Tutorial and Lab Problems # 4
MATH3871/MATH5970

1. **Simulation from Normal via Rejection from Exponential.** Consider the pdf of $X \sim \mathsf{N}(0,1)$, conditional on $X > 0$:
$$f(x) = \frac{2}{\sqrt{2\pi}} \exp(-x^2/2), \quad x > 0 .$$
Suppose that in the rejection algorithm (see the lectures) we use $g(x) = \exp(-x), x > 0$ as proposal density. Find the enveloping constant c such that $f(x) \leq cg(x)$, then write pseudo-code to describe the rejection sampling algorithm. How can you use this rejection to simulate $X \sim \mathsf{N}(0,1)$ (without any conditioning on $X > 0$, so that X can be positive or negative)? What is the acceptance probability of the algorithm?

2. **Simulation from Truncated Normal.** We wish to simulate from the pdf of $X \sim \mathsf{N}(0,1)$, conditional on $X > a$ (for some given a),
$$f(x) = \frac{\exp(-x^2/2)}{\overline{\Phi}(a)\sqrt{2\pi}}, \quad x > a ,$$
using a rejection algorithm with proposal density $g(x;\lambda) = \lambda \exp(-\lambda(x - a)), x > a$ (the pdf of $X \sim \mathsf{Exp}(\lambda)$, conditional on $X > a$). a) Give a formula for the acceptance probability of the rejection algorithm. b) How can we improve the efficiency of the rejection algorithm? c) Using the most efficient version of the rejection algorithm, find the (limiting) probability of acceptance as $a \uparrow \infty$?

3. **Simulation from Normal-Laplace Density.** The normal-Laplace density is defined as:
$$\mathrm{nl}(z;\lambda,\alpha) \propto \phi(z) \exp(-\lambda|z-\alpha|),$$
where $\phi(z)$ is the standard normal pdf. Show that it can be written as a mixture of two truncated Gaussian densities. This makes it possible to easily simulate from the pdf. Write pseudo code and then implement in Matlab and R a function that samples from $\mathrm{nl}(z;\lambda,\alpha)$. To simulate from the truncated normal you may use the function `trandn.R` (in the R package `TruncatedNormal`) and `trandn.m` (available from Moodle).

Answers:

1. $c = \sqrt{2\exp(1)/\pi}$, so that acceptance probability is $1/c \approx 0.76$. The

Algorithm 1 : Simulating from $\mathsf{N}(0, 1)$ using $\mathsf{Exp}(1)$ proposal.

repeat
  $E_1, E_2 \sim_{\text{iid}} \mathsf{Exp}(1)$
until $2E_1 \geq (E_2 - 1)^2$
$U \sim \mathsf{U}(0, 1)$
return $X \leftarrow E_2 \times \text{sign}(2U - 1)$

2. Ans: a) The formula is

$$\lambda\sqrt{2\pi}\exp(a\lambda - \lambda^2/2)\overline{\Phi}(a) \ . \tag{1}$$

b) Select $\lambda^*(a) = (a + \sqrt{a^2 + 4})/2$ to maximize the acceptance probability. c) Substitute $\lambda^*(a)$ into (1) and apply L'Hopital's rule twice to obtain the limiting value of 1. To simplify computations note that $\lambda^*/a \to 1$ as $a \uparrow \infty$.

3. The pdf has a mixture form:

$$w_1 \frac{\phi(z + \lambda)\mathbb{I}\{z > \alpha\}}{\overline{\Phi}(\lambda + \alpha)} + w_2 \frac{\phi(z - \lambda)\mathbb{I}\{z < \alpha\}}{\overline{\Phi}(\lambda - \alpha)}$$

with weights $w_{1,2} \propto \overline{\Phi}(\lambda \pm \alpha)/\phi(\lambda \pm \alpha)$.

Hence, we have the following simulation algorithm.

Algorithm 2 : Simulating an normal-Laplace variable

Require: parameters $\lambda > 0$ and $\alpha \in \mathbb{R}$
$U \sim \mathsf{U}(0, 1)$
$w_1 \leftarrow \overline{\Phi}(\lambda + \alpha)/\phi(\lambda + \alpha)$
$w_2 \leftarrow \overline{\Phi}(\lambda - \alpha)/\phi(\lambda - \alpha)$
if $U < w_2/(w_1 + w_2)$ **then**
  Simulate $Z \sim \mathsf{N}(\lambda, 1)$, conditional on $Z < \alpha$
else
  Simulate $Z \sim \mathsf{N}(-\lambda, 1)$, conditional on $Z > \alpha$
return Z as distributed from the pdf $\text{nl}(z; \lambda, \alpha)$