Tutorial Problems 2
MATH3871/MATH5970
Bayesian Inference and Computation

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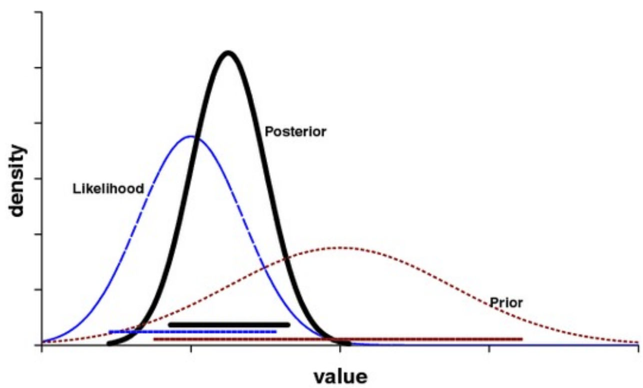

1. Derive the posterior if $X_1, \ldots, X_n$ are a random sample from the distribution with probability mass function (Geometric distribution with $x > 0$)

$$p(x \mid \theta) = \theta^{x-1}(1-\theta); \quad x = 1, 2, \ldots$$

with the $\mathsf{Beta}(p, q)$ prior distribution

$$p(\theta) = \frac{\theta^{p-1}(1-\theta)^{q-1}}{\mathrm{B}(p,q)}, \quad 0 < \theta < 1$$

2. Derive the posterior if $X_1, \ldots, X_n$ are a random sample from the distribution with probability mass function

$$p(x \mid \theta) = \exp(-\theta)\frac{\theta^x}{x!}, \quad x = 0, 1, \ldots$$

with the prior distribution

$$p(\theta) = \exp(-\theta), \quad 0 < \theta$$

3. The proportion, θ, of defective items in a large shipment is unknown, but expert assessment assigns θ the $\mathsf{Beta}(2, 200)$ prior distribution. If 100 items are selected at random from the shipment, and 3 are found to be defective, what is the posterior distribution of θ?

4. If a different statistician, having observed the 3 defectives, calculated her posterior distribution as being a Beta distribution with mean $4/102$ and variance 0.0003658, then what prior distribution had she used?

5. The diameter of a component from a long production run varies according to a $\mathsf{N}(\theta, 1)$ distribution. An Engineer specifies that the prior distribution for θ is $\mathsf{N}(10, 0.25)$. In one production run 12 components are sampled and found to have a sample mean diameter of $31/3$. Use this information to calculate the probability that the mean component diameter is at least 10 units.

Example of inverse transform method. Suppose you are given the density $f_R(r) = r\,\mathrm{e}^{-r^2/2}, r > 0$ and we wish to simulate from it. We can use the inverse-transform method for this. The cdf of R is, by integration of the pdf,

$$F_R(r) = 1 - \mathrm{e}^{-\frac{1}{2}r^2}, \quad r > 0,$$

and its inverse is found by solving $u = F_R(r)$ in terms of r, giving

$$F_R^{-1}(u) = \sqrt{-2\log(1-u)}, \quad u \in (0, 1)\ .$$

Thus R has the same distribution as $\sqrt{-2\log(1-U)}$, with $U \sim \mathsf{U}(0, 1)$. Since $1-U$ also has a $\mathsf{U}(0, 1)$ distribution, R has also the same distribution as $\sqrt{-2\ln U}$.