A choice of basis of holomorphic differentials

Classically, for hyperelliptic surfaces, one uses the basis

$$\frac{\lambda^\ell}{w}, \quad \ell = 0, 1, 2, \ldots, g-1$$

where

$$w^2 = (\lambda - \alpha_{g+1}) \prod_{j=1}^{g} [(\lambda - \alpha_j)(\lambda - \beta_j)].$$

But this results in terribly ill-conditioned period matrices. It is much better to use something like

$$\frac{1}{w} \prod_{\substack{j=1 \\ j \neq \ell}}^{g} (\lambda - \beta_j), \quad \ell = 1, 2, \ldots, g.$$

Some notes on complexity

Consider computing a genus g solution. Suppose ϵ is the tolerance parameter.

Initialization:

- We first have to compute the periods of the basis of differentials: $O(g^2 \text{polylog}(\epsilon))$ complexity
- Setting up the Riemann–Hilbert problem requires solving 3 $g \times g$ linear systems — $O(g^3)$ in the current code.

Solve at (x, t):

- We adaptively choose the number of collocation nodes per contour (i.e., gap) based on the known analyticity of the true solution.
- We use a block-diagonal preconditioner — $O(g \,\text{polylog}(\epsilon))$ to apply.
- We solve with GMRES using this preconditioner in $O(\text{polylog}(\epsilon))$ iterations, a total cost of $O(g^2 \text{polylog}(\epsilon))$