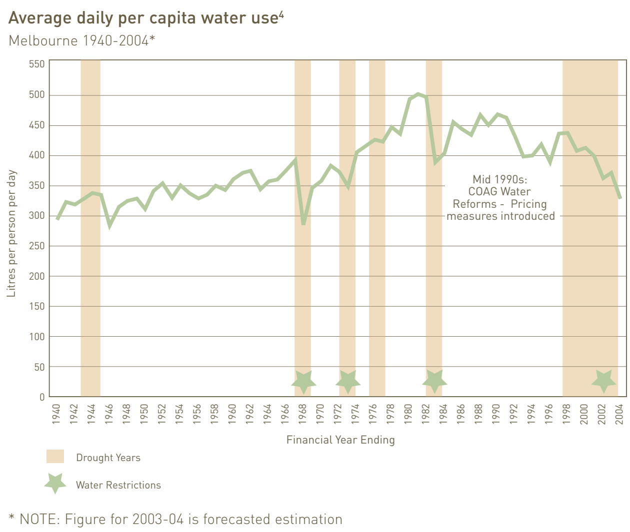
MATH3871/MATH5970
Bayesian Inference and Computation

# Tutorial Problems 1 (Solutions)

*1) Water consumption*




(a) The mean of a $\text{Gamma}(\theta|a,b)$ distribution is given by:

$$\begin{aligned}\mathbb{E}[\theta] = \int \theta \text{Gamma}(\theta|a,b)d\theta &= \int \theta \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-b\theta)d\theta \\ &= \frac{b^a}{\Gamma(a)}\int \theta^{(a+1)-1}\exp(-b\theta)d\theta \\ &= \frac{b^a}{\Gamma(a)}\frac{\Gamma(a+1)}{b^{a+1}} \\ &= \frac{a}{b}\end{aligned}$$

as $\Gamma(a+1) = a\Gamma(a)$. Hence, the mean of a $\text{Gamma}(\alpha + \sum_i x_i, \beta + n)$ distribution is $(\alpha + \sum_i x_i, \beta + n)$.

(b) The variance of a $\text{Gamma}(\theta|a,b)$ distribution is given by $\mathbb{E}[\theta^2] - (\mathbb{E}[\theta])^2$. Now

$$\begin{aligned}\mathbb{E}[\theta^2] = \int \theta^2 \text{Gamma}(\theta|a,b)d\theta &= \int \theta^2 \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-b\theta)d\theta \\ &= \frac{b^a}{\Gamma(a)}\int \theta^{(a+2)-1}\exp(-b\theta)d\theta \\ &= \frac{b^a}{\Gamma(a)}\frac{\Gamma(a+2)}{b^{a+2}} \\ &= \frac{a(a+1)}{b^2}\end{aligned}$$

as $\Gamma(a+2) = (a+1)\Gamma(a+1) = (a+1)a\Gamma(a)$.

Hence, the variance is given by

$$\frac{a(a+1)}{b^2} - \left(\frac{a}{b}\right)^2 = \frac{a^2+a-a^2}{b^2} = \frac{a}{b^2}.$$

Hence, the variance of a $\text{Gamma}(\theta|\alpha + \sum_i x_i, \beta + n)$ distribution is given by $\frac{\alpha+\sum_i x_i}{(\beta+n)^2}$.

(c) The predictive distribution for a future observation $y \sim \text{Poisson}(\theta)$ where $\theta \sim \pi(\theta|x)$ is given by

$$\begin{aligned}
p(y) &= \int \pi(y|\theta)\pi(\theta|x)d\theta \\
&= \int \left(\frac{\theta^y}{y!}\exp(-\theta)\right)\left(\frac{(\beta+n)^{\alpha+\sum_i x_i}}{\Gamma(\alpha+\sum_i x_i)}\theta^{\alpha+\sum_i x_i - 1}\exp[-(\beta+n)\theta]\right)d\theta \\
&= \frac{(\beta+n)^{\alpha+\sum_i x_i}}{y!\Gamma(\alpha+\sum_i x_i)}\int \theta^{y+\alpha+\sum_i x_i-1}\exp[-(\beta+n+1)\theta]d\theta \\
&= \frac{(\beta+n)^{\alpha+\sum_i x_i}}{y!\Gamma(\alpha+\sum_i x_i)}\frac{\Gamma(y+\alpha+\sum_i x_i)}{(\beta+n+1)^{y+\alpha+\sum_i x_i}} \\
&= \frac{\Gamma(y+\alpha+\sum_i x)}{y!\Gamma(\alpha+\sum_i x_i)}\frac{(\beta+n)^{\alpha+\sum_i x_i}}{(\beta+n+1)^{y+\alpha+\sum_i x_i}} \\
&= \binom{y+\alpha+\sum_i x_i - 1}{y}\left(\frac{1}{(\beta+n+1)}\right)^y\left(\frac{\beta+n}{\beta+n+1}\right)^{\alpha+\sum_i x_i} \\
&= \binom{y+\alpha+\sum_i x_i - 1}{y}\left(\frac{1}{(\beta+n+1)}\right)^y\left(1-\frac{1}{\beta+n+1}\right)^{\alpha+\sum_i x_i}
\end{aligned}$$

which is the probability mass function of a negative binomial distribution $\text{NegBin}\left(y \mid \alpha + \sum_i x_i, \frac{1}{\beta+n+1}\right)$.

*2) Rock Strata*

(a) The posterior probability of $A \mid F$ is

$$\begin{aligned}
\mathbb{P}(A \mid F) &= \frac{\mathbb{P}(F|A)\mathbb{P}(A)}{\mathbb{P}(F|A)\mathbb{P}(A) + rP(F|B)P(B)} \\
&= \frac{\frac{9}{10}*\frac{4}{5}}{\frac{9}{10}*\frac{4}{5}+0.2*\frac{1}{5}} \\
&= 18/19 \approx 0.947.
\end{aligned}$$

Similarly, $\mathbb{P}(B \mid F) = 1/19 \approx 0.053$.

(b)

$$\mathbb{P}(\text{correct classification}) = \mathbb{P}(A|F)\mathbb{P}(F) + \mathbb{P}(B \mid \bar{F})\mathbb{P}(\bar{F})$$

We know the first term from (a), but need to compute the others.

$$\begin{aligned}
\mathbb{P}(F) &= \mathbb{P}(F|A)\mathbb{P}(A) + \mathbb{P}(F|B)\mathbb{P}(B) \\
&= \frac{9}{10}\frac{4}{5} + \frac{2}{10}\frac{1}{5} = 19/25 = 0.76. \\
\mathbb{P}(\bar{F}) &= 1 - 19/25 = 6/25 = 0.24. \\
\mathbb{P}(B|\bar{F}) &= \frac{\mathbb{P}(\bar{F}|B)\mathbb{P}(B)}{\mathbb{P}(\bar{F})} \\
&= \frac{\frac{8}{10}\frac{1}{5}}{\frac{6}{25}} = 2/3 \approx 0.667.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(\text{correct classification}) &= \mathbb{P}(A|F)\mathbb{P}(F) + \mathbb{P}(B \mid \bar{F})\mathbb{P}(\bar{F}) \\
&= \frac{18}{19}\frac{19}{25} + \frac{2}{3}\frac{6}{25} = 22/25 = 0.88.
\end{aligned}$$