Singular integral equations: The KdV RH problem $t = 0$

$$\mathcal{C}_{\mathbb{R}}^{\pm} u(s) = \lim_{\epsilon \downarrow 0} \mathcal{C}_{\mathbb{R}} u(z \pm \mathrm{i}\epsilon)$$

$$\mathcal{C}_{\mathbb{R}} u(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{u(s)}{s - z} \mathrm{d}s$$

The RH problems we consider are all equivalent to singular integral equations

$$\mathcal{C}_{\mathbb{R}}^{+} \mathbf{u}(s) - \mathcal{C}_{\mathbb{R}}^{-} \mathbf{u}(s) \begin{bmatrix} 1 - |R(s)|^2 & -\overline{R}(s) \mathrm{e}^{-2\mathrm{i}sx} \\ R(s) \mathrm{e}^{2\mathrm{i}sx} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -|R(s)|^2 & -\overline{R}(s) \mathrm{e}^{-2\mathrm{i}sx} \\ R(s) \mathrm{e}^{2\mathrm{i}sx} & 0 \end{bmatrix}.$$

A preconditioned version:

$$\mathcal{C}_{\mathbb{R}}^{+} \mathbf{u}(s) \begin{bmatrix} 1 & 0 \\ -R(s) \mathrm{e}^{2\mathrm{i}sx} & 1 \end{bmatrix} - \mathcal{C}_{\mathbb{R}}^{-} \mathbf{u}(s) \begin{bmatrix} 1 & -\overline{R}(s) \mathrm{e}^{-2\mathrm{i}sx} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -R(s) \mathrm{e}^{-2\mathrm{i}sx} & \overline{R}(s) \mathrm{e}^{-2\mathrm{i}sx} \end{bmatrix}$$

$$r_{j,\alpha}(s) = \left[\left(\frac{s+\mathrm{i}\nu}{s-\mathrm{i}\nu}\right)^j - 1\right]\mathrm{e}^{\mathrm{i}\alpha s}$$

If $\alpha j \leq 0$ then

$$\mathscr{C}_{\mathbb{R}}^{+} r_{j,\alpha}(s) = \begin{cases} r_{j,\alpha}(s) & j > 0, \\ 0 & j < 0, \end{cases}$$

$$\mathscr{C}_{\mathbb{R}}^{-} r_{j,\alpha}(s) = \begin{cases} 0 & j > 0, \\ -r_{j,\alpha}(s) & j < 0. \end{cases}$$

There exists a function $\eta_{j,n}(\alpha)$, $\alpha j > 0$ such that

$$\mathscr{C}_{\mathbb{R}}^{+} r_{j,\alpha}(s) = \begin{cases} -\sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{n,0}(s) & j > 0, \\ r_{j,\alpha}(s) + \sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{-n,0}(s) & j < 0, \end{cases}$$

$$\mathscr{C}_{\mathbb{R}}^{-} r_{j,\alpha}(s) = \begin{cases} -r_{j,\alpha}(s) - \sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{n,0}(s) & j > 0, \\ \sum_{n=1}^{j} \eta_{j,n}(\alpha) r_{-n,0}(s) & j < 0. \end{cases}$$