Tutorial Problems 3
MATH3871/MATH5970
Bayesian Inference and Computation

1. Find Jeffreys' prior for θ in the geometric model

$$\mathbb{P}(X = x \mid \theta) = (1-\theta)^{x-1}\theta; \quad x = 1, 2, \ldots$$

for a random sample $X_1, \ldots, X_n$.

2. Suppose that $X_1, \ldots, X_n \sim \mathsf{N}(\mu, \tau^{-1})$ with a known μ, where $\tau = 1/\sigma^2$ is the precision parameter. If we specify the prior $\tau \sim \mathsf{Gamma}(\alpha, \beta)$, since $\tau = 1/\sigma^2$, then σ^2 is distributed as Inverse Gamma. Show that the Gamma distribution is conjugate for the posterior of τ, and that the Inverse Gamma distribution is the conjugate for the posterior of σ^2.

Solutions

1. The likelihood for this model is

$$L(x|\theta) = (1-\theta)^{\sum x_i - n}\theta^n.$$

Noting that $E(x) = 1/\theta$, then the information matrix is computed as

$$\begin{aligned}
I(\theta) &= -E(\frac{\partial^2 \log L(x|\theta)}{\partial\theta^2}) \\
&= -E(\frac{\partial^2((\sum x_i - n)\log(1-\theta) + n\log\theta)}{\partial\theta^2}) \\
&= -E(\frac{\partial[-\frac{\sum x_i - n}{1-\theta} + \frac{n}{\theta}]}{\partial\theta}) \\
&= E(\frac{\sum x_i - n}{(1-\theta)^2} + n/\theta^2) \\
&= \frac{nE(x_i) - n}{(1-\theta)^2} + n/\theta^2 \\
&= n[\frac{1/\theta - 1}{(1-\theta)^2} + 1/\theta^2] \\
&\propto \frac{1}{\theta^2(1-\theta)}
\end{aligned}$$

Hence Jeffreys' prior is proportional to $|I(\theta)|^{1/2} = \theta^{-1}(1-\theta)^{-1/2}$.

2. The likelihood is

$$L(x\,|\,\tau^{-1}) = \prod_{i=1}^n \sqrt{\tfrac{\tau}{2\pi}}\exp(-\tau/2(x-\mu)^2)$$

and the prior is a Gamma distribution

$$\pi(\tau) \propto \tau^{\alpha-1}\exp(-\beta\tau).$$

Hence the posterior distribution is ($S_\mu^2 := \sum_{i=1}^n (x_i-\mu)^2/n$)

$$\begin{aligned}
\pi(\tau\,|\,x) &\propto \tau^{n/2}\exp(-nS_\mu^2\tau/2) \times \tau^{\alpha-1}\exp(-\beta\tau) \\
&= \tau^{\alpha+n/2-1}\exp(-\tau[\beta + nS_\mu^2]) \\
\tau\,|\,x &\sim \text{Gamma}(n/2+\alpha, \beta + 1/2nS_\mu^2).
\end{aligned}$$

Similarly, if we use the σ^2 parameterisation, then

$$L(x\,|\,\sigma^2) = \prod_{i=1}^n \tfrac{1}{\sqrt{2\pi\sigma^2}}\exp(-\tfrac{1}{2\sigma^2}(x-\mu)^2),$$

and the prior is an Inverse Gamma distribution

$$\pi(\sigma^2) \propto \sigma^{2(-\alpha-1)}\exp(-\beta/\sigma^2).$$

Hence the posterior distribution is

$$\begin{aligned}
\pi(\sigma^2\,|\,x) &\propto \sigma^{-n/2}\exp\left(-\frac{nS_\mu^2}{2\sigma^2}\right) \times \sigma^{2(-\alpha-1)}\exp(-\beta/\sigma^2) \\
&= \sigma^{-n/2-\alpha-1}\exp\left(-\frac{1}{\sigma^2}[\beta + \frac{nS_\mu^2}{2}]\right) \\
\sigma^2\,|\,x &\sim \text{IGamma}(n/2+\alpha, \beta + nS_\mu^2/2)).
\end{aligned}$$