Chebyshev polynomials of the third and fourth kind

The Chebyshev polynomials $(V_n(x))_{n\geq 0}$ of the third kind are the orthonormal polynomials on $[-1,1]$ with respect to the weight

$$\sqrt{\frac{x+1}{1-x}}.$$

The Chebyshev polynomials $(W_n(x))_{n\geq 0}$ of the fourth kind are the orthonormal polynomials on $[-1,1]$ with respect to the weight

$$\sqrt{\frac{1-x}{1+x}}.$$

For $j > 0$, $U_j(x)$ should be well approximated by a shifted and scaled third-kind Chebyshev series. And $U_{-j}(x)$ should be well approximated by a shifted and scaled fourth-kind Chebyshev series.

The coefficients in this series are approximated using collocation — by imposing that the jump condition for the RHP should hold exactly at a set of points.

Cauchy integrals of orthogonal polynomials

It turns out that

$$\frac{1}{2\pi \mathrm{i}}\int_{-1}^{1}\frac{V_n(x)}{x-z}\sqrt{\frac{x+1}{1-x}}\mathrm{d}x$$

can be computed explicitly.

But it is often easier to use the elegant fact that the Cauchy integrals of weighted OPs satisfy the same three-term recurrence as the OPs themselves. One only needs to compute the Cauchy integral of the weight to get the recurrence started. Adaptive QR can be employed to alleviate stability issues.

S Olver, R M Slevinsky, and A Townsend. Fast algorithms using orthogonal polynomials. Acta Numerica, 29:573–699, may 2020