Fixed-time theory

Fix a normalization point $x_0 \in \mathbb{R}$. Then there exists a fundamental set of solutions $s(x;\lambda)$ and $c(x;\lambda)$ of $-\psi_{xx} - q(x,0)\psi = \lambda\psi$:

$$c(x_0;\lambda) = 1, \quad c_x(x_0;\lambda) = 0,$$
$$s(x_0;\lambda) = 0, \quad s_x(x_0;\lambda) = 1.$$

As in the case of the whole-line IST, the inverse transform requires the determination of a solution ψ that can be normalized at infinity. We want the normalization process to introduce only mild singularities. So, we look for

$$\psi(x;\lambda) = \mathrm{e}^{\mathrm{i}(x-x_0)\sqrt{\lambda}}(1 + o(1)), \quad |\lambda| \to \infty.$$

Fixed-time theory

Define the monodromy matrix

$$T(\lambda)=\begin{bmatrix} c(x_0+L;\lambda) & s(x_0+L;\lambda) \\ c_x(x_0+L;\lambda) & s_x(x_0+L;\lambda) \end{bmatrix}.$$

Then

$$\psi_\pm(x;\lambda)=c(x;\lambda)+\frac{\pm\sqrt{\Delta^2(\lambda)-1}+\frac{1}{2}(T_{22}(\lambda)-T_{11}(\lambda))}{T_{12}(\lambda)}s(x;\lambda),$$

$$\Delta(\lambda)=\frac{1}{2}\mathrm{Tr}\,T(\lambda).$$

For x fixed, knowing $\psi_\pm$ as a function of λ is enough to determine $q(x,0)$.

B A Dubrovin. Integrable Systems and Riemann Surfaces Lecture Notes. http://people.sissa.it/~dubrovin/r
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