Myers: Holographic entanglement entropy (Easy/Medium)

We will apply the Ryu-Takayanagi (RT) prescription [1] to evaluate the entanglement entropy in the boundary CFTs dual to certain interesting bulk solutions. According to this prescription, the entanglement entropy of a certain region V in the boundary theory is given by

$$S(V) = \underset{m \sim V}{\mathrm{ext}} \left[\frac{\mathcal{A}(m)}{4G} \right] , \tag{1}$$

where m are codimension-2 bulk surfaces which are *homologous* to V in the boundary (and in particular $\partial m = \partial V$), and $\mathcal{A}(m)$ denotes the 'area' of m. We will be studying extremal surfaces in a few asymptotically AdS geometries. So consider the (d+1)-dimensional anti-de Sitter vacuum written in global coordinates:

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega_{d-1} \qquad \text{with} \quad f(r) = f_0(r) = \frac{r^2}{L^2} + 1\,, \tag{2}$$

where $d\Omega_{d-1}$ is the line element on a round unit (d–1)-sphere and L is the AdS curvature scale. This metric is useful to study the boundary CFT in a conformal frame where it is placed on the cylinder $R \times S^{d-1}$. This metric becomes a black hole if we replace

$$f_0(r) \to f_{bh}(r) = \frac{r^2}{L^2} + 1 - \frac{\mu}{r^{d-2}}\,, \tag{3}$$

which is dual to the CFT at finite temperature.[1] Note that the black hole function vanishes at some radius r_h, which is the position of the event horizon. It may be useful to write the mass parameter μ in terms of the horizon radius r_h with

$$\mu = r_h^{d-2} \left(\frac{r_h^2}{L^2} + 1 \right) . \tag{4}$$

Now as noted above, we consider the boundary CFT as living on the cylinder $R \times S^{d-1}$ with metric[2]

$$d\tilde{s}^2 = -dt^2 + L^2\, d\Omega_{d-1} = -dt^2 + L^2 \left(d\theta^2 + sin^2\theta\, d\Omega_{d-2} \right) . \tag{5}$$

[1] Let us add that the temperature of the boundary theory is given by $T = \partial_r f_{bh}(r)|_{r=r_h}/(4\pi)$. Throughout the following, we will assume that we are in a high temperature phase with $TL \gg 1$, where the black hole solution is the appropriate geometry to describe the boundary CFT. Interested students may see [2] to understand what happens at low temperatures.

[2] The latter may be determined by considering the limit $r \to \infty$ in eq. (2) and removing the Weyl factor r^2/L^2 from the d-dimensional boundary metric describing the CFT directions, *i.e.,* drop dr^2.

We would like to use the RT prescription (1) to evaluate the entanglement entropy for a spherical region in this geometry. In particular, we focus our attention on a fixed time slice, *e.g.,* $t = 0$ and then choose the entangling surface as $\theta = \theta_0$.

Hence we must identify the extremal area surface in the bulk geometry (2) – we leave $f(r)$ unspecified and d general for the moment. Since the bulk geometry is static, the extremal surface will remain at $t = 0$ everywhere. Further the geometry of the region of interest is spherically symmetric, *i.e.,* the entangling surface is invariant under rotations in the remaining angular directions in $d\Omega_{d-2}$, and hence the radial profile of the RT surface in the bulk can be written as $r(\theta)$. Therefore the holographic entanglement entropy (1) becomes

$$S(\theta_0) = \frac{\mathcal{A}_{ext}}{4G} = \frac{\Omega_{d-2}}{4G} \int_0^{\theta_0} d\theta \, \sin^{d-2}\theta \, r^{d-2}(\theta) \sqrt{r^2(\theta) + \frac{\partial_\theta r(\theta)^2}{f(r(\theta))}} \,. \tag{6}$$

Here $\Omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma(\frac{d-1}{2})$ is the volume of a unit $(d$–$2)$-sphere. Further we have implicitly assumed that the surface runs from $r = r_0$ – this will be the minimum radius – at $\theta = 0$ to $r \to \infty$ at $\theta = \theta_0$. Finally, the RT prescription must find the profile which yields the extremal area. Hence we can treat the above expression for the area as an action and extremize it by taking the functional variation with respect to $r(\theta)$.

- Use Mathematica to find the equation determining the bulk profile for the RT surface. Simplify the equation by writing it in the form $0 = r''(\theta) + \dots$.

Now as the next step let us focus our attention on two-dimensional boundary theories. Hence we set $d = 2$. In this case, the extremal surfaces are actually (space-like) geodesics in the three-dimensional bulk geometry.

- Use the equation of motion derived above to evaluate the geodesics. The suggested approach is to choose boundary conditions at $r = r_0$ and $\partial_\theta r = 0$[3] at $\theta = 0$ and integrate outwards, *i.e.,* eventually the geodesic will head off to $r \to \infty$ at some finite value of θ. This will allow you to fix θ_0 as a function of r_0. Hint: You might want to use `WhenEvent` and `Sow` inside `NDSolve` to stop the integration when r is too big and to record the corresponding value of θ_0 at the boundary.

As a warm-up, consider the geodesics in the vacuum AdS_3 spacetime. In this case, the solutions may be found analytically as

$$r^2(\theta) = \frac{L^2 \cos^2\theta_0}{\sin^2\theta_0 - \sin^2\theta} \,, \tag{7}$$

[3] This ensures that the RT surface is smooth at $\theta = 0$.

and then $r_0 = L \tan\theta_0$.

- a) Verify that eq. (7) solves the equation which you derived after substituting $d = 2$ and $f(r) = f_0(r)$. Hint: Recall that to substitute a solution into a differential equation pure functions are very useful.

- b) Substitute this profile into eq. (6) and evaluate the entanglement entropy. However, since the extremal surface extends all the way to the asymptotic boundary (at $\theta = \theta_0$, this area is diverges. This divergence reflects the UV divergence that one finds in the entanglement entropy evaluated in the boundary CFT. We can regulate by introducing a radial cut-off at $r = L^2/\delta$ where δ is a short distance cut-off in the boundary theory. With this regulated integral, you should recover the well-known result of Calabrese and Cardy [3]
$$S(\theta_0) = \frac{c}{3}\,\log\left[\frac{2L}{\delta}\,\sin\theta_0\right]\,, \tag{8}$$
where $c = 3L/(2G)$ is the central charge of the two-dimensional boundary CFT.

Now let us turn to the AdS black hole background, still with $d = 2$.

- Evaluate the geodesics in this background for various values of the initial radius r_0. You should find that for large r_0, the geodesics are not very different from those in the pure AdS$_3$ spacetime. However, as r_0 starts to get closer to the horizon at r_h, you should find that the geodesics begin wind around the full range of $\theta = 0$ to 2π a number of times before reaching the asymptotic boundary.

Hence there are actually many different extremal surfaces/geodesics with the precisely the same boundary condition: $r \to \infty$ at $\theta = \theta_0$. In such a situation, the RT prescription is that we should choose the extremal surface with the minimal area. By considering a few examples, verify that for relatively small values of θ_0, *i.e.,* $\theta_0 < \pi/2$, that the minimal area surface is the simplest surface, *i.e.,* the one which does not wind around the black hole. Of course, to determine the minimal area surface, we must again evaluate the area in eq. (6) for the various geodesics. As above, this requires introducing a radial cut-off at $r = L^2/\delta$ where δ is a short distance cut-off in the boundary CFT.

- Compare your numerical results against the analytic answer:
$$S(\theta_0) = \frac{c}{3}\,\log\left[\frac{\sinh\left(2\pi T\,L\,\theta_0\right)}{\pi T\,\delta}\right]\,, \tag{9}$$

where again $c = 3L/(2G)$ and the temperature is determined by the formula in footnote 1.[4]

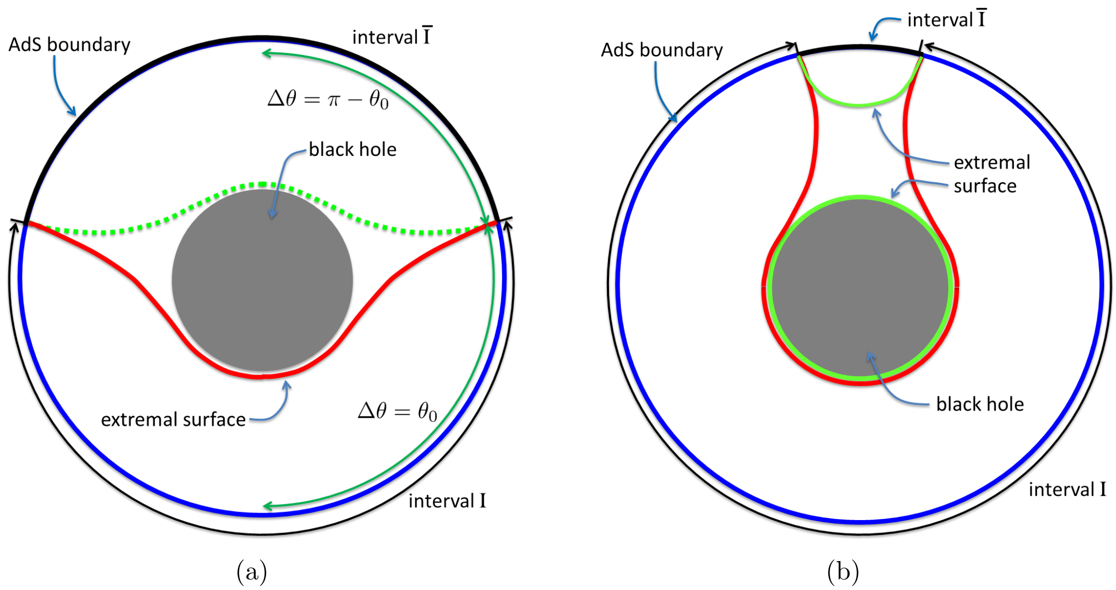

Figure 1: Extremal surfaces in the high temperature phase. The figures show a cross-section of the AdS_3 black hole at constant t. (a) For sufficiently small intervals, the holographic entanglement entropy is evaluated with the red geodesic. The dashed green geodesic passing on the other side of the black hole is not homologous to the interval V, however, it would yield the entanglement entropy for the complementary interval $\bar{V}$. (b) For large $\Delta\theta_0$, the dominant saddle-point (in green) has two disconnected components, *i.e.,* the geodesic homologous to $\bar{V}$ and the geodesic wrapping around the horizon.

In fact, the above analysis would indicate that the nonwinding geodesic is the minimal area geodesic for all values of $0 \leq \theta_0 \leq \pi$. However, we have not considered all of the geodesics with the specified boundary condition: $r \to \infty$ at $\theta = \theta_0$. All of the geodesics considered begin at the minimum radius at $\theta = 0$. However, as illustrated in figure 1(a), there is another set of geodesics which reach the boundary at the same point but which begin on the other side of the black hole at $\theta = \pi$. Alternatively, by symmetry, these are equivalent to the geodesics that start at $\theta = 0$ but reach $r \to \infty$ at $\theta = \pi - \theta_0$. However, read on because there is a subtlety here.

[4]Note that this expression is again reproducing a familiar result for two-dimensional CFTs [3], however, for general CFT's, this expression only applies in the limit that $L \to \infty$ with $L\theta_0$ fixed.

- Using your previous analysis, you should be able to verify that for large boundary intervals, *e.g.,* $\theta_0 \simeq \pi$, that the area/length of the geodesic spanning $\Delta\theta = \pi - \theta_0$ is much less than that of the geodesic spanning $\Delta\theta = \theta_0$.

However, the former geodesic fails to provide the appropriate RT surface because of the following. Another aspect of the RT surface is that it must be homologous to the boundary region for which we are evaluating the entanglement entropy [4]. That is, if E denotes the extremal RT surface and R, the boundary region for which we are evaluating the holographic entanglement entropy. Note that the boundaries of these two codimension-two surfaces match, *i.e.,* $\partial E = \partial R$. The holomology constraint requires that there must be a codimension-one surface B in the bulk such that $\partial B = E \cup R$. Alternatively, $E \cup R$ forms a closed surface and we should be able to shrink this surface to a point in the bulk geometry.

The present case provides a good example of where this holomology constraint comes into play in a nontrivial way. In general, entanglement entropies must satisfy a variety of nontrivial inequalities. In particular, one such inequality is known as the Araki-Lieb inequality [5],

$$|S(A) - S(B)| \leq S(A \cup B)\,. \tag{10}$$

In the present case, we may apply this inequality to the original interval I and its complement $\overline{I}$ to find

$$|S(I) - S(\overline{I})| \geq S_{therm}\,, \tag{11}$$

where S_{therm} denotes the thermal entropy of the full boundary CFT. Now for large θ_0, the 'short' geodesic, *i.e.,* the geodesic spanning $\Delta\theta = \pi - \theta_0$, in the previous discussion provides the holographic entanglement entropy for the complementary interval $\overline{I}$. If the same geodesic also described the entanglement entropy of the original interval I, we would have $|S(I) - S(\overline{I})| = 0$ and then the Araki-Lieb inequality (11) would demand that the thermal entropy must vanish. However, we know that S_{therm} does not vanish! Rather it is given by the horizon entropy of the black hole, *i.e.,*

$$S_{therm} = S_{\rm BH} = \frac{A(r_h)}{4G} = \frac{2\pi r_h}{4G}\,. \tag{12}$$

However, this also suggests the resolution of our problem. As illustrated in figure 1(b), rather than evaluating $S(I)$ by considering the 'short' geodesic alone, we take the extremal surface that is defined by the combination of the 'short' geodesic and the black hole horizon, which is in fact also an extremal surface in the bulk space time. This combination satisfies the desired homology constraint and one can show that for sufficiently large θ_0 that it provides the minimal area surface for the desired boundary conditions. Further, in this case, we will have $|S(I) - S(\overline{I})| = S_{therm}$ and so the Araki-Lieb inequality (11) is saturated. For certain fixed values of r_h, find the critical value of θ_0 at which the dominant saddle-point goes from being the red surface to the green surface in figure 1(b).

Here again, there is an analytic answer, which is given by

$$\sinh\left(2\pi L\,T\,\theta_0\right) \;=\; \sinh\left(2\pi L\,T(\pi-\theta_0)\right)\,\exp\left(2\pi^2 L\,T\right)\;. \tag{13}$$

- Compare your numerical results with the corresponding solution of the above equation.

- Repeat the above steps but now for $d = 3$ or higher. The biggest difference is that for the black hole background in $d \geq 3$, there are multiple saddle-points but the surfaces never cross over themselves and instead, the corresponding extremal surfaces fold back on themselves [6].[5] As a consequence for sufficiently large θ_0, there are not extremal surfaces with the desired boundary condition which begin at $\theta = 0$.

References

[1] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96**, 181602 (2006) [hep-th/0603001];
S. Ryu and T. Takayanagi, "Aspects of Holographic Entanglement Entropy," JHEP **0608**, 045 (2006) [hep-th/0605073].

[2] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [hep-th/9803131].

[3] P. Calabrese and J. L. Cardy, "Entanglement entropy and quantum field theory," J. Stat. Mech. **0406** (2004) P06002, [arXiv:hep-th/0405152].

[4] See, for example:
M. Headrick and T. Takayanagi, "A Holographic proof of the strong subadditivity of entanglement entropy," Phys. Rev. D **76**, 106013 (2007) [arXiv:0704.3719 [hep-th]].

[5] H. Araki and E. H. Lieb, "Entropy inequalities," Commun. Math. Phys. **18** (1970), 160.

[6] V. E. Hubeny, H. Maxfield, M. Rangamani and E. Tonni, "Holographic entanglement plateaux," JHEP **1308**, 092 (2013) [arXiv:1306.4004 [hep-th]].

[5] The source of this folding is the extra measure factor $\sin^{d-2}\theta$ in eq. (6). Geometrically, the folding occurs because the extrinsic curvature would diverge if the surface was to cross over itself at a point.