Broken Symmetries and the Goldstone Theorem

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Contents

I. Introduction

The Goldstone theorem, because it makes an exact statement concerning the excitation spectrum of a physical system, occupies what is essentially a unique position in quantum field theory. Briefly stated, the theorem asserts that for systems in which the vacuum is not an eigenstate of a time-independent operator (in general, the integral of a local operator over a given spacelike surface), there must exist massless particle excitations. Physically such a statement is eminently plausible inasmuch as such massless particles (like the vacuum) can have zero energy and thus provide a mechanism whereby an arbitrarily large number of null eigenstates of the energy momentum four-vector can be constructed. Since these new "vacuum states" will not, in general, be eigenstates of the conserved operator, one can by this heuristic device produce what is generally referred to as a "broken symmetry." This approach is potentially useful since it provides a way in which one can obtain solutions possessing a lower symmetry than a given Lagrangian without the necessity of explicitly introducing a symmetry-breaking interaction. On the other hand, the price one pays for this apparent simplicity, aside from the problem of accommodating a massless particle within a given physical scheme, generally consists of a rather complicated set of dynamical constraints on the theory, which must be carefully handled in all perturbative calculations.

We initiate this study of broken symmetries in Section II with a presentation and comparison of a number of the presently known proofs of the Goldstone theorem. It is furthermore shown how so-called failures of the theorem can occur in acausal theories. The question of the existence of integral charge operators is discussed and related to the inequivalent representations which underlie all broken-symmetry theories.

Section III continues this line of development by a detailed consideration of relativistic theories in which one has the particularly simple situation in which the relevant conserved current is linear in one of the canonical variables. Although it is possible to illustrate a number of interesting aspects of broken symmetries within the framework of such models, unfortunately they invariably consist of generalizations of free fields (with a gauge variable) and the broken symmetry is consequently never observable.

In Section IV the connection between broken symmetries and gauge invariance is discussed within the context of electrodynamics as well as the more complicated non-Abelian gauge theories. In such cases it is demonstrated that for gauges which are not manifestly covariant, the Goldstone theorem need not apply and, furthermore, that the use of manifestly covariant formulations allows one only to infer the presence of unphysical zero-mass gauge modes. In addition, the problem of counting the number of distinct Goldstone bosons associated with the breaking of a group symmetry is considered.

The nonrelativistic applications of the Goldstone theorem are discussed in Section V in a manner which is intended to emphasize the correspondence to the relativistic case. The examples considered here are of particular interest since one has, in the domain of nonrelativistic quantum mechanics, the important advantage that there exist examples of physically interesting broken-symmetry theories which are amenable to calculation.

The corresponding relativistic case (in which the conserved current is bilinear in the canonical fields) is discussed in Section VI. A soluble model in two dimensions is considered with its four-dimensional counterpart. A relativistic analog of superconductivity is considered in perturbation theory, and nontrivial symmetry-breaking effects are displayed.

Finally, in Section VII, various versions of the scalar theory and the Lee model are considered with the inclusion of a symmetry-breaking effect. The high degree of solubility of these models, along with the fact that the symmetry-breaking effects are nontrivial, makes them particularly interesting. Although they are not fully relativistic, they have more elementary particle aspects than the usual nonrelativistic models and consequently are quite suggestive.

II. The Goldstone Theorem

It is customary to begin any attempt to describe a physical situation within the framework of quantum field theory with the construction of a suitable interaction Lagrangian. In the event that there are several basic fields involved, this problem may often be simplified by a physical observation indicating a symmetry among these fields thereby restricting the choice of possible interaction terms to those particular combinations of fields which respect the symmetry. Frequently, however, the symmetry in question is not exact and it is consequently necessary to guess the form of the corrections to the symmetric interaction Lagrangian. This procedure is usually carried out under the assumption that the solutions of a given physical system will display the same degree of symmetry as the original Lagrangian. However, it has become evident (particularly in the domain of nonrelativistic quantum field theory) that the solutions to a set of field equations are often not unique and that perfectly acceptable solutions exist which have less symmetry than that displayed by the Lagrange function. Consequently, it might be possible to avoid the introduction of a symmetry-breaking interaction

Lagrangian and proceed instead by attempting to find an appropriate broken-symmetry solution to the symmetrical Lagrangian.

The reason this possibility even occurs is that, although the field equations and commutation relations respect the relevant invariance operation, under appropriate circumstances the states can be modified in such a way as to induce a breakdown of the symmetry. This is somewhat reminiscent of an interaction picture where the states are given in terms of the complete set $|a'\rangle$ of the symmetry-preserving Hamiltonian modified by exponential factors so as to form the new states

$$\left(\exp\left\{i\int_{t_0}^{t} dx\, H_{\text{s.b.}}(x)\right\}\right)_+ |a'\rangle$$

In fact, however, the actual structure of the states involves a number of rather subtle problems (including, for example, the requirement that such a modification not affect the usual time independence of this complete set of states). Although the equations for the Green's functions of the broken symmetry case are functionally identical to the fully symmetric situation, nonetheless the possibility exists of asymmetric solutions as a consequence of the different boundary conditions which can be imposed on the solutions to the Green's function equations. In particular, the usual assumption that the state of lowest energy is an eigenstate of the group generators can be dropped in order to obtain asymmetric solutions. Although such an assumption is an external one, it is also quite natural, and dispensing with it often leads to constraints on the spectrum and parameters of the problem. The spectrum constraint is known as the Goldstone theorem[(1)] and requires that, under fairly general circumstances, the currents associated with the broken symmetry have zero-mass excitations. Insofar as the theorem is valid and the massless excitations are identifiable as physical particles, it is clear that these massless modes make it unlikely that the method of broken symmetries will be valid in the case of strongly interacting particles. A considerable amount of energy and ingenuity has been devoted to various attempts to avoid the consequences of the theorem for this reason. Indeed, until fairly recently, the apparent failure of the theorem in the case of certain many-body problems was frequently invoked to suggest a rather general breakdown of the theorem (which would, of course, greatly enhance the possibility of consistently applying the broken-symmetry approach to strong interactions).

The purpose of this section is to carefully state and prove the above-mentioned theorem. Having done this, the reason for any so-called failures will be clear. It will also follow that the only way to avoid the consequences of the theorem within the context of causal theories with conserved currents lies in the possibility that the predicted zero-mass modes need not be associated with physical particles, but may refer to decoupled nonphysical excitations. Unfortunately, however, such a decoupling cannot usually be demonstrated in the absence of a complete solution.

The next few paragraphs are devoted to what is essentially the classical proof of the Goldstone theorem with a clear statement concerning its domain of validity. We begin by emphasizing the important role of the causality condition in guaranteeing that commutators of the charge operator relevant to the broken symmetry are independent of the spacelike surface on which this charge is evaluated. This observation is essential to explain apparent failures of the theorem referred to above. In the process of developing these arguments, we shall observe that the integral charge cannot be well defined when the symmetry is broken. This is a fundamental aspect of the theory and is discussed in some detail. In particular, the exponentiated generators of the symmetry group are discussed. It is shown that the mappings they describe are to be interpreted in a limiting sense and that these operators serve to map from one representation of the commutation relations to another inequivalent representation in a manner consistent with the original field equations. Our treatment does not aspire to the degree of rigor which may be required by the most mathematically inclined. It is, however, basically correct, and, if desired, one may refine the arguments used here without encountering any fundamental difficulties. [An informative early discussion of these problems with greater emphasis on mathematical rigor may be found in a work by Streater.[(2)]]

We begin with the assumption that there exists a conserved current[(3)] $j^\mu(x)$, i.e.,

$$\partial_\mu j^\mu(x) = 0$$

Although for convenience we display only one spatial index and suppress all internal indices which label the current operator, there is no essential difficulty introduced by considering more complex operators. If, for example, Lorentz symmetry were to be broken, one would consider the operator

$$F^{\mu\nu\lambda}(x) = x^\mu T^{\nu\lambda}(x) - x^\nu T^{\mu\lambda}(x)$$

which obeys the differential conservation law

$$\partial_\lambda F^{\mu\nu\lambda}(x) = 0$$

Now let us consider any combination of operators A of the field theory being considered. We emphasize that A is not necessarily local and, indeed, it is sufficient for our purpose that A exist only in a formal sense (provided that certain of its commutators are well defined). We define the generator[4,5]

$$Q_R(t) \equiv \int_{|\mathbf{x}|<R} d^3x\, j^0(\mathbf{x}, t)$$

and denote the surface of the sphere $|\mathbf{x}| = R$ as $\sigma(R)$. In order to incorporate the effect of current conservation, we consider the relation

$$\int_{|\mathbf{x}|<R} d^3x[\partial_\mu j^\mu(x), A] = 0$$

or

$$[\partial_0 Q_R(t), A] + \left[\int_{\sigma(R)} d\boldsymbol{\sigma}\cdot\mathbf{j}, A\right] = 0$$

If for some sufficiently large value of R (say L), we find that

$$\left[\int_{\sigma(R>L)} d\boldsymbol{\sigma}\cdot\mathbf{j}, A\right] = 0 \tag{2.1}$$

it follows that

$$\partial_0[Q_{R>L}(t), A] = 0$$

or

$$[Q_{R>L}(t), A] = B \tag{2.2}$$

with

$$dB/dt = 0$$

It might happen, of course, that these equations are only valid in the limit $R\to\infty$, in which case we find

$$\lim_{R\to\infty}\left[\int d\boldsymbol{\sigma}\cdot\mathbf{j}, A\right] = 0 \tag{2.3}$$

and thus

$$\lim_{R\to\infty}[Q_R(t), A] = B \tag{2.4}$$

with

$$dB/dt = 0$$

Note that, if we denote in the usual fashion

$$\lim_{R\to\infty} Q_R(t) = Q$$

Equation (2.4) becomes $[Q, A] = B$. Strictly speaking, however, though the commutator exists, we shall see that when the symmetry is broken, Q does not and thus the limit Eq. (2.4) cannot be evaluated inside the commutator.

It should be pointed out that there are many theories in which Eq. (2.1) is valid for a large class of operators A. If we deal with a theory which is locally causal and A is localized in a finite region of space time, Eq. (2.1) follows at once for sufficiently large R. If A is not localized to a finite volume of space-time but has rapidly decreasing weights for large values of coordinates, Eq. (2.3) may still be valid in a causal theory. As we shall see in Section V, Eq. (2.3) is valid for non-relativistic problems involving rapidly decreasing potentials. How rapid this decrease must be depends explicitly on the structure of j^k as well as the potential $V(r)$.

It is clear that the above remarks apply to a wide class of theories with no particular reference to symmetry breaking. We may now make the basic broken-symmetry assumption by setting

$$\langle 0|B|0\rangle \neq 0 \tag{2.5}$$

where $|0\rangle$ is a translationally invariant vacuum state. Before we prove the Goldstone theorem in the classical manner, we will examine some of the more general aspects of the very strong assumption (2.5). Through this study it will be much easier to make the connection to other forms of the proof. In particular, (2.5) implies that $|0\rangle$ cannot be an eigenstate of Q, and it follows from $\exp\{i\lambda Q\}|0\rangle \neq |0\rangle$ that Q is not a unitary operator. In fact, as shown by Fabri and Picasso,[6] the operator Q does not exist even in the sense of a weak limit. This follows from the observation that since $|0\rangle$ is translationally invariant, so is $Q|0\rangle$. Consequently,

$$\langle 0|QQ|0\rangle = \int d^3x\, \langle 0|j^0(x)Q|0\rangle$$

diverges unless Q annihilates the vacuum. Since this is contrary to hypothesis, it follows that whenever the symmetry is broken, the associated generator does not exist. Such a result (which has been recognized in various forms since the early times of broken symmetries) is rarely, if ever, of any concern when doing general calculations with broken symmetries.[7] This is because Q usually does not occur by itself, but only in commutators with some other operator. In that case, it is to be interpreted as the limiting form of (2.4) and no difficulty arises if Q is sufficiently localizable and causal.

As a slight generalization of these statements, note that in a causal theory with any sufficiently localized operator L, the operator

$$\begin{aligned}\tilde{L}_R &\equiv \exp\{i\eta Q_R(t)\}L\exp\{-i\eta Q_R(t)\}\\ &= L + i\eta[Q_R(t), L] + \tfrac{1}{2}(i\eta)^2[Q_R(t), [Q_R(t), L]] + \cdots\end{aligned}$$

considered term by term has no divergences in the limit $R \to \infty$ because of the assumed causality condition. We may assume that this power series expansion exists for sufficiently small η and we shall formally denote $\lim_{R\to\infty} L_R$ as $e^{i\eta Q}Le^{-i\eta Q}$. From the exponential structure of the above transformation, it follows that, for the class of operators

$$\tilde{L}_R{}^i \equiv \exp\{i\eta Q_R(t)\}L^i\exp\{-i\eta Q_R(t)\}$$

we have for any matrix element

$$\begin{aligned}&\langle a'|\tilde{L}_R{}^1\tilde{L}_R{}^2\ldots\tilde{L}_R{}^n|b'\rangle\\ &\qquad = \langle a'|\exp\{i\eta Q_R(t)L^1L^2\ldots L^n\exp\{-i\eta Q_R(t)\}|b'\rangle\end{aligned}$$

and so, to the extent that $\lim_{R\to\infty}\tilde{L}_R{}^i$ is well defined, so also is the limit of this matrix element as $R \to \infty$. This suggests that formally a new set of states $|\eta, a'\rangle \equiv \lim_{R\to\infty}\exp\{-i\eta Q_R(t)\}|a'\rangle$ be introduced.[8] This is a useful procedure and we shall outline its consequences here, noting however that some care must be exercised for this identification to be meaningful since the separation implied by the above equation is not mathematically well defined. As has been shown, the operator $e^{-i\eta Q}$ can only be given meaning when it appears in combination with $e^{i\eta Q}$. Indeed, in the following chapters it will be shown for several models that

$$\lim_{R\to\infty}\langle a'|e^{-i\eta Q_R(t)}|b'\rangle = 0$$

for all states $|a'\rangle$ and $|b'\rangle$. Consequently, $|\eta, a'\rangle$ is orthogonal to all members of the original Hilbert space and, if it were taken to be a member of the same space, it would necessarily be a null vector. This, of course, is inadmissible since then it would follow that all matrix elements of the form of (2.5) vanish.

We now are prepared to return to the proof of the Goldstone theorem which follows as a direct consequence of (2.4) upon imposition of (2.5). Taking the vacuum expectation and inserting a complete set of states, (2.4) becomes

$$\lim_{R\to\infty}\sum_n [\langle 0|Q_R(t)|n\rangle\langle n|A|0\rangle - \langle 0|A|n\rangle\langle n|Q_R(t)|0\rangle] = \langle 0|B|0\rangle \neq 0$$

The operator $j^0(x)$ is assumed to be local with the usual translational behavior

$$j^0(x) = e^{-iPx} j^0(0) e^{iPx}$$

and, further, it is assumed that $e^{iPx}|0\rangle = |0\rangle$. Thus, the preceding equation becomes

$$\begin{aligned}\lim_{R\to\infty}\sum_n \int_R d^3x[\langle 0|j^0(0)|n\rangle\langle n|A|0\rangle e^{ip_n x} - \langle 0|A|n\rangle\langle n|j^0(0)|0\rangle e^{-ip_n x}]\\ = \sum_n (2\pi)^3\delta(\mathbf{p}_n)[\langle 0|j^0(0)|n\rangle\langle n|A|0\rangle e^{-ip_n{}^0x^0}\\ - \langle 0|A|n\rangle\langle n|j^0(0)|0\rangle e^{ip_n{}^0x^0}]\\ = \langle 0|B|0\rangle\end{aligned}$$

Now this equation is valid for all times x^0 and, since $dB/dx^0 = 0$, it follows that the left-hand side of this equation must not depend on x^0. Clearly these conditions are consistent only if the left-hand side vanishes except for those states where $p_n{}^0|_{\mathbf{p}_n\to 0} = 0$. Furthermore, there must exist such states since the right-hand side does not vanish. Thus we have shown that if a symmetry is broken (in a "sufficiently" causal theory) there must be excitation modes in the spectrum of the generator of that symmetry whose energy vanishes in the limit that the momentum of these modes vanishes. The corresponding statement in the relativistic case is the assertion of the existence of zero mass particles. This is the celebrated Goldstone theorem. Note that the actual proof is quite trivial and the only real assumption, besides the translational behavior of the vacuum and current, has been that $dB/dt = 0$. It has been pointed out,[(9)] however, that this assumption could fail to hold in cases of some physical significance and was, indeed, the basis of some apparent breakdowns of the Goldstone theorem.

We have repeatedly emphasized above that if a symmetry is broken, the operator $e^{i\lambda Q}$ cannot be unitary. Now, having proved that the Goldstone theorem is applicable to causal theories, it is easy to exploit this result to demonstrate the converse statement, namely, that if $j^0(x)$ does not have a massless particle in its spectrum, the operator

$$\begin{aligned}U(\eta) &= \lim_{R\to\infty} \exp\left\{i\eta\int_R d^3x j^0(x, t)\right\}\\ &= \lim_{R\to\infty} U_R(\eta, t)\end{aligned}$$

is unitary. Since the exponential form guarantees that

$$U(\eta)U^+(\eta) = 1$$

and

$$U(\eta_1)U(\eta_2) = U(\eta_1 + \eta_2)$$

it is only necessary to demonstrate that

$$\lim_{R\to\infty} U_R(\eta, t)|0\rangle = |0\rangle$$

This, in turn, can be guaranteed if, for all Wightman functions, the relation

$$\lim_{R\to\infty} \langle 0|U_R(\eta, t)\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)U_R{}^+(\eta, t)|0\rangle = \langle 0|\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)|0\rangle$$

is valid. The proof is performed by observing that if, for an operator A of the form $\phi(x_1)\phi(x_2)\ldots\phi(x_n)$, it can be shown that

$$\frac{d}{d\eta}\langle 0|U(\eta)AU^+(\eta)|0\rangle = 0$$

for all λ, then

$$\begin{aligned}\langle 0|U(\eta)AU^+(\eta)|0\rangle &= \langle 0|U(0)AU^+(0)|0\rangle \\ &= \langle 0|A|0\rangle\end{aligned}$$

which is the required result. Explicitly evaluating the derivative yields

$$\frac{d}{d\eta}\langle 0|U_R(\eta, t)AU_R{}^+(\eta, t)|0\rangle = i\langle 0|[Q_R(t), U_R(\eta, t)AU_R{}^+(\eta, t)]|0\rangle \tag{2.6}$$

Now, if A is localized in a finite volume of space-time as is the case for $\phi(x_1)\phi(x_2)\ldots\phi(x_n)$ which is localized to a region of radius $L \leq \sum_i [|x_i{}^0| + |\mathbf{x}_i|]$ it follows by causality that, for $R > L + |t|$

$$U_R(\eta, t)AU_R{}^+(\eta, t)$$

is independent of t and R. By the same reasoning,

$$\left.\frac{d}{d\eta}\langle 0|U_R(\eta, t)AU_R{}^+(\eta, t)|0\rangle\right|_{R>L+t}$$

is independent of t and R. Consequently, the conditions required for the application of the Goldstone theorem to the right-hand side of

Eq. (2.6) are satisfied for sufficiently large R. Since by assumption no zero mass modes are present, it follows that this commutator vanishes and

$$\frac{d}{d\eta} \lim_{R\to\infty} \langle 0| U_R(\eta, t) A U_R^+(\eta, t)|0\rangle = 0$$

which demonstrates that

$$\langle 0| U(\eta) A U^+(\eta)|0\rangle = \langle 0|A|0\rangle$$

Thus, if $j^0(x)$ excites no massless modes, the transformation $U(\eta)$ is unitarily implementable.

What has been presented so far is the more or less traditional theoretical formulation of the Goldstone theorem. A reformulation in the mathematically more rigorous context of axiomatic quantum field theory has been given by Streater.[2] Kastler, Robinson, and Swieca[10] have also presented an axiomatic proof, which involves a reversal of the chain of argument. Without attempting to maintain their level of rigor we shall now present a version of their argument which displays in somewhat simpler form the underlying physical ideas. From this it will be clear that, despite the very different language used, the physical content of their proof is essentially the same as in the earlier and more straightforward proofs. It will also be apparent that the arguments can, with a little effort, be made fully rigorous.

Following their method in outline we shall show (without assuming the Goldstone theorem itself) that if a causal theory has a smallest mass different from zero and a conserved current $j^\mu(x)$ such that $\partial_\mu j^\mu(x) = 0$, then

$$\lim_{R\to\infty} \exp\,[i\eta Q_R(t)]$$

is a unitary operator. This statement is equivalent to the Goldstone theorem for, as shown in the preceding discussion, when the symmetry is broken (i.e., when $\langle 0|[Q, A]|0\rangle \neq 0$), the vacuum is not an eigenstate of Q and hence $\exp\,[i\eta Q]$ is not unitary. Since it is assumed that $\partial_\mu j^\mu = 0$, it follows in that case that the lowest mass of the theory must vanish.

As before, let us choose an operator A depending on strictly local operators "smeared" over a finite region of space-time. We may choose A to be bounded. We shall only be interested in the part of A which has vanishing vacuum expectation value, so we replace A by

$$A - \langle 0|A|0\rangle$$

Now from A we can form yet another operator B which has the property that

$$\langle n|B|0\rangle = (1/E_n^2)\langle n|A|0\rangle \tag{2.7}$$

This is a sensible equation as, by assumption, E_n can never vanish. Clearly one such operator [namely, $B = (1/H^2)A$] exists, although we shall need a somewhat more complicated structure for B. Once again all conclusions will be drawn from the usual Goldstone commutator

$$G_R \equiv \int_R d^3x \langle 0|[j^0(x), A]|0\rangle$$

We shall extract the spectral information that $\lim_{R\to\infty} G_R = 0$ if there are no massless modes in a much more complicated manner than before, even though no fewer assumptions are made. Straightforward manipulation and the use of Eq. (2.7) show that

$$\begin{aligned} G_R &= \int_R d^3x \langle 0|[j^0(x)H^2B - BH^2j^0(x)]|0\rangle \\ &= \int_R d^3x \langle 0|\big[[[j^0(x), H], H], B\big]|0\rangle \\ &= -\partial_0{}^2 \int_R d^3x \langle 0|[j^0(x), B]|0\rangle \end{aligned}$$

Now we are confronted with the classical Goldstone problem, namely, under what conditions does the fact that $\partial_\mu j^\mu(x) = 0$ insure that certain specific matrix elements of $Q = \lim_{R\to\infty} \int d^3x j^0(x, t)$ are independent of the time t. If B were a localized operator, we would immediately have $\lim_{R\to\infty} G_R = 0$. However, assuming A is localizable, $(1/H^2)A$ is not generally localizable. On the other hand, it is not difficult to construct from A an operator B which obeys Eq. (2.7) and is still sufficiently local for our purposes.

From the operator A we use the unitary generators of time translation to form $A(x^0) = e^{iHx^0}Ae^{-iHx^0}$. In addition, a function $g(p^0)$ is defined so that, if m is the smallest mass of the theory under consideration, $g(p^0) = 1/p_0{}^2$ if $|p^0| \geq m$ and, for $|p^0| \leq m$, $g(p^0)$ is chosen so as to be infinitely differentiable, even in p^0, and real. From this, the Fourier transform

$$g(x^0) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} e^{-ip^0x^0} g(p^0)$$

is formed. Using the usual theorems of Fourier analysis, it follows that $g(x^0)$ decreases faster than any power of $1/x^0$ as $x^0 \to \infty$. Having made these definitions, we finally make the identification

$$B = \int_{-\infty}^{\infty} dx^0 g(x^0) A(x^0)$$

This is consistent with the previous assumptions about the behavior of B since

$$\begin{aligned}\langle n|B|0\rangle &= \int_{-\infty}^{\infty} dx^0 g(x^0)\langle n|A(x^0)|0\rangle \\ &= \int_{-\infty}^{\infty} dx^0 \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, g(p^0) e^{-i(p^0 - p^0{}_n)x^0} \langle n|A|0\rangle \\ &= \frac{1}{(p_n{}^0)^2}\langle n|A|0\rangle\end{aligned}$$

Because of current conservation, in order to demonstrate that

$$\lim_{R\to\infty} \langle 0|[Q_R(t), B]|0\rangle$$

is independent of the time t, it is only necessary to show that

$$\lim_{R\to\infty} \langle 0|\left[\int d^3x\, \nabla\cdot\mathbf{j}(\mathbf{x}, t), B\right]|0\rangle = 0$$

It is assumed that A is confined to a space–time region of radius L so that $A(t)$ is confined to a region of radius $L + |t|$. It is further convenient to decompose the function $g(x^0)$ into two continuous functions $g(x^0) = g_1{}^R(x^0) + g_2{}^R(x^0)$ so that $g_1(x^0) = 0$ if $|x^0| > R - L - |t|$ and where $g_2(x^0)$ has the property that, for each n,

$$\lim_{R\to\infty} (R - L - |t|)^n \|g_2{}^R\| = 0 \tag{2.8}$$

where

$$\|g_2{}^R\| \equiv \int_{-\infty}^{\infty} dx^0 |g_2{}^R(x^0)|$$

Clearly, Eq. (2.8) is consistent with the extremely bounded behavior of $g(x^0)$ for large x^0. Now we find that

$$\lim_{R\to\infty} |\langle 0|\left[\int_R d^3x\, \nabla\cdot\mathbf{j}(\mathbf{x}, t), \int_{-\infty}^{\infty} dx^0 g(x^0) A(x^0)\right]|0\rangle|$$
$$\leq \lim_{R\to\infty} 2\|g_2{}^R\|\; \|\langle 0| \int_R d^3x\, \nabla\cdot\mathbf{j}\|\; \|A(x^0)\| \tag{2.9}$$

Since $\|A(x^0)\| \equiv \sup \|A|a'\rangle\|$ is finite by assumption while we know that

$$\|g_2{}^R\| < 1/R^n \tag{2.10}$$

for any n and large R, we will have the required result if it can be shown that $\|\langle 0| \int_R d^3x \nabla \cdot \mathbf{j}\|$ is bounded by some power of R. It is quite easy to give an heuristic derivation of this. First, note that

$$\left\|\langle 0| \int_R d^3x\, \Delta \cdot \mathbf{j}\right\| = \left|\langle 0| \int_R d^3x\, \nabla \cdot \mathbf{j}(\mathbf{x}, t) \int_R d^3x'\, \nabla \cdot \mathbf{j}(\mathbf{x}', t)|0\rangle\right|^{1/2}$$

Now, in this example, it is useful to introduce the type of smoothing function mentioned in reference (4) to make the identification

$$\int_R d^3x\, \nabla \cdot \mathbf{j}(\mathbf{x}, t) \to \int_{-\infty}^{\infty} d^3x f_R(\mathbf{x}) \nabla \cdot \mathbf{j}(\mathbf{x}, t)$$

We assume that $f_R(\mathbf{x})$ is differentiable and that $f_R(\mathbf{x}) = 0$ for $|\mathbf{x}| > R + \epsilon$ (see Fig. 1).

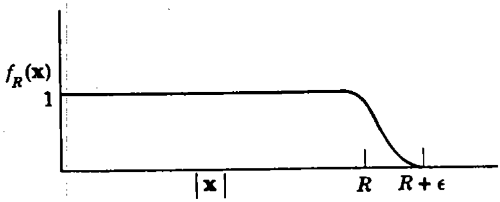


FIGURE 1

Using the fact that $\partial_\mu j^\mu(x) = 0$, the usual Lehmann representation shows that

$$\|\langle 0 \int_{-\infty}^{\infty} d^3x\, \nabla \cdot \mathbf{j} f_R(\mathbf{x})\|$$
$$= \left|\iint\int d^3x\, d^3x' f_R(\mathbf{x}) f_R(\mathbf{x}')\, \partial_0{}^2 \nabla^2 \Delta^{(+)}(x - x'; \kappa^2) B(\kappa) d\kappa^2\right|^{1/2}_{x^0 \neq x^{0\prime}}$$

In turn, we infer from this that

$$\left\|\langle 0| \int d^3x\, \nabla \cdot \mathbf{j} f_R(\mathbf{x})\right\|$$
$$= \left|\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} d^3x\, d^3x' e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \nabla f(\mathbf{x}) \cdot \nabla f(\mathbf{x}') \int d\kappa^2\, (\mathbf{p}^2 + \kappa^2)^{1/2} B(\kappa)\right|^{1/2}$$

Since $f(\mathbf{x})$ is nonzero only for large values of $x \sim R$ as $R \to \infty$, the above integral only has contributions for $\mathbf{p}^2 \to 0$. Thus, we have

$$\left\|\langle 0| \int d^3x\, \nabla \cdot \mathbf{j} f_R \right\| = \left| \frac{1}{2} \int d^3x (\nabla f(x))^2 \int d\kappa^2\, \kappa B(\kappa) \right|^{1/2}$$
$$\sim \left[\frac{2\pi R^2}{\epsilon} \int d\kappa^2\, \kappa B(\kappa) \right]^{1/2}$$

Assuming that $B(\kappa)$ is sufficiently bounded so the above integral exists and that $\epsilon \sim R$, we find $\|\langle 0| \int d^3x\, \nabla \cdot \mathbf{j} f_R\| \sim R^{1/2}$. Inserting this into Eq. (2.9) and using Eq. (2.10), it follows that

$$\lim_{R\to\infty} \langle 0| \left[\int_R d^3x\, \nabla \cdot \mathbf{j}(\mathbf{x}, t), B \right] |0\rangle = 0$$

and, consequently, $\lim G_R = 0$. Thus, we have established that if A is any localized operator and if the spectrum of the operator j^0 does not extend to zero, then

$$\langle 0| \left[\int d^3x\, j^0(\mathbf{x}, t), B \right] |0\rangle = 0 \tag{2.11}$$

in a causal theory. This is equivalent to the Goldstone theorem. Proceeding in exactly the same manner as before, it can be shown that the operator $\lim_{R\to\infty} \exp\{i\eta Q_R\}$ is unitary. Thus, through a somewhat more complicated (but basically identical) procedure, we have again established all the results proved with the classical formulation of the theorem.

To this point we have kept our discussion on a relatively abstract level and have not made direct contact with the local operators used for the quantitative calculations that have been performed in broken-symmetry theories. In these usually simple cases it is often assumed that $Q^i = \int d^3x\, j^{0i}(\mathbf{x}, t)$ are the generators of a Lie Group and that the local field operators $\phi^i(\mathbf{x}, t)$ form a representation of the group so that $i[Q^i, \phi^j(x)] = \sum f_{ijk} \phi^k(\mathbf{x}, t)$. The broken-symmetry assumption made is that

$$\langle 0|\phi^i(\mathbf{x}, t)|0\rangle = \langle 0|\phi^i(0)|0\rangle$$
$$= \eta^i \neq 0$$

for at least one i. The Goldstone theorem then asserts that there is a ϕ^l for which $\langle n|\phi^l|0\rangle \neq 0$ where $\langle n|$ is an excitation mode such that $\lim_{R\to 0} E_n(\mathbf{k}) = 0$. In a relativistic problem $E_n = (\mathbf{k}^2 + m^2)^{1/2}$ so one concludes that $m = 0$, that is, that massless particles with the quantum numbers of ϕ^l relative to the vacuum are present. In general, one would further anticipate that the propagator

$$G_{ll}(x, x') = i\langle 0|(\phi_l(x)\phi_l(x'))_+|0\rangle - i\langle 0|\phi_l(0)|0\rangle^2$$

has a pole in momentum space corresponding to this massless excitation. A third proof (historically, the second) of the Goldstone theorem, based on the occurrence of this pole at $p^2 = 0$ in the propagator, was given by Bludman and Klein.[11] Since its content is essentially the same as that of our previous proofs and it depends heavily on field theoretic detail not used there, we shall give it only cursory examination. It should be noted, however, that the methods used in deriving this theorem are very similar to those used in self-consistent calculation of Green's functions for broken-symmetry field theories. This will become apparent in Section VI.

Assume that the set of operators $\phi_i(\mathbf{x}, t)$ transform as an irreducible representation of the group generated by Q_i and satisfy the field equation

$$(-\partial^2 + \mu_0{}^2)\phi_i = j_i + J_i$$

where J_i is an external source. We assume that

$$\langle 0|\phi_i(\mathbf{x}, t)|0\rangle|_{J=0} = \eta_i \neq 0$$

so, for consistency, the above equation must satisfy the condition

$$\mu_0{}^2\langle 0|\phi_i|0\rangle|_{J=0} = \langle 0|j_i|0\rangle|_{J=0}$$

In addition, we define

$$\sum_\alpha [\lambda_\alpha Q_\alpha, \phi_j] = \sum_{\alpha,k} \lambda_\alpha f_{\alpha jk}\phi_k = \delta\phi_j$$

where λ_α is arbitrary. Applying this to the above field equation, we may write the vacuum expectation of the transformed equation as

$$\left[(-\partial^2 + \mu_0{}^2)\delta_{ij} - \frac{\bar{\delta}\langle 0|j_i|0\rangle}{\bar{\delta}\langle 0|\phi_j|0\rangle}\right]_{J=0} \langle 0|\delta\phi_j|0\rangle = 0$$

If $\langle 0|\delta\phi_j|0\rangle \neq 0$ for one index, j' (in accord with the broken symmetry assumption), we must have

$$\mu_0{}^2\delta_{ij'} - \left.\frac{\bar{\delta}\langle 0|j_i|0\rangle}{\bar{\delta}\langle 0|\phi_{j'}|0\rangle}\right|_{J=0} = 0 \tag{2.12}$$

Now, using conventional techniques of quantum field theory, we find that the propagator of the field ϕ_i is given by

$$D_{ij}(x, x') = \left.\frac{\delta\langle 0|\phi_i(x)|0\rangle}{\delta J_j(x')}\right|_{J=0}$$

which, in accord with the field equation, satisfies the momentum space equation

$$\left[\mu_0{}^2\delta_{ik} - \frac{\delta\langle 0|j_i|0\rangle}{\delta\langle 0|\phi_k|0\rangle}\right] D_{kl}(0) = \delta_{il} \tag{2.13}$$

at $p^2 = 0$. Clearly, if $D_{kl}(p^2 \to 0) \to \infty$, as is the case if a zero mass excitation is present, then

$$\mu_0{}^2\delta_{ik} - \frac{\delta\langle 0|j_i|0\rangle}{\delta\langle 0|\phi_k|0\rangle} = 0 \tag{2.14}$$

Thus Eqs. (2.12) and (2.14) are very similar and, in examining a particular problem we need only deduce from Eq. (2.12) when Eq. (2.14) is valid, to demonstrate the presence of a massless particle. The reader is referred to Bludman and Klein for examples.

Because the formulations discussed so far are not stated directly in terms of observable quantities but only vacuum expectations of operators, Streater[12] was led to formulate a corollary[13] of the Goldstone theorem in terms of the mass spectrum of the fields which form an irreducible representation of the group generated by $j^0{}_\alpha(x)$. For simplicity we confine ourselves to one conserved current $j^\mu(x)$ with time-independent charge operator $Q = \int d^3x\, j^0(x)$ and two fields, ϕ_1 and ϕ_2, such that $i[Q, \phi_1] = \phi_2$ and $i[Q, \phi_2] = -\phi_1$. With these assumptions it is possible to prove that if ϕ_1 and ϕ_2 excite particles of different mass m_1 and m_2, then $j^0(x)$ has a massless excitation in its spectrum. This can be directly generalized to show that if the two-point functions of ϕ_1 and ϕ_2 are not identical, then $j^0(x)$ excites a massless particle and, indeed, the same holds true if the whole excitation spectra of ϕ_1 and ϕ_2 are not identical. Starting with the proof for the two-point function, one notes

$$i[Q, \phi_1(x)\phi_2(x')] = \phi_2(x)\phi_2(x') - \phi_1(x)\phi_1(x')$$

from which it follows that

$$i\langle 0|[Q, \phi_1(x)\phi_2(x')]|0\rangle = \langle 0|\phi_2(x)\phi_2(x')|0\rangle - \langle 0|\phi_1(x)\phi_1(x')|0\rangle$$

Now, if for any value of x and x' the right-hand side does not vanish, it follows at once by the Goldstone theorem that $j^0(x)$ excites a massless particle. Further, it is clear that if the Lehmann spectral weights for the two Wightman functions on the right-hand side of the above equation are different for any value of the Lehmann weight parameter, then for some value of x and x' the right-hand side is nonvanishing and Streater's corollary follows. Note that the massless particle has the quantum numbers of $\phi_1\phi_2$. The generalization of this is evident. Take any polynomial $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}$ in the fields ϕ_1 and ϕ_2 and form

$$[Q, \phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}] = \sum_{j=1}^{n} [Q, \phi_{i_j}]\phi_{i_1}\phi_{i_2}\cdots\phi_{i_{j-1}}\phi_{i_{j+1}}\cdots\phi_{i_n}$$

If

$$\langle 0|[Q, \phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}]|0\rangle \neq 0$$

it follows that $j^0(x)$ excites a massless particle and all the nth order Green's functions are not identical. Thus this corollary, which is easily generalized to include more complicated group structures, is clearly a straightforward application of the theorem as formulated in our previous discussions.

Finally, we wish to analyze in very direct terms the conditions under which the theorem need not apply. As stated previously, a manifestly Lorentz-invariant theory is locally causal, and there can be no failure of the theorem per se if the currents used to form the generators are actually conserved. We emphasize that this does not mean that the massless modes are physically observable. Thus, to see a failure we must examine theories which are not manifestly covariant. To indicate that the time component of a vector is now allowed to appear independently of whether or not the corresponding spatial components do, we introduce the vector $n^\mu = (\mathbf{0}, 1)$. Then a straightforward analysis[14] with $\partial_\mu j^\mu(x) = 0$ shows that the Fourier transform of $\langle 0|[j^\mu(x), \phi(0)]|0\rangle$ has the form

$$[A + B\epsilon(k^0)]k^\mu\delta(k^2) + Cn^\mu\delta(\mathbf{k})\delta(n\cdot k) + [k^\mu(n\cdot k) - n^\mu k^2]D + En^\mu\delta(n\cdot k)$$

where A and B are functions of $n\cdot k$, E of k^2, and D of both variables. If the term proportional to $k^\mu\delta(k^2)$ were the only one present, as in the case of manifestly covariant theories, the Goldstone theorem would hold and there would be no difficulties. Historically, the term proportional to C was first held responsible for the nonrelativistic "failures" of the Goldstone theorem.[15] However, this is not the case, inasmuch as such a term represents an isolated state that might be interpreted as a transition between the various possible vacuum states which are implied by the broken symmetry. The fact that such a state is not a limit of a branch of the excitation spectrum as **k** tends to zero rules out this term as indicated by Eq. (2.4). That it is not relevant can be seen in a rather direct fashion by noting that, in general, the equation

$$\langle 0|[j^0(x), \phi(x')]|0\rangle|_{x^0=x^{0'}} \sim \delta(\mathbf{x}-\mathbf{x}')$$

must be valid, but

$$\int dk\, e^{ik(x-x')} C n^0 \delta(\mathbf{k})\delta(n\cdot k)|_{x^0=x^{0'}} = C$$

Thus the term proportional to C does not have the correct structure and no escape from the Goldstone theorem can occur in this manner. The term proportional to E has an unconventional spectral behavior but cannot be ruled out on general principles. Its presence, however, provides no contradiction to the assumptions of the theorem. The term proportional to D is the important one inasmuch as it appears by explicit calculation in radiation gauge electrodynamics. It must be realized, however, that in no sense should it be related to the spurions of the type implied by C. Structures of the form $D \sim \delta(k^0)$ are unable to contribute, so they cannot be responsible for the nonvanishing of the Goldstone commutator. Consequently, if D is to contribute and have a normal, well-defined spectral structure, we have

$$D(\mathbf{k}\to 0) \sim \int_{m^2>0} dm^2\, \sigma(m^2)\delta(k_0{}^2 - m^2)$$

From this it follows at once that

$$\frac{d}{dt}\langle 0|\left[\int d^3x\, j^0(\mathbf{x}, t), \phi(x')\right]|0\rangle \neq 0$$

Thus the theory is not sufficiently causal to allow the term

$$\lim_{V\to\infty} \langle 0|\left[\int_{\sigma(V)} d\sigma_k\, j^{k0}(\mathbf{x}, \mathrm{t}), \phi(x')\right]|0\rangle$$

to vanish and the assumptions used to prove the theorem are not valid. We will discuss this behavior in depth through several examples in the following sections.

III. Naturally Occurring Broken Symmetries

In recent years a considerable insight into some of the more tractable problems associated with broken symmetries has been acquired through the study of theories which possess a rather remarkable set of gauge properties. Characteristic of such theories is their invariance under a class of *c*-number gauge transformations, the existence of which allows one to demonstrate the occurrence of a zero mass particle without any reference to a broken symmetry or the precise details of the dynamics. These massless quanta will generally be referred to in this work as naturally occurring[16,17] to distinguish them from their counterparts in theories which display a spontaneous breakdown of symmetry. Although the existence of zero mass particles in both types of theories may be inferred from a broken-symmetry condition, it can be shown that in the former case this approach invariably turns out to be merely another way of imposing a particular invariance property on the relevant Lagrangian. Since the usual procedure employed in broken-symmetry calculations appears to somewhat obscure the relative simplicity of systems whose massless boson excitation follows purely from gauge invariance arguments, it is our intention in the subsequent discussion of various field theoretical models to emphasize this crucial role of gauge invariance by displaying first the zero mass particle implied by the relevant Lagrangian and only then proceeding to demonstrate the possibility of constructing an associated broken symmetry. This procedure contrasts, of course, with that which has customarily been employed in the more conjectural domain of spontaneous symmetry breaking where one generally postulates at the outset a broken symmetry condition and subsequently deduces the associated zero mass particle implied by the Goldstone theorem. Since there is neither any *a priori* reason for assuming the internal consistency of these constraints expressing the spontaneous breaking of the symmetry (a point to be discussed more fully in Sect. VII) nor a dynamical mechanism for the massless boson, it is appropriate that this study of possible applications of the Goldstone theorem should begin with a consideration of massless particles of the naturally occurring variety. Despite the somewhat trivial nature of this type of broken symmetry, there is a considerable amount of contact with the features characteristic of the more general problem, while at the same time one has the advantage of being able to avoid some of the difficult consistency questions peculiar to spontaneous symmetry breaking.

In order to fix ideas more firmly, we consider as the simplest example of a naturally occurring massless particle a free spin-zero Hermitian field. We make use of the well-known result that a second-order differential equation can always be reduced to two first-order equations and thus describe this field by the vector $\phi^\mu(x)$ with the scalar $\phi(x)$. This leads to the Lagrangian density

$$\mathscr{L} = \phi^\mu \partial_\mu \phi + \tfrac{1}{2}\phi^\mu \phi_\mu - \tfrac{1}{2}\mu_0{}^2\phi^2 \tag{3.1}$$

which for the case $\mu_0{}^2 = 0$ is readily seen to be invariant under the constant gauge transformation

$$\phi \to \phi + \eta \tag{3.2}$$

This invariance is, of course, equivalent to the local conservation law

$$\partial_\mu \phi^\mu = 0$$

a result which formally leads to the conclusion that the operator

$$Q = \int \phi^0(x)\, d^3x$$

is conserved and the generator of the transformation (3.2). It is indeed true that (3.2) is unitarily implementable in any finite volume Ω and that the operator

$$Q_\Omega = \int_\Omega \phi^0(x)\, d^3x$$

by virtue of the commutation relation

$$[\phi^0(x), \phi(x')]\delta(x^0 - x^{0\prime}) = -i\delta(x - x') \tag{3.3}$$

is the generator of Eq. (3.2) in Ω, i.e.,

$$U_\Omega(\eta)\phi(x)U_\Omega{}^+(\eta) = \phi(x) + \eta$$

where

$$U_\Omega(\eta) = \exp\{i\eta Q_\Omega\}$$

The unitarity of the operator $U_\Omega(\eta)$ clearly implies the equivalence of the Hilbert spaces constructed on the two states $\langle 0|$ and $\langle 0|U_\Omega(\eta)$. To state this somewhat differently, since $\phi^0(x)$ and $\phi(x)$ are a complete set of operators, any state constructed from these operators and the state $\langle 0|U_\Omega(\eta)$ can be expanded in terms of the complete set of states generated by the same set of operators acting on the vacuum $\langle 0|$. Thus the set of states constructed on $\langle 0|U_\Omega(\eta)$ may be said to provide a representation of the canonical commutation relations which is fully equivalent to that constructed on $\langle 0|$. However, as has already been observed in the preceding chapter, in the limit $\Omega \to \infty$ such a result no longer obtains. This may be seen by a review of the simple calculation[18] performed in Section II of the norm of the state

$$|Q\rangle \equiv \lim_{\Omega\to\infty} Q_\Omega|0\rangle$$

Since $\lim_{\Omega\to\infty} Q_\Omega$ is translationally invariant, so also is $|Q\rangle$. Thus,

$$\begin{aligned}\langle Q|Q\rangle &= \langle Q| \int_{\Omega\to\infty} \phi^0(x)\, d^3x|0\rangle \\ &= \int d^3x \langle Q|\phi^0(0)|0\rangle\end{aligned}$$

so that $\langle Q|Q\rangle$ is either zero or infinite. However, the first alternative implies

$$\int d^3x \phi^0(x)|0\rangle = 0$$

in contradiction with the canonical commutation relation (3.3), whereas the second implies that $\lim_{\Omega\to\infty} Q_\Omega$ cannot be defined on the vacuum. The net result of this discussion is that $U_{\Omega\to\infty}(\eta)$ does not exist and consequently the set of states constructed on $\langle 0|U_\infty(\eta)$ provides a representation of the canonical commutation relations which is inequivalent to that constructed on $\langle 0|$.

The inequivalence of these representations can be made somewhat more transparent by using the usual decomposition into creation and annihilation operators. Thus we write

$$\begin{aligned}\phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{(2\omega)^{1/2}} [a(\mathbf{k})\, e^{ikx} + a^*(\mathbf{k})\, e^{-ikx}] \\ \phi^0(x) &= -i \int \frac{d^3k}{(2\pi)^{3/2}} \left(\frac{\omega}{2}\right)^{1/2} [a(\mathbf{k})\, e^{ikx} - a^*(\mathbf{k})\, e^{-ikx}]\end{aligned} \tag{3.4}$$

where $a(\mathbf{k})$ and $a^+(\mathbf{k})$ satisfy the usual commutation relations

$$\begin{aligned}[a(\mathbf{k}), a^*(\mathbf{k}')] &= \delta(\mathbf{k} - \mathbf{k}') \\ [a(\mathbf{k}), a(\mathbf{k}')] &= [a^*(\mathbf{k}), a^*(\mathbf{k}')] = 0\end{aligned} \tag{3.5}$$

and we choose that representation of the commutation relations for which

$$a(\mathbf{k})|0\rangle = 0 \tag{3.6}$$

Since the field $\phi(x)$ has only a zero mass particle in its spectrum, the state formally defined by $\langle 0|U_\infty(\eta)$ can conveniently be viewed as a coherent superposition of states containing various numbers of zero-momentum, zero-energy quanta. In terms of the operators $a(\mathbf{k})$ and $a^*(\mathbf{k})$ these states can be written as

$$\langle a'| = \langle 0| \exp\{a'^*a(\mathbf{0}) - a^*(\mathbf{0})a'\}$$

thereby implying the broken-symmetry conditions

$$\langle a'|a(\mathbf{k})|a'\rangle = a'\delta(\mathbf{k})$$
$$\langle a'|a^*(\mathbf{k})|a'\rangle = a'^*\delta(\mathbf{k})$$

One can see more precisely the necessary inequivalence of the Hilbert spaces based on $\langle 0|$ and $\langle a'|$ in the limit $\Omega \to \infty$ upon performing the quantization of $\phi(x)$ in a box of volume Ω rather than the infinite volume implied by Eqs. (3.4) and (3.5). Using the identity

$$e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]}$$

for $[A, B]$ a c-number, it then follows that

$$\langle 0|a'\rangle = \exp\left[-\frac{1}{2}\frac{a'a'^*}{(2\pi)^3}\int_\Omega d^3x\right]$$

which clearly displays the orthogonality of the states $\langle 0|$ and $\langle a'|$ in the limit $\Omega \to \infty$. Since it requires only a trivial generalization to show that any state in the Hilbert space constructed on $\langle 0|$ is orthogonal to $\langle a'|$, one has the result that the Hilbert spaces corresponding to different values of a' are all mutually orthogonal.

Having now displayed some of the mathematical complications associated with the implementation of the gauge group of the massless scalar field, it is well to emphasize here that the importance of the inequivalent representations of the canonical commutation relations is by no means confined to this particular example but is an essential ingredient of any broken symmetry, independent of whether it is of the naturally occurring or spontaneous variety. Since the basic characteristic of a broken symmetry is the noninvariance of the vacuum under the operations of the group, it follows in complete analogy to the massless scalar field that the generators of the group acting on the vacuum must lead to a state of infinite norm, i.e., they take one out of the original Hilbert space. Provided that these important features are kept in mind, it is instructive to formally construct, from the original vacuum, the new vacuum states

$$\langle\eta| = \langle 0|U_\infty(\eta)$$

for which one derives the broken symmetry condition

$$\langle\eta|\phi(x)|\eta\rangle = \eta \tag{3.7}$$

The relation (3.7) may, of course, be viewed without any reference to an operator construction as a statement of the existence of other vacuum states $\langle\eta|$ (i.e., null eigenstates of P^μ) for which Eq. (3.6) fails to hold. Thus the general characteristic of naturally occurring broken symmetries arises from the possibility of replacing Eq. (3.6) by

$$a(\mathbf{k})|\eta\rangle = \eta|\eta\rangle\delta(\mathbf{k})(2\pi)^{3/2}(\tfrac{1}{2}\omega)^{1/2} \tag{3.8}$$

which by Eq. (3.4) insures that

$$\langle\eta|\phi(x)|\eta\rangle = \eta$$

Although the existence of such a representation is obviously in no way dependent upon the specific dynamics, the states $|\eta\rangle$ can be identified with the physical vacuum only for a special class of theories. Thus for the case of a free field whose equation of motion is of the form

$$(-\partial^2 + \mu_0{}^2)\phi(x) = 0$$

the consistency condition

$$\mu_0{}^2\langle\eta|\phi(x)|\eta\rangle = 0$$

is clearly compatible with Eq. (3.7) only for the case of vanishing bare mass.

It may be somewhat instructive to consider at this point some general properties of the vacuum expectation values associated with this massless scalar field. Upon application of the field equations one is readily led to the conclusion that this theory is undefined only to the extent that it does not contain any prescription for the calculation of the expectation value of the field operator. Although the equations of motion and the canonical commutation relations enable one to express the time-ordered product of an arbitrary number of such operators in terms of the vacuum expectation value of $\phi(x)$, this latter number must remain completely undetermined. However, one can consistently choose it to be any real number merely by recognizing the fact that Eq. (3.8) defines an infinite number of inequivalent representations of the commutation relations, each of which corresponds to a different numerical value of $\langle 0|\phi(x)|0\rangle$. Although the class (3.8) by no means includes all possible representations,[19] it is sufficiently general to allow a complete discussion of the naturally occurring type of broken symmetry.

Thus far the only explicit example of a broken-symmetry theory which has been discussed in this section has consisted of a free field. However, it is not to be concluded from the considerable emphasis which it has received here that the free field is the only mathematically well-defined, broken-symmetry theory. In fact, of the two field equations

$$\partial_\mu \phi^\mu = 0 \tag{3.9}$$

$$\phi^\mu = -\partial^\mu \phi \tag{3.10}$$

implied by Eq. (3.1) only the first is relevant to the construction of the broken symmetry.[17] That this is indeed a sufficient condition becomes apparent upon recalling that the application of the Goldstone theorem requires only the existence of a conserved current such that the equal-time commutator of the fourth component with some other operator of the theory has a nonvanishing expectation value. Thus the conservation law [Eq. (3.9)] along with the commutation relation [Eq. (3.3)] immediately leads to the existence of a massless particle for any Lagrangian of the form

$$\mathscr{L} = \phi^\mu \, \partial_\mu \phi + {}^1\!/_2 \phi^\mu \phi_\mu + \mathscr{L}'(\phi^\mu) \tag{3.11}$$

where $\mathscr{L}'(\phi^\mu)$ does not contain $\phi(x)$. To carry out the explicit construction one computes

$$\langle 0|[\phi^\mu(x), \phi(0)]|0\rangle$$

which is readily found from Eqs. (3.3) and (3.9) to have the form

$$-\partial^\mu \Delta(x, 0)$$

where

$$\Delta(x, \mu^2) = 2\pi \int \frac{dp}{(2\pi)^4} \, e^{ipx} \epsilon(p^0) \delta(p^2 + \mu^2)$$

A Goldstone theorem of this type is thus based on the elements ϕ^0, ϕ, and 1 of the canonical group. The essential point here which distinguishes this case from theories which display a spontaneous symmetry breakdown is that the commutator is proportional to unity (which necessarily has a nonvanishing expectation value) and therefore the consistency of requiring the nonvanishing of the equal time commutator is assured.

As an example of one of the simpler Lagrangians of the form of (3.11), we consider the system[17] described by

$$\mathscr{L} = \phi^\mu \partial_\mu \phi + \tfrac{1}{2}\phi^\mu\phi_\mu + \phi^{\mu\prime} \partial_\mu \phi' + \tfrac{1}{2}\phi^{\mu\prime}\phi_\mu' + \tfrac{1}{2}\mu_0{}^2\phi'^2 + \lambda\phi^\mu\phi_\mu'$$

In addition to the conservation law $\partial_\mu\phi^\mu = 0$ implied by the invariance under

$$\phi \rightarrow \phi + \eta$$

one has the equations of motion

$$\begin{aligned} \phi^\mu &= -\partial^\mu\phi - \lambda\phi^{\mu\prime} \\ \partial_\mu\phi^{\mu\prime} &= -\mu_0{}^2\phi' \\ \phi^{\mu\prime} &= -\partial^\mu\phi' - \lambda\phi^\mu \end{aligned}$$

These can be readily shown to yield for the Green's functions

$$\begin{aligned} G(x) &= i\langle 0|(\phi(x)\phi(0))_+|0\rangle \\ G'(x) &= i\langle 0|(\phi'(x)\phi'(0))_+|0\rangle \end{aligned}$$

the momentum space representation

$$\begin{aligned} G(p) &= \frac{1-\lambda^2}{p^2} + \frac{\lambda^2}{p^2+\mu_0{}^2} \\ G'(p) &= \frac{1}{p^2+\mu_0{}^2} \end{aligned} \tag{3.12}$$

thus explicitly confirming the existence of the massless particle implied by the invariance of the theory under c-number translation of $\phi(x)$.

It is clear from the form of Eq. (3.12) that the restriction $\lambda^2 \leq 1$ is essential to preserve the positive definite metric of the Hilbert space and that the limiting value $\lambda^2 = 1$ (corresponding to $Z = 0$) corresponds to the disappearance of the massless particle from the spectrum of $\phi(x)$. However, this latter limit is a rather delicate one, as is shown by a calculation of the two-point function of the field $\phi^{\mu}(x)$. One finds that the residue at the zero-mass pole of this latter function diverges as $(1 - \lambda^2)^{-1}$, a result which is in fact essential to the consistency of the canonical commutation relations. The fact that the massless particle appears in the spectrum of $\phi^{\mu}(x)$ rather than $\phi(x)$ in the limit is of some importance in itself if one considers the possibility of including less trivial interactions consistent with (3.11). If, for example, such couplings allow the $Z = 0$ limit, it is nonetheless true that ϕ^{μ}, in accord with the Goldstone theorem, will create massless quanta. Since it is ϕ^{μ} rather than ϕ which appears in the coupling terms, these quanta will therefore be physical particles rather than mere gauge excitations. This result could be of some significance inasmuch as it suggests that one can anticipate considerable difficulty in attempting to decouple the unwanted massless particles which invariably occur in broken-symmetry models of elementary particles.

Because of the appreciable extent to which a number of soluble field theoretical examples of symmetry breaking have recently been investigated, we shall for the moment defer a discussion of some of the less trivial applications of naturally occurring zero mass particles in order to direct attention to these various models. It is our intent to show that, regardless of the context in which they have been considered, these models are all basically of the naturally occurring type. Although this may not be particularly obvious from an inspection of the relevant Lagrangian, it is useful to carry out this demonstration in order to sharpen the distinction between the two fundamental types of broken-symmetry theories.

One manner in which the relation of a broken-symmetry theory to the c-number translation group has been somewhat obscured is due to a frequently used, though basically irrelevant, increase in the number of degrees of freedom of the canonical fields. Thus, while it has been emphasized here that the existence of such a gauge group implies a Goldstone theorem based on the elements of the canonical group, this may be considerably less apparent if one constructs the argument using the conserved current operator implied by the introduction of additional degree of freedom. If, for example, one generalizes (3.1) with $\mu_0^2 = 0$ to the case of a free field described by $\phi_i(x)$ ($i = 1, 2$), the Lagrangian is invariant under

$$\phi(x) \rightarrow (1 + iq\delta\lambda)\phi(x) \tag{3.13}$$

the matrix q in the two-dimensional charge space being given by

$$q = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

so that the operator

$$j^{\mu} = i\phi^{\mu} q \phi$$

is a conserved current. However, despite the fact that a consistent Goldstone theorem can be constructed using the equal-time commutator

$$\begin{aligned} \langle\eta|\left[\int j^0(x')\, d^3x', \phi(x)\right]|\eta\rangle &= q\langle\eta|\phi(x)|\eta\rangle \\ &= q\eta \end{aligned} \tag{3.14}$$

it is somewhat misleading to assert that the symmetry generated by

$$Q = \int j^0(x)\, d^3x$$

has been broken. Though formally true, such a statement is merely a trivial consequence of the existence (in a finite volume) of a generator of the transformation

$$\phi \rightarrow \phi + \eta$$

This can be clarified (at the expense of some mathematical rigor) by using the ill-defined operators $U_{\infty}(\eta)$ relating the different vacuums to write Eq. (3.14) in the form

$$\begin{aligned} \langle 0|U_{\infty}(\eta)[Q, \phi(x)]U_{\infty}{}^{+}(\eta)|0\rangle &= \langle 0|[Q, \phi(x)]|0\rangle \\ &\quad + i\langle 0|\left[\int d^3x'\phi^0(x')q\eta, \phi(x)\right]|0\rangle \\ &= q\eta + q\langle 0|\phi|0\rangle \end{aligned}$$

a result which emphasizes the fact that the broken symmetry is supported by the generators ϕ^0, ϕ, and 1 of the canonical group rather than the corresponding elements of the gauge group of (3.13).

Another point which frequently leads to some confusion concerning broken-symmetry solutions in soluble models arises from the possibility that the invariances of a given Lagrangian under field translation may be considerably less than obvious. However, as has already been emphasized, this in no way alters the basic fact that the broken symmetry supported by such a system is of the naturally occurring variety. Perhaps the simplest example of a theory of this type (in which the invariance is only thinly disguised) is a slight variation of a model considered by Hellman and Roman.[20] It is described by the Lagrangian

$$\mathscr{L} = \phi^\mu\, \partial_\mu\phi + \tfrac{1}{2}\phi^\mu\phi_\mu - \tfrac{1}{2}\mu_0{}^2\phi^2 - \tfrac{1}{2}g\phi q'\phi \tag{3.15}$$

where $\phi(x)$ is again a two-component Hermitian field and

$$q' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Although this theory clearly does not possess an invariance under $\phi \to \phi + \eta$, the field equation

$$(-\partial^2 + \mu_0{}^2 + gq')\phi = 0 \tag{3.16}$$

shows that

$$\langle 0|\phi(x)|0\rangle \neq 0$$

is consistent provided that

$$\mu_0{}^4 = g^2 \tag{3.17}$$

This, of course, allows the two choices $\pm\mu_0{}^2$ for g, the significance of these roots being best displayed by a diagonalization of q'. Defining

$$\begin{aligned} \phi^{(+)} &= (2)^{-1/2}(\phi_1 + \phi_2) \\ \phi^{(-)} &= (2)^{-1/2}(\phi_1 - \phi_2) \end{aligned}$$

the equation of motion [Eq. (3.16)] becomes

$$(-\partial^2 + \mu_0{}^2 \pm g)\phi^{(\pm)} = 0$$

a result which shows that the two values $\mu_0{}^2$ and $-\mu_0{}^2$ correspond respectively to the vanishing of the bare mass of the fields $\phi^{(-)}$ and $\phi^{(+)}$. Thus for the case $g = \mu_0{}^2$ the Lagrangian is invariant under

$$\phi^{(-)} \to \phi^{(-)} + \eta$$

and for $g = -\mu_0{}^2$ one has the corresponding invariance under

$$\phi^{(+)} \to \phi^{(+)} + \eta$$

This conclusion can, of course, be reached in a somewhat different manner by directly considering the change induced in the Lagrangian by

$$\phi_{1,2} \to \phi_{1,2} + \eta_{1,2}$$

Thus one finds

$$\delta\mathscr{L} = -\mu_0^2(\eta_1\phi_1 + \eta_2\phi_2 + {}^1\!/_2\eta_1^2 + {}^1\!/_2\eta_2^2) - g(\phi_1\eta_2 + \phi_2\eta_1 + \eta_1\eta_2)$$

which clearly vanishes for

$$\begin{aligned} \eta_1\mu_0^2 &= -g\eta_2 \\ \eta_2\mu_0^2 &= -g\eta_1 \end{aligned}$$

whence

$$\begin{aligned} \mu_0^4 &= g^2 \\ \eta_1^2 &= \eta_2^2 \end{aligned}$$

in agreement with Eq. (3.17).

It is, of course, possible to introduce some additional complexities into the two-field system described by (3.15) without in any way altering our general results. In particular one can consider a more general mass matrix and thus replace Eq. (3.15) by

$$\mathscr{L} = \phi^\mu\,\partial_\mu\phi + {}^1\!/_2\phi^\mu\phi_\mu - {}^1\!/_2 g\phi q'\phi - {}^1\!/_2\mu_1^2\phi_1^2 - {}^1\!/_2\mu_2^2\phi_2^2 \tag{3.18}$$

In this case the field equations

$$\begin{aligned} (-\partial^2 + \mu_1^2)\phi_1 &= -g\phi_2 \\ (-\partial^2 + \mu_2^2)\phi_2 &= -g\phi_1 \end{aligned}$$

imply that the condition

$$\langle 0|\phi_i(x)|0\rangle \neq 0$$

is consistent only for

$$\mu_1^2\mu_2^2 = g^2 \tag{3.19}$$

Although one could once again diagonalize the mass term in the Lagrangian, it is somewhat simpler in the present case to calculate the lowest order Green's function. For the field $\phi_1(x)$ one finds the momentum space representation

$$G^{-1}(p) = p^2 + \mu_1^2 - \frac{g^2}{p^2 + \mu_2^2}$$

which has zeros at

$$-p^2 = \frac{\mu_1^2 + \mu_2^2}{2} \pm \left[\left(\frac{\mu_1^2 - \mu_2^2}{2}\right)^2 + g^2\right]^{1/2}$$

Upon inclusion of the constraint (3.19) one finds that the propagator has a pole at zero as well as $\mu_1^2 + \mu_2^2$. Since the $\phi_i(x)$ are still free fields, however, it clearly must be possible to rewrite (3.18) as the sum of two free-particle Lagrangians of masses zero and $\mu_1^2 + \mu_2^2$ and to thus determine a conservation law corresponding to the zero mass particle. Choosing the root $g = \pm\mu_1\mu_2$ of Eq. (3.19) and using the equations

$$\begin{aligned}\partial_\mu\phi_1{}^\mu &= -\mu_1{}^2\phi_1 - g\phi_2\\ \partial_\mu\phi_2{}^\mu &= -\mu_2{}^2\phi_2 - g\phi_1\end{aligned}$$

one readily deduces that

$$\partial_\mu(\mu_2\phi_1{}^\mu \mp \mu_1\phi_2{}^\mu) = 0$$

thereby establishing the existence of the c-number translation group

$$\mu_2\phi_1 \mp \mu_1\phi_2 \rightarrow \mu_2\phi_1 \mp \mu_1\phi_2 + \eta$$

associated with the massless free field part of (3.18).

Although there is, in principle, no limit to the number of additional complexities one can introduce into the study of the broken symmetries associated with a finite number of coupled free fields, it is easy to convince oneself that there are no real conceptual advances to be derived from any of these generalizations. On the other hand, the case in which the number of such fields is allowed to become infinite, while not requiring any substantially new techniques, has been the subject of such considerable interest in this application as to merit a somewhat detailed discussion. We refer, of course, to the well-known Zachariasen model[21] which, despite its relatively simple structure, has in recent years proved to be an invaluable tool of the theoretical physicist in the testing of some of the new techniques employed in particle physics. Although this model was first discovered in the context of dispersion theory, it was subsequently shown by Thirring[22] that there is an alternative formulation within the framework of Lagrangian field theory, and it is, of course, this latter approach which provides the basis for a broken-symmetry application. It is to be emphasized at the outset that the Lagrangian of this theory includes only bilinear coupling terms and as such consists only of free fields. The fact that a nontrivial S-matrix can be defined for the Zachariasen model is a direct consequence of the introduction of a continuum of fields into the Lagrangian and therefore should not be taken to imply any deep physical content in the theory.

Although many authors have considered the model to provide a "reasonable" approximation to Yukawa and quartic boson interactions, this is a highly conjectural view and completely irrelevant to its application as a broken-symmetry theory. We shall therefore be content to show that the Zachariasen model can support a naturally occurring type of broken symmetry[23] at the same time completely avoiding any reference to its possible value as an approximation to a less trivial theory.

The Zachariasen model can be conveniently described as a theory in which the interaction of a B particle with $A\bar{A}$ pairs is restricted by the important condition that the A and $\bar{A}$ particles only be created in pairs. Because of this latter feature (i.e., the absence of an asymptotic condition for the A particles), a Lagrangian formulation of the theory requires that $A\bar{A}$ pairs be described by a single field operator with a continuous mass parameter. This result is made more transparent upon observing that if one restricts a more conventional type of field theory which contains an A particle in the manner described here, then the free Green's function $G_0(x)$ never appears alone but always in the combination $G_0{}^2(x)$. One thus need only refer to the identity

$$G_0{}^2(x, M_A) = \int_{4M_A{}^2}^{\infty} ds f^2(s) G_0(x, s) \tag{3.20}$$

where

$$f^2(s) = \frac{1}{16\pi^2}\left(\frac{s - 4M_A{}^2}{s}\right)^{1/2}$$

and

$$G_0(p, s) = \frac{1}{p^2 + s}$$

to show the equivalence to a theory in which the interaction is mediated by a continuous mass field. It must be remarked that since $f^2(s)$ asymptotically approaches a constant, the integral in Eq. (3.20) does not really exist. However, this fact merely reflects the well-known, self-energy divergence and is completely irrelevant within the context of the present application. All the essential features of the model are retained upon replacing $f(s)$ by a function which vanishes sufficiently rapidly at infinity; we therefore freely assume such a regularization in the subsequent discussion.

One can now proceed to write the Lagrangian describing the interaction between the B particle and the $A\bar{A}$ pairs [associated, respectively, with the fields $\phi(x)$ and $\phi(x, s)$] in the form

$$\mathscr{L} = \phi^\mu \partial_\mu \phi + \tfrac{1}{2}\phi^\mu \phi_\mu - \tfrac{1}{2}\mu_0{}^2\phi^2 + \int_{4M_A{}^2}^{\infty} ds\{\phi^\mu(s)\,\partial_\mu\phi(s) + \tfrac{1}{2}\phi^\mu(s)\phi_\mu(s)$$
$$- \tfrac{1}{2}s\phi^2(s)\} + g_0\phi \int_{4M_A{}^2}^{\infty} ds f(s)\phi(s) \tag{3.21}$$

The equations of motion implied by (3.21),

$$(-\partial^2 + \mu_0{}^2)\phi(x) = g_0 \int_{4M_A{}^2}^{\infty} ds f(s)\phi(x, s)$$
$$(-\partial^2 + s)\phi(x, s) = g_0 f(s)\phi(x)$$

together with the only nonvanishing equal-time commutation relations

$$[\partial_0\phi(x), \phi(x')] = -i\delta(\mathbf{x} - \mathbf{x}')$$
$$[\partial_0\phi(x, s), \phi(x', s')] = -i\delta(\mathbf{x} - \mathbf{x}')\delta(s - s')$$

yield a set of coupled integral equations for the Green's functions

$$G(x) = i\langle 0|(\phi(x)\phi(0))_+|0\rangle$$
$$G(x; s, s') = i\langle 0|(\phi(x, s)\phi(x', s'))_+|0\rangle$$

These integral equations can be solved by elementary techniques to yield the momentum space representation

$$G(p) = \left[p^2 + \mu_0{}^2 - g_0{}^2 \int_{4M_A{}^2}^{\infty} ds \frac{f^2(s)}{p^2 + s}\right]^{-1} \tag{3.22}$$
$$G(p; s, s') = \frac{\delta(s - s')}{p^2 + s} + g_0{}^2 \frac{f(s)}{p^2 + s}\, G(p) \frac{f(s')}{p^2 + s'}$$

Since it is usual to require the existence of a stable particle of mass μ in the theory, we set

$$\mu_0{}^2 = \mu^2 + g_0{}^2 \int_{4M_A{}^2}^{\infty} ds \frac{f^2(s)}{s - \mu^2} \tag{3.23}$$

which, upon insertion in Eq. (3.22), yields the once-subtracted form of the propagator

$$G(p) = (p^2 + \mu^2)^{-1}\left[1 + g_0{}^2 \int_{4M_A{}^2}^{\infty} ds \frac{f^2(s)}{(p^2 + s)(s - \mu^2)}\right]^{-1}$$

In addition to this formal mass renormalization procedure one can carry out the corresponding coupling constant renormalization by introducing the definitions

$$Z_3^{-1} = 1 + g_0^2 \int_{4M_A^2}^{\infty} ds \frac{f^2(s)}{(s - \mu^2)^2}$$

and

$$g^2 = Z_3 g_0^2$$

by means of which one writes the convenient alternative form for $G(p; s, s')$,

$$G(p; s, s') = \frac{\delta(s - s')}{p^2 + s} + \frac{g^2}{p^2 + \mu^2} \frac{f(s)}{p^2 + s} \times \frac{1}{Z_3 + g^2 \int_{4M_A^2}^{\infty} ds \frac{f^2(s)}{(p^2 + s)(s - \mu^2)}} \frac{f(s')}{p^2 + s'}$$

Since the field $\phi(x, s)$ creates $A\bar{A}$ pairs, the $A\bar{A}$ scattering amplitude is readily extracted from $G(p; s, s')$ by straightforward techniques, thereby yielding the result

$$\begin{aligned} e^{i\delta} \sin \delta &= \pi \frac{g^2}{p^2 + \mu^2} f^2(-p^2) \left[Z_3 + g^2 \int_{4M_A^2}^{\infty} \frac{f^2(s)\, ds}{(p^2 + s)(s - \mu^2)} \right]^{-1} \\ &= \pi \frac{g^2}{p^2 + \mu^2} f^2(-p^2) \left[1 - (p^2 + \mu^2) g^2 \int_{4M_A^2}^{\infty} ds \frac{f^2(s)}{(s - \mu^2)^2 (p^2 + s)} \right]^{-1} \end{aligned}$$

which we note contains no reference whatever to unrenormalized quantities.

With this brief summary of the solution as obtained by direct application of the equations of motion, one can now proceed to examine the possibility of a broken symmetry approach to the model by imposing on the solution the constraint

$$\langle 0|\phi(x)|0\rangle = \eta \neq 0$$

It follows immediately that the equations of motion require the consistency conditions

$$\mu_0^2 \langle 0|\phi(x)|0\rangle = g_0 \int_{4M_A^2}^{\infty} ds f(s) \langle 0|\phi(x, s)|0\rangle \tag{3.24}$$

$$s\langle 0|\phi(x, s)|0\rangle = g_0 f(s) \langle 0|\phi(x)|0\rangle \tag{3.25}$$

the first of which may be eliminated in favor of the equation

$$\mu_0^2 = g_0^2 \int_{4M_A^2}^{\infty} ds \frac{f^2(s)}{s} ds \tag{3.26}$$

Thus the broken symmetry implies a relation [Eq. (3.25)] between $\langle 0|\phi(x)|0\rangle$ and $\langle 0|\phi(x, s)|0\rangle$ as well as a condition [Eq. (3.26)] on the parameters of the theory. It is furthermore obvious upon comparison with Eq. (3.23) that the broken-symmetry solution of the model is merely a particular solution which is derivable from Eq. (3.22) by choosing the bare mass μ_0^2 in accord with Eq. (3.26). The broken-symmetry condition serves only to ensure the existence of the required zero-mass particle and, as is invariably the case for a naturally occurring broken symmetry, the Green's functions [Eq. (3.22)] contain no reference to the broken-symmetry parameter η. The crucial point to be emphasized is that once the condition (3.26) is incorporated into the Lagrangian (by suitable choice of μ_0^2 and/or g_0^2) there is no further physical content in the broken-symmetry condition. This situation is in sharp contrast to the case of spontaneous symmetry breaking where the constraints cannot merely be incorporated into the Lagrangian and subsequently ignored in all higher order calculations.

Before leaving the Zachariasen model in favor of more complex theories, it is well to display here the conservation law and associated gauge group which supports the broken symmetry. The conserved operator must clearly be of the form

$$\phi^\mu + g_0 \int_{4M_A^2}^{\infty} ds\alpha(s)\phi^\mu(x, s)$$

where $\alpha(s)$ is to be determined. Using the equations of motion one finds

$$\partial_\mu \left[\phi^\mu(x) + g_0 \int_{4M_A^2}^{\infty} ds\alpha(s)\phi^\mu(x, s)\right] = g_0 \int_{4M_A^2}^{\infty} [f(s) - s\alpha(s)]\phi(x, s)$$
$$- \phi(x)\left[\mu_0^2 - g_0^2 \int_{4M_A^2}^{\infty} f(s)\alpha(s)\, ds\right]$$

which by Eq. (3.26) vanishes for

$$\alpha(s) = (1/s)f(s)$$

It is now easy to show that the Lagrangian is invariant under the c-number gauge group

$$\phi(x) \to \phi(x) + \eta$$
$$\phi(x, s) \to \phi(x, s) + g_0(1/s)f(s)\eta$$

a result which could have been anticipated from Eq. (3.25).

For the sake of completeness it should perhaps be remarked that although we have considered here only one particular limit of the more general model discussed in Zachariasen's paper, no essentially new features emerge upon introduction of a direct coupling of the field $\phi(x, s)$ to itself. It has been pointed out that with this additional interaction one can decrease $\mu_0{}^2$ to zero (thereby requiring also the vanishing of $g_0{}^2$) and consequently effect the decoupling of the massless boson even though the scattering amplitude fails to vanish.[24] Since, however, the massless particle in the limit refers to a free field, this is entirely equivalent to the statement that it is always possible to introduce extraneous massless free fields into any field theory. It does not provide any insight into the more interesting question of the possible decoupling of the massless quanta of a field with a nontrivial interaction. The results for the complete Zachariasen model therefore seem not to be of sufficient interest to warrant inclusion here.

Thus far the examples considered in this chapter have dealt exclusively with instructive, but essentially trivial, cases of broken-symmetry theories. In an effort to avoid leaving the reader with the impression that this exhausts the class of naturally occurring broken symmetries, we briefly discuss a somewhat more complex theory of the type suggested by (3.11).[17,20] The model consists of a pseudoscalar field $\phi(x)$ coupled to the pseudovector current of a Hermitian spinor field $\psi(x)$ and is described by the Lagrangian

$$\begin{aligned}\mathscr{L} = (i/2)\psi\beta\,\gamma^\mu\,\partial_\mu\psi - (m/2)\psi\beta\psi + \phi^\mu\,\partial_\mu\phi + \tfrac{1}{2}\phi^\mu\phi_\mu \\ + \tfrac{1}{2}g_0\psi\beta\,\gamma_5\,\gamma_\mu\psi[\phi^\mu + \tfrac{1}{4}g_0\psi\beta\,\gamma_5\,\partial^\mu\psi]\end{aligned} \tag{3.27}$$

where the term quadratic in g_0 has been included in order to preserve equivalence with the well-known derivative coupling theory. Because of the translational invariance of (3.27) under

$$\phi(x) \to \phi(x) + \eta$$

it follows that one has the local conservation law

$$\partial_\mu\phi^\mu = 0 \tag{3.28}$$

The remaining field equations implied by (3.27) are

$$\gamma^\mu[(1/i)\,\partial_\mu - g_0\,\gamma_5\,\partial_\mu\phi]\cdot\psi + m\psi = 0$$

$$\phi_\mu = -\,\partial_\mu\phi - g_0\,j_{5\mu}$$

where a dot notation has been introduced to denote a symmetrical operator product and

$$j_5^{\mu} = \tfrac{1}{2}\psi\beta\,\gamma_5\,\gamma^{\mu}\psi$$

It is now easy to verify that the condition

$$\langle 0|\phi(x)|0\rangle \neq 0$$

is consistent with the equations of motion without the introduction of any constraints upon the parameters of the system. From Eq. (3.28) one readily shows that the two-point function

$$G^{\mu}(x) = i\langle 0|(\phi^{\mu}(x)\phi(0))_{+}|0\rangle$$

has the form

$$G^{\mu}(x) = -\partial^{\mu}\int\frac{dp}{(2\pi)^4}\,e^{ipx}\,\frac{1}{p^2 - i\epsilon}$$

thereby demonstrating the existence of the zero-mass particle required by the Goldstone theorem which is based on the elements of the canonical group. It is to be noted that although the theory is soluble for the case $m = 0$ this point is entirely irrelevant to the present discussion. Indeed the interest of the model stems largely from the independence of the broken-symmetry aspects from the detailed dynamics.

The above discussion of the derivative coupling model has served to illustrate in a more concrete example our assertion concerning the occurrence of a zero-mass particle in theories of the type described by (3.11). The significance of this result becomes all the more remarkable when one realizes that upon leaving the rather limited domain of soluble field theories there exists no further class of field theories (other than the higher spin generalizations of the field translation group) which is known to admit such a precise statement concerning the physical mass spectrum. In this context it must be recalled that although it was long thought that the usual type of gauge invariance associated with electrodynamics implied the vanishing of the photon mass, this view has relatively recently been discredited.[25-27] It should, however, be noted that the objections raised against the massless photon do not apply to at least one type of electromagnetic coupling.[17] We refer to the spin 1 generalization of the field translation group which rigorously allows the conclusion of a vanishing photon mass. Such a theory is characterized by the fact that the coupling is mediated through the electromagnetic field tensor rather than the vector potential, so that one requires a Lagrangian of the form

$$\mathscr{L} = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathscr{L}'(F^{\mu\nu})$$

Since $\mathscr{L}'(F^{\mu\nu})$ does not contain the vector potential A^μ, one clearly has the local conservation law

$$\partial_\nu F^{\mu\nu} = 0 \tag{3.29}$$

associated with the invariance of the Lagrangian under

$$A^\mu(x) \to A^\mu(x) + \eta^\mu \tag{3.30}$$

for η^μ being arbitrary constants. As in the usual radiation gauge formulation one can take $A_k(x)$ to be transverse

$$\partial_k A_k = 0$$

so that using Eq. (3.29) leads to the result

$$\langle 0|[F^{\mu\nu}(x), A^\lambda(0)]|0\rangle = a\left[g^{\mu\lambda}\,\partial^\nu - g^{\nu\lambda}\,\partial^\mu - \frac{n^\mu\,\partial^\nu - n^\nu\,\partial^\mu}{\nabla^2}(n\partial)\,\partial^\lambda\right]\Delta(x;0)$$

where $n^\mu = (\mathbf{0}, 1)$. From the commutation relation

$$[F_T{}^{0k}(x), A^l(x')] = i\left(\delta_{kl} - \frac{\partial_k\,\partial_l}{\nabla^2}\right)\delta(\mathbf{x} - \mathbf{x}')$$

for the transverse of $F^{0k}(x)$, one readily finds

$$a = 1$$

thereby establishing the existence of a massless photon solely from the invariance of $\mathscr{L}$ under the gauge transformation Eq. (3.30). It is clear from the brief discussion given here of electromagnetic couplings which are mediated entirely through the field tensor that theories of this type support a naturally occurring broken-symmetry condition of the form

$$\langle 0|A^\mu(x)|0\rangle \neq 0$$

One further notes that this does not require a nonvanishing $\mathscr{L}'(F^{\mu\nu})$ so that it applies equally well to the free field case ($\mathscr{L}' = 0$) and to a Pauli moment interaction ($\mathscr{L}' = \tfrac{1}{2}\lambda F^{\mu\nu}M_{\mu\nu}$).

Although the electromagnetic field in the context of broken symmetries merits a much more detailed consideration than that which has been given here, this properly belongs to a study of the usual gauge properties of electrodynamics and we consequently defer such a discussion to the following section.

IV. Gauge Theories

As we have seen in Section II, any manifestly covariant broken-symmetry theory must exhibit massless particles in the spectrum of states associated with the corresponding current. There is, however, one class of relativistic theory which lacks manifest covariance and which may therefore escape the conclusions of the Goldstone theorem, namely gauge theories such as electrodynamics in the radiation gauge. This section is devoted to a discussion of a variety of broken-symmetry theories involving vector gauge fields.

If we wish to attribute any of the observed approximate symmetries of relativistic particle physics to spontaneous symmetry breaking, and at the same time avoid the appearance in the theory of unobserved massless particles, then there are really only two choices available. Either we couple in gauge fields, as described in this section, or we have to suppose that the massless particles required by the Goldstone theorem are in fact completely uncoupled, and therefore unobservable. In the latter case, however, the Hilbert space of the system may always be written as the direct product of a physical Hilbert space, free of massless particles, and a free-particle Fock space describing the Goldstone particles. The broken symmetry appears only in the latter, and no trace of it remains in the physical predictions of the theory, which must in fact exhibit complete symmetry.

There is another reason for considering the possible introduction of gauge fields. The success of the "gauge principle" in electrodynamics,[28] whereby the electromagnetic field is introduced in the course of extending the symmetry group from global to local transformations, and of its analog in gravitation,[29] led to the idea that other interactions might perhaps be understood in a similar fashion.[30] A major obstacle in the way of this hypothesis was the fact that the vector particles associated with the gauge fields are apparently required by the theory to be massless, like the photon and the graviton.

However, it was suggested some time ago by Anderson[31] that the two problems posed by the vanishing of the masses of the "Goldstone bosons" and the "Yang-Mills bosons" might in certain circumstances "cancel out." In fact, as we shall see explicitly below, in a theory with *both* gauge fields *and* symmetry breaking the two polarization states of the massless vector particles and the single state of the massless scalar may combine to yield the three states appropriate to a massive vector boson. This idea provides what is, at least at first sight, an extremely attractive escape from the problem of predicted but unobserved massless bosons. It has been discussed by Higgs,[32,33] Brout and Englert,[34] Guralnik, Hagen, and Kibble,[35] and Kibble.[36]

At least in the case of the electromagnetic field it is always possible to maintain manifest covariance, by using the Lorentz gauge and, for example, the Gupta-Bleuler formalism.[37] Then the Goldstone theorem certainly applies, and requires the existence of massless states. However, it says nothing about whether these states are physical, and indeed we shall see explicitly in cases discussed below that the corresponding massless fields are generally pure gauge parts whose matrix elements between physical states vanish identically. In the case of non-Abelian gauges, the Gupta-Bleuler formalism in inapplicable, but one can work in terms of Schwinger's extended-operator formalism,[38] with similar results.

Let us begin by examining the simplest possible model of a broken symmetry theory—a free massless scalar field described by the Lagrangian density

$$L = \phi^\mu\, \partial_\mu \phi + {}^1\!/_2 \phi^\mu \phi_\mu \tag{4.1}$$

This Lagrangian is invariant under the field translations

$$\phi(x) \rightarrow \phi(x) + g\lambda \tag{4.2}$$

where g is a positive coupling constant introduced for later convenience, and λ is a real parameter. Obviously, $\langle\phi\rangle = \langle 0|\phi(x)|0\rangle$ cannot be invariant under (4.2), so this symmetry is always broken. As discussed in Section III, the various degenerate vacuums distinguished by different values of $\langle\phi\rangle$ belong to unitarily inequivalent (but physically indistinguishable) representations of the canonical commutation relations.

Now let us couple the conserved current $g\phi^\mu$ to a gauge vector field. We obtain[39]

$$L = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + {}^1\!/_4 F^{\mu\nu}F_{\mu\nu} + \phi^\mu\, \partial_\mu \phi + {}^1\!/_2 \phi^\mu \phi_\mu + g\phi^\mu A_\mu \tag{4.3}$$

which is invariant not only under Eq. (4.2), but also under the gauge transformations of the second kind,

$$\begin{aligned} \phi(x) &\rightarrow \phi(x) + g\lambda(x) \\ A_\mu(x) &\rightarrow A_\mu(x) + \partial_\mu \lambda(x) \end{aligned} \tag{4.4}$$

Although this is an essentially trivial model, it will be helpful to analyze it in some detail in order to bring out a number of points of more general applicability, particularly in view of the fact that many other models reduce to this one in an appropriate linear approximation. In particular, we wish to discuss the relationship between the Lorentz gauge and Coulomb gauge methods.

In the Coulomb gauge we remove the arbitrariness corresponding to the transformations (4.4) by imposing on A_μ the condition $\partial_k A^k = 0$. This may be achieved, for example, by adding to the Lagrangian a Lagrange multiplier term

$$-C\,\partial_k A^k \tag{4.5}$$

The field equations are then

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.6}$$

$$\partial_\mu F^{\mu\nu} = -g\phi^\nu + \delta_k{}^\nu\, \partial^k C \tag{4.7}$$

$$\partial_k A^k = 0 \tag{4.8}$$

$$\phi_\mu = -\partial_\mu \phi - gA_\mu \tag{4.9}$$

$$\partial_\mu \phi^\mu = 0 \tag{4.10}$$

The independent dynamical variables are ϕ and ϕ^0, and the two transverse components of each of A_k and F^{0k}. The dependent variables are determined in terms of these by constraint equations. In particular, from Eqs. (4.6) and (4.7) one finds for A_0 and the longitudinal part of F^{0k} the equations

$$-\nabla^2 A^0 = \partial_k F^{0k} = g\phi^0 \tag{4.11}$$

Eliminating these variables, one finds the dynamical equations of motion

$$\partial_0 A_k{}^T = F^T_{0k} \qquad \partial_0 F^{0kT} = -\nabla^2 A^{kT} + g^2 A^{kT} \tag{4.12}$$

$$\partial_0 \phi = -\phi_0 + g^2(\nabla^2)^{-1}\phi_0 \qquad \partial_0 \phi^0 = \nabla^2 \phi \tag{4.13}$$

It follows that A_k^T, F_{0k}^T, and ϕ^0 satisfy the Klein-Gordon equation for mass g. In fact, combining Eqs. (4.12) and (4.13), and introducing the new variables $V_k = A_k + g^{-1}\,\partial_k\phi$, we recover the standard equations describing a vector field of mass g. Note, however, that for ϕ we can only derive the equation

$$(-\partial^2 + g^2)\,\partial_\mu\phi = 0$$

One can verify these conclusions directly from the Lagrangian (4.3). Let us introduce in place of A_μ the new variables

$$V_\mu = A_\mu + g^{-1}\,\partial_\mu\phi \tag{4.14}$$

Then Eq. (4.9) becomes an explicit solution for ϕ_μ,

$$\phi_\mu = -gV_\mu \tag{4.15}$$

which may be used to eliminate ϕ_μ from the Lagrangian, yielding

$$L = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu V_\nu - \partial_\nu V_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}g^2V^\mu V_\mu \\ - C\,\partial_k(V^k - g^{-1}\,\partial^k\phi) \tag{4.16}$$

The independent dynamical variables are now V_k and all three components of F^{0k}. In this form it is clear that V_μ describes particles of spin one and mass g.

The Lagrangian (4.16) is still invariant under the transformations (4.2), though not of course under (4.4), but in a completely trivial way. In fact, from Eqs. (4.10) and (4.14) we see that ϕ is determined in terms of V by the equation

$$\nabla^2\phi = g\,\partial_k V^k \tag{4.17}$$

and (4.2) represents merely the arbitrariness in the solution of this equation. (Explicit dependence on x is ruled out by the requirement of translational invariance.) Note that the equation obtained by variation of ϕ, namely

$$\nabla^2 C = 0$$

shows similarly that C is at most a constant.

It is useful to reexamine the proof of the Goldstone theorem, in the context of this particular model, to see explicitly why it fails. The conserved current, whose integrated time component is the formal generator of the transformations (4.2) is simply $g\phi^\mu$. Thus the expectation value we have to consider in examining the proof of the theorem is

$$f^\mu(x) = -ig\langle[\phi^\mu(x), \phi(0)]\rangle \tag{4.18}$$

By the canonical commutation relations, we have

$$\int d^3x f^0(\mathbf{x}, 0) = g$$

If this equation were true not only for $t = 0$, but for all times, then the Goldstone theorem would follow. However, reexpressing f^μ in terms of the vector field V^μ, using Eqs. (4.15) and (4.17), we have

$$f^\mu(x) = +ig^3\langle[V^\mu(x), (\nabla^2)^{-1}\,\partial_k V^k]\rangle$$

But the commutator function of a free vector field has the form

$$i[V^\mu(x), V_\nu(y)] = (\delta_\nu{}^\mu - g^{-2}\,\partial^\mu\,\partial_\nu)\Delta(x - y; g^2)$$

Hence we find that f^0 has a causal structure,

$$f_0(x) = g\,\partial_0\Delta(x; g^2)$$

but that f^k has not:

$$f_k(x) = g(1 - g^2/\nabla^2)\,\partial_k\Delta(x; g^2)$$

It is this nonlocal structure which makes the proof of the Goldstone theorem fail. We note that by explicit calculation one finds the time dependence

$$\int d^3x f^0(\mathbf{x}, t) = g \cos gt \tag{4.19}$$

In any broken symmetry theory, the integral over all space of the current operator,

$$Q = \int d^3x j^0(x)$$

fails to exist. However, if we evaluate the commutator of j^0 with any local operator *before* doing the spatial integration, the result is always well defined. Formally, if the theory is manifestly covariant, the "operator" Q is time independent, in the sense that all its commutators evaluated in this manner are time independent. However, in the same sense for our model it satisfies the equation

$$(\partial_0{}^2 + g^2)Q = 0$$

in agreement with Eq. (4.19). This was noted by Guralnik, Hagen, and Kibble.[35]

Now let us turn to the Lorentz gauge. We shall find it convenient to impose the Lorentz gauge condition by adding to the Lagrangian Eq. (4.3) a Lagrange multiplier term analogous to Eq. (4.5), namely,

$$-G\,\partial_\mu A^\mu + \tfrac{1}{2}\alpha GG \tag{4.20}$$

where α is an arbitrary constant introduced to allow direct comparison both with Schwinger's formalism[38] ($\alpha = 0$), and with the more conventional Fermi Lagrangian[37] ($\alpha = 1$). Note that in a second-order form Eq. (4.20) would correspond to a term

$$-(1/2\alpha)(\partial_\mu A^\mu)^2$$

The advantage of the first-order form lies precisely in the possibility of taking $\alpha = 0$.

With this addition to the Lagrangian, the only equations of motion which are changed are Eqs. (4.7) and (4.8) which are now replaced by

$$\partial_\mu F^{\mu\nu} = -g\phi^\nu + \partial^\nu G \tag{4.21}$$

and

$$\partial_\mu A^\mu = \alpha G \tag{4.22}$$

Note that from Eqs. (4.10) and (4.21), it follows that G is a massless field,

$$\partial^2 G = 0 \tag{4.23}$$

The important difference between the Lorentz gauge and the Coulomb gauge derives from the fact that (4.20), unlike (4.5), contains a time derivative, so that G, unlike C, is formally a dynamical variable. The equations are no longer automatically equivalent to the gauge-invariant equations with $G = 0$, but have additional solutions with nonvanishing G which must be eliminated by imposing some subsidiary condition on the physical states.

We note that the generator of the gauge transformations (4.4) is

$$\begin{aligned} G(\lambda) &= \int d^3x[G(x)\,\partial_0\lambda(x) - \lambda(x)\,\partial_0 G(x)] \\ &= \int d^3x[G\,\partial_0\lambda - \lambda(\partial_k F^{0k} - g\phi^0)] \end{aligned} \tag{4.24}$$

There are two alternative procedures at this point. We can adopt either the Gupta-Bleuler formalism, or the extended-operator formalism of Schwinger. In the Gupta-Bleuler formalism, the fields are represented by operators in a Hilbert space with indefinite metric, and the subspace of physical states is selected by imposing on them the subsidiary condition

$$G^{(+)}(x)|\rangle = 0 \qquad (\partial_k F^{0k(+)} - g\phi^{0(+)})|\rangle = 0 \tag{4.25}$$

where the superscript plus denotes the positive-frequency part. Equation (4.23) is essential here in allowing us to draw an invariant distinction between positive and negative frequencies.

On the other hand, in Schwinger's formalism the field variables are not represented in a Hilbert space, but in a more general functional space, and the physical states are distinguished by the requirement of gauge invariance,

$$G(\lambda)\Psi = 0$$

or equivalently

$$G\Psi = 0 \qquad (\partial_k F^{0k} - g\phi^0)\Psi = 0 \tag{4.26}$$

We shall consider both these formalisms in the following.

The canonically conjugate pairs of field variables are now (F^{0k}, A_k), (G, A^0), and (ϕ, ϕ^0). As before it is convenient to make a change of variables. We introduce a canonical transformation to the pairs (F^{0k}, V_k), (G, χ^0), and (ϕ, π^0), where

$$\begin{aligned} V_k &= A_k + g^{-1}\,\partial_k\phi - g^{-2}\,\partial_k G \\ \chi^0 &= A^0 + g^{-2}\,\partial_k F^{0k} \\ \pi^0 &= \phi^0 - g^{-1}\,\partial_k F^{0k} \end{aligned} \tag{4.27}$$

To exhibit the Lagrangian in manifestly covariant form, it is convenient to introduce also the dependent variables V_0, χ^k, π^k, so that we may write

$$\begin{aligned} L = -\tfrac{1}{2}F^{\mu\nu}\,(\partial_\mu V_\nu - \partial_\nu V_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}g^2 V^\mu V_\mu \\ + \chi^\mu\,\partial_\mu G + \pi^\mu\,\partial_\mu\phi + \tfrac{1}{2}\pi^\mu\pi_\mu + g\pi^\mu\chi_\mu + \tfrac{1}{2}\alpha GG \end{aligned}$$

Here we clearly have a vector field V_μ describing particles of mass g, and two scalar fields. No coupling remains between the vector and scalar fields.

Let us first consider Schwinger's formalism, which in this case is somewhat simpler. To do this, we may adopt a representation in which states are labelled by the eigenvalues of, for example, ϕ, χ^0, and V_k, all at a given time t. On the space of functionals of these variables, the canonically conjugate fields are represented by functional differential operators, namely

$$\pi^0 = -i\frac{\delta}{\delta\phi'} \qquad G = i\frac{\delta}{\delta\chi^{0'}} \qquad F^{0k} = i\frac{\delta}{\delta V_{k'}}$$

The gauge-invariance requirements (4.26) become

$$G\Psi = 0 \qquad \pi^0\Psi = 0$$

or

$$\frac{\delta}{\delta\chi_0{}'}\Psi = 0 \qquad \frac{\delta}{\delta\phi'}\Psi = 0 \tag{4.28}$$

They show that the physical states are in fact independent of the scalar fields $\chi_0{}'$ and ϕ', and may be represented by functionals of the vector field $V_k{}'$ alone. These states form a Hilbert space under the scalar product defined by functional integration over all functionals of $V_k{}'$. No massless particles remain among the physical states, in complete agreement with the results of the Coulomb gauge treatment.

In the formalism of Gupta and Bleuler, on the other hand, the fields are supposed to be represented in a Hilbert space with indefinite metric. The subspace of physical states is selected by imposing the subsidiary conditions

$$G^{(+)}\Psi = 0 \qquad \pi^{0(+)}\Psi = 0 \tag{4.29}$$

Here the scalar fields do not annihilate the physical states, but their matrix elements between pairs of physical states are nevertheless zero in virtue of Eq. (4.29). The large Hilbert space with indefinite metric may be written, because of the absence of coupling between the vector and scalar fields, as a direct product of a physical Hilbert space with positive definite metric (which is in fact the Fock space for the vector particles), and an unphysical Hilbert space describing the scalar particles. In this latter space, the subspace selected by the conditions (4.29) consists exclusively of states with zero norm with the sole exception of the vacuum state.

It is interesting to examine more closely the unphysical fields ϕ and G. The corresponding field equations are

$$\begin{aligned} \partial_\mu G &= -g\pi_\mu \qquad & \partial_\mu\phi &= -\pi_\mu - g\chi_\mu \\ \partial_\mu\chi^\mu &= \alpha G \qquad & \partial_\mu\pi^\mu &= 0 \end{aligned} \tag{4.30}$$

It follows that

$$\partial^2 G = 0 \qquad \partial^2\phi = \alpha g G \tag{4.31}$$

The massless particles required by the Goldstone theorem are described by the field G. In the special case $\alpha = 0$ there are two independent massless fields, corresponding to the fact that in that case the Lagrangian is also invariant (up to a divergence) under the transformation $G(x) \to G(x) + \lambda$, which is generated formally by the spatial integral of A_0 or χ_0. In general, however, there is only one invariance, whose formal generator is the spatial integral of $g\phi^0$ or $g\pi^0$. Note that it is not the canonical conjugate ϕ of this density which describes the Goldstone bosons, but rather the field G whose time derivative is $g\pi^0$. In fact the verification of the Goldstone theorem for this case rests on an examination of the commutator function $-ig[\pi_\mu, \phi] = i[\partial_\mu G, \phi]$, which must be nonzero and have a time-independent spatial integral.

The covariant commutation relations of the scalar fields are easy to derive from the equations of motion [Eqs. (4.30)] and the canonical commutation relations. We find that the mutual consistency of the subsidiary conditions [Eq. (4.28) or (4.29)] for different times is assured by the commutator

$$i[G(x), G(y)] = 0 \tag{4.32}$$

The commutator function which is important for the discussion of the Goldstone theorem is

$$i[\phi(x), G(y)] = gD(x-y) = \frac{g}{2\pi}\epsilon(x^0 - y^0)\delta[(x-y)^2] \tag{4.33}$$

It is easily seen to have the requisite locality property, which ensures that

$$i\int d^3x[\partial_0 G(x), \phi(0)] = g \tag{4.34}$$

Finally we note also that

$$\begin{aligned} i[\phi(x), \phi(y)] &= D(x-y) - \alpha g^2\left[\frac{\partial}{\partial m^2}\Delta(x-y; m^2)\right]_{m^2=0} \\ &= \frac{1}{2\pi}\epsilon(x^0-y^0)\{\delta[(x-y)^2] + \tfrac{1}{4}\alpha g^2\theta[-(x-y)^2]\} \end{aligned} \tag{4.35}$$

These covariant commutation relations are valid for both Schwinger's formalism and for that of Gupta and Bleuler. However, it is important to recall that they play significantly different roles in the two cases. In Schwinger's formalism they express the effect of reversing the order of a pair of functional differential operators. In that context their form is neither surprising nor troublesome. In the formalism of Gupta and Bleuler, however, they describe the commutation properties of Hilbert-space operators and are therefore required to satisfy rather more stringent requirements. In particular, let us take the vacuum expectation value of Eq. (4.35) and insert a complete set of states in the usual way. (Note that this operation has no meaning in Schwinger's formalism.) Denoting by ρ_n the diagonal element, ± 1, of the metric operator, we find

$$\sum_n \rho_n |\langle n|\phi(0)|0\rangle|^2 (2\pi)^4 \delta(p_n - k) = \theta(k^0) 2\pi\delta(k^2) - \alpha g^2 \left[\frac{\partial}{\partial m^2} \{\theta(k^0) 2\pi\delta(k^2 + m^2)\} - \pi^2 ln\left(\frac{m^2}{M^2}\right) \delta(k) \right]_{m^2=0} \quad (4.36)$$

where M is an arbitrary constant mass associated with the arbitrary constant in the field ϕ. It is evident that unless $\alpha = 0$ the matrix elements of ϕ must be extremely singular. The type of structure which appears on the right-hand side of Eq. (4.36) can in fact be produced only by a cancellation between infinite positive and negative terms on the left-hand side. The root of this problem may be traced to Eq. (4.31). In momentum space, it is clear that the field G is proportional to $\delta(k^2)$. Thus, in order to satisfy the second of this pair of equations, it is necessary that ϕ should be proportional to the derivative of the δ function, $\delta'(k^2)$. Now, the quantity which appears in the commutator function, namely the derivative with respect to m^2 of the odd function $\epsilon(k^0)\delta(k^2 + m^2)$, evaluated at $m^2 = 0$, is a perfectly well-defined Lorentz-invariant distribution; however, the derivative of the even function $\delta(k^2 + m^2)$ is not,[40] but it is arbitrary to the extent of an additive multiple of the four-dimensional δ function. [This is represented in Eq. (4.36) by the arbitrariness in M.]

These difficulties may be avoided either by using Schwinger's extended-operator formalism, in which Eq. (4.35) makes perfect sense but Eq. (4.36) cannot be written down, or alternatively by making the particular choice $\alpha = 0$, for which no problems arise. It is worth recalling that this is precisely the case for which the second-order form of the Lagrangian cannot be used.

Finally, before leaving this model we wish to point out that in the Gupta-Bleuler formalism it provides an example of the decoupling of the massless modes, leaving no trace of broken symmetry among the physical states. Here, the physical states are described by the vector field V_μ, which is completely invariant under the symmetry transformations (4.2). In a sense, therefore, we have not so much broken the symmetry as eliminated it completely from the theory. As in the case of "naturally occurring" broken symmetries, the symmetry-breaking parameter is physically irrelevant.

As a rather less trivial example let us now consider the model of a self-interacting two-component scalar field

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

described by the Lagrangian[41]

$$L = \phi^\mu \,\partial_\mu \phi + \tfrac{1}{2}\phi^\mu \phi_\mu - V(\phi\phi) \tag{4.37}$$

It is clearly invariant under the rotations

$$\phi \to e^{i\Lambda q}\phi \tag{4.38}$$

where q is the antisymmetric 2×2 matrix defined in Section III.

If V has a maximum at $\phi\phi = 0$ and a minimum at some other value, e.g., if

$$V = -\tfrac{1}{2}a^2\phi\phi + \tfrac{1}{2}b^2(\tfrac{1}{2}\phi\phi)^2 \tag{4.39}$$

then we may expect that in the ground state the expectation value of ϕ will not be zero, but rather will approximate to the value at which V has a minimum. Clearly, because of the invariance, there must be an infinitely degenerate set of ground states, with expectation values corresponding to all the points round the circle in the $\phi_1 - \phi_2$ space on which V has its minimum value. From the equations of motion, and the requirement of translational invariance of the ground state, one easily derives the consistency requirement

$$\left\langle \frac{\partial V}{\partial \phi} \right\rangle = \langle \phi(\tfrac{1}{2}b^2\phi\phi - a^2)\rangle = 0 \tag{4.40}$$

which serves to fix the magnitude of $\langle\phi\rangle$. The various degenerate ground states are labeled by a phase angle α, and characterized by the expectation values

$$\langle\phi\rangle = \eta = e^{i\alpha q}\eta_0 \tag{4.41}$$

where η_0 may be chosen to be, say,

$$\eta_0 = \begin{bmatrix} 0 \\ |\eta| \end{bmatrix} \tag{4.42}$$

with $|\eta|$ determined by Eq. (4.40). (By translational invariance, η is of course independent of x.) In lowest order,

$$|\eta|^2 = 2a^2/b^2$$

The current corresponding to the invariance (4.38) is

$$j^\mu = i\phi^\mu q\phi \tag{4.43}$$

It satisfies

$$\partial_\mu j^\mu = 0 \tag{4.44}$$

Formally, the transformation between one member of the set of degenerate ground states and another is performed by the unitary operator

$$U(\lambda) = \exp\left\{i\lambda \int d^3x j^0(x)\right\} \tag{4.45}$$

In fact, however, this integral does not converge to a well-defined operator in the limit of infinite volume, and the various degenerate ground states belong to unitarily inequivalent representations of the commutation relations (compare Section III).

The equations of motion obtained from the Lagrangian Eq. (4.37), with V given by Eq. (4.39), are

$$\partial_\mu \phi = -\phi_\mu \tag{4.46}$$

$$\begin{aligned} \partial_\mu \phi^\mu &= -\partial V/\partial\phi \\ &= \phi(a^2 - \tfrac{1}{2}b^2\phi\phi) \end{aligned} \tag{4.47}$$

To analyze the structure of the theory, it is useful to begin by making a linear approximation. We write $\phi = \eta + \phi'$, and retain only terms linear in ϕ' in the equations of motion. Then Eqs. (4.46) and (4.47) yield the equation

$$\partial^2\phi' = b^2\eta(\eta\phi') \tag{4.48}$$

It is clear from the structure of the mass matrix $b^2\eta\eta$ that this theory describes two particles, one with mass zero, and the other with mass m given by

$$m^2 = b^2\eta\eta = 2a^2$$

In higher approximations, there are, naturally, interactions between the two scalar fields, which will have the effect, among others, of renormalizing the mass of the massive mesons. However, we are assured by the Goldstone theorem that no such renormalization occurs for the massless mesons. These are the Goldstone particles, and their mass must remain zero. It should be noted that the value of η must change in higher orders to take account more precisely of the consistency condition (4.40).

Now let us consider the effect of coupling the current [Eq. (4.43)] to the electromagnetic field. (We are now interpreting the two components of ϕ as the real components of a charged field.) The Lagrangian then takes the form

$$L = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \phi^\mu(\partial_\mu - ieqA_\mu)\phi \\ + \tfrac{1}{2}\phi^\mu\phi_\mu - V(\phi\phi) \tag{4.49}$$

As in our previous discussion of the simple free-field model, we could use either the Lorentz gauge or the Coulomb gauge to discuss this theory. However, since no new points of principle emerge from the use of the Lorentz gauge in this problem, we shall be content with the Coulomb gauge. Moreover, we shall not write explicitly the Langrange multiplier term analogous to Eq. (4.5).

The field equations derived from the Lagrangian Eq. (4.49) are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.50}$$

$$\partial_\mu F^{\mu\nu} = ie\phi^\nu q\phi \tag{4.51}$$

$$\phi_\mu = -\partial_\mu\phi + ieA_\mu q\phi \tag{4.52}$$

$$\begin{aligned}(\partial_\mu - ieA_\mu q)\phi^\mu &= -\partial V/\partial\phi \\ &= \phi(a^2 - \tfrac{1}{2}b^2\phi\phi)\end{aligned} \tag{4.53}$$

If we now make the same linearizing approximation as before, the last two equations become

$$\phi_\mu = -\partial_\mu\phi' + ieq\eta A_\mu \tag{4.54}$$

$$\partial_\mu\phi^\mu = b^2\eta(\eta\phi') \tag{4.55}$$

The components of these equations in the direction of η are independent of A_μ, and yield

$$(-\partial^2 + m^2)(\eta\phi') = 0 \tag{4.56}$$

with $m^2 = b^2|\eta|^2$ as before. Thus the massive scalar particles are unaffected by the coupling in of the electromagnetic field. To simplify the remaining equations, it is convenient to introduce the new variables

$$V_\mu = A_\mu + \frac{i}{e\eta\eta}\eta q\,\partial_\mu\phi' = -\frac{i}{e\eta\eta}\eta q\phi_\mu \tag{4.57}$$

Then Eqs. (4.50) and (4.51) become

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu \\ \partial_\mu F^{\mu\nu} &= M^2 V^\nu \end{aligned} \tag{4.58}$$

with $M^2 = e^2|\eta|^2$. These are of course the standard equations describing a massive vector field. Thus we see that the massless Goldstone bosons have combined with the electromagnetic field to produce a massive vector field. No massless particles remain in the theory.

It is interesting to examine an alternative approach to this theory. This approach has the advantage of not being restricted to a perturbation treatment based on the linearized approximation, but correspondingly it has the defect of involving complicated algebraic transformations of the dynamical variables which are not obviously well-defined for operator fields.

We introduce the polar decomposition[42]

$$\phi = \begin{bmatrix} \rho \sin\theta \\ \rho\cos\theta \end{bmatrix} \tag{4.59}$$

where the field ρ has a nonvanishing expectation value, $\langle\rho(x)\rangle = |\eta|$. Then, ignoring problems of operator ordering (which may in fact be rather severe), we find that the Lagrangian Eq. (4.49) takes the form

$$\begin{aligned} L = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu + {}^1\!/\!_4 F^{\mu\nu}F_{\mu\nu} + \rho^\mu\,\partial_\mu\rho \\ + \rho\sigma^\mu(\partial_\mu\theta - eA_\mu) + {}^1\!/\!_2\rho^\mu\rho_\mu + {}^1\!/\!_2\sigma^\mu\sigma_\mu - V(\rho^2) \end{aligned} \tag{4.60}$$

where we have written

$$\begin{aligned} \rho^\mu &= \phi_1{}^\mu \sin\theta + \phi_2{}^\mu\cos\theta \\ \sigma^\mu &= \phi_1{}^\mu\cos\theta - \phi_2{}^\mu\sin\theta \end{aligned} \tag{4.61}$$

We then introduce the new variables

$$V_\mu = A_\mu - (1/e)\,\partial_\mu\theta \tag{4.62}$$

to which Eq. (4.57) is a linear approximation, and eliminate σ^μ using the relation

$$\sigma^\mu = e\rho V^\mu \tag{4.63}$$

This yields our final Lagrangian

$$L = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu V_\nu - \partial_\nu V_\mu + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}e^2\rho^2 V^\mu V_\mu + \rho^\mu \partial_\mu \rho + \tfrac{1}{2}\rho^\mu \rho_\mu - V(\rho^2) \tag{4.64}$$

This Lagrangian evidently describes a scalar field ρ and a vector field V^μ whose squared masses are determined in lowest approximation by the second derivative of $V(\rho^2)$ and by $e^2\langle\rho^2\rangle \approx e^2|\eta|^2$, respectively. Once again, we see that no massless particles remain in the theory—nor, of course, does any obvious trace of the broken symmetry.

That the Lagrangian (4.49) is really equivalent, in the quantized field theory, to (4.64) has been verified in low orders of perturbation theory by Higgs.[33] We conclude, therefore, that there is every indication that the exact theory, like its linearized approximation, is free of massless particles. Thus the coupling of the electromagnetic field to the current associated with the broken symmetry suffices to eliminate the massless Goldstone bosons.

We now turn to a very different class of theories, which involve only vector fields. It is instructive to examine the electromagnetic field itself from the point of view of broken symmetry, particularly in view of the relationship which has often been assumed between gauge invariance and the vanishing of the photon mass.[43,44]

Let us first recall that the Lagrangian (4.1) for the massless scalar field is invariant under the transformations (4.2), generated by the integrated time component of the conserved current ϕ^μ. The corresponding conserved quantity in the case of the free electromagnetic field is simply $F^{\mu\nu}$ itself. In view of Maxwell's equations, it satisfies the conservation law (for any constant η_k)

$$\partial_\mu(F^{\mu k}\eta_k) = 0 \tag{4.65}$$

From the canonical commutation relations

$$[F^{T0k}(\mathbf{x}, t), A_l(\mathbf{y}, t)] = i\left(\delta_l{}^k - \frac{\partial^k \partial_l}{\nabla^2}\right)\delta(\mathbf{x} - \mathbf{y}) \tag{4.66}$$

(where the superscript T denotes the transverse part of a vector), we find

$$-i\int_V d^3x[(F^{0k}\eta_k)(\mathbf{x}, 0), A_l(0)] = \eta_l - \eta_k \int_{\sigma(V)} d\sigma^k \frac{x_l}{4\pi|\mathbf{x}|^3} = \tfrac{2}{3}\eta_l \tag{4.67}$$

Thus the integrated time component of (4.65) generates the constant field translations (or "constant gauge transformations")

$$A_k(x) \rightarrow A_k(x) + \tfrac{2}{3}\eta_k \tag{4.68}$$

which are the analog of (4.1).

In this free-field case, the integrated commutator (4.67) is actually independent of time, and there is no difficulty in applying the Goldstone theorem to deduce the existence of massless particles in the theory. However, it is clear that this result reflects only our choice of a free field and is essentially trivial.

A physically much more interesting model is interacting electrodynamics. We easily find that the continuity equation (4.65) may be generalized to[44]

$$\partial_\mu[(F^{\mu k} + ex^k j^\mu)\eta_k] = 0 \tag{4.69}$$

Using the general form of the two-point function for the electromagnetic field in the Coulomb gauge,

$$\begin{aligned} i\langle 0|A^\mu(x)A^\nu(0)|0\rangle &- \frac{1}{4\pi|\mathbf{x}|}\,\delta(x^0)\delta_0{}^\mu\delta_0{}^\nu \\ &= \left(g^{\mu\nu} - \frac{\delta_0{}^\mu\,\partial^\nu + \delta_0{}^\nu\,\partial^\mu}{\nabla^2}\,\partial_0 - \frac{\partial^\mu\,\partial^\nu}{\nabla^2}\right)\int_0^\infty dm^2\rho(m^2)\Delta^{(+)}(x;m^2) \end{aligned}$$

we may verify that

$$\langle 0|[j^0(x), A^k(0)]|0\rangle = 0$$

for all values of x^0. This remarkable result, together with the canonical commutation relations (which of course remain valid in the interacting theory) enable us to derive the equal-time commutation relation

$$-i\int d^3x\langle 0|[(F^{0k} + ex^k j^0)(\mathbf{x}, 0)\eta_k, A_l(0)] = \tfrac{2}{3}\eta_l \tag{4.70}$$

Thus we see that the continuity equation (4.69) is still associated with the constant gauge transformations (4.68).

However, it is no longer true in general that the integral in Eq. (4.70) is independent of the time. Indeed, if we evaluate it explicitly for an arbitrary time difference t, we obtain the result

$$\tfrac{2}{3}\eta_l\int_0^\infty dm^2\rho(m^2)\cos(mt)$$

Thus the Goldstone theorem cannot be applied to deduce the existence of massless particles in the theory. Indeed, there is no contradiction in assuming that $\rho(m^2)$ vanishes for values of m^2 less than some finite threshold mass.

In the Lorentz gauge, of course, the Goldstone theorem must apply. In that case, the analogous continuity equation to Eq. (4.69) is

$$\partial_\mu[(\partial^\mu A^\nu + ex^\nu j^\mu)\eta_\nu] = 0 \tag{4.71}$$

In this case, the two-point function of the theory takes the form

$$i\langle 0|A^\mu(x)A^\nu(0)|0\rangle = \left(g^{\mu\nu} - \frac{\partial^\mu\,\partial^\nu}{\partial^2}\right)\left[Z_3\Delta^{(+)}(x;0) + \int_0^\infty dm^2\rho(m^2)\Delta^{(+)}(x;m^2)\right] + \frac{\partial^\mu\,\partial^\nu}{\partial^2}\Delta^{(+)}(x;0)$$

in which we have explicitly separated out the contribution of the massless particles to the spectral function. It follows that

$$-i\int d^3x\langle 0|[(\partial^0 A^\nu + ex^\nu j^0)(x)\eta_\nu, A_\mu(0)]|0\rangle = \eta_\mu \tag{4.72}$$

independent of time. Thus the Goldstone theorem applies and there are massless particles in the theory. However, it is clear that (4.72) is independent of the value of Z_3 and, in particular, is perfectly consistent with the assumption that $Z_3 = 0$. The only term in the two-point function which is actually relevant is the longitudinal term which represents a pure gauge part. This again emphasizes the fact that the massless particles required to exist by the Goldstone theorem need have no connection with physical massless particles. They are pure gauge parts.

It is interesting to examine the special case of electrodynamics in two dimensions, when the electron mass is taken to be zero. As has been shown by Schwinger,[45] this theory is exactly soluble.

The field equations for the electromagnetic field in this model are

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= ej^\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{4.73}$$

However, these are not really equations of motion for dynamical degrees of freedom of the system, for in two dimensions there are no independent degrees of freedom associated with the electromagnetic field. Indeed, if we choose the Coulomb gauge, then $A_1 = 0$, and A_0 is purely a constraint variable, given as an explicit function of the current by the equation

$$-\nabla^2 A^0 = ej^0 \tag{4.74}$$

The electromagnetic field itself has only a single component

$$F_{01} = -\partial_1 A_0 = -F_{10}$$

It can be shown by careful evaluation of the relevant commutators[46] that in addition to these equations the current satisfies the condition

$$\varepsilon^{\mu\nu}\,\partial_\nu j_\mu = -\frac{e}{\pi}F_{01} = -\frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}$$

where $\varepsilon^{\mu\nu}$ is the antisymmetric tensor with $\varepsilon^{01} = 1$. It follows that

$$(-\partial^2 + e^2/\pi)j^\mu = 0 \tag{4.75}$$

Thus the excitations of vector type in the theory are massive. (It should be remarked in passing that there is no physical distinction between scalar and vector particles in two dimensions.)

This theory is interesting from the point of view of broken symmetries, because there exists a symmetry, namely, that under the chiral (γ_5) gauge transformations of the massless fermion field, such that the corresponding current $\varepsilon^{\mu\nu}j_\nu$ is not conserved. It follows, of course, that it cannot support a Goldstone theorem. We also note that although the expression $F^{\mu 1} + ex^1 j^\mu$ does indeed obey a continuity equation, as in Eq. (4.69), its spatial integral

$$\int dx^1(F^{01} + ex^1 j^0) \tag{4.76}$$

does not generate a transformation analogous to Eq. (4.70).

In all these models, we see that the Goldstone theorem is essentially irrelevant to the question of whether or not the photon mass is zero. That is a dynamical question which cannot be answered on grounds of symmetry alone. Any attempt to apply the Goldstone theorem is defeated by the long-range character of the interaction, which ensures that the spatial integral of the time component of the appropriate current is not in fact time independent or, in the Lorentz gauge, by the fact that the resulting Goldstone bosons need have no connection with physical photons.

We now wish to return to models of the Goldstone type and consider the generalization to non-Abelian symmetry groups. Our aim will be to show that the earlier discussion of the Goldstone model may be extended to this case with essentially no change.

Let us consider an n-component real scalar field ϕ which transforms according to a given representation of a compact g-parameter Lie group G,

$$\phi \to e^{i\lambda\cdot T}\phi \tag{4.77}$$

where

$$\lambda\cdot T = \lambda^A T_A$$

Here the λ^A are g real parameters, and the T_A are g real antisymmetric $n \times n$ matrices obeying the commutation relations of the Lie algebra of G,

$$[T_A, T_B] = T_C t^C_{AB}$$

These relations are satisfied in particular by the matrices

$$(t_A)^C_B = t^C_{AB}$$

of the adjoint representation.

We take as our Lagrangian an obvious generalization of (4.37),

$$L = \phi^\mu\,\partial_\mu\phi + \tfrac{1}{2}\phi^\mu\phi_\mu - V(\phi) \tag{4.78}$$

in which, of course, $V(\phi)$ is assumed to be invariant under the transformations (4.77). However, we shall not in this case assume a particular form like Eq. (4.39).

From the invariance of the Lagrangian we may infer the existence of currents

$$j_A{}^\mu = -i\phi^\mu T_A\phi \tag{4.79}$$

satisfying the continuity equations

$$\partial_\mu j_A{}^\mu = 0 \tag{4.80}$$

Formally, the transformations (4.77) are generated by the integrated time components of these currents. However, as in Eq. (4.45), these operators do not in fact exist in the limit of infinite volume, except in the case where the symmetry is unbroken and the vacuum is invariant.

From the assumed translational invariance of the vacuum state, with the equations of motion

$$\begin{aligned}\partial_\mu\phi &= -\phi_\mu \\ \partial_\mu\phi^\mu &= -\partial V/\partial\phi\end{aligned} \tag{4.81}$$

we obtain the consistency condition

$$\left\langle \frac{\partial V}{\partial \phi} \right\rangle = 0 \tag{4.82}$$

analogous to Eq. (4.40).

If $\langle \phi \rangle = \eta$ is a consistent broken-symmetry solution of Eq. (4.82), then so also is $\langle \phi \rangle = e^{i\lambda \cdot T}\eta$ for any λ. Thus, although $\langle \phi \rangle$ has n components, and there are n Eqs. (4.82), $\langle \phi \rangle$ is not completely determined, and these equations cannot all be independent. In fact, the number of algebraically independent equations is just the number of group invariants which can be constructed from the left-hand side of Eq. (4.82). Since these equations transform according to the same n-dimensional representation of G as does $\langle \phi \rangle$, this number is equal to the number of group invariants constructible from $\langle \phi \rangle$. It has been called the *canonical number* ν by Bludman and Klein[47] and is completely determined by the representation to which ϕ belongs. We see, therefore, that there are just enough independent equations in (4.82) to fix the invariants formed from $\langle \phi \rangle$. Conceivably, there might be several distinct solutions for these invariants. However, for simplicity we shall assume that the solution is unique.

Let us now choose a particular canonical value of η, so that all other physically equivalent solutions of Eq. (4.82) may be written in the form $\langle \phi \rangle = e^{i\lambda \cdot T}\eta$. In general, not all values of the parameters λ will yield distinct values of $\langle \phi \rangle$. For there may be a nontrivial subgroup G^η of elements of G which leave the canonical η invariant,

$$e^{i\kappa \cdot T}\eta = \eta$$

This is the stability subgroup of G at η. Let its dimension be $g - r$. Then every element of G may be written in the form

$$e^{i\lambda \cdot T} = e^{i\mu \cdot T} e^{i\kappa \cdot T}$$

where $e^{i\kappa \cdot T}$ is an element of G^η, and the r remaining parameters μ parameterize the factor space G/G^η. (This is the space of cosets of G^η, and is not in general a group.) Then

$$\langle \phi \rangle = e^{i\mu \cdot T} e^{i\kappa \cdot T}\eta = e^{i\mu \cdot T}\eta$$

so that these same parameters μ serve to label the various degenerate ground states distinguished by different values of $\langle \phi \rangle$. Moreover, distinct values of μ correspond to distinct values of $\langle \phi \rangle$, since otherwise the relevant group element should belong to G^η. (At any rate, this is true in the neighborhood of the identity. There could in fact be a discrete set of different values of μ corresponding to the same $\langle\phi\rangle$. However, this is irrelevant for our purposes.) The ground states therefore form an r-dimensional manifold. Since we have already seen that the n components of $\langle\phi\rangle$ are restricted by ν conditions, we have the equality $r = n - \nu$.

To clarify the meaning of these numbers, it may be helpful to consider some simple examples. Let us take the group G to be $SU(3)$. If ϕ belongs to the adjoint representation, then $n = g = 8$. By appropriate choice of axes, we can arrange that the only nonvanishing components of η are η_3 and η_8, in the usual notation. Thus $\nu = 2$, corresponding to the fact that just two invariants can be formed from $\langle\phi\rangle$. Here the dimensionality of the manifold of equivalent values of $\langle\phi\rangle$ is $r = 6$. Moreover, η is left invariant by a two-dimensional subgroup $U(1) \times U(1)$ of G, so that $g - r = 2$, as it should be.

As a second example, suppose that ϕ belongs to the fundamental three-dimensional representation of G. Since this representation is necessarily complex, while n denotes the number of real components, we have $n = 6$. By appropriate choice of axes, η may be brought to a form in which only one of its three components is nonzero, and this one is real. Thus $\nu = 1$, corresponding to the fact that only one group invariant can be formed from $\langle\phi\rangle$. It follows that $r = 5$. Moreover, η is left invariant by a three-dimensional subgroup $SU(2)$ of $SU(3)$, so that $g - r = 3$.

It should be noted that when $\langle\phi\rangle = \eta$, the subgroup G^η of transformations which leave η unchanged is the subgroup of unbroken symmetry transformations. We have, therefore, r broken components of the symmetry, and $g - r$ unbroken components. It will be convenient in the following discussion to distinguish these components by different types of indices. We shall use labels $I, J, \ldots = 1, \ldots, r$ for the parameters μ of the coset space G/G^η, and $P, Q, \ldots = r + 1, \ldots, g$ for the parameters κ of the subgroup G^η of unbroken symmetry transformations. By definition, we have

$$T_P\eta = 0 \tag{4.83}$$

for these latter components. Thus $T_A\eta$ is nonzero only when A is one of the first r indices. Moreover, we can show that the $n \times r$ matrix

$$(T_I\eta)^a \tag{4.84}$$

has its maximal rank r. For if not, it has an eigenvector c^I with eigenvalue zero, so that we can find a linear combination of the generators such that

$$c^I T_I \eta = 0$$

But then this linear combination $c^I T_I$ would have to belong to the subalgebra corresponding to G^η, which by definition it does not.

Now, to obtain a first approximation to the solution of the field equations [Eqs. (4.81)] we may, as in the Abelian case, make the substitution

$$\phi = \eta + \phi' \tag{4.85}$$

and retain only linear terms in ϕ'. Defining the mass matrix

$$(m^2)_{ab} = \left(\frac{\partial^2 V}{\partial \phi^a \, \partial \phi^b}\right)_{\phi = \eta} \tag{4.86}$$

we then have

$$(-\partial^2 + m^2)\phi' = 0 \tag{4.87}$$

On grounds of stability it is reasonable to assume that m^2 has no negative eigenvalues. Moreover, since V is an invariant, it is a function of the ν invariants which can be constructed from ϕ. Therefore the rank of the matrix m^2 is at most equal to ν. It could happen in special cases that the rank was in fact less than ν, for example if V were actually independent of certain of the invariants. However, we shall assume that this is not the case, and that m^2 has its maximal rank ν. Thus we have ν massive fields, and $r = n - \nu$ massless ones, as one might expect from the fact that there are r broken-symmetry components.

As in the case of the group parameters, it is now convenient to distinguish two sets of indices among the components of ϕ^a. By an orthogonal transformation, we can arrange that m^2 consists of a nonsingular $\nu \times \nu$ matrix surrounded by zeros. To match the decomposition of the indices labeling the group parameters, we shall use $i, j, \ldots = 1, \ldots, r$ for the indices corresponding to the subspace annihilated by m^2, and $p, q, \ldots = r + 1, \ldots, n$ for the indices of the orthogonal ν-dimensional subspace on which m^2 is nonsingular.

Now, the invariance of V requires that

$$\frac{\partial V}{\partial \phi} T_A \phi = 0$$

Hence, differentiating with respect to ϕ and setting $\phi = \eta$ we obtain

$$m^2 T_A \eta = 0$$

Using the coordinates we have chosen this equation may be written as

$$(T_I \eta)^p = 0 \tag{4.88}$$

Thus only the first r components of $T_I\eta$ are different from zero. But, we showed earlier that the matrix (4.84) has rank r. Thus it follows that the $r \times r$ matrix

$$X_I{}^j = (T_I \eta)^j \tag{4.89}$$

is nonsingular. Moreover, from the antisymmetry of the matrices T_I it follows that

$$\eta T_I \eta = \eta_j X_I{}^j = 0$$

whence

$$\eta_j = 0 \tag{4.90}$$

Thus the only nonvanishing components of η are the ν components η^p in the subspace on which m^2 is nonsingular.

Let us now examine the effect of coupling the conserved currents $j_A{}^\mu$ to a set of g gauge vector fields $A_\mu{}^A$. We take in place of Eq. (4.78) the Lagrangian

$$\begin{aligned} L = -\tfrac{1}{2} F_A{}^{\mu\nu}(\partial_\mu A_\nu{}^A - \partial_\nu A_\mu{}^A - i t^A_{BC} A_\mu{}^B A_\nu{}^C) + \tfrac{1}{4} F_A{}^{\mu\nu} F^A_{\mu\nu} \\ + \phi^\mu(\partial_\mu - i A_\mu{}^A T_A)\phi + \tfrac{1}{2}\phi^\mu \phi_\mu - V(\phi) \end{aligned} \tag{4.91}$$

We now make the same linearizing approximation as before, writing $\phi = \eta + \phi'$ and retaining only linear terms in the equations of motion, which thus reduce to

$$\begin{aligned} F^A_{\mu\nu} &= \partial_\mu A_\nu{}^A - \partial_\nu A_\mu{}^A \\ \partial_\mu F^{\mu\nu}_A &= i\phi^\nu T_A \eta \\ \phi_\mu &= -\partial_\mu \phi' + i A_\mu{}^A T_A \eta \\ \partial_\mu \phi^\mu &= -m^2 \phi' \end{aligned}$$

with m^2 given as before by Eq. (4.86).

Using the fact that the only nonvanishing components of the matrix $(T_A\eta)^b$ are those of the nonsingular submatrix $X_I{}^j$, given by Eq. (4.89), we find that these equations separate into three distinct sets. For the vector fields $A_\mu{}^P$ corresponding to the $g - r$ unbroken components of the symmetry, we have simply the equations for free massless vector fields. Similarly, for the scalar fields ϕ^p corresponding to the subspace on which m^2 is nonsingular, we recover the corresponding members of the set of equations (4.87). Finally, for the first r indices, it is convenient to introduce in place of A^I and ϕ^i the new fields

$$V_\mu{}^I = A_\mu{}^I + i(X^{-1})_j{}^I\,\partial_\mu\phi^j$$

Then $\phi_\mu{}^j = X_I{}^j V_\mu{}^I$, and so the equations reduce to

$$\begin{aligned} F^I_{\mu\nu} &= \partial_\mu V_\nu{}^I - \partial_\nu V_\mu{}^I \\ \partial_\mu F_I{}^{\mu\nu} &= (M^2)_{IJ} V^{J\nu} \end{aligned} \tag{4.92}$$

where the vector-particle mass matrix

$$M^2 = -\,\tilde{X}X$$

is positive definite because of the nonsingularity of X. More explicitly, it is given by

$$(M^2)_{IJ} = -\,(T_I\eta)_j(T_J\eta)^j \tag{4.93}$$

We conclude therefore that at least in this linear approximation the effect of introducing vector gauge fields is to eliminate all the massless scalar particles from the theory. The r massless scalar fields originally present combine with r of the vector gauge fields to produce r massive vector fields, with masses given in terms of the symmetry-breaking parameters η by the eigenvalues of the mass matrix [Eq. (4.93)]. There remain in the theory $g - r$ massless vector fields, corresponding to the unbroken components of the symmetry group.

Finally, let us consider briefly the problem of going beyond the linear approximation. To do this, it is convenient, as in the Abelian gauge case, to adopt a polar decomposition of ϕ, analogous to Eq. (4.59). We write

$$\phi = e^{i\theta\cdot T}\rho \tag{4.94}$$

where ρ has only ν independent components ρ^p, while the r variables $\theta = (\theta^I)$ correspond to the parameters μ of the coset space G/G^η.

We also introduce new fields related to $A_\mu{}^A$ and $F_A{}^{\mu\nu}$ by an operator gauge transformation determined by the variables θ, namely

$$V_\mu{}^A = (e^{-i\theta\cdot t})_B{}^A A_\mu{}^B + \left(\frac{1 - e^{-i\theta\cdot t}}{i\theta\cdot t}\right)_I{}^A \partial_\mu\theta^I$$

and

$$G_A^{\mu\nu} = F_B^{\mu\nu}(e^{i\theta\cdot t})_A{}^B$$

Then, using the gauge invariance of the Lagrangian (4.91), we may write it as

$$
\begin{aligned}
L = &-\tfrac{1}{2}G_A^{\mu\nu}(\partial_\mu V_\nu{}^A - \partial_\nu V_\mu{}^A - it_{AC}^B V_\mu{}^B V_\nu{}^C) \\
&+ \tfrac{1}{4}G_A^{\mu\nu}G_{\mu\nu}^A + \rho^\mu(\partial_\mu - iV_\mu{}^A T_A)\rho \\
&- \sigma^\mu V_\mu{}^A T_A \rho + \tfrac{1}{2}\rho^\mu\rho_\mu + \tfrac{1}{2}\sigma^\mu\sigma_\mu - V(\rho)
\end{aligned}
$$

where we have written

$$
\begin{aligned}
\rho_p{}^\mu &= (\phi^\mu\, e^{\theta\cdot T})_p \\
\sigma_i{}^\mu &= (\phi^\mu\, e^{\theta\cdot T})_i
\end{aligned}
$$

We may then eliminate the fields σ^μ. Writing $\rho = \eta + \rho'$, we finally obtain

$$
\begin{aligned}
L = &-\tfrac{1}{2}G_A^{\mu\nu}(\partial_\mu V_\nu{}^A - \partial_\nu V_\mu{}^A - it_{BC}^A V_\mu{}^B V_\nu{}^C) \\
&+\tfrac{1}{4}G_A^{\mu\nu}G_{\mu\nu}^A + \rho^\mu\,\partial_\mu\rho' + \tfrac{1}{2}\rho^\mu\rho_\mu - V(\eta + \rho') \\
&- i\rho^\mu T_A\rho' V_\mu{}^A + \tfrac{1}{2}[V_\mu{}^A T_A(\eta + \rho')]_j\,[V^{B\mu}T_B(\eta + \rho')]^j
\end{aligned}
\tag{4.95}
$$

Note that although the term

$$- i\rho^\mu T_A \eta V_\mu{}^A$$

vanishes identically in virtue of Eqs. (4.83) and (4.88), it is not generally true that the corresponding term involving ρ', which we have included in Eq. (4.95), is necessarily zero.

The Lagrangian Eq. (4.95) clearly exhibits the same structure as its linearized approximation. There are ν massive scalar fields ρ', with masses given in lowest order by the mass matrix (4.86). Of the vector fields, r have a bare mass term in the Lagrangian, given by Eq. (4.93), while the remaining $g - r$ have only cubic or quartic interaction terms. There is, however, no guarantee in this case that the masses will remain unrenormalized, and it is quite conceivable that the physical particles corresponding to even these fields could be massive (just as there is no known requirement which forces the photon to have zero mass).

We see, therefore, that whether the broken symmetry group is Abelian or non-Abelian, the introduction of gauge vector fields serves to remove all the unwanted massless scalar particles from the theory. There remain ν massive scalar and r massive vector fields, and also a number $g - r$ of massless vector fields, equal to the dimensionality of the subgroup of unbroken symmetry transformations. In particular, no such particles are left if all the components of the symmetry group are broken.

V. Nonrelativistic Broken Symmetries

Broken symmetry theories (though not always recognized as such) have played a major role in our understanding of many phenomena in nonrelativistic physics, particularly ferromagnetism, superconductivity, and superfluidity. These theories can often be studied with much greater mathematical rigor than relativistic ones and, moreover, the physical significance of the mathematical formalism is often much clearer. It is therefore particularly valuable to study nonrelativistic broken-symmetry theories.

It will be useful to begin with some general remarks applicable to all broken-symmetry theories and later to concentrate on particular cases. We are in each case concerned with a theory whose basic equations of motion and commutation relations are invariant under some given group of transformations of the fundamental dynamical variables. So long as the total volume Ω of the system is finite, these transformations are induced by unitary operators of the form

$$U_\Omega(\eta) = \exp(i\eta Q_\Omega)$$

where the generator Q_Ω may be expressed either as an integral

$$Q_\Omega = \int_\Omega d^3x j^0(\mathbf{x}, t) \tag{5.1}$$

or a sum over lattice sites

$$Q_\Omega = \sum_{\mathbf{x}\in\Omega} j_{\mathbf{x}}^0(t) \tag{5.2}$$

where j^0 is an appropriately defined density. In either case it satisfies a conservation law

$$\frac{d}{dt} Q_\Omega = 0 \tag{5.3}$$

or, provided that Q_Ω is not explicitly time dependent,

$$[Q_\Omega, H_\Omega] = 0 \tag{5.4}$$

where H_Ω is the Hamiltonian in volume Ω. The only case we shall be concerned with in which Eq. (5.4) does not hold is that of Galilean transformations. For the moment, however, we assume that Eq. (5.4) is satisfied.

We say that the symmetry is broken if, in the limit of infinite volume, the ground state is not itself invariant under these transformations. We may distinguish two classes of broken-symmetry theories. In the simpler case, of which the isotropic ferromagnet is a good example, the ground state $|0\rangle_\Omega$ in a finite volume Ω is already nonsymmetric. Then there must exist a set of states

$$|\eta\rangle_\Omega = U_\Omega{}^+(\eta)|0\rangle_\Omega \tag{5.5}$$

which, because of Eq. (5.4), are degenerate in energy with the ground state. However it may also happen that the ground state is completely symmetric in any finite volume, but that in the limit of infinite volume there is a degenerate set of ground states. We defer consideration of this case for the moment.

To show that the ground state $|0\rangle_\Omega$ is not invariant it is sufficient to find some operator A whose ground-state expectation value is not invariant,

$${}_\Omega\langle 0|U_\Omega(\eta)AU_\Omega{}^+(\eta)|0\rangle_\Omega \neq {}_\Omega\langle 0|A|0\rangle_\Omega$$

or, in infinitesimal form,

$$-i_\Omega\langle 0|[A, Q_\Omega]|0\rangle_\Omega = \tilde{\eta} \neq 0 \tag{5.6}$$

In the ground state of a ferromagnet, for example, all the spins are aligned parallel to some given direction. The degenerate ground states, obtained by spin rotations, are characterized by different orientations of the overall magnetization vector. Here the operator A may be chosen, for example, to be a component of the spin operator at a particular lattice site.

In any finite volume Ω the degenerate ground states $|0\rangle_\Omega$ are related by unitary transformations; however, it is easy to see that in the limit of infinite volume this can no longer be true. Suppose that there exist limiting states $|\eta\rangle$, related by unitary transformations,

$$|\eta\rangle = e^{i\eta Q}|0\rangle$$

with

$$Q = \int d^3x j^0(\mathbf{x}, t)$$

and that these states belong to the domain of definition of the operator Q. Then the scalar product $\langle 0|\eta\rangle$ must be a continuous, differentiable function of η. But

$$\begin{aligned}\frac{d}{d\eta}\,{}_\Omega\langle 0|\eta\rangle_\Omega &= -i_\Omega\langle 0|Q_\Omega|\eta\rangle_\Omega \\ &= -i\Omega_\Omega\langle 0|j^0(\mathbf{0}, t)|\eta\rangle_\Omega\end{aligned}$$

by translational invariance. In the limit $\Omega \to \infty$ this expression in either zero (in which case all the states are identical) or infinite, which is impossible. What happens, in fact, is that the scalar products ${}_\Omega\langle 0|\eta\rangle_\Omega$ tend to zero as $\Omega \to \infty$. (We shall verify this statement in detail for specific examples later.) The limiting states $|\eta\rangle$ therefore constitute an uncountably infinite set of orthonormal states, which evidently cannot all belong to a single separable Hilbert space. Each of them actually belongs to a distinct unitarily inequivalent representation of the canonical commutation relations.

Let us now turn to the second class of broken-symmetry theories. These have the property that in any finite volume the degeneracy of the ground state is only approximate. That is to say, there is a unique nondegenerate ground state, but there are other states with only slightly larger energy which become degenerate with it in the limit of infinite volume. For example, the state of a crystal whose center of mass is localized to within some distance R of the origin has slightly greater energy than the true ground state, but in the limit of infinite volume keeping the density fixed this energy difference tends to zero. Thus an infinite crystal has translationally noninvariant states degenerate with the ground state. Another example is provided by the condensed Bose gas. In any finite volume there is a unique nondegenerate ground state for any given particle number N. However in the limit of infinite volume keeping the density fixed, we find a degenerate set of ground states which may be labeled by the phase of the wave function. These states may be obtained as limits of finite volume states in which the total particle number is somewhat uncertain. However in the limit of infinite volume the particle density tends to a definite limit.

A very convenient way to treat such problems [see Bogoliubov[48]] is to add a small symmetry-breaking perturbation νH_1 to the Hamiltonian. [A recent discussion from a point of view similar to that adopted here has been given by Wagner.[49]] This is useful even in the first class of broken-symmetry theories if we want to discuss the effects of finite temperature. In a ferromagnet, for example, at finite temperature we should normally find a uniform population of all the degenerate ground states. It is, however, much more convenient for many purposes to deal with a state in which there is a definite magnetization direction. To achieve this we add a small external magnetic field. We then compute expectation values in an appropriate ensemble for volume Ω,

$$\langle A\rangle_{\nu,\Omega} = \frac{\mathrm{tr}_\Omega\, A e^{-\beta(H+\nu H_1)}}{\mathrm{tr}_\Omega\, e^{-\beta(H+\nu H_1)}} \tag{5.7}$$

where tr_Ω denotes the trace over the Hilbert space of states of the system in volume Ω. Then we take the limit of infinite volume, and finally let $\nu \to 0$, obtaining

$$\langle A \rangle = \lim_{\nu\to 0} \lim_{\Omega\to\infty} \langle A \rangle_{\nu,\Omega} \tag{5.8}$$

It is characteristic of broken symmetry theories that the order of these limits is not reversible. Indeed from the present point of view this may be taken as the definition of a broken symmetry. If we set $\nu = 0$ first we obtain for any Ω an ensemble in which all the degenerate ground states are equally populated. Thus the expectation values are fully symmetric under the transformations induced by the unitary operators $U_\Omega(\eta)$. This symmetry is naturally preserved in the infinite-volume limit. On the other hand, if we retain the symmetry-breaking interaction we find that for large Ω the energy differences it produces between the otherwise degenerate (or nearly degenerate) states are roughly proportional to Ω. Thus for large Ω one particular member of the degenerate set will be much more heavily populated than any other. In the limit $\Omega \to \infty$ we select a single state. Thus the resulting expectation values will not be invariant under the transformations, and this noninvariance will persist even when we finally set $\nu = 0$.

This method can, of course, be used even at zero temperature. We introduce the interaction νH_1 which breaks the degeneracy of the ground states, or which artificially depresses the energy of the state we wish to select below that of the true ground state. Then in the limit of infinite volume a single member of the set of degenerate states is selected.

A very important question is the following. Are these various degenerate ground states in the infinite-volume limit *physically* equivalent or not? In other words, can we make a physical measurement which will distinguish between them? The answer to this question is closely bound up with measurement theory, and in fact depends on our choice of operators to represent observables. If we require that all observables should themselves be invariant under the group, then the answer must be that they are indistinguishable. But if we allow interactions with measuring apparatus which break the symmetry, then of course we can distinguish them. For example, it is usual to admit as observables operators which are not translationally invariant, but localized in some region of space. Naturally, the interaction with a measuring apparatus designed to measure such a quantity breaks the translational symmetry of the system, although viewed by an observer external to both system and apparatus there is complete translational symmetry. What we are measuring is of course position relative to the apparatus. In such a case, states distinguished only by the position of the center of mass are distinguishable. Similar remarks apply to the measurement of the magnetization direction of a ferromagnet and, less obviously perhaps, to the phase of a superfluid or superconductor. None of these quantities has any absolute significance, but all can be measured relative to an externally defined standard, provided that we allow interactions which break the appropriate symmetry. (The Josephson effect provides a possible realization of such a measuring device for a superconductor.)

It should be noted that if we allow symmetry-breaking interactions with external systems then these same interactions may be identified with the νH_1 above. An interaction which permits an observable distinction to be drawn between the degenerate ground states also provides a mechanism which can physically determine which ground state is selected by the system. Thus, for example, an external magnetic field can be used to measure the magnetization direction in a ferromagnet, and also to determine in advance along which direction the spins will align themselves. If such interactions are allowed, then the addition of a small symmetry-breaking interaction νH_1 is more than a mathematical device. It is a correct description of physical reality.

Now let us examine the application of the Goldstone theorem to nonrelativistic broken-symmetry theories. The theorem states that if there are no long-range interactions, then such a theory must possess excitation modes whose frequency tends to zero as $\mathbf{k} \to \mathbf{0}$. We shall follow in outline the method of Lange.[50] [See also Kibble.[51]]

We consider a particular "ground state" $|0\rangle_\Omega$, which need not in fact be the state of lowest energy, except in the infinite-volume limit. A similar discussion applies if we replace the expectation value in this state by an ensemble average of appropriate type.

We define the quantity

$$f_\Omega(\mathbf{k}, \omega) = -i \int_\Omega d^3x \int dt\, e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\, {}_\Omega\langle 0|[A, j^0(\mathbf{x}, t)]|0\rangle_\Omega \qquad (5.9)$$

and its infinite-volume limit,

$$f(\mathbf{k}, \omega) = \lim_{\Omega\to\infty} f_\Omega(\mathbf{k}, \omega)$$

where A is the operator in Eq. (5.6) whose ground-state expectation value is noninvariant. We assume that this operator is a quasi-local function of the dynamical variables at time $t = 0$. Let us suppose first that it is a completely local function, localized at the origin. Then Eq. (5.6) with the locality of the commutation relations would imply that

$$-i_{\Omega}\langle 0|[A, j^0(\mathbf{x}, 0)]|0\rangle_{\Omega} = \tilde{\eta}\delta(\mathbf{x}) \tag{5.10}$$

whence we obtain a sum rule

$$\int \frac{d\omega}{2\pi} f_{\Omega}(\mathbf{k}, \omega) = \tilde{\eta} \tag{5.11}$$

valid for all values of $\mathbf{k}$.

If A is only approximately local, but depends on the field operators within some finite volume V_0, then the commutator in Eq. (5.10) is not proportional to a delta function. Nevertheless we still have

$$-i \int_V d^3x_{\Omega}\langle 0|[A, j^0(\mathbf{x}, 0)]|0\rangle_{\Omega} = \tilde{\eta} \tag{5.12}$$

for any volume V containing V_0. In this case the integral in Eq. (5.11) is no longer independent of $\mathbf{k}$. However in the limit $\Omega \to \infty$ it is a continuous (indeed, entire) function of $\mathbf{k}$ satisfying the limiting condition

$$\lim_{\mathbf{k}\to 0} \int \frac{d\omega}{2\pi} f(\mathbf{k}, \omega) = \tilde{\eta} \tag{5.13}$$

which is in fact all we require. This is equivalent to the condition

$$\lim_{V\to\infty} \lim_{\Omega\to\infty} -i \int_V d^3x_{\Omega}\langle 0|[A, j^0(\mathbf{x}, 0)]|0\rangle_{\Omega} = \tilde{\eta} \tag{5.14}$$

This is the broken-symmetry condition in its weakest form and is sufficient to prove Eq. (5.13).

Equation (5.13), obtained by integrating over a large volume V and taking the limit $V \to \infty$, should be carefully distinguished from the condition which results from integrating over the entire volume Ω. Because of Eq. (5.3) this yields a time-independent result

$$-i \int_{\Omega} d^3x_{\Omega}\langle 0|[A, j^0(\mathbf{x}, t)]|0\rangle_{\Omega} = \tilde{\eta}$$

It follows that

$$\int \frac{d\omega}{2\pi} e^{-i\omega t} f_{\Omega}(\mathbf{0}, \omega) = \tilde{\eta}$$

whence

$$f_{\Omega}(\mathbf{0}, \omega) = 2\pi\tilde{\eta}\delta(\omega) \tag{5.15}$$

It is clear from the definition [Eq. (5.9)] that whenever $f_\Omega(\mathbf{k}, \omega) \neq 0$ there must be intermediate states of energy ω and momentum $\mathbf{k}$ (relative to the ground-state energy and momentum) which couple to the ground state via the operator A, and also via j^0. Thus from Eq. (5.15) we can conclude that for $\mathbf{k} = \mathbf{0}$ there are such states with $\omega = 0$. These are, however, simply the degenerate ground states $|\eta\rangle_\Omega$, and the fact that the energy of these "excitations" vanishes is simply a restatement of the broken-symmetry condition.

For example, for a ferromagnet the condition (5.15) tells us that no energy is required to rotate all the spins together. Clearly the rotated ground states may be regarded as formed from the original ground state by adding a number of $\mathbf{k} = \mathbf{0}$ spin waves. However the fact that $\omega = 0$ for these modes in fact tells us nothing about the physically interesting question of the frequency spectrum of long-wavelength spin waves.

These degenerate ground states may be eliminated from the problem by considering the limit of infinite total volume, $\Omega \to \infty$. For it is easy to see that they contribute to the sum over intermediate states in Eq. (5.9) only for the isolated value $\mathbf{k} = \mathbf{0}$. (This argument assumes translational invariance and requires modification when it is the translational symmetry itself that is being broken.) This contribution is proportional to a Kronecker delta $\delta_{\mathbf{k}0}$ rather than to a Dirac delta function. Thus in Eq. (5.10) or Eq. (5.12) their contribution falls off like Ω^{-1} and vanishes in the limit $\Omega \to \infty$ [see Lange[50]].

In fact, in the infinite volume limit the degenerate ground states cannot be coupled to each other by any operator A, since they belong to quite distinct Hilbert spaces. Physically, in a ferromagnet for example, this is an expression of the fact that to go from one ground state to another in the case of infinite volume requires the flipping of an infinite number of spins. This cannot be achieved by operating with any finite operator.

Thus if we let $\Omega \to \infty$ then the condition (5.13) gives us genuine information about the frequency spectrum of long-wavelength excitations. The degenerate ground states are irrelevant. Let us then consider the expression (5.14). If we could show that it was time independent, then we could prove not only Eq. (5.15), but also the much more interesting result

$$\lim_{\mathbf{k}\to\mathbf{0}} f(\mathbf{k}, \omega) = 2\pi\bar{\eta}\delta(\omega) \tag{5.16}$$

This is the Goldstone theorem. It shows that there are excitations for which $\omega \to 0$ as $\mathbf{k} \to \mathbf{0}$.

To do this [see Lange[50]] it is sufficient to show that for all n

$$\lim_{\mathbf{k}\to\mathbf{0}} \int \frac{d\omega}{2\pi}\, \omega^n f(\mathbf{k}, \omega) = \tilde{\eta}\delta_{n0} \tag{5.17}$$

For $n = 0$ we already have the required result in Eq. (5.13). In many cases one can prove that f (or some closely related function) must be positive. Then it is sufficient to prove Eq. (5.17) for one other value of n. For $n > 0$ it is equivalent to

$$i^{n+1} \lim_{V\to\infty} \int_V d^3x \langle 0| \left[A, \frac{d^n j^0}{dt^n}(\mathbf{x}, 0)\right] |0\rangle = 0 \tag{5.18}$$

Now corresponding to the global conservation law [Eq. (5.3)] there must be a microscopic conservation law expressible in the form

$$\frac{d}{dt} \int_V d^3x j^0(\mathbf{x}, t) = -S_V(t) \tag{5.19}$$

where S_V is a surface integral over the bounding surface of V, or at any rate has contributions only from a small region close to this surface. Thus it is sufficient to show that

$$\begin{aligned} \lim_{V\to\infty} \langle 0|[A, S_V(0)]|0\rangle &= 0 \\ \lim_{V\to\infty} \langle 0|[A, [H, S_V(0)]]|0\rangle &= 0 \end{aligned} \tag{5.20}$$

and so on. Often (as noted above) it is enough to prove one or two of these relations. The rest follow by positivity of the spectral function.

Since A is assumed localized at or near the origin, and S_V is localized on or near the boundary of V, the locality of the equal-time commutation relations will normally ensure that the first of these equations [Eqs. (5.20)] is satisfied. Whether the remaining equations are satisfied or not depends on the range of the interactions. If the interactions are of any given finite range R, then the commutator of H with any given local function will depend on field operators only within a distance R of the original point of localization. In that case it is clear that for every value of n we can choose a volume large enough to make the multiple commutator vanish. In that case we have therefore established Eq. (5.16). An only slightly more complex argument serves to prove the same result for interactions which fall off exponentially with distance. However for those which fall off as an inverse power one must look at the magnitudes of the quantities involved in more detail. For this case a general discussion is rather difficult since as we shall see later the limiting power of r is not the same in all cases but depends on the type of system being considered.

Let us then recapitulate the theorem briefly. In any broken symmetry theory with only short-range interactions there must be Goldstone excitations, whose frequency vanishes continuously ($\omega \to 0$) in the long-wavelength limit ($\mathbf{k} \to \mathbf{0}$). These may be pictured as space-dependent oscillations in the parameter which distinguishes the various degenerate ground states. (Recall that for finite volume the $\mathbf{k} = \mathbf{0}$ modes correspond to a constant change in this parameter.) However, when long-range interactions are present, the theorem does not follow, and indeed in practice, the introduction of long-range forces often serves to eliminate these zero-energy-gap modes. What precisely is meant by "long-range" and "short-range" is a question which can only be answered in the context of more specific theories.

Before turning to the discussion of the various examples of physical interest it may be helpful to consider a simple soluble model which serves to illustrate some of these ideas [see Kibble(51)]. This is, in essence, a nonrelativistic form of the Boulware-Gilbert model discussed in the preceding section, in which the electromagnetic field has been replaced by an instantaneous interaction. The model is described by the Hamiltonian

$$H = \tfrac{1}{2} \int d^3x[\pi^2 + (\nabla\phi)^2] + \tfrac{1}{2} \int d^3x \int d^3y \pi(\mathbf{x}) V(\mathbf{x} - \mathbf{y}) \pi(\mathbf{y}) \quad (5.21)$$

along with the canonical commutation relation

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y}) \quad (5.22)$$

It is invariant under the field translations

$$\phi(\mathbf{x}) \to \phi(\mathbf{x}) + \eta \quad (5.23)$$

which are generated in any volume Ω by the unitary operators

$$\exp\left[-i\eta \int_\Omega d^3x \pi(\mathbf{x})\right] \quad (5.24)$$

However, it is clear that the expectation value

$$\langle\phi\rangle = \langle 0|\phi(\mathbf{x}, t)|0\rangle \quad (5.25)$$

cannot be invariant under (5.23). We therefore have a broken symmetry of what we termed the "naturally occurring" type. There is a degenerate set of ground states labeled by different values of $\langle\phi\rangle$.

For this model $j^0 = -\pi$, and there is a microscopic conservation law of the form

$$-\dot{\pi} + \nabla\cdot(\nabla\phi) = 0 \tag{5.26}$$

so that the surface contribution defined in Eq. (5.15) is

$$S_V = \int_{\sigma(V)} d\boldsymbol{\sigma}\cdot\nabla\phi \tag{5.27}$$

integrated over the boundary $\sigma(V)$ of V.

Equation (5.8) is, in this case, obtained simply by taking the expectation value of the canonical commutation relation (5.22). It is easy starting from this equation to follow through the discussion of the Goldstone theorem in detail. Instead of doing this, however, we shall simply solve the model and verify its conclusions.

Since we are interested in the limit of infinite volume, $\Omega \to \infty$, it will be convenient always to define Fourier components by

$$\phi_{\mathbf{k}} = \int_{\Omega} d^3x\phi(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{5.28}$$

so that

$$\phi(\mathbf{x}) = \frac{1}{\Omega}\sum_{\mathbf{k}} \phi_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \to \int \frac{d^3k}{(2\pi)^3}\,\phi_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

Our Hamiltonian is then

$$H = \frac{1}{2\Omega}\sum_{\mathbf{k}} \{\mathbf{k}^2\phi_{\mathbf{k}}{}^*\phi_{\mathbf{k}} + [1 + V_{\mathbf{k}}]\pi_{\mathbf{k}}{}^*\pi_{\mathbf{k}}\} \tag{5.29}$$

The transformations (5.23) affect only the $\mathbf{k} = \mathbf{0}$ component:

$$\phi_{\mathbf{k}} \to \phi_{\mathbf{k}} + \eta\Omega\delta_{\mathbf{k},0}$$

From the equations of motion

$$\dot{\pi}_{\mathbf{k}} = -\mathbf{k}^2\phi_{\mathbf{k}}$$
$$\dot{\phi}_{\mathbf{k}} = [1 + V_{\mathbf{k}}]\pi_{\mathbf{k}}$$

it is clear that the excitation spectrum is

$$\omega^2 = \mathbf{k}^2[1 + V_{\mathbf{k}}] \tag{5.30}$$

If Ω is finite then V_0 is finite. (We assume that the singularity at $\mathbf{x} = \mathbf{0}$, if any, is integrable.) Thus it follows that there is always a mode with $\omega = 0$ at $\mathbf{k} = \mathbf{0}$, in agreement with our earlier conclusion. In this mode $\pi_0 = 0$, so that the mode consists merely of a constant change in ϕ, which is of course a transformation from one degenerate ground state to another.

What is really interesting, however, is not this single mode but the behavior of the modes near $\mathbf{k} = \mathbf{0}$ in the infinite volume limit. The condition that $\omega \to 0$ as $\mathbf{k} \to \mathbf{0}$ is clearly

$$\lim_{\mathbf{k}\to\mathbf{0}} \mathbf{k}^2 V_{\mathbf{k}} = 0 \tag{5.31}$$

which is satisfied if the potential falls off at large distances faster than $1/r$. For a Coulomb potential, however,

$$V(\mathbf{x}) = g^2/4\pi|\mathbf{x}|$$

we have

$$V_{\mathbf{k}} = g^2/\mathbf{k}^2$$

and therefore

$$\lim_{\mathbf{k}\to\mathbf{0}} \omega^2(\mathbf{k}) = g^2 \tag{5.32}$$

Thus we have verified the general conclusions reached in the discussion of the Goldstone theorem. When only short-range interactions are admitted, the frequency spectrum tends to zero in the limit of zero wave number. However in the presence of long-range interactions (meaning in this case a $1/r$ potential) there is a finite energy gap.

It is also worth noting that for stability the potential must satisfy the inequality

$$1 + V_{\mathbf{k}} \geq 0 \tag{5.33}$$

for all values of $\mathbf{k}$. This means, in effect, that the potential must be repulsive or at least not too strongly attractive. For example, if we choose a Yukawa potential

$$V(\mathbf{x}) = \gamma e^{-\mu|\mathbf{x}|}/4\pi|\mathbf{x}|$$

then condition (5.33) becomes

$$\gamma \geq -\mu^2 \tag{5.34}$$

It is interesting to note that in the limiting case $\gamma = -\mu^2$ we no longer have $\omega \sim k$ near $\mathbf{k} = \mathbf{0}$ but rather $\omega \sim k^2$. This is an illustration of the fact that although the Goldstone theorem can guarantee for us the existence of modes whose frequency tends to zero, it is not in itself sufficient to tell us the details of the frequency spectrum.

Now let us turn to what is in many ways the simplest of the nonrelativistic broken-symmetry theories—the isotropic ferromagnet. This is the system which was chosen by Lange[50] to exemplify his discussion of the nonrelativistic Goldstone theorem. We shall use subscripts to label the lattice sites, and superscripts to denote vector components. In total volume Ω the Hamiltonian is

$$H = -\tfrac{1}{2} \sum_{\mathbf{x},\mathbf{y}\in\Omega} J_{\mathbf{x}-\mathbf{y}} \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}} \tag{5.35}$$

The commutation relations are

$$[S_{\mathbf{x}}{}^{i}(t), S_{\mathbf{y}}{}^{j}(t)] = i\delta_{\mathbf{x},\mathbf{y}}\epsilon^{ijk} S_{\mathbf{x}}{}^{k}(t) \tag{5.36}$$

This model is invariant under the spin rotations generated by the unitary operators

$$U_\Omega(\boldsymbol{\theta}) = \exp(i\boldsymbol{\theta}\cdot\mathbf{S})$$

where $\mathbf{S}$ is the total spin

$$\mathbf{S} = \sum_{\mathbf{x}\in\Omega} \mathbf{S}_{\mathbf{x}} \tag{5.37}$$

However the ground state is clearly not invariant under these transformations. The broken-symmetry condition is expressed by the fact that there is a nonvanishing total magnetization, so that

$$\langle \mathbf{S}_{\mathbf{x}} \rangle = {}_{\Omega}\langle 0|\mathbf{S}_{\mathbf{x}}(t)|0\rangle_{\Omega} = \mathbf{s} \neq \mathbf{0} \tag{5.38}$$

(The antiferromagnet requires a different treatment.) The other ground states degenerate with $|0\rangle_\Omega$ are obtained by spin rotations,

$$|\boldsymbol{\theta}\rangle_\Omega = U_\Omega(\boldsymbol{\theta})|0\rangle_\Omega \tag{5.39}$$

They are characterized by different orientations of $\mathbf{s}$ (whose length is of course fixed).

If we choose the direction of $\mathbf{s}$ to be the z axis then the ground state is the one in which every spin has its maximum z component, so that

$$S_{\mathbf{x}}{}^{+}|0\rangle_\Omega = 0 \tag{5.40}$$

where

$$S_{\mathbf{x}}{}^{+} = S_{\mathbf{x}}{}^{x} + iS_{\mathbf{x}}{}^{y} \tag{5.41}$$

The corresponding ground-state energy is

$$E = {}_\Omega\langle 0|H|0\rangle_\Omega = -\tfrac{1}{2}s^2 \sum_{\mathbf{x},\mathbf{y}} J_{\mathbf{x}-\mathbf{y}} = -\tfrac{1}{2}Ns^2\tilde{J}_0 \tag{5.42}$$

where N is the total number of lattice sites and $\tilde{J}$ is the Fourier transform of J,

$$\tilde{J}_{\mathbf{k}} = \sum_{\mathbf{x}\in\Omega} J_{\mathbf{x}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{5.43}$$

Now let us consider the limit of infinite volume $\Omega \to \infty$. We wish to show that in this limit the degenerate ground states are no longer related by unitary operators and in fact belong to distinct Hilbert spaces. To this end we evaluate the scalar product

$${}_\Omega\langle 0|\boldsymbol{\theta}\rangle_\Omega = {}_\Omega\langle 0|\, e^{i\boldsymbol{\theta}\cdot\mathbf{S}}|0\rangle_\Omega \tag{5.44}$$

We have here the same factor for each lattice site. If we use the Euler-angle representation (ϕ, θ, ψ) for $\boldsymbol{\theta}$, this factor is

$$D^s_{ss}(\phi, \theta, \psi) = e^{is\phi} \cos^{2s} \tfrac{1}{2}\theta\, e^{is\psi}$$

Hence

$${}_\Omega\langle 0|\phi\theta\psi\rangle_\Omega = [e^{is(\phi+\psi)} \cos^{2s} \tfrac{1}{2}\theta]^N \tag{5.45}$$

which tends to zero as $N \to \infty$, unless $\theta = 0$. It follows immediately that in this limit the various ground states belong to different (orthogonal) Hilbert spaces, since they form an uncountably infinite set of orthonormal states and therefore cannot belong to a single separable Hilbert space.

From the Hamiltonian (5.35) and the commutation relations (5.36) we obtain the equations of motion

$$\dot{\mathbf{S}}_{\mathbf{x}} = \sum_{\mathbf{y}\in\Omega} J_{\mathbf{x}-\mathbf{y}}\mathbf{S}_{\mathbf{x}} \times \mathbf{S}_{\mathbf{y}} \tag{5.46}$$

In this particular case the densities corresponding to the group generators are in fact identical with the basic dynamical variables. Thus Eq. (5.46) plays a dual role. It is at the same time the fundamental equation of motion and also the microscopic conservation law. That it has the appropriate structure may be seen by summing over a finite volume V. We then obtain

$$\frac{d}{dt}\sum_{\mathbf{x}\in V} \mathbf{S}_{\mathbf{x}} = \sum_{\mathbf{x}\in V}\ \sum_{\mathbf{y}\in\Omega-V} J_{\mathbf{x}-\mathbf{y}} S_{\mathbf{x}} \times \mathbf{S}_{\mathbf{y}} \tag{5.47}$$

The right-hand side of this equation is the surface term. Clearly if the interactions have a finite range R, which means that

$$J_{\mathbf{x}-\mathbf{y}} = 0 \qquad \text{if} \qquad |\mathbf{x} - \mathbf{y}| > R \tag{5.48}$$

then it has contributions only from points within a distance R of the boundary of V.

To discuss the application of the Goldstone theorem to this particular case we may define, as in Eq. (5.9), the expression

$$f_\Omega{}^{ij}(\mathbf{k}, \omega) = -i \sum_{\mathbf{x}\in\Omega} \int dt\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}{}_\Omega\langle 0|[S_0{}^i(0), S_\mathbf{x}{}^j(t)]|0\rangle_\Omega$$

The equal-time commutation relations (5.36), with the broken-symmetry condition (5.38), then yield the sum rule

$$\int \frac{d\omega}{2\pi} f_\Omega{}^{ij}(\mathbf{k}, \omega) = \epsilon^{ijk} s^k \tag{5.49}$$

as the analog of Eq. (5.11). Moreover the conservation of total spin yields

$$f_\Omega{}^{ij}(\mathbf{0}, \omega) = 2\pi\epsilon^{ijk} s^k \delta(\omega) \tag{5.50}$$

Like Eq. (5.15) this relation refers to transitions between different degenerate ground states. However if the forces are short-range then we can also prove that

$$\lim_{\mathbf{k}\to\mathbf{0}} f^{ij}(\mathbf{k}, \omega) = 2\pi\epsilon^{ijk} s^k \delta(\omega) \tag{5.51}$$

which shows that the frequency of the spin waves tends to zero as $\mathbf{k} \to \mathbf{0}$. [This follows rigorously from the positivity of if^{+-} with Eq. (5.52) below.]

It is interesting to verify this (well-known) conclusion directly, and also to investigate precisely what "short-range" means in this context. Let us for convenience take the z axis in the direction of s. Using Eq. (5.41), we obtain from Eq. (5.46) the equation of motion

$$i\dot{S}_\mathbf{x}{}^+ = \sum_{\mathbf{y}\in\Omega} J_{\mathbf{x}-\mathbf{y}}(S_\mathbf{x}{}^+ S_\mathbf{y}{}^z - S_\mathbf{x}{}^z S_\mathbf{y}{}^+) \tag{5.52}$$

Now to determine the spin-wave frequency spectrum let us linearize this equation by replacing S^z by s. (Actually this linearization is unnecessary if we confine our attention to the states with single spin waves, which are in fact exact eigenstates of the Hamiltonian; however, it will simplify the discussion.) Then taking the Fourier transform we find for the frequency spectrum of S^+ the equation

$$\begin{aligned} \omega &= s \sum_{\mathbf{x}\in\Omega} J_{\mathbf{x}}(1 - e^{-i\mathbf{k}\cdot\mathbf{x}}) \\ &= s[\tilde{J}_{\mathbf{0}} - \tilde{J}_{\mathbf{k}}] \end{aligned} \tag{5.53}$$

When Ω is finite, it is immediately obvious that $\omega = 0$ for the isolated mode at $\mathbf{k} = \mathbf{0}$ which corresponds to a transition between one degenerate ground state and another. In the limit $\Omega \to \infty$ it is necessary to restrict $J_{\mathbf{x}}$ by the condition that $\tilde{J}_{\mathbf{0}}$ be finite, which means that $J_{\mathbf{x}}$ must decrease at infinity at least like $1/r^{3+\epsilon}$. (Conventionally, of course, one considers only nearest-neighbor interactions, so that $J_{\mathbf{x}}$ is of strictly finite range.) Provided this condition is satisfied, it immediately follows that $\omega \to 0$ in the limit $\mathbf{k} \to \mathbf{0}$. Thus the frequency of the spin waves indeed tends to zero as the wave number tends to zero.

In the context of this model, therefore, the interaction is "short-range" if $J_{\mathbf{x}}$ behaves better than $1/r^3$ at large distances. In contrast to the previously considered example, the limiting case in which $J_{\mathbf{x}}$ behaves like $1/r^3$ does not lead to a finite frequency limit as $\mathbf{k} \to \mathbf{0}$, but rather to a more pathological theory in which ω is infinite for all values of $\mathbf{k}$. One could achieve a finite, but directionally dependent, limiting value of ω by allowing a directional dependence in $J_{\mathbf{x}}$. For example, if $J_{\mathbf{x}}$ behaves at large distances like $P_2(\cos\theta)/r^3$, then near $\mathbf{k} = \mathbf{0}$, ω is proportional to $P_2(\cos\theta)$. However, in all such cases the limiting value of ω is negative in some directions, which means that the ground state is not truly the state of lowest energy.

The stability criteria are also somewhat different in this model. From Eq. (5.40) it is clear that S^+ is the *annihilation* operator for spin waves. It should therefore have positive frequency, and indeed if it does not then there must exist states to which it can couple with energy lower than the ground state. Thus the stability condition is that

$$\tilde{J}_{\mathbf{0}} - \tilde{J}_{\mathbf{k}} \geq 0 \tag{5.54}$$

for all $\mathbf{k}$. This is satisfied in particular if

$$J_{\mathbf{x}} \geq 0$$

for all $\mathbf{x}$. (Note, however, that this latter condition is not necessary.)

In this model, violation of the stability condition (5.47) does not lead to ω becoming complex as it did in our previous model. If, for some specific value of $\mathbf{k}$, $\tilde{J}_{\mathbf{k}} > \tilde{J}_0$, then the addition of spin waves with this wave number will lower the energy. Hence the true ground state will be obtained essentially by adding in as many such spin waves as possible. The most important case is that of antiferromagnetism. Here, for a simple cubic lattice of lattice constant a the relevant vector is the one with all three components equal to π/a. Addition of as many spin waves as possible with this wave number is equivalent to flipping over every alternate spin.

Now we turn to what is in many ways the most interesting of all nonrelativistic broken-symmetry theories—the Bose gas.

This system is described by the Hamiltonian

$$H = \frac{1}{2m}\int_\Omega d^3x\nabla\psi^*\cdot\nabla\psi + U\int_\Omega d^3x\psi^*\psi + \tfrac{1}{2}\int_\Omega d^3x\int_\Omega d^3y\psi^*(\mathbf{x})\psi^*(\mathbf{y})V(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x}) \quad (5.55)$$

where U is a constant external potential introduced mainly to allow us to treat the case of Coulomb interactions, with U representing a uniform distribution of opposite charge. The canonical commutation relation is

$$[\psi(\mathbf{x}, t), \psi^*(\mathbf{y}, t)] = \delta(\mathbf{x}-\mathbf{y}) \quad (5.56)$$

This model is invariant under several different groups of transformations. Those we shall be concerned with are the following:

a. Phase transformations

$$\psi(\mathbf{x}) \to \psi(\mathbf{x})\, e^{i\lambda} \qquad \psi_{\mathbf{k}} \to \psi_{\mathbf{k}}\, e^{i\lambda} \quad (5.57)$$

These are induced by the unitary operators $e^{i\lambda N}$ with

$$N = \int_\Omega d^3x\psi^*(\mathbf{x})\psi(\mathbf{x}) = \frac{1}{\Omega}\sum_{\mathbf{k}}\psi_{\mathbf{k}}{}^*\psi_{\mathbf{k}} \quad (5.58)$$

b. Space translation

$$\psi(\mathbf{x}) \to \psi(\mathbf{x}+\mathbf{a}) \qquad \psi_{\mathbf{k}} \to \psi_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{a}} \quad (5.59)$$

These are induced by the unitary operators $e^{i\mathbf{a}\cdot\mathbf{P}}$, where

$$\begin{aligned}\mathbf{P} &= -\tfrac{1}{2}i\int_\Omega d^3x[\psi^*(\nabla\psi) - (\nabla\psi^*)\psi] \\ &\equiv \int_\Omega d^3x\mathbf{j}(\mathbf{x}) = \frac{1}{\Omega}\sum_{\mathbf{k}}\mathbf{k}\psi_{\mathbf{k}}{}^*\psi_{\mathbf{k}}\end{aligned} \quad (5.60)$$

c. Galilean transformations

For a system with a finite volume Ω the Galilean invariance group is, of course, discrete, because of the quantization of momentum. If Ω is a cube of edge a, then the allowable velocity vectors are those each of whose components is a multiple of $2\pi/am$. The transformations are

$$\begin{aligned} \psi(\mathbf{x}) &\to \psi(\mathbf{x} - \mathbf{v}t)\, e^{im\mathbf{v}\cdot\mathbf{x} - i\frac{1}{2}m\mathbf{v}^2 t} \\ \psi_{\mathbf{k}} &\to \psi_{\mathbf{k}-m\mathbf{v}}\, e^{i\frac{1}{2}m\mathbf{v}^2 t - i\mathbf{k}\cdot\mathbf{v}t} \end{aligned} \tag{5.61}$$

They are induced by unitary operators $U_\Omega(\mathbf{v})$ which may be written in a purely formal sense as

$$U_\Omega(\mathbf{v}) = e^{i\mathbf{v}\cdot\mathbf{G}} \tag{5.62}$$

with

$$\mathbf{G} = m \int_\Omega d^3x \psi^* \mathbf{x} \psi - \mathbf{P}t \tag{5.63}$$

However, it should be noted that Eq. (5.62) does *not* define $U_\Omega(\mathbf{v})$ for all values of $\mathbf{v}$; for in a system with periodic boundary conditions $\mathbf{x}$ is only really defined modulo a translation of amount a in any of the three coordinate directions. If we make a change $z \to z + a$ then $U_\Omega(\mathbf{v})$ changes by the amount $e^{iav_z Nm}$, which is equal to unity for the allowable values of v_z (since the eigenvalues of N are integers).

The operator $\mathbf{G}$ can of course be defined by making a specific choice of the range of $\mathbf{x}$ (say $-a/2$ to $a/2$ for each coordinate). But it is not time independent and does not represent a symmetry of the system. In fact, explicit calculation shows that, for example,

$$\frac{dG^z}{dt} = -ma \int_{z=a/2} dxdy\, j^z \tag{5.64}$$

Now let us examine the possibility of constructing a representation in which the expectation value $\langle\psi\rangle$ is nonzero. In one important respect the situation here is different from that in the relativistic models we have examined, for we have generally considered assigning a non-vanishing expectation value to an operator which transforms as a scalar under the Lorentz group. But ψ does not transform according to a true representation of the Galilei group, but according to a projective representation. It would be natural to impose the requirement that the vacuum or ground state of our representation should be translationally invariant, so that $\langle\psi\rangle$ would necessarily be independent of $\mathbf{x}$ and t. However, if we can indeed construct such a representation then we can apply a Galilean transformation to it and obtain another representation in which $\langle\psi\rangle$ depends on $\mathbf{x}$ and t via factors of the type that appear in Eq. (5.61). At first sight it would appear the ground state in such a representation could not be translationally invariant. In fact, however, it is essentially so in the limit of infinite volume, as we shall see.

The equation of motion obtained from the Hamiltonian (5.55) is

$$i\frac{\partial}{\partial t}\psi(\mathbf{x}) = -\frac{1}{2m}\nabla^2\psi(\mathbf{x}) + U\psi(\mathbf{x}) + \int_\Omega d^3y\psi^*(\mathbf{y})\psi(\mathbf{y})V(\mathbf{x}-\mathbf{y})\psi(\mathbf{x}) \tag{5.65}$$

We now take the expectation value of this equation and make the approximation of replacing $\langle\psi^*\psi\psi\rangle$ by $\langle\psi^*\rangle\langle\psi\rangle\langle\psi\rangle$. Then, assuming the form

$$\langle\psi(\mathbf{x}, t)\rangle = \alpha\, e^{i\mathbf{p}\cdot\mathbf{x} - iEt} \tag{5.66}$$

or equivalently

$$\langle\psi_{\mathbf{k}}(t)\rangle = \alpha\Omega\delta_{\mathbf{k},\mathbf{p}}e^{-iEt} \tag{5.67}$$

we obtain

$$E = \mathbf{p}^2/2m + E_0 \tag{5.68}$$

with

$$E_0 = U + \alpha^*\alpha V_0 \tag{5.69}$$

We note that to the same approximation we have used, the number density is

$$\langle n\rangle = \langle N\rangle/\Omega = \langle\psi^*\rangle\langle\psi\rangle = \alpha^*\alpha \tag{5.70}$$

Hence E_0 is the potential energy due to the external potential and due to the uniform distribution of particles with density $\alpha^*\alpha$.

Thus if we can break the symmetry in this way, we find as usual a number of different "ground states" which are transformed into each other under the operations of the group. Since the Galilean transformations are explicitly time dependent, and do not commute with time translations, it is no longer true, however, that all our ground states are degenerate in energy.

The ground states are labeled by two parameters, the complex number α and the momentum vector $\mathbf{p}$ (which must form one of the allowed set of momentum vectors). It is interesting to see how these states are transformed under the various transformations we have considered. The magnitude of α is invariant under all of them. Galilean transformations, of course, change the value of $\mathbf{p}$ but leave α unaltered, whereas under the other two classes of transformations (and also under time translations) $\mathbf{p}$ does not change but the phase of α does. Under phase transformations (A) we have

$$\alpha \to \alpha e^{i\lambda} \tag{5.71}$$

Under space translations

$$\alpha \to \alpha e^{i\mathbf{p}\cdot\mathbf{a}} \tag{5.72}$$

and finally under the time translation $t \to t + \tau$

$$\alpha \to \alpha e^{-iE\tau} \tag{5.73}$$

Thus for $\alpha \neq 0$ the phase symmetry and the Galilean symmetry are always broken. So is the space translation symmetry except in the case $\mathbf{p} = 0$. The time translation symmetry is broken unless $E = 0$.

Of course we have not yet established that representations with these properties exist at all. Let us consider first the case of the noninteracting Bose gas, for which $V(\mathbf{x}) = 0$. For convenience we shall also set $U = 0$. [This system has been studied in considerable detail by Araki and Woods.[52]] This noninteracting system has a further invariance, not shared by the interacting gas, under the constant field translations

$$\psi(\mathbf{x}) \to \psi(\mathbf{x}) + \alpha \tag{5.74}$$

or

$$\psi_{\mathbf{k}} \to \psi_{\mathbf{k}} + \alpha\Omega\delta_{\mathbf{k},0} \tag{5.75}$$

(We could also consider translations of any other Fourier component, but these other cases can be obtained by applying a Galilean transformation.) These transformations are generated by the unitary operators

$$U_\Omega(\alpha) = \exp \int_\Omega d^3x[\alpha\psi^*(\mathbf{x}) - \alpha^*\psi(\mathbf{x})] = \exp[\alpha\psi_0^* - \alpha^*\psi_0] \tag{5.76}$$

Now the vacuum state of the conventional Fock representation $|0\rangle_\Omega$ is defined by

$$\psi(\mathbf{x})|0\rangle_\Omega = 0 \tag{5.77}$$

Applying the operators (5.76) to this state we obtain a set of states

$$|\alpha\rangle_\Omega = U_\Omega(\alpha)|0\rangle_\Omega \tag{5.78}$$

characterized by the eigenvalue equations

$$\psi(\mathbf{x})|\alpha\rangle_\Omega = |\alpha\rangle_\Omega\alpha$$

or

$$\psi_{\mathbf{k}}|\alpha\rangle_\Omega = |\alpha\rangle_\Omega\alpha\Omega\delta_{\mathbf{k},0} \tag{5.79}$$

Using the operator identity

$$U_\Omega(\alpha) = \exp(\alpha\psi_0{}^*)\exp(-\alpha^*\psi_0)\exp(-\tfrac{1}{2}\Omega\alpha^*\alpha)$$

we can write these states in terms of more familiar ones. In fact,

$$|\alpha\rangle_\Omega = \sum |N\rangle_\Omega(N!)^{-\frac{1}{2}}(\Omega^{\frac{1}{2}}\alpha)^N \exp(-\tfrac{1}{2}\Omega\alpha^*\alpha) \tag{5.80}$$

where $|N\rangle_\Omega$ is the N-particle ground state in volume Ω,

$$|N\rangle_\Omega = (\Omega^{\frac{1}{2}}\psi_0{}^*)^N|0\rangle_\Omega(N!)^{-\frac{1}{2}}$$

Since these states contain only particles of zero momentum and energy, they are of course all degenerate with the vacuum state. The states $|\alpha\rangle_\Omega$ constitute the set of degenerate ground states associated with the broken symmetry under the transformations (5.74).

We note that the scalar product of two such states is

$$_\Omega\langle\beta|\alpha\rangle_\Omega = \exp[\Omega(\beta^*\alpha - \tfrac{1}{2}\beta^*\beta - \tfrac{1}{2}\alpha^*\alpha)] \tag{5.81}$$

For $\beta \neq \alpha$ this tends to zero as $\Omega \to \infty$. Hence we see that in the limit of infinite volume the states $|\alpha\rangle_\Omega$ go over into the ground states of a set of unitarily inequivalent representations. It is easy to characterize these representations, for they are clearly obtained from the Fock representation simply by a constant field translation. They may be realized in the Hilbert space of the infinite-volume Fock representation by operators of the form

$$\psi(\mathbf{x}) = \psi_F(\mathbf{x}) + \alpha \tag{5.82}$$

where $\psi_F(\mathbf{x})$ is the field operator in the Fock representation. However, the transformation from ψ_F to ψ is not implementable by a unitary transformation.

The states $|\alpha\rangle_\Omega$ are of course invariant under space translations. However under the phase transformations generated by the number operator they transform according to

$$e^{i\lambda N}|\alpha\rangle_\Omega = |\alpha e^{i\lambda}\rangle_\Omega \tag{5.83}$$

Thus the phase transformations cannot be unitarily implemented in any of these representations except the Fock representation.

By applying a Galilean transformation to the states $|\alpha\rangle_\Omega$ we obtain a set of states

$$|\alpha; \mathbf{v}\rangle_\Omega = U_\Omega(\mathbf{v})|\alpha\rangle_\Omega \tag{5.84}$$

characterized by the relations

$$\begin{aligned} \psi(\mathbf{x})|\alpha; \mathbf{v}\rangle_\Omega &= |\alpha; \mathbf{v}\rangle_\Omega \alpha e^{im\mathbf{v}\cdot\mathbf{x} - i\frac{1}{2}m\mathbf{v}^2 t} \\ \psi_{\mathbf{k}}|\alpha; \mathbf{v}\rangle_\Omega &= |\alpha; \mathbf{v}\rangle_\Omega \alpha \Omega \delta_{\mathbf{k}, m\mathbf{v}} e^{-i\frac{1}{2}m\mathbf{v}^2 t} \end{aligned} \tag{5.85}$$

Evidently, the state $|\alpha; \mathbf{v}\rangle_\Omega$ contains only particles of momentum $m\mathbf{v}$ and energy $\frac{1}{2}m\mathbf{v}^2$. However since it is not an eigenstate of the number operator it does not have definite total momentum or energy. Under space translations, for example, it transforms according to

$$e^{i\mathbf{a}\cdot\mathbf{P}}|\alpha; \mathbf{v}\rangle_\Omega = |\alpha e^{im\mathbf{v}\cdot\mathbf{a}}; \mathbf{v}\rangle_\Omega \tag{5.86}$$

Hence in the infinite volume limit space translations are not unitarily implementable in these representations. Nor of course are time translations.

We note that the expectation value of the particle density $n = N/\Omega$ in one of the states $|\alpha\rangle_\Omega$ is

$$\langle n\rangle = {}_\Omega\langle\alpha|N|\alpha\rangle_\Omega/\Omega = \alpha^*\alpha \tag{5.87}$$

and that

$$\langle n^2\rangle - \langle n\rangle^2 = \alpha^*\alpha/\Omega$$

Hence in the limit $\Omega \to \infty$ the number density takes on the definite value $\alpha^*\alpha$. So too in the states $|\alpha; \mathbf{v}\rangle_\Omega$ do the momentum density and energy density.

So far this discussion has been mainly mathematical. A clearer idea of the physical significance of these representations may be gained from considering the problem of how we should describe an infinite Bose gas of finite density, whose states evidently do not belong to the Fock representation since the total particle number is infinite. To specify the representation completely it is sufficient to give the ground state expectation values of all quasi-local operators A (those which can be written as limits of sequences of operators A_Ω, with each A_Ω a function only of the field operators within Ω). It is natural to start with the N-particle ground-state in volume Ω, with $N = n\Omega$ and take the limit $\Omega \to \infty$. Thus we define[52]

$$\langle A\rangle_n = \lim_{\Omega\to\infty} {}_\Omega\langle n\Omega|A_\Omega|n\Omega\rangle_\Omega \tag{5.88}$$

This formula defines a representation of the commutation relations (5.56). It is, however, a reducible representation; that is to say there exist operators other than multiples of the identity which commute with every operator in the representation. To see this, we note that in the limit of infinite volume the addition or removal of any finite number of particles in the ground state makes no difference. Thus for example we arrive at the same value of $\langle A\rangle_n$ if we replace the states $|n\Omega\rangle_\Omega$ in Eq. (5.88) by

$$|n\Omega - 1\rangle_\Omega = (\psi_0/n^{1/2}\Omega)|n\Omega\rangle_\Omega$$

Hence if we define an operator ϕ by

$$\phi = \lim_{\Omega\to\infty} (\psi_0/n^{1/2}\Omega) \tag{5.89}$$

then we must have

$$\langle\phi^* A\phi\rangle_n = \langle A\rangle_n \tag{5.90}$$

It follows that the operator ϕ must be unitary, and must commute with every other operator. Hence the representation $\langle A\rangle_n$ is reducible, and can be decomposed into irreducible representations in each of which ϕ is represented by a number of modulus unity.

The explicit reduction may be accomplished by first noting that we can write instead of Eq. (5.88) the equation

$$\langle A\rangle_n = \lim_{\Omega\to\infty} \sum_N \frac{(n\Omega)^N}{N!} e^{-n\Omega}{}_\Omega\langle N|A_\Omega|N\rangle_\Omega \tag{5.91}$$

for, when Ω is large, the Poisson distribution here yields contributions significantly different from zero only for values of N in the neighborhood of $n\Omega$ (within a range of order $(n\Omega)^{1/2}$). By the preceding argument, each of these gives in the limit the same contribution. Now we can reexpress the right-hand side of Eq. (5.91) in terms of the states $|\alpha\rangle_\Omega$ using the identity

$$\int \frac{d\theta}{2\pi} (|\alpha\rangle_{\Omega\Omega}\langle\alpha|)_{\alpha=n^{1/2}e^{i\theta}} = \sum_N |N\rangle_\Omega \frac{(n\Omega)^N}{N!} e^{-n\Omega}{}_\Omega\langle N| \tag{5.92}$$

(which is easy to verify by taking matrix elements). This yields the decomposition

$$\langle A\rangle_n = \int \frac{d\theta}{2\pi} \langle A\rangle_{n^{1/2}e^{i\theta}} \tag{5.93}$$

where

$$\langle A\rangle_\alpha = \lim_{\Omega\to\infty} {}_\Omega\langle\alpha|A_\Omega|\alpha\rangle_\Omega \tag{5.94}$$

Note that in these representations ϕ is represented simply by $e^{i\theta}$.

It should be noted that in the representation defined by $\langle A\rangle_n$, the phase transformations are unitarily implementable. In fact they are represented by rotations of θ; however, this does not contradict our earlier discussion, since in the present case the degenerate ground states with definite values of θ are not normalizable states.

The representation described by the expectation value $\langle A\rangle_\alpha$ is of course that given by the operators (5.82). It is irreducible since the Fock representation is. Thus we have completed the reduction of the representation $\langle A\rangle_n$ into irreducible components.

If we were to consider as observables only operators invariant under the constant field-translations (5.74) then we should obtain exactly the same expectation values $\langle A\rangle_\alpha$ for any different values of α. This is, however, too strong a restriction, for we clearly want to be able to regard the particle density as an observable. If we allow as observables operators which are not symmetric under these transformations but are nevertheless symmetric under the phase transformations (5.57), then we would still obtain the same expectation values by using the representation $\langle A\rangle_n$, or $\langle A\rangle_\alpha$ for any value of α satisfying $\alpha^*\alpha = n$. To this extent these representations are physically equivalent.

Is there then any physical sense in which we can distinguish between the various $\langle A\rangle_\alpha$ differing only in the phase of α? For a ferromagnet, the analogous quantity is the magnetization direction, and it seems physically obvious that this *is* an observable, and that we should normally represent a ferromagnet by a state with definite magnetization direction, rather than by an ensemble with all possible magnetization directions. The particular direction is in this case selected by interaction with any small residual magnetic field. At first sight, it is much less clear that the phase of a condensed Bose gas should be treated as an observable. However, the difference between the two situations is more apparent than real. For any real Bose gas, there undoubtedly are interactions capable of changing the total particle number of our system by absorption, emission, or interchange with the surroundings. (Indeed, for a *photon* gas this is obvious, and there is surely no doubt that the phase of an electromagnetic wave has an observational significance.)

It is therefore perfectly reasonable to include a small external interaction term which breaks the number symmetry. As we shall see below, such a term can lead to a preferential selection of a state $|\alpha\rangle_\Omega$ with a particular phase. Of course, the absolute phase of our system has no physical significance. We can measure it only relative to some externally defined standard. But the same is equally true of magnetization direction.

In summary, if we do not accept the phase as an observable, we can choose it arbitrarily; if we do accept it, and allow symmetry-breaking interactions, then the phase will be determined by them. In either case, we can adequately describe the system by a state with well-defined phase.

Let us then examine the effect of including a small symmetry-breaking interaction of the form

$$\nu H_1 = -\nu \int_\Omega d^3x[\psi(\mathbf{x}) + \psi^*(\mathbf{x})] = -\nu(\psi_0 + \psi_0^*) \tag{5.95}$$

One may regard the choice of real phase factor here analogous to the choice of the z axis for the direction of the external magnetic field.

Since the interaction breaks the number symmetry, it is most convenient to discuss this problem using the grand canonical ensemble, in which the expectation value of any observable A is given by

$$\langle A\rangle = \operatorname{tr}(\rho A) \tag{5.96}$$

where ρ is the density operator

$$\rho = \exp^{-\beta(H+\nu H_1-\mu N)}/\operatorname{tr}[e^{-\beta(H+\nu H_1-\mu N)}] \tag{5.97}$$

Here $\beta = 1/kT$, T is the temperature, and μ is the chemical potential. Because of the independence of the modes described by different values of $\mathbf{k}$ we can write

$$\rho = \Pi_{\mathbf{k}}\rho_{\mathbf{k}}$$

For any mode with $\mathbf{k} \neq \mathbf{0}$ the distribution is completely unaffected by the interaction term νH_1, and is given by

$$\rho_{\mathbf{k}} = [1 - e^{\beta(\mu-\mathbf{k}^2/2m)}]^{-1}\exp\left[\frac{\beta}{\Omega}\left(\mu - \frac{\mathbf{k}^2}{2m}\right)\psi_{\mathbf{k}}^*\psi_{\mathbf{k}}\right] \tag{5.98}$$

(Recall that with our normalization the number operator for the mode $\mathbf{k}$ is $\psi_{\mathbf{k}}^*\psi_{\mathbf{k}}/\Omega$.) On the other hand, for the mode $\mathbf{k} = \mathbf{0}$ the only term in the energy is that arising from the interaction νH_1. Thus,

$$\rho_0 = c \exp \beta[(\mu/\Omega)\psi_0^*\psi_0 + \nu(\psi_0^* + \psi_0)]$$

where c is a normalization constant. Recognizing that the effect of the term in ν is to translate the field operator, we can therefore write

$$\rho_0 = [1 - e^{\beta\mu}] \exp\left[\frac{\beta\mu}{\Omega}\left(\psi_0{}^* + \frac{\nu\Omega}{\mu}\right)\left(\psi_0 + \frac{\nu\Omega}{\mu}\right)\right] \tag{5.99}$$

Thus the only effect of the interaction νH_1 is to translate ψ_0. All expectation values are obtainable from those in the ensemble with $\nu = 0$ by this translation. In particular we find

$$\langle\psi_0\rangle = \langle\psi_0{}^*\rangle = -\nu\Omega/\mu \tag{5.100}$$

or equivalently

$$\langle\psi(\mathbf{x})\rangle = \langle\psi^*(\mathbf{x})\rangle = -\nu/\mu \tag{5.101}$$

and also

$$\langle\psi_0{}^*\psi_0\rangle = \nu^2\Omega^2/\mu^2 + \Omega/(e^{-\beta\mu} - 1) \tag{5.102}$$

Note that the mean value of the number density of particles in the ground state is

$$n_0 = \langle N_0\rangle/\Omega = \langle\psi_0{}^*\psi_0\rangle/\Omega^2 \tag{5.103}$$

The chemical potential must of course be negative in order that the expectation value of the number of particles in each mode be positive. It is determined in terms of the mean value of the total number density by the implicit equation

$$n = \nu^2/\mu^2 + 1/\Omega(e^{-\beta\mu} - 1) + n(\beta, \mu) \tag{5.104}$$

where $n(\beta, \mu)$ is the number density of particles in the excited states, given by

$$n(\beta, \mu) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\exp[\beta(\mathbf{k}^2/2m - \mu)] - 1} \tag{5.105}$$

We note that for $\mu < 0$ each term on the right-hand side of Eq. (5.104) is a monotonically increasing function of μ. Since the value of the right-hand side ranges from 0 at $\mu = -\infty$ to $+\infty$ at $\mu = 0$, there is therefore always a unique root for μ.

Let us examine the limit $\Omega \to \infty$. Here we find very different behavior for $\nu = 0$ and $\nu \neq 0$. If $\nu = 0$ and

$$n > n_c(\beta) = n(\beta, 0) \tag{5.106}$$

then $\mu \to 0$ and, in fact, for large Ω,

$$\mu = -[\beta\Omega(n - n_c)]^{-1} \tag{5.107}$$

[Of course, if the inequality (5.106) is not satisfied the gas is not condensed and no problem arises.]

On the other hand, if $\nu \neq 0$, then it is clear that μ remains finite as $\Omega \to \infty$; for otherwise the first term in Eq. (5.104) becomes infinite, hence the second term is negligible and so in the limit

$$\mu = -\nu[n - n(\beta, \mu)]^{1/2} \tag{5.108}$$

or, for small ν,

$$\mu \approx -\nu[n - n_c]^{1/2} \tag{5.109}$$

Thus from Eq. (5.87) we find the expectation value

$$\langle\psi(\mathbf{x})\rangle = (n - n_c)^{1/2} \tag{5.110}$$

It is important to note that this expression is independent of ν for small ν, and remains finite even in the limit $\nu \to 0$.

It is apparent that the limits $\nu \to 0$ and $\Omega \to \infty$ do not commute. By taking the limit $\Omega \to \infty$ first we have obtained a distribution which is just the usual one for the condensed Bose gas, but with the particles in the macroscopically occupied condensate being represented by the ground state $|\alpha\rangle$ with $\alpha = (n - n_c)^{1/2}$. We have, in fact, constructed an ensemble of states in the Hilbert space built on this particular ground state.

Let us now return to the problem of an interacting Bose gas described by the Hamiltonian (5.55).

From the Goldstone theorem we learn that if the potential is of sufficiently short range there must be a branch of the excitation spectrum which has zero energy gap. That the necessary short-range condition is satisfied, for potentials which fall off at infinity faster than Coulomb potential, has been proved by Swieca[53] on the basis of certain plausible assumptions about the behavior of the Green's functions. It is easy to verify the conclusion in the Bogoliubov approximation.[54]

In view of our discussion of the noninteracting Bose gas, it is reasonable to suppose that in the interacting case the condensed Bose gas should also be described (in the infinite volume limit) by an ensemble in a Hilbert space corresponding to a representation with $\langle\psi\rangle \neq 0$, and to treat $\psi - \langle\psi\rangle$ as small in comparison to $\langle\psi\rangle$.

From Eq. (5.66) we learn that the time dependence of $\langle\psi\rangle$ is given by

$$\langle\psi\rangle = \alpha e^{-iE_0 t} \tag{5.111}$$

It is convenient to absorb this phase factor by writing

$$\psi(\mathbf{x}, t) = [\alpha + \psi'(\mathbf{x}, t)]\, e^{-iE_0 t} \tag{5.112}$$

We now linearize the equation of motion [Eq. (5.65)] by substituting Eq. (5.112) and keeping only terms linear in ψ'. This yields in momentum space the equation

$$i\frac{\partial}{\partial t}\psi_{\mathbf{k}}' = (\mathbf{k}^2/2m)\,\psi_{\mathbf{k}}' + \alpha V_{\mathbf{k}}(\alpha^*\psi_{\mathbf{k}}' + \alpha\psi'^*_{-\mathbf{k}}) \tag{5.113}$$

It follows that the excitation spectrum is given by

$$\omega^2 = \frac{\mathbf{k}^2}{2m}\left(\frac{\mathbf{k}^2}{2m} + 2\alpha^*\alpha V_{\mathbf{k}}\right) \tag{5.114}$$

We note that $\omega \to 0$ as $\mathbf{k} \to \mathbf{0}$ provided that

$$\lim_{\mathbf{k}\to\mathbf{0}} \mathbf{k}^2 V_{\mathbf{k}} = 0 \tag{5.115}$$

the same condition as in the case of the soluble model we discussed earlier [see Eq. (5.31)]. This is in conformity with the Goldstone theorem. There are modes with zero energy gap provided that the forces are of short range; specifically, provided that the potential falls off faster than $1/r$ at large distances. Note that the behavior of ω near $\mathbf{k} = \mathbf{0}$ is given by

$$\omega = c|\mathbf{k}| \tag{5.116}$$

where c, the velocity of sound, is determined by the integral of the potential,

$$c^2 = (\alpha^*\alpha)V_{\mathbf{0}}/m \tag{5.117}$$

(It can be shown that this agrees with the classical formula.)

On the other hand, for a repulsive Coulomb potential,

$$V(\mathbf{x}) = e^2/4\pi|\mathbf{x}|$$
$$V_{\mathbf{k}} = e^2/\mathbf{k}^2$$

we find that as $\mathbf{k} \to \mathbf{0}$ the frequency tends to a finite plasma frequency,

$$\lim_{\mathbf{k}\to\mathbf{0}} \omega^2 = (\alpha^*\alpha)\, e^2/m = \omega_p{}^2 \tag{5.118}$$

For long-range interactions we find an energy gap whose magnitude is determined by the strength of the potential and by the density.

An interesting fact which emerges from this model is that it is possible to have a number of distinct broken symmetries, all yielding the same Goldstone excitations. We recall that the Goldstone modes may be regarded as long-wavelength oscillations in the parameter which distinguishes the various degenerate ground states. In our case, the effects of both phase transformations and spatial translations as described by Eq. (5.83) and (5.86) are to change the phase of α. It is therefore not surprising that the corresponding Goldstone modes are identical. On the other hand, the Galilean transformations change the value of the other symmetry-breaking parameter, $\mathbf{v}$. However, from Eq. (5.85) one sees that $m\mathbf{v}$ may be identified with the spatial gradient of the phase of $\langle\psi\rangle$. In fact, this is the conventional definition of the superfluid velocity. Thus an oscillation in $\mathbf{v}$ is actually no different from an oscillation in the phase of α. Both, of course, represent phonons.

There is another situation in which translational symmetry is broken, namely, that of a crystal. In a finite volume the ground state $|0\rangle_\Omega$ of a crystal is translationally invariant, and, despite the periodic structure the particle density, for example, is completely uniform. However, it is obviously convenient for many purposes to consider a different type of state in which the particle density has the periodicity of the lattice. A state in which the atoms of the crystal are localized near a specific set of lattice points necessarily has some uncertainty in momentum, and therefore a higher energy than the ground state. However, we have seen in the previous example that it is possible in the infinite-volume limit to obtain a state which has definite values of both phase and number density, and in exactly the same way one can construct states with a definite center-of-mass and simultaneously definite momentum and energy density. Here too the Goldstone modes are phonons.

Finally we wish to consider a fermion gas. The system is described by a two-component field

$$\psi(\mathbf{x}) = \begin{bmatrix} \psi_1(\mathbf{x}) \\ \psi_2(\mathbf{x}) \end{bmatrix} \tag{5.119}$$

satisfying the canonical anticommutation relation

$$\{\psi_i(\mathbf{x}, t), \psi_j^*(\mathbf{y}, t)\} = \delta_{ij}\delta(\mathbf{x} - \mathbf{y}) \tag{5.120}$$

For simplicity we choose a spin-independent interaction, described by the Hamiltonian

$$H = \frac{1}{2m}\sum_i \int_\Omega d^3x \nabla\psi_i{}^* \nabla\psi_i + U\sum_i \int_\Omega d^3x \psi_i{}^*\psi_i$$
$$+ \tfrac{1}{2}\sum_{i,j}\int_\Omega d^3x \int_\Omega d^3y \psi_i{}^*(\mathbf{x})\psi_j{}^*(\mathbf{y}) V(\mathbf{x}-\mathbf{y})\psi_j(\mathbf{y})\psi_i(\mathbf{x}) \quad (5.121)$$

The corresponding equation of motion is

$$i\frac{\partial}{\partial t}\psi(\mathbf{x}) = -\frac{1}{2m}\nabla^2\psi(\mathbf{x}) + U\psi(\mathbf{x})$$
$$+ \int d^3y [\psi^*(\mathbf{y})\psi(\mathbf{y})] V(\mathbf{x}-\mathbf{y})\psi(\mathbf{x}) \quad (5.122)$$

This system possesses essentially the same symmetries as the boson gas. In particular, it is symmetric under the phase transformations generated by the total number operator

$$N = \int_\Omega d^3x \psi^*(\mathbf{x})\psi(\mathbf{x}) \quad (5.123)$$

In addition it is symmetric under spin rotations, generated by the total spin operators

$$\mathbf{S} = \tfrac{1}{2}\int_\Omega d^3x \psi^*(\mathbf{x})\boldsymbol{\sigma}\psi(\mathbf{x}) \quad (5.124)$$

We shall consider the possibility of breaking the phase symmetry corresponding to the conserved particle number, while maintaining the symmetry under spin rotations, as is done in the Bardeen–Cooper–Schrieffer model of superconductivity.(55) [This model has been discussed from the point of view of broken symmetry by Haag,(56) Ezawa,(57) and Emch and Guenin.(58)] To do this we look for a ground state in which the expectation value $\langle\psi\psi\rangle$ is nonvanishing. If this state is assumed translationally invariant, then

$$\langle\psi_i(\mathbf{x})\psi_j(\mathbf{y})\rangle = f_{ij}(\mathbf{x}-\mathbf{y})$$

If, in addition, we impose the requirement of invariance under spin rotations, we must have

$$\langle\psi_i(\mathbf{x})\psi_j(\mathbf{y})\rangle = \epsilon_{ij} f(\mathbf{x}-\mathbf{y}) \quad (5.125)$$

where

$$\epsilon = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Of course, $f(\mathbf{x})$ must be a function only of $|\mathbf{x}|$.

As before, if we can find one such state then we can find a whole family of them. In particular, by applying Galilean transformations, we would obtain translationally noninvariant states characterized by an overall velocity. However, for our purposes it will be sufficient to consider only one such state which for convenience we choose to be at rest.

In addition to Eq. (5.125) we shall also need

$$\langle \psi_i^*(\mathbf{x})\psi_j(\mathbf{y})\rangle = \delta_{ij}g(\mathbf{x} - \mathbf{y}) \tag{5.126}$$

Note that

$$2g(\mathbf{0}) = n$$

the particle density.

If $V(\mathbf{x})$ is assumed to be of short range, then we know from the Goldstone theorem that there must be a branch of the excitation spectrum for which $\omega \to 0$ as $\mathbf{k} \to \mathbf{0}$. However, unlike the situation in the case of the boson gas which we considered earlier these modes are not the elementary quasi-particle excitations. Indeed if we make the linearizing approximation in Eq. (5.122) replacing

$$[\psi^*(\mathbf{y})\psi(\mathbf{y})]\psi(\mathbf{x})$$

by

$$2g(\mathbf{0})\psi(\mathbf{x}) - g(\mathbf{y} - \mathbf{x})\psi(\mathbf{y}) - f(\mathbf{y} - \mathbf{x})\epsilon\psi^*(\mathbf{y})$$

then we obtain an energy spectrum with a gap of the form

$$\omega^2 = \epsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2 \tag{5.127}$$

[see, for example, Valatin[59]].

In fact, the zero-energy-gap modes in this system are phononlike as before, and we must expect them to be excited only by bosonlike operators, constructed from pairs of fermi field operators, for example $\psi\psi$ or $\psi^*\psi$. The Goldstone theorem which establishes the existence of such modes rests on an examination of commutators of the form

$$[\psi^*\psi, \psi\epsilon\psi]$$

It therefore follows that these modes can be excited by the operators $\psi^*\psi$ or $\psi\epsilon\psi$. However, a direct verification of this conclusion by solving the equations of motion in an appropriate approximation would be too lengthy to include here. [In this connection see Gorkov[60] and Anderson.[61]]

VI. Goldstone Bosons and Composite Fields

Until now our considerations have essentially been confined to examples of broken-symmetry theories in which the Goldstone bosons are directly associated with an elementary field operator obeying canonical commutation relations, rather than with the more complicated situation in which the field operator associated with the Goldstone particle is composite. However, the discussion in Section II demonstrated that in general we should anticipate that the Goldstone particle may be identifiable with a composite field and that this is particularly likely to be the case when the symmetry breaking is related to a physical attribute such as particle mass. However, it turns out as a practical matter that, as a consequence of divergences and fairly involved constraints, composite particle broken-symmetry theories are very difficult to handle in a consistent manner, even in the lowest orders of perturbation theory.

Because of these difficulties we choose to defer to a later point the examination of more realistic theories and shall begin this discussion by investigating the Thirring model (a soluble two-dimensional field theory), under the assumption that the vacuum expectation of the vector current j^μ associated with this model has a nonvanishing value. This problem has been considered previously[62] but, because of the imposition of certain arbitrary assumptions, only specialized and somewhat misleading solutions were obtained. Since the basic problem here is to maintain consistency with Lorentz invariance, it must be borne in mind that if a field ϕ carried intrinsic spin, it transforms under the generators of the Lorentz group $J^{\mu\nu}$ according to

$$-i[J^{\mu\nu}, \phi(x)] = (x^\mu \partial^\nu - x^\nu \partial^\mu)\phi(x) + \sigma^{\mu\nu}\phi(x)$$

where $\sigma^{\mu\nu}$ is a matrix associated with the spin of the field ϕ. Thus in this case the requirement $\langle 0|\phi|0\rangle \neq 0$ becomes $\langle 0|[J^{\mu\nu}, \phi]|0\rangle \neq 0$ which, in turn, means that in general Lorentz invariance is broken. [Note that in the case $\phi^\lambda = j^\lambda$ we have $(\sigma^{\mu\nu})^{\lambda\sigma} = g^{\lambda\mu}g^{\nu\sigma} - g^{\lambda\nu}g^{\mu\sigma}$.] This procedure is thus obviously more likely to give nonsensical results than in the case of spinless fields. However, because of the special structure of the

Thirring model,[63] one is able to obtain meaningful and interesting results with this procedure, as we shall now demonstrate.

We take the Lagrangian to be

$$\mathscr{L} = (i/2)\psi\alpha^\mu \partial_\mu\psi + (\lambda/2)j^\mu j_\mu + j^\mu A_\mu \tag{6.1}$$

where $\psi(x)$ is a Hermitian field and the Dirac algebra has the two-dimensional representation

$$\alpha^0 = -\alpha_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\alpha^1 = \alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The current j^μ is defined so that

$$\frac{\delta j^\mu(x)}{\delta\psi(x)} = \alpha^\mu q\psi(x)$$

where we have introduced the usual matrix $q = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ which acts in the internal charge space of the field ψ. It is to be emphasized that this implies $j^\mu(x) = \frac{1}{2}\psi(x)\alpha^\mu q\psi(x)$ only if such a quantity is well defined. Since this is not the case (as has frequently been observed in the literature), we note that the most general definition of $j^\mu(x)$ in the presence of the external source $A_\mu(x)$ is[64]

$$j^\mu(x) = \lim_{\mathbf{x}\to\mathbf{x}'} \tfrac{1}{2}\psi(x)\alpha^\mu q \exp\left\{-iq\int_{x'}^{x} dx_\mu''[\xi A^\mu(x'') - \eta\gamma_5 A_5{}^\mu(x'') + \lambda\xi j^\mu(x'') - \lambda\eta\gamma_5 j_5{}^\mu(x'')]\right\}\psi(x') \tag{6.2}$$

γ_5 being the pseudoscalar matrix $\alpha^0\alpha^1$ while $A_5{}^\mu$ and $j_5{}^\mu$ are defined by

$$A_5{}^\mu(x) = \epsilon^{\mu\nu}A_\nu(x)$$
$$j_5{}^\mu(x) = \epsilon^{\mu\nu}j_\nu(x)$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ and $\epsilon^{01} = +1$. It is understood that the limit is to be taken with $x^0 = x^{0'}$ subject to the condition $\xi + \eta = 1$ which is necessary for Lorentz invariance.[64] Returning to (6.1) and using the action principle, we find the field equation

$$\alpha^\mu\left(\frac{1}{i}\partial_\mu - qA_\mu - \lambda q j_\mu\right)\psi = 0 \tag{6.3}$$

and the equal-time commutation relation

$$\{\psi(x), \psi(x')\} = \delta(\mathbf{x} - \mathbf{x}')$$

In the limit of vanishing source, it follows as a consequence of the solution of the detailed dynamics[64] that $\partial^2 j^\mu = 0$. This result would also follow from gauge invariance if the naive single-point description of the current operator were correct. Thus j^μ satisfies a massless free field equation invariant under the transformation

$$j^\mu(x) \rightarrow j^\mu(x) + \eta^\mu \tag{6.4}$$

If $j^\mu(x)$ were an elementary field we could proceed directly as in the naturally occurring cases considered earlier. For example, Eq. (6.3) would appear to be invariant under the transformation (6.4) combined with the transformation

$$\psi(x) \rightarrow \exp\{i\lambda q \eta^\mu x_\mu\}\psi(x) \tag{6.5}$$

and, if this were the case, it would be trivial to establish the consistency of the broken symmetry. In fact, however, the situation is far more complicated because of the composite field nature of $j^\mu(x)$. Thus a transformation of type (6.5), when fed into the definition of the current, as given by Eq. (6.2), will induce additional broken symmetry effects since $j^\mu(x)$ is not, in general, invariant under the transformation (6.5). Thus the consistency of the broken symmetry is not immediately evident.

We now consider this theory using a Green's function approach much like that used for more complicated problems. This discussion will rely heavily on techniques previously used to solve the model and the reader is consequently referred to these earlier works on the Thirring model for many of the details not supplied here.[63,64] We begin by considering the case $\lambda = 0$. Then, defining

$$G(x, x') = i\epsilon(x, x')\,\frac{\langle 0|(\psi(x)\psi(x'))_+|0\rangle_A}{\langle 0|0\rangle_A}$$

we find from Eq. (6.3) and the commutation relations that

$$\alpha^\mu\left(\frac{1}{i}\,\partial_\mu - qA_\mu\right)G(x, x') = \delta(x - x')$$

It follows that

$$G(x, x') = \tilde{G}_0(x - x')\exp\{iq[F(x) - F(x')]\} \tag{6.6}$$

if

$$\alpha^\mu \frac{1}{i} \partial_\mu \tilde{G}_0(x - x') = \delta(x - x')$$

and

$$\alpha^\mu \partial_\mu F(x) = \alpha^\mu A_\mu(x)$$

The solutions to these equations are not unique and, indeed, if $\tilde{G}_0(x - x')$ satisfies the above equations, so does $\tilde{G}_0(x - x') + \text{constant}$. We thus write

$$\tilde{G}_0(x - x') = G_0(x - x') + \gamma$$

γ being a constant matrix with $G_0(x)$ defined by

$$G_0(x) = i(\partial_0 - \alpha^1 \partial_1)D(x)$$

where $D(x)$ satisfies the equation

$$-\partial^2 D(x) = \delta(x)$$

subject to the usual causal boundary conditions. These same boundary conditions imply, however, a unique structure for $F(x)$, i.e.,

$$F(x) = -i \int dx' G_0(x - x')\alpha^\mu A_\mu(x')$$

The extra terms in $\tilde{G}_0(x - x')$ which are proportional to q are associated with the broken symmetry and would normally be ruled out by requirements of Lorentz invariance. This can be seen explicitly by using the definition Eq. (6.2) of $j^\mu(x)$ and the result

$$G_0(x - x') = -\frac{\alpha^1}{2\pi} \frac{1}{\mathbf{x} - \mathbf{x}'} \qquad x^0 = x^{0'}$$

from which one infers

$$\begin{aligned}\frac{\langle 0|j^\mu(x)|0\rangle_A}{\langle 0|0\rangle_A} &= \frac{i}{2} \lim_{\mathbf{x}\to\mathbf{x}'} \mathrm{Tr} q\alpha^\mu G(x, x') \exp\left[-iq \int_{x'}^{x} dx_\mu''(\xi A^\mu - \eta\gamma_5 A_5{}^\mu)\right] \\ &= \left.\frac{\langle 0|j^\mu(x)|0\rangle_A}{\langle 0|0\rangle_A}\right|_{\gamma=0} + \eta^\mu \qquad (6.7)\end{aligned}$$

where

$$\eta^\mu = \mathrm{Tr} \frac{i}{2} q\alpha^\mu\gamma$$

and

$$\frac{\langle 0|j^\mu(x)|0\rangle_A}{\langle 0|0\rangle_A} = \int dx' D^{\mu\nu}(x - x')A_\nu(x')$$

The current correlation function $D^{\mu\nu}(x - x')$ has the structure

$$D^{\mu\nu}(x - x') = -\frac{1}{\pi}[\xi\epsilon^{\mu\sigma}\epsilon^{\nu\tau} + \eta g^{\mu\sigma}g^{\nu\tau}]\,\partial_\sigma\,\partial_\tau D(x - x')$$

It is noteworthy that the simplicity of this model, rather than any general principle, accounts for the fact that this function is independent of the external source.

Having determined the vacuum expectation value of $j^\mu(x)$ in the presence of a source, we may explicitly evaluate the function $\langle 0|0\rangle_A$ by using

$$\begin{aligned}\frac{1}{i}\frac{\delta}{\delta A_\mu(x)}\langle 0|0\rangle_A &= \langle 0|j^\mu(x)|0\rangle_A \\ &= \langle 0|0\rangle_A\left[\eta^\mu + \int dx' D^{\mu\nu}(x - x')A_\nu(x')\right]\end{aligned}$$

which upon integration yields

$$\langle 0|0\rangle_A = \exp\left\{\frac{i}{2}\int dxdx' A_\mu(x)D^{\mu\nu}(x - x')A_\nu(x') + i\int \eta^\mu A_\nu(x)\,dx\right\} \quad (6.8)$$

In writing Eq. (6.8) we have included η^μ within the integral since it will be convenient for subsequent calculations to give η^μ a space–time behavior which allows one to freely carry out integration by parts. Such a procedure is admissible, provided that one ultimately proceeds to the limit η^μ = constant in the calculation of all matrix elements.

We are now equipped to consider broken symmetries in the fully interacting theory. To do this we observe that any matrix element $\langle a'|b'\rangle_{A,\lambda}$ in consequence of the action principle satisfies the relation

$$\frac{\delta}{\delta\lambda}\langle a'|b'\rangle_{A,\lambda} = -\frac{i}{2}\int dx\,\frac{\delta}{\delta A_\mu(x)}\,\frac{\delta}{\delta A^\mu(x)}\langle a'|b'\rangle_{A,\lambda}$$

or

$$\langle a'|b'\rangle_{A,\lambda} = \exp\left\{-\frac{i\lambda}{2}\int dx\,\frac{\delta}{\delta A^\mu(x)}\,\frac{\delta}{\delta A_\mu(x)}\right\}\langle a'|b'\rangle_{A,0}$$

Thus, knowing the source dependence for $\lambda = 0$ makes it possible to find the matrix elements for the fully interacting theory. Referring to Eq. (6.8) we obtain

$$\langle 0|0\rangle_{A,\lambda} = \exp\left\{-\frac{i\lambda}{2}\int dx \frac{\delta}{\delta A^\mu(x)}\frac{\delta}{\delta A_\mu(x)}\right\}\exp\left\{\frac{i}{2}\int A_\mu D^{\mu\nu}A_\nu\, dx\, dx' + i\int \eta^\mu A_\mu\, dx\right\}$$

Using the commutation relation

$$\left[\frac{\delta}{\delta A^\mu(x)}, A^\nu(x')\right] = \delta_\mu{}^\nu \delta(x - x')$$

the above expression is directly evaluated to yield

$$\langle 0|0\rangle_{A,\lambda} = C\exp\left\{i\int \eta^\mu A_\mu\, dx\right\}\exp\left\{\frac{i}{2}\int (A_\mu + \lambda\eta_\mu)D_\lambda^{\mu\nu}(A_\nu + \lambda\eta_\nu)\right\} \tag{6.9}$$

where the constant C is independent of A^μ and η^μ and so will not appear in any of the appropriately defined matrix elements of the theory. The current correlation function, including the effects of the interaction, is found to be[64]

$$D_\lambda^{\mu\nu}(x) = -\frac{1}{\pi}\left(\epsilon^{\mu\sigma}\epsilon^{\nu\tau}\frac{\xi}{1 + \lambda\xi/\pi} + g^{\mu\sigma}g^{\nu\tau}\frac{\eta}{1 - \lambda\eta/\pi}\right)\partial_\sigma\, \partial_\tau D(x)$$

From Eq. (6.9) we find directly

$$\frac{\langle 0|j^\mu(x)|0\rangle_{0,\lambda}}{\langle 0|0\rangle_{0,\lambda}} = \eta^\mu + \lambda\int D_\lambda^{\mu\nu}(x)\, dx\eta_\nu \tag{6.10}$$

Upon making the decomposition

$$\eta^\mu = \eta_1{}^\mu + \eta_2{}^\mu$$

where

$$\partial_\mu \eta_1{}^\mu = 0$$
$$\epsilon_{\mu\nu}\partial^\mu \eta_2{}^\nu = 0$$

one can readily evaluate the integral (6.10) to find in the limit η^μ = constant

$$\frac{\langle 0|j^\mu|0\rangle_{0,\lambda}}{\langle 0|0\rangle_{0,\lambda}} = \eta_1{}^\mu\frac{1}{1 + \lambda\xi/\pi} + \eta_2{}^\mu\frac{1}{1 - \lambda\eta/\pi} \tag{6.11}$$

which clearly illustrates the renormalization of the symmetry-breaking parameters $\eta_1{}^\mu$ and $\eta_2{}^\mu$ induced by the interaction. Before discussing Eq. (6.11) any further, we calculate the remaining Green's functions in the presence of interaction. It is easily established that

$$G_{A,\lambda}(x_1, x_2, \ldots x_{2n}) = \frac{1}{\langle 0|0\rangle_{A,\lambda}} \exp\left\{-\frac{i\lambda}{2}\int dx \frac{\delta^2}{\delta A^\mu(x)\delta A_\mu(x)}\right\} \times \langle 0|0\rangle_A G_A(x_1, x_2, \ldots x_{2n})$$

where

$$G_A(x_1, x_2, \ldots x_{2n}) = \tilde{G}_0(x_1, x_2, \ldots x_{2n}) \exp\left\{i \sum_{i=1}^{2n} q_i F(x_i)\right\}$$

$\tilde{G}_0(x_1, x_2, \ldots x_{2n})$ being the free field Green's function which satisfies the equation

$$\left(\alpha^\mu \frac{1}{i}\partial_\mu\right)_1 \tilde{G}_0(x_1, x_2, \ldots x_{2n}) = \sum_{i=2}^{2n} (-1)^i \delta(x_1 - x_i) \times \tilde{G}_0(x_2, \ldots x_{i-1}, x_{i+1}, \ldots x_{2n})$$

and, as before,

$$F(x_i) = -i\int dx' G_0(x_i - x')\alpha^\mu A_\mu(x')$$

After a straightforward calculation it is found that

$$G_{A,\lambda}(x_1, x_2, \ldots x_{2n}) = \exp\left\{i \sum_{i,j} q_i \int N_\mu(x' - x_i)[A^\mu(x' - x_i) + \lambda\eta^\mu]\, dx'\right\} \times G_{0,\lambda}(x_1, x_2, \ldots x_{2n}) \quad (6.12)$$

where

$$G_{0,\lambda}(x_1, x_2, \ldots x_{2n}) = \tilde{G}_0(x_1, x_2, \ldots x_{2n}) \exp\left\{i\frac{\lambda}{2}\sum_{i,j} q_i q_j D(x_i - x_j) \times \left(\frac{1}{1 - \lambda\eta/\pi} - \frac{\gamma_{5i}\gamma_{5j}}{1 + \lambda\xi/\pi}\right)\right\}$$

and

$$N_\mu(x' - x_i) = \left(\frac{1}{1 - \lambda\eta/\pi}\partial_\mu - \frac{1}{1 + \lambda\xi/\pi}\epsilon_{\mu\nu}\gamma_5\,\partial^\nu\right) D(x' - x_i)$$

From the form of Eq. (6.12) it is immediately evident that the broken symmetry does not affect the commutation relations of the current j^μ with the field ψ and we have the usual results[64]

$$[j^0(x), \psi(x')] = -q \frac{1}{1 - \lambda\eta/\pi} \psi(x)\delta(\mathbf{x} - \mathbf{x}')$$

$$[j_5{}^0(x), \psi(x')] = -q \frac{1}{1 + \lambda\xi/\pi} \gamma_5\psi(x)\delta(\mathbf{x} - \mathbf{x}')$$

Having displayed the Fermi Green's functions, it is necessary to check the consistency of definition (6.2) with the result (6.11) through the use of Eq. (6.12). A straightforward calculation shows that the combination of Eq. (6.12) with Eq. (6.2) yields Eq. (6.10) so that no constraint whatever is placed on the theory for consistency of the broken symmetry. This is essentially the same situation as in naturally occurring broken symmetries and, indeed, it is clear that, because the dependence of the theory on the symmetry-breaking parameter [as exemplified by the Green's function (6.12)] is relatively simple, this theory is a natural although intrinsically more complex extension of the models discussed in Section III to composite particle field theories. We may now make contact in the case of the Thirring model with composite particle theories which do not support a broken symmetry in the free field limit by letting $\eta^\mu \to 0$ in such a way that $\langle 0|j^\mu|0\rangle_{A=0}$ does not vanish.[62] We then get a constraint on the coupling constant since this can only occur if

$$\lambda = \pi/\eta$$

or

$$\lambda = -\pi/\xi$$

These, however, are not very satisfactory solutions since pathological singularities are introduced into the Green's functions by such a choice.

In concluding this discussion of the Thirring model, it is amusing to make one further observation about the structure of this theory which will be of use in the consideration of the current–current model in four dimensions. In particular, we note that the equation for the Green's function $G(x - x')$ in the presence of the interaction and the source A_μ has the structure

$$\alpha^\mu\left[\frac{1}{i}\,\partial_\mu - q\left(A_\mu + \lambda\frac{\langle 0|j_\mu|0\rangle}{\langle 0|0\rangle}\right) - \lambda q\,\frac{1}{i}\,\frac{\delta}{\delta A^\mu}\right]G = \delta(x - x')$$

which, in turn, may be written as

$$\alpha^{\mu}\left[\frac{1}{i}\,\partial_{\mu} - q\mathscr{A}_{\mu}(x)\right]G(x, x') + i\int \lambda q\alpha_{\mu}{}'D^{\mu\nu}(x - \xi)\,d\xi G(x, x'') \times \Gamma_{\nu}(x'', x'''; \xi)G(x''', x')\,dx''\,dx''' = \delta(x - x') \quad (6.13)$$

where we have made the identifications

$$\mathscr{A}_{\mu}(x) \equiv A_{\mu}(x) + \lambda\,\frac{\langle 0|j_{\mu}(x)|0\rangle}{\langle 0|0\rangle}$$

$${}'D^{\mu\nu}(x, x') \equiv \frac{\delta\mathscr{A}^{\mu}(x)}{\delta A_{\nu}(x')} = g^{\mu\nu}\delta(x - x') + \lambda D^{\mu\nu}_{\lambda}(x, x')$$

and

$$q\Gamma_{\mu}(x', x'; \xi) = -\frac{\delta}{\delta\mathscr{A}^{\mu}(\xi)}\,G^{-1}(x, x')$$

Equation (6.13) corresponds formally to the Green's function equation for electrodynamics in the presence of an external source. However, in this two-dimensional case, the analogy is purely formal since the singularities of ${}'D^{\mu\nu}(x, x')$ correspond to those of $D^{\mu\nu}_{\lambda}(x, x')$ and occur at zero mass while the boson in two-dimensional electrodynamics is massive. It is, of course, straightforward to check the consistency of Eq. (6.13) with previous forms displayed for this propagation function.

As the obvious generalization of the preceding discussion, we now consider broken-symmetry solutions of the current–current interaction in four dimensions. The natural procedure at this point would be to extend the Lagrangian (6.1) to four dimensions and to calculate the effect of the broken-symmetry condition on its solutions. This problem has been discussed in the literature[65,66] and solutions consistent with current conservation[66] have been presented. However, these solutions have been obtained in a fairly formal way without detailed reference to the definition of the current as the limit of field operators as was done for the Thirring model and as is also necessary here if $[j^0, j^k] \neq 0$ is to be consistent with the definition of $j^{\mu}(x)$. Such considerations are extremely difficult to handle in four dimensions since we anticipate that

$$j^{\mu}(x) = \lim_{\epsilon\to 0} {}^1\!/\!_2\psi(x + \epsilon)\exp\left\{i\lambda q\int_{x-\epsilon}^{x+\epsilon} dx_{\mu}{}''j^{\mu}(x'')\right\}q\,\alpha^{\mu}\psi(x - \epsilon)$$

and

$$\lim_{\epsilon\to 0}\psi(x + \epsilon)\psi(x - \epsilon)$$

is cubically divergent as $\varepsilon \to 0$, as opposed to the linear divergence in the case of one spatial dimension. Since it is probably desirable at this point to avoid detailed consideration of these problems, we shall first discuss the boson current–current interaction in four dimensions. One can readily show in this case that at equal times $[j^0, j^k] \neq 0$ even with a strictly local definition of the current. Consequently, for the level of rigor used here, it is not necessary to worry about the problems of Lorentz invariance, redefinition of the energy operator, etc., which occur when one is forced to define the current as a nonlocal limit. The Lagrangian is assumed to be

$$\mathscr{L} = \phi^\mu \, \partial_\mu \phi + {}^1\!/\!_2 \phi^\mu \phi_\mu - {}^1\!/\!_2 m^2 \phi^2 - {}^1\!/\!_2 g_0 [i\phi q \phi^\mu][i\phi q \phi_\mu] + J^\mu i\phi q \phi_\mu \quad (6.14)$$

Here J is an external source and the two component fields ϕ and ϕ^μ are Hermitian and obey the usual equal–time commutation relation

$$[\phi^0(x), \phi(x')] = -i\delta(\mathbf{x} - \mathbf{x}')$$

Introducing the new operator

$$D'^\mu(j^\mu) = \partial^\mu + ig_0 q j^\mu - iqJ^\mu$$

(6.14) results in the field equations

$$D'^\mu(j^\mu)\phi = -\phi^\mu \quad (6.15)$$

$$D'^\mu(j^\mu)\phi_\mu + m^2\phi = 0 \quad (6.16)$$

It follows from these equations that the current $j^\mu(x) = i\phi(x)q\phi^\mu(x)$ is conserved in the case of vanishing source or if the source current is conserved. Combining Eqs. (6.15) and (6.16), we arrive at the familiar second-order field equation

$$[-D^{\mu\prime}(j^\mu)D_\mu{}'(j^\mu) + m^2]\phi = 0$$

In order to insure current conservation we tacitly assume here that $J^\mu(x)$ is confined to transverse sources. This actually entails no loss of generality in the final answer, but simplifies some calculations.

It is convenient to study the two boson propagators

$$G(x, x') = i\,\frac{\langle 0|(\phi(x)\phi(x'))_+|0\rangle}{\langle 0|0\rangle}$$

and

$$G^\mu(x, x') = i\,\frac{\langle 0|(\phi^\mu(x)\phi(x'))_+|0\rangle}{\langle 0|0\rangle}$$

which in consequence of the field equations, commutation relations, and action principle satisfy the equations

$$G^{\mu}(x, x') = D'^{\mu}\left(\frac{1}{i}\frac{\delta}{\delta J_{\mu}}\right)G(x, x') \tag{6.17}$$

$$D'^{\mu}\left(\frac{1}{i}\frac{\delta}{\delta J_{\mu}}\right)G_{\mu}(x, x') = \delta(x - x') - m^2 G(x, x') \tag{6.18}$$

These may be combined to yield the second-order form

$$\left[-D'^{\mu}\left(\frac{1}{i}\frac{\delta}{\delta J_{\mu}}\right)D_{\mu}'\left(\frac{1}{i}\frac{\delta}{\delta J_{\mu}}\right) + m^2\right]G(x, x') = \delta(x - x')$$

Following the technique outlined in our discussion of the Thirring model, we define a new conserved quantity

$$\begin{aligned}\mathscr{A}^{\mu}(x) &= g_0 \frac{\langle 0|j^{\mu}(x)|0\rangle}{\langle 0|0\rangle} - J^{\mu}(x)\\ &\equiv g_0 \eta^{\mu}(x) - J^{\mu}(x)\end{aligned}$$

and the corresponding propagator function

$$\begin{aligned}D^{\mu\nu}(x, x') &= \frac{\delta}{\delta J_{\mu}(x)}\mathscr{A}^{\nu}(x')\\ &= -g^{\mu\nu}\delta(x - x') + g_0 G^{\mu\nu}(x, x')\end{aligned}$$

where

$$G^{\mu\nu}(x, x') = i\frac{\langle 0|(j^{\mu}(x)j^{\nu}(x'))_{+}|0\rangle}{\langle 0|0\rangle} - \eta^{\mu}(x)\eta^{\nu}(x') + \frac{\left\langle 0\left|\dfrac{\delta j^{\mu}(x)}{\delta J_{\nu}(x')}\right|0\right\rangle}{\langle 0|0\rangle}$$

Then using the chain rule

$$\begin{aligned}\frac{\delta}{\delta J_{\mu}(x)} &= \int \frac{\delta \mathscr{A}^{\nu}(x')}{\delta J_{\mu}(x)}\frac{\delta}{\delta \mathscr{A}^{\nu}(x')}\,dx'\\ &= \int D^{\mu\nu}(x, x')\frac{\delta}{\delta \mathscr{A}^{\nu}(x')}\,dx'\end{aligned}$$

one obtains

$$G^{\mu}(x, x') = -\left[\partial^{\mu} + iq\mathscr{A}^{\mu} + g_0 q\int D^{\mu\nu}(x, x'')\,dx''\frac{\delta}{\delta \mathscr{A}^{\nu}(x'')}\right]G(x, x')$$

and

$$\left[\partial^\mu + iq\mathscr{A}^\mu + g_0 q \int D^{\mu\nu}(x, x'')\, dx'' \frac{\delta}{\delta\mathscr{A}^\nu(x'')}\right] G_\mu(x, x') = \delta(x - x') - m^2 G(x, x')$$

Writing the Green's functions in this way serves to emphasize that upon replacement of g_0 by e_0 they become formally identical to the corresponding functions in electrodynamics.

Of course, to actually establish the equivalence of this theory to electrodynamics, we must demonstrate that the structure of $D^{\mu\nu}$ corresponds exactly to the photon propagator in electrodynamics with an appropriate identification of the renormalized charge and some choice of gauge. It is possible to establish this identification in general with D describing a massless particle if the broken symmetry condition $\eta^\mu(x)|_{J=0} = \eta^\mu \neq 0$ is satisfied. We shall, for the sake of simplicity, examine only the lowest-order approximation in which $\delta G/\delta\mathscr{A}^\nu$ is neglected in the above equations. In this case we have

$$G^\mu(x, x') = -D^\mu(x) G(x, x')$$
$$D^\mu(x) G_\mu(x, x') + m^2 G(x, x') = \delta(x - x')$$

which may be solved to yield

$$G = 1/(-D^\alpha D_\alpha + m^2) \tag{6.19}$$

and

$$G^\mu = -D^\mu[1/(-D^\alpha D_\alpha + m^2)] \tag{6.20}$$

where we have introduced the operator

$$D^\mu = \partial^\mu - iqJ^\mu(x) + ig_0 q\eta^\mu(x)$$

It is now possible to determine the structure of $D^{\mu\nu}(x, x')$. To this end, note that since $j^\mu = i\phi q\phi^\mu$, it follows that

$$\eta^\mu = \frac{\langle 0|j^\mu(x)|0\rangle}{\langle 0|0\rangle} = \mathrm{Tr}\, qG^\mu(x, x)$$

Using this result one obtains

$$\frac{1}{i}\frac{\delta}{\delta J_\nu(x')} D^\mu(x) = qD^{\mu\nu}(x, x')$$

so that differentiation of Eq. (6.19) yields

$$\frac{1}{i}\frac{\delta}{\delta J_\nu} G\bigg|_{J=0} = -D^{\nu\alpha}[GqG_\alpha + G_\alpha qG] \tag{6.21}$$

while differentiation of Eq. (6.20) now results in

$$\frac{1}{i}\frac{\delta}{\delta J_\nu}G_\mu\bigg|_{J=0} = -D^{\nu\alpha}\{g_{\alpha\mu}qG - D_\mu[GqG_\alpha + G_\alpha qG]\} \tag{6.22}$$

Since

$$G^{\mu\nu}(x,x') = \operatorname{Tr} q\,\frac{\delta}{\delta J_\nu(x')}G^\mu(x,x)$$

Eq. (6.22) shows that

$$G_\mu{}^\nu = -iD^{\nu\alpha}[g_{\alpha\mu}\operatorname{Tr}G - \operatorname{Tr}D_\mu(GG_\alpha + G_\alpha G)]$$

which, when inserted into the equation

$$g_0{}^{-1}(D^{\mu\nu} + g^{\mu\nu}) = G^{\mu\nu}$$

with $J^\mu = 0$ yields

$$\int D^{\nu\alpha}(x - x'')\,dx''\{g_{\alpha\mu}\delta(x' - x'') + ig_0[g_{\alpha\mu}\delta(x' - x'')\operatorname{Tr}G(x'',x'') + \operatorname{Tr}G_\mu(x' - x'')G_\alpha(x'' - x') - \int \operatorname{Tr}[D_\mu(x' - x''')dx''' \, G_\alpha(x''' - x'')G(x'' - x')]]\} = -\delta_\mu{}^\nu\delta(x - x') \tag{6.23}$$

If we introduce Fourier transforms so that

$$G(x - x') = \int\frac{dp}{(2\pi)^4}e^{ip(x-x')}G(p)$$

and, similarly for $D^{\nu\alpha}$, we find from Eq. (6.23) that

$$[D^{\alpha\mu}(k)]^{-1} = -g^{\alpha\mu} + \Pi'^{\alpha\mu}(k) \tag{6.24}$$

where

$$\Pi'^{\alpha\mu}(k) \equiv -ig_0\int\frac{dp}{(2\pi)^4}[g^{\alpha\mu}\operatorname{Tr}G(p) + \operatorname{Tr}\{G^\mu(p)G^\alpha(p + k) - D^\mu(p)G^\alpha(p)G(p + k)\}] \tag{6.25}$$

Up to this point, we have not specified that the symmetry be broken. We now make this requirement explicit by setting

$$\frac{\langle 0|j^\mu|0\rangle}{\langle 0|0\rangle}\bigg|_{J=0} = \eta^\mu \neq 0 \tag{6.26}$$

We then find that

$$G(p)|_{J=0} = [(p^\alpha + g_0 q\eta^\alpha)(p_\alpha + g_0 q\eta_\alpha) + m^2]^{-1} \tag{6.27}$$

which has the alternative expression

$$G(p) = \frac{p^2 + m^2 + g_0{}^2\eta^2 - 2g_0 q\eta p}{[p^2 + m^2 + g_0{}^2\eta^2]^2 - 4g_0{}^2(\eta p)^2} \tag{6.28}$$

The consistency of relation (6.26) with the Green's functions as given by Eqs. (6.27) and (6.28) requires that $\eta^\mu = \operatorname{Tr} qG^\mu(0)$ which is given explicitly by

$$\eta^\mu = -2ig_0\eta_\nu \int \frac{dp}{(2\pi)^4} \frac{g^{\mu\nu}(p^2 + m^2 + g_0{}^2\eta^2) - 2p^\mu p^2}{[p^2 + m^2 + g_0{}^2\eta^2]^2 - 4g_0{}^2(\eta p)^2} \tag{6.29}$$

Since the right-hand side of this equation is quadratically divergent, we must introduce a cutoff. Using a Euclidean cutoff, we find for large Λ that Eq. (6.29) reduces to

$$\eta^\mu = g_0\Lambda^2(1/16\pi^2)\eta^\mu \tag{6.30}$$

or

$$1 = g_0{}^2\Lambda^2/16\pi^2 \tag{6.31}$$

Thus in this case, unlike the Thirring model, the consistency of the theory hinges upon placing a constraint on the parameters of the theory. That this constraint arises is due to the nonvanishing boson bare mass which makes it impossible for the free current to have massless excitations and, hence, a broken symmetry.

With this information we may carry out the inversion of Eq. (6.24). It is easily found by direct calculation that

$$\Pi'^{\alpha\mu}(0) = \frac{\partial}{\partial\eta_\alpha} \operatorname{Tr} qG^\mu(0) \tag{6.32}$$

It turns out that this equation is valid for all orders of perturbation theory and assures that $D^{\alpha\mu}(k)$ always has a pole at $k^2 = 0$. From the right-hand side of Eq. (6.30) with the use of Eq. (6.31), Eq. (6.32) becomes

$$\Pi'^{\alpha\mu}(0) = g^{\alpha\mu}$$

so that

$$[D^{\alpha\mu}(k)]^{-1} = [\Pi'^{\alpha\mu}(k) - \Pi'^{\alpha\mu}(0)]$$

Direct evaluation shows that

$$\Pi'^{\alpha\mu}(k) - \Pi'^{\alpha\mu}(0) = [g^{\alpha\mu}k^2 - k^\mu k^\alpha]\bar{I}'(k^2)$$

where $\bar{I}'(k^2)$ when evaluated, using a Euclidean cutoff, is given as

$$\bar{I}'(k^2) = \frac{1}{24}\left[\frac{g_0}{\pi^2}\ln\frac{\Lambda}{m} + \frac{g_0 k^2}{2}\int_{4m^2}^{\infty} d\kappa^2 \frac{(1-4m^2/\kappa^2)^{3/2}}{\kappa^2(k^2+\kappa^2-i\epsilon)}\right]$$

From this we find that

$$D^{\alpha\mu}(k) = \frac{1}{k^2\bar{I}'(k^2)}\left[g^{\alpha\mu} - \frac{k^\alpha k^\mu}{k^2}\right]$$

Thus we may conclude that massless electrodynamics of a scalar field in the presence of a constant external field A^μ is reproduced if the identification $\alpha_0 = (24\pi^2)/\ln(\Lambda/m)$ is made. Consequently, the broken symmetry has no physically measurable effect and, although Lorentz symmetry is broken, the result has not been catastrophic for giving this theory physical meaning.

If one considers broken Lorentz symmetries of interactions with fewer invariance properties, the situation is very different. For example, in the case of a self-interacting vector meson field, B^μ with mass $\mu_0{}^2$ and an interaction of the form $g_0(B^\mu B_\mu)^2$ or a self-interacting Fermi field[67] ψ with broken symmetry conditions $\langle 0|B^\mu|0\rangle \neq 0$, $\langle 0|\psi|0\rangle \neq 0$ one finds, using the same techniques as in the Goldstone model,[68] that the masses of the particles depend upon the Lorentz frame of the observation. Since this is not the basis for an acceptable theory within the scope of our present knowledge, one cannot take the broken symmetry seriously in these cases.

Since we have already verified that $j^\mu(x)$ excites a massless particle, we must verify that this particle contributes with the proper weight to the consistency of the Goldstone commutator, using our approximations in the current–current model. We start with the usual relation

$$-i[J^{\mu\nu}, j^\lambda(x)] = (x^\mu\,\partial^\nu - x^\nu\,\partial^\mu)j^\lambda(x) + g^{\mu\lambda}j^\nu(x) - g^{\lambda\nu}j^\mu(x) \quad (6.33)$$

where, in terms of the energy momentum tensor $T^{\mu\nu}(x)$,

$$J^{\mu\nu} = \int d^3x[x^\mu T^{0\nu}(x) - x^\nu T^{0\mu}(x)] \quad (6.34)$$

Upon taking the vacuum expectation of Eq. (6.33), we find that

$$-i\langle 0|[J^{\mu\nu}, j^\lambda(x)]|0\rangle = g^{\mu\lambda}\eta^\nu - g^{\lambda\nu}\eta^\mu \quad (6.35)$$

which serves to emphasize that the Goldstone theorem guarantees, independently of perturbation theory, the presence of a massless particle in the spectrum of $j^{\lambda}(x)$ and, hence, that the photon of this theory is massless. To ascertain that Eq. (6.35) is consistent with our solutions for this theory, we observe that because of Eq. (6.34) we need only to evaluate the quantity

$$C_{\eta}^{\mu\nu\lambda}(x' - x) \equiv i\langle 0|[T^{\mu\nu}(x'), j^{\lambda}(x)]|0\rangle$$

$C_{\eta}^{\mu\nu\lambda}$ being related in the usual manner to the function

$$T_{\eta}^{\mu\nu\lambda} = \frac{\delta}{\delta J^{\lambda}} \frac{\langle 0|T^{\mu\nu}|0\rangle}{\langle 0|0\rangle}\bigg|_{J=0}$$

To evaluate $T_{\eta}^{\mu\nu\lambda}$ explicitly, the energy-momentum tensor is observed to be

$$T^{\mu\nu} = \phi^{\mu}\phi^{\nu} - g_0 j^{\mu}j^{\nu} + J^{\mu}j^{\nu} + J^{\nu}j^{\mu} - \tfrac{1}{2}g^{\mu\nu} \times [\phi^{\alpha}\phi_{\alpha} - g_0 j^{\alpha}j_{\alpha} + 2J^{\alpha}j_{\alpha} + m^2\phi^2]$$

Then, to the same order of approximation as in the Green's function calculations, we find

$$\begin{aligned}\frac{\langle 0|T^{\mu\nu}(x)|0\rangle}{\langle 0|0\rangle} = &-ig_0 \operatorname{Tr} qG^{\mu}(x, x) \operatorname{Tr} qG^{\nu}(x, x) + iJ^{\mu} \operatorname{Tr} qG^{\nu}(x, x) \\ &+ iJ^{\nu} \operatorname{Tr} qG^{\mu}(x, x) + \tfrac{1}{2}g^{\mu\nu} \\ &\times [-ig_0 \operatorname{Tr} qG^{\alpha}(x, x) \operatorname{Tr} qG_{\alpha}(x, x) + 2iJ^{\alpha} \operatorname{Tr} qG_{\alpha}(x, x)] \\ &+ \text{irrelevant terms}\end{aligned}$$

It is straightforward but tedious to justify the neglect of the terms not explicitly displayed here and, indeed, the consistency of our result will indirectly confirm the validity of this procedure. From the above it follows that

$$T_{\eta}^{\mu\nu\lambda}(k) = \tfrac{1}{2}D^{\lambda\alpha}(k)\{-2\eta^{\nu}[\Pi'^{\mu}_{\alpha}(k) - \Pi'^{\mu}_{\alpha}(0)] + g^{\mu\nu}\eta_{\beta}[\Pi'^{\beta}_{\alpha}(k) - \Pi'^{\beta}_{\alpha}(0)] + (\mu \leftrightarrow \nu)\}$$

Using the forms previously derived for these functions and converting from the time-ordered product to the commutator form, we find

$$C_{\eta}^{\mu\nu\lambda}(k) = -2\pi i k^{\lambda}(\eta k)\left[g^{\mu\nu} - \frac{k^{\mu}\eta^{\nu} + \eta^{\mu}k^{\nu}}{\eta k}\right]\epsilon(k^0)\delta(k^2)$$

Using Eq. (6.34), it is finally verified that the approximations of this model are consistent with Eq. (6.35) with no further constraint on the theory.

Now we will use the same Green's function techniques[69] to consider a theory in which the symmetry-breaking parameter is a mass and, hence, appears in the solutions in a nontrivial physically significant way. We consider the Lagrangian of the Nambu–Jona-Lasinio theory[70]

$$\mathscr{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + g_0[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\tau_i\psi)^2] + \lambda_5{}^i i\bar{\psi}\gamma_5\tau_i\psi + S\bar{\psi}\psi \quad (6.36)$$

where $\lambda_5{}^i$ and S are c-number sources. When the sources are turned off, we find from the usual invariance arguments that the axial current $j^{\mu}_{i5} = \bar{\psi}\gamma^{\mu}\gamma_5\tau_i\psi$ is conserved. We shall not concern ourselves here with the technically correct definition of the current obtained by separating points of the two fields involved in defining j^{μ}_{i5}.

In the source-free limit it is usually argued that as a consequence of the chiral invariance associated with the vanishing of the bare mass term in Eq. (6.36), the physical mass of the particle excited by ψ vanishes. However, such a statement can only be valid if the vacuum is required to be an eigenstate of the pseudoscalar charge operator

$$Q_{5i} = \int d^3x\, j^0_{i5}(x)$$

To see this, we examine the Goldstone commutator

$$[Q_5{}^i, i\bar{\psi}\gamma_5\tau_j\psi] = -2i\delta^i{}_j\bar{\psi}\psi$$

from which it follows that

$$\langle 0|[Q_5{}^i, i\bar{\psi}\gamma_5\tau_i\psi]|0\rangle = -2\,\mathrm{Tr}\,G \quad (6.37)$$

where

$$G(x, x') = i\epsilon(x, x')\left.\frac{\langle 0|(\psi(x)\bar{\psi}(x'))_+|0\rangle}{\langle 0|0\rangle}\right|_{\lambda=\lambda_5{}^i=0}$$

According to the Lehmann representation

$$G(x - x') = \int \frac{dp}{(2\pi)^4} e^{ip(x-x')}\int_{-\infty}^{\infty} d\kappa\, \frac{A(\kappa)}{\gamma p + \kappa}$$

so

$$\mathrm{Tr}\,G(x, x) = 8\int \frac{dp}{(2\pi)^4}\int_{-\infty}^{\infty} d\kappa\, \frac{\kappa A(\kappa)}{p^2 + \kappa^2}$$

Since $A(\kappa)$ generally has the structure

$$A(\kappa) = Z\delta(\kappa - m) + A'(\kappa)$$

one can obtain $\mathrm{Tr}\,G = 0$ if the renormalized mass vanishes and $A'(\kappa) = A'(-\kappa)$. On the other hand, we see that if G is dominated by a single massive excitation such that

$$A(\kappa) = \delta(\kappa - m)$$

then

$$\operatorname{Tr} G = 8 \int \frac{dp}{(2\pi)^4} \frac{m}{p^2 + m^2} \neq 0 \tag{6.38}$$

and the vacuum is not an eigenstate of the chiral current $Q_5{}^i$. Of course, in practice Eq. (6.38) must be given meaning by the introduction of a cutoff and, therefore, we shall choose a simple Euclidean cutoff when this expression is to be explicitly evaluated. The condition described by Eq. (6.38) will be of prime concern in our analysis of the Lagrangian (6.36). Applying the Goldstone theorem to Eq. (6.37) combined with Eq. (6.38) illustrates that massless pseudoscalar bosons are excited by the "composite field operator" $i\bar{\psi}\gamma_5\tau_j\psi$.

Returning to more explicit considerations resulting from Eq. (6.36), we note that the field equations are

$$[i\gamma^\mu\, \partial_\mu + S + \lambda_5{}^i i\gamma_5\tau_i]\psi + 2g_0[(\bar{\psi}\psi)\psi - (\bar{\psi}\gamma_5\tau_i\psi)\gamma_5\tau^i\psi] = 0$$

and

$$-i\,\partial_\mu\bar{\psi}\gamma^\mu + \bar{\psi}S + \bar{\psi}i\gamma_5\tau_i\lambda_5{}^i + 2g_0[\bar{\psi}(\bar{\psi}\psi) - \bar{\psi}\gamma_5\tau^i(\bar{\psi}\gamma_5\tau^i\psi)] = 0$$

from which we find that G in the presence of the sources satisfies the equation

$$\Big\{i\,\gamma^\mu\,\partial_\mu + 2ig_0[\operatorname{Tr} G - \gamma_5\tau^i \operatorname{Tr} \gamma_5\tau_i G] + S + \lambda_5{}^i i\gamma_5\tau_i + 2g_0\Big[\frac{1}{i}\frac{\delta}{\delta S} + i\frac{\delta}{\delta\lambda_5{}^i}\gamma_5\tau^i\Big]\Big\}G(x, x') = \delta(x - x') \tag{6.39}$$

For the first approximation to G, we neglect the variational derivatives and make the identification

$$m = -2ig_0[\operatorname{Tr} G - \gamma_5\tau^i \operatorname{Tr} \gamma_5\tau_i G] \tag{6.40}$$

to find that

$$G = \Big[\gamma^\mu \frac{1}{i}\,\partial_\mu + m - S - \lambda_5{}^i i\gamma_5\tau_i\Big]^{-1} \tag{6.41}$$

We now make the simplifying (but unnecessary) assumption that m is pure scalar so that to this order $A(\kappa) = \delta(\kappa - m)$. Equation (6.40) with the sources off becomes, through the use of Eq. (6.41),

$$\begin{aligned} m &= -2ig_0 \operatorname{Tr} G \\ &= -16ig_0 m \int \frac{dp}{(2\pi)^4} \frac{1}{p^2 + m^2} \end{aligned}$$

and, since we assume m to be nonvanishing, these lead to the constraint equation

$$\begin{aligned} 1 &= -16ig_0 \int \frac{dp}{(2\pi)^4} \frac{1}{p^2 + m^2} \\ &= \frac{g_0}{\pi^2} \left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right] \end{aligned} \tag{6.42}$$

with Λ as the usual Euclidean cutoff.

Now let us make a more careful analysis of Eq. (6.39) in order to see the role played by the Goldstone particle. We make the convenient definitions

$$A_5{}^i = \lambda_5{}^i - 2g_0 \operatorname{Tr} \gamma_5 \tau^i G$$

and

$$A = S + 2ig_0 \operatorname{Tr} G$$

From these and the action principle it is possible to construct Green's functions involving four field operators. Those which are useful here are the meson propagators

$$D_5{}^{ij}(x, x') = i \frac{\delta}{\delta \lambda_{5i}(x)} A_5{}^j(x') \tag{6.43}$$

and

$$-iD(x, x') = \frac{1}{i} \frac{\delta}{\delta S(x)} A(x') \tag{6.44}$$

It is easily established that

$$\frac{1}{i} \frac{\delta}{\delta \lambda_{5i}(x)} A(x') = \frac{1}{i} \frac{\delta}{\delta S(x)} A_5{}^i(x') = 0$$

With these new functions and the chain rule for functional derivatives, Eq. (6.39) becomes

$$\left\{\gamma^\mu \frac{1}{i} \partial_\mu + m - \lambda_5{}^i i\gamma_5 \tau_i - S + 2ig_0\left[\gamma_5 \tau_i D_5{}^{ij} \frac{1}{i} \frac{\delta}{\delta A_5{}^j} - D \frac{1}{i} \frac{\delta}{\delta A}\right]\right\} G = 1$$

This equation formally has the same structure as the propagator for a fermion of bare mass m interacting with a pseudoscalar isovector meson and a scalar meson. To establish equivalence to this order we must, of course, confirm that D_5^{ij} and D have the correct structure. The calculation is handled most easily by introducing the function

$$G^i(x, x'; \xi) = \frac{1}{i} \frac{\delta}{\delta \lambda_{5i}(\xi)} G(x, x')$$

so that

$$D_5^{ij}(\xi, \xi') = \delta^{ij}\delta(\xi - \xi') - 2g_0 i \operatorname{Tr} \gamma_5\tau^j G^i(\xi', \xi'; \xi) \tag{6.45}$$

since

$$G^i(x, x'; \xi) = -\int G(x, x'')\left[\frac{1}{i}\frac{\delta}{\delta\lambda_{5i}(\xi)} G^{-1}(x'', x''')\right]G(x''', x')\, dx''\, dx'''$$

We find from the approximation (6.41) with the use of Eq. (6.40) that

$$\begin{aligned} -G^i(\xi', \xi'; \xi) = G(\xi', \xi)\gamma_5\tau^i G(\xi, \xi') - \int G(\xi', \xi'')\, d\xi'' \\ \times \{-2ig_0 \operatorname{Tr} G^i(\xi'', \xi''; \xi) \\ + 2ig_0 \operatorname{Tr} (\gamma_5\tau_\alpha G^i(\xi'', \xi''; \xi))\gamma_5\tau_\alpha\}G(\xi'', \xi') \end{aligned} \tag{6.46}$$

Taking the trace of Eq. (6.46) on $\gamma_5\tau_j$ and inserting the results into Eq. (6.45), we find that, to this order

$$[D_5^{ij}(\xi, \xi')]^{-1} = \delta_{ij}\delta(\xi - \xi') + 2ig_0 \operatorname{Tr} G(\xi, \xi')\gamma_5\tau_j G(\xi', \xi)\gamma_5\tau_i \tag{6.47}$$

It is of interest to note (as is easily shown) that to all orders we may write

$$[D_5^{ij}(\xi, \xi')]^{-1} = \delta_{ij}\delta(\xi - \xi') + 2ig_0 \frac{1}{i}\frac{\delta}{\delta A_5^i(\xi)} \operatorname{Tr} \gamma_5\tau_j G(\xi', \xi')$$

With the sources off, Eq. (6.47) in momentum space becomes

$$[D_5^{ij}(k)]^{-1} = \delta_{ij} + 2ig_0 \int \frac{dp}{(2\pi)^4} \operatorname{Tr} G(p)\gamma_5\tau_j G(p + k)\gamma_5\tau_i$$

Using the consistency condition (6.42) to handle the most divergent part of the above integral, it follows that

$$D_5^{ij}(k) = \frac{1}{k^2 L}\, \frac{\delta^{ij}}{1 - \dfrac{k^2 g_0}{L 2\pi^2}\displaystyle\int_{4m^2}^{\infty} \frac{d\kappa^2}{\kappa^2}\, \frac{(1 - 4m^2/\kappa^2)^{1/2}}{k^2 + \kappa^2}}$$

where

$$L = \frac{8g_0}{i}\int \frac{dp}{(2\pi)^4}\, \frac{1}{(p^2 + m^2)^2} = \frac{g_0}{\pi^2} \ln \frac{\Lambda}{m}$$

This form clearly displays the massless excitation. Further, we see that if we use an elementary particle interaction between a massless meson and a massive nucleon, the results are identical if one makes the identification

$$\frac{G^2}{2\pi^2} = \frac{g_0}{2\pi^2 L}$$

where G is the renormalized meson–nucleon coupling constant.

Using essentially identical techniques, we find that for the scalar boson the propagator to this order is

$$D(\xi, \xi')^{-1} = \delta(\xi - \xi') - 2ig_0 \operatorname{Tr} G(\xi, \xi')G(\xi', \xi) \tag{6.48}$$

and, to all orders, is given as

$$D(\xi, \xi')^{-1} = \delta(\xi - \xi') - 2ig_0 i \frac{\delta}{\delta A(\xi)} G(\xi', \xi')$$

Explicit evaluation of Eq. (6.48) results in the momentum space representation

$$D(k) = \frac{1}{L(k^2 + 4m^2)} \frac{1}{1 - \dfrac{k^2}{L2\pi^2} \displaystyle\int_{4m^2}^{\infty} \frac{d\kappa^2}{\kappa^2} \frac{(1 - 4m^2/\kappa^2)^{1/2}}{k^2 + \kappa^2}}$$

so that the scalar particle of this theory has a mass $2m$.

It is to be noted that, in the case in which the symmetry is not broken, it is required that the pseudoscalar meson and scalar meson have the same mass. Indeed, the proof of this statement provides an interesting example of the "bad" behavior of the generator Q_5^i. If $|\pi\rangle$ is a state of a pseudoscalar meson such that $H|\pi\rangle = E|\pi\rangle$ (H being the Hamiltonian) and Q_5 is assumed to be well defined and time independent, we see that the scalar particle state $Q_5|\pi\rangle$ has the same mass since

$$\begin{aligned} HQ_5|\pi\rangle &= Q_5 H|\pi\rangle \\ &= EQ_5|\pi\rangle \end{aligned}$$

That this does not follow here just confirms that the operator

$$Q_5^V = \int_V j_5^0(x)\, d^3x$$

does not exist in the limit $V \to \infty$. For finite volume, of course, we expect the matrix element of this operator to depend on x^0 and, hence, $[Q_5^V, H] \neq 0$.

It is of some interest to examine the other two-point functions of this theory. In particular, it is found that the matrix element

$$G_{ij}^{\mu\nu}(x, x') = i\langle 0|\,(j_{5i}^{\mu} j_{5j}^{\nu})_{+}\,|0\rangle$$

has the Fourier transform

$$G_{ij}^{\mu\nu}(k) = i\int \frac{dp}{(2\pi)^4}\,\mathrm{Tr}\,\gamma^{\mu}\gamma_5\tau_i\,\frac{1}{\gamma p + m}\,\gamma^{\nu}\gamma_5\tau_j\,\frac{1}{\gamma(p+k)+m}$$
$$-2\delta_{ij}\,\frac{m^2}{\pi^2}\ln\frac{\Lambda}{m}\,\frac{k^{\mu}k^{\nu}}{k^2}$$

which again exhibits the massless boson. Similarly, a straightforward calculation shows that

$$\langle 0|\,[\bar{\psi}(x)\,\gamma^{\mu}\gamma_5\tau_i\psi(x),\, i\bar{\psi}(x')\gamma_5\tau_j\psi(x')]\,|0\rangle$$
$$= -2\pi\int \frac{dp}{(2\pi)^4}\, e^{ip(x-x')}\,\epsilon(p^0)\delta(p^2)2p^{\mu}\delta_{ij}\,\mathrm{Tr}\,G(x, x)$$

which is consistent with Eq. (6.37).

It is remarkable to note that in this model we have performed operations on a nonrenormalizable theory that have demonstrated its equivalence to a renormalizable theory. This peculiar result corresponds to the fact that different rearrangements of a power series expansion of such a divergent theory can lead to different answers. The broken-symmetry condition and the associated Green's function technique of solutions used here have served to pick out a particular familiar solution. As is well known, there may be other solutions.

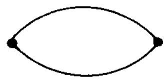
It is of some interest to consider what happens to this model(69) in the presence of the electromagnetic interaction

$$\mathscr{L}_{\text{int}}^{\text{E.M.}} = e\bar{\psi}\,\gamma^{\mu}[(1-\tau_3)/2]\psi A_{\mu}$$

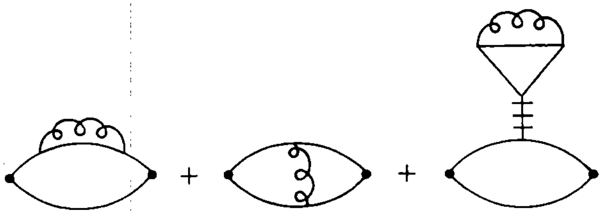
A simple calculation shows that

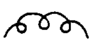
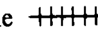
$$\partial_{\mu} j_{5i}^{\mu} = e\epsilon_{3ij}A^{\mu}\bar{\psi}\,\gamma_{\mu}\gamma_5\tau_j\psi$$

so only j_{53}^{μ} is now conserved. Using the Goldstone theorem it is seen that the π^0 meson alone is required to remain massless if the fermion mass $m \neq 0$. The mass of the charged mesons may be calculated to second order in e using the propagation functions derived as above as the $e = 0$ limit of the theory. These calculations are very tedious, but it is amusing to note the type of graphs that contribute. Using Eq. (6.47) we may give the pion propagator the pictorial representation

$$(D_5^{ij})^{-1} = 1 +$$ 

where the solid lines are nucleon propagators G. This clearly points out the composite nature of the pion and shows that the electromagnetic mass splitting will be given as corrections to the nucleon bubble. The contributing terms are of the form



Here  represents the photon propagator while  is the propagator of the pseudoscalar meson. The structure of the corrections serves to emphasize the composite nature of the pion in this theory.

There are, of course, many other theories which have been considered in the context of nonsoluble relativistic broken symmetries but, for the most part, these theories are either modifications of the above theories or are much more complicated. In the latter category are the electrodynamics of Johnson, Baker, and Willey,[72] and the fundamental four-Fermi interactions of Heisenberg et al.[71] Within the framework of broken symmetries as discussed here, both of these theories appear to have serious faults. However, until such time as more complete solutions are presented, it is perhaps desirable to defer final judgment on the possible inconsistency of the broken-symmetry aspect of these theories.

VII. Symmetry Breaking Effects in Some Noncovariant Field Theories

Despite the appreciable number of examples of relativistic theories displaying various forms of symmetry breakdown which have been presented in the preceding sections, none of the models considered can be described as being entirely satisfactory as a physically meaningful broken-symmetry theory of elementary particles. In particular, it has been noted that although the discussion of naturally occurring broken symmetries demonstrated how it may be possible in certain theories to solve the problem of how to consistently induce a nonvanishing expectation value for a certain field operator, one was nonetheless led to the conclusion that a symmetry breaking of this type is purely formal. Thus the fact that one can in such cases generate a nonvanishing vacuum expectation value for a field $\phi(x)$ has been seen to arise entirely from the cyclic (or ignorable) property of $\phi(x)$ and the consequent feature that $\phi(x)$ is undefined to within an additive constant. Since this ambiguity does not have any effect upon the (appropriately truncated) Green's functions, there can never be any observable consequence of choosing $\langle 0|\phi(x)|0\rangle \neq 0$ regardless of how complex the interactions may be. This criticism, of course, does not apply to what we have called spontaneous symmetry breaking where the breakdown of symmetry will generally manifest itself in all measurement processes. On the other hand, theories of this latter type have a considerably less certain mathematical foundation, there being little basis for judging the consistency of constraints of the form (2.5).

This then naturally leads one to ask whether there is not a class of theories in which the desired features of mathematical tractability and physical nontriviality may be simultaneously realized. It has in fact recently been shown[73,74] that in the domain of nonrelativistic field theories there are a number of models which combine both of these attributes and we shall consequently devote the entirety of this chapter to a discussion of these rather remarkable theories.

In order to clarify the connection to our previous discussion as much as possible, we recall that the momentum space representation of the Hamiltonian for a simple scalar field has the form

$$H = \int d^3k\omega_k a^*(\mathbf{k})a(\mathbf{k}) \tag{7.1}$$

where

$$\omega_k = (\mathbf{k}^2 + \mu^2)^{1/2}$$

It was shown in Section III that for such a system one can formally construct from the vacuum $\langle 0|$ the state

$$\begin{aligned}\langle a'| &= \langle 0|U \\ &= \langle 0| \exp\{a'^*a(\mathbf{0}) - a^*(\mathbf{0})a'\}\end{aligned}$$

such that

$$\langle a'|a(\mathbf{k})|a'\rangle = a'\delta(\mathbf{k})$$

thus providing an inequivalent representation of the commutation relations by means of the improper unitary operator U. Now although it is readily shown in the case $\mu^2 = 0$ that it is consistent to take $\langle a'|$ to be a zero-energy eigenstate of the Hamiltonian (7.1), it is precisely the difficulty in generalizing this result to the interacting case which is responsible for the highly conjectural nature of most broken-symmetry theories. This problem can be avoided in the case of naturally occurring broken symmetries by virtue of the cyclic nature of the field $\phi(x)$ but only at the expense of accepting what is essentially a trivial type of symmetry breaking. It is easy to convince oneself that the problem is basically related to the fact that upon introducing a coupling of the field $\phi(x)$ of the form

$$J(x)\phi(x)$$

[where $J(x)$ is generally bilinear in the elementary field operators of the theory], it is not possible to solve the eigenvalue problem for those states containing an arbitrary number of massless bosons, and consequently one cannot retain the state $\langle a'|$ as a zero-energy eigenstate of H. Since this circumstance is a consequence of the fact that in a Lorentz-invariant theory J will generally be capable of creating an arbitrarily large number of particles, it is clear that only when one gives up the requirement of relativistic covariance (and thereby the antiparticle concept) does it become possible to identify $\langle a'|$ with the state of lowest energy and at the same time produce a physically nontrivial effect.

The simplest example of a theory which is capable of displaying nontrivial symmetry-breaking effects is the well-known neutral scalar theory. In view of its unusually simple structure a detailed study of this model is quite feasible and at the same time can be expected to provide considerable insight into some aspects of broken symmetries. The relevant Hamiltonian is

$$H = mN^*N + \int d^3k\omega_k\theta^*(\mathbf{k})\theta(\mathbf{k}) - N^*N\int d^3k\alpha(|\mathbf{k}|)[\theta(\mathbf{k}) + \theta^*(\mathbf{k})]$$

where

$$\alpha(|\mathbf{k}|) = g_0\frac{u(|\mathbf{k}|)}{(2\pi)^{3/2}}\frac{1}{(2\omega)^{1/2}}$$

$u(|\mathbf{k}|)$ being the usual form factor inserted to guarantee the convergence of the theory. The commutation relations are taken to be of the form

$$\{N, N^*\} = 1 \tag{7.2}$$

$$[\theta(\mathbf{k}), \theta^*(\mathbf{k})] = \delta(\mathbf{k} - \mathbf{k}') \tag{7.3}$$

It is customary to choose that particular representation of the commutation relations defined by the requirement that the vacuum be annihilated by the operators N and $\theta(\mathbf{k})$, i.e.,

$$N|0\rangle = 0 \tag{7.4}$$

$$\theta(\mathbf{k})|0\rangle = 0 \tag{7.5}$$

Because of the fact that the fermion field refers to only a single degree of freedom, it will turn out that there are no representations of Eq. (7.2) which are not unitarily equivalent to Eq. (7.4), and we shall consequently focus our attention entirely on the construction of representations of Eq. (7.3) other than that described by Eq. (7.5).

Let us assume for the moment the existence of a state $\langle\theta'|$ which is degenerate with the vacuum in the sense that $\langle\theta'|H = 0$ and such that it satisfies the condition

$$\langle\theta'|\theta(\mathbf{k})|\theta'\rangle \neq 0 \tag{7.6}$$

It follows from the equations of motion

$$i\frac{\partial\theta(\mathbf{k}, t)}{\partial t} = [\theta(\mathbf{k}), H] = \omega_k\theta(\mathbf{k}) - \alpha(|\mathbf{k}|)N^*N$$

$$i\frac{\partial N(t)}{\partial t} = [N, H] = mN - N\int d^3k\alpha(|\mathbf{k}|)[\theta(\mathbf{k}) + \theta^*(\mathbf{k})]$$

that Eq. (7.6) is consistent only if

$$\omega_k\langle\theta'|\theta(\mathbf{k})|\theta'\rangle = 0 \tag{7.7}$$

i.e., $\mu = 0$ and $\langle\theta'|\theta(\mathbf{k})|\theta'\rangle \sim \delta(\mathbf{k})$. Since θ' can be defined to be the constant of proportionality in this relation, it follows that in the limit of vanishing θ mass one can take

$$\langle\theta'|\theta(\mathbf{k})|\theta'\rangle = \theta'\delta(\mathbf{k}) \tag{7.8}$$

It is well to emphasize at this point that the theory we consider here is indeed intermediate in complexity between the naturally occurring and the spontaneous types of symmetry breaking. Thus by requiring $\alpha(0) \neq 0$ (which condition we shall impose throughout this section) the zero-momentum mode of the $\theta(\mathbf{k})$ field is not decoupled and $\theta(\mathbf{k}) \to \theta(\mathbf{k}) + \theta'\delta(\mathbf{k})$ is consequently not an invariance property of H. On the other hand, despite the nontrivial coupling of the zero-momentum mode, the constraint (7.8) is clearly consistent at the kinematical level and, in marked contrast to the cases considered in the preceding section, does not require any further constraints on the higher order vacuum expectation values. It is, of course, this latter circumstance which is responsible for the somewhat uncertain foundations of the usual theories of spontaneous symmetry breaking.

Here again it is possible to carry out the formal construction of $\langle\theta'|$ in terms of the usual improper transformation on $\langle\theta|$, i.e.,

$$\begin{aligned}\langle\theta'| &= \langle 0|U(\theta') \\ &= \langle 0|\exp\{\theta'^*\theta(\mathbf{0}) - \theta^*(\mathbf{0})\theta'\}\end{aligned}$$

Despite the frequently mentioned fact that the operator $U(\theta')$ is not really well defined, one finds that in the calculation of all quantities of any possible physical interest one may conveniently ignore this slight subtlety. Thus for the present $U(\theta')$ will be freely employed as if it were an ordinary unitary operator with the implicit understanding that we shall subsequently return to reformulate the solution entirely in terms of the vacuum expectation values of the theory. Thus this discussion will consider $U(\theta')$ to be essentially only a heuristic tool which, though quite invaluable for purposes of displaying the physical content of the theory, can be entirely eliminated in the event that mathematical rigor may be preferred to simplicity of formulation.

In order to be able to discuss the effect of Eq. (7.8) on the solution it is necessary to briefly comment on the usual approach to the theory. The most direct treatment consists in defining the operator

$$U = \exp\left\{-N^*N\int d^3k\,\frac{\alpha(|\mathbf{k}|)}{\omega_k}\,[\theta(\mathbf{k}) - \theta^*(\mathbf{k})]\right\}$$

which clearly generates the transformation.

$$\begin{aligned}\tilde{\theta}(\mathbf{k}) &\equiv U\theta(\mathbf{k})U^* = \theta(\mathbf{k}) - \frac{\alpha(|\mathbf{k}|)}{\omega_k}N^*N \qquad (7.9)\\ \tilde{N} &\equiv UNU^* = \exp\left\{\int d^3k\,\frac{\alpha(|\mathbf{k}|)}{\omega_k}\,[\theta(\mathbf{k}) - \theta^*(\mathbf{k})]\right\}N\end{aligned}$$

In terms of the set (7.9) the Hamiltonian assumes the form

$$H = \left(m - \int d^3k \frac{\alpha^2(|\mathbf{k}|)}{\omega}\right)\tilde{N}^*\tilde{N} + \int d^3k\omega_k\tilde{\theta}(\mathbf{k})\tilde{\theta}(\mathbf{k})$$

a result which immediately displays the well-known equivalence of the model to a theory describing uncoupled bosons and fermions of masses μ and

$$m - \int d^3k \frac{\alpha^2(|\mathbf{k}|)}{\omega_k}$$

respectively.

Although the above transformation appears at first sight to exhaust the entire physical content of the theory, there is one singular case which requires particular attention. This special circumstance occurs when $\alpha(0)$ is finite and $\omega(\mathbf{k} = 0) \doteq 0$, i.e., in the case of a massless θ particle in the theory with a nonvanishing coupling of the zero-momentum mode of the field $\theta(\mathbf{k})$. In this case the transformation (7.9) clearly becomes singular and there arises the possibility of constructing additional solutions based on the inequivalent representations of the commutation relations.

Since our discussion of these additional representations has already remarked upon the consistency of the condition

$$\langle\theta'|\theta(\mathbf{k})|\theta'\rangle = \theta'\delta(\mathbf{k}) \tag{7.10}$$

there remains only the task of demonstrating whether this constraint has a nontrivial effect upon the eigenvalue spectrum of the theory. To this end it is again convenient to refer to the operator $U(\theta')$ which, since it formally generates c-number translations on $\theta(\mathbf{k})$, can be used to provide a realization of Eq. (7.10). This result is made most transparent upon considering the vacuum expectation value

$$\langle\theta'|N(t)N^*(t')|\theta'\rangle$$

Now although one can (formally) show that the operator $U(\theta')$ in the case of a naturally occurring broken symmetry is time independent, it follows from the nontrivial coupling of $\theta(\mathbf{0})$ in this model that $U(\theta')$ cannot be assumed to be a constant of the motion. Thus we use the more explicit notation $U(\theta', t = 0)$ to denote the fact that we have chosen to construct the state $\langle\theta'|$ in terms of the operators $\theta(\mathbf{k}, t = 0)$ and $\theta^*(\mathbf{k}, t = 0)$. One thus has

$$\begin{aligned}\langle\theta'|N(t)N^*(t')|\theta'\rangle &= \langle 0|U(\theta', t = 0)\, e^{iHt}N(0)\, e^{-iH(t-t')} \\ &\qquad\times N^*(0)\, e^{-iHt'}U^*(\theta', t = 0)|0\rangle \\ &= \langle 0|N(0)\, e^{-iH(N,\theta+\theta'\delta(\mathbf{k}))(t-t')}N^*(0)|0\rangle\end{aligned}$$

where we have used the notation $H[N, \theta + \theta'\delta(\mathbf{k})]$ to indicate that in the expression for H the operator $\theta(\mathbf{k})$ is to be replaced by $\theta(\mathbf{k}) + \theta'\delta(\mathbf{k})$. The above establishes the equivalence of the neutral scalar theory with the broken-symmetry condition (7.8) to a theory in which one chooses the usual representation

$$\theta(\mathbf{k})|0\rangle = 0$$

of the commutation relation and incorporates the symmetry-breaking effect directly into the Hamiltonian. It is perhaps unnecessary to remark that the same calculations can be performed for the case of a naturally occurring broken symmetry with, however, the important modification that because of the cyclic nature of the zero mass field the Hamiltonian (and consequently all the Green's functions) of the model are invariant under this transformation.

Before going on to the next higher stage of complexity in theories which possess a nontrivial symmetry breaking, we shall provide a more rigorous derivation of the above results, thereby demonstrating the existence of the alternative approach alluded to earlier in this chapter. Thus we seek to verify for this model the statement that the operator $U(\theta', t)$ is indeed inessential for purposes of establishing the equivalence of the broken-symmetry condition to the c-number translation of $\theta(\mathbf{k})$ in the Hamiltonian. Since the net result of the replacement of $\theta(\mathbf{k})$ by $\theta(\mathbf{k}) + \theta'\delta(\mathbf{k})$ in the neutral scalar theory is the addition of the term

$$-2\alpha(0)\, N^*N \operatorname{Re} \theta'$$

it is to be shown directly from the equations for the Green's functions that the only nontrivial symmetry-breaking effect consists of a mass renormalization. Although it is also possible in some of the other soluble models which we shall discuss here to carry out the proof of equivalence by direct calculation, we shall only provide the alternative derivation in the neutral scalar case and refer the reader elsewhere for a discussion of additional examples.[73] However, it is well to emphasize at the outset the obvious impossibility of giving a completely general proof of this result, a circumstance which follows from the fact that there is no guarantee that the transformed Hamiltonian describes a meaningful theory except in the relatively few cases where an exact solution can be obtained for a wide class of inequivalent representations of the canonical commutation relations.

To carry out the proof for the neutral scalar theory we shall focus attention on the two-point function

$$G(t) = i\langle 0|(N(t)N^*(0))_+|0\rangle$$

the extension of this discussion to the most general vacuum expectation value being fairly straightforward. It is convenient to introduce a source function $J(\mathbf{k})$ by adding to the Hamiltonian the term

$$-\int d^3k[\theta(\mathbf{k}) + \theta^*(\mathbf{k})]J(\mathbf{k})$$

thereby enabling one to write the equation for $G(t)$ in terms of functional derivatives with respect to $J(\mathbf{k})$. We shall consider this source to be essentially an arbitrary function in the interval (t_2, t_1) subject only to the condition that it vanishes at t_1 and t_2 $(t_1 > t_2)$. One then anticipates from our previous discussion that the physical vacuum $|\theta' t_2\rangle$ at time t_2 is to be defined by

$$\theta(\mathbf{k}, t_2)|\theta' t_2\rangle = \theta'\delta(\mathbf{k})|\theta' t_2\rangle \tag{7.11}$$

with the corresponding result

$$\langle\theta' t_1|\theta^*(\mathbf{k}, t_1) = \theta'^*\delta(\mathbf{k})\langle\theta' t_1| \tag{7.12}$$

used to define the vacuum $\langle\theta' t_1|$, after the source has been turned off. In fact Eqs. (7.11) and (7.12) need not be imposed as eigenvalue equations, since it is sufficient to merely impose the broken-symmetry condition in the weaker form

$$\langle\theta' t_1|\theta(\mathbf{k}, t_2)|\theta' t_2\rangle = \theta'\delta(\mathbf{k})\langle\theta' t_1|\theta' t_2\rangle \tag{7.13}$$

$$\langle\theta' t_1|\theta^*(\mathbf{k}, t_1)|\theta' t_2\rangle = \theta'^*\delta(\mathbf{k})\langle\theta' t_1|\theta' t_2\rangle \tag{7.14}$$

In the presence of the source $J(\mathbf{k})$ the appropriate Green's function $G(t)$ is

$$G(t, J) = i\frac{\langle\theta' t_1|(N(t)N^*(0))_+|\theta' t_2\rangle}{\langle\theta' t_1|\theta' t_2\rangle}$$

which, in terms of the amplitude

$$\langle \theta' t_1 | \theta' t_2 \rangle = e^{iw(J)}$$

satisfies the equation

$$e^{-iw(J)}\left[m - i\frac{\partial}{\partial t} - \int d^3k\alpha(|\mathbf{k}|)\frac{1}{i}\frac{\delta}{\delta J(\mathbf{k})}\right] e^{wi(J)}G(t) = \delta(t)$$

In order to evaluate $w(J)$ we note that

$$\frac{\delta}{\delta J(\mathbf{k})}\langle \theta' t_1 | \theta' t_2 \rangle = i\langle \theta' t_1 | \theta(\mathbf{k}) + \theta^*(\mathbf{k}) | \theta' t_2 \rangle \tag{7.15}$$

Upon using the equation of motion

$$i\frac{\partial}{\partial t}\theta(\mathbf{k}) = \omega_k \theta(\mathbf{k}) - \alpha(|\mathbf{k}|)N^*N - J(\mathbf{k})$$

one easily deduces that

$$\frac{\langle \theta' t_1 | \theta(\mathbf{k}, t) | \theta' t_2 \rangle}{\langle \theta' t_1 | \theta' t_2 \rangle} = \theta' \delta(\mathbf{k}) + \int_{t_2}^{t_1} \mathscr{G}_r(\mathbf{k}, t - t')J(\mathbf{k}, t')\, dt'$$

where

$$\begin{aligned} \mathscr{G}_r(\mathbf{k}, t - t') &= \int \frac{dE}{2\pi} e^{-iE(t-t')} \frac{1}{\omega_k - E - i\epsilon} \\ &= i\theta_+(t - t_1)\, e^{-i\omega_k(t-t')} \end{aligned}$$

and we have used Eq. (7.14). From the corresponding result for the matrix element of $\theta^*(\mathbf{k})$ one has the result

$$\frac{\langle \theta' t_1 | \theta(\mathbf{k}, t) + \theta^*(\mathbf{k}, t) | \theta' t_2 \rangle}{\langle \theta' t_1 | \theta' t_2 \rangle} = 2\,\mathrm{Re}\,\theta' \delta(\mathbf{k}) + \int_{t_2}^{t_1} \mathscr{G}(\mathbf{k}, t - t')J(\mathbf{k}, t')\, dt' \tag{7.16}$$

where

$$\mathscr{G}(\mathbf{k}, t) = \int \frac{dE}{2\pi} e^{-iEt}\left[\frac{1}{\omega_k - E - i\epsilon} + \frac{1}{\omega_k + E - i\epsilon}\right]$$

One can thus integrate Eq. (7.15) to obtain the explicit expression for $w(J)$

$$\begin{aligned} w(J) = \tfrac{1}{2}\int d^3k \iint_{t_2}^{t_1} dt\, dt' J(\mathbf{k}, t)\mathscr{G}(\mathbf{k}, t - t')J(\mathbf{k}, t') \\ + 2\,\mathrm{Re}\,\theta' J(\mathbf{k} = 0, E = 0) \end{aligned}$$

where

$$J(\mathbf{k}, E) = \int e^{iEt} J(\mathbf{k}, t)\, dt$$

To facilitate the solution of the integro-differential equation for $G(t, J)$ we introduce at this point the operator[75]

$$I(t) = \exp\left\{\int d^3k\, dE\, J(\mathbf{k}, E)(e^{-iEt} - 1); \frac{\delta}{\delta J(\mathbf{k}, E)}\right\}$$

where the semicolon indicates that in the expansion of $I(t)$ all variational derivatives appear to the right of all J's. One can readily deduce from this definition the property

$$\frac{\delta}{\delta J(\mathbf{k}, t)} I(t') = I(t') \frac{\delta}{\delta J(\mathbf{k}, t + t')}$$

which leads to the alternative representation of $I(t)$

$$I(t) = \exp\left\{-it \int d^3k\, dE\, E J(\mathbf{k}, E) \frac{\delta}{\delta J(\mathbf{k}, E)}\right\}$$

Defining $I(t)G(t) \equiv \bar{G}(t)$ and taking the Fourier transform of equation for $G(t)$, there follows

$$e^{-iw(J)}\left[m - E + \int E'\, d^3k'\, dE' J(\mathbf{k}', E') \frac{\delta}{\delta J(\mathbf{k}', E')} + \int d^3k'\, dE' \frac{\alpha(|\mathbf{k}'|)}{i} \frac{\delta}{\delta J(\mathbf{k}', E')}\right] e^{iw(J)} \bar{G}(E) = 1$$

from which one can immediately deduce the formal solution

$$G(E) = i\int_0^\infty dx\, e^{-ix(m-E)} \exp\left\{-ix \int E'\, d^3k'\, dE' J(\mathbf{k}', E') \frac{\delta}{\delta J(\mathbf{k}', E')} + ix \int d^3k'\, dE' \alpha(|\mathbf{k}'|) \frac{1}{i} \frac{\delta}{\delta J(\mathbf{k}', E')}\right\} e^{iw(J)}\Big|_{J=0} \tag{7.17}$$

where we have observed that $\bar{G}(E; J = 0) = G(E; J = 0)$. One can now make use of the operator identity[75]

$$e^{A+B} = e^B \exp\left(A \frac{1 - e^{-\lambda}}{\lambda}\right)$$

λ being a c-number defined by the commutator condition

$$[A, B] = -\lambda A$$

to rewrite Eq. (7.17) as

$$G(E) = i\int_0^\infty dx\, e^{-ix(m-E)} \exp\left\{\int d^3k'\, dE'\, \frac{1-e^{iE'x}}{-iE'} \times \alpha(|\mathbf{k}'|)\frac{\delta}{\delta J(\mathbf{k}', E')}\right\} e^{iw(J)}\Big|_{J=0}$$

Using Eq. (7.16) one readily obtains the complete Green's function

$$\begin{aligned} G(E) &= i\int_0^\infty dx\, e^{-ix(m-E-2\alpha(0)\,\mathrm{Re}\,\theta')} \exp\left\{\frac{i}{2}\int\frac{d^3k'\,dE'}{2\pi}\,\frac{1-\cos E'x}{E'^2}\right. \\ &\qquad \left.\times\, \alpha^2(|\mathbf{k}'|)\left(\frac{1}{\omega'-E'-i\epsilon}+\frac{1}{\omega'+E'-i\epsilon}\right)\right\} \qquad (7.18) \\ &= i\int_0^\infty dx\, e^{-ix(m'-E)} \exp\left\{-\int d^3k\, \alpha^2(|\mathbf{k}|)\,\frac{1-e^{-i\omega x}}{\mathbf{k}^2}\right\} \end{aligned}$$

where we have defined

$$m' = m - 2\alpha(0)\,\mathrm{Re}\,\theta' - \int d^3k\,\frac{\alpha^2(|\mathbf{k}|)}{|\mathbf{k}|}$$

It is now relatively straightforward to verify that the solution given by Eq. (7.18) has a simple pole at $E = m'$, a result which is in complete agreement with the somewhat more heuristic derivation of the symmetry-breaking effect quoted earlier in this chapter. Thus, in consequence of the fact that the quantity θ' appears only in the expression for m' it follows that the symmetry-breaking effect consists entirely of a mass renormalization, thereby verifying the results of the more direct approach utilizing $U(\theta', t)$. It is also possible to extract from Eq. (7.18) some of the familiar properties of the neutral scalar theory which are customarily obtained by the somewhat less elegant approach in which one specifies the representation (7.5) at the outset and subsequently carries out the explicit construction of the single fermion state. In particular, the wave function renormalization constant Z is found by inspection to be given by

$$Z = \exp\left[-\int d^3k\,\frac{\alpha^2(|\mathbf{k}|)}{\mathbf{k}^2}\right] \qquad (7.19)$$

It may be of interest to remark that this result can also be obtained directly by using the canonical variables defined by Eq. (7.9). Thus the definition

$$\begin{aligned} Z &= |\langle 0|N|N\rangle|^2 \\ &= |\langle 0|N\tilde{N}^*|0\rangle|^2 \end{aligned}$$

with the identity

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

for $[A,B]$ a c-number, immediately implies

$$\begin{aligned} Z &= |\langle 0| \exp\left\{-\int d^3k \frac{\alpha(|\mathbf{k}|)}{|\mathbf{k}|} [\theta(\mathbf{k}) - \theta^*(\mathbf{k})]\right\}|0\rangle|^2 \\ &= \exp\left[-\int d^3k \frac{\alpha^2(|\mathbf{k}|)}{\mathbf{k}^2}\right] \end{aligned} \tag{7.20}$$

in agreement with Eq. (7.19).

Before leaving this interesting model it would be well to comment in a more precise fashion upon the meaning we attach to the phrase "nontrivial symmetry breaking" within the context of this section. Thus we attempt to anticipate the possible objection that since the sole effect of the condition

$$\langle 0|\theta(\mathbf{k})|0\rangle = \theta'\delta(\mathbf{k}) \tag{7.21}$$

is to induce a mass renormalization term (which, of course, is not strictly speaking an observable quantity), the symmetry-breaking effect is in fact a trivial one. Without going into great detail on this point it is sufficient to note that a simple generalization of the neutral scalar theory shows that such an objection may be easily discounted. This can be accomplished by simply doubling the number of degrees of freedom of the N field and replacing the interaction term in the Hamiltonian by

$$-N^*\tau_i N \int d^3k\alpha(|\mathbf{k}|)[\theta(\mathbf{k}) + \theta^*(\mathbf{k})] \tag{7.22}$$

where τ_i is any one of the three Pauli spin matrices. Thus the fermion field in this variant of the model is described by two distinct operators N_1 and N_2 which for $\mu \neq 0$ are easily shown to possess identical excitation spectra. For vanishing θ mass it immediately follows in complete analogy to our previous discussion that one can consistently require

$$\langle 0|\theta(\mathbf{k})|0\rangle \neq 0$$

In this case, however, one sees that the single fermion states are associated with the two eigenvalues

$$m \pm 2\alpha(0) \operatorname{Re} \theta' - \int d^3k \frac{\alpha^2(|\mathbf{k}|)}{|\mathbf{k}|}$$

a result which clearly shows that for nonvanishing symmetry breaking one has an observable mass difference $4\alpha(0)$ Re θ' between the two single fermion states.

It is also interesting to note that this simple extension of the neutral scalar theory is particularly useful inasmuch as it actually requires one to break an invariance of the Hamiltonian in order to guarantee the consistency of Eq. (7.21). We refer to the fact that in the strict sense it is not entirely accurate in the case of the neutral scalar theory to refer to Eq. (7.21) as a broken-symmetry condition inasmuch as there is no invariance property which requires the vanishing of the expectation value of $\theta(\mathbf{k})$. On the other hand, it is clear that with the interaction term (7.22) the Hamiltonian is invariant under

$$\begin{aligned} \theta(\mathbf{k}) &\rightarrow -\theta(\mathbf{k}) \\ N &\rightarrow -i\tau_j N \qquad j \neq i \end{aligned} \tag{7.23}$$

a transformation which is effected by the operator

$$U = \exp\left\{i\frac{\pi}{2}N^*\tau_j N + i\pi\int \theta^*(\mathbf{k})\theta(\mathbf{k})\,d^3k\right\}$$

Since the requirement that the set of physical states in this extension of the neutral scalar theory fail to respect the invariance (7.23) may be imposed in the form (7.21), it is by no means inappropriate to designate this a broken-symmetry theory. It should be remarked that although the invariance property described by (7.23) represents a discrete symmetry of the Hamiltonian rather than the usual case of a continuous symmetry group, one can easily construct theories in which the symmetry breaking occurs in this latter type of invariance group by replacing (7.22) with an SU_2-invariant term describing the interaction of two fermions with a triplet of massless mesons. By requiring that one of the fields $\theta_i(\mathbf{k})$ satisfy a condition of the type (7.21) one can consistently break the SU_2 symmetry, but unfortunately the model is no longer exactly soluble for nonvanishing symmetry breaking. Thus, the fact that the preceding discussion has dealt with the breakdown of a discrete invariance group has not been motivated by any indication that the techniques described here cannot be applied to continuous groups but is rather a consequence of the fact that we lack sufficiently powerful mathematical tools to deal with these more complex theories.

There does exist, however, at least one well-known model with a continuous invariance group in which the solubility of the theory is not destroyed by imposing a condition of the type (7.21). This theory describes the interaction of a fixed fermion with a θ meson as described by the Hamiltonian

$$H = mN^*N + \int d^3k\omega_k\theta^*(\mathbf{k})\theta(\mathbf{k}) - \lambda \int d^3k\alpha(|\mathbf{k}|)\theta^*(\mathbf{k}) \int d^3k'\alpha(|\mathbf{k}'|)\theta(\mathbf{k}')N^*N \quad (7.24)$$

where $\alpha(|\mathbf{k}|)$ has the form

$$\alpha(|\mathbf{k}|) = \frac{u(|\mathbf{k}|)}{(2\omega_k)^{1/2}(2\pi)^{3/2}}$$

and the commutation relations are given by Eqs. (7.2) and (7.3).

It immediately follows from Eq. (7.24) that the operators

$$Q_N = N^*N$$
$$Q_\theta = \int d^3k\theta^*(\mathbf{k})\theta(\mathbf{k})$$

are separately conserved so that in the usual representation [Eq. (7.5)] of the commutation relations the set of physical states will be the eigenvectors of these two operators. However, we note that as in the neutral scalar theory there arises the possibility of having physically meaningful inequivalent representations in the event that $\mu = 0$. This is seen to be an immediate consequence of the fact that the condition which breaks the conservation law for the number of θ particles

$$\langle\theta'|\theta(\mathbf{k})|\theta'\rangle = \theta'\delta(\mathbf{k}) \quad (7.25)$$

is consistent with the equation of motion

$$\left(\omega - i\frac{\partial}{\partial t}\right)\theta(\mathbf{k}) = \lambda N^*N\alpha(|\mathbf{k}|) \int d^3k'\alpha(|\mathbf{k}'|)\theta(k')$$

if and only if the boson has vanishing mass. We again require that $\alpha(0) \neq 0$ so that the zero-momentum component of the field will have a nonvanishing coupling, thereby providing the basis for a nontrivial symmetry breaking. As in the neutral scalar theory we anticipate that the broken-symmetry condition may be eliminated by the replacement of $\theta(\mathbf{k})$ by $\theta(\mathbf{k}) + \theta'\delta(\mathbf{k})$ in the Hamiltonian. One thus obtains the equivalent description of this theory in terms of

$$\begin{aligned} H(\theta') = {} & [m - \lambda\alpha^2(0)]N^*N + \int d^3k\omega\theta^*(\mathbf{k})\theta(\mathbf{k}) \\ & - \lambda \int d^3k\alpha(|\mathbf{k}|)\theta^*(\mathbf{k}) \int d^3k'\alpha(|\mathbf{k}'|)\theta(k') \\ & - \lambda\alpha(0) \int d^3k\alpha(|\mathbf{k}|)[\theta'^*\theta(\mathbf{k}) + \theta^*(\mathbf{k})\theta']N^*N \end{aligned}$$

and the usual representation Eq. (7.5) of the commutation relations. It is to be noted that the condition (7.25) leads to a mass renormalization of the N field and has the further effect of introducing a Yukawa type coupling into the theory.

The physical content of the model can be conveniently displayed as in the neutral scalar theory by a canonical transformation

$$\begin{aligned} \tilde{\theta}(\mathbf{k}) &= \theta(\mathbf{k}) - A(\mathbf{k})N^*N \\ \tilde{N} &= \exp\left\{\int d^3k[A^*(\mathbf{k})\theta(\mathbf{k}) - A(\mathbf{k})\theta^*(\mathbf{k})]\right\}N \end{aligned} \tag{7.26}$$

where $A(\mathbf{k})$ is to be determined by the requirement that the Hamiltonian, when expressed in terms of the new variables (7.26), should have no terms linear in the meson field operators. This leads to the form

$$A(\mathbf{k}) = \frac{\lambda\alpha(0)\theta'\alpha(|\mathbf{k}|)/\omega}{1 - \lambda \int d^3k[\alpha^2(|\mathbf{k}|)/|\mathbf{k}|]}$$

and the Hamiltonian

$$\begin{aligned} H(\theta') = {} & \left(m - \lambda \frac{\alpha^2(0)|\theta'|^2}{1 - \lambda \int d^3k[\alpha^2(|\mathbf{k}|)/|\mathbf{k}|]}\right)\tilde{N}^*\tilde{N} \\ & + \int d^3k\omega\tilde{\theta}^*(\mathbf{k})\tilde{\theta}(\mathbf{k}) - \lambda \int d^3k\alpha(|\mathbf{k}|)\tilde{\theta}^*(\mathbf{k}) \int d^3k'\alpha(|\mathbf{k}'|)\tilde{\theta}(\mathbf{k}) \end{aligned}$$

Although the symmetry breaking once again has only a mass renormalization effect on the physical spectrum, one can further anticipate that in this model the wave function renormalization constant also should depend upon $|\theta'|^2$. In particular, one readily infers from Eqs. (7.20) and (7.26) the result

$$Z = \exp\left[-\lambda^2\alpha^2(0)|\theta'|^2\left(1 - \lambda \int d^3k\frac{\alpha^2(|\mathbf{k}|)}{|\mathbf{k}|}\right)^{-2} \int d^3k \frac{\alpha^2(|\mathbf{k}|)}{|\mathbf{k}|^2}\right]$$

which differs from unity if, and only if, the symmetry-breaking parameter $|\theta'| \neq 0$.

In contrast to the case of the neutral scalar theory, the condition (7.25) in this model allows an immediate application of the Goldstone theorem. Thus we note that the commutator condition

$$\begin{aligned}\langle\theta'|[Q_\theta, \theta(\mathbf{k})]|\theta'\rangle &= -\langle\theta'|\theta(\mathbf{k})|\theta'\rangle \\ &= -\theta'\delta(\mathbf{k})\end{aligned}$$

shows directly the existence of a physical excitation with the quantum numbers of the θ particle which has zero energy and momentum. Since we have noted in detail some of the arguments which show the "nonexistence" of the operator Q_θ in the event that it fails to annihilate the vacuum, it is interesting to note that this result, while entirely correct, is quite harmless in the case under consideration. To see this we note that it is more convenient to discuss the operator Q_θ in terms of the usual representation (7.5). This, of course, requires that one transform Q_θ to the form

$$Q_\theta = \int d^3k\theta^*(\mathbf{k})\theta(\mathbf{k}) + [\theta'^*\theta(\mathbf{0}) + \theta^*(\mathbf{0})\theta'] + \theta'^*\theta'\delta(0)$$

which makes quite clear the nonexistence of Q_θ as a well-defined operator. Despite the fact that the models under consideration here are somewhat too simple to expect an exact analogy with the relativistic theories discussed in the preceding section, the above explanation of the divergences known to occur in the definition of the charge operator in a broken-symmetry theory is sufficiently model-independent to suggest that the nonexistence of such operators will cause no difficulty in a proper formulation of the theory. Not only do these infinities fail to represent any possible objection to the breaking of a symmetry, but they have been further seen to follow as an entirely natural consequence of the requirement that a given operator have a nonvanishing expectation value.

Although the preceding discussion of the direct interaction model has not proved the equivalence of the broken-symmetry condition to the replacement of $\theta(\mathbf{k})$ by $\theta(\mathbf{k}) + \theta'\delta(\mathbf{k})$, this can (by virtue of the solubility of the theory for all θ') be carried out by straightforward though laborious techniques.[73] If, however, we are to succeed in extending our results in the direction of more physically significant theories, it is necessary to allow the discussion of models whose mathematical structure is not quite so simple. Both the neutral scalar theory and the direct interaction model are mathematically tractable by virtue of the fact that the physical effect of the symmetry breaking consists entirely of a mass renormalization. In the more general case one finds that as soon as allowance is made for more complicated symmetry-breaking effects, the solubility of the theory is destroyed.

An outstanding example of such a theory which is soluble only for the usual representation of the canonical commutation relations is the Lee model.(76) The relevant Hamiltonian is

$$H = m_V V^*V + m_N N^*N + \int d^3k\omega_k \theta^*(\mathbf{k})\theta(\mathbf{k}) - \int d^3k\alpha(|\mathbf{k}|)[V^*N\theta(\mathbf{k}) + N^*V\theta^*(\mathbf{k})]$$

where the only nonvanishing canonical commutation relations are

$$\{V, V^*\} = 1$$
$$\{N, N^*\} = 1$$
$$[\theta(\mathbf{k}), \theta^*(\mathbf{k})] = \delta(\mathbf{k} - \mathbf{k}')$$

It is well known that there exist in this theory two conserved generators of gauge transformations

$$Q_1 = V^*V + N^*N$$
$$Q_2 = \int d^3k\theta^*(\mathbf{k})\theta(\mathbf{k}) - N^*N$$

corresponding, respectively, to the conservation of fermion number and the difference between the number of θ and N particles. It is, of course, the existence of these two conservation laws which, in the usual representation of the commutation relations, is ultimately responsible for the solubility of the theory. Since in the customary approach to the Lee model one attempts to solve the eigenvalue problem sector by sector, the choice of representation must be specified at the outset, thereby requiring the adoption of a procedure which automatically dismisses the possibility of finding additional solutions of the theory. We shall, however, show that there can exist additional solutions for which the θ field has a nonvanishing vacuum expectation value.

It is clear that as in the direct interaction model a condition of the type

$$\langle\theta'|\theta(\mathbf{k})|\theta'\rangle = \theta'(\mathbf{k}) \tag{7.27}$$

corresponds to the breaking of the θ–N conservation law inasmuch as the consistency requirement

$$\langle\theta'|[\theta(\mathbf{k}), Q_2]|\theta'\rangle = \theta'(\mathbf{k})$$

implies that the vacuum $\langle\theta'|$ no longer is an eigenstate of the operator Q_2. Furthermore the time independence of Q_2 (which is, of course, unaltered by the broken symmetry) requires by the Goldstone theorem that $\theta(\mathbf{k})$ excite a zero-mass particle. Although allowance has been made in Eq. (7.27) for a result which is somewhat more general than Eq. (7.6), it is readily shown from the equation of motion

$$\begin{aligned} i\frac{\partial}{\partial t}\,\theta(\mathbf{k}, t) &= [\theta(\mathbf{k}), H] \\ &= \omega_k\theta(\mathbf{k}) - \alpha(|\mathbf{k}|)N^*N \end{aligned}$$

and the condition

$$V|\theta'\rangle = N|\theta'\rangle = 0$$

that

$$\left(-i\frac{\partial}{\partial t} + \omega_k\right)\langle\theta'|\theta(\mathbf{k})|\theta'\rangle = 0$$

which is consistent with the required time independence of $\theta'(\mathbf{k})$ only if $\omega_k\theta'(\mathbf{k}) = 0$. This clearly demands that μ vanish and $\theta'(\mathbf{k})$ be of the form $\theta'\delta(\mathbf{k})$, in complete analogy to the results obtained in the case of the somewhat simpler models which have been discussed. The fact that the consistency requirements on the symmetry-breaking condition can be satisfied entirely at the kinematical level without placing any further constraints on the theory thus allows one to infer immediately the internal consistency of Eq. (7.27). Strictly speaking the extension of our preceding results to the Lee model cannot be accomplished with the degree of rigor which characterized the discussion of the neutral scalar and direct interaction theories. This is a consequence of the fact that the Lee model is not soluble except in the usual representation of the commutation relations so that one cannot prove the existence of a well-defined solution in the case of a broken symmetry. On the other hand, the replacement of $\theta(\mathbf{k})$ by $\theta(\mathbf{k}) + \theta'\delta(\mathbf{k})$ can be formally shown to be equivalent to the broken-symmetry condition, and the Hamiltonian thereby generated

$$H(\theta') = H - \alpha(0)(V^*N\theta' + N^*V\theta'^*) \qquad (7.28)$$

furthermore fails to suggest the occurrence of any pathological features in the theory. We shall therefore be content to assume that there is no essential difficulty in passing to the infinite volume limit in the commutators of the charge operator [which is, of course, essential in order to demonstrate the equivalence of the broken-symmetry condition to the replacement of $\theta(\mathbf{k})$ by $\theta(\mathbf{k}) + \theta'\delta(\mathbf{k})$]. Stated somewhat differently, the implementation of the transformation generated by the operator Q_2 in a finite volume can now be extended [in direct analogy to the discussion following Eq. (7.10)] to infinite volume by virtue of our assumption concerning the existence of meaningful solutions to the theory described by Eq. (7.28).

The breakdown of Q_2 conservation and the increased structural complexity of the model can be illustrated more forcefully by diagonalizing all the terms in $H(\theta')$ which are bilinear in the canonical fields. Thus we define the new operators V' and N' by the unitary transformation

$$\begin{bmatrix} V' \\ N' \end{bmatrix} = \begin{bmatrix} \cos\gamma & e^{-i\beta}\sin\gamma \\ -e^{i\beta}\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} V \\ N \end{bmatrix}$$

which leads to a diagonal form of the mass matrix

$$\begin{bmatrix} m_V & -\alpha(0)\theta'^* \\ -\alpha(0)\theta' & m_N \end{bmatrix}$$

if one sets

$$\theta' = |\theta'|\, e^{i\beta}$$

$$\tan 2\gamma = \frac{2\alpha(0)|\theta'|}{m_N - m_V}$$

The eigenvalues are readily found to be

$$m_V' = \frac{m_N + m_V}{2} + \frac{m_V - m_N}{2}\left(1 + 4\,\frac{|\alpha(0)\theta'|_2}{(m_N - m_V)^2}\right)^{1/2}$$

$$m_N' = \frac{m_N + m_V}{2} + \frac{m_N - m_V}{2}\left(1 + 4\,\frac{|\alpha(0)\theta'|^2}{(m_N - m_V)^2}\right)^{1/2}$$

in terms of which one has

$$\begin{aligned} H(\theta') = {} & m_V'V^{*\prime}V' + m_N'N^{*\prime}N' + \int d^3k\omega_k\theta^*(\mathbf{k})\theta(\mathbf{k}) \\ & - \cos^2\gamma \int d^3k\alpha(|\mathbf{k}|)[V^{*\prime}N'\theta(\mathbf{k}) + N^{*\prime}V'\theta^*(\mathbf{k})] \qquad (7.29) \\ & + \sin^2\gamma \int d^3k\alpha(|\mathbf{k}|)[V^{*\prime}N'\theta^*(\mathbf{k})\,e^{2i\beta} + N^{*\prime}V'\theta(\mathbf{k})\,e^{-2i\beta}] \\ & - \sin\gamma\cos\gamma \int d^3k\alpha(|\mathbf{k}|)[V^{*\prime}V' - N^{*\prime}N'][e^{-i\beta}\theta(\mathbf{k}) + e^{i\beta}\theta^*(\mathbf{k})] \end{aligned}$$

for the Hamiltonian of the broken-symmetry theory. It is to be noted that as a consequence of the breakdown of Q_2 conservation the additional processes $N \rightarrow V + \theta$, $N \rightarrow N + \theta$, and $V \rightarrow V + \theta$ are all allowed by Eq. (7.29) in addition to the usual transition $V \rightarrow N + \theta$ of the Lee model. Although the resultant theory is not soluble, it is clear that the broken-symmetry condition must imply a dependence of the N and V masses on the parameter θ', as can be readily verified in perturbation theory.

Although the only way in which one can generate nontrivial symmetry-breaking effects in the Lee model is accomplished by treating the θ meson as a Goldstone particle, it is instructive to carry out the analogous procedure for the fermions of the theory in order to illustrate the failure of such an approach for systems containing only a finite number of degrees of freedom. In order to carry out the construction it is essential to find a state which has the quantum numbers of the N particle and is degenerate in energy with the vacuum. From the equation of motion

$$
\begin{aligned}
i\frac{\partial}{\partial t} N &= [N, H] \\
&= m_N N - \int d^3k \alpha(|\mathbf{k}|)\theta^*(\mathbf{k}) V
\end{aligned}
$$

one trivially calculates the two-point function

$$G(t) = i\epsilon(t)\langle 0|(N(t)N^*(0))_+|0\rangle$$

in the usual representation of N, V, and $\theta(\mathbf{k})$ to have the form

$$G(t) = \theta_+(t)\, e^{-im_N t} \tag{7.30}$$

It is clear from Eq. (7.30) that the state $N^*(t)|0\rangle$ is time independent if, and only if, $m_N = 0$, for which case one is led to attempt a construction of a set of broken-symmetry states by defining

$$\langle N'| = \langle 0| \exp\{N'^*N - N^*N'\}$$

The parameters N' and N'^* are c-numbers which commute with boson field operators and anticommute with fermion operators, the only nonvanishing commutator containing N' or N'^* required by consistency to be nonzero being

$$\{N', N'^*\} = \xi^2$$

where ξ is a real c-number. It is easy to show that $\langle N'|$ is a zero-energy eigenstate of H (thereby allowing its identification as a vacuum state) and has the further property that

$$\theta(\mathbf{k})|N'\rangle = V|N'\rangle = 0$$

On the other hand, the fact that the N field is associated with only a single degree of freedom means that, unlike the cases previously discussed, the operator

$$U = \exp\{N'^*N - N^*N'\}$$

is unitary.

By introduction of the parameter λ one can calculate the operator

$$\tilde{N}(\lambda) = U(\lambda)NU^*(\lambda)$$

where

$$U(\lambda) = \exp\{\lambda(N'^*N - N^*N')\}$$

One readily verifies that $\tilde{N}(\lambda)$ satisfies the differential equation

$$\frac{\partial^2\tilde{N}}{\partial\lambda^2} = -\xi^2\tilde{N}$$

so that $\tilde{N} = \tilde{N}(\lambda = 1)$ has the form

$$\tilde{N} = N\cos\xi + N'(\sin\xi/\xi) \tag{7.31}$$

This result immediately leads to the broken-symmetry condition

$$\langle N'|N|N'\rangle = N'(\sin\xi/\xi)$$

which clearly displays the noninvariance of the new set of vacuum states $\langle N'|$ under gauge transformations on the fermion fields N and V. It is to be noted that in consequence of the Fermi statistics the operator U does not merely affect a c-number translation of the field and that the result [Eq. (7.31)] is entirely in accord with the canonical commutation relation

$$\{\tilde{N}, \tilde{N}^*\} = 1$$

Using the same technique as before, one can express the state $\langle N'|$ as a linear combination of the vacuum $\langle 0|$ and the single N particle state $\langle 0|N$. Thus one readily derives the result[74]

$$\langle N'| = \langle 0|\cos\xi + \langle 0|N'N'^*\frac{1-\cos\xi}{\xi^2} + \langle 0|N'^*N\frac{\sin\xi}{\xi} \tag{7.32}$$

Since each term on the right-hand side of Eq. (7.32) represents a zero-energy eigenstate of H, it is clear that any state $\langle N'|$ may be chosen as the physical vacuum. Thus in the usual way one can build up a complete Hilbert space constructed from this chosen vacuum state. The relation between the Hilbert spaces constructed from the different vacuums can be readily found in direct analogy to our earlier results. In particular one deduces that the effect of the broken-symmetry states can be entirely simulated by the replacement of H by

$$\begin{aligned} UHU^* &= H\left(N\cos\xi + N'\frac{\sin\xi}{\xi}, V, \theta\right) \\ &= H(N, V, \theta) + 2\sin^2\xi/2\int d^3k\alpha(|\mathbf{k}|)(V^*N\theta(\mathbf{k}) + N^*\theta^*(\mathbf{k})V) \\ &\qquad - \frac{\sin\xi}{\xi}\int d^3k\alpha(|\mathbf{k}|)(V^*N'\theta(\mathbf{k}) + N'^*\theta^*(\mathbf{k})V) \end{aligned}$$

where we have denoted the usual Hamiltonian of the Lee model by $H(N, V, \theta)$. It must be strongly emphasized, however, that because U is a unitary operator this transformation must leave the eigenvalue spectrum of H unaltered, a result which can be readily verified by direct calculation in the lowest sectors. Thus the sole effect of U is to mix states of different fermion number, a circumstance which is in marked contrast with the case in which the Goldstone particle is the θ meson and the c-number translation of the θ field is effected by a nonunitary operator which generates inequivalent theories. In the latter case the symmetry breaking effected by the θ meson can change the basic physics of the model (even to the point of rendering the theory insoluble) whereas the symmetry breaking induced by a massless N particle is entirely formal. This distinction is, of course, a consequence of the well-known fact that the inequivalent representations of the commutation relations can only occur in theories which possess an infinite number of degrees of freedom.

Since the models which we have considered here cannot be readily generalized to the relativistic case, it is well to conclude this section by briefly commenting upon the utility of the results which have been obtained. Certainly they must serve to dispel much of the pessimism which may have been generated in our discussion of physically trivial broken-symmetry theories with regard to the possibility of inducing real effects by means of a symmetry-breaking condition. On the other hand, a crucial ingredient in this consideration of static theories has been the exact solubility of the states containing no fermions but an arbitrary number of θ particles. Since it was precisely this feature which made possible the demonstration of the existence of additional vacuum states in the theory (corresponding to the inequivalent representations), it is clear that the attribute of partial solubility has been an indispensable tool throughout the present section. The fact that there is no Lorentz-invariant theory in which such a circumstance occurs appears, at first sight, to exclude the possibility of finding a correspondingly simple example in the relativistic domain. On the other hand, it is interesting to note that even in a covariant theory with a scalar zero-mass excitation (assuming one can establish the existence of such a particle) it might well be possible to proceed in a fashion somewhat analogous to that followed here. In particular the "in" and "out" field operators which create the zero-mass quanta can be used to assert the existence of additional zero-energy eigenstates of P^μ which are related to the vacuum by the usual improper operator transformation. Such a device enables one to anticipate the precise fashion in which a nontrivial broken-symmetry effect may arise from the noninvariance of a Lagrangian under translation of the boson field operator. By this heuristic argument the results of the present section are seen to imply the possibility of carrying out the construction of theories such as those proposed in Section VI with considerably greater confidence. At present, however, a more rigorous argument does not appear to lie within the realm of feasibility.

VIII. Conclusion

In the preceding discussions we have attempted to outline in detail some of the most relevant and interesting aspects of the theory of broken symmetries. In so doing, it has been found that, under conditions met by manifestly covariant theories or nonrelativistic theories with sufficiently damped potentials, the Goldstone theorem is a rigorous result and the broken symmetry consequently requires the presence of a massless boson. On the other hand, we have also been able to supply interesting examples of acausal systems in which the "Goldstone bosons" become massive. Such theories might even be of physical relevance in the event that vector gauge fields of the type discussed in Section IV are found to have application to the real world. In addition, these examples serve to emphasize the fact that, in problems involving causal theories, there exists the possibility that the Goldstone bosons might completely decouple so as not to appear in any physically measurable amplitude.

In the case of nonrelativistic problems, we have been able to consider Hamiltonians which are physically significant and in which the broken symmetry plays a fundamental and understandable role. On the other hand, in the case of the corresponding relativistic problems, our success in practical terms has been rather limited even though it has been possible to demonstrate that broken symmetries actually occur in relativistic theories which possess a special type of gauge invariance. In such cases, the broken-symmetry argument makes no basic change in the physical content of the theory and is merely equivalent to the usual argument concerning the relation between gauge invariance and masslessness. Unfortunately, however, more complex and realistic, relativistic, broken-symmetry models can at present only be studied within the context of perturbation theory so that, despite the fact that a great deal is understood in general terms, the methods of broken symmetries have not thus far achieved any considerable success in the realm of relativistic particle physics.

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   The construction of bilinear currents in field theory is a major problem and must be approached through the use of limiting processes. Because of singularities, it can turn out that the current correctly constructed does not behave as expected from naive arguments. For example, in the Schwinger model [J. Schwinger, *Phys. Rev.*, **130**, 406 (1963)], which has formally conserved vector and axial-vector currents, it is seen upon solution of the field equations that the axial current is not conserved, a phenomenon which might also occur in more physical models. A model which supposedly exhibits this behavior has been presented by Th. A. J. Maris and G. Jacob, *Phys. Rev. Letters*, **17**, 1300 (1966). [See also Th. A. J. Maris, *Nuovo Cimento*, **45A**, 223

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4. To be more explicit, we may introduce a C^∞ test function $f_R(x)$ which has the property that

$$f_R(\mathbf{x}) = 1 \qquad \text{if} \qquad |\mathbf{x}| \leq R$$

while

$$f_R(\mathbf{x}) = 0 \qquad \text{if} \qquad |\mathbf{x}| > R + \epsilon$$

and define

$$Q_R^i(t) = \int d^3x f_R(\mathbf{x}) j_i^0(\mathbf{x}, t)$$

5. We confine our considerations to charges that are integrals of local densities. For a discussion of charges not of this type, see, for example, A. Katz and Y. Frishman, *Nuovo Cimento*, **42A**, 1009 (1966).
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