Integrating Out a Heavy Quark: Explicit Sketch

1. Setting Up the Problem

Consider QCD with all quark flavors, including a heavy top quark of mass m_t. In (Euclidean-signature) notation, the relevant part of the action is:

$\mathrm{S}_{\mathrm{QCD}} \;=\; \int d^4x \left[\tfrac{1}{4}\, G^a_{\mu\nu}\, G^a_{\mu\nu} \;+\; \bar{t}\left(\gamma^\mu D_\mu + m_t\right) t \;+\; \dots\right],$

where

$\mathrm{G}^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s\, f^{abc}\, A^b_\mu\, A^c_\nu$

$isthenon-Abelianfieldstrengthtensor(withgaugecoupling\mathrm{g}_s$ and structure constants f^{abc}), and $D_\mu = \partial_\mu - i\, g_s\, A^a_\mu\, T^a$ is the covariant derivative in the quark representation. Throughout, $\dots$ indicates terms involving the lighter quarks, ghosts, etc., which for our purposes are not crucial to show explicitly.

The partition function (generating functional) is written schematically as:

$\mathrm{Z} \;=\; \int \mathcal{D}A_\mu\, \mathcal{D}(\bar{t},t)\; \exp\!\big[-\,S_{\mathrm{QCD}}[A_\mu,\bar{t},t,\dots]\big].$

$Whenwe``integrateout"thetopquark, weperformthepathintegralover\mathrm{t},$:

$\int \mathcal{D}(\bar{t},t)\; \exp\!\big[-\int d^4x\; \bar{t}\left(\gamma^\mu D_\mu + m_t\right) t\big] \;=\; \det\!\big(\gamma^\mu D_\mu + m_t\big).$

$Hencetheeffectiveactionobtainsacontribution$

$\mathrm{S}_{\mathrm{eff}}[A_\mu] \;=\; -\,\ln\,\det\!\big(\gamma^\mu D_\mu + m_t\big) \;+\; \dots\;,$

$wherethe``\dots"includestheoriginalgauge-fieldactionandotherquarkterms.Weseekthelocaloperator(s)genera$

expansion of $\ln\det(\gamma^\mu D_\mu + m_t)$.

2. Factor Out the Large Mass and Expand

Define the Dirac operator

$\mathrm{O} \;=\; \gamma^\mu D_\mu + m_t.$

$Then$

$\ln\det(\mathcal{O}) \;=\; \mathrm{Tr}\;\ln(\mathcal{O}).$

$Wecanwrite$

$\mathrm{O} \;=\; \mathrm{m}_t\left(1 + X\right), \quad \text{where} \quad X \;=\; \frac{\gamma^\mu D_\mu}{m_t}.$

$Thus,$

$\ln\det(\mathcal{O}) \;=\; \mathrm{Tr}\;\ln\!\big(m_t\big) \;+\; \mathrm{Tr}\;\ln\!\big(1 + X\big).$

$Thepiece\mathrm{Tr}\ln(m_t)$ is field-independent and does not affect the gluon dynamics, so we can drop it. Hence,

$\ln\det(\mathcal{O}) \;=\; \mathrm{Tr}\;\ln(1+X) \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,\mathrm{Tr}\!\big(X^n\big), \quad X \;=\; \frac{\gamma^\mu D_\mu}{m_t}.$

3. Power-Series Terms and Gauge Invariance

Each X^n is $(\gamma^\mu D_\mu)^n/m_t^n$. To obtain a local gauge-invariant operator of *dimension four* in the fields, we need to examine terms of a certain total dimension in derivatives and fields. Since D_μ is of mass dimension 1, dividing by the large mass m_t reduces that dimension by 1 each time.

It turns out that the first nontrivial local operator at dimension four (beyond a constant) is

$\mathrm{G}^a_{\mu\nu}\, G^a_{\mu\nu}.$

$Thisarisesinpartbecausethefieldstrength\mathrm{G}^a_{\mu\nu}$ appears through commutators of covariant derivatives, $[D_\mu, D_\nu] \propto G^a_{\mu\nu}$, and the gamma matrices γ^μ can produce a factor of $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ in the expansion.

4. Schwinger–DeWitt (Heat Kernel) Perspective

A more systematic method uses the Schwinger–DeWitt expansion in (Euclidean) proper time:

$$\Gamma[A_\mu] \;=\; -\,\mathrm{ln det}\big(\gamma^\mu D_\mu + m_t\big) \;=\; -\int_0^\infty \frac{ds}{s}\, e^{-\,m_t^2\, s}\,\mathrm{Tr}\Big[e^{-\,s\,(\gamma^\mu D_\mu)^2}\Big].$$

$One then expands e^{-s\,(\gamma^\mu D_\mu)^2}$ in a heat-kernel series. The coefficient of s^0 (in 4D) turns out to include the term $\propto G^a_{\mu\nu}\,G^a_{\mu\nu}$, so after integrating over s, one obtains a logarithmic dependence on m_t multiplying $\int d^4x\, G^a_{\mu\nu}G^a_{\mu\nu}$.

5. Evaluating the Gamma and Color Traces

For a more explicit look, recall

$$(\gamma^\mu D_\mu)^2 \;=\; \gamma^\mu\gamma^\nu\, D_\mu D_\nu \;=\; D^2 \;+\; \tfrac{1}{2}\,\sigma^{\mu\nu}\,\big[D_\mu, D_\nu\big].$$

$Since [\mathrm{D}_\mu, D_\nu] = i\, g_s\, T^a\, G^a_{\mu\nu}$, we get

$$(\gamma^\mu D_\mu)^2 \;=\; D^2 \;+\; \tfrac{i\,g_s}{2}\,\sigma^{\mu\nu}\,T^a\,G^a_{\mu\nu}.$$

$The Dirac traces of combinations of \sigma^{\mu\nu}$ yield factors that pick out $G^a_{\mu\nu}\,G^a_{\mu\nu}$, while color traces over T^aT^b yield factors proportional to δ^{ab}. Altogether, this leads to a local expression of the form

$$\int d^4x\, G^a_{\mu\nu}\,G^a_{\mu\nu},$$

$with a calculable coefficient (often containing \ln m_t$ or similar).

6. Final Form of the Effective Action

After dropping constant terms and higher-dimensional operators (those suppressed by additional factors of $1/m_t$), one finds the low-energy effective action:

$$\mathrm{S_{eff}}[A_\mu] \;=\; \int d^4x\,\Big[\tfrac{1}{4}\,G^a_{\mu\nu}\,G^a_{\mu\nu} \;+\; \delta c\,\big[G^a_{\mu\nu}\,G^a_{\mu\nu}\big] \;+\; \dots\Big],$$

$where \delta c$ is a calculable coefficient depending on m_t, renormalization scheme, etc. Physically, this manifests as a shift in the effective gauge coupling at scales below m_t—the heavy quark no longer appears as a real degree of freedom but has left behind a local operator in the effective theory.

7. Summary

- Integrating out a heavy quark in QCD amounts to computing the determinant $\det(\gamma^\mu D_\mu + m_t)$ in the background of the gluon fields.
- At low energies ($p \ll m_t$), one expands this determinant in powers of $1/m_t$.
- The first nontrivial, gauge-invariant local operator at dimension four is $G^a_{\mu\nu}\,G^a_{\mu\nu}$, which corrects the gluon kinetic term via a coefficient $\propto \ln(m_t)$.
- This is the standard result in effective field theory: heavy colored fermions "decouple" at low energies, leaving behind local operators in the effective action.