Tutorial and Lab Problems # 7
MATH3871/MATH5970

1. **Hammersley-Clifford result.** Recall Result 1.2 in the MCMC lecture notes. Show that the transition density of the Gibbs sampler satisfies:

$$\pi(\boldsymbol{y})\kappa_{d\to 1}(\boldsymbol{x}\,|\,\boldsymbol{y}) = \pi(\boldsymbol{x})\kappa_{1\to d}(\boldsymbol{y}\,|\,\boldsymbol{x}) \;.$$

   Hence conclude that the transition density $\kappa_{d\to 1}$ satisfies the global balance equation: $\int \pi(\boldsymbol{x})\kappa_{1\to d}(\boldsymbol{y}\,|\,\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \pi(\boldsymbol{y})$.

2. **Detailed Balance for MH sampler.** Consider Algorithm 1.1 in the notes and equations (1.5) and (1.6). Explain why (1.5) gives the transition density of the Matropolis-Hastings sampler. Show that the detailed balance equations in (1.6) are satisfied.

3. **Examples 1.2 and 1.3.** Re-run Examples 1.2 and 1.3 in the notes and reproduce the autocorrelations plots on Figure 1.3 (for Matlab you can use the `acf.m` function on Moodle). Can you select a value for the scaling constant ς of the random-walk sampler such that the random-walk sampler becomes the preferred sampler?

4. **Example 1.6.** Run the Gibbs sampler for Example 1.6 in R/Matlab. Using the output of the Gibbs sampler, reproduce the scatterplot on Figure 1.4.

   For this simple example, we can do the simulation of (μ, σ^2) exactly by first simulating $\sigma^2\,|\,\tau$ and then $\mu\,|\,\sigma^2, \tau$. Implement this exact sampler and produce a scatterplot like the one on Figure 1.4. Comment on whether the Gibbs sampler approximation agrees with the exact simulation.

Answers:

1. We have

$$\begin{aligned}
\frac{\kappa_{1\to d}(\boldsymbol{y}\,|\,\boldsymbol{x})}{\kappa_{d\to 1}(\boldsymbol{x}\,|\,\boldsymbol{y})} &= \prod_{i=1}^{d} \frac{\pi(y_i\,|\,y_1,\ldots,y_{i-1},x_{i+1},\ldots,x_n)}{\pi(x_i\,|\,y_1,\ldots,y_{i-1},x_{i+1},\ldots,x_n)} \\
&= \prod_{i=1}^{d} \frac{\pi(y_1,\ldots,y_i,x_{i+1},\ldots,x_n)}{\pi(y_1,\ldots,y_{i-1},x_i,\ldots,x_n)} \\
&= \frac{\pi(\boldsymbol{y})}{\pi(\boldsymbol{x})} \frac{\prod_{i=1}^{d-1} \pi(y_1,\ldots,y_i,x_{i+1},\ldots,x_n)}{\prod_{i=2}^{d} \pi(y_1,\ldots,y_{i-1},x_i,\ldots,x_n)} \\
&= \frac{\pi(\boldsymbol{y})}{\pi(\boldsymbol{x})} \frac{\prod_{i=1}^{d-1} \pi(y_1,\ldots,y_i,x_{i+1},\ldots,x_n)}{\prod_{i=1}^{d-1} \pi(y_1,\ldots,y_i,x_{i+1},\ldots,x_n)} = \frac{\pi(\boldsymbol{y})}{\pi(\boldsymbol{x})}
\end{aligned}$$

2. Done in the notes, but will go over it in tute as well. To derive the formula (1.5) for the transition density, you can write: $\mathbb{P}(\boldsymbol{X}_{k+1} \in A\,|\,\boldsymbol{X}_k = \boldsymbol{x}_k) =$

$$\begin{aligned}
&= \mathbb{P}(\boldsymbol{X}_{k+1} \in A, U < \alpha(\boldsymbol{Y},\boldsymbol{X}_k)\,|\,\boldsymbol{X}_k = \boldsymbol{x}_k) + \mathbb{P}(\boldsymbol{X}_{k+1} \in A, U > \alpha(\boldsymbol{Y},\boldsymbol{X}_k)\,|\,\boldsymbol{X}_k = \boldsymbol{x}_k) \\
&= \mathbb{P}(\boldsymbol{Y} \in A, U < \alpha(\boldsymbol{Y},\boldsymbol{x}_k)) + \mathbb{P}(\boldsymbol{X}_k \in A, U > \alpha(\boldsymbol{Y},\boldsymbol{X}_k)\,|\,\boldsymbol{X}_k = \boldsymbol{x}_k) \\
&= \int_A g(\boldsymbol{y}\,|\,\boldsymbol{x}_k)\alpha(\boldsymbol{y},\boldsymbol{x}_k)\mathrm{d}\boldsymbol{y} + \mathbb{I}\{\boldsymbol{x}_k \in A\}\mathbb{P}(U > \alpha(\boldsymbol{Y},\boldsymbol{x}_k)) \\
&= \int_A g(\boldsymbol{y}\,|\,\boldsymbol{x}_k)\alpha(\boldsymbol{y},\boldsymbol{x}_k)\mathrm{d}\boldsymbol{y} + \mathbb{I}\{\boldsymbol{x}_k \in A\}\left(1 - \alpha^*(\boldsymbol{x}_k)\right),
\end{aligned}$$

where

$$\alpha^*(\boldsymbol{x}) := \int g(\boldsymbol{y}\,|\,\boldsymbol{x})\alpha(\boldsymbol{y},\boldsymbol{x})\mathrm{d}\boldsymbol{y}.$$

Hence, the density $\kappa(\boldsymbol{y}\,|\,\boldsymbol{x})$ is given by

$$\kappa(\boldsymbol{y}\,|\,\boldsymbol{x}) = g(\boldsymbol{y}\,|\,\boldsymbol{x})\alpha(\boldsymbol{y},\boldsymbol{x}) + \delta(\boldsymbol{y} - \boldsymbol{x}_k)(1 - \alpha^*(\boldsymbol{x})),$$

where $\delta(\boldsymbol{y} - \boldsymbol{x}_k)$ covers the case when the chain stays in the same state.