```
<< Local`QFTToolKit2`
"Local notational definitions";
rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := |a|;
it[a_] := Style[a, Italic]
iD := it[D]
iI := it["I"]
C∞ := C"∞"
Bx_ := T[B, "d", {x}]
("∇"S)n_ := T["∇"S, "d", {n}]

Clear[expandDC];
expandDC[sub_: {}] := tuRepeat[{sub, tuOpDistribute[Dot],
     tuOpSimplify[Dot], tuOpDistribute[CircleTimes], tuOpSimplify[CircleTimes]}];
Clear[expandCom]
expandCom[subs_: {}][exp_] := Block[{tmp = exp},
    tmp = tmp //. tuCommutatorExpand // expandDC[];
    tmp = tmp /. toxDot //. Flatten[{subs}];
    tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
    tmp
   ];
$sgeneral :=
 {T[γ, "d", {5}].T[γ, "d", {5}] → 1, ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
  T["∇", "d", {_}][1n_] → 0, a_ . 1n_ → a, 1n_ . a_ → a}
$sgeneral // ColumnBar
Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] :→ CommutatorP[b, a] /; OrderedQ[{b, a}],
    CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
    tt : T[γ, "u", {μ}].T[γ, "d", {5}] :> -Reverse[tt]
   };
$symmetries // ColumnBar
```

```
γ5.γ5 → 1
(γ5)† → γ5
∇_[1n_] → 0
(a_).1n_ → a
1n_.(a_) → a
```

```
tt : gμ_ν_ :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : Fμ_ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
[a_, b_]_ :→ -[b, a]_ /; OrderedQ[{b, a}]
{a_, b_}+ :→ {b, a}+ /; OrderedQ[{b, a}]
{γ5, γμ}+ → 0
tt : γμ.γ5 :→ -Reverse[tt]
```

1204.0328: Particle Physics From Almost Commutative Spacetime

2. Almost Commutative Manifolds and Gauge Theories -- Canonical Triple

```
$defall = {};(*accumulator for all definitions*)
accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
   ""];
selectDef[heads_, with_:{}] := tuRuleSelect[$defall][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] // Last;
PR[CO["We use equivalence symbol for isomorphism, and
    Mod[] symbol for quotient group?"]
]
```

We use equivalence symbol for isomorphism, and Mod[] symbol for quotient group?

2.1 Spin manifolds in noncommutative geometry

```
PR["● M is 4-dim manifold with canonical triple ",
  {𝒜 -> C"∞"[M], ℋ -> L²[M, S], 𝒟 -> slash[iD]},
  NL, "The connection: ", $connection = "∇"^S[S[]],
  NL, "Dirac operator: ",
  {slash[D][ψ] → -I T[γ, "u", {μ}].T["∇"^S, "d", {μ}][ψ], ψ ∈ Γ[M, S],
    T["∇"^S, "d", {μ}][f ψ] → f "∇"^S[ψ] + tuPartialD[f, μ] ψ,
    CommutatorM[slash[iD], f].ψ → -I T[γ, "u", {μ}]. tuPartialD[f, μ].ψ
   } // ColumnBar,
  NL, "Have ℤ2-grading(chirality): ",
  $s = {T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
    T[γ, "d", {5}].T[γ, "d", {5}] → 1,
    ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
    CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
    T[γ, "d", {5}][L²[M, S]] -> L²[M, S]⁺ ⊕ L²[M, S]⁻}; $s // ColumnBar,
  accumDef[$s];
  NL, "Charge conjugation: ", $ =
   J_M[{J_M.J_M → -1, CommutatorM[J_M, slash[iD]] → 0, CommutatorM[J_M, T[γ, "d", {5}]] → 0}];
  accumDef[$];
  $ // ColumnForms
 ];
```

● M is 4-dim manifold with canonical triple $\{\mathcal{A}\to C^{\infty}[M],\ \mathcal{H}\to L^2[M, S],\ \mathcal{D}\to \not{D}\}$

The connection: $\nabla^S[S[]]$

Dirac operator: $\left|\begin{array}{l} (\not{D})[\psi] \to -i\,\gamma^{\mu}.\nabla^S{}_{\mu}[\psi] \\ \psi \in \Gamma[M, S] \\ \nabla^S{}_{\mu}[f\,\psi] \to f\,\nabla^S[\psi] + \psi\,\underset{-\mu}{\partial}[f] \\ [\not{D}, f]_-.\psi \to -i\,\gamma^{\mu}.\underset{-\mu}{\partial}[f].\psi \end{array}\right.$

Have $\mathbb{Z}_2$-grading(chirality): $\left|\begin{array}{l} \gamma_5 \to \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4 \\ \gamma_5.\gamma_5 \to 1 \\ (\gamma_5)^{\dagger} \to \gamma_5 \\ \{\gamma_5, \gamma^{\mu}\}_+ \to 0 \\ \gamma_5[L^2[M, S]] \to L^2[M, S]^+ \oplus L^2[M, S]^- \end{array}\right.$

Charge conjugation: $J_M\left[\left|\begin{array}{l} J_M.J_M \to -1 \\ [J_M, \not{D}]_- \to 0 \\ [J_M, \text{Tensor}[\gamma, \text{|Void}, \text{|5}]]_- \to 0 \end{array}\right.\right]$

2.2 Almost-commutative manifolds

```
PR["● F→finite space triple: ", F → {𝒜F, ℋF, 𝒟F},
  " where ", {𝒜F[CG[MN[ℂ]]], ℋF[CG["N-dim complex Hilbert space"]],
    𝒟F[CG["hermitian MN[ℂ]"]], MN[ℂ][CG["NxN matrix"]]} // ColumnBar,
  NL, "•ℋF is ℤ2 graded (even) if ∃ a grading operator: ",
  $ = γF[{ConjugateTranspose[γF] → γF, γF.γF → 1F, γF[ℋF] -> ℋF⁺ ⊕ ℋF⁻,
      {γF[ψ ∈ ℋF] → ±ψ},
      CommutatorM[γF, a ∈ AF] → 0,
      CommutatorP[γF, 𝒟F] → 0
    }]; accumDef[$]; $ // ColumnForms
 ];
```

● F→finite space triple: $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F\}$ where

$\mathcal{A}_F[M_N[\mathbb{C}]]$
$\mathcal{H}_F[$N-dim complex Hilbert space$]$
$\mathcal{D}_F[$hermitian $M_N[\mathbb{C}]]$
$M_N[\mathbb{C}][$NxN matrix$]$

•$\mathcal{H}_F$ is $\mathbb{Z}_2$ graded (even) if ∃ a grading operator: $\gamma_F[$

$(\gamma_F)^\dagger \to \gamma_F$
$\gamma_F.\gamma_F \to 1_F$
$\gamma_F[\mathcal{H}_F] \to (\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^-$
$\gamma_F[\psi \in \mathcal{H}_F] \to \pm\psi$
$[\gamma_F, a \in A_F]_- \to 0$
$\{\gamma_F, \mathcal{D}_F\}_+ \to 0$

$]$

```
εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1,}}},
   {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
  ];
PR["Almost-commutative spin manifold: ",
 $ = M × F → {C"∞"[M, 𝒜F], L²[M, S] ⊗ ℋF, 𝒟 -> slash[𝒟] ⊗ 1N + T[γ, "d", {5}] ⊗ 𝒟F};
 ColumnForms[$],
 NL, "with grading: ", γ -> T[γ, "d", {5}] ⊗ γF,
 NL, "•Distance: ", {d𝒟[x, y] → sup[‖a[x] - a[y]‖], a ∈ 𝒜 && ‖CommutatorM[𝒟, a]‖ ≤ 1},
 NL, "●Charge conjugation for F: even space F is real if ∃ ",
 $J = JF[ℋF] -> { JF. JF -> ε, JF.𝒟F -> ε'. 𝒟F.JF, JF.γF -> ε''.γF.JF};
 ColumnForms[$J], accumDef[$J];
 NL, "where the routine εRule[KOdim_] is provided ",
 CR[" What is the meaning of ε's?"],
 NL, "•", $ = ForAll[{a, b}, a | b ∈ 𝒜F, {CommutatorM[a, rghtA[b]] → 0, rghtA[b] →
      JF.ConjugateTranspose[b].ConjugateTranspose[JF]}][CG["Order-0 condition"]],
 accumDef[$];
 NL, "•",
 $ = ForAll[{a, b}, a | b ∈ 𝒜F, {CommutatorM[CommutatorM[𝒟F, a], rghtA[b]] → 0, rghtA[b] →
      JF.ConjugateTranspose[b].ConjugateTranspose[JF]}][CG["Order-1 condition"]],
 accumDef[$]; ""
]
```

Almost-commutative spin manifold: $M\times F\to \left|\begin{array}{l} C^{\infty}[M,\ \mathcal{A}_F] \\ L^2[M,\ S]\otimes\mathcal{H}_F \\ \mathcal{D}\to(\not{\mathcal{D}})\otimes 1_N+\mathrm{Tensor}[\gamma,\ |\mathrm{Void}\ ,\ |5]\otimes\mathcal{D}_F \end{array}\right.$

with grading: $\gamma\to\gamma_5\otimes\gamma_F$

•Distance: $\{d_{\mathcal{D}}[x,\ y]\to \sup[\|a[x]-a[y]\|],\ a\in\mathcal{A}\ \&\&\ \|[\mathcal{D},\ a]_-\|\le 1\}$

●Charge conjugation for F: even space F is real if ∃ $J_F[\mathcal{H}_F]\to \left|\begin{array}{l} J_F.J_F\to\varepsilon \\ J_F.\mathcal{D}_F\to\varepsilon'.\mathcal{D}_F.J_F \\ J_F.\gamma_F\to\varepsilon''.\gamma_F.J_F \end{array}\right.$

where the routine εRule[KOdim_] is provided What is the meaning of ε's?

•$(\forall_{\{a,b\},a|b\in\mathcal{A}_F}\ \{[a,\ b^o]_-\to 0,\ b^o\to J_F.b^\dagger.(J_F)^\dagger\})$[Order-0 condition]

•$(\forall_{\{a,b\},a|b\in\mathcal{A}_F}\ \{[[\mathcal{D}_F,\ a]_-,\ b^o]_-\to 0,\ b^o\to J_F.b^\dagger.(J_F)^\dagger\})$[Order-1 condition]

```
PR["●Lemma2.7. Definition 2.5: ", $J[[2]],
  NL, "Where γF decomposes ", $h = ℋ → Table[ℋi,j, {i, 2}, {j, 2}];
  MatrixForms[$h],
  " into ", ℋ → ℋ⁺ ⊕ ℋ⁻ , " i.e. ", $gh = γF.ℋ → {{ℋ⁺, 0}, {0, ℋ⁻}};
  MatrixForms[$gh],
  $gh0 = $gh /. {ℋ⁺ -> ℋ1,1, ℋ⁻ -> ℋ2,2};
  Yield, $gh1 = γF.{{a_, b_}, {c_, d_}} -> DiagonalMatrix[{a, d}];
  MatrixForms[$gh1],
  NL, "Represent ", $j = JF → Table[ji,j, {i, 2}, {j, 2}];
  MatrixForms[$j], " of the same dimensions.",
  NL, "•For: ",
  $JF = {JF → U.cc, U.ConjugateTranspose[U] → 1N, U ∈ U[ℋ"±"], cc → Conjugate},
  NL, "where: ",
  $cc = {ConjugateTranspose[cc] → cc,
    Conjugate[cc] → cc, cc . cc → 1, cc.a_ :→ Conjugate[a].cc},
  Imply, $0 = $ = JF.ConjugateTranspose[JF],
  yield, $ = $0 → ($ // tuRepeat[{tuRule[$JF[[1 ;; 3]]], $cc}, tuOpSimplifyF[Dot]]);
  Framed[$],
  Yield, $ = $ /. ConjugateTranspose → SuperDagger /. Dot → xDot /. $j /.
    SuperDagger[a_] :→ Map[Thread[SuperDagger[#]] &, Transpose[a]] /; MatrixQ[a];
  MatrixForms[$],
```

```
   Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
   Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
   Yield, $JJ = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
   FramedColumn[$JJ], CK
  ];
PR[
 line, "•For ", $s = n → 0; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0;
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] → γF.xDot[a];
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$];
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$];
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. 1 → 1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "•Then we have: ", $ = {$JJ1, $JJ, $Jg}; ColumnForms[$],
 Yield, $ = $ /. j1,2 | j2,1 → 0 // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply, {ConjugateTranspose[j1,1] -> j1,1, ConjugateTranspose[j2,2] -> j2,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 2; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
    γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply,
 {ConjugateTranspose[j1,2] → -j2,1, ConjugateTranspose[j2,1] → -j1,2} // FramedColumn
]
PR[
```

```
 line, "•For ", $s = n → 4; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
     γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /.  1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,2 | j2,1 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply,
 {ConjugateTranspose[j1,1] → -j1,1, ConjugateTranspose[j2,2] → -j2,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 6; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
     γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. 1 → 1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /.  1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply, {ConjugateTranspose[j1,2] → j2,1, ConjugateTranspose[j2,1] → j1,2} // FramedColumn
]
```

●Lemma2.7. Definition 2.5: $\{J_F.J_F \to \varepsilon,\ J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F,\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F\}$

Where γ_F decomposes $\mathcal{H} \to \begin{pmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} \end{pmatrix}$ into $\mathcal{H} \to \mathcal{H}^+ \oplus \mathcal{H}^-$ i.e. $\gamma_F.\mathcal{H} \to \begin{pmatrix} \mathcal{H}^+ & 0 \\ 0 & \mathcal{H}^- \end{pmatrix}$

→ $\gamma_F.\begin{pmatrix} a\_ & b\_ \\ c\_ & d\_ \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Represent $J_F \to \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}$ of the same dimensions.

•For: $\{J_F \to U.cc,\ U.U^\dagger \to 1_N,\ U \in U[\mathcal{H}^\pm],\ cc \to \text{Conjugate}\}$
where: $\{cc^\dagger \to cc,\ cc^* \to cc,\ cc.cc \to 1,\ cc.(a\_) \mapsto a^*.cc\}$

⇒ $J_F.(J_F)^\dagger \longrightarrow \boxed{J_F.(J_F)^\dagger \to 1_N}$

→ $\text{xDot}\left[\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}, \begin{pmatrix} (j_{1,1})^\dagger & (j_{2,1})^\dagger \\ (j_{1,2})^\dagger & (j_{2,2})^\dagger \end{pmatrix}\right] \to 1_N$

→ $\begin{pmatrix} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger & j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger & j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \end{pmatrix} \to 1_N$

→ $\{\{j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger,\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger\},$
$\{j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger,\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger\}\} \to \{\{1_{N^+}, 0\}, \{0, 1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \to 0 \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger \to 0 \\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \to 1_{N^-} \end{array}}$ ←CHECK

•For $\boxed{n \to 0}$ ⟶ $J_F . J_F \to 1$
$J_F . \gamma_F \to \gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to \gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to \gamma_F . J_F . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to 1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → 1_N

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to 1_N$

→ {{$j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1}$, $j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}$}, {$j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1}$, $j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}$}} →
{{1_{N^+}, 0}, {0, 1_{N^-}}}

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \end{array}}$ ⟵CHECK

•Then we have:
$$\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{array}$$

→
$$\begin{array}{l} j_{1,1} . j_{1,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . j_{2,2} \to 1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{array}$$

⇒ $\boxed{\begin{array}{l} (j_{1,1})^\dagger \to j_{1,1} \\ (j_{2,2})^\dagger \to j_{2,2} \end{array}}$

•For $\boxed{n \to 2}$ ⟶ $J_F . J_F \to -1$
$J_F . \gamma_F \to -\gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to -\gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to (-\gamma_F . J_F) . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to -1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → -1_N

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to -1_N$

→ $\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1},\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\},\ \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1},\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to$
$\{\{-1_{N^+},\ 0\},\ \{0,\ -1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}}$ ⟵CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions:
$$\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,2} . \mathcal{H}_{2,2} \end{array}$$

⇒ $\boxed{j_{1,1} \mid j_{2,2} \to 0}$

→
$$\begin{array}{l} j_{1,2} . j_{2,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . j_{1,2} \to -1_{N^-} \\ j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . (j_{2,1})^\dagger \to 1_{N^-} \\ 0 \to 0 \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{array}$$

⇒ $\boxed{\begin{array}{l} (j_{1,2})^\dagger \to -j_{2,1} \\ (j_{2,1})^\dagger \to -j_{1,2} \end{array}}$

•For $\boxed{n \to 4}$ ⟶ $J_F . J_F \to -1$
$J_F . \gamma_F \to \gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to \gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to \gamma_F . J_F . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to -1_N}$

→ xDot[$\{\{j_{1,1}, j_{1,2}\}, \{j_{2,1}, j_{2,2}\}\}, \{\{j_{1,1}, j_{1,2}\}, \{j_{2,1}, j_{2,2}\}\}$] $\to -1_N$

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to -1_N$

→ $\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1}, j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\}, \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1}, j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to \{\{-1_{N^+}, 0\}, \{0, -1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}}$ ⟵CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions:
$$\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{array}$$

⇒ $\boxed{j_{1,2} \mid j_{2,1} \to 0}$

→
$$\begin{array}{l} j_{1,1} . j_{1,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . j_{2,2} \to -1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{array}$$

⇒ $\boxed{\begin{array}{l} (j_{1,1})^\dagger \to -j_{1,1} \\ (j_{2,2})^\dagger \to -j_{2,2} \end{array}}$

•For $\boxed{n \to 6}$ $\longrightarrow$ $\begin{matrix} J_F \cdot J_F \to 1 \\ J_F \cdot \gamma_F \to -\gamma_F \cdot J_F \end{matrix}$

$\to$ $\boxed{J_F \cdot \gamma_F \to -\gamma_F \cdot J_F}$ $\longrightarrow$ $J_F \cdot \gamma_F \cdot \mathcal{H} \to (-\gamma_F \cdot J_F) \cdot \mathcal{H}$

....

$\to \begin{pmatrix} j_{1,1} \cdot \mathcal{H}_{1,1} \to -j_{1,1} \cdot \mathcal{H}_{1,1} - j_{1,2} \cdot \mathcal{H}_{2,1} & j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 & j_{2,2} \cdot \mathcal{H}_{2,2} \to -j_{2,1} \cdot \mathcal{H}_{1,2} - j_{2,2} \cdot \mathcal{H}_{2,2} \end{pmatrix}$

$\to \boxed{\begin{matrix} j_{1,1} \cdot \mathcal{H}_{1,1} \to -j_{1,1} \cdot \mathcal{H}_{1,1} - j_{1,2} \cdot \mathcal{H}_{2,1} \\ j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to -j_{2,1} \cdot \mathcal{H}_{1,2} - j_{2,2} \cdot \mathcal{H}_{2,2} \end{matrix}}$ $\longleftarrow$CHECK

•For $\boxed{J_F \cdot J_F \to 1_N}$

$\to$ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] $\to 1_N$

$\to \begin{pmatrix} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} & j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} & j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \end{pmatrix} \to 1_N$

$\to$ {{$j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1}$, $j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2}$}, {$j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1}$, $j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2}$}} $\to$ {{1_{N^+}, 0}, {0, 1_{N^-}}}

$\to \boxed{\begin{matrix} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} \to 1_{N^+} \\ j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \to 0 \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} \to 0 \\ j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \to 1_{N^-} \end{matrix}}$ $\longleftarrow$CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions:

$$\begin{matrix} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} \to 1_{N^+} \\ j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \to 0 \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} \to 0 \\ j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \to 1_{N^-} \\ j_{1,1} \cdot (j_{1,1})^\dagger + j_{1,2} \cdot (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} \cdot (j_{2,1})^\dagger + j_{1,2} \cdot (j_{2,2})^\dagger \to 0 \\ j_{2,1} \cdot (j_{1,1})^\dagger + j_{2,2} \cdot (j_{1,2})^\dagger \to 0 \\ j_{2,1} \cdot (j_{2,1})^\dagger + j_{2,2} \cdot (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} \cdot \mathcal{H}_{1,1} \to -j_{1,1} \cdot \mathcal{H}_{1,1} \\ j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to -j_{2,2} \cdot \mathcal{H}_{2,2} \end{matrix}$$

$\Rightarrow \boxed{j_{1,1} \mid j_{2,2} \to 0}$

$$\to \begin{matrix} j_{1,2} \cdot j_{2,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} \cdot j_{1,2} \to 1_{N^-} \\ j_{1,2} \cdot (j_{1,2})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} \cdot (j_{2,1})^\dagger \to 1_{N^-} \\ 0 \to 0 \\ j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{matrix}$$

$\Rightarrow \boxed{\begin{matrix} (j_{1,2})^\dagger \to j_{2,1} \\ (j_{2,1})^\dagger \to j_{1,2} \end{matrix}}$

Commutative Subalgebras

```
PR["● Define subalgebra of 𝒜: ",
 $sAt = 𝒜̃_J → {a ∈ 𝒜, a.J → J.ConjugateTranspose[a], rghtA[a] → a}, accumDef[$sAt];
 NL, "•Unitary group: ",
 U[𝒜] → {u ∈ 𝒜, u.ConjugateTranspose[u] | ConjugateTranspose[u].u → 1_N},
 Imply, ForAll[x ∈ M,
  u[x].ConjugateTranspose[u[x]] | ConjugateTranspose[u[x]].u[x] → 1_N],
 Imply, u ∈ U[𝒜] ⇔ u[x] ∈ U[𝒜_F],
 NL, "•Lie algebra: ", 𝔲[𝒜] → {X ∈ 𝒜, ConjugateTranspose[X] → -X} → C^"∞"[M, 𝔲[𝒜_F]],
 NL, "•Special unitary group: ", SU[𝒜_F] → {𝓊 ∈ U[𝒜_F], Det[u] → 1},
 NL, "•Lie algebra SU[𝒜_F]: ", 𝔰𝔲[𝒜_F] → {X ∈ 𝒜_F, ConjugateTranspose[X] → -X, Tr[X] → 0},
 line,
 "●Adjoint action. space: ", $F = F → Table[Subscript[i, F], {i, {𝒜, ℋ, 𝒟, γ, J}}],
 NL, "Define: for ", ξ ∈ $F[[2, 2]],
 Yield, $ = {Ad[U[𝒜_F]] → Endo[$F[[2, 2]]], ad[𝔲[$F[[2, 1]]]] → Endo[$F[[2, 2]]]};
 Column[$],
 yield, $ = {Ad[u][ξ] → u.ξ.ConjugateTranspose[u] → u.rghtA[ConjugateTranspose[u]].ξ,
   ad[A][ξ] → A.ξ - ξ.A → (A - rghtA[A]).ξ}; accumDef[$]; Column[$]
]
```

● **Define subalgebra of** $\mathcal{A}$**:** $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A},\ a.J \to J.a^\dagger,\ a^o \to a\}$

•**Unitary group:** $U[\mathcal{A}] \to \{u \in \mathcal{A},\ u.u^\dagger \mid u^\dagger.u \to 1_N\}$

⇒ $\forall_{x\in M} (u[x].u[x]^\dagger \mid u[x]^\dagger.u[x] \to 1_N)$

⇒ $u \in U[\mathcal{A}] \Leftrightarrow u[x] \in U[\mathcal{A}_F]$

•**Lie algebra:** $\mathfrak{u}[\mathcal{A}] \to \{X \in \mathcal{A},\ X^\dagger \to -X\} \to C^\infty[M, \mathfrak{u}[\mathcal{A}_F]]$

•**Special unitary group:** $SU[\mathcal{A}_F] \to \{\mathcal{U} \in U[\mathcal{A}_F],\ Det[u] \to 1\}$

•**Lie algebra SU[$\mathcal{A}_F$]:** $\mathfrak{su}[\mathcal{A}_F] \to \{X \in \mathcal{A}_F,\ X^\dagger \to -X,\ Tr[X] \to 0\}$

●**Adjoint action. space:** $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F, J_F\}$

Define: for $\xi \in \mathcal{H}_F$

$$\to \begin{array}{l} Ad[U[\mathcal{A}_F]] \to Endo[\mathcal{H}_F] \\ ad[\mathfrak{u}[\mathcal{A}_F]] \to Endo[\mathcal{H}_F] \end{array} \longrightarrow \begin{array}{l} Ad[u][\xi] \to u.\xi.u^\dagger \to u.u^{\dagger o}.\xi \\ ad[A][\xi] \to A.\xi - \xi.A \to (A - A^o).\xi \end{array}$$

```
PR["●Gauge symmetry. ", {ϕ[M] → M, "diffeomorphism of C∞[M]"},
 NL, "define automorphism: ", {α_ϕ[f] → f.inv[ϕ], f ∈ (C^"∞") [M]},
 NL, "define diffeomorphism: ", Diff[M×F] → Aut[(C^"∞") [M, 𝒜_F]],
 Imply, {a ∈ (C^"∞") [M, 𝒜_F], α_ϕ[a] → a.inv[ϕ], α_ϕ[a][x] → a.inv[ϕ][x]} // Column,
 NL, "•Define for ", Inn[a] ->
   {u ∈ (C^"∞") [M, U[𝒜_F]], α_u[a] → u.a.ConjugateTranspose[u] → Inn[𝒜]} // ColumnForms,
 NL, "•Define outer automorphism: ", Out[𝒜] → Mod[Aut[𝒜], Inn[𝒜]],
 NL, "•Define kernel: ", Ker[ϕ] → {ϕ[U[𝒜]] → Inn[𝒜], ϕ[u → α_u],
    u ∈ U[𝒜], ForAll[a ∈ 𝒜, u.a.ConjugateTranspose[u] → a]} // ColumnForms
]
```

●**Gauge symmetry.** $\{\phi[M] \to M$, diffeomorphism of $C^\infty[M]\}$

define automorphism: $\{\alpha_\phi[f] \to f.\phi^{-1},\ f \in C^\infty[M]\}$

define diffeomorphism: $Diff[M \times F] \to Aut[C^\infty[M, \mathcal{A}_F]]$

$$\Rightarrow \begin{array}{l} a \in C^\infty[M, \mathcal{A}_F] \\ \alpha_\phi[a] \to a.\phi^{-1} \\ \alpha_\phi[a][x] \to a.\phi^{-1}[x] \end{array}$$

•**Define for** $Inn[a] \to \left| \begin{array}{l} u \in C^\infty[M, U[\mathcal{A}_F]] \\ \alpha_u[a] \to u.a.u^\dagger \to Inn[\mathcal{A}] \end{array} \right.$

•**Define outer automorphism:** $Out[\mathcal{A}] \to Mod[Aut[\mathcal{A}], Inn[\mathcal{A}]]$

•**Define kernel:** $Ker[\phi] \to \left| \begin{array}{l} \phi[U[\mathcal{A}]] \to Inn[\mathcal{A}] \\ \phi[u \to \alpha_u] \\ u \in U[\mathcal{A}] \\ \forall_{a\in\mathcal{A}} (u.a.u^\dagger \to a) \end{array} \right.$

2.3 Subgroups and subalgebras

```
PR["●Unitary transform. Given a triple: ", {𝒜, ℋ, 𝒟},
  " the representation π of 𝒜 on ℋ: ", π[a][ℋ] ,
  NL, "•Define unitary transform: ",
  $0 = U -> {U[ℋ] → ℋ, {𝒜, ℋ, 𝒟} -> {𝒜, ℋ, U. 𝒟.ConjugateTranspose[U]},
     (a ∈ 𝒜) -> U. π[a].ConjugateTranspose[U],
     γ -> U. γ.ConjugateTranspose[U], J -> U. J.ConjugateTranspose[U]};
  ColumnForms[$0],
  NL, "•EG1. ", {U → π[u], u ∈ U[𝒜]},
  NL, "•EG2. (adjoint action) ", $s = {U → Ad[u] → u.J.u.ConjugateTranspose[J]},
  Yield, $ = U.π[a].ConjugateTranspose[U], "POFF",
  Yield, $ = $ /. ($s[[1, 1]] -> $s[[1, 2, 2]] /. u → π[u]) // ConjugateCTSimplify1[{}],
  Yield, $ = $ /. aa_ . bb_ . π[a] → aa . π[a].bb, (*could be more specific*)
  Yield, $ = $ // tuRepeat[{ConjugateTranspose[J_] . J_ → 1,
      J_ .ConjugateTranspose[J_] → 1}, tuDotSimplify[]],
  Yield, $ = $ /. π[a_].π[b_]. ConjugateTranspose[π[c_]] →
     π[a. b. ConjugateTranspose[c]], "PONdd",
  Yield, $ = $ /. u_ .a_ .ConjugateTranspose[u_] → αu[a]
 ];
```

●Unitary transform. Given a triple:
{𝒜, ℋ, 𝒟} the representation π of 𝒜 on ℋ: π[a][ℋ]

•Define unitary transform: $U \to \left| \begin{array}{l} U[\mathcal{H}] \to \mathcal{H} \\ \left| \begin{array}{l} \mathcal{A} \\ \mathcal{H} \\ \mathcal{D} \end{array} \right. \to \left| \begin{array}{l} \mathcal{A} \\ \mathcal{H} \\ U.\mathcal{D}.U^\dagger \end{array} \right. \\ a \in \mathcal{A} \to U.\pi[a].U^\dagger \\ \gamma \to U.\gamma.U^\dagger \\ J \to U.J.U^\dagger \end{array} \right.$

•EG1. {U → π[u], u ∈ U[𝒜]}
•EG2. (adjoint action) $\{U \to Ad[u] \to u.J.u.J^\dagger\}$
→ $U.\pi[a].U^\dagger$
....
→ $\pi[\alpha_u[a]]$

```
PR["•Define Gauge group: ", 𝒢[M×F] → {u.J.u.ct[J], u ∈ U[𝒜]},
 NL, "Consider: ", {Ad[U[𝒜]] → 𝒢[M×F], Ad[u] → u.rghtA[ct[u]]} // Column,
 Imply, Ker[Ad] → {u ∈ U[𝒜], (u.J.u.ct[J] → 1) ⇒ (u.J → ct[J].u)},
 NL, "•Define finite gauge group for finite space F: ",
 𝒢[F] → {ℋF → U[(𝒜̃F)JF], hF → u[(𝒜̃F)JF]} // ColumnForms,
 NL, "•Proposition 2.13. ",
 e213 = {𝒢[F] ≃ Mod[SU[𝒜F], SHF], 𝒜F → "complex algebra", SHF → {g ∈ HF, Det[g] → 1}};
 Column[e213],
 NL, "●Proof 2.13: ",
 NL, "•define UH-equivalence: ", $su = u_ ⇔ u_ . h_ -> ForAll[h, h ∈ HF, (u | u.h ∈ U[𝒜F])],
 Yield, $G = {𝒢[F] ≃ Mod[U[𝒜F], HF]} → {u ⇔ u.h},
 Yield, $ = $G /. $su,
 NL, "•define SUSH equivalence: ",
 $su = su_ ⇔ su_ . g_ -> ForAll[g, g ∈ SHF, (su | su.g ∈ SU[𝒜F])],
 Yield, $SU = {Mod[SU[𝒜F], SHF]} → {su ⇔ su.g},
 Yield, $0 = $SU /. $su,
 NL, "(1)•Is SHF a normal subgroup of SU[𝒜F]?: ",
 $ = ForAll[{g, v}, g ∈ SHF && v ∈ SU[𝒜F], (v.g. inv[v]) ∈ SHF],

 NL, "•Evaluate: ", $ = Det[$0 = v.g. inv[v] ∈ HF],
 yield, $ = $ /. a_ ∈ b_ → a,
 yield, $ = Thread[$, Dot] /. Det[inv[a_]] → 1 / Det[a] /. Dot → Times,
 NL, "Since: ", g ∈ SHF,
 imply, $s = Det[g] → 1,
 imply, $0 ∈ SHF,
 imply, "SHF Normal Subgroup of SU[𝒜F]" // Framed
]
```

•Define Gauge group: $\mathcal{G}[M\times F]\to\{u.J.u.J^\dagger,\ u\in U[\mathcal{A}]\}$

Consider: $\begin{array}{l} Ad[U[\mathcal{A}]]\to\mathcal{G}[M\times F] \\ Ad[u]\to u.u^{\dagger o} \end{array}$

⇒ $Ker[Ad]\to\{u\in U[\mathcal{A}],\ (u.J.u.J^\dagger\to 1)\Rightarrow(u.J\to J^\dagger.u)\}$

•Define finite gauge group for finite space F: $\mathcal{G}[F]\to\left|\begin{array}{l}\mathcal{H}_F\to U[\tilde{\mathcal{A}}_{F\,J_F}] \\ h_F\to u[\tilde{\mathcal{A}}_{F\,J_F}]\end{array}\right.$

•Proposition 2.13. $\begin{array}{l}\mathcal{G}[F]\simeq Mod[SU[\mathcal{A}_F],\ SH_F] \\ \mathcal{A}_F\to \text{complex algebra} \\ SH_F\to\{g\in H_F,\ Det[g]\to 1\}\end{array}$

●Proof 2.13:

•define UH-equivalence: $(u\_).(h\_)\Leftrightarrow u\_\to\forall_{h,h\in H_F}(u\mid u.h\in U[\mathcal{A}_F])$

→ $\{\mathcal{G}[F]\simeq Mod[U[\mathcal{A}_F],\ H_F]\}\to\{u\Leftrightarrow u.h\}$

→ $\{\mathcal{G}[F]\simeq Mod[U[\mathcal{A}_F],\ H_F]\}\to\{\forall_{h,h\in H_F}(u\mid u.h\in U[\mathcal{A}_F])\}$

•define SUSH equivalence: $(su\_).(g\_)\Leftrightarrow su\_\to\forall_{g,g\in SH_F}(su\mid su.g\in SU[\mathcal{A}_F])$

→ $\{Mod[SU[\mathcal{A}_F],\ SH_F]\}\to\{su\Leftrightarrow su.g\}$

→ $\{Mod[SU[\mathcal{A}_F],\ SH_F]\}\to\{\forall_{g,g\in SH_F}(su\mid su.g\in SU[\mathcal{A}_F])\}$

(1)•Is SH_F a normal subgroup of $SU[\mathcal{A}_F]$?: $\forall_{\{g,v\},g\in SH_F\&\&v\in SU[\mathcal{A}_F]}\ v.g.v^{-1}\in SH_F$

•Evaluate: $Det[v.g.v^{-1}\in H_F]\longrightarrow Det[v.g.v^{-1}]\longrightarrow Det[g]$

Since: $g\in SH_F\Rightarrow Det[g]\to 1\Rightarrow(v.g.v^{-1}\in H_F)\in SH_F\Rightarrow$ $\boxed{SH_F\ \text{Normal Subgroup of}\ SU[\mathcal{A}_F]}$

```
PR["•Property of unitary matrix u: ",
 {Abs[Det[u]] → 1,
   {"Eigenvalues of u", λu ∈ U[1],
    Exists[{u, u'}, u ∈ U[𝒜F] && u' ∈ U[N], u'.u.ct[u'] -> λu 1N]}} // FramedColumn,
 Imply, Exists[λu, λu ∈ U[1] && λu^N → Det[u] && N → dim[ℋF] && U[1] ≤ U[𝒜F]],
 Imply, $ = ($0 = inv[λu].u ∈ SU[𝒜F]) ⟸ {$ = Det[$0[[1]]], $ = Thread[$, Dot],
     $ = $ /. Det[inv[λu]] → λu^(-N), $ = $ /. Det[u] → λu^N, SU[𝒜F]} // ColumnForms,

 NL, "■define group homomorphism from UH->SUSH: ",
 $ph = {φ[$G[[1, 1]]] → Mod[SU[𝒜F], SHF], φ[{u}] → {inv[λu].u}};
 Column[$ph],
 NL, "□Check if φ is independent of representative ", λu,
 NL, "•suppose: ", Implies[Exists[λu', (λu')^N → Det[u]],
  inv[λu].λu' ∈ μN["multiplicative group Nth root of unity"]],
 NL, "•", Implies[Implies[Implies[U[1] ≤ HF, μN ≤ SHF], {inv[λu].u} == {inv[λu'].u}],
  Framed[φ["independent of λu"]]],
 NL, "□Check if φ is independent of representative ", u ∈ U[𝒜F],
 NL, "?: ", $0 = $ = ForAll[u, u ∈ HF, φ[{u}]],
 Yield, $ = $ /. $ph, "POFF",
 NL, "For ", $s = (g -> inv[λh].h) ∈ SHF,
 Yield, $ = $ /. dd : HoldPattern[Dot[a_]] → dd .g,
 Yield, $ = $ /. $s[[1]],
 Yield, $ = $ /. dd : HoldPattern[Dot[_]] :→ tuDotTermLeft[inv[_], {inv[λu]}][dd],
 Yield, $ = $ /. inv[a_]. inv[b_] → inv[b.a],
 Yield, $[[3]] = φ[{u.h}]; $, "PONdd",
 yield, $[[3]] == $0[[3]] // Framed,
 NL, "•Suppose ", $ = ForAll[{u1, u2}, {u1 | u2 ∈ U[𝒜F]}, φ[{u1}] == φ[{u2}]],
 Yield, $ = $ /. φ[{a_}] → {inv[λa].a.ga} /. gu1 → 1 /. gu2 → (g ∈ SHF),
 Yield, $ = $ /. HoldPattern[Dot[a_]] → Dot[λu1, a],
 Yield, $ = $ /. a_ . inv[a_] → 1 /. g ∈ SHF → g // tuDotSimplify[],
 Yield, $3 = $ = $[[3]],
 " for some: ", $ = $[[2]] // First // DeleteCases[#, u2] & // tuDotSimplify[];
 $ ∈ SHF,
 imply, "φ is injective.",
 Imply, $ = $3 /. Thread[Apply[List, $] → 1] // tuDotSimplify[]; Framed[$]
]
```

• Property of unitary matrix u:

$$\boxed{\begin{array}{l}\text{Abs[Det[u]]} \to 1 \\ \{\text{Eigenvalues of u, } \lambda_u \in U[1],\ \exists_{\{u,u'\},u\in U[\mathcal{A}_F]\&\&u'\in U[N]}\ (u'.u.(u')^\dagger \to 1_N\,\lambda_u)\}\end{array}}$$

⇒ $\exists_{\lambda_u}\ (\lambda_u \in U[1]\ \&\&\ \lambda_u^N \to \text{Det}[u]\ \&\&\ N \to \text{dim}[\mathcal{H}_F]\ \&\&\ U[1] \le U[\mathcal{A}_F])$

⇒ $(\lambda_u^{-1}.u \in SU[\mathcal{A}_F]) \Longleftarrow \left|\begin{array}{l}\text{Det}[\lambda_u^{-1}.u] \\ \text{Det}[\lambda_u^{-1}].\text{Det}[u] \\ \lambda_u^{-N}.\text{Det}[u] \\ \lambda_u^{-N}.\lambda_u^{N} \\ SU[\mathcal{A}_F]\end{array}\right.$

■define group homomorphism from UH->SUSH: $\begin{array}{l}\varphi[\mathcal{G}[F] \simeq \text{Mod}[U[\mathcal{A}_F], H_F]] \to \text{Mod}[SU[\mathcal{A}_F], SH_F] \\ \varphi[\{u\}] \to \{\lambda_u^{-1}.u\}\end{array}$

□Check if φ is independent of representative λ_u

• suppose: $\exists_{\lambda_u'}\ ((\lambda_u')^N \to \text{Det}[u]) \Rightarrow \lambda_u^{-1}.\lambda_u' \in \mu_N$[multiplicative group Nth root of unity]

• $((U[1] \le H_F \Rightarrow \mu_N \le SH_F) \Rightarrow \{\lambda_u^{-1}.u\} == \{(\lambda_u')^{-1}.u\}) \Rightarrow \boxed{\varphi[\text{independent of } \lambda_u]}$

□Check if φ is independent of representative $u \in U[\mathcal{A}_F]$

?: $\forall_{u,u\in H_F}\ \varphi[\{u\}]$

→ $\forall_{u,u\in H_F}\ \{\lambda_u^{-1}.u\}$

.... ⟶ $\boxed{\varphi[\{u.h\}] == \varphi[\{u\}]}$

• Suppose $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \varphi[\{u_1\}] == \varphi[\{u_2\}]$

→ $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \{\lambda_{u_1}^{-1}.u_1.1\} == \{\lambda_{u_2}^{-1}.u_2.(g \in SH_F)\}$

→ $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \{\lambda_{u_1}.\lambda_{u_1}^{-1}.u_1.1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.(g \in SH_F)\}$

→ $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \{u_1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$

→ $\{u_1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$ for some: $\lambda_{u_1}.\lambda_{u_2}^{-1}.g \in SH_F$ ⇒ φ is injective.

⇒ $\boxed{\{u_1\} == \{u_2\}}$

```
PR["●Full symmetry group. ",
 NL, "•Homomorphic action Θ of a group H on group N: ", Θ[H] → Aut[N],
 NL, "•semi-direct product ", $ = N ▷ H → {{n, h}, n ∈ N && h ∈ H},
 NL, "Properties: ", $sdg = {
   {"product", {n_, h_} · {n1_, h1_} → {n . Θ[h]. n1, h . h1}},
   {"unit", {1, 1}},
   {"inverse", invSDG[{n_, h_}] → {Θ[inv[h]].inv[n], inv[h]}
   }}; FramedColumn[$sdg],
 "POFF",
 NL, "•Check inverse: ",
 NL, "Let: ", $n = {n, h},
 and, "inverse: ", $i = invSDG[$n],
 NL, "For ", $ = $n · $i,
 Yield, $ = $ /. $sdg[[3, 2]] /. $sdg[[1, 2]],
 NL, "If ", $s = {inv[a_] .a_ → 1, a_ .inv[a_] → 1, Θ[a_]. Θ[inv[a_]] → 1,
   Θ[a_] . n1_ . Θ[a_] . n2_ → Θ[a].n1.n2, (*homomorphic property*)
   {Θ[a_], b_} -> {1, b}(* Is Θ[h].1→1? *)
  },
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], OK,
 NL, "For ", $ = $i · $n,
 Yield, $ = $ /. $sdg[[3, 2]] /. $sdg[[1, 2]],
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], OK,
 "PONdd",
 NL, "•Invariance under Diff[M]: ", Exists[Θ, Θ -> "homomorphism",
  {Θ[Diff[M]] → Aut[𝒢[M×F]] ↦ Θ[ϕ].U → U∘inv[ϕ], ϕ ∈ Diff[M], U ∈ 𝒢[M×F]}],
 Yield, "Full symmetry group: ", 𝒢[M×F] ▷ Diff[M]
]
```

●Full symmetry group.
•Homomorphic action Θ of a group H on group N: $\Theta[H] \to \mathrm{Aut}[N]$
•semi-direct product $N \triangleright H \to \{\{n, h\}, n \in N \,\&\&\, h \in H\}$

Properties:
$$\boxed{\begin{array}{l} \{\text{product}, \{n\_, h\_\} \cdot \{n1\_, h1\_\} \to \{n.\Theta[h].n1, h.h1\}\} \\ \{\text{unit}, \{1, 1\}\} \\ \{\text{inverse}, \text{invSDG}[\{n\_, h\_\}] \to \{\Theta[h^{-1}].n^{-1}, h^{-1}\}\} \end{array}}$$

....
•Invariance under Diff[M]:
$\exists_{\Theta,\Theta\to\text{homomorphism}} \{\Theta[\mathrm{Diff}[M]] \to \mathrm{Aut}[\mathcal{G}[M\times F]] \mapsto \Theta[\phi].U \to U\circ\phi^{-1}, \phi \in \mathrm{Diff}[M], U \in \mathcal{G}[M\times F]\}$
→ Full symmetry group: $\mathcal{G}[M\times F] \triangleright \mathrm{Diff}[M]$

```
PR["●Principal bundles. ",
 NL, "Let ", $ = {{G → "Lie group", P → "principal G-bundle"} ↦ (π[P] → M),
   Aut[P] -> {f[P] → P, ForAll[{p, g}, p ∈ P && g ∈ G, f[p.g] → f[p].g]},
   Implies[f, Exists[f̄, {(f̄[M] → M) ↦ (f̄[π[p]] → π[f[p]]), f̄ → "diffeomorphism"}]]
  }; ColumnBar[$],
 NL, "•Gauge transformation of P: ",
 𝒢[P] → ForAll[g, g ∈ Aut[P], {ḡ == 1_M, π[g[p]] → π[p]}],
 NL, "?Is 𝒢[P] a normal subgroup: ",
 NL, "Since ", $ = f̄[π[p]] → π[f[p]],
 Yield, $ = $ /. f → f∘g∘inv[f],
 NL, "Since: ", $s = {(c__∘a_∘b_)[p_] → (c∘a)[b[p]], (a_ ∘ b_)[p_] → a[b[p]]},
 Yield, $ = MapAt[# //. $s &, $, 2],
 NL, "Using: ", $s = {π[f_[p_]] -> f̄[π[p]], a_[b_[π[p]]] :→ Flatten[a∘b][π[p]]},
 Yield, $ = MapAt[# //. $s &, $, 2]; Framed[Head /@ $],
 NL, "For ", $s = {ḡ → 1_M, f_ ∘ 1_M ∘ f1_ → f∘f1, f̄_ ∘ inv[f_] → 1_M},
 Yield, $ = $ //. $s; $ = Head /@ $,
 imply, $ = $[[1, 1]] ∈ 𝒢[P]; Framed[$ ≤ Aut[P]],
 NL, "Quotient: ", Quotient[Aut[P], 𝒢[P]] ≃ Diff[M]

]
```

●Principal bundles.

Let $\left|\begin{array}{l} \{G\to \text{Lie group},\ P\to \text{principal G-bundle}\}\mapsto(\pi[P]\to M) \\ \text{Aut}[P]\to\{f[P]\to P,\ \forall_{\{p,g\},p\in P\&\&g\in G}\ (f[p.g]\to f[p].g)\} \\ f\Rightarrow \exists_{\bar f}\{(\bar f[M]\to M)\mapsto(\bar f[\pi[p]]\to\pi[f[p]]),\ \bar f\to \text{diffeomorphism}\}\end{array}\right.$

•Gauge transformation of P: $\mathcal{G}[P]\to\forall_{g,g\in \text{Aut}[P]}\{\bar g == 1_M,\ \pi[g[p]]\to\pi[p]\}$

?Is $\mathcal{G}[P]$ a normal subgroup:

Since $\bar f[\pi[p]]\to\pi[f[p]]$

→ $\overline{f\circ g\circ f^{-1}}[\pi[p]]\to\pi[(f\circ g\circ f^{-1})[p]]$

Since: $\{(c\_\_\circ a\_\circ b\_)[p\_]\to(c\circ a)[b[p]],\ (a\_\circ b\_)[p\_]\to a[b[p]]\}$

→ $\overline{f\circ g\circ f^{-1}}[\pi[p]]\to\pi[f[g[f^{-1}[p]]]]$

Using: $\{\pi[f\_[p\_]]\to\bar f[\pi[p]],\ a\_[b\_[\pi[p]]]:\to \text{Flatten}[a\circ b][\pi[p]]\}$

→ $\boxed{\overline{f\circ g\circ f^{-1}}\to\bar f\circ\bar g\circ\overline{f^{-1}}}$

For $\{\bar g\to 1_M,\ f\_\circ 1_M\circ f1\_\to f\circ f1,\ \bar f\_\circ\overline{f\_^{-1}}\to 1_M\}$

→ $\overline{f\circ g\circ f^{-1}}\to 1_M$ ⇒ $\boxed{(f\circ g\circ f^{-1}\in\mathcal{G}[P])\le \text{Aut}[P]}$

Quotient: $\text{Quotient}[\text{Aut}[P],\ \mathcal{G}[P]]\simeq \text{Diff}[M]$

2.5 Inner fluctuations and gauge transformations

```
PR["● Definition 2.15:  For a Real ACM: ", M×F → {𝒜, ℋ, 𝒟, J},
  NL, "•Define: ", $O = Ω𝒟"1" → {xSum[aj. CommutatorM[𝒟, bj], {j}], aj | bj ∈ 𝒜},
  NL, "•inner fluctuations: ",
  𝒜f -> {ForAll[𝒜, 𝒜 ∈ $O[[1]], ConjugateTranspose[𝒜] == 𝒜]},
  NL, "•fluctuated Dirac operator: ", $DA = 𝒟𝒜 → 𝒟 + 𝒜f + ε'.J.𝒜f.ConjugateTranspose[J],
  NL, "■Calculate on inner fluctuations: ",
  NL, $A = $0 = {𝒜 → a.CommutatorM[slash[𝒟], b],
     a | b ∈ C"∞"[M], slash[𝒟] → -I T[γ, "u", {μ}] tuDs["∇"S][_, μ]},
  Yield, $ = $0[[1]] /. $0[[-1]] /. CommutatorM → MCommutator //
    tuDotSimplify[{T[γ, "u", {μ}]}],
  yield, $0 = $ = $ /. tuDs["∇"S][_, μ].b -> tuDs["∇"S][b, μ] + b.tuDs["∇"S][_, μ] //
     tuDotSimplify[{T[γ, "u", {μ}]}],
  NL, "Define ", $Am = $ = I T[𝒜, "d", {μ}] -> $[[2]] /. T[γ, "u", {μ}] → I;
  $ = -I # & /@ $;
  Framed[$ ∈ Real[C"∞"[M]]],
  NL, "Proof:",
  "POFF",
  NL, $0;
  $1 = ConjugateTranspose /@ $0 // ConjugateCTSimplify1[{}, {}, {T[γ, "u", {μ}]}];
  $2 = 𝒜 → ConjugateTranspose[𝒜];
  $ = {$0, $1, $2},
  Yield, $ = tuEliminate[$, {𝒜}],
  yield, $ = Implies[$[[-1]], $[[-1, 2]] ∈ Reals] /. T[γ, "u", {μ}] → 1;
  Framed[$],
  "PONdd",
  NL, "For ", $ = slash[𝒟]𝒜 -> slash[𝒟] + 𝒜 + JM.𝒜.ConjugateTranspose[JM],
  NL, "Since: ", $s = {jj : JM.𝒜 :> -Reverse[jj], JM.ConjugateTranspose[JM] → 1},
  imply, $ = slash[𝒟]𝒜 -> slash[𝒟] + 𝒜 + JM.𝒜.ConjugateTranspose[JM]
    // tuRepeat[$s, tuDotSimplify[]]
 ];
```

● Definition 2.15: For a Real ACM: $M \times F \to \{\mathcal{A}, \mathcal{H}, \mathcal{D}, J\}$

•Define: $\Omega^1_{\mathcal{D}} \to \{\sum_{\{j\}} [a_j.[\mathcal{D}, b_j]_-], a_j \mid b_j \in \mathcal{A}\}$

•inner fluctuations: $\mathcal{A}_f \to \{\forall_{\mathcal{A},\mathcal{A}\in\Omega^1_{\mathcal{D}}} \mathcal{A}^\dagger == \mathcal{A}\}$

•fluctuated Dirac operator: $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \varepsilon'.J.\mathcal{A}_f.J^\dagger + \mathcal{A}_f$

■Calculate on inner fluctuations:

$\{\mathcal{A} \to a.[\not{\mathcal{D}}, b]_-, a \mid b \in C^\infty[M], \not{\mathcal{D}} \to -i\, \gamma^\mu\, \underline{\nabla}^S_\mu[\_]\}$

$\to\ \mathcal{A} \to i\, a.b.\underline{\nabla}^S_\mu[\_]\, \gamma^\mu - i\, a.\underline{\nabla}^S_\mu[\_].b\, \gamma^\mu \ \longrightarrow\ \mathcal{A} \to -i\, a.\underline{\nabla}^S_\mu[b]\, \gamma^\mu$

Define $\boxed{(\mathcal{A}_\mu \to -i\, a.\underline{\nabla}^S_\mu[b]) \in \mathrm{Real}[C^\infty[M]]}$

Proof:

....

For $\not{\mathcal{D}}_{\mathcal{A}} \to \mathcal{A} + J_M.\mathcal{A}.(J_M)^\dagger + \not{\mathcal{D}}$

Since: $\{jj : J_M.\mathcal{A} :\to -\mathrm{Reverse}[jj], J_M.(J_M)^\dagger \to 1\}$ $\Rightarrow$ $\not{\mathcal{D}}_{\mathcal{A}} \to \not{\mathcal{D}}$

```
PR["●Inner fluctuations. ",
 NL, "•Dirac operator: ", $d = 𝒟 → slash[𝒟] ⊗ 1F + T[γ, "d", {5}] ⊗ 𝒟F, accumDef[$d];
 NL, "•Examine: ", $ = $A[[1]] /. slash[𝒟] → 𝒟; Framed[$],
 yield, $0 = $ /. $d // simpleCommutator[{}] // tuDotSimplify[],

 NL, "Evaluate[1]: ", $1 = $ = $0[[2, 1]],
 yield, $ = $ /. tuCommutatorExpand // tuDotSimplify[],
 NL, "Use: ", $s = {(slash[𝒟] ⊗ 1n_).b -> slash[𝒟] ⊗ b + b. (slash[𝒟] ⊗ 1n)}, accumDef[$s];
 Yield, $ = $ /. $s // tuDotSimplify[],
 NL, "Use: ", $slashD =
  $s = $sD = {$A[[-1]], a_ .(( c_ tuDs["∇"^S][_, μ]) ⊗ b_) → c ⊗ ( a.tuDs["∇"^S][b, μ]),
     (-I a_) ⊗ b_ → a ⊗ (-I b)}, accumDef[$s];
 Yield, $1 = $1 → ($ //. $s); Framed[$1], CK,
 accumDef[$slashD];
 NL, "Evaluate[2]: ", $2 = $ = $0[[2, 2]],
 NL, "Since: ", CommutatorM[T[γ, "d", {5}], b] → 0,
 NL, "Use: ", $s = {$[[2]] → ($[[2]] /. CommutatorM[a_ ⊗ b_, c_] → a ⊗ CommutatorM[b, c]),
   a_ . ((tt : T[γ, "d", {5}]) ⊗ b_) → tt ⊗ (a.b)},
 Yield, $ = $ /. $s /. $s ; Framed[$2 = $2 -> $],
 yield, "define: ", Framed[$2a = $[[2]] → ϕ],
 NL, "with ", Reverse[$Am],
 Imply, $A1 = $ = $0 /. $2 /. $1 /. $2a /. Reverse[$Am] // Simplify; Framed[$]
]
```

●Inner fluctuations.

•Dirac operator: $\mathcal{D}\to(\not{D})\otimes 1_F+\gamma_5\otimes\mathcal{D}_F$

•Examine: $\boxed{\mathcal{A}\to a.[\mathcal{D},b]_-}$ $\longrightarrow$ $\mathcal{A}\to a.[(\not{D})\otimes 1_F,b]_- + a.[\gamma_5\otimes\mathcal{D}_F,b]_-$

Evaluate[1]: $a.[(\not{D})\otimes 1_F,b]_-$ $\longrightarrow$ $-a.b.((\not{D})\otimes 1_F)+a.((\not{D})\otimes 1_F).b$

Use: $\{((\not{D})\otimes 1_{n\_}).b\to(\not{D})\otimes b+b.((\not{D})\otimes 1_n)\}$

$\to$ $a.((\not{D})\otimes b)$

Use: $\{\not{D}\to -i\,\gamma^\mu\,\underline{\nabla}^S_\mu[\_],\ (a\_).((c\_\ \underline{\nabla}^S_\mu[\_])\otimes b\_)\to c\otimes a.\underline{\nabla}^S_\mu[b],\ (-i\ a\_)\otimes b\_\to a\otimes(-i\ b)\}$

$\to$ $\boxed{a.[(\not{D})\otimes 1_F,b]_-\to\gamma^\mu\otimes(-i\ a.\underline{\nabla}^S_\mu[b])}$ ⟵CHECK

Evaluate[2]: $a.[\gamma_5\otimes\mathcal{D}_F,b]_-$

Since: $[\gamma_5,b]_-\to 0$

Use: $\{[\gamma_5\otimes\mathcal{D}_F,b]_-\to\gamma_5\otimes[\mathcal{D}_F,b]_-,\ (a\_).((tt:\gamma_5)\otimes b\_)\to tt\otimes a.b\}$

$\to$ $\boxed{a.[\gamma_5\otimes\mathcal{D}_F,b]_-\to\gamma_5\otimes a.[\mathcal{D}_F,b]_-}$ $\longrightarrow$ define: $\boxed{a.[\mathcal{D}_F,b]_-\to\phi}$

with $a.\underline{\nabla}^S_\mu[b]\to i\,\mathcal{A}_\mu$

$\Rightarrow$ $\boxed{\mathcal{A}\to\gamma_5\otimes\phi+\gamma^\mu\otimes\mathcal{A}_\mu}$

```
PR["•Fluctuated Dirac operator: ", $ = $DA,
 Yield, $ = $ /. 𝒜f → 𝒜;
 Yield, $0 = MapAt[# /. $A1 &, $, 2] // tuDotSimplify[],

 NL, "■Examine[𝒜]: ", $ = Select[$0[[2]], ! FreeQ[#, 𝒜] &],
 NL, "J Anticommutes: ",
 $s = e_ .J.(T[γ, "u", {μ}] ⊗ a_).b_ → -T[γ, "u", {μ}] ⊗ (e.J. a.b),
 Yield, $ = $ /. $s,
 yield, $ = $ /. a_ ⊗ b_ + (-a_ ⊗ c_) -> a ⊗ (b - c); Framed[$],
 NL, "Define ", $e216B = e216 = {Bμ -> $[[2]], Bμ ∈ Γ[Endo["E"]]};
 Framed[$e216B], CG[" (2.16)"],
 NL, "Define twisted connection: ",
 $ = T["∇"^"E", "d", {μ}] -> T["∇"^S, "d", {μ}] ⊗ Id + I Id ⊗ Bμ;
 Framed[$],
 Yield, $ = -I T[γ, "u", {μ}].# & /@ $ // tuDotSimplify[],
 $ = $ /. T[γ, "u", {μ}].(Id ⊗ b_) -> T[γ, "u", {μ}] ⊗ b;
 Yield, $ = $ /. -I a_ . (b_ ⊗ c_) → (-I a b) ⊗ c,
 NL, "Using: ", $s = (I # & /@ Reverse[$A[[-1]]] /. tuDDown[a_][_, m_] → T[a, "d", {m}]),
 Yield, e216a = $ /. $s; Framed[$],

 NL, "■Examine[ϕ]: ",
 NL, "Define ", Φ ∈ Γ[Endo["E"]] ∋
  ($ = T[γ, "d", {5}] ⊗ Φ -> Select[$0[[2]], ! FreeQ[#, ϕ] &] + T[γ, "d", {5}] ⊗ 𝒟F),
 Imply, e218 = 𝒟A → e216a[[1]] + $[[1]]; Framed[e218]
]
```

•Fluctuated Dirac operator: $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \varepsilon' . J . \mathcal{A}_f . J^\dagger + \mathcal{A}_f$

→

→ $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \gamma_5 \otimes \phi + \gamma^\mu \otimes \mathcal{A}_\mu + \varepsilon' . J . (\gamma_5 \otimes \phi) . J^\dagger + \varepsilon' . J . (\gamma^\mu \otimes \mathcal{A}_\mu) . J^\dagger$

■Examine[𝒜]: $\gamma^\mu \otimes \mathcal{A}_\mu + \varepsilon' . J . (\gamma^\mu \otimes \mathcal{A}_\mu) . J^\dagger$

J Anticommutes: $(e\_) . J . (\gamma^\mu \otimes a\_) . (b\_) \to -\gamma^\mu \otimes e . J . a . b$

→ $-\gamma^\mu \otimes \varepsilon' . J . \mathcal{A}_\mu . J^\dagger + \gamma^\mu \otimes \mathcal{A}_\mu$ ⟶ $\boxed{\gamma^\mu \otimes (-\varepsilon' . J . \mathcal{A}_\mu . J^\dagger + \mathcal{A}_\mu)}$

Define $\boxed{\{B_\mu \to -\varepsilon' . J . \mathcal{A}_\mu . J^\dagger + \mathcal{A}_\mu,\ B_\mu \in \Gamma[\mathrm{Endo}[E]]\}}$ (2.16)

Define twisted connection: $\boxed{\nabla^E{}_\mu \to i\, Id \otimes B_\mu + \nabla^S{}_\mu \otimes Id}$

→ $-i\, \gamma^\mu . \nabla^E{}_\mu \to \gamma^\mu . (Id \otimes B_\mu) - i\, \gamma^\mu . (\nabla^S{}_\mu \otimes Id)$

→ $-i\, \gamma^\mu . \nabla^E{}_\mu \to \gamma^\mu \otimes B_\mu + (-i\, \nabla^S{}_\mu\, \gamma^\mu) \otimes Id$

Using: $\nabla^S{}_\mu\, \gamma^\mu \to i\, (\not{D})$

→ $\boxed{-i\, \gamma^\mu . \nabla^E{}_\mu \to \gamma^\mu \otimes B_\mu + (-i\, \nabla^S{}_\mu\, \gamma^\mu) \otimes Id}$

■Examine[ϕ]:

Define $\Phi \in \Gamma[\mathrm{Endo}[E]] \ni (\gamma_5 \otimes \Phi \to \gamma_5 \otimes \phi + \gamma_5 \otimes \mathcal{D}_F + \varepsilon' . J . (\gamma_5 \otimes \phi) . J^\dagger)$

⇒ $\boxed{\mathcal{D}_A \to \gamma_5 \otimes \Phi - i\, \gamma^\mu . \nabla^E{}_\mu}$

```
$hermitian = {𝒜μ};
PR["Since: ",
 $ = Implies[Inactive[tuMemberQ[𝒜μ, $hermitian]], ConjugateTranspose[𝒜μ] == 𝒜μ],
 imply, $ = -I # & /@ Activate[$] /. -I ConjugateTranspose[a_] → SuperDagger[I a],
 imply, Framed[I $[[2]] ∈ I u],
 NL, "For ", I 𝔤[F] → I Mod[u[F], 𝔥[F]],
 imply, $ = $e219 = 𝒜μ ∈ C"∞"[M, I 𝔤[F]]; $ // Framed
]
```

```
Since: Inactive[tuMemberQ[𝒜μ, $hermitian]] ⇒ (𝒜μ)† == 𝒜μ ⇒ (ⅈ 𝒜μ)† == -ⅈ 𝒜μ ⇒ [𝒜μ ∈ ⅈ u]

For ⅈ 𝔤[F] → ⅈ Mod[u[F], 𝔥[F]] ⇒ [𝒜μ ∈ C∞[M, ⅈ 𝔤[F]]]
```

```
PR["Gauge transformation on fluctuating Dirac operator. ",
  Yield, $00 = $0 = 𝒟𝒜 → 𝒟 + 𝒜 + ε'.J.𝒜.ConjugateTranspose[J],
  NL, "Expanding Rules: ",
  $s0 = {U → u.J.u.ConjugateTranspose[J], CommutatorM[a, rghtA[b]] → 0,
    CommutatorM[𝒜, J.u.ConjugateTranspose[J]] → 0,
    CommutatorM[CommutatorM[𝒟, a], rghtA[b]] → 0,
    J.𝒟 → ε'.𝒟.J, rghtA[b] → J.ConjugateTranspose[b].ConjugateTranspose[J],
    JJ_ .ConjugateTranspose[JJ_] :> 1 /; MemberQ[{J, u}, JJ],
    ConjugateTranspose[JJ_].JJ_ :→ 1 /; MemberQ[{J, u}, JJ],
    ε^2 → 1};
  Yield, $s0x = $s0 /. CommutatorM → MCommutator // tuDotSimplify[{ε'}] //
    tuRuleEliminate[{rghtA[b]}];
  FramedColumn[$s0x],
  NL, "Evaluate: ",
  $0a = $ = U.#.ConjugateTranspose[U] & /@ $0 // tuDotSimplify[{ε', ε}],

  Yield,
  $1 = $ = $[[2]] // tuRepeat[$s0x, tuDotSimplify[]] // ConjugateCTSimplify1[{ε', ε}];
  $1 = $1 // tuRepeat[$s0x, tuDotSimplify[]], (*Need to repeat for some reason*)
  NL, "From commutation rules: ",
  $s = tuRuleSolve[$s0x[[5]], Dot[𝒟, J]],

  NL, "■Simplify the term: ",
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  yield, $ = $ /. $s0x[[7]] // tuDotSimplify[{ε', ε}],
  NL, "From ", $s = u.CommutatorM[𝒟, ConjugateTranspose[u]] ->
    u.MCommutator[𝒟, ConjugateTranspose[u]],
  $s = $s // tuDotSimplify[];
  yield, $s = $s /. $s0 // tuDotSimplify[],
  yield, $s = tuRuleEliminate[{u.𝒟.ConjugateTranspose[u]}][{$s}];
  Framed[$s],
  Imply, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  Yield, $ = $ //. $s0 // tuDotSimplify[{ε', ε}],
  yield, $1a = $ = $ /. $s; Framed[$], CK
 ];
PR[
  "■Simplify the term: ",
  Yield, $0 = $ = $1[[1]]; Framed[$],
  NL, "Use: ", $s = tuRuleSolve[$s0x /. u → ConjugateTranspose[u], 𝒜._],
  Yield, $ = $ /. $s // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1b = $]
 ];
```

```
$s0x /. xu → ConjugateTranspose[u];
PR[
 "■Simplify the term: ",
 Yield, $0 = $ = $1[[3]]; Framed[$],
 NL, "Append 1→ ", $s = J.ConjugateTranspose[J],
 imply, $ = $. $s // tuDotSimplify[{ε', ε}],
 NL, "Use ",
 $s = tuRuleSolve[$s0x /. u → ConjugateTranspose[u], 𝒜._],
 " with ConjugateTranspose: ", $sa = aa : a | J → ConjugateTranspose[aa],
 Yield, $s = $s /. ConditionalExpression[a_, b_] → a /. $sa //
   tuAddPatternVariable[{a, b}],
 NL, "The Rule applies to: ", $sa = 𝒜 → u.𝒜.ConjugateTranspose[u],
 yield, $s = $s /. $sa,
 Imply, $ = $ /. $s,
 yield, $ = $ // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1c = $]
]
PR["■Check if equal to (2.20). Our calculation: ",
  $ = $0a[[1]] -> $1a + $1b + $1c; Framed[$],
  NL, "Evaluate (2.20) with ", $ = $00 /. 𝒜 → 𝒜ᵘ, CK,
  Yield, $[[2]] =
   $[[2]] /. 𝒜ᵘ → u.𝒜.ConjugateTranspose[u] + u.CommutatorM[𝒟, ConjugateTranspose[u]] //
    tuDotSimplify[{ε'}];
  Framed[$],
  NL, CR["Almost equal."]
 ];
```

Gauge transformation on fluctuating Dirac operator.

→ $\mathcal{D}_{\mathcal{A}} \to \mathcal{A} + \mathcal{D} + \varepsilon'.J.\mathcal{A}.J^\dagger$

Expanding Rules:

$$\to \boxed{\begin{array}{l} U \to u.J.u.J^\dagger \\ a.J.b^\dagger.J^\dagger - J.b^\dagger.J^\dagger.a \to 0 \\ -J.u.J^\dagger.\mathcal{A} + \mathcal{A}.J.u.J^\dagger \to 0 \\ -a.\mathcal{D}.J.b^\dagger.J^\dagger + J.b^\dagger.J^\dagger.a.\mathcal{D} - J.b^\dagger.J^\dagger.\mathcal{D}.a + \mathcal{D}.a.J.b^\dagger.J^\dagger \to 0 \\ J.\mathcal{D} \to \mathcal{D}.J\,\varepsilon' \\ (JJ\_).JJ\_^\dagger :\to 1\ /;\ \text{MemberQ}[\{J, u\}, JJ] \\ JJ\_^\dagger.(JJ\_) :\to 1\ /;\ \text{MemberQ}[\{J, u\}, JJ] \\ \varepsilon^2 \to 1 \end{array}}$$

Evaluate: $U.\mathcal{D}_{\mathcal{A}}.U^\dagger \to U.\mathcal{A}.U^\dagger + U.\mathcal{D}.U^\dagger + U.J.\mathcal{A}.J^\dagger.U^\dagger\,\varepsilon'$

→ $u.J.u.J^\dagger.\mathcal{A}.J.u^\dagger.J^\dagger.u^\dagger + u.J.u.J^\dagger.\mathcal{D}.J.u^\dagger.J^\dagger.u^\dagger + u.J.u.\mathcal{A}.u^\dagger.J^\dagger.u^\dagger\,\varepsilon'$

From commutation rules: $\{\mathcal{D}.J \to \frac{J.\mathcal{D}}{\varepsilon'}\}$

■Simplify the term:

$$\to \boxed{u.J.u.J^\dagger.\mathcal{D}.J.u^\dagger.J^\dagger.u^\dagger} \longrightarrow \frac{u.J.u.J^\dagger.J.\mathcal{D}.u^\dagger.J^\dagger.u^\dagger}{\varepsilon'} \longrightarrow \frac{u.J.u.\mathcal{D}.u^\dagger.J^\dagger.u^\dagger}{\varepsilon'}$$

From $u.[\mathcal{D}, u^\dagger]_- \to u.(\mathcal{D}.u^\dagger - u^\dagger.\mathcal{D}) \longrightarrow u.[\mathcal{D}, u^\dagger]_- \to -\mathcal{D} + u.\mathcal{D}.u^\dagger \longrightarrow \boxed{\{u.\mathcal{D}.u^\dagger \to \mathcal{D} + u.[\mathcal{D}, u^\dagger]_-\}}$

$$\Rightarrow \frac{u.J.\mathcal{D}.J^\dagger.u^\dagger}{\varepsilon'} + \frac{u.J.u.[\mathcal{D}, u^\dagger]_-.J^\dagger.u^\dagger}{\varepsilon'}$$

$$\to u.\mathcal{D}.u^\dagger + \frac{u.J.u.[\mathcal{D}, u^\dagger]_-.J^\dagger.u^\dagger}{\varepsilon'} \longrightarrow \boxed{\mathcal{D} + u.[\mathcal{D}, u^\dagger]_- + \frac{u.J.u.[\mathcal{D}, u^\dagger]_-.J^\dagger.u^\dagger}{\varepsilon'}} \longleftarrow \text{CHECK}$$

■Simplify the term:

→ $\boxed{u.J.u.J^{\dagger}.\mathcal{A}.J.u^{\dagger}.J^{\dagger}.u^{\dagger}}$

Use: $\{\mathcal{A}.J.u^{\dagger}.J^{\dagger} \to J.u^{\dagger}.J^{\dagger}.\mathcal{A}\}$

→ $\boxed{u.\mathcal{A}.u^{\dagger}}$

■Simplify the term:

→ $\boxed{u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}\ \varepsilon'}$

Append $1 \to J.J^{\dagger}$ ⇒ $u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}.J.J^{\dagger}\ \varepsilon'$

Use $\{\mathcal{A}.J.u^{\dagger}.J^{\dagger} \to J.u^{\dagger}.J^{\dagger}.\mathcal{A}\}$ with ConjugateTranspose: $aa : a \mid J \to aa^{\dagger}$

→ $\{\mathcal{A}.J^{\dagger}.u^{\dagger}.J \to J^{\dagger}.u^{\dagger}.J.\mathcal{A}\}$

The Rule applies to: $\mathcal{A} \to u.\mathcal{A}.u^{\dagger} \longrightarrow \{u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}.J \to J^{\dagger}.u^{\dagger}.J.u.\mathcal{A}.u^{\dagger}\}$

⇒ $u.J.J^{\dagger}.u^{\dagger}.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon' \longrightarrow \boxed{J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$

■Check if equal to (2.20). Our calculation:

$$\boxed{U.\mathcal{D}_{\mathcal{A}}.U^{\dagger} \to \mathcal{D} + u.[\mathcal{D},\ u^{\dagger}]\_ + u.\mathcal{A}.u^{\dagger} + \frac{u.J.u.[\mathcal{D},\ u^{\dagger}]\_.J^{\dagger}.u^{\dagger}}{\varepsilon'} + J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$$

Evaluate (2.20) with $\mathcal{D}_{\mathcal{A}^u} \to \mathcal{A}^u + \mathcal{D} + \varepsilon'.J.\mathcal{A}^u.J^{\dagger}$ ⟵CHECK

→ $\boxed{\mathcal{D}_{\mathcal{A}^u} \to \mathcal{D} + u.[\mathcal{D},\ u^{\dagger}]\_ + u.\mathcal{A}.u^{\dagger} + J.u.[\mathcal{D},\ u^{\dagger}]\_.J^{\dagger}\ \varepsilon' + J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$

Almost equal.

```
PR["●Define bilinear form: ", $0 = $ = 𝔘𝒟[ξ, ξp] → BraKet[J.ξ, 𝒟.ξp](*⟨J.ξ,𝒟.ξp⟩*),
  Yield, $ = $ /. dd : 𝒟.ξp → -J.J.dd //. simpleBraKet[],
  Yield, $ = $ /. BraKet[J.a_, J.b_] → BraKet[b, a] /. J.𝒟 → 𝒟.J,
  Yield, $ = $ /. BraKet[𝒟.a_, b_] → BraKet[a, 𝒟.b](*𝒟 is Hermitian*),
  Yield, $s = Reverse[$0] // tuAddPatternVariable[{ξp, ξ}],
  Yield, $ = $ /. $s; Framed[$]
 ];
```

●Define bilinear form: $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to \langle J.\xi \mid \mathcal{D}.\xi p\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\langle J.\xi \mid J.J.\mathcal{D}.\xi p\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\langle \mathcal{D}.J.\xi p \mid \xi\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\langle J.\xi p \mid \mathcal{D}.\xi\rangle$

→ $\langle J.(\xi\_) \mid \mathcal{D}.(\xi p\_)\rangle \to \mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p]$

→ $\boxed{\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\mathfrak{U}_{\mathcal{D}}[\xi p,\ \xi]}$

```
PR["●Define classical fermions: ", (ℋ⁺)cl → {ξ̃ → Grassmann, ξ ∈ ℋ⁺},
  NL, "●Define action functional: ", $S = S → Sb + Sf → Tr[f[𝒟𝒜 / Λ]] + BraKet[J.ξ̃, 𝒟𝒜.ξ̃] / 2
 ];
PR["●Invariance of action functional under ",
  $s = {𝒟𝒜 → U.𝒟𝒜.ConjugateTranspose[U], xx : ξ̃ → U.xx},
  NL, "■Boson ", $0 = $ = tuExtractPattern[Tr[_]][$S] // First,
  yield, $ = $ /. $s,
  yield, xSum[f[λn / Λ], n], CG[" Invariant"],
  NL, "■Fermion ", $0 = $ = tuExtractPattern[BraKet[_, _]][$S] // First,
  Yield, $ = $ /. $s,
  NL, "Apply ",
  $s = {J.U → U.J, ConjugateTranspose[u_].u_ → 1, BraKet[U . a_, U.b_] -> BraKet[a, b]},
  Yield, $ = $ //. $s // tuDotSimplify[], CG[" Invariant"]
 ];
```

●Define classical fermions: $\mathcal{H}^{+}{}_{cl} \to \{\tilde{\xi} \to \text{Grassmann},\ \xi \in \mathcal{H}^{+}\}$

●Define action functional: $S \to S_b + S_f \to \frac{1}{2} \left\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi} \right\rangle + \mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

●Invariance of action functional under $\{\mathcal{D}_{\mathcal{A}} \to U.\mathcal{D}_{\mathcal{A}}.U^{\dagger},\ xx : \tilde{\xi} \to U.xx\}$

■Boson $\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \longrightarrow \mathrm{Tr}[f[\frac{U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}}{\Lambda}]] \longrightarrow \sum_{n}[f[\frac{\lambda_n}{\Lambda}]]$ Invariant

■Fermion $\left\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi} \right\rangle$

$\to \left\langle J.U.\tilde{\xi} \mid U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}.U.\tilde{\xi} \right\rangle$

Apply $\{J.U \to U.J,\ u\_^{\dagger}.(u\_) \to 1,\ \left\langle U.(a\_) \mid U.(b\_) \right\rangle \to \left\langle a \mid b \right\rangle\}$

$\to \left\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi} \right\rangle$ Invariant

```
Clear[i];
PR["●Theorem 2.19. A real even almost-commutative manifold M×F describes
    a gauge theory on M with gauge group 𝒢[M×F]->C∞[M,𝒢[F]]. ",
  NL, "•Sketch of Proof: ",
  $t219 = $ = {{"(2.19)" → {I 𝒜μ[x] ∈ 𝔤[F] → Mod[𝓊[𝒜F], 𝒽F]},
      𝒜[CG["Total algebra"]] → C"∞"[M, 𝒜F] → xSum[section[ii, Γ[M×𝒜F]], {ii}],
      {ω → I T[𝒜, "d", {μ}] . DifForm[T[x, "u", {μ}]], ω[CG["𝔤[F]-valued 1-form"]]},
      P[CG["Principal bundle"]] → M×𝒢[F],
      "(2.22)" → ω[CG["connection form on P"]],
      "group of gauge transform"[P] -> C"∞"[M, 𝒢[F]],
      "(2.12)" ⇒ 𝒢[M×F][CG["group of gauge transform"]][P],
      "(2.11)" ⇒ 𝒢[M×F] -> {U → u.J.u.ConjugateTranspose[J], u ∈ U[𝒜]},
      (rep[𝒜F[ℋF]] ⇒ rep[𝒢[F][ℋF]])
       ⇒ (M×ℋF ↔ "vector bundle of"[P → M×𝒢[F]])
    }}; Grid[Transpose[$], Frame → All],
  NL, "Note: ", {("E" → M×ℋF) ↔
     (P[CG["Principal bundle"]] → M×𝒢[F])⟹CG["action of gauge group on fermions"],
    ℋ["ACM"] → L²[M, S]⊗ℋF → L²[M, S⊗"E"],
    "⇒ particle fields"⟶section[S⊗"E"]} // ColumnBar
 ];
tuSaveAllVariables[]
```

●Theorem 2.19. A real even almost-commutative manifold $M\times F$ describes a gauge theory on M with gauge group $\mathcal{G}[M\times F]\text{->}C^\infty[M,\mathcal{G}[F]]$.

•Sketch of Proof:

$(2.19)\to\{\mathbb{i}\,\mathcal{A}_\mu[x]\in\mathfrak{g}[F]\to \mathrm{Mod}[\mathcal{u}[\mathcal{A}_F],\,\mathcal{h}_F]\}$
$\mathcal{A}[\text{Total algebra}]\to C^\infty[M,\,\mathcal{A}_F]\to \sum_{\{ii\}}[\mathrm{section}[ii,\,\Gamma[M\times\mathcal{A}_F]]]$
$\{\omega\to\mathbb{i}\,\mathcal{A}_\mu.\underline{d}[x^\mu],\,\omega[\mathfrak{g}[F]\text{-valued 1-form}]\}$
$P[\text{Principal bundle}]\to M\times\mathcal{G}[F]$
$(2.22)\to\omega[\text{connection form on P}]$
$\text{group of gauge transform}[P]\to C^\infty[M,\,\mathcal{G}[F]]$
$(2.12)\Rightarrow\mathcal{G}[M\times F][\text{group of gauge transform}][P]$
$(2.11)\Rightarrow\mathcal{G}[M\times F]\to\{U\to u.J.u.J^\dagger,\,u\in U[\mathcal{A}]\}$
$(\mathrm{rep}[\mathcal{A}_F[\mathcal{H}_F]]\Rightarrow\mathrm{rep}[\mathcal{G}[F][\mathcal{H}_F]])\Rightarrow M\times\mathcal{H}_F\leftrightarrow\text{vector bundle of}[P\to M\times\mathcal{G}[F]]$

Note:
$(E\to M\times\mathcal{H}_F)\leftrightarrow(P[\text{Principal bundle}]\to M\times\mathcal{G}[F])\Longrightarrow\text{action of gauge group on fermions}$
$\mathcal{H}[ACM]\to L^2[M,\,S]\otimes\mathcal{H}_F\to L^2[M,\,S\otimes E]$
$\Rightarrow$ particle fields $\longrightarrow$ section$[S\otimes E]$