A singular integral equation

But due to the singularities at $\pm\sqrt{\alpha_j}$, $j>1$, $U(z)$ must be singular there. So, define

$$w_j(z)=\sqrt{\frac{z-\sqrt{\beta_j}}{\sqrt{\alpha_{j+1}}-z}}.$$

And we look for $U_j(z)$ such that

$$N(z)=\begin{bmatrix}1 & 1\end{bmatrix}+\sum_j \frac{1}{2\pi \mathrm{i}}\int_{\sqrt{\beta_j}}^{\sqrt{\alpha_{j+1}}}\frac{U_j(z')w_j(z')}{z'-z}\mathrm{d}z'-\sum_j \frac{1}{2\pi \mathrm{i}}\int_{-\sqrt{\beta_j}}^{-\sqrt{\alpha_{j+1}}}\frac{U_{-j}(z')w_j(-z')}{z'-z}\mathrm{d}z'.$$

We expect that

$$N(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{U(z')}{z' - z} \mathrm{d}z' = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathscr{C}_\Gamma U(z).$$

Owing to the Plemelj lemma, U will then satisfy

$$\begin{aligned} U(z) - \mathscr{C}_\Gamma^- U(z)(J(z) - I) &= \begin{bmatrix} 1 & 1 \end{bmatrix}(J(z) - I), \\ \mathscr{C}_\Gamma^- U(z) &:= \lim_{\epsilon \downarrow 0} \mathscr{C}_\Gamma U(z - \mathrm{i}\epsilon) \end{aligned}$$

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Chebyshev polynomials of the third and fourth kind

The Chebyshev polynomials $(V_n(x))_{n\geq 0}$ of the third kind are the orthonormal polynomials on $[-1,1]$ with respect to the weight

$$\sqrt{\frac{x+1}{1-x}}.$$

The Chebyshev polynomials $(W_n(x))_{n\geq 0}$ of the fourth kind are the orthonormal polynomials on $[-1,1]$ with respect to the weight

$$\sqrt{\frac{1-x}{1+x}}.$$

For $j > 0$, $U_j(x)$ should be well approximated by a shifted and scaled third-kind Chebyshev series. And $U_{-j}(x)$ should be well approximated by a shifted and scaled fourth-kind Chebyshev series.

The coefficients in this series are approximated using collocation — by imposing that the jump condition for the RHP should hold exactly at a set of points.