Tutorial and Lab Problems # 12
MATH3871/MATH5970

Approximate simulation via Importance Sampling. (Example 1.9 in Lecture notes) We wish to simulate from $\pi(x) \propto f(x) = x^{\alpha-1}\exp(-x), x > 0$ using the importance sampling density $g(x) = \beta\exp(-\beta x)$. We can use the following algorithm to approximately simulate from π.

Algorithm 1 Importance Sampler for simulating from $\mathsf{Gamma}(\alpha, 1)$

Require: α, β, number of steps/time t
  $N \leftarrow 0$
  repeat
    $N \leftarrow N + 1$
    $X_N \leftarrow -\ln(U)/\beta$, where $U \sim \mathsf{U}(0, 1)$
    $W_N \leftarrow X_N^{\alpha-1}\exp(-(1-\beta)X_N)$
  until $W_1 + \cdots + W_N > t$
  return X_N as an approximate draw from $\mathsf{Gamma}(\alpha, 1)$

We make the following observations.

- Note that W need only be known up to a constant of proportionality. For example, we could have used $X^{\alpha-1}\exp(-(1-\beta)X)/\beta$ instead of $X^{\alpha-1}\exp(-(1-\beta)X)$.

- The code below not only implements the algorithm, but checks how accurate the sampling is based on n independent repetitions/runs of the Algorithm 1 using with the same t.

  We check the accuracy visually by plotting a $q - q$ plot and by comparing a kernel density estimator with the density of the true $\mathsf{Gamma}(\alpha, 1)$ distribution. In addition, we compute the p-value of a Kolmogorov-Smirnov test statistic, which quantifies the strength of the evidence that the data follows the $\mathsf{Gamma}(\alpha, 1)$ distribution.

- A simple way to select β is to pick one that maximizes the effective sample size, that is, minimizes an estimate of $\mathbb{E}W^2/(\mathbb{E}W)^2$. Try it yourself.

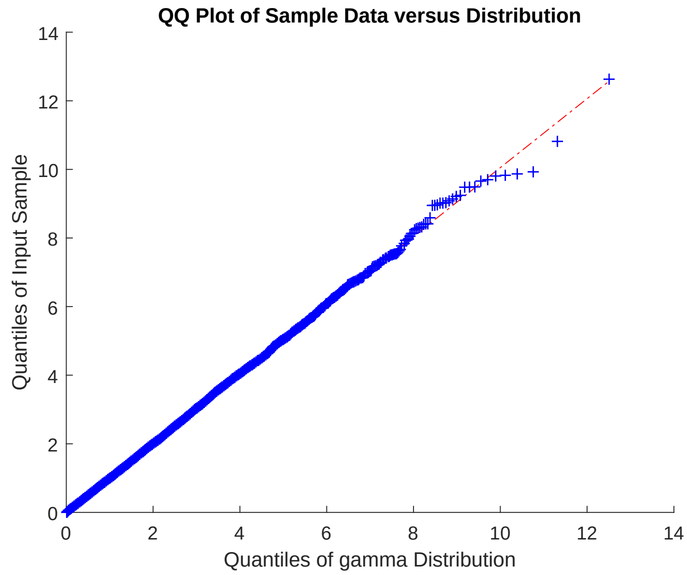

Figure 1: The plot compares the theoretical quantiles (computed from knowledge of the true distr.) with the estimated quantiles (computed from the output of the sampler).

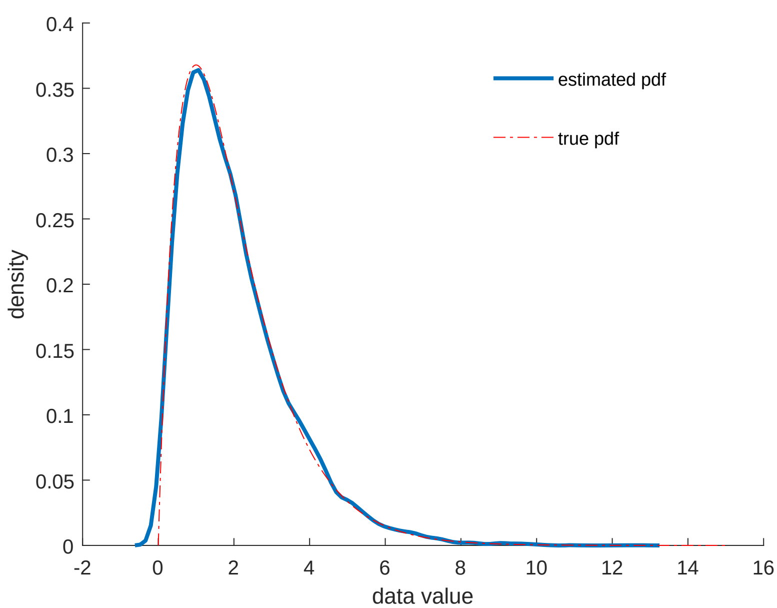

Figure 2: A visual comparison of the estimated density (from output of sampler) with the true density.

```
clear all, clf
b=1;a=2;  % problem parameters
t=10^2;   % algorithmic parameters
n=10^4;   % number of times algorithm repeated
for iter=1:n
    T=0; N=0;     % initialization of alg
    while T<t
        N=N+1;
        X=-log(rand)/b;
        W(N)=X^(a-1)*exp(-(1-b)*X);
        T=T+W(N);
    end
    data(iter)=X; % collect draws
    ef(iter)=mean(W.^2)/mean(W); % compute reciprical of ESS
end
mean(ef) % average ESS indicated quality of sample

figure(1) % compare if the data agrees with exact pdf
ksdensity(data),hold on,x=[eps:0.01:15];
plot(x,x.^(a-1).*exp(-x)/gamma(a),'r')
figure(2)
pd=makedist('Gamma','a',a);
qqplot(data,pd) % make a 1-1 plot
% and test the hypothesis that data comes from correct distr.
[h,p] = kstest(data,'CDF',pd)
```