MATH3871/MATH5960
Assignment 1

Note: This assignment is due in Week 6 at the start of the lecture.

Name(s): _____

I (We) declare that this assessment item has not been submitted for academic credit elsewhere, and acknowledge that the assessor of this item may, for the purpose of assessing this item:

- Reproduce this assessment item and provide a copy to another member of the University; and/or,
- Communicate a copy of this assessment item to a plagiarism checking service (which may then retain a copy of the assessment item on its database for the purpose of future plagiarism checking).

I (We) certify that I (We) have read and understood the University Rules in respect of Student Academic Misconduct.

Signed: _____

Date: _____

Note the following instructions for completing the assignment.

1. You can work in groups of up to 12 people and submit a single (typeset) assignment (with the names of these 12 people). All members of the group will get the same marks (except for 4Glory questions, which are individually awarded), however, group members must state clearly their contribution to the overall assignment submission.

2. You can split the work amongst the members of the group in any way you like, but free-riders should note that all of the assignment questions are examinable and potentially on the final exam. Thus, it is in everybody's interest to understand the assignment problems as part of the preparation for the final exam.

3. Individuals who do not like working in groups are still free to submit an individual assignment.

4. To typeset the theoretical part of the assignment, you may work with the package `http://www.latex-project.org/`. There is a large amount of information available on Latex at this URL including the installation guides for the compiler (miktex) and the editors (free of charge). You can use either Lyx `http://www.lyx.org/` or Texnic `http://www.texniccenter.org/`. Both are free latex editors that you can easily install and get started with. Once you have completed installation, you may use the latex template provided in the Assignment folder on Moodle.

5. Questions marked as 4GLORY are not compulsory. Any person(s) solving these questions will (at the discretion of the lecturer) attract some credit points and goodwill that may offset any inadvertent mistakes in the assignments.

1. Suppose X has the Pareto distribution $\mathsf{Pareto}(a, b)$, where $a > 0$ is known but $b > 0$ is unknown, so that

$$\pi(x \mid b) = ba^b x^{-b-1} \quad \text{for } x > a.$$

Find Jeffreys' prior. What makes this prior special?

2. Consider the following Bayesian model:

- $\pi(\mu, \sigma^2) \propto 1/\sigma^2$;
- $(X_i \mid \mu, \sigma^2) \sim \mathsf{N}(\mu, \sigma^2)$, $i = 1, \ldots, n$ independently.

Let $\tau = \{x_1, \ldots, x_n\}$ represent the observed data (the training set).

(a) Write down the posterior pdf $\pi(\mu, \sigma^2 \mid \tau)$ up to a constant of proportionality.

(b) Show that $(\mu \mid \sigma^2, \tau) \sim \mathsf{N}(\bar{x}_n, \sigma^2/n)$, where $\bar{x}_n$ is the sample mean.

(c) Show that $(1/\sigma^2 \mid \mu, \tau) \sim \mathsf{Gamma}(n/2, nV_\mu/2)$, where $V_\mu = \sum_i (x_i - \mu)^2/n$ is the classical sample variance for known μ.

(d) Show that the posterior pdf of μ given the data is given by

$$\pi(\mu \mid \tau) \propto \left((\mu - \bar{x})^2 + V\right)^{-n/2},$$

where $V = \sum_i (x_i - \bar{x})^2/n$.
Hint: Read Lecture 3.

3. (4GLORY) Suppose that in the previous question we have

$$\tau = \{-0.43, -1.67, 0.12, 0.28, -1.14\}\ .$$

Given this "training data", draw 10^4 new measurements $Y_1, Y_2, \ldots$ from the posterior predictive pdf,

$$\pi(y \mid \tau) = \iint \pi(y \mid \mu, \sigma^2)\pi(\mu, \sigma^2 \mid \tau)\, \mathrm{d}\mu \mathrm{d}\sigma^2\ ,$$

and plot their kernel density estimate (use `ksdensity.m` or `density.R`).

(a) What is the sample (or estimated) variance of this new sample?

(b) Comment on how it compares to your Monte Carlo estimate of the posterior mean $\mathbb{E}[\sigma^2 \mid \tau]$?

(c) The posterior predictive density can be calculated analytically. Derive the formula for the pdf $\pi(y \mid \tau)$ and plot it on the same graph as the kernel density estimate.