Cauchy integrals of orthogonal polynomials

It turns out that

$$\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{V_n(x)}{x-z} \sqrt{\frac{x+1}{1-x}} \mathrm{d}x$$

can be computed explicitly.

But it is often easier to use the elegant fact that the Cauchy integrals of weighted OPs satisfy the same three-term recurrence as the OPs themselves. One only needs to compute the Cauchy integral of the weight to get the recurrence started. Adaptive QR can be employed to alleviate stability issues.

S Olver, R M Slevinsky, and A Townsend. Fast algorithms using orthogonal polynomials. Acta Numerica, 29:573–699, may 2020

Incomplete list of other possible approaches

- “Classical" numerics with exponential integrators. This requires discretization in both space and time.

  C Klein. Fourth order time-stepping for low dispersion Korteweg–de Vries and nonlinear Schroedinger equations. Electronic Transactions on Numerical Analysis, 29:116–135, 2008

- Compute the spectral data (bands and Dirichlet spectrum) and solve the Dubrovin equations. This requires discretization in both space and time.

  B A Dubrovin. Integrable Systems and Riemann Surfaces Lecture Notes. http://people.sis 2009

- Compute the spectral data (bands and Dirichlet spectrum), compute the periods of the basis of differentials and evaluate the theta function formula.

  B Deconinck, M Heil, A Bobenko, M van Hoeij, and M Schmies. Computing Riemann Theta Functions. Mathematics of Computation, 73(247):1417–1442, 2004

  B Deconinck and J N Kutz. Computing spectra of linear operators using the Floquet-Fourier-Hill method. Journal of Computational Physics, 219:296–321, 2006

  `Approxfun.jl, Chebfun`

  A R Osborne. Nonlinear Fourier Analysis for the Korteweg-de Vries Equation I : An Algorithm for the Direct Scattering Transform. Journal of Computational Physics, 313:284–313, 1991

  S Wahls, S Chimmalgi, and P J Prins. FNFT: A Software Library for Computing Nonlinear Fourier Transforms. Journal of Open Source Software, 3(23):597, mar 2018

- Others?