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Infrared Limits for the N^{3}LO Era

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Abstract: We collect all necessary soft and collinear limits for the construction of a next-to-next-to-next-to-leading order subtraction scheme. We recompute and test all known infrared limits. Additionally, we identify any missing ingredients and evaluate them. New results include squared currents for the double-soft one loop asymptotic and the evanescent parts of the two loop soft current.

Keywords: QCD, Scattering Amplitudes, Higher-Order Perturbative Calculations

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$$\langle a_1\lambda_1, \dots, a_n\lambda_n | M(q_1, \dots, q_n, \{p_i\}_{i=1}^m)\rangle \quad \sim$$
$$\epsilon_{\lambda_1}^{\alpha_1\,*}(q_1) \dots \epsilon_{\lambda_n}^{\alpha_n\,*}(q_n)\, \mathbf{J}^{a_1\dots a_n}_{\alpha_1\dots\alpha_n}(q_1, \dots, q_n, \{p_i\}_{i=1}^m)\, |M(\{p_i\}_{i=1}^m)\rangle\ , \tag{3.1}$$

$$\mathbf{J} = \left(g_s^B\right)^n \left(\mathbf{J}^{(0)} + \frac{\mu^{-2\epsilon}\alpha_s^B}{(4\pi)^{1-\epsilon}}\mathbf{J}^{(1)} + \dots\right)\ , \qquad \alpha_s^B \equiv \frac{\left(g_s^B\right)^2}{4\pi}\ , \tag{3.2}$$

$$\mathbf{J}^{(0)a}_{\ \ \alpha}(q, \{p_i\}_{i=1}^m) = \sum_i \mathbf{T}_i^a j_\alpha(q; p_i), \quad j_\alpha(q; p_i) = -\frac{p_{i\alpha}}{p_i \cdot q}\ . \tag{3.3}$$

$$\mathbf{J}^{(1)a}_{\ \ \alpha}(q, \{p_i\}_{i=1}^m) = \sum_{i\neq j} i f^{abc}\mathbf{T}_i^b\mathbf{T}_j^c \gamma_\alpha^{(1)}(q; p_i, p_j), \tag{3.4}$$

$$\gamma_\alpha^{(1)}(q; p_i, p_j) = r_\Gamma \frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\epsilon^2}\left(\frac{p_{i\alpha}}{p_i\cdot q} - \frac{p_{j\alpha}}{p_j \cdot q}\right)\left(\frac{\mu^2 s_{ij}}{s_{qi}s_{qj}}\right)^\epsilon e^{i\pi\epsilon\sigma_{ij}} \tag{3.5}$$

$$r_\Gamma = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \tag{3.6}$$

$$\sigma_{ij} = \begin{cases} -1 & \text{if } i \text{ and } j \text{ incoming} \\ +1 & \text{otherwise} \end{cases} \tag{3.7}$$

$$\mathbf{J}^{(0)a_1a_2}_{\ \ \alpha_1\alpha_2}(q_1, q_2, \{p_i\}_{i=1}^m) = \frac{1}{2}\left\{\mathbf{J}^{(0)a_1}_{\ \ \alpha_1}(q_1, \{p_i\}_{i=1}^m), \mathbf{J}^{(0)a_2}_{\ \ \alpha_2}(q_1, \{p_i\}_{i=1}^m)\right\} + \Gamma^{(0)a_1a_2}_{\ \ \alpha_1\alpha_2}(q_1, \{p_i\}_{i=1}^m) \tag{3.8}$$

$$\Gamma^{(0)a_1a_2}_{\ \ \alpha_1\alpha_2}(q_1, \{p_i\}_{i=1}^m) = \sum_i i f^{a_1a_2c}\mathbf{T}_i^c \gamma^{(0)}_{\alpha_1\alpha_2}(q_1, q_2; p_i) \tag{3.9}$$

$$\gamma^{(0)}_{\alpha_1\alpha_2}(q_1, q_2; p_i) = \frac{p_{i\alpha_1}q_{1\alpha_2} - p_{i\alpha_2}q_{2\alpha_1}}{(q_1\cdot q_2)\big(p_i\cdot(q_1+q_2)\big)} - \frac{p_i\cdot(q_1-q_2)}{2p_i\cdot(q_1+q_2)}\left(\frac{p_{i\alpha_1}p_{i\alpha_2}}{(p_i\cdot q_1)(p_i\cdot q_2)} + \frac{g_{\alpha_1\alpha_2}}{q_1\cdot q_2}\right) \tag{3.10}$$

$$
\begin{aligned}
&\mathbf{J}^{(1)a_1a_2}_{\alpha_1\alpha_2}\big(q_1,q_2,\{p_i\}_{i=1}^m\big) && (3.11)\\
&\quad=\mathbf{J}^{(1)a_1}_{\alpha_1}\big(q_1,\{p_i\}_{i=1}^m\big)\mathbf{J}^{(0)a_2}_{\alpha_2}\big(q_2,\{p_i\}_{i=1}^m\big)+\mathbf{J}^{(1)a_2}_{\alpha_2}\big(q_2,\{p_i\}_{i=1}^m\big)\mathbf{J}^{(0)a_1}_{\alpha_1}\big(q_1,\{p_i\}_{i=1}^m\big) && (3.12)\\
&\qquad+\Gamma^{(1)a_1a_2}_{\alpha_1\alpha_2}\big(q_1,q_2,\{p_i\}_{i=1}^m\big) && (3.13)
\end{aligned}
$$

$$
\Gamma^{(1)a_1a_2}_{\alpha_1\alpha_2}\big(q_1,q_2,\{p_i\}_{i=1}^m\big)=\frac{1}{q_1\cdot q_1}f^{a_1bd}f^{a_2cd}\sum_{i\neq j}\mathbf{T}_i^b\mathbf{T}_j^c\gamma^{(1)}_{\alpha_1\alpha_2}(q_1,q_2,p_i,p_j) \tag{3.14}
$$

$$
\begin{aligned}
\gamma^{(1)}_{\alpha_1\alpha_2}(q_1,q_2,p_i,p_j)\equiv{}&\tilde{J}(q_1,q_2,p_i,p_j)\big[\tilde{g}_{\alpha_1\alpha_2}\big]+\hat{J}_{++}(q_1,q_2,p_i,p_j)\big[\hat{g}_{\perp\alpha_1\alpha_2}\big]\\
&+J_{++}(q_1,q_2,p_i,p_j)\left[\frac{q_1\cdot q_2}{(p_i\cdot q_1)(p_j\cdot q_2)}\big(p_{i\perp\alpha_1}p_{j\perp\alpha_2}-p_{j\perp\alpha_1}p_{i\perp\alpha_2}\big)\right]\\
&+J_{+-}(q_1,q_2,p_i,p_j)\left[\hat{g}_{\perp\alpha_1\alpha_2}-\frac{2\,p_{i\perp\alpha_1}p_{i\perp\alpha_2}}{p_{i\perp}^2}\right]+J_{+-}(q_2,q_1,p_j,p_i)\left[\hat{g}_{\perp\alpha_1\alpha_2}-\frac{2\,p_{j\perp\alpha_1}p_{j\perp\alpha_2}}{p_{j\perp}^2}\right].
\end{aligned} \tag{3.15}
$$

3.1 Squared Currents

$$
|\mathbf{J}^{(0)}\big(q,\{p_i\}_{i=1}^m\big)|^2=-\sum_{i,j}j(q,p_i)\cdot j(q,p_j)\mathbf{T}_i^a\mathbf{T}_j^a \tag{3.16}
$$

$$
\begin{aligned}
&\mathbf{J}^{(0)\dagger}\big(q,\{p_i\}_{i=1}^m\big)\cdot\mathbf{J}^{(1)}\big(q,\{p_i\}_{i=1}^m\big)+\text{h.c.}\\
&\quad=\sum_{i\neq j}\mathbf{T}_i\cdot\mathbf{T}_j\;2\,\mathrm{Re}\big(j(q,p_i)\cdot\gamma^{(1)}(q,p_i,p_j)\big)+\sum_{(i,j,k)}f^{abc}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\;2\,\mathrm{Im}\big(j(q,p_i)\cdot\gamma^{(1)}(q,p_j,p_k)\big)
\end{aligned} \tag{3.17}
$$

$$
|\mathbf{J}^{(0)}\big(q_1,q_2,\{p_i\}_{i=1}^m\big)|^2=\Big(\mathbf{J}^{(0)}\big(q_1,\{p_i\}_{i=1}^m\big)\mathbf{J}^{(0)}\big(q_2,\{p_i\}_{i=1}^m\big)\Big)_{sym}+W^{(0)}\big(q_1,q_2,\{p_i\}_{i=1}^m\big) \tag{3.18}
$$

$$
W^{(0)}\big(q_1,q_2,\{p_i\}_{i=1}^m\big)=-C_A\sum_{i,j}\mathbf{T}_i\cdot\mathbf{T}_jS^{(0)}(q_1,q_2,p_i,p_j) \tag{3.19}
$$

$$
\begin{aligned}
S^{(0)}(q_1,q_2,p_i,p_j)={}&-\gamma^{(0)}_{\alpha_1\alpha_2}(q_1,q_2,p_i)\gamma^{(0)\alpha_1\alpha_2}(q_1,q_2,p_j)-\gamma^{(0)}_{\alpha_1\alpha_2}(q_1,q_2,p_i)j^{\alpha_1}(q_1,p_i)j^{\alpha_2}(q_2,p_j)\\
&+\gamma^{(0)}_{\alpha_1\alpha_2}(q_1,q_2,p_i)j^{\alpha_1}(q_1,p_i)j^{\alpha_2}(q_2,p_j)-\frac{3}{4}\left(j(q_1,p_i)\cdot j(q_1,p_j)\right)\left(j(q_2,p_i)\cdot j(q_2,p_j)\right)\\
&+\frac{1}{2}j(q_1,p_i)^2\left(j(q_2,p_i)\cdot j(q_2,p_j)\right)+\frac{1}{2}j(q_2,p_j)^2\left(j(q_1,p_i)\cdot j(q_1,p_j)\right)
\end{aligned} \tag{3.20}
$$

3.1.1 One-Loop Double-Soft Limit

$$
\begin{aligned}
&\mathbf{J}^{(0)\dagger}\big(q_1,q_2,\{p_i\}_{i=1}^m\big)\cdot\mathbf{J}^{(1)}\big(q_1,q_2,\{p_i\}_{i=1}^m\big)+\text{h.c.}\\
&\quad=\bigg\{\Big[|\mathbf{J}^{(0)}\big(q_1,\{p_i\}_{i=1}^m\big)|^2\Big(\mathbf{J}^{(0)\dagger}\big(q_2,\{p_i\}_{i=1}^m\big)\cdot\mathbf{J}^{(1)}\big(q_2,\{p_i\}_{i=1}^m\big)+\text{h.c.}\Big)\Big]_{sym}+(q_1\longleftrightarrow q_2)\bigg\}\\
&\qquad+\sum_{i,j,k,l}\Big(f^{ade}f^{bce}\mathbf{T}_i^a\{\mathbf{T}_j^b,\mathbf{T}_k^c\}\mathbf{T}_l^d+\text{h.c.}\Big)\,S^{(1)}(q_1,q_2,p_i,p_j,p_k,p_l)\\
&\qquad+\sum_{(i,j,k)}f^{abc}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\;S^{(1)}(q_1,q_2,p_i,p_j,p_k)+\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_j\;S^{(1)}(q_1,q_2,p_i,p_j)
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
&S^{(1)}(q_1,q_2,p_i,p_j,p_k,p_l)\\
&=2\,\mathrm{Re}\bigg\{-\frac{1}{4}\Big(\gamma^{(1)}(q_1,p_k,p_l)\cdot j(q_1,p_i)\Big)\left(j(q_2,p_i)\cdot j(q_2,p_j)\right)-\frac{1}{4}\gamma^{(1)}_{\alpha}(q_1,p_j,p_k)\gamma^{(0)\alpha\beta}(q_1,q_2,p_i)j_\beta(q_2,p_l)\\
&\quad+\frac{1}{8}\Big(\gamma^{(1)}_{\alpha\beta}(q_1,q_2,p_i,p_j)-\gamma^{(1)}_{\alpha\beta}(q_1,q_2,p_i,p_i)\Big)\,j^{\alpha}(q_1,p_l)j^{\beta}(q_2,p_k)+(q_1\longleftrightarrow q_2)\bigg\}
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&S^{(1)}(q_1,q_2,p_i,p_j,p_k)\\
&=2C_A\,\mathrm{Im}\Big\{\frac{1}{2}\gamma^{(1)}_{\alpha}(q_1,p_j,p_k)\gamma^{(0)\alpha\beta}(q_1,q_2,p_i)j_{\beta}(q_2,p_i)-\frac{1}{2}\gamma^{(1)}_{\alpha}(q_1,p_j,p_k)\gamma^{(0)\alpha\beta}(q_1,q_2,p_i)j_{\beta}(q_2,p_j)\\
&\quad-\frac{1}{2}\gamma^{(1)}_{\alpha}(q_1,p_j,p_k)\gamma^{(0)\alpha\beta}(q_1,q_2,p_j)j_{\beta}(q_2,p_i)+\frac{1}{4}\gamma^{(0)}_{\alpha\beta}(q_1,q_2,p_i)\gamma^{(1)\alpha\beta}(q_1,q_2,p_j,p_k)\\
&\quad-\frac{3}{4}\Big(\gamma^{(1)}(q_1,p_j,p_k)\cdot j(q_1,p_i)\Big)\left(j(q_2,p_i)\cdot j(q_2,p_j)\right)-\frac{1}{2}\Big(\gamma^{(1)}(q_1,p_j,p_k)\cdot j(q_1,p_i)\Big)\left(j(q_2,p_j)\cdot j(q_2,p_k)\right)\\
&\quad+\frac{1}{4}\Big(\gamma^{(1)}(q_1,p_j,p_k)\cdot j(q_1,p_j)\Big)\left(j(q_2,p_i)\cdot j(q_2,p_j)\right)+\frac{1}{2}\gamma^{(1)}_{\alpha\beta}(q_1,q_2,p_i,p_j)j^{\alpha}(q_1,p_i)j^{\beta}(q_2,p_k)\\
&\quad-\frac{1}{4}\gamma^{(1)}_{\alpha\beta}(q_1,q_2,p_i,p_j)j^{\alpha}(q_1,p_j)j^{\beta}(q_2,p_k)+(q_1\longleftrightarrow q_2)\Big\}
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&S^{(1)}(q_1,q_2,p_i,p_j)\\
&=2C_A^2\,\mathrm{Re}\Big\{\frac{1}{8}\gamma^{(1)}_{\alpha\beta}(q_1,q_2,p_i,p_j)j^{\alpha}(q_1,p_i)j^{\beta}(q_2,p_i)-\frac{1}{8}\gamma^{(1)}_{\alpha\beta}(q_1,q_2,p_i,p_j)j^{\alpha}(q_1,p_i)j^{\beta}(q_2,p_j)\\
&\quad-\Big(\gamma^{(1)}(q_1,p_i,p_j)\cdot j(q_1,p_i)\Big)\left(j(q_2,p_i)\cdot j(q_2,p_j)\right)+\frac{1}{4}\gamma^{(1)}_{\alpha\beta}(q_1,q_2,p_i,p_j)\gamma^{(0)\alpha\beta}(q_1,q_2,p_i)\\
&\quad+(q_1\longleftrightarrow q_2)\Big\}
\end{aligned}
\tag{3.24}
$$

4 Soft Quark Limits

$$
\mathbf{J}^{(0)}_{a_1a_2}(q_1,q_2,\{p_i\}_{i=1}^m)=-\sum_i T^c_{a_1a_2}\mathbf{T}^c_i\,\frac{\bar{u}(q_1,\lambda_1)\,\not{p}_i\,v(q_2,\lambda_2)}{2(q_1\cdot q_2)\big(p_i\cdot(q_1+q_2)\big)}\ . \tag{4.1}
$$

$$
\begin{aligned}
&\mathbf{J}^{(1)}_{a_1a_2}(q_1,q_2,\{p_i\}_{i=1}^m)=\frac{1}{q_1\cdot q_2}\sum_{i\neq j}(T^bT^c)_{a_1a_2}\mathbf{T}^b_i\mathbf{T}^c_j\\
&\quad\times\left[\bar{u}(q_1,\lambda_1)\left(\frac{\not{p}_i}{p_i\cdot(q_1+q_2)}\,J_{q\bar{q}}(q_1,q_2,p_i,p_j)-\frac{\not{p}_j}{p_j\cdot(q_1+q_2)}\,J_{q\bar{q}}(q_2,q_1,p_j,p_i)\right)v(q_2,\lambda_2)\right]
\end{aligned}
\tag{4.2}
$$

4.1 Squared Currents

$$
|\mathbf{J}^{(0)}(q_1,q_2)|^2=T_F\sum_{i,j}\mathbf{T}_i\cdot\mathbf{T}_j\mathcal{I}(q_1,q_2,p_i,p_j) \tag{4.3}
$$

$$
\mathcal{I}(q_1,q_2,p_i,p_j)=\frac{p_i\cdot q_1p_j\cdot q_2+p_i\cdot q_2p_j\cdot q_1-p_i\cdot p_jq_1\cdot q_2}{(q_1\cdot q_2)^2p_i\cdot(q_1+q_2)p_j\cdot(q_1+q_2)} \tag{4.4}
$$

$$
\begin{aligned}
&\mathbf{J}^{(0)\dagger}(q_1,q_2,\{p_i\}_{i=1}^m)\mathbf{J}^{(1)}(q_1,q_2,\{p_i\}_{i=1}^m)+\text{h.c.}\\
&=-2T_F\sum_{(i,j,k)}d^{abc}\mathbf{T}^a_i\mathbf{T}^b_j\mathbf{T}^c_k\mathcal{I}(q_1,q_2,p_i,p_j)\mathrm{Re}\big\{J_{q\bar{q}}(q_1,q_2,p_j,p_k)-J_{q\bar{q}}(q_2,q_1,p_j,p_k)\big\}\\
&\quad-2T_F\sum_{(i,j,k)}f^{abc}\mathbf{T}^a_i\mathbf{T}^b_j\mathbf{T}^c_k\mathcal{I}(q_1,q_2,p_i,p_j)\mathrm{Im}\big\{J_{q\bar{q}}(q_1,q_2,p_j,p_k)+J_{q\bar{q}}(q_2,q_1,p_j,p_k)\big\}
\end{aligned}
\tag{4.5}
$$