambda u'e^{\lambda t} + \lambda^2 ue^{\lambda t} + au'e^{\lambda t} + a\lambda ue^{\lambda t} + bue^{\lambda t} &= 0 \\
u'' + 2\lambda u' + \lambda^2 u + au' + a\lambda u + bu &= 0
\end{aligned}$$

However note that $\lambda = \dfrac{-a}{2}$, $4b = a^2 \implies b = \dfrac{a^2}{4}$

$$u'' - au' + \frac{a^2}{4} + au' - \frac{a^2}{2}u + \frac{a^2}{4}u = 0$$

$$\implies u'' = 0$$
$$\implies u(t) = c_1 t + c_2$$

for some c_1, c_2. Thus the general solution is

$$y(t) = u(t)e^{\lambda t} = (c_1 t + c_2)e^{\lambda t}$$

If we obtain no solutions ($4b > a^2$) in the real set, we get two complex conjugate roots.

$$\lambda_{1,2} = \frac{-a}{2} \pm \frac{\sqrt{4b - a^2}}{2} i = \alpha \pm \beta i$$

We get the general solution

$$\begin{aligned} y(t) &= c_1 e^{(\alpha+\beta i)t} + c_2 e^{(\alpha-\beta i)} \\ &= c_1 e^{\alpha t} e^{\beta i t} + c_2 e^{\alpha t} e^{-\beta i t} \end{aligned}$$

However, $e^{i\beta t} = \cos(\beta t) + i\sin(\beta t) \implies y(t) = e^{\alpha t}(c_1\cos(\beta t) + c_1 i\sin(\beta t) + c_2\cos(-\beta t) + i\sin(-\beta t)$

$$y(t) = (c_1 + c_2 e^{\alpha t}\cos(\beta t) + (c_1 - c_2)ie^{\alpha t}\sin(\beta t)$$

Let $A = c_1 + c_2$ and $B = (c_1 - c_2)i$

$$\begin{aligned} y(t) &= Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t) \\ &= e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t)) \end{aligned}$$

1.16 Lecture 16 - Constant-coefficient Non-Homogenous

Suppose we have an ODE $y'' + p(t)y' + q(t)y = r(t)$ and we are attempting to solve for $y(t)$. This is a constant coefficient non homogenous linear order 2^{nd} oder ODE. (if $p(t)$, $q(t)$ are constant). Let $L(x) = x'' + px' + qx$. Suppose we have two linearly independent solutions $y_1(t), y_2(t)$. Thus $L(y_1) = L(y_2) = r \implies L(y_1) - L(y_2) = 0$. As the ODE is linear $\implies$ L is linear, $L(y_1) - L(y_2) = L(y_1 - y_2) = 0$. This is a homogenous ODE and can be easily solved

Steps:

1. Solve the corresponding homogenous ODE ($r = 0$) $\to y_n = c_1 y_n + c_2 y_k$
2. Find solution y_p (reduction of order) $\to y_p = u(t)y_{k_1}$
3. General case is $y(t) = y_p + y_n$

Example 10. Solve for $y(t)$ given $y'' + y' - 2y = t^2 - 2t + 3$

1. Solve $y'' + y' - 2y = 0$, characteristic equation is $\lambda^2 + \lambda - 2 = 0$. Implying

$$(\lambda + 2)(\lambda - 2) = 0$$

Thus $\lambda_1 = -2$ and $\lambda_2 = 1$, therefore

$$y_n(t) = c_1 e^{-2t} + c_2 e^t$$

2. Reduction of Order: as $y_1(t) = e^t$ is a solution we have a linearly independent solution $y_p(t) = u(t)e^t$. Subbing in

$$\begin{aligned} (ue^t)'' + (ue^t)' - 2ue^t &= t^2 - 2t + 3 \\ u''e^t + 2u'e^t t + ut^2 e^t + u'e^t + ute^t - 2ue^t &= t^2 - 2t + 3 \\ u'' + 2u't + ut^2 + u' + u5 - 2u &= (t^2 - 2t + 3)e^{-t} \end{aligned}$$

$$\implies y(t) = c_1 e^{-2t} + c_2 e^t - \frac{t^2}{2} + \frac{t}{2} - \frac{7}{4}$$

1.17 Lecture 17 -

1.18 Lecture 18 - Applications of 2nd order ODEs

1.18.1 Mechanical/Electrical Ossiclators

SMALL ANGLE TIME

$$\sin(\theta) = \theta$$

$$m\frac{dy}{dt} = -mg + kdy$$

Example 11. Given a pendulum,

$$m\frac{d^2y}{dt^2} = -ky$$

$$y(0) = y_0,\ y'(0) = 0$$

$$my'' + ky = 0$$

$$y'' + \frac{k}{m}y = 0$$

Denote $a = \dfrac{k}{m}$

$$y'' + ay = 0 \implies \lambda^2 + a = 0$$

$$\therefore y = e^{\lambda t}$$

For $\lambda = \pm i\sqrt{a}$

$$y(t) = A\cos(\sqrt{a}t) + B\sin(\sqrt{a}t)$$

However with this model there is no dampining over time.

$$m\frac{d^2y}{dt^2} = -ky - \gamma y'$$

$$y'' + \left(\frac{\gamma}{m}\right)y' + \left(\frac{k}{m}\right)y = 0$$

Denoting $\dfrac{\gamma}{m} = b$ and $\dfrac{m}{k} = a$

$$y'' + by' + ay = 0$$

If $b^2 > 4a$, $\lambda_1, \lambda_2 < 0$, if $b^2 < 4a$, $\lambda_{1,2} = \dfrac{-b}{2} \pm i\sqrt{4a - b^2}$

Chapter 2

Multivariable Calculus

Foreword: From around this point onwards I began to lost track on the exact Lectures that these notes come from, so they are going to be referred by weeks.

Week 7

2.1 Introduction to Multivariate Calculus

2.1.1 Review of one-variable case

Let $f : D \to \mathbb{R}$ be a function with a domain D an open subset of $\mathbb{R}$. For $a \in D$ we say that the limit $\lim\limits_{x \to a} f(x)$ exists if and only if

1. The limit from the left exists
2. the limit from the right exists
3. these two limits coincide etc.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Furthermore if the limit exists and is equal to the actual value of f at a.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

We say that f is continuous at $x = a$. If f is continuous on all D, we say that f is continuous function on D

2.1.2 The two-variable case

When f is a function of more than one variable, the situation is more subtle. There are more than two ways to approach a given point of interest.

Example 12. Consider the function

$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

with domain given by $\mathbb{R}^2 \setminus (0, 0)$

Approaching the origin along $y = 0$, if $x \neq 0 f(x, 0) = \dfrac{x^2}{x^2 + 0} = 1$. Then

$$\lim_{x \to 0} f(x, 0) = 1$$

If $y \neq 0$, $f(0, y) = \dfrac{0}{0 + y^2} = 0$. Then,

$$\lim_{y \to 0} f(0, y) = 0$$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, as $\lim_{x\to 0} f(x,0) \neq \lim_{y\to 0} f(0,y)$. In general, for the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, it is neccessary that every parth in D approaching (a,b) gives the same limiting value ((a,b) may not neccesarily be inbluetooth usbc D). This gives a method for finding if a limit does not exist for multivariate limits.

$$\text{If } \begin{cases} f(x,y) \to L_1 \text{ as } (x,y) \to (a,b) \text{ along the path } C_1 \in D \\ f(x,y) \to L_2 \text{ as } (x,y) \to (a,b) \text{ along the path } C_2 \in D \end{cases}$$

Remark. The above notation is somewhat deficient and perhaps one should write

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

to indicate that only paths D terminating in (a,b) (which itself may or may not be in D) are considered. For instance, $f(x,y) = x^2 + y^2$ with $D = \{(x,y) : x^2 + y^2 < 1\}$, then $\lim_{(x,y)\to(1,0)} f(x,y)$ exists and is 1. However if

$$\begin{cases} x^2 + y^2 & \text{for } D = \{(x,y) : x^2 + y^2 < 1\} \\ 0 & \text{for } D = \{(x,y) : x^2 + y^2 > 1\} \end{cases}$$

then $\lim_{(x,y)\to(1,0)} f(x,y)$ does not exist.

2.2 Lecture 20 - Continuation of Functions of Multiple Variables

Generally, we write $\lim_{(x,y)\to(a,b)} f(x,y) = L$ to mean the values of $f(x,y)$ approach L as the point (x,y) approaches (a,b) along any path in the domain f. That is, we can make the value of $f(x,y)$ as close to L as we like by taking (x,y) sufficiently close to (a,b). This is formalised through the following definition

Definition 1. Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b).

Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$, $\exists\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$. Where $|f(x,y) - L|$ can be described as the distance between $f(x,y)$ and L in $\mathbb{R}$. The $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between (x,y) and (a,b) in $\mathbb{R}^2$. The definition is essentially saying that the distance between $f(x,y)$ and L can be made arbitrarily small by making the distance between (x,y) and (a,b) sufficiently small, **but not 0**.

2.3 Lecture 23 - Partial Derivatives

Consider the surface $z = f(x,y) = 1 - x^2 - y^2$ and the point $P = (1,-1,-1)$ on the surface. Use the "y is constant" cross-section through P to find the slope in the x-direction at P. The slope in the x-direction with y help fixed, is called the **partial derivative** of f with respect to x at the point (a,b).

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly for the slope in the y-direction, with x held fixed, is called the **partial derivative** of f with respect to y at point (a,b).

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Example 13. Find the partial derivative of $f(x,y) = x\sin y + y\cos x$

$$f(x,y) = x\sin y + y\cos x$$

$$\frac{\partial f}{\partial x} = \sin y - y\cos x, \quad \frac{\partial f}{\partial y} = x\cos y + \cos x$$

Higher order of partial derivatives of f (given they exist) would look something like

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

We also have this thing

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big)$$

Pretty trivial. Note that if f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Example 14. It is possible that for some point P, the partial derivatives are well defined but the function is not continuous at P. Consider

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{xy} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

$$\begin{aligned} &\lim_{x\to 0} f(x,0) = 0 \\ &\lim_{x\to 0} f(x,x) = \frac{1}{2} \neq \lim_{x\to 0} f(x,0) \end{aligned}$$

So $f(x,y)$ is not continuous at $(0,0)$

$$\begin{aligned} \frac{\partial f(0,0)}{\partial x} &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= 0 \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} \\ &= 0 \end{aligned}$$

2.4 The Chain Rule

For one variable functions, for instance $y = f(u)$ and $u = g(x)$, we can use the (one-variable) chain rule to compute $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{or} \qquad y'(x) = f'(u)g'(x).$$

An example:

Example 15. Find $y'(x)$ where $y(x) = (x^2+1)^5$

Let $u = x^2 + 1 \implies \frac{du}{dx} = 2x$, therefore $y = u^5$ and thus

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4 \cdot (2x) = 10x(x^2+1)^4$$

2.4.1 Chain rule for $f(x,y)$

The chain rule for $f(x,y)$, for x and y being functions of t is usually written as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Sometimes $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are written as f_x and f_y respectively. We can derive this equation.

Given $f(x,y)$ with $x(t) = x$ and $y(t) = y$,

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}$$

Assuming f is "smooth" and Δf is small, we can make this following approximation

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

Furthermore

$$\frac{\Delta f}{\Delta t} \approx f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t}$$

As $\Delta t \to 0$ assuming $x(t), y(t)$ are smooth, we obtain

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

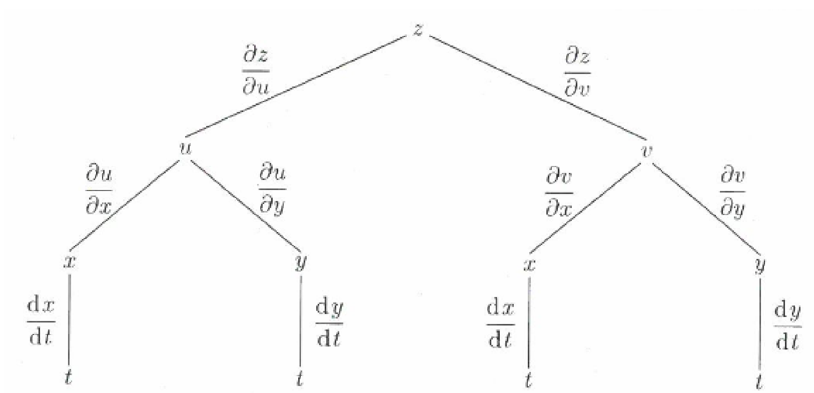
Note 1. You can extrapolate this definition to any number of dimensions. Given a function $f(a_1(t), a_2(t), a_3(t) \ldots, a_n(t))$, you can represent its derivative as

$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial a_i} \frac{da_i}{dt} = \frac{\partial f}{\partial a_1} \frac{da_1}{dt} + \frac{\partial f}{\partial a_2} \frac{da_2}{dt} + \cdots + \frac{\partial f}{\partial a_n} \frac{da_n}{dt}$$

Where n is the highest dimension of the function f.

2.5 Extended Chain Rules with Tree Diagrams

For more complex functions, such as $z = f(u(x(t), y(t)), v(x(t), y(t))$ (sub-sub-functions). We can use a tree diagram representation to figure out the chain rule.



So we have

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial u}\Big(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\Big) + \frac{\partial z}{\partial v}\Big(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\Big) \\ &= \Big(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\Big)\frac{dx}{dt} + \Big(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\Big)\frac{dy}{dt}\end{aligned}$$

2.6 Tangent Planes

2.6.1 Equations for a tangent plane

Generally for $z = f(x,y)$ at $(a, b, f(a,b))$, we define the **tangent plane** to be

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Or

$$(x,y,z) = (a, b, f(a,b)) + \lambda(1, 0, f_x(a,b)) + \mu(0, 1, f_y(a,b)), \quad \lambda, \mu \in \mathbb{R}$$

Note that $\lambda = x - a$ and $\mu = y - b$ in the latter expression, you can substitute to check.

2.6.2 Directional Derivative

Define

$$f_u(x,y) = \lim_{h\to 0} \frac{f(x+hu, y+hu_2) - f(x,y)}{h}$$

Where $||u|| = 1$. Set $x = a + hu_1$, $y = b + hu_2$ such that $\lim_{h\to 0}(x,y) = (a,b)$ and set $z(h) = f(x(h), y(h))$

From the chain rule

$$\begin{aligned}\frac{dz}{dh} &= \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} \\ &= \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot u\end{aligned}$$

Example 16. Compute the directional derivative of $f(x,y) = 4 - x^2 - 4y^2$at (3,-1) in the (1,1) direction.

$$\begin{aligned}f(x,y) &= f - x^2 - 4y^2 \\ f_x(x,y) &= -2x \\ f_y(x,y) &= -8y \\ f_{(1,1)}(3,-1) &= (f_x(3,-1), f_y(3,-1))\cdot\frac{(1,1)}{||(1,1)||} \\ &= \frac{1}{\sqrt{2}}(-6,8)\cdot(1,1) = \sqrt{2}\end{aligned}$$

2.6.3 Gradient vector ∇f

The gradient vector of f is a vector with partial derivative compnents

$$\nabla f = (f_x, f_y) = f_x i + f_y j$$

Example 17. Find the gradient of $f(x,y) = x^2 - 3(y-1)^2 + 3$:

$$\begin{aligned}\nabla f &= f_x i + f_y j \\ &= 2xi - 6(y-1)j\end{aligned}$$

The gradient $\nabla f(a,b)$ is perpendicual to the countour line throguh (a,b) and in points increasing f. The direction/magnitude of steepest slope at (a,b) are given by $\nabla f(a,b)$ and $||\nabla f(a,b)||$. We can understand these two facts by

2.7 Differentials

2.7.1 Estimating error

If the error in x is most E_1 and in y is at most E_2 then a reasonanle estimate of the worst case error in the linear approximation of f at (a,b) is

$$|E| \approx |f_x(a,b)E)_1| + |f_y(a,b)E_2|$$

Example 18. Suppose when making up a barrel ofbase radius 1m and height 2m, you allow an error of 5% in radius and height. Estimate the worst case error in volume

$$V(r,h) = \pi r^2 h \;, V(1,2) = 2\pi$$
$$V_r(r,h) = \pi r h,\; V_r(1,2) = 4\pi$$
$$V_h(r,h) = \pi r^2,\; V_h(1,2) = \pi$$

$$E_r = \frac{5}{100} r = \frac{5}{100},\; |E_h| = \frac{5}{100} \cdot 2 = \frac{10}{100}$$

$$\begin{aligned} |E| &= |V_r(1,2)E_r| + |V_h(1,2)E_h| \\ &= 4\pi \cdot \frac{5}{100} + \pi \cdot \frac{10}{100} = \frac{30\pi}{100} \end{aligned}$$

Now, since $V(1,2) = 2\pi$ the percentage error is $\frac{|E|}{V(1,2)} = \frac{30\pi}{100} \cdot \frac{1}{2\pi} = \frac{15}{100} \equiv 15\%$. EXACT WORSE CASE SCENARIOS. $r = 1.05,\; h = 2.1,$

$$\frac{V(1.05, 21)}{V(1,2)} = 1.1576$$

, $r = 0.95,\; h = 1.9,$

$$\frac{V(0.95, 1.9)}{V(1,2)} = 0.8579...$$

2.7.2 Quadratic Approximation

Let $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$. Then

$$Q(a) = f(a) \tag{2.1}$$
$$Q'(x) = f'(a) + f''(a)(x-a) \tag{2.2}$$
$$Q'(a) = f'(a) \tag{2.3}$$
$$Q''(x) = f''(a) \tag{2.4}$$
$$Q''(a) = f''(a) \tag{2.5}$$
$$\tag{2.6}$$

Eqs (2.1), (2.3) and (2.6) show that $f(x)$ and $Q(x)$ are equal at $x = a$ as are their first and second derivatives.

Example 19. Compute $e^{0.1}$ using a linear and quadratic approximation

For $f(x) = e^x$, we have the quadratic approximation about $a = 0$,

$$\begin{aligned} Q(x) &= f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 \\ &= 1 + x + \frac{1}{2}x^2 \end{aligned}$$

Then $e^{0.1} = f(0.1) \approx Q(0.1) = 1.105$

Example 20. In the same graph, sketch $f(x) = \cos x$ as well as its linear and quadratic approximations around 0. (Obviously can't do sketch real time in LaTeXbut take photo of graph or watch lecture recording at 9:21am (21 mins)) Given $f(x) = \cos x,\; f'(x) = -\sin x,\; f''(x) = -\cos x,\; f(0) = 1,\; f'(0) = 0,\; f''(0) = -1.$

$$\text{Linear approximation, } L(x) = f(0) + f'(0)x = 1$$

$$\text{Quadratic approximation, } Q(x) = f(0) + f'(0) + \frac{1}{2} f''(0)x^2 = 1 - \frac{1}{2}x^2$$

2.7.3 Quadratic approximations

$$Q(x,y) = c + mx + ny + Ax^2 + Bxy + Cy^2$$

such that $Q(a,b) = f(a,b)$ and such that all first order and second order partial derivatives of f and Q agree at (a,b). To verify

$$\begin{aligned} Q(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2 \end{aligned}$$

Example 21. Check that $Q_{xx}(a,b) = f_{xx}(a,b)$ and $Q_{xy}(a,b) = f_{xy}(a,b)$

$$\begin{aligned} Q_x(x,y) &= f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b) \\ Q_{xx}(x,y) &= f_{xx}(a,b),\ Q_{xx}(a,b) = f_{xx}(a,b) \\ Q_{xy}(x,y) &= f_{xy}(a,b),\ Q_{xy}(a,b) = f_{xy}(a,b) \end{aligned}$$

A more specific example. Bruh moment.

Example 22. Find the quadratic approximation around $(0,0)$ of

$$f(x,y) = 1 - x^2 - y^2 + xy + x^3 + x^2y^2$$

$$\begin{aligned} f(x,y) &= 1 - x^2 - y^2 + xy + x^3 + x^2y^2,\ f(0,0) = 1 \\ f_x(x,y) &= -2x + y + 3x^2 + 2xy^2,\ f_x 0,0 = 0 \\ f_y(x,y) &= 2y + x + 2x^2y,\ f_y(0,0) = 0 \\ f_{xx}(x,y) &= -2 + 6x + 2y^2,\ f_{xx}(0,0) = -2 \\ f_{yy}(x,y) &= 2 + 2x^2,\ f_{yy}(0,0) = 2 \\ f_{xy}(x,y) &= 1 + 4xy,\ f_{xy}(0,0) = 1 \end{aligned}$$

The quadratic approximation is

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 \\ &\quad + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2 \\ &\quad = 1 - x^2 + xy = y^2 = f(x,y) - x^3 - x^2y^2 \end{aligned}$$

2.8 Constrained optimisation and Lagrange multipliers

2.8.1 Lagrange multipliers

One often needs to maximise or minimise a function subject to certain constraints.

Example 23. Find the minimum value of $x^2 + y^2$ subject to the constraint $x + y = 1$.

We want to minimise the function $f$subject to the constraint $x + y = 1$

We can explicitly solve the **constraint-equation:** $y = 1 - x$, so

$$F(x) := f(x, 1-x) = x^2 + (1-x)^2 = 2x^2 - 2x + 1$$

Which is a function of x alone. The critical points of F occur when $F'(x) = -2 + 4x = 0$, implying $x = \frac{1}{2}$. Since $F''(x) = 4 > 0$, it follows that F has a minimum at $x = \frac{1}{2}$. Therefore f subject to the constraint $x + y = 1$ achieves its minimum value at $x = y = \frac{1}{2}$ with the actual value also $\frac{1}{2}$.

You can solve this problem much better without requiring the explicit solution of the constraint-equation.

Define a second function g such that the constraint-equation corresponds to $g(x, y) = 0$.
In our example, $g(x, y) = x + y - 1$. Graphing the contour polot $f(x, y) = x^2 + y^2$ and add a graph of the single contour $g(x, y) = 0$.

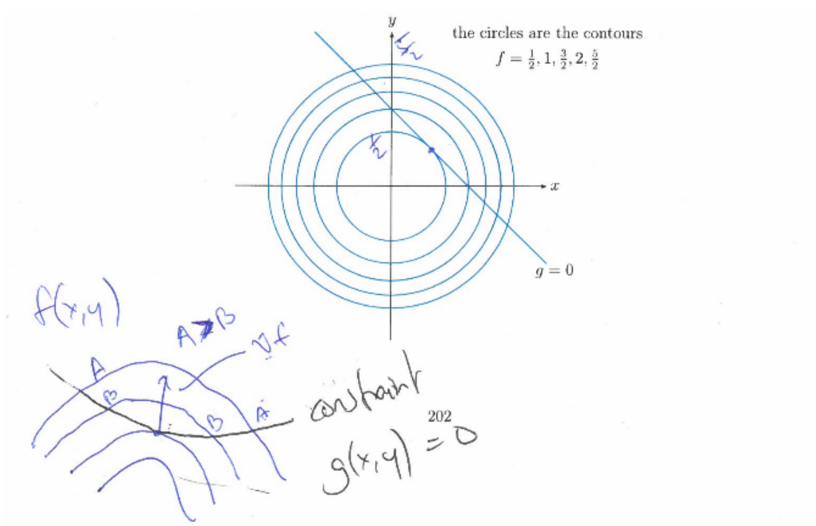


Figure 2.1:

The minimum occurs where the contour $g(x, y) = 0$ touches one of the contours of f, which means that ∇f and ∇g are parralel. Hence

$$\nabla f = \lambda \nabla g$$

for some λ, known as the **Langrange multiplier**.

Example 24. Find the minimum value of $x^2 + y^2$ subject to the constraint $x + y = 1$

Let $f(x, y) = x^2 + y^2$ and $g(x, y) = x + y - 1$. We want to minimise $f(x, y)$ subject to the constraint $g(x, y) = 0$. Next, $\nabla f = (2x, 2y)$ and $\nabla g = (1, 1)$. Set $\nabla f = \lambda \nabla g$, so $2x = \lambda$, $2y = \lambda$. Including the constraint $g(x, y) = 0$, this gives us 3 equations in 3 unknowns. The solution is $x = y = \frac{1}{2}$, $\lambda = 1$. To verify that this is a minimum, set $x = \frac{1}{2} + \varepsilon$, $y = \frac{1}{2} - \varepsilon$ such that $x + y = 1$.

$$\begin{aligned} f(\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon) &= \left(\frac{1}{2} + \varepsilon\right)^2 + \left(\frac{1}{2} - \varepsilon\right)^2 \\ &= \frac{1}{2} + 2\varepsilon^2 \end{aligned}$$

We see that $f(\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon)$ is minimised when $\varepsilon = 0$, so $(\frac{1}{2}, \frac{1}{2})$ gives the constrained minimum.

Note 2. The following two remarks are important and are direct references to the workbook.

Remark. When solving a constrained optimisation problem with Langrange multipliers, it is typically not necessary to find the value of λ. So when solving your equations obtained from $\nabla f = \lambda \nabla g$ and the constraint-equation, your aim is really to **eliminate** λ in order to solve for variables in your problem.

If upon using the method of Langrange multipliers you obtain simple, linear equations to solve then consider yourself lucky. In some cases you will in fact obtain nonlinear equations that you need to solve. If this happens we suggest the first thing you to is try taking the **ratio** of your equations obtained from $\nabla f = \lambda \nabla g$. This allows you to immediately eliminate λ from your equations and proceed from there. The next example nicely illustrates this point.

2.9 Line Integrals

2.9.1 Work done by a constant force

In a single dimension, work is done by a constant force F moving an object along a straight line of length d is represented as

$$W = Fd$$

In two or three dimensions, work done by a constant force in a moving particle alng a straight line from $P \to Q$ is

$$W = F \cdot \overrightarrow{PQ} = mgd\cos\theta$$

2.9.2 Work done over a curve

We now consider the more general case of the work done by a force field

$$F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$$

which moves an object along a curve C.

First give an approximation, dividing C into n arcs such that the i'th arc has length δs_i. We approximate δs_i by evaluating F at a specified point P_i on the arc.

2.10 Central forces

A force $F(x,y,z)$ is called central if it has the form:

$$\mathbf{F} = F(r)\hat{\mathbf{r}} = \frac{F(r)}{r}\mathbf{r}$$

where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$ Central forces act towards or away from the origin. The maginitude is dependant on the distance from the origin.

Consider the gradient of the radial distance,

$$\begin{aligned}\nabla r &= \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right)\\ &= (x^2+y^2+z^2)^{\frac{1}{2}}(x,y,z)\\ &= \frac{1}{r}\,\underline{r} = \underline{\hat{r}}\end{aligned}$$

Let $F(r) = -\dfrac{dV}{dr}$ for some function V. We show that the central force $\mathbf{F}(r) = F(r)\hat{r}$ is conserved with potential function $V(r)$.

Proof.

$$\begin{aligned}
-\nabla V &= \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right) \\
&= \left(-\frac{dV}{dr}\frac{\partial r}{\partial x}, -\frac{dV}{dr}\frac{\partial r}{\partial y}, -\frac{dV}{dr}\frac{\partial r}{\partial z}\right) \quad \text{(By the chain rule)} \\
&= F(r)\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = F(r)\hat{r} \\
\frac{dE}{dt} &= \frac{1}{2}m\underline{\dot{v}} \cdot \underline{v} + \frac{1}{2}m\underline{v} \cdot \underline{\dot{v}} + \frac{dV}{dr}\dot{r} = m\underline{a} \cdot \underline{v} + \frac{dV}{dr}\dot{r}
\end{aligned}$$

Therefore all **central forces are conservative** (provided that $\mathbf{F}(r)$ is integrable). Hence a particle moving in a central force $\mathbf{F}(r) = -V'(r)\hat{r}$ has energy

$$E = \frac{1}{2}mv^2 + V(r)$$

which remains constant. □

2.11 Angular Momentum

A particle with mass m, velocity $\mathbf{v}$ and position $\mathbf{r}$, then the angular momentum is given by

$$L = m(\mathbf{r} \times \mathbf{v})$$

A force exerted on the particle, by Newton's Second Law.

$$\begin{aligned}
\underline{\dot{L}} &= m(\underline{\dot{\mathbf{r}}} \times \underline{\mathbf{v}}) + m(\underline{\mathbf{r}} \times \underline{\dot{\mathbf{v}}}) \\
&= m(\underline{\mathbf{v}} \times \underline{\mathbf{v}}) + m(\underline{\mathbf{r}} \times \underline{\mathbf{a}}) \\
&= \underline{\mathbf{r}} \times \underline{\mathbf{F}}
\end{aligned}$$

We call $\mathbf{r} \times \mathbf{F}$ the **torque** for a central force $\mathbf{F} = \dfrac{F(r)}{r}\mathbf{r}$. So the time derivative is

$$\begin{aligned}
\underline{\dot{\mathbf{L}}} = \ \underline{\mathbf{r}} \times \underline{\mathbf{F}} &= \frac{F(r)}{r}(\underline{\mathbf{r}} \times \underline{\ \mathbf{r}} \\
&= 0
\end{aligned}$$

As $\mathbf{L}$ is constant (it's not gonna change anytime soon), we have **conservation of angular momentum**. Also as angular momentum is a vector quantity $\implies$ that both the magnitude and direction is constant as well.

By the definition of the cross product, the direction of the angular momentum is perpendicular to the position and velocity vectors.

This means that motion is confined to a plane. **Without any loss of generality**, we can choose this plane to be the $x - y$ plane.

2.12 Polar Coordinates

Its natural to represent central force problems in polar coordinates in $\mathbb{R}^2$. These are an example of **curvlinear coordinates**. The basic vectors used in curvlinear coordinates are **not** fixed in space. They change according to a position of a particle.

Example 25. One basis vector is given by

$$\hat{\mathbf{r}} = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$$

We can check that a perpendicular unit vector is given by

$$\hat{\theta} = -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}$$

The vectors $\underline{\mathbf{r}}$ provide an orthogonal basis for $\mathbb{R}^2$. We need to to check if the following two properties hold

1.
$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$

2.
$$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

We can obtain the following expressions for the velocity and acceleration

$$\begin{aligned}
\underline{r} &= r\underline{\hat{r}} \\
\underline{v} = \underline{\dot{r}} &= \dot{r}\underline{\hat{r}} + r\frac{d\underline{\hat{r}}}{dt} \\
&= \dot{r}\underline{\hat{r}} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} \\
&= \dot{r}\underline{\hat{r}} + r\dot{\theta}\underline{\hat{\theta}}
\end{aligned}$$

For the acceleration

$$\begin{aligned}
\underline{a} &= \frac{d\underline{v}}{dt} \\
&= \ddot{r}\underline{\hat{r}} + \dot{r}\frac{d\underline{\hat{r}}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= \ddot{r}\underline{\hat{r}} + \dot{r}\frac{d\underline{\hat{r}}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\underline{\hat{\theta}} + r\dot{\theta}\frac{d\underline{\hat{\theta}}}{d\theta}\dot{\theta} \\
&= (\ddot{r} - r\dot{\theta}^2)\underline{\hat{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{\hat{\theta}}
\end{aligned}$$

We are dealing with a central force $\mathbf{F} = -V'(r)\hat{r}$ such that the equation of motion is

$$-V'(r)\hat{r} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Equating the radial and angular components. For the radial part.

$$\ddot{r} - r\dot{\theta}^2 = -\frac{V'(r)}{m}$$

And for the angular equation

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0;$$

Rewriting the angular equation we have

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

This is saying that the magnitude of angular momentum is constant over time. I.e

$$L = mr^2\dot{theta}$$

Where L is constant. We can describe the motion of the particle by the radial equation

$$\ddot{r} = \frac{L^2}{m^2r^3} - \frac{V'(r)}{m}$$

2.13 Circular Motion

Attractive central forces are capable of producing circular motion moving at constant speed. Given for circular motion, r is constant $\implies \dot{r} = \ddot{r} = 0$. We have

$$\begin{aligned}
v^2 &= (\dot{r}\underline{\hat{r}} + r\dot{\theta}\underline{\hat{\theta}}) \cdot (\dot{r}\underline{\hat{r}} + r\dot{\theta}\underline{\hat{\theta}}) \\
&= \dot{r}^2 + r^2\dot{\theta}^2 \\
&= r^2\dot{\theta}^2 = \frac{L^2}{mr^2}
\end{aligned}$$

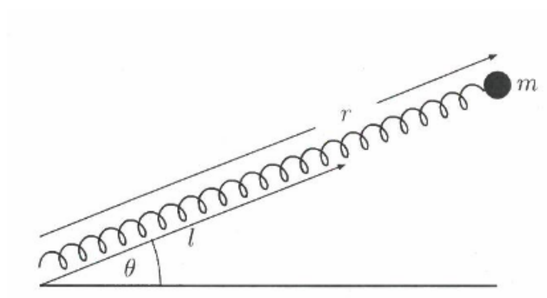
The value of L is given by the explicit knowledge of the potential. Since $\ddot{r} = 0$,

$$0 = \frac{L^2}{m^2 r^3} - \frac{V'(r)}{m}$$
$$L^2 = mr^3 V'(r)$$

Which gives $v^2 = \dfrac{rV'(r)}{m}$

Example 26 (A rotation spring without damping)**.** Consider a spring of length l and spring constant k, with a mass attached at one end and fixed at the origin. The system sits on a frictionless table and is able to rotate about the origin in the horizontal plane. This is a central force problem where

$$F(r) = -k(r-l) \quad V(r) = \frac{k}{2}(r-l)^2$$



For $r > l$, circular motion occurs with $\dot{r} = 0$ and $v = \sqrt{\dfrac{kr(r-l)}{m}}$. Otherwise the motion is determined by the solution to the ODE.

$$\dot{r} = \frac{L^2}{m^2 r^3} - \frac{k(r-l)}{m}$$

2.14 The envelope of a family of functions

Consider a function of two variables $f(x,t)$, we can intepret this funcition as a family of one variable functions by setting

$$y^{(t)}(x) = f(x,t)$$

...

How does the concept of the envelope arises in economic modelling? Suppose that a bakery's weekly bread roll production is described by $f(x)$ (given the conditions for $x \leq 0$ that $f(0) = 0, f(x) \leq 0$, $f'(x) > 0$ and the cost of one kg of flour be c, the selling price for one kilogram of bread rolls is p. We will consider p to be a variable, while c is a constant. The profit $P(p,x)$ is given by

$$P(p,x) = pf(x) - cx$$

It is straightfoward to verify that the cross section of $P(p,x)$ in $p$havve a local maximum when

$$f'(x) = cp^{-1}$$

Specifically

$$P_x(p,x) = pf'(x) - c \tag{2.7}$$
$$P_{xx}(p,x) = pf''(x) \tag{2.8}$$

When $f'(x) = cp^{-1}$, Eq. (3.1) shows that a cross-section of P in p has a critical point. Also since, $f''(x) < 0$, Eq. (3.2) shows that the critical point is al local maximum.

Let $x = g(p)$ be the solution for (3). We define the maximised profit function f^{max} by

$$f^{max}(p) = pf(g(p)) - cg(p)$$

This is the envelope of $P(p, x)$ with the property

$$\frac{dp^{max}}{dp} = f(g(p))$$

1. $f(x) = 4x^{1/2}$
2. $f(x) = \frac{x}{x+1}$

both of which satisfy the required conditions for $x \leq 0 \mid f(0) = 0, f(x) \leq 0,\ f'(x) > 0\ f''(x) < 0$.

$$\begin{aligned}
f(x) &= 4x^{\frac{1}{2}} \\
f'(x) &= 2x^{-\frac{1}{2}} = cp^{-1} \implies x = g(p) = \frac{4p^2}{c^2} \\
f(g(p)) &= 4\sqrt{\frac{4p^2}{c^2}} = \frac{8p}{c}
\end{aligned}$$

Then

$$p^{max}(p) = \frac{8p}{c} \times p - c\Big(\frac{4p^2}{c^2}\Big) = \frac{4p^2}{c}$$