MATH3871/MATH5960 Bayesian Inference and Computation, Lab 3 Exercises

1. Suppose that Y is an exponentially distributed random variable with mean $\lambda > 0$, and density $f(y) = \frac{1}{\lambda}\exp\left(-\frac{y}{\lambda}\right)$ for $y > 0$.

   Derive the inverse-transform method to simulate from this density. Implement the sampling in R/Matlab and verify its correctness by plotting a histogram and the true pdf on the same figure.

2. Using R/Matlab approximate the following integrals via Monte Carlo integration:

   (a)
   $$\int_0^1 \exp[\exp(x)]\mathrm{d}x.$$

   (b)
   $$\int_{-\infty}^{\infty} \exp(-x^2)\mathrm{d}x.$$

   (c)
   $$\int_0^1\int_0^1 \exp\left[(x+y)^2\right]\mathrm{d}y\mathrm{d}x.$$

   (d)
   $$\int_0^{\infty}\int_0^{x} \exp\left[-(x+y)\right]\mathrm{d}y\mathrm{d}x,$$

   where you can use the indicator function

   $$\mathbb{I}\{y < x\} = \begin{cases} 1 & \text{if } y < x \\ 0 & \text{otherwise} \end{cases}$$

   to equate the integral to one in which the domains of integration both go from 0 to ∞.

Solutions

1. Then the distribution function of an exponential random variable is

   $$F(y) = \begin{cases} 1 - \exp\left(-\frac{y}{\lambda}\right) & y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

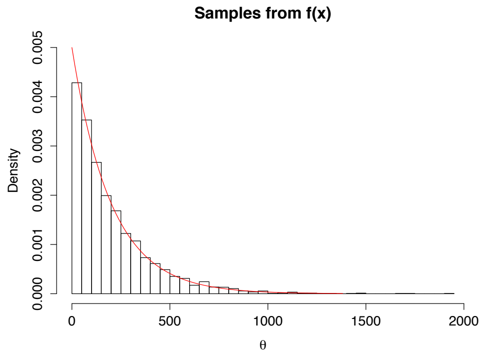
   To implement the inverse transform method of simulation, we need to find $F^{-1}(p)$, $p \in (0,1)$. For $p \in (0,1)$, $F^{-1}(p)$ satisfies $F(F^{-1}(p)) = p$, or

   $$\begin{aligned} 1 - \exp\left(-F^{-1}(p)/\lambda\right) &= p \\ \exp\left(-F^{-1}(p)/\lambda\right) &= 1 - p \\ -F^{-1}(p)/\lambda &= \ln(1-p) \\ F^{-1}(p) &= -\lambda\ln(1-p) \end{aligned}$$

   So to simulate an exponential random variable with mean λ, we simulate a uniformly distributed random variable U on $[0,1]$, and then compute $Y = -\lambda\ln(1-U)$. Note that if U is uniformly distributed on $[0,1]$, then $1-U$ is also uniformly distributed on $[0,1]$ so that we could simply compute $Y = -\lambda\ln U$ (simpler and faster).

   Some *R* code to implement this is:

```
func=function(n,lambda) {
  return( -lambda*log(runif(n)) )
}
lambda=200
hist(func(5000,lambda),probability=T,xlab=expression(theta),ylab="Density",
main="Samples from f(x)",nclass=31,ylim=c(0,1/lambda))
xx=seq(0,qexp(0.999,1/lambda),length=100)
lines(xx,dexp(xx,1/lambda),col=2)
```



2. (a) Because the function $\exp(-x^2)$ is symmetric about the origin, $\int_{-\infty}^{\infty}\exp(-x^2)dx = 2\int_0^{\infty}\exp(-x^2)\mathrm{d}x$. Now transform the integral from $[0,\infty)$ to $[0,1]$ by the transformation $u = 1/(1+x)$:

$$\int_0^{\infty}\exp(-x^2)\mathrm{d}x = \int_0^1 \frac{1}{u^2}\exp\left(-\left(\frac{1}{u}-1\right)^2\right)\mathrm{d}u$$

We can now approximate the integral as:

```
> u=runif(10000)
> 2*mean(1/u^2*exp(-(1/u-1)^2))
[1] 1.785992
```

(The true answer is $\sqrt{\pi}\approx 1.77$).

(b) $\int_0^1\int_0^1 \exp((x+y)^2)\mathrm{d}y\mathrm{d}x = \mathbb{E}\left[\exp((U_1+U_2)^2)\right]$, where U_1, U_2 are independent uniformly distributed random variables on $[0,1]$. Hence we can approximate the integral as:

```
> u1=runif(10000)
> u2=runif(10000)
> mean(exp((u1+u2)^2))
[1] 4.832706
```

(c) We can write

$$\int_0^{\infty}\int_0^{x}\exp(-(x+y))\mathrm{d}y\mathrm{d}x = \int_0^{\infty}\int_0^{\infty}\exp(-(x+y))\mathbb{I}\{y<x\}\mathrm{d}y\mathrm{d}x.$$

Now make the transformations $u_1 = 1/(1+x)$ and $u_2 = 1/(1+y)$ so that the integrals on $[0,\infty)$ are transformed to $[0,1]$. Hence we have

$$\int_0^{\infty}\int_0^{\infty}\exp(-(x+y))\mathbb{I}_y\{y<x\}\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^1 \frac{1}{u_1^2u_2^2}\exp\left(-\left(\frac{1}{u_1}+\frac{1}{u_2}-2\right)\right)\mathbb{I}\{u_1<u_2\}\mathrm{d}u_1\mathrm{d}u_2.$$

So now we can approximate the integral by:

```
> u1=runif(10000)
> u2=runif(10000)
> z1=1/u1-1
> z2=1/u2-1
> mean(1/u1^2/u2^2*exp(-(z1+z2))*(u1<u2))
[1] 0.5084041
```

(The true answer is 0.5).