# Tutorial and Lab Problems # 8
# MATH3871/MATH5970

1. **Skew-Normal Distribution.** The skew-normal density (see `https://en.wikipedia.org/wiki/Skew_normal_distribution`) has pdf:

$$\pi(x) = 2\phi(x)\Phi(\alpha x),$$

where $\alpha$ is real, $\phi$ is the standard normal pdf, and $\Phi$ is the corresponding cumulative distribution function (cdf). This pdf arises in the Probit model from lectures. One way to simulate from this density is to introduce an auxiliary variable $z$ and consider the joint pdf:

$$\pi(x, z) = 2\phi(x)\phi(z)\mathbb{I}\{z \leq \alpha x\}$$

Show that the marginal pdf $\int \pi(x, z)\mathrm{d}z$ is the skew-normal pdf. Then, write pseudo-code implementing the Gibbs sampler to simulate from $\pi(x, z)$. Then, implement the algorithm in Matlab/R/Python and simulate $10^5$ approximate draws.

2. **Skew-Normal Probit-Style Sampling.** Yet another way to simulate from the skew-normal is to introduce an auxiliary variable $z$ and consider the joint pdf:

$$\pi(x, z) = 2\phi(x)\phi(z + \alpha x)\mathbb{I}\{z \leq 0\}$$

Show that the marginal pdf $\int \pi(x, z)\mathrm{d}z$ is the skew-normal pdf. Then, write pseudo-code implementing the Gibbs sampler to simulate from $\pi(x, z)$. After examining the MCMC output, which Gibbs sampler do you think is more efficient?

# Answers:

**Algorithm 1** : Simulating from skew-normal pdf with parameter $\alpha > 0$.

1. $\boldsymbol{Y}_0 \leftarrow (0,0)$
   **for** $k = 1, \ldots, t$ **do**
   $(X, Z) \leftarrow \boldsymbol{Y}_{k-1}$
   $X \,|\, Z \sim \mathsf{TN}_{(Z/\alpha,\infty)}(0,1)$
   $Z \,|\, X \sim \mathsf{TN}_{(-\infty,\alpha X)}(0,1)$
   $\boldsymbol{Y}_k \leftarrow (X, Z)$
   **return** $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_t$

Some sample code.

```
clear all,clc
x=-5:0.03:5;
a=4;
y=2*normcdf(a*x).*normpdf(x);

t=10^5;
data=nan(2*t,2);
Z=0;X=0;
for k=1:t
    X=trandn(Z/a,Inf);
    data(2*k-1,:)=[X,Z];
    Z=trandn(-Inf,a*X);
    data(2*k,:)=[X,Z];
end

ksdensity(data(:,1)),
hold on
plot(x,y,'r')

for k=11:2*t
   plot(data(1:k,1),data(1:k,2),'r.'), axis([-3,3,-3,3])
   pause(.1),hold on
   comet(data(k-1:k,1),data(k-1:k,2))
   pause(.1)
   hold off
end
```

2. Pseudo-code for this Probit-style sampling is as follows.

**Algorithm 2** : Simulating from skew-normal ala Probit.

$\boldsymbol{Y}_0 \leftarrow (0,0)$
**for** $k = 1, \ldots, t$ **do**
$(X, Z) \leftarrow \boldsymbol{Y}_{k-1}$
$Z \,|\, X \sim \mathsf{TN}_{(-\infty,0)}(-\alpha X, 1)$
$X \,|\, Z \sim \mathsf{N}(-\alpha Z/(1+\alpha^2), 1/(1+\alpha^2))$
$\boldsymbol{Y}_k \leftarrow (X, Z)$
**return** $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_t$