Quiz 1 Formula Sheet

Singular Value Decomposition

For a matrix with n rows and p columns (taking $n \geq p$), the SVD of X is:

$$X = UDV^T, \quad \text{where}$$

U holds the left singular vectors of X,
V holds the right singular vectors of X, and
D holds the singular values of X.

Least Squares Estimators

Under the OLS regression model,

$$Y = X\beta + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I_n),$$

the LS estimator for the coefficients is:

$$\hat{\beta} = (X^\top X)^{-1} X^\top Y$$

and the ridge estimator for the coefficients is:

$$\hat{\beta}_\lambda = (X^\top X + \lambda I)^{-1} X^\top Y.$$

Principal Components Analysis

For an $n \times p$ design matrix:

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix} = \begin{bmatrix} - & x_1 & - \\ - & x_2 & - \\ & \vdots & \\ - & x_n & - \end{bmatrix}$$

with mean vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)^T$, PCA is done by eigendecomposing the sample covariance:

$$S = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T = \Phi \Lambda \Phi^T,$$

where the columns of $\Phi = [\phi_1 \ \phi_2 \ \ldots \ \phi_p]$ give us the unit vectors
in the directions of our new coordinate system.

Recommender Systems

The Funk-SVD matrix factorization model for a potentially sparse utility matrix Y is with K-dimensional embeddings for n_u users and n_m items is:

$$\hat{Y} \approx UV^T = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1K} \\ u_{21} & u_{22} & \ldots & u_{2K} \\ u_{31} & u_{32} & \ldots & u_{3K} \\ \vdots & \vdots & \ldots & \vdots \\ u_{n_u 1} & u_{n_u 2} & \ldots & u_{n_u K} \end{bmatrix} \times \begin{bmatrix} v_{11} & v_{21} & v_{31} & \ldots & v_{n_m 1} \\ v_{12} & v_{22} & v_{32} & \ldots & v_{n_m 2} \\ \vdots & & & & \vdots \\ v_{1K} & v_{2K} & v_{3K} & \ldots & v_{n_m K} \end{bmatrix}$$