Question3: Gibbs sampling for Truncated Normal.

(a). Replicate the numerical results (the two figures) for Example 1.7

We wish to simulate from bivariate normal density $X \sim N(o, \sum)$ conditional on $X \geq o$. In other words, the target is

$$\pi(x) \propto \exp\left(-x^T \Sigma^{-1} x/2\right) I\{x \geq o\},$$

where we may assume (without loss of generality) that Σ is a correlation matrix:

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

In my model, I assume $\rho = 0.9$. So $\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$.

Let$\mathrm{TN}_I(\mu, \sigma^2)$be notation for the distribution of $X \sim N(\mu, \sigma^2)$,conditional on $X \in I$. The conditional densities are $X|Y = y \sim \mathrm{T}N_{[0,\infty)}(\rho y, 1-\rho^2)$ and $Y|X = x \sim \mathrm{T}N_{[0,\infty)}(\rho x, 1-\rho^2)$, which are simple to simulate from. Figure 1 shows the output of this sampler using 10^4 iterations, starting with $X_1 = 0$.

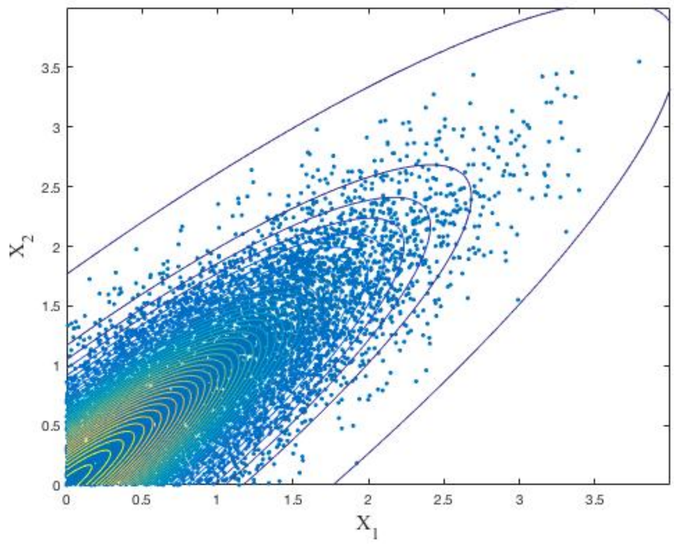

Figure 1: Contours of the target π and (approximate) random realizations from π using Gibbs sampling.

We can make a transformation to possibly speed up the convergence of the sampler. Let $(x, y)^T = Lz$, where

$$L = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.9 & \sqrt{1-0.9^2} \end{bmatrix}$$

$$\pi(z) \propto \exp\left(-||z||^2/2\right) I\{Lz \geq o\}.$$

The first conditional for Gibbs sampling becomes $Z_1|Z_2 = z_2 \sim TN_{[\alpha,\infty)}(0,1)$, where

$$\alpha = \max\{0, -\sqrt{1-\rho^2}z_2/\rho\},$$

where $\rho = 0.9$. The second conditional becomes $Z_2|Z_1 = z_1 \sim TN_{[\alpha,\infty)}(0,1)$ with $\alpha = -z_1\rho/\sqrt{1-\rho^2}$. Once we have simulated $Z = (Z_1, Z_2)^T$ with (approximate) pdf π(z), we can obtain an X from π(x) via the back-transformation X = LZ. Figure 2 suggests that this transformation is effective in accelerating the convergence of the Gibbs sampler. Note that the figures show an error bars (horizonal lines) such that anything below the error bars may be considered to be zero.

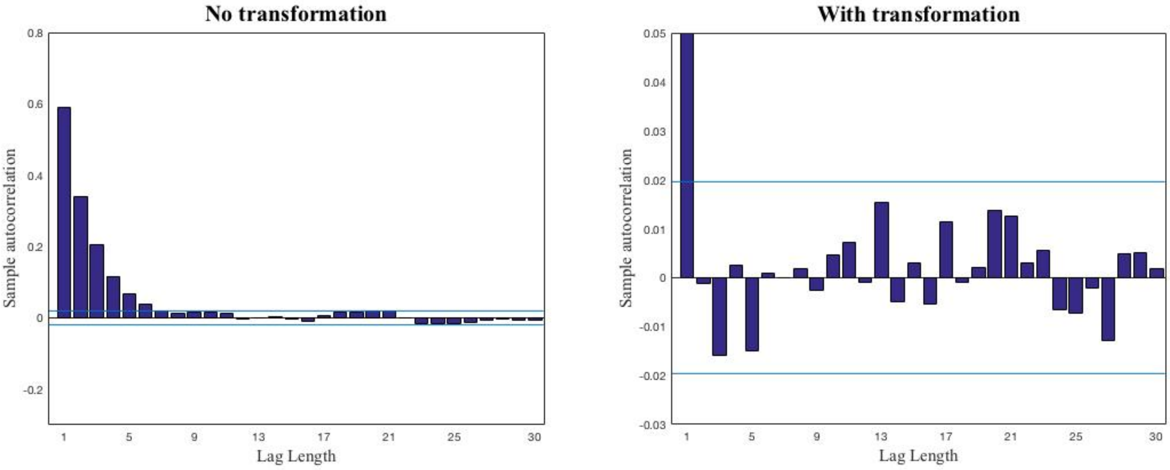

Figure 2: Left panel: autocorrelations of X1 for Gibbs sampling without the transformation. Right panel: autocorrelations with the transformation.

The code has been uploaded on Moodle.

(b). Estimate $E_\pi X1$ and compute the error of the estimator using the batch-means variance estimate (equation (1.2) in the notes) with m = 10^4.

In this question, assume t-large is a very large number where I assume t-large is $5 * 10^5$. Then, we use Gibbs sampling to obtain X_1 and X_2 with t-large iterations. Next, I plot the autocorrelation figure of X_1 which is shown in figure3. By figure3, we could identify which lag has least correlation with the first X_1, then we could select this lag as burnin. After we obtain the burnin, we could identify a $t = m \times b$ where $m = 10^4, b =$ burin. Assume burnin=b=13, then the new $t = 1.3 * 10^5$, next we could take the first t X_1 from original X_1 set. When we obtain the new set of X_1 with t elements, we could calculate the $E_\pi X1$ by mean formula. Then I use Z_k where $Z_k = \frac{1}{b}\sum_{j=(k-1)b+1}^{kb} \emptyset\left(X_j\right), k = 1 \ldots m.$ to calculate the variance estimator according to the formula:

$$V = \frac{b * \frac{1}{m} \sum_{k=1}^{m} (z_k - \overline{z_m})^2}{t}$$

The detailed code has been uploaded on Moodle.

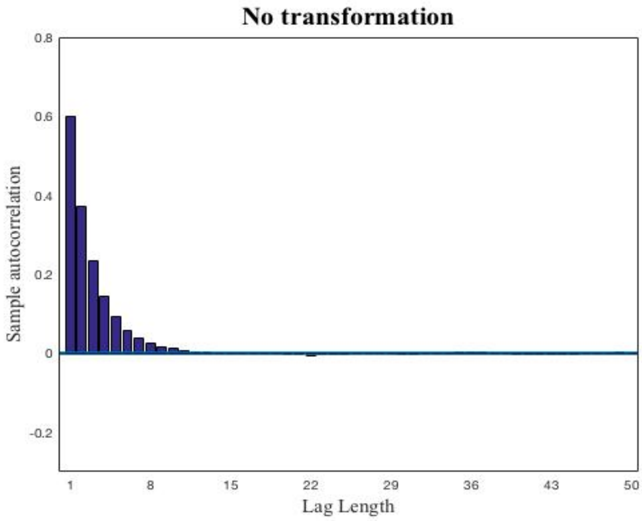

Figure3: the autocorrelation of X_1 with t elements

There is an example for expectation and batch-means variance of X_1 with burnin =15.

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E_x1 =

    0.8843


variance_z =

   8.6068e-06
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