The KdV equation

The entirety of this talk will concern the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0, \quad q(x,0) = q_0(x), \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

We will initially assume that q_0 decays rapidly but that will be relaxed later.

The IST for the KdV equation is derived by considering the Lax pair

$$\mathcal{L}(t)\psi = -\psi_{xx} - q(x,t)\psi = \lambda\psi, \quad \psi_t = M(\lambda,q)\psi.$$

The specific form of $M(\lambda,q)$ is not important for this talk.

The existence of the Lax pair is really the statement that if $q(x,t)$ solves the KdV equation then one can find simultaneous solutions $\psi_\pm(x,t,\lambda)$ of these two equations.

The KdV equation

The evolution of the KdV flow represents an isospectral deformation of the Schrödinger operator

$$\mathscr{L}(t)\psi = -\psi_{xx} - q(x,t)\psi.$$

The "spectrum" of $\mathscr{L}$ is constant in time.

The forward scattering transform amounts to determining linearly independent solutions $\psi_\pm(x,t,\lambda)$ of the Lax pair, $\mathscr{L}(t)\psi_\pm = \lambda\psi_\pm$, that are:

1. analytic on the complement of the spectrum of $\mathscr{L}$,
2. are uniquely determined by jump/residue conditions on the spectrum, and
3. satisfy $\psi_\pm(x,t,\lambda) = \mathrm{e}^{\pm\mathrm{i}\sqrt{\lambda}x \pm 4\mathrm{i}\sqrt{\lambda}^3 t}(1+o(1))$, $\lambda \to \infty$.