If $\alpha j \le 0$ then

$$\mathscr{C}_{\mathbb{R}}^{+} r_{j,\alpha}(s) = \begin{cases} r_{j,\alpha}(s) & j > 0, \\ 0 & j < 0, \end{cases}$$

$$\mathscr{C}_{\mathbb{R}}^{-} r_{j,\alpha}(s) = \begin{cases} 0 & j > 0, \\ -r_{j,\alpha}(s) & j < 0. \end{cases}$$

There exists a function $\eta_{j,n}(\alpha)$, $\alpha j > 0$ such that

$$\mathscr{C}_{\mathbb{R}}^{+} r_{j,\alpha}(s) = \begin{cases} -\sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{n,0}(s) & j > 0, \\ r_{j,\alpha}(s) + \sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{-n,0}(s) & j < 0, \end{cases}$$

$$\mathscr{C}_{\mathbb{R}}^{-} r_{j,\alpha}(s) = \begin{cases} -r_{j,\alpha}(s) - \sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{n,0}(s) & j > 0, \\ \sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{-n,0}(s) & j < 0. \end{cases}$$

$$r_{j,\alpha}(s) = \left[\left(\frac{s + \mathrm{i}\nu}{s - \mathrm{i}\nu} \right)^{j} - 1 \right] \mathrm{e}^{\mathrm{i}\alpha s}$$

$$Y_{j,n}(\alpha) = -e^{-|\alpha|^{\nu}} L^{(n)}_{|j|-n}(2|\alpha|)$$

$$\sum_{n=1}^{j} \eta_{j,n}(\alpha)\, r_{\sigma n,0}(x) = \sum_{n=1}^{|j|} \gamma_{j,n}(\alpha) \left(\frac{-2\mathrm{i}\sigma}{x+\sigma \mathrm{i}} \right)^{n}, \sigma = \nu\, \mathrm{sign}(j),$$

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$$\sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{\sigma n,0}(x) = \sum_{n=1}^{|j|} \gamma_{j,n}(\alpha) \left(\frac{-2\mathrm{i}\sigma}{x+\sigma\mathrm{i}}\right)^n, \sigma = \nu\,\mathrm{sign}(j),$$

$$\gamma_{j,n}(\alpha) = -\mathrm{e}^{-|\alpha|\nu} L_{|j|-n}^{(n)}(2|\alpha|)$$

$$\text{p.v.}\int_{\mathbb{R}} r_{j,-\alpha}(s)\mathrm{d}s = \begin{cases} -4\pi e^{-|\alpha|}L^{(1)}_{|j|-1}(2|\alpha|\sigma) & j\alpha > 0, \\ -2\pi|j| & \alpha = 0, \\ 0 & j\alpha < 0, \end{cases}$$

With this formula we can compute $L^2(\mathbb{R})$ inner products.

$$\begin{array}{c} \text{Apply the approximate operator exactly to the basis} \\ \text{Compute inner products exactly} \end{array} \implies \text{apply GMRES}$$

To do this in high precision, the only tool that is needed is a high-precision enabled FFT.

This also necessary since we will need ≈ 4000 basis functions to adequately approximate the reflection coefficient —- 8000×8000 dense `BigFloat` matrix. And this is supposing we knew a priori which α's to choose.