For the KdV equation with decaying data, the chosen solutions $\psi_\pm$ are assembled into a vector

$$\begin{aligned}
\Phi(z) &= \Phi(x,t,z) = \begin{bmatrix} \psi_+(x,t,z^2) & \psi_-(x,t,z^2) \end{bmatrix} \mathrm{e}^{-\mathrm{i}\theta(z)\sigma_3}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
\theta(z) &= \theta(x,t,z) = xz + 4tz.
\end{aligned}$$

This vector is meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies

$$\Phi^+(s) = \Phi^-(s) \begin{bmatrix} 1 - |R(s)|^2 & -\overline{R}(s)\mathrm{e}^{-2\mathrm{i}\theta(s)}, \\ R(s)\mathrm{e}^{2\mathrm{i}\theta(s)} & 1 \end{bmatrix}, \quad s \in \mathbb{R}, \quad \Phi(\infty) = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

where superscripts indicate boundary values from above (+) and below (−).

The KdV equation

This is a Riemann–Hilbert (RH) problem for the unknown Φ. For a given (x,t), once Φ is known as a function of z, the solution of $q(x,t)$ can be recovered via

$$q(x,t) = 2\mathrm{i} \lim_{z\to\infty} z\partial_x \Phi_1(z).$$

There are many interesting methods for computing the forward scattering transform:

S Chimmalgi, P J Prins, and S Wahls. Fast Nonlinear Fourier Transform Algorithms Using Higher Order Exponential Integrators. IEEE Access, 7:145161–145176, 2019

T T. Scattering and inverse scattering for the AKNS system: A rational function approach. Studies in Applied Mathematics, page sapm.12434, aug 2021

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The KdV equation

For the KdV equation with decaying data, the chosen solutions $\psi_\pm$ are assembled into a vector

$$\Phi(z) = \Phi(x,t,z) = \begin{bmatrix} \psi_+(x,t,z^2) & \psi_-(x,t,z^2) \end{bmatrix} e^{-i\theta(z)\sigma_3}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

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$$q(x,t) = 2i \lim_{z\to\infty} z\partial_x \Phi_1(z).$$

Benefits and complications

$$\Phi^+(s) = \Phi^-(s)\begin{bmatrix} 1-|R(s)|^2 & -\overline{R}(s)\mathrm{e}^{-2\mathrm{i}\theta(s)}, \\ R(s)\mathrm{e}^{2\mathrm{i}\theta(s)} & 1 \end{bmatrix}, \quad \Phi(\infty) = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The (x,t) dependence is explicit via $\theta(s) = xs + 4ts$ — plug in (x,t) and solve numerically!

This is not free: $\mathrm{e}^{2\mathrm{i}\theta(s)}$ is highly oscillatory.

The current method to treat this involves contour deformations.