Tutorial and Lab Problems # 9
MATH3871/MATH5970

Given the data $\tau = \{x_1, \ldots, x_n\}$, suppose that we use the Bayesian likelihood $(X \,|\, \boldsymbol{\theta}) \sim \mathsf{N}(\mu, \sigma^2)$ with parameter $\boldsymbol{\theta} = (\mu, \sigma^2)^\top$ and wish to compare the following two nested models.

1. Model $m = 1$, where $\sigma^2 = \sigma_0^2$ is known and this is incorporated via the prior

$$g(\boldsymbol{\theta} \,|\, m = 1) = g(\mu \,|\, \sigma^2, m = 1) g(\sigma^2 \,|\, m = 1) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(\mu - x_0)^2}{2\sigma^2}} \times \delta(\sigma^2 - \sigma_0^2) \; .$$

2. Model $m = 2$, where both mean and variance are unknown with prior

$$g(\boldsymbol{\theta} \,|\, m = 2) = g(\mu \,|\, \sigma^2) g(\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(\mu - x_0)^2}{2\sigma^2}} \times \frac{b^t (\sigma^2)^{-t-1} \mathrm{e}^{-b/\sigma^2}}{\Gamma(t)} \; .$$

Show that the prior $g(\boldsymbol{\theta} \,|\, m = 1)$ can be viewed as the limit of the prior $g(\boldsymbol{\theta} \,|\, m = 2)$ when $t \to \infty$ and $b = t\sigma_0^2$. Hence, conclude that

$$g(\tau \,|\, m = 1) = \lim_{\substack{t \to \infty \\ b = t\sigma_0^2}} g(\tau \,|\, m = 2)$$

and use this result to calculate $B_{1\,|\,2}$. Check that the formula for $B_{1\,|\,2}$ agrees with the Savage-Dickey density ratio:

$$\frac{g(\tau \,|\, m = 1)}{g(\tau \,|\, m = 2)} = \frac{g(\sigma^2 = \sigma_0^2 \,|\, \tau)}{g(\sigma^2 = \sigma_0^2)} \; ,$$

where $g(\sigma^2 \,|\, \tau)$ and $g(\sigma^2)$ are the posterior and prior, respectively, under model $m = 2$.

Answers:

First, if $1/\sigma^2 =: X_t \sim \mathsf{Gamma}(t, t\sigma_0^2)$, then

$$\mathbb{P}(|X_t - 1/\sigma_0^2| > \epsilon) \leq \frac{\mathbb{V}\mathrm{ar}(X_t)}{\epsilon^2} = \frac{t}{(t\sigma_0^2)^2 \epsilon^2} \longrightarrow 0$$

gives the convergence in probability $X_t \xrightarrow{\mathbb{P}} 1/\sigma_0^2$.

Next, let $\bar{x}_n := \sum_{i\geq 0} x_i/(n+1)$ and $S_n^2 := \sum_{i\geq 0}(x_i - \bar{x}_i)^2/(n+1)$. We have

$$\begin{aligned}
g(\tau) = g(\tau \,|\, m=2) &= \frac{b^t}{\Gamma(t)} \iint \frac{\mathrm{e}^{-\frac{b}{\sigma^2} - \frac{\sum_{i=0}^n (x_i-\mu)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{(n+1)/2}(\sigma^2)^{t+1}} \mathrm{d}\mu \mathrm{d}\sigma^2 \\
&= \frac{b^t}{\Gamma(t)} \int \frac{\mathrm{e}^{-\frac{b}{\sigma^2} - \frac{(n+1)S_n^2}{2\sigma^2}}}{(2\pi\sigma^2)^{(n+1)/2}(\sigma^2)^{t+1}} \frac{\sigma\sqrt{2\pi}}{\sqrt{n+1}} \mathrm{d}\sigma^2 \\
&= \frac{b^t}{\Gamma(t)(2\pi)^{n/2}\sqrt{n+1}} \int \frac{\mathrm{e}^{-\frac{b+(n+1)S_n^2/2}{\sigma^2}}}{(\sigma^2)^{t+n/2+1}} \mathrm{d}\sigma^2 \\
&= \frac{b^t}{\Gamma(t)(2\pi)^{n/2}\sqrt{n+1}} \frac{\Gamma(t+n/2)}{(b+(n+1)S_n^2/2)^{t+n/2}}
\end{aligned}$$

Since $\Gamma(t+n/2)/\Gamma(t) \simeq t^{n/2}$ and

$$\frac{t^{t+n/2}\sigma_0^{2t}}{(t\sigma_0^2 + (n+1)S_n^2/2)^{t+n/2}} \longrightarrow \frac{1}{\sigma_0^n} \mathrm{e}^{-\frac{(n+1)S_n^2}{2\sigma_0^2}},$$

we have

$$\lim_{\substack{t\to\infty \\ b=t\sigma_0^2}} g(\tau \,|\, m=2) = \frac{1}{(2\pi)^{n/2}\sqrt{n+1}} \times \frac{1}{\sigma_0^n} \mathrm{e}^{-\frac{(n+1)S_n^2}{2\sigma_0^2}}$$

Therefore,

$$B_{1\,|\,2} = \frac{(b+(n+1)S_n^2/2)^{t+n/2}}{b^t \sigma_0^n} \frac{\Gamma(t)}{\Gamma(t+n/2)} \mathrm{e}^{-\frac{(n+1)S_n^2}{2\sigma_0^2}}.$$

To show that this agrees with Savage-Dickey, we first find the posterior:

$$g(\sigma^2 \,|\, \tau) = \frac{\alpha^{t+n/2}}{\Gamma(t+n/2)} \frac{\mathrm{e}^{-\alpha/\sigma^2}}{(\sigma^2)^{t+n/2+1}},$$

where $\alpha = b + (n+1)S_n^2/2$. Dividing by the prior $g(\sigma^2)$ gives

$$\frac{\Gamma(t)\alpha^{t+n/2}}{b^t\Gamma(t+n/2)} \frac{\mathrm{e}^{-(\alpha-b)/\sigma^2}}{(\sigma^2)^{n/2}},$$

which agrees with $B_{1\,|\,2}$ when evaluated at $\sigma^2 = \sigma_0^2$.