Physics 234A: String Theory

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Homework 2.

1 Problem

Consider a particle of unit mass moving in a harmonic oscillator potential.

1.1

What is the Hamiltonian H for this system? Find the spectrum of the Hamiltonian and the corresponding basis of eigenstates.

1.2

Compute the partition function $Z(\beta) = \mathrm{Tr}\exp(-\beta H)$.

1.3

Formulate a path integral computation which is equivalent to computing $Z(\beta)$ of the previous part.

1.4

Evaluate the path-integral by doing a saddle point expansion near the stationary points of the action . You will encounter an infinite sum in $\log Z(beta)$ that needs regularization.

1.5

Show the two expressions for $Z(\beta)$ one obtains are the same, provide one uses ζ function regularization to regulate the infinite sum in the previous part.

2 Problem

In class, we studied quantization of the $1+1$ dimensional sigma model,

$$S = \frac{1}{4\pi}\int d\tau d\theta \left((\partial_\tau X)^2 - (\partial_\theta X)^2\right).$$

For simplicity we will focus on a single real boson X, on the cylinder, $\theta \sim \theta + 2\pi$. In this problem we will focus on the QFT on a fixed background – the metric on Σ is not dynamical for now.

2.1

Quantize the theory, deriving the mode expansion for X, and the commutation relations.

2.2

Derive the energy H and the momentum operator P for this theory. H here is the operator generating time translations, P generates space translations.

2.3

What is the Hilbert space $\mathcal{H}$ of the theory?

2.4

Let

$$H_R = \frac{1}{2}(H - P), \qquad H_L = \frac{1}{2}(H + P)$$

Show that H_R and H_L involve only the right and the left moving oscillators, respectivelly.

2.5

Give a geometric interpretation to

$$Z(\tau_1, \tau_2) = \mathrm{Tr}_{\mathcal{H}}\, e^{-2\pi\tau_2 H} e^{-2\pi\tau_1 P}$$

Letting $\tau = \tau_1 + i\tau_2$, compute

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, q^{H_L} \bar{q}^{H_R}$$

where $q = e^{2\pi i\tau}$. (Hint: The spectrum is discrete, except for the zero mode part, which is continuous, as it corresponds to a free particle propagating on $\mathbb{R}$. Define $\mathrm{Tr}_{\mathcal{H}_0} e^{-\beta p^2/2}$ as an integral, $V \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\beta p^2/2}$, obtained via putting the theory in a finite volume V.)

2.6

Show that the theory is invariant under finite diffeomorphisms of $\Sigma = T^2$, acting on τ, as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

where $ad - bc = 1$, and $a, b, c, d \in \mathbb{Z}$. (Note the leading power of q in the partition function comes from ζ function regularization; without it, modular invariantce would have been lost).

3

Problem 2.12, BBS, page 57.