Fixed-time theory

Consider $t = 0$

$$\mathscr{L}(0) = -\partial_x - q(\diamond, 0)$$

Then it is well-known that the Bloch spectrum, for sufficiently regular periodic potentials,

$$\sigma_{\mathrm{B}}(q) := \left\{ \lambda \in \mathbb{C} : \text{ there exists a solution } \psi(\diamond; \lambda) \text{ of } \mathscr{L}\psi = \lambda\psi \text{ such that } \sup_{x \in \mathbb{R}} |\psi(x;\lambda)| < \infty \right\}$$

is a countable union of real intervals

$$\sigma_{\mathrm{B}}(q) = \bigcup_{k=1}^{g+1} [\alpha_k, \beta_k], \quad \text{where } g \in \mathbb{Z}_{>0} \; (\beta_{g+1} = \infty) \text{ or } g = \infty.$$

with

$$\alpha_k < \beta_k < \alpha_{k+1}, \quad k = 1, 2, \ldots$$

S P Novikov, S V Manakov, L P Pitaevskii, and V E Zakharov. Theory of Solitons. Constants Bureau, New York, 1984

Fixed-time theory

Fix a normalization point $x_0 \in \mathbb{R}$. Then there exists a fundamental set of solutions $s(x;\lambda)$ and $c(x;\lambda)$ of $-\psi_{xx} - q(x,0)\psi = \lambda\psi$:

$$\begin{aligned} c(x_0;\lambda) &= 1, \quad & c_x(x_0;\lambda) &= 0, \\ s(x_0;\lambda) &= 0, \quad & s_x(x_0;\lambda) &= 1. \end{aligned}$$

As in the case of the whole-line IST, the inverse transform requires the determination of a solution ψ that can be normalized at infinity. We want the normalization process to introduce only mild singularities. So, we look for

$$\psi(x;\lambda) = \mathrm{e}^{\mathrm{i}(x-x_0)\sqrt{\lambda}}(1+o(1)), \quad |\lambda| \to \infty.$$