The KdV equation with decaying data

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The KdV equation

For the KdV equation with decaying data, the chosen solutions $\psi_\pm$ are assembled into a vector

$$\Phi(z) = \Phi(x,t,z) = \begin{bmatrix} \psi_+(x,t,z^2) & \psi_-(x,t,z^2) \end{bmatrix} e^{-i\theta(z)\sigma_3}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\theta(z) = \theta(x,t,z) = xz + 4tz.$$

This vector is meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies

$$\Phi^+(s) = \Phi^-(s) \begin{bmatrix} 1 - |R(s)|^2 & -\overline{R}(s) e^{-2i\theta(s)}, \\ R(s) e^{2i\theta(s)} & 1 \end{bmatrix}, \quad s \in \mathbb{R}, \quad \Phi(\infty) = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

where superscripts indicate boundary values from above (+) and below (−).

This is a Riemann–Hilbert (RH) problem for the unknown Φ. For a given (x,t), once Φ is known as a function of z, the solution of $q(x,t)$ can be recovered via

$$q(x,t) = 2i \lim_{z \to \infty} z \partial_x \Phi_1(z).$$