```
<< Local`QFTToolKit2`;
Get[$HomeDirectory <> "/Mathematica/NonCommutative/1204.0328
     ParticlePhysicsFromAlmostCommutativeSpacetime.1.redo.out"];

"Local notational definitions";
rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := |a|;
it[a_] := Style[a, Italic]
iD := it[D](*italics refers to M or F space, script refers to M×F space.*)
iA := it[A]
iI := it["I"]
C∞ := C^"∞"
Bx_ := T[B, "d", {x}]
("▽"^S)n_ := T["▽S", "d", {n}]

$noArg := tuDDown[a_][b_, c_] → a
Clear[clearArgBlank]
clearArgBlank :=
 (# /. Blank → xBlank /. a_ . xBlank[] . Longest[c__] → a.c /. xBlank → Blank &)

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
   ""];
selectDef[heads_, with_: {}, all_: Null] := tuRuleSelect[$defall][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;

Clear[expandDC];
expandDC[sub_: {}, scalar_: {}] :=
 tuRepeat[{sub, tuOpDistribute[Dot], tuOpSimplify[Dot, scalar],
   tuOpDistribute[CircleTimes], tuOpSimplify[CircleTimes, scalar]}]
$sgeneral := {T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
  T[γ, "d", {5}].T[γ, "d", {5}] → 1, ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
  CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  CommutatorP[T[γ, "u", {μ}], T[γ, "u", {ν}]] → 2 T[g, "uu", {μ, ν}],
  T["▽", "d", {_}][1n_] → 0, a_ . 1n_ → a, 1n_ . a_ → a}
$sgeneral // ColumnBar
accumDef[$sgeneral]

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] ⧴ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "dd", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] ⧴ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] ⧴ CommutatorP[b, a] /; OrderedQ[{b, a}],
    tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
  };
$symmetries // ColumnBar
```

```
γ5 → γ^1 γ^2 γ^3 γ^4
γ5.γ5 → 1
(γ5)^† → γ5
{γ5, γ^μ}+ → 0
{γ^μ, γ^ν}+ → 2 g^μ ν
∇_[1n_] → 0
(a_).1n_ → a
1n_.(a_) → a
```

```
tt : g^μ_ ν_ :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : F^μ_ ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : F_μ_ ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
[a_, b_]- :→ -[b, a]- /; OrderedQ[{b, a}]
{a_, b_}+ :→ {b, a}+ /; OrderedQ[{b, a}]
tt : γ^μ.γ5 :→ Reverse[tt]
```

1204.0328: Particle Physics From Almost Commutative Spacetime

3. The Spectral Action of AC-manifold

3.1 The heat expansion of the spectral action

3.1.1 A generalized Lichnerowicz formula

```
PR["●Lichnerowicz formula.",
  NL, "• ", "E"[CG["vector bundle"]] → M,
  NL, "• ", △^"E"[CG["Laplacian of connection on E, ∇^E"]],
  NL, "• generalized Laplacian has form ", {△^"E" - F, F ∈ Γ[Endo["E"]]},
  NL, "• ",
  $ = {iD[CG[
       "generalized Dirac operator on ℤ2-graded vector bundle E with odd parity"]],
    iD[Γ[M, "E"^"±"]] -> Γ[M, "E"^"∓"],
    iD · iD ∈ {△^"E" - F, F ∈ Γ[Endo["E"]]}}; $ // ColumnBar,
  NL, CR["There may be an interchange of symbols ", 𝒟 ↔ iD, " in the following."]
 ];
```

```
●Lichnerowicz formula.
• E[vector bundle] → M
• △^E[Laplacian of connection on E, ∇^E]
• generalized Laplacian has form {-F + △^E, F ∈ Γ[Endo[E]]}
  | D[generalized Dirac operator on ℤ2-graded vector bundle E with odd parity]
• | D[Γ[M, E^±]] → Γ[M, E^∓]
  | D · D ∈ {-F + △^E, F ∈ Γ[Endo[E]]}
There may be an interchange of symbols 𝒟 ↔ D in the following.
```

```
PR["■Show ", $ = {iD_iA[
      CG["generalized Dirac operator on ℤ2-graded vector bundle E with odd parity"]],
     iD_iA · iD_iA ∈ {△"E" - F, F ∈ Γ[Endo["E"]]}};
  $ // ColumnBar,
  line,
  next, "Compute ", $[[2, 1]],
  " where from (2.18) ",
  NL, "Given ",
  $sDA = $s = {iD_iA -> -I T[γ, "u", {μ}]. tuDDown["∇E"][_, μ] + T[γ, "d", {5}] ⊗ Φ,
      tuDDown["∇E"][_, μ][CG["S[spinor]⊗E[vector bundle]"]].a_ →
       (tuDDown["∇S"][_, μ] ⊗ 1_ℋF).a + I (1_N ⊗ B_μ).a,
      Φ ∈ Γ[CG["Endo[E],Higg's field"]],
      "∇E" → "∇S" ⊗ 1_ℋF + I (1_N ⊗ B_μ)
    }; $s // ColumnBar,

  NL, "Define adjoint Rule[]s: ",
  $ad = ad[aa_][bb_] → aa.bb - bb.aa;
  $ad = {$ad, $ad // tuPatternRemove // Reverse // tuAddPatternVariable[{aa, bb}]};
  $ad // ColumnBar,
  NL, "Define: ",
  $d = {tuDDown[iD][a_, μ] → ad[tuDDown["∇E"][_, μ]][a],
     a_ . _ .Longest[c__] → a.c}; $d // ColumnBar,
  NL, "Explicit Operator formalism with ",
  Φ ∈ ℋF, " Incorporate Φ into CircleTimes expression.",
  Yield,
  $00 = $ = tuDDown[iD][Φ, μ],
  Yield, $ = $ /. $d,
  Yield, $ = $ /. $ad,
  NL, "Add rhs to aid in manipulation: ", $ = #.rhs & /@ $ // expandDC[],
  Yield, $ = $ //. tuRule[$sDA] // expandDC[] // Inactivate[#, Plus] &,
  next, "Gather Φ∈ℋF into appropriate space: ",
  $s = {(a_ ⊗ b_).Φ :> a ⊗ (b.Φ) /; ! FreeQ[b, B], Φ.(a_ ⊗ b_) :> a ⊗ (Φ.b) /; ! FreeQ[b, B]};
  $s // ColumnBar,
  Yield, $ = $ /. $s // Activate,

  NL, "Apply Leibnitz rule: ",
  $s = {a_ . Φ . rhs → (a [ Φ]) . rhs + Φ.(a.rhs), h_[Φ] ⧴ (h /. Blank -> Φ /. Φ[] → Φ)};
  $s // ColumnBar,
  " on ", $0 = $1 = $ // tuTermSelect[_ . Φ . rhs] // First,
  Yield, $1 = $1 //. $s,
  Imply, $ = $ - $0 + $1,
  NL, "In operator form (Drop rhs). ",
  Yield, $ = $ /. rhs → 1 //. tuOpSimplify[Dot],
  NL, "From the property: ", $s = selectDef[tuDDown["∇S"][_, _]],
  Yield, $s = $s /. {f → Φ, ψ → 1_N} // tuDerivativeExpand[{1_}] // (# /. 1_n_ → 1 &) //
    tuOpSimplifyF[Dot],
  Yield, $ = $ /. $s,
  Yield, $ = $ /. (c1_ : 1) a_ ⊗ b_ + (c2_ : 1) a_ ⊗ c_ -> a ⊗ (c1 b + c2 c) /.
      tuCommutatorSolve[2][CommutatorM[Φ, T[B, "d", {μ}]]] // Simplify,
  NL, "Put Φ in proper space ", $s = (aa : tuDPartial[Φ, _]) ⊗ 1_ℋF → 1_N ⊗ aa,
  Yield, $ = $00 -> $ /. $s; $ // Framed, accumDef[{$, $ad, $d, $sDA}]
   CG[" (3.1)"]
 ];
```

■Show
D_A[generalized Dirac operator on $\mathbb{Z}_2$-graded vector bundle E with odd parity]
$D_A \cdot D_A \in \{-F + \triangle^E,\ F \in \Gamma[\text{Endo}[E]]\}$

◆Compute $D_A \cdot D_A$ where from (2.18)

Given
$D_A \to \gamma_5 \otimes \Phi - i\,\gamma^\mu . \nabla^E_{\_\mu}[\_]$
$\nabla^E_{\_\mu}[\_]$[S[spinor]⊗E[vector bundle]].(a_) $\to i\,(1_N \otimes B_\mu).a + (\nabla^S_{\_\mu}[\_] \otimes 1_{\mathcal{H}_F}).a$
$\Phi \in \Gamma$[Endo[E],Higg's field]
$\nabla^E \to \nabla^S \otimes 1_{\mathcal{H}_F} + i\,1_N \otimes B_\mu$

Define adjoint Rule[]s:
ad[aa_][bb_] → aa.bb – bb.aa
(aa_).(bb_) – (bb_).(aa_) → ad[aa][bb]

Define:
$D_{\_\mu}$[a_] → ad[$\nabla^E_{\_\mu}$[_]][a]
(a_).(_).Longest[c__] → a.c

Explicit Operator formalism with
$\Phi \in \mathcal{H}_F$ Incorporate Φ into CircleTimes expression.

→ $\underline{D}_\mu[\Phi]$
→ ad[$\underline{\nabla}^E_\mu$[_]][Φ]
→ $-\Phi.\underline{\nabla}^E_\mu[\_] + \underline{\nabla}^E_\mu[\_].\Phi$

Add rhs to aid in manipulation: $-\Phi.\underline{\nabla}^E_\mu[\_].\text{rhs} + \underline{\nabla}^E_\mu[\_].\Phi.\text{rhs}$

→ $-i\,\Phi.(1_N \otimes B_\mu).\text{rhs} + -\Phi.(\underline{\nabla}^S_\mu[\_] \otimes 1_{\mathcal{H}_F}).\text{rhs} + i\,(1_N \otimes B_\mu).\Phi.\text{rhs} + (\underline{\nabla}^S_\mu[\_] \otimes 1_{\mathcal{H}_F}).\Phi.\text{rhs}$

◆Gather $\Phi \in \mathcal{H}_F$ into appropriate space:
(a_⊗b_).Φ :→ a⊗b.Φ /; ! FreeQ[b, B]
Φ.(a_⊗b_) :→ a⊗Φ.b /; ! FreeQ[b, B]

→ $-i\,(1_N \otimes \Phi.B_\mu).\text{rhs} + i\,(1_N \otimes B_\mu.\Phi).\text{rhs} - \Phi.(\underline{\nabla}^S_\mu[\_] \otimes 1_{\mathcal{H}_F}).\text{rhs} + (\underline{\nabla}^S_\mu[\_] \otimes 1_{\mathcal{H}_F}).\Phi.\text{rhs}$

Apply Leibnitz rule:
(a_).Φ.rhs → a[Φ].rhs + Φ.a.rhs
h_[Φ] :→ (h /. Blank → Φ /. Φ[] → Φ)
on $(\underline{\nabla}^S_\mu[\_] \otimes 1_{\mathcal{H}_F}).\Phi.\text{rhs}$

→ $(\underline{\nabla}^S_\mu[\Phi] \otimes 1_{\mathcal{H}_F}).\text{rhs} + \Phi.(\underline{\nabla}^S_\mu[\_] \otimes 1_{\mathcal{H}_F}).\text{rhs}$
⇒ $-i\,(1_N \otimes \Phi.B_\mu).\text{rhs} + i\,(1_N \otimes B_\mu.\Phi).\text{rhs} + (\underline{\nabla}^S_\mu[\Phi] \otimes 1_{\mathcal{H}_F}).\text{rhs}$

In operator form (Drop rhs).

→ $-i\,1_N \otimes \Phi.B_\mu + i\,1_N \otimes B_\mu.\Phi + \underline{\nabla}^S_\mu[\Phi] \otimes 1_{\mathcal{H}_F}$

From the property: $\underline{\nabla}^S_\mu[f.\psi] \to f.\underline{\nabla}^S_\mu[\psi] + \psi\,\underline{\partial}_\mu[f]$

→ $\underline{\nabla}^S_\mu[\Phi] \to \underline{\partial}_\mu[\Phi]$
→ $-i\,1_N \otimes \Phi.B_\mu + i\,1_N \otimes B_\mu.\Phi + \underline{\partial}_\mu[\Phi] \otimes 1_{\mathcal{H}_F}$
→ $1_N \otimes (-i\,[\Phi, B_\mu]_-) + \underline{\partial}_\mu[\Phi] \otimes 1_{\mathcal{H}_F}$

Put Φ in proper space $(aa : \underline{\partial}\,[\Phi]) \otimes 1_{\mathcal{H}_F} \to 1_N \otimes aa$

→ $\boxed{D_{-\mu}[\Phi] \to 1_N \otimes (-i\,[\Phi, B_\mu]_-) + 1_N \otimes \partial_{-\mu}[\Phi]}$ (3.1)

```
$sconvert :=
  {T["∇E", "d", {X_}] → tuDDown["∇E"][_, X], T[iD, "d", {μ_}] -> tuDDown[iD][_, μ]};
```

```
PR["◆Define curvature of Bμ: ",
  $F = T[F, "dd", {μ, ν}] → tuDPartial[Bν, μ] - tuDPartial[Bμ, ν] + I CommutatorM[Bμ, Bν],
  NL, "•Define curvature of ∇E: ", $O = {Ω"E"[X, Y] → T["∇E", "d", {X}].T["∇E", "d", {Y}] -
      T["∇E", "d", {Y}].T["∇E", "d", {X}] - T["∇E", "d", {CommutatorM[X, Y]}],
    {tuDPartial[_, μ] → X, tuDPartial[_, ν] → Y}[CG["vector fields"]]};
  $O = $O /. {T["∇E", "d", {X_}] → tuDDown["∇E"][_, X]};
  $O // ColumnBar, accumDef[{$F, $O}];
  (*indexed version*)
  $s = selectDef[Ω"E"[_, _], {"∇S"}] /. Ω"E"[a_, b_] → T[Ω"E", "dd", {a, b}];
  $s = {$s // tuIndicesRaise[{μ, ν}], $s}; accumDef[{$s}];

  NL, CO["▪For local coordinates(cartesian): "],
  CommutatorM[tuDPartial[_, μ], tuDPartial[_, ν]] → 0,
  NL, "Use: ", $s = {CommutatorM[X, Y] → 0, X -> μ, Y → ν, T["∇E", "d", {0}] → 0} /.
    {T["∇E", "d", {X_}] → tuDDown["∇E"][_, X]},
  Imply, $e33 = $ = $O[[1]] //. $s,
  Yield,
  $ = (#.rhs & /@ $ // expandDC[]) //. (tuRule[$sDA] // tuAddPatternVariable[{μ}]);
  Yield, $ = $ // expandDC[]; $ // ColumnSumExp,
  NL, "Using Liebnitz rule for differential terms: ",
  $s = {a_ . b_ . c_ :> a[b] . c + b.a.c /; ! FreeQ[a, "∇S"],
    (tuDDown["∇S"][_, n_] ⊗ 1_{H_F})[1_N ⊗ b_] ⧴ 1_N ⊗ tuDDown["∇S"][b, n]
   }; $s // ColumnBar,
  Yield, $ = $ /. $s /. $s // Expand; $ // ColumnSumExp,
  NL, "In operator form(drop RHS arguement) ",
  $ = $ // expandDC[rhs → 1]; $ // ColumnSumExp;
  Yield, $ = $ // tuCircleTimesExpand // tuOpSimplifyF[Dot] // tuCircleTimesGather[];
  $ // ColumnSumExp,
  NL, "Express in Commutators ", $sdd = {tuDDown["∇S"][_, n_].tuDDown["∇S"][_, m_] ->
     tuDDown["∇S"][tuDDown["∇S"][_, m], n],
    tuCommutatorSolve[][CommutatorM[T[B, "d", {μ}], T[B, "d", {ν}]]],
    tuDDown["∇S"][b_, a_] → tuDPartial[b, a]},
  Yield, $ = $ /. $sdd; $ // ColumnSumExp,
  NL, "Apply (3.2) ",
  $s = tuRuleSolve[$F, CommutatorM[_, _]] // First // Map[-# &, #] &,
  Yield, $s34 = $ = $ /. $s // ExpandAll;
  $ // ColumnSumExp // Framed, CG["(3.4)"],
  accumDef[{$e33, $s34}],

  NL, "Since ", $e33s = $e33 /. {"E" → "S", "∇E" -> "∇S"} /. $sdd,
  Yield, $ = $ /. Reverse[$e33s]; $ // Framed, CG["(3.4)"],
  $s = $ /. ΩS_[a_, b_] → T[ΩS, "dd", {a, b}];
  $s = {$s // tuIndicesRaise[{μ, ν}], $s};
  accumDef[{$e33s, $, $s}];
  (*indexed version*)
  $s = selectDef[Ω"S"[_, _], {}] /. Ω"S"[a_, b_] → T[Ω"S", "dd", {a, b}];
  $s = {$s // tuIndicesRaise[{μ, ν}], $s},
  accumDef[$s]
 ];
```

◆Define curvature of B_μ: $F_{\mu\nu} \to i\,[B_\mu, B_\nu]_- - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

•Define curvature of ∇^E: $\Omega^E[X,Y] \to \nabla^E_{-X}[\_].\nabla^E_{-Y}[\_] - \nabla^E_{-Y}[\_].\nabla^E_{-X}[\_] - \nabla^E_{-[X,Y]_-}[\_]$

$\{\partial_{-\mu}[\_] \to X,\ \partial_{-\nu}[\_] \to Y\}$[vector fields]

▪For local coordinates(cartesian): $[\underline{\partial}_\mu[\_], \underline{\partial}_\nu[\_]]_- \to 0$

Use: $\{[X,Y]_- \to 0,\ X \to \mu,\ Y \to \nu,\ \underline{\nabla}^E{}_0[\_] \to 0\}$

⇒ $\Omega^E[\mu,\nu] \to \underline{\nabla}^E{}_\mu[\_].\underline{\nabla}^E{}_\nu[\_] - \underline{\nabla}^E{}_\nu[\_].\underline{\nabla}^E{}_\mu[\_]$

→

$$\to\ \Omega^E[\mu,\nu].\mathrm{rhs} \to \sum\left[\begin{array}{l} i\,(i\,(1_N\otimes B_\mu).(1_N\otimes B_\nu).\mathrm{rhs} + (1_N\otimes B_\mu).(\nabla^S_{-\nu}[\_]\otimes 1_{\mathcal{H}_F}).\mathrm{rhs}) \\ -i\,(i\,(1_N\otimes B_\nu).(1_N\otimes B_\mu).\mathrm{rhs} + (1_N\otimes B_\nu).(\nabla^S_{-\mu}[\_]\otimes 1_{\mathcal{H}_F}).\mathrm{rhs}) \\ i\,(\nabla^S_{-\mu}[\_]\otimes 1_{\mathcal{H}_F}).(1_N\otimes B_\nu).\mathrm{rhs} \\ (\nabla^S_{-\mu}[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S_{-\nu}[\_]\otimes 1_{\mathcal{H}_F}).\mathrm{rhs} \\ -i\,(\nabla^S_{-\nu}[\_]\otimes 1_{\mathcal{H}_F}).(1_N\otimes B_\mu).\mathrm{rhs} \\ -(\nabla^S_{-\nu}[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S_{-\mu}[\_]\otimes 1_{\mathcal{H}_F}).\mathrm{rhs} \end{array}\right]$$

Using Liebnitz rule for differential terms:

$$\begin{array}{l} (a\_).(b\_).(c\_) \mapsto a[b].c + b.a.c\ /;\ !\,\mathrm{FreeQ}[a, \nabla^S] \\ (\nabla^S_{-n\_}[\_]\otimes 1_{\mathcal{H}_F})[1_N\otimes b\_] \mapsto 1_N\otimes\nabla^S_{-n}[b] \end{array}$$

$$\to\ \Omega^E[\mu,\nu].\mathrm{rhs} \to \sum\left[\begin{array}{l} -i\,(1_N\otimes\nabla^S_{-\nu}[B_\mu]).\mathrm{rhs} \\ i\,(1_N\otimes\nabla^S_{-\mu}[B_\nu]).\mathrm{rhs} \\ -(1_N\otimes B_\mu).(1_N\otimes B_\nu).\mathrm{rhs} \\ (1_N\otimes B_\nu).(1_N\otimes B_\mu).\mathrm{rhs} \\ (\nabla^S_{-\mu}[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S_{-\nu}[\_]\otimes 1_{\mathcal{H}_F}).\mathrm{rhs} \\ -(\nabla^S_{-\nu}[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S_{-\mu}[\_]\otimes 1_{\mathcal{H}_F}).\mathrm{rhs} \end{array}\right]$$

In operator form(drop RHS arguement)

$$\to\ \Omega^E[\mu,\nu] \to \sum\left[\begin{array}{l} (\nabla^S_{-\mu}[\_].\nabla^S_{-\nu}[\_] - \nabla^S_{-\nu}[\_].\nabla^S_{-\mu}[\_])\otimes 1_{\mathcal{H}_F} \\ 1_N\otimes(-B_\mu.B_\nu + B_\nu.B_\mu - i\,\nabla^S_{-\nu}[B_\mu] + i\,\nabla^S_{-\mu}[B_\nu]) \end{array}\right]$$

Express in Commutators

$\{\underline{\nabla}^S{}_{n\_}[\_].\underline{\nabla}^S{}_{m\_}[\_] \to \underline{\nabla}^S{}_n[\underline{\nabla}^S{}_m[\_]],\ B_\mu.B_\nu \to [B_\mu, B_\nu]_- + B_\nu.B_\mu,\ \underline{\nabla}^S{}_{a\_}[b\_] \to \underline{\partial}_a[b]\}$

$$\to\ \Omega^E[\mu,\nu] \to \sum\left[\begin{array}{l} 1_N\otimes(-[B_\mu, B_\nu]_- - i\,\partial_{-\nu}[B_\mu] + i\,\partial_{-\mu}[B_\nu]) \\ (-\nabla^S_{-\nu}[\nabla^S_{-\mu}[\_]] + \nabla^S_{-\mu}[\nabla^S_{-\nu}[\_]])\otimes 1_{\mathcal{H}_F} \end{array}\right]$$

Apply (3.2) $-[B_\mu, B_\nu]_- \to -i\,(-F_{\mu\nu} - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu])$

$$\to\ \boxed{\Omega^E[\mu,\nu] \to \sum\left[\begin{array}{l} 1_N\otimes(i\,F_{\mu\nu}) \\ (-\nabla^S_{-\nu}[\nabla^S_{-\mu}[\_]] + \nabla^S_{-\mu}[\nabla^S_{-\nu}[\_]])\otimes 1_{\mathcal{H}_F} \end{array}\right]}\ (3.4)$$

Since $\Omega^S[\mu,\nu] \to -\underline{\nabla}^S{}_\nu[\underline{\nabla}^S{}_\mu[\_]] + \underline{\nabla}^S{}_\mu[\underline{\nabla}^S{}_\nu[\_]]$

$$\to\ \boxed{\Omega^E[\mu,\nu] \to 1_N\otimes(i\,F_{\mu\nu}) + \Omega^S[\mu,\nu]\otimes 1_{\mathcal{H}_F}}\ (3.4)$$

$\{\Omega^{S\mu\nu} \to -\underline{\nabla}^{S\nu}[\underline{\nabla}^{S\mu}[\_]] + \underline{\nabla}^{S\mu}[\underline{\nabla}^{S\nu}[\_]],\ \Omega^S{}_{\mu\nu} \to -\underline{\nabla}^S{}_\nu[\underline{\nabla}^S{}_\mu[\_]] + \underline{\nabla}^S{}_\mu[\underline{\nabla}^S{}_\nu[\_]]\}$

```
PR["◆Calculate ", $0 = $ = CommutatorM[T[iD, "d", {μ}], T[iD, "d", {ν}]].Φ /. $sconvert,
  yield, $ = $ /. tuCommutatorExpand // tuDotSimplify[],
  Yield, $ = $ //. a_[_, μ_] . b_ → a[b, μ],
  NL, "From the definition: ",
  $d = {selectDef[tuDDown[iD][_, _], {ad}], selectDef[ad[_][_], {ad}]};
  $d // ColumnBar,
  NL, "Use ",
  $s = $d /. $ad // tuAddPatternVariable[{μ}];
  $s // ColumnBar, accumDef[$s];
  Yield, $ = $ //. $s //. $ad /. $d // expandDC[],
  NL, "Collect terms:",
  Yield, $ = $ //. tuOpCollect[];
  NL, "Convert to Commutators: ",
  $sc = {a_ .Φ - b_ .Φ → (a - b).Φ, Φ.a_ - Φ. b_ → Φ.(a - b), a_ .b_ - b_ .a_ →
      CommutatorM[a, b], CommutatorM[a_, b_] ⧴ -CommutatorM[b, a] /; OrderedQ[{b, a}]};
  $sc // ColumnBar,
  Yield, $ = $ // tuRepeat[$sc, {expandDC[]}];
  Yield, $ = $0 -> $; $ // Framed,

  NL, "From (3.3): ", $s1 = $e33;
  yield, $s1 = $s1 /. $sc // Reverse // tuAddPatternVariable[{μ, ν}],
  Imply, $ = $ /. $s1,
  Yield, $ = $ /. tuCommutatorExpand /. $sc;
  Yield, $ = $ /. ($ad /. $sc); $ // Framed, CG[" (3.4+)"],
  line,
  (**)
  $s = selectDef[Ω^"E"[_, _], {"∇^E"}] /. "E" → "S" /. "∇^E" -> "∇^S" /. tuDDown["∇^S"][_, n_].
       tuDDown["∇^S"][_, m_] -> tuDDown["∇^S"][tuDDown["∇^S"][_, m], n] // Reverse,
  NL, "Since ", $ = Flatten[{$s34, CommutatorM[$s34[[1]], Φ] → 0}] /. $s;
  $ // ColumnBar,
  Yield, $ = CommutatorM[#, Φ] & /@ $[[1]] /. $[[2]],
  Yield, $ = $ // tuCommutatorSimplify[],
  NL, "Using the definition for ad[]: ",
  Yield, $ = $ /. ($ad /. $sc); $ // Framed,
  CR["Puzzling role of operator product."]
 ];
```

◆Calculate $[\underline{D}_\mu[\_], \underline{D}_\nu[\_]]_-.\Phi \longrightarrow \underline{D}_\mu[\_].\underline{D}_\nu[\_].\Phi - \underline{D}_\nu[\_].\underline{D}_\mu[\_].\Phi$

→ $-\underline{D}_\nu[\underline{D}_\mu[\Phi]] + \underline{D}_\mu[\underline{D}_\nu[\Phi]]$

From the definition: $\left| \begin{array}{l} \underline{D}_\mu[a\_] \to ad[\underline{\nabla}^E_\mu[\_]][a] \\ ad[aa\_][bb\_] \to aa.bb - bb.aa \end{array} \right.$

Use $\left| \begin{array}{l} \underline{D}_{\mu\_}[a\_] \to -a.\underline{\nabla}^E_\mu[\_] + \underline{\nabla}^E_\mu[\_].a \\ (aa\_).(bb\_) - (bb\_).(aa\_) \to ad[aa][bb] \end{array} \right.$

→ $-\Phi.\underline{\nabla}^E_\mu[\_].\underline{\nabla}^E_\nu[\_] + \Phi.\underline{\nabla}^E_\nu[\_].\underline{\nabla}^E_\mu[\_] + \underline{\nabla}^E_\mu[\_].\underline{\nabla}^E_\nu[\_].\Phi - \underline{\nabla}^E_\nu[\_].\underline{\nabla}^E_\mu[\_].\Phi$

Collect terms:

→

Convert to Commutators: $\left| \begin{array}{l} (a\_).\Phi - (b\_).\Phi \to (a - b).\Phi \\ \Phi.(a\_) - \Phi.(b\_) \to \Phi.(a - b) \\ (a\_).(b\_) - (b\_).(a\_) \to [a, b]_- \\ [a\_, b\_]_- :\to -[b, a]_- /; OrderedQ[\{b, a\}] \end{array} \right.$

→

→ $\boxed{[\underline{D}_\mu[\_], \underline{D}_\nu[\_]]_-.\Phi \to -[\Phi, [\underline{\nabla}^E_\mu[\_], \underline{\nabla}^E_\nu[\_]]_-]_-}$

From (3.3): $\longrightarrow [\underline{\nabla}^E_{\mu\_}[\_], \underline{\nabla}^E_{\nu\_}[\_]]_- \to \Omega^E[\mu, \nu]$

⇒ $[\underline{D}_\mu[\_], \underline{D}_\nu[\_]]_-.\Phi \to -[\Phi, \Omega^E[\mu, \nu]]_-$

→

→ $\boxed{ad[\underline{D}_\mu[\_]][\underline{D}_\nu[\_]].\Phi \to ad[\Omega^E[\mu, \nu]][\Phi]}$ (3.4+)

$-\underline{\nabla}^S_\nu[\underline{\nabla}^S_\mu[\_]] + \underline{\nabla}^S_\mu[\underline{\nabla}^S_\nu[\_]] \to \Omega^S[\mu, \nu]$

Since $\left| \begin{array}{l} \Omega^E[\mu, \nu] \to 1_N \otimes (i F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F} \\ [\Omega^E[\mu, \nu], \Phi]_- \to 0 \end{array} \right.$

→ $0 \to [1_N \otimes (i F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}, \Phi]_-$

→ $0 \to [1_N \otimes (i F_{\mu\nu}), \Phi]_- + [\Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}, \Phi]_-$

Using the definition for ad[]:

→ $\boxed{0 \to ad[1_N \otimes (i F_{\mu\nu})][\Phi] + ad[\Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}][\Phi]}$ Puzzling role of operator product.

```
PR["Calculate (3.5) from local coordinate Laplacian: ",
 $0 = $ = "△E" → -T[g, "uu", {μ, ν}].(T["∇E", "d", {μ}].T["∇E", "d", {ν}] -
         T[Γ, "udd", {ρ, μ, ν}].T["∇E", "d", {ρ}]) /. $sconvert,
 accumDef[$];
 NL, "Use definition ",
 $s = tuRule[($sDA[[2]] // tuPatternRemove) /. a → 1 // expandDC[]] //
   tuAddPatternVariable[{μ}],
 Yield, $ = $ // expandDC[$s]; $ // ColumnSumExp,
 Yield, $ = $ // expandDC[]; $ // ColumnSumExp,
 NL, "Expand derivative operators ",
 $ = $ /. Longest[f_] . a_ . b_ :→ f.(a[b] + b.a) /; ! FreeQ[a, DerivOps] // expandDC[];
 $ // ColumnSumExp,
 NL, "Simplify using Scalars ", $scal = {Tensor[g | Γ, _, _]},
 Yield, $ = $ //. tuOpSimplify[Dot, $scal] // Expand;
 $ // ColumnSumExp;
 NL, "Evaluate  ", $s = $ // tuExtractPattern[a_[b_ ⊗ c_]],
 Yield,
 $s[[1]] =
  $s[[1]] → ($s[[1, 1]] /. bb : $s[[1, 1, 2]] → ($s[[1, 0, 1]][bb] // tuOpBlankFill));
 $s[[2]] = $s[[2]] → 0;
 $s = $s,
 Yield, $ = $ /. $s;
 $ // ColumnSumExp;
 NL, "Simplifying: ", $1b0 = $1b = $ // tuTermExtract[{B}, {"∇S", Γ}];
 $1b = $1b // tuCircleTimesGather[] // tuIndexContractUpDn[g, {μ, ν}];
 $1b = $1b0 → $1b,
 Yield, $ = $ /. $1b; $ // ColumnSumExp;
 NL, "Simplifying: ", $1b0 = $1b = $ // tuTermExtract[{}, {B, Γ}];
 $1b = $1b // tuCircleTimesGather[];
 $1b = $1b0 → $1b,
 Yield, $ = $ /. $1b; $ // ColumnSumExp;
 NL, "Simplifying ", $1b0 = $1b = $ // tuTermExtract[(1N ⊗ Tensor[B, _, _]).__];
 $1b = $1b0 → ($1b /. (1N ⊗ (bb : Tensor[B, _, _])).a_ (gg : Tensor[g, _, _]) :>
       (1N ⊗ (tuIndexContractUpDn[g, {μ, ν}][bb gg])).a /. ν → μ),
 Yield, $ = $ /. $1b; $ // ColumnSumExp;
 NL, "Using ",
 $s = $0 //. tuOpSimplify[Dot, $scal] /. {"∇E" -> "∇S", "△E" -> "△S"} // Expand;
 accumDef[{$s}];
 $s = tuRuleSolve[$s, tuTermSelect[Γ][$s][[1]]], accumDef[{$s}], CK,
 NL, "Simplifying ", $1b0 = $1b = $ // tuTermExtract[{1ℋF}, {B}];
 Yield, $1b = $1b // Simplify // tuCircleTimesGather[];
 Yield, $1b = $1b /. g_ (a_ ⊗ b_) → (g a) ⊗ b // ExpandAll;
 Yield, $1b = $1b0 → ($1b /. $s /. Dot[a_, b_] :→ Dot[b, a] /; OrderedQ[{b, a}]),
 Yield, $e35 = $ = $ /. $1b;
 $ // ColumnSumExp // Framed, CG[" (3.5)"], accumDef[$]
]
```

Calculate (3.5) from local coordinate Laplacian:

$\Delta^E \to -g^{\mu\nu}.(-\Gamma^\rho{}_{\mu\nu}.\nabla^E{}_\rho[\_] + \nabla^E{}_\mu[\_].\nabla^E{}_\nu[\_])$

Use definition $\{\nabla^E{}_{\mu\_}[\_] \to i\, 1_N \otimes B_\mu + \nabla^S{}_\mu[\_] \otimes 1_{\mathcal{H}_F}\}$

$$\to \Delta^E \to \sum\left[\begin{array}{l} g^{\mu\nu}.(1_N\otimes B_\mu).(1_N\otimes B_\nu) \\ -i\, g^{\mu\nu}.(1_N\otimes B_\mu).(\nabla^S{}_{\_\nu}[\_]\otimes 1_{\mathcal{H}_F}) \\ -i\, g^{\mu\nu}.(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F}).(1_N\otimes B_\nu) \\ -g^{\mu\nu}.(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S{}_{\_\nu}[\_]\otimes 1_{\mathcal{H}_F}) \\ i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N\otimes B_\rho) \\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(\nabla^S{}_{\_\rho}[\_]\otimes 1_{\mathcal{H}_F}) \end{array}\right]$$

$$\to \Delta^E \to \sum\left[\begin{array}{l} g^{\mu\nu}.(1_N\otimes B_\mu).(1_N\otimes B_\nu) \\ -i\, g^{\mu\nu}.(1_N\otimes B_\mu).(\nabla^S{}_{\_\nu}[\_]\otimes 1_{\mathcal{H}_F}) \\ -i\, g^{\mu\nu}.(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F}).(1_N\otimes B_\nu) \\ -g^{\mu\nu}.(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S{}_{\_\nu}[\_]\otimes 1_{\mathcal{H}_F}) \\ i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N\otimes B_\rho) \\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(\nabla^S{}_{\_\rho}[\_]\otimes 1_{\mathcal{H}_F}) \end{array}\right]$$

Expand derivative operators

$$\Delta^E \to \sum\left[\begin{array}{l} -g^{\mu\nu}.(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F})[\nabla^S{}_{\_\nu}[\_]\otimes 1_{\mathcal{H}_F}] \\ g^{\mu\nu}.(1_N\otimes B_\mu).(1_N\otimes B_\nu) \\ -i\, g^{\mu\nu}.(1_N\otimes B_\mu).(\nabla^S{}_{\_\nu}[\_]\otimes 1_{\mathcal{H}_F}) \\ -i\,(g^{\mu\nu}.(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F})[1_N\otimes B_\nu] + g^{\mu\nu}.(1_N\otimes B_\nu).(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F})) \\ -g^{\mu\nu}.(\nabla^S{}_{\_\nu}[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F}) \\ i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N\otimes B_\rho) \\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(\nabla^S{}_{\_\rho}[\_]\otimes 1_{\mathcal{H}_F}) \end{array}\right]$$

Simplify using Scalars {Tensor[g | Γ, _, _]}

→

Evaluate $\{(\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})[1_N\otimes B_\nu],\ (\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})[\nabla^S{}_\nu[\_]\otimes 1_{\mathcal{H}_F}]\}$

$\to \{(\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})[1_N\otimes B_\nu] \to 1_N\otimes\nabla^S{}_\mu[B_\nu],\ (\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})[\nabla^S{}_\nu[\_]\otimes 1_{\mathcal{H}_F}] \to 0\}$

→

Simplifying: $(1_N\otimes B_\mu).(1_N\otimes B_\nu)\, g^{\mu\nu} \to 1_N\otimes B^\nu.B_\nu$

→

Simplifying: $-(\nabla^S{}_\nu[\_]\otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})\, g^{\mu\nu} \to -(\nabla^S{}_\nu[\_].\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})\, g^{\mu\nu}$

→

Simplifying

$-i\,(1_N\otimes B_\mu).(\nabla^S{}_\nu[\_]\otimes 1_{\mathcal{H}_F})\, g^{\mu\nu} - i\,(1_N\otimes B_\nu).(\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})\, g^{\mu\nu} \to -2\,i\,(1_N\otimes B^\mu).(\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})$

→

Using $\{g^{\mu\nu}\,\Gamma^\rho{}_{\mu\nu}\,\nabla^S{}_\rho[\_] \to \Delta^S + \nabla^S{}_\mu[\_].\nabla^S{}_\nu[\_]\, g^{\mu\nu}\}$ ⟵CHECK

Simplifying

→

→

$\to -(\nabla^S{}_\nu[\_].\nabla^S{}_\mu[\_]\otimes 1_{\mathcal{H}_F})\, g^{\mu\nu} + \nabla^S{}_\rho[\_]\otimes 1_{\mathcal{H}_F}\, g^{\mu\nu}\,\Gamma^\rho{}_{\mu\nu} \to \Delta^S\otimes 1_{\mathcal{H}_F}$

$$\to \boxed{\Delta^E \to \sum\left[\begin{array}{l} \Delta^S\otimes 1_{\mathcal{H}_F} \\ 1_N\otimes B^\nu.B_\nu \\ -2\,i\,(1_N\otimes B^\mu).(\nabla^S{}_{\_\mu}[\_]\otimes 1_{\mathcal{H}_F}) \\ -i\, 1_N\otimes\nabla^S{}_{\_\mu}[B_\nu]\, g^{\mu\nu} \\ i\, 1_N\otimes B_\rho\, g^{\mu\nu}\,\Gamma^\rho{}_{\mu\nu} \end{array}\right]} \quad (3.5)$$

Proposition 3.1

```
PR["•Given the Lichnerowicz formula: ",
 $ = $L = {slash[iD].slash[iD] → "△S" + s / 4,
    "△S"[CG["Laplacian of spin connection ∇S"], s[CG["scalar curvature of M"]]],
    slash[iD][CG["compact Riemannian spin manifold"]]};
 $ // ColumnBar,
 accumDef[$L];
 NL, "Prove(prop.3.1):",
 NL, $31 = $0 =
   $ = {𝒟𝒜.𝒟𝒜 → △"E" - Q, Q → -( s ⊗ 1ℋF) / 4 - 1N ⊗ (Φ . Φ) + I / 2 (T[γ, "u", {μ}].T[γ, "u", {ν}]) ⊗
         T[F, "dd", {μ, ν}] - I T[γ, "u", {μ}]. T[γ, "d", {5}] ⊗ T[iD, "d", {μ}].Φ};
 $ // ColumnBar,
 line,

 NL, "●Compute: ", $ = $0[[1, 1]],
 NL, "with simplifying Rules: ",
 $s1 = {(tt : Tensor[γ, _, _]) .(a_ ⊗ b_) ⧴ tt.a ⊗ b, (*γ act on M space*)
   T[γ, "d", {5}].T[γ, "d", {5}] → 1N,
   1n_ . a_ → a, a_ . 1n_ → a
  }; $s1 // ColumnBar;
 next, "Apply and Relabel dummy index of 2nd term(if needed)",
 $s = {selectDef[𝒟𝒜, {Φ}], a_ . b_ ⧴ a. (b /. μ → ν) /; ! FreeQ[a, μ]},
 Yield, $ = $ //. $s[[1]] // expandDC[] // (# /. $s[[2]] &);
 $ // ColumnSumExp, OK,

 next, "Apply Lichnerowicz formula: ",
 {$1a, $} = $ // tuTermApply[{(slash[iD] ⊗ 1F).(slash[iD] ⊗ 1F)},
    {}, {$L}, {tuCircleTimesGather[]}, 1]; $1a,
 Yield, $ // ColumnSumExp; OK,

 next, "Expand Dirac derivative operator: ",
 $s = {slash[iD] -> slash[iD][_],
   selectDef[slash[iD][_], {γ}] /. μ → ν // tuAddPatternVariable[ψ]},
 $ = $ //. $s // expandDC[];
 Yield, $ // ColumnSumExp; OK,

 next, "Combine non-derivative terms and move and separate γ's to M-space: ",
 Yield, {$s, $} = $ // tuTermApply[{}, {"∇S", s, "△S"}, {$s1}, {tuCircleTimesExpand}, 1];
 $s,
 Yield, $ // ColumnSumExp; OK,

 and,
 $sg = {
   (a_ ⊗ b_ ).((tt : Tensor[γ, _, _]) ⊗ d_) -> (a.tt ⊗ b).(1N ⊗ d),
   ((tt : Tensor[γ, _, _]) ⊗ b_ ).(c_ ⊗ d_) -> (1N ⊗ b).(tt.c ⊗ d),
   tt : (tuDDown[a_][b_, c_]).(gg : Tensor[γ, _, _]) ⧴ Reverse[tt],
   (g1 : Tensor[γ, _, _]).(g2 : Tensor[γ, _, _]).(dd : tuDDown[a_][b_, c_]) ⊗ 1F →
    (g1.g2 ⊗ 1F).(dd ⊗ 1F)
  }; $sg // ColumnBar,
 Yield, {$s, $} = $ // tuTermApply[{"∇S"}, {s, "△S"}, {$sg}, {}, 1]; $s,
 Yield, $ // ColumnSumExp; OK,

 next, "Expand derivative operators: ",
 $s = {a_ .((dd : tuDDown[_][_, _]) ⊗ 1F).(1N ⊗ b_) → a. (1N ⊗ dd[b]) + a.(1N ⊗ b).(dd ⊗ 1F),
   tuDDown[a_][_, c_][d_] -> tuDDown[a][d, c],
   tt : T[γ, "u", {_}].T[γ, "d", {5}] → -Reverse[tt]},
```

```
{$s, $} = $ // tuTermApply[{"∇S"}, {s, "△S"}, {$s}, {}, 1]; $s;
$ = $ //. $s;
Yield, $ // ColumnSumExp; OK,

next, "Expand covariant derivatives: ",
$s = {tuDDown["∇S"][(bb : T[B, "d", {μ_}]), ν_] →
    tuDPartial[bb, ν] - T[Γ, "udd", {ρ, μ, ν}] T[B, "d", {ρ}],
   tuDDown["∇S"][(bb : Φ), ν_] → tuDPartial[bb, ν]
  },
{$s, $} = $ // tuTermApply[{"∇S"}, {s, "△S"}, {$s}, {expandDC[], Expand}, 1];
$s;
Yield, $ // ColumnSumExp; OK,

next, "Order terms and γ's and use: ",
$s = {
   (bb : 1N ⊗ b_).(tt : Tensor[γ, _, _].Tensor[γ, _, _] ⊗ 1F) → tt.bb,
   ((tt : Tensor[γ, _, _].Tensor[γ, _, _]) ⊗ 1F).(1N ⊗ b_) → tt ⊗ b,
   tt : T[γ, "u", {_}].T[γ, "d", {5}] → -Reverse[tt],
   $sgg = tuRuleSelect[$sgeneral][CommutatorP[_, _]][[-1]] /. tuCommutatorExpand},
Yield, {$s, $} = $ // tuTermApply[{Tensor[γ, _, _].Tensor[γ, _, _]},
    {}, {$s, $s1, $sgg, tuOpCollect[CircleTimes], tuOpCollect[Dot],
     tuOpSimplify[CircleTimes], tuOpSimplify[Dot]}, {Expand, Simplify}, 1];
$s;
$ // ColumnSumExp; OK,

next, "Using ", $s =
  I # & /@ (selectDef[tuDDown[iD][Φ, _]] /. 1N ⊗ a_ → a /. tuCommutatorExpand // Reverse) //
   Expand,
Yield, {$s, $} = $ // tuTermApply[{Φ, γ}, {},
    {ν → μ, tuOpCollect[CircleTimes], $s}, {}, 1];
$s,
Yield, $ // ColumnSumExp; OK,

next, "Specify g as scalar: ",
$scal = Tensor[g, _, _];
$s = {(gg : $scal ) ⊗ b_ → gg (1N ⊗ b)},
{$s, $} = $ /. $s // tuTermApply[{g}, {},
    {$s, tuOpSimplify[CircleTimes], tuOpSimplify[Dot, {$scal}]}, {}, 1];
$s,
Yield, $ // ColumnSumExp; CK,

next, "Apply ",
$1 = $sgg[[1, 1]];
$1 = $1 - $sgg[[1, 2]];
$sgg1 = $sgg[[1, 1]] -> $sgg[[1]] / 2 + $1 / 2;
$sgg1 = $sgg1 /. $sgg // tuAddPatternVariable[{μ, ν}],
$ = $ /. $sgg1;
NL, "and order indices ",
$s = {tt : T[γ, "u", {ν}].T[γ, "u", {μ}] ⊗ a_ :→ tuIndexSwap[{μ, ν}][tt],
   T[Γ, "udd", {ρ, ν, μ}] -> T[Γ, "udd", {ρ, μ, ν}]},
Yield,
{$s, $} = $ // tuTermApply[{Tensor[γ, _, _]}, {T[γ, "d", {5}]}, {$s}, {expandDC[]}, 1];
$s;
$ = $ // Expand;
Yield, $ // ColumnSumExp, CK,

next, "Gather γ's and substitute ", $s = {
```

```
   (ca_:1) (tt:T[γ, "u", {μ}].T[γ, "u", {ν}]) ⊗ a_ +
     (cb_:1) T[γ, "u", {μ}].T[γ, "u", {ν}] ⊗ b_ → tt ⊗ (ca a + cb b)},
 NL, "and substitute: ",
 $sf =
  tuRuleSolve[(selectDef[Tensor[F, _, _]]), CommutatorM[_, _]] /. tuCommutatorExpand,
 {$s, $} = $ // tuTermApply[{T[γ, "u", {μ}].T[γ, "u", {ν}]}, {}, {$s, $sf}, {Simplify}, 1];
 $s;
 $ // ColumnSumExp;

 next, "Contract g's ", $s = { (gg:T[g, "uu", {_, _}]) ⊗ a_ → (1N ⊗ a) gg} // Flatten,
 Yield, {$s, $} = $ // tuTermApply[{g}, {}, {$s}, {}, 1]; $s;
 $pass = $ = tuIndexContractUpDn[g, {ν}][#] & /@ $ // expandDC[{}, {Tensor[Γ, _, _]}];
 Yield, $ // ColumnSumExp
]
PR["The definitions for ", $s =
  {$sa = selectDef["△E", {B}] /. ℋF → F, $sb = # ⊗ 1F & /@ selectDef["△S", {}]} // expandDC[];
 $s // ColumnBar,
 Yield, $s = tuRuleSolve[$sb, $sb[[2, 1, 2]]];
 $s = tuRuleSolve[$sa /. $s, $sb[[1]]][[1]],
 Yield, $s = $s /. tuIndexOrderPairs;
 $s = $s //
   tuTermApply[{ν}, {μ}, {ν → μ, T[Γ, "udd", {ρ, ν, μ}] -> T[Γ, "udd", {ρ, μ, ν}]}, {}];
 $s = $s /. tuDDown["∇S"][(bb:T[B, "d", {μ_}]), ν_] →
      tuDPartial[bb, ν] - T[Γ, "udd", {ρ, μ, ν}] T[B, "d", {ρ}] //
    expandDC[{}, {Tensor[Γ, _, _]}] // Expand,
 Yield,
 $s = $s // (# /. T[Γ, "udd", {ρ, ν, μ}] -> T[Γ, "udd", {ρ, μ, ν}] &),

 next, "Adjust indices ",
 $ = $pass /. $s //. tuIndexOrderPairs;
 $ = tuIndexContractUpDn[g, {ν}][#] & /@ $ // expandDC[{}, {Tensor[Γ, _, _]}];
 Yield, {$s, $} = $ // tuTermApply[{tuDUp}, {}, {}, {tuIndexSwapUpDown[{μ}]}, 1];
 $ // ColumnSumExp // Framed, CR["Extra Γ term: Check △S definition."]
]
```

•Given the Lichnerowicz formula:

$(\not{D}).(\not{D}) \to \triangle^S + \frac{s}{4}$

$\triangle^S$[Laplacian of spin connection ∇^S, s[scalar curvature of M]]

$(\not{D})$[compact Riemannian spin manifold]

Prove(prop.3.1):

$\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to -Q + \triangle^E$

$Q \to -\frac{1}{4} s \otimes 1_{\mathcal{H}_F} - i\,\gamma^\mu.\gamma_5 \otimes D_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi$

●Compute: $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}}$

with simplifying Rules:

◆Apply and Relabel dummy index of 2nd term(if needed)

$\{\mathcal{D}_{\mathcal{A}} \to (\not{D}) \otimes 1_F + \gamma_5 \otimes \Phi + \gamma^\mu \otimes B_\mu,\ (a\_).(b\_) \mapsto a.(b\ /.\ \mu \to \nu)\ /;\ !\,\mathrm{FreeQ}[a, \mu]\}$

$$\to \sum\left[\begin{array}{l} ((\not{D})\otimes 1_F).((\not{D})\otimes 1_F) \\ ((\not{D})\otimes 1_F).(\gamma_5\otimes\Phi) \\ ((\not{D})\otimes 1_F).(\gamma^\mu\otimes B_\mu) \\ (\gamma_5\otimes\Phi).((\not{D})\otimes 1_F) \\ (\gamma_5\otimes\Phi).(\gamma_5\otimes\Phi) \\ (\gamma_5\otimes\Phi).(\gamma^\mu\otimes B_\mu) \\ (\gamma^\mu\otimes B_\mu).((\not{D})\otimes 1_F) \\ (\gamma^\mu\otimes B_\mu).(\gamma_5\otimes\Phi) \\ (\gamma^\mu\otimes B_\mu).(\gamma^\nu\otimes B_\nu) \end{array}\right] \text{ OK}$$

◆Apply Lichnerowicz formula: $((\not{D})\otimes 1_F).((\not{D})\otimes 1_F) \to (\triangle^S + \frac{s}{4})\otimes 1_F$

→ OK

◆Expand Dirac derivative operator: $\{\not{D} \to (\not{D})[\_], (\not{D})[\psi\_] \to -i\,\gamma^\nu.\underline{\nabla}^S{}_\nu[\psi]\}$

→ OK

◆Combine non-derivative terms and move and separate γ's to M-space:

→ $(\gamma_5\otimes\Phi).(\gamma_5\otimes\Phi) + (\gamma_5\otimes\Phi).(\gamma^\mu\otimes B_\mu) + (\gamma^\mu\otimes B_\mu).(\gamma_5\otimes\Phi) + (\gamma^\mu\otimes B_\mu).(\gamma^\nu\otimes B_\nu) \to$
$\gamma_5.\gamma^\mu\otimes\Phi.B_\mu + \gamma^\mu.\gamma_5\otimes B_\mu.\Phi + \gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu + 1_N\otimes\Phi.\Phi$

→ OK

and

$(a\_\otimes b\_).((tt:\text{Tensor}[\gamma,\_,\_])\otimes d\_) \to (a.tt\otimes b).(1_N\otimes d)$
$((tt:\text{Tensor}[\gamma,\_,\_])\otimes b\_).(c\_\otimes d\_) \to (1_N\otimes b).(tt.c\otimes d)$
$tt:a\_{}_{-c\_}[b\_].(gg:\text{Tensor}[\gamma,\_,\_]) :\to \text{Reverse}[tt]$
$(g1:\text{Tensor}[\gamma,\_,\_]).(g2:\text{Tensor}[\gamma,\_,\_]).(dd:a\_{}_{-c\_}[b\_])\otimes 1_F \to (g1.g2\otimes 1_F).(dd\otimes 1_F)$

→ $-i\,(\gamma^\nu.\underline{\nabla}^S{}_\nu[\_]\otimes 1_F).(\gamma_5\otimes\Phi) - i\,(\gamma^\nu.\underline{\nabla}^S{}_\nu[\_]\otimes 1_F).(\gamma^\mu\otimes B_\mu) -$
$i\,(\gamma_5\otimes\Phi).(\gamma^\nu.\underline{\nabla}^S{}_\nu[\_]\otimes 1_F) - i\,(\gamma^\mu\otimes B_\mu).(\gamma^\nu.\underline{\nabla}^S{}_\nu[\_]\otimes 1_F) \to$
$-i\,(\gamma^\nu.\gamma_5\otimes 1_F).(\underline{\nabla}^S{}_\nu[\_]\otimes 1_F).(1_N\otimes\Phi) - i\,(\gamma^\nu.\gamma^\mu\otimes 1_F).(\underline{\nabla}^S{}_\nu[\_]\otimes 1_F).(1_N\otimes B_\mu) -$
$i\,(1_N\otimes\Phi).(\gamma_5.\gamma^\nu\otimes 1_F).(\underline{\nabla}^S{}_\nu[\_]\otimes 1_F) - i\,(1_N\otimes B_\mu).(\gamma^\mu.\gamma^\nu\otimes 1_F).(\underline{\nabla}^S{}_\nu[\_]\otimes 1_F)$

→ OK

◆Expand derivative operators:

$\{(a\_).((dd:\underline{\ }\,[\_])\otimes 1_F).(1_N\otimes b\_) \to a.(1_N\otimes dd[b]) + a.(1_N\otimes b).(dd\otimes 1_F),$
$\underline{a}_{=c\_}[\_][d\_] \to \underline{a}_c[d],\ tt:\gamma\_.\gamma_5 \to -\text{Reverse}[tt]\}$

→ OK

◆Expand covariant derivatives: $\{\underline{\nabla}^S{}_{\nu\_}[bb:B_{\mu\_}] \to -B_\rho\,\Gamma^\rho{}_{\mu\nu} + \underline{\partial}_\nu[bb],\ \underline{\nabla}^S{}_{\nu\_}[bb:\Phi] \to \underline{\partial}_\nu[bb]\}$

→ OK

◆Order terms and γ's and use:

$\{(bb:1_N\otimes b\_).(tt:\text{Tensor}[\gamma,\_,\_].\text{Tensor}[\gamma,\_,\_]\otimes 1_F) \to tt.bb,$
$((tt:\text{Tensor}[\gamma,\_,\_].\text{Tensor}[\gamma,\_,\_])\otimes 1_F).(1_N\otimes b\_) \to tt\otimes b,$
$tt:\gamma\_.\gamma_5 \to -\text{Reverse}[tt],\ \gamma^\mu.\gamma^\nu + \gamma^\nu.\gamma^\mu \to 2\,g^{\mu\nu}\}$

→ OK

◆Using $\Phi.B_\mu - B_\mu.\Phi + i\,\underline{\partial}_\mu[\Phi] \to i\,\underline{D}_\mu[\Phi]$

→ $\gamma_5.\gamma^\mu\otimes(\Phi.B_\mu - B_\mu.\Phi) + i\,\gamma_5.\gamma^\nu\otimes\underline{\partial}_\nu[\Phi] \to \gamma_5.\gamma^\mu\otimes(i\,\underline{D}_\mu[\Phi])$

→ OK

◆Specify g as scalar: $\{(gg:\text{Tensor}[g,\_,\_])\otimes b\_ \to gg\,1_N\otimes b\}$

$-2\,i\,(1_N\otimes B_\mu\,g^{\mu\nu}).(\underline{\nabla}^S{}_\nu[\_]\otimes 1_F) \to -2\,i\,(1_N\otimes B_\mu).(\underline{\nabla}^S{}_\nu[\_]\otimes 1_F)\,g^{\mu\nu}$

→ ⟵CHECK

◆Apply $\gamma^{\mu}\_.\gamma^{\nu}\_ \to \frac{1}{2}(\gamma^\mu.\gamma^\nu - \gamma^\nu.\gamma^\mu) + g^{\mu\nu}$

and order indices $\{tt:\gamma^\nu.\gamma^\mu\otimes a\_ :\to \text{tuIndexSwap}[\{\mu,\nu\}][tt],\ \Gamma^\rho{}_{\nu\mu} \to \Gamma^\rho{}_{\mu\nu}\}$

→

→ $\sum[$
$\triangle^S\otimes 1_F$
$\frac{s\otimes 1_F}{4}$
$\gamma_5.\gamma^\mu\otimes(i\,D_{-\mu}[\Phi])$
$\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu$
$-\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes B_\nu.B_\mu$
$\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes\partial_{-\nu}[B_\mu]$
$-\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes\partial_{-\mu}[B_\nu]$
$1_N\otimes\Phi.\Phi$
$g^{\mu\nu}\otimes B_\mu.B_\nu$
$i\,g^{\nu\mu}\otimes(B_\rho\,\Gamma^\rho{}_{\mu\nu})$
$-i\,g^{\nu\mu}\otimes\partial_{-\nu}[B_\mu]$
$-2\,i\,(1_N\otimes B_\mu).(\nabla^S_{-\nu}[\_]\otimes 1_F)\,g^{\mu\nu}$
$]$⟵CHECK

◆Gather γ's and substitute

$\{(tt:\gamma^{\mu}.\gamma^{\nu})\otimes a\_\ (ca\_:1)+\gamma^{\mu}.\gamma^{\nu}\otimes b\_\ (cb\_:1)\to tt\otimes(a\,ca+b\,cb)\}$

and substitute: $\{B_{\mu}.B_{\nu}-B_{\nu}.B_{\mu}\to i\,(-F_{\mu\nu}-\underline{\partial}_{\nu}[B_{\mu}]+\underline{\partial}_{\mu}[B_{\nu}])\}$

◆Contract g's $\{(gg:g\_\_)\otimes a\_\to gg\,1_N\otimes a\}$

→

$$\to \sum\left[\begin{array}{l} \triangle^S\otimes 1_F \\ \frac{s\otimes 1_F}{4} \\ i\,\gamma_5.\gamma^{\mu}\otimes \underset{-\mu}{D}[\Phi] \\ -\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\ 1_N\otimes\Phi.\Phi \\ 1_N\otimes B_{\mu}.B^{\mu} \\ -i\,1_N\otimes\underline{\partial}^{\mu}[B_{\mu}] \\ -2\,i\,(1_N\otimes B_{\mu}).(\underline{\nabla}^{S\mu}[\_]\otimes 1_F) \\ i\,1_N\otimes B_{\rho}\,\Gamma^{\rho}{}_{\mu}{}^{\mu} \end{array}\right]$$

The definitions for

$$\triangle^E\to\triangle^S\otimes 1_F+1_N\otimes B^{\nu}.B_{\nu}-2\,i\,(1_N\otimes B^{\mu}).(\underset{-\mu}{\nabla^S}[\_]\otimes 1_F)-i\,1_N\otimes\underset{-\mu}{\nabla^S}[B_{\nu}]\,g^{\mu\nu}+i\,1_N\otimes B_{\rho}\,g^{\mu\nu}\,\Gamma^{\rho}{}_{\mu\nu}$$

$$\triangle^S\otimes 1_F\to-((\underset{-\mu}{\nabla^S}[\_].\underset{-\nu}{\nabla^S}[\_]\,g^{\mu\nu})\otimes 1_F)+(g^{\mu\nu}\,\Gamma^{\rho}{}_{\mu\nu}\,\underset{-\rho}{\nabla^S}[\_])\otimes 1_F$$

$$\to \triangle^S\otimes 1_F\to\triangle^E-1_N\otimes B^{\nu}.B_{\nu}+i\,(2\,(1_N\otimes B^{\mu}).(\underline{\nabla}^S{}_{\mu}[\_]\otimes 1_F)+1_N\otimes\underline{\nabla}^S{}_{\mu}[B_{\nu}]\,g^{\mu\nu}-1_N\otimes B_{\rho}\,g^{\mu\nu}\,\Gamma^{\rho}{}_{\mu\nu})$$

→

$$\triangle^S\otimes 1_F\to\triangle^E-1_N\otimes B_{\mu}.B^{\mu}+2\,i\,(1_N\otimes B^{\mu}).(\underline{\nabla}^S{}_{\mu}[\_]\otimes 1_F)+i\,1_N\otimes\underline{\partial}_{\mu}[B_{\nu}]\,g^{\mu\nu}-i\,1_N\otimes B_{\rho}\,g^{\mu\nu}\,\Gamma^{\rho}{}_{\mu\nu}-i\,1_N\otimes B_{\rho}\,g^{\mu\nu}\,\Gamma^{\rho}{}_{\nu\mu}$$

$$\to \triangle^S\otimes 1_F\to\triangle^E-1_N\otimes B_{\mu}.B^{\mu}+2\,i\,(1_N\otimes B^{\mu}).(\underline{\nabla}^S{}_{\mu}[\_]\otimes 1_F)+i\,1_N\otimes\underline{\partial}_{\mu}[B_{\nu}]\,g^{\mu\nu}-2\,i\,1_N\otimes B_{\rho}\,g^{\mu\nu}\,\Gamma^{\rho}{}_{\mu\nu}$$

◆Adjust indices

$$\to \boxed{\sum\left[\begin{array}{l} \triangle^E \\ \frac{s\otimes 1_F}{4} \\ i\,\gamma_5.\gamma^{\mu}\otimes \underset{-\mu}{D}[\Phi] \\ -\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\ 1_N\otimes\Phi.\Phi \\ -i\,1_N\otimes B_{\rho}\,\Gamma^{\rho}{}_{\mu}{}^{\mu} \end{array}\right]}$$ Extra Γ term: Check $\triangle^S$ definition.

```
PR["●Compare calculation of 4th and 5th terms on p.29: ",
 NL, $ = (slash[iD].T[γ, "u", {μ}] ⊗ 1_F).(1_N ⊗ T[B, "d", {μ}]) +
   (1_N ⊗ T[B, "d", {μ}]).(T[γ, "u", {μ}].slash[iD] ⊗ 1_F),
 next, $s = {slash[iD] -> slash[iD][_],
   selectDef[slash[iD][_], {γ}] /. μ → ν // tuAddPatternVariable[ψ]},
 Yield, $ = $ //. $s // expandDC[],
 next, $s = tt : (tuDDown[a_][b_, c_]).(gg : Tensor[γ, _, _]) ⧴ Reverse[tt],
 Yield, $ = $ /. $s,
 next, $s = {(a_ .(dd : tuDDown[_][_, _]) ⊗ 1_F).(1_N ⊗ b_) → (a ⊗ 1_F).(1_N ⊗ dd[b]) +
     (a ⊗ 1_F).(1_N ⊗ b).(dd ⊗ 1_F), tuDDown[a_][_, c_][d_] -> tuDDown[a][d, c]};
 $s // ColumnBar,
 Yield, $ = $ //. $s,
 next, $s = {tuDDown["∇^S"][(bb : T[B, "d", {μ_}]), ν_] →
    tuDPartial[bb, ν] - T[Γ, "udd", {ρ, μ, ν}] T[B, "d", {ρ}]},
 Yield, $ = $ /. $s // expandDC[]; $ // ColumnSumExp,
 next,
 $s = {tt : (gg_ ⊗ 1_F).(1_N ⊗ T[B, "d", {μ}]) → Reverse[tt], (gg_ ⊗ 1_F).(dd_ ⊗ 1_F) → gg.dd ⊗ 1_F},
 Yield, $ = $ //. $s // Simplify;
 $ = $ //. tuOpCollect[Dot] //. tuOpCollect[CircleTimes] //. tuOpCollect[Dot];
 $ // ColumnSumExp,
 next, $s = {tuRuleSelect[$sgeneral][CommutatorP[_, _]][[-1]] /. tuCommutatorExpand,
   (a_ ⊗ b_).(c_ ⊗ (d_ (gg : Tensor[Γ, _, _]))) -> ((a gg) ⊗ b).(c ⊗ d )},
 Yield, $ = $ /. $s // expandDC[] // (# /. $s &),
 next, $s = {T[γ, "u", {ν}].T[γ, "u", {μ}] tt : Tensor[Γ, _, _] →
    (T[γ, "u", {ν}].T[γ, "u", {μ}] + T[γ, "u", {μ}].T[γ, "u", {ν}]) / 2 tt,
   tuRuleSelect[$sgeneral][CommutatorP[_, _]][[-1]] /. tuCommutatorExpand},
 Yield, $ = $ //. $s,
 next, {$s, $} = $ // tuTermApply[{"∇^S"}, {}, {}, {tuIndexContractUpDn[g, {μ}]}, 1];
 $s,
 Yield, $ // Framed, CG[" Same as text"]
]
```

●Compare calculation of 4th and 5th terms on p.29:

$((\not{D}).\gamma^{\mu}\otimes 1_F).(1_N\otimes B_\mu)+(1_N\otimes B_\mu).(\gamma^{\mu}.(\not{D})\otimes 1_F)$

◆$\{\not{D}\to(\not{D})[\_],\ (\not{D})[\psi\_]\to -i\,\gamma^{\nu}.\underline{\nabla}^{S}{}_{\nu}[\psi]\}$

→ $-i\,(\gamma^{\nu}.\underline{\nabla}^{S}{}_{\nu}[\_].\gamma^{\mu}\otimes 1_F).(1_N\otimes B_\mu)-i\,(1_N\otimes B_\mu).(\gamma^{\mu}.\gamma^{\nu}.\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F)$

◆tt : $a\_{}_{\underline{\ }c\_}[b\_].(gg:\mathrm{Tensor}[\gamma,\_,\_])$:→ Reverse[tt]

→ $-i\,(\gamma^{\nu}.\gamma^{\mu}.\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F).(1_N\otimes B_\mu)-i\,(1_N\otimes B_\mu).(\gamma^{\mu}.\gamma^{\nu}.\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F)$

◆ $((a\_).(dd:\underline{\ }_{-\_}[\_])\otimes 1_F).(1_N\otimes b\_)\to(a\otimes 1_F).(1_N\otimes dd[b])+(a\otimes 1_F).(1_N\otimes b).(dd\otimes 1_F)$

$a\_{}_{-c\_}[\_][d\_]\to a_{-c}[d]$

→ $-i\,(1_N\otimes B_\mu).(\gamma^{\mu}.\gamma^{\nu}.\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F)-i\,((\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes\underline{\nabla}^{S}{}_{\nu}[B_\mu])+(\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes B_\mu).(\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F))$

◆$\{\underline{\nabla}^{S}{}_{\nu\_}[bb:B_{\mu\_}]\to -B_\rho\,\Gamma^{\rho}{}_{\mu\nu}+\underline{\partial}_{\nu}[bb]\}$

→

$\sum[$ $-i\,(1_N\otimes B_\mu).(\gamma^{\mu}.\gamma^{\nu}.\nabla^{S}_{-\nu}[\_]\otimes 1_F)$

$-i\,(-(\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes(B_\rho\,\Gamma^{\rho}{}_{\mu\nu}))+(\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes\partial_{-\nu}[B_\mu])+(\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes B_\mu).(\nabla^{S}_{-\nu}[\_]\otimes 1_F))$ $]$

◆$\{$tt : $(gg\_\otimes 1_F).(1_N\otimes B_\mu)\to$ Reverse[tt], $(gg\_\otimes 1_F).(dd\_\otimes 1_F)\to gg.dd\otimes 1_F\}$

→ $-i\sum[$ $(\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes(-B_\rho\,\Gamma^{\rho}{}_{\mu\nu}+\partial_{-\nu}[B_\mu]))$

$(1_N\otimes B_\mu).((\gamma^{\mu}.\gamma^{\nu}+\gamma^{\nu}.\gamma^{\mu}).\nabla^{S}_{-\nu}[\_]\otimes 1_F)$ $]$

◆$\{\gamma^{\mu}.\gamma^{\nu}+\gamma^{\nu}.\gamma^{\mu}\to 2\,g^{\mu\nu},\ (a\_\otimes b\_).(c\_\otimes(d\_\ (gg:\mathrm{Tensor}[\Gamma,\_,\_])))\to((a\,gg)\otimes b).(c\otimes d)\}$

→ $-i\,((\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes\underline{\partial}_{\nu}[B_\mu])+2\,(1_N\otimes B_\mu).(g^{\mu\nu}.\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F)-((\gamma^{\nu}.\gamma^{\mu}\,\Gamma^{\rho}{}_{\mu\nu})\otimes 1_F).(1_N\otimes B_\rho))$

◆$\{\gamma^{\nu}.\gamma^{\mu}\ (tt:\mathrm{Tensor}[\Gamma,\_,\_])\to\frac{1}{2}tt\,(\gamma^{\mu}.\gamma^{\nu}+\gamma^{\nu}.\gamma^{\mu}),\ \gamma^{\mu}.\gamma^{\nu}+\gamma^{\nu}.\gamma^{\mu}\to 2\,g^{\mu\nu}\}$

→ $-i\,((\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes\underline{\partial}_{\nu}[B_\mu])+2\,(1_N\otimes B_\mu).(g^{\mu\nu}.\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F)-((g^{\mu\nu}\,\Gamma^{\rho}{}_{\mu\nu})\otimes 1_F).(1_N\otimes B_\rho))$

◆$-2\,i\,(1_N\otimes B_\mu).(g^{\mu\nu}.\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F)\to -2\,i\,(1_N\otimes B^{\nu}).(\underline{\nabla}^{S}{}_{\nu}[\_]\otimes 1_F)$

→ $\boxed{-i\,(\gamma^{\nu}.\gamma^{\mu}\otimes 1_F).(1_N\otimes\partial_{-\nu}[B_\mu])-2\,i\,(1_N\otimes B^{\nu}).(\nabla^{S}_{-\nu}[\_]\otimes 1_F)+i\,((g^{\mu\nu}\,\Gamma^{\rho}{}_{\mu\nu})\otimes 1_F).(1_N\otimes B_\rho)}$

Same as text

3.1.4 The heat expansion

```
PR["●Theorem 3.2. ",
  $t32 = {Tr[Exp[-t H]] -> xSum[t^((k - n) / 2) a_k[H], {k ≥ 0}],
    H → "Laplacian"["E"],
    n → dim[M],
    a_k[H] → xIntegral[a_k[x, H] √Det[g] , x ∈ M]
   }; Column[$t32]
 ];
```

●Theorem 3.2.

$\mathrm{Tr}[e^{-H\,t}]\to\sum_{\{k\geq 0\}}[t^{\frac{k-n}{2}}\,a_k[H]]$

$H\to\mathrm{Laplacian}[E]$

$n\to\dim[M]$

$a_k[H]\to\int_{x\in M}\sqrt{\mathrm{Det}[g]}\,a_k[x,H]$

```
PR["●Theorem 3.3. ",
  $t33 = {a0[x, H] → (4 π)^(-n / 2) Tr"E"x[1N],
    a2[x, H] -> (4 π)^(-n / 2) Tr"E"x[s / 6 1N + F],
    a4[x, H] -> (4 π)^(-n / 2) (1 / 360)
      Tr"E"x[(-12 △[s] + 5 s.s - 2 T[R, "dd", {μ, ν}].T[R, "uu", {μ, ν}] +
          2 T[R, "dddd", {μ, ν, ρ, σ}]. T[R, "uuuu", {μ, ν, ρ, σ}] + 60 s. F +
          180 F . F - 60 △[F] + 30 T[Ω"E", "dd", {μ, ν}]. T[Ω"E", "uu", {μ, ν}])],
    H → "∇""E" - F,
    s → "scalar curvature of ∇",
    △ → "scalar Laplacian",
    T[Ω"E", "dd", {μ, ν}] → "curvature of connnection ∇E"
   }; Column[$t33]
 ];
```

●Theorem 3.3.

$a_0[x, H] \to 2^{-n}\, \pi^{-n/2}\, \mathrm{Tr}_{E_x}[1_N]$

$a_2[x, H] \to 2^{-n}\, \pi^{-n/2}\, \mathrm{Tr}_{E_x}[F + \frac{s\, 1_N}{6}]$

$a_4[x, H] \to$

$\frac{1}{45} 2^{-3-n}\, \pi^{-n/2}\, \mathrm{Tr}_{E_x}[180\, F.F + 60\, s.F + 5\, s.s - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[F] - 12\, \triangle[s]]$

$H \to \nabla^{E} - F$

s → scalar curvature of ∇

△ → scalar Laplacian

$\Omega^{E}{}_{\mu\nu} \to$ curvature of connection ∇^{E}

```
Clear[$s]
PR["●Proposition 3.4. ",
  $t34 =
   {Tr[f[𝒟𝒜 / Λ]] -> a4[𝒟𝒜^2] f[0] + 2 xSum[f4-k Λ^(4-k) ak[𝒟𝒜^2] / Γ[(4 - k) / 2], {k, 0, 4, even}],
    fj_ → xIntegral[v^(j-1) f[v], v]},
  Yield, $t34 = $t34 /. {k, 0, 4, even} → {k, {0, 2}} /. xSum → Sum,
  line,
  NL, "¶ Proof: Let: ", $g = $ = g[v] → xIntegral[Exp[-s v] h[s], s],
  Yield, $ = $ /. v → t iDA^2,
  Yield, $ = Tr /@ $ /. Tr[xIntegral[a_ h[s], b_]] → xIntegral[Tr[a] h[s], b],
  Yield,
  $ = $ /. (tuRule[$t32] // First // tuAddPatternVariable[t] // (# /. H → iDA^2 &)),
  Yield, $0 = $ = $ /. a_ xSum[b_, c_] → xSum[a b, c] /. tuOpSwitch[xIntegral, xSum] //
      PowerExpand // tuIntegralSimplify,
  NL, "Assume ", $s = t ≪ 1, imply, "keep only terms k≤4 ",
  NL, "● For: ", $s = {k → 4, n → 4, xSum[a_, _] → a, xIntegral[h[s], s] → g[0]},
  yield, $ = $0[[2]] //. $s,
  NL, "● For: ", $s = {k → 2, n → 4, xSum[a_, _] → a},
  yield, $ = $0[[2]] //. $s,
  line
 ];
```

●Proposition 3.4. $\{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \to 2 \sum_{\{k,0,4,\mathrm{even}\}} [\frac{\Lambda^{4-k} f_{4-k}\, a_k[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[\frac{4-k}{2}]}] + f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\ f_{j\_} \to \int v^{-1+j} f[v]\, dv\}$

$\to \{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \to 2\, (\frac{\Lambda^4 f_4\, a_0[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2\, a_2[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[1]}) + f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\ f_{j\_} \to \int v^{-1+j} f[v]\, dv\}$

¶ Proof: Let: $g[v] \to \int e^{-s v} h[s]\, ds$

$\to g[t\, D_A^2] \to \int e^{-s t D_A^2} h[s]\, ds$

$\to \mathrm{Tr}[g[t\, D_A^2]] \to \int h[s]\, \mathrm{Tr}[e^{-s t D_A^2}]\, ds$

$\to \mathrm{Tr}[g[t\, D_A^2]] \to \int h[s] \sum_{\{k \geq 0\}} [(s\, t)^{\frac{k-n}{2}} a_k[D_A^2]]\, ds$

$\to \mathrm{Tr}[g[t\, D_A^2]] \to \sum_{\{k \geq 0\}} [t^{\frac{k-n}{2}} \int s^{\frac{k-n}{2}} h[s]\, ds\, a_k[D_A^2]]$

Assume $t \ll 1$ ⇒ keep only terms k≤4

● For: $\{k \to 4,\ n \to 4,\ \sum_{\_}[a\_] \to a,\ \int h[s]\, ds \to g[0]\}$ ⟶ $g[0]\, a_4[D_A^2]$

● For: $\{k \to 2,\ n \to 4,\ \sum_{\_}[a\_] \to a\}$ ⟶ $\frac{\int \frac{h[s]}{s} ds\, a_2[D_A^2]}{t}$

```
PR["Calculation following from (3.11). Start with: ",
  Yield, $0 /. {n → 4},
  NL, "From (3.11): ", $ = Γ[z] -> xIntegral[r^(z-1) Exp[-r], {r, 0, ∞}], "POFF",
  Yield, $ = $ //. {r → s v, z → (4 - k) / 2},
  Yield, $ = $ /. xIntegral[a_, {v s, 0, ∞}] → xIntegral[a s, v] // PowerExpand,
  Yield, $ = $ // tuIntegralSimplify, "PONdd",
  yield, $ss = tuRuleSolve[$, $[[2, 1]]] // First // Map[1 / # &, #] &,
  NL, "● Apply to: ", $ = $0 /. {n → 4},
  Yield, $p = $ // tuExtractIntegrand,
  Yield, $p = $p /. $ss /. h[s] xIntegral[a_, b_] → xIntegral[h[s] a, b],
  Yield, $ = tuReplacePart[$, {$p}] // tuIntegralSimplify,
  Yield, $ = $ /. tuOpMerge[xIntegral],
  NL, "● Apply: ", $s = (Reverse[$g] // xIntegral[#, v] & /@ # & // tuIntegralSimplify),
  Yield, $s = $s /. tuOpMerge[] /. xIntegral[a_, b__] → xIntegral[A a, b] /.
     xIntegral[a_, b_, c_] → xIntegral[a, c, b] // tuAddPatternVariable[A],
  Yield, $pass5 = $ = $ /. $s; $ // Framed,
  NL, "Substitute: ", $s = {g[u_] → f[√u], v → u^2},
  Yield, $ = $ //. $s /. ii : xIntegral[_, _] :→ tuIntegralSwitchVar[d[u^2] → 2 u d[u]][ii] //
     PowerExpand // Simplify;
  $ // Framed,
  NL, "Substitute: ", $s = t → Λ^-2,
  Yield, $ = $ /. $s // PowerExpand // Simplify; $ // Framed
 ];
```

Calculation following from (3.11). Start with:

$\rightarrow \mathrm{Tr}[g[t\, D_A^2]] \rightarrow \sum_{\{k\geq 0\}} [t^{\frac{1}{2}(-4+k)} \int s^{\frac{1}{2}(-4+k)}\, h[s]\, ds\, a_k[D_A^2]]$

From (3.11): $\Gamma[z] \rightarrow \int_0^\infty e^{-r}\, r^{-1+z}\, dr$

.... $\longrightarrow s^{\frac{1}{2}(-4+k)} \rightarrow \frac{\int e^{-s v}\, v^{-1+\frac{4-k}{2}}\, dv}{\Gamma[\frac{4-k}{2}]}$

● Apply to: $\mathrm{Tr}[g[t\, D_A^2]] \rightarrow \sum_{\{k\geq 0\}} [t^{\frac{1}{2}(-4+k)} \int s^{\frac{1}{2}(-4+k)}\, h[s]\, ds\, a_k[D_A^2]]$

$\rightarrow \{2, 1, 2, 1\} \rightarrow s^{\frac{1}{2}(-4+k)}\, h[s]$

$\rightarrow \{2, 1, 2, 1\} \rightarrow \frac{\int e^{-s v}\, v^{-1+\frac{4-k}{2}}\, h[s]\, dv}{\Gamma[\frac{4-k}{2}]}$

$\rightarrow \mathrm{Tr}[g[t\, D_A^2]] \rightarrow \sum_{\{k\geq 0\}} [\frac{t^{\frac{1}{2}(-4+k)} \int\int e^{-s v}\, v^{-1+\frac{4-k}{2}}\, h[s]\, dv\, ds\, a_k[D_A^2]}{\Gamma[\frac{4-k}{2}]}]$

$\rightarrow \mathrm{Tr}[g[t\, D_A^2]] \rightarrow \sum_{\{k\geq 0\}} [\frac{t^{\frac{1}{2}(-4+k)} \int\int e^{-s v}\, v^{-1+\frac{4-k}{2}}\, h[s]\, dv\, ds\, a_k[D_A^2]}{\Gamma[\frac{4-k}{2}]}]$

● Apply: $\int\int e^{-s v}\, h[s]\, ds\, dv \rightarrow \int g[v]\, dv$

$\rightarrow \int\int e^{-s v}\, h[s]\, A\_\, dv\, ds \rightarrow \int A\, g[v]\, dv$

$\rightarrow \boxed{\mathrm{Tr}[g[t\, D_A^2]] \rightarrow \sum_{\{k\geq 0\}} [\frac{t^{\frac{1}{2}(-4+k)} \int v^{-1+\frac{4-k}{2}}\, g[v]\, dv\, a_k[D_A^2]}{\Gamma[\frac{4-k}{2}]}]}$

Substitute: $\{g[u\_] \rightarrow f[\sqrt{u}],\ v \rightarrow u^2\}$

$\rightarrow \boxed{\mathrm{Tr}[f[\sqrt{t}\ D_A]] \rightarrow \sum_{\{k\geq 0\}} [\frac{t^{-2+\frac{k}{2}} \int 2\, u^{3-k}\, f[u]\, du\, a_k[D_A^2]}{\Gamma[2-\frac{k}{2}]}]}$

Substitute: $t \rightarrow \frac{1}{\Lambda^2}$

$\rightarrow \boxed{\mathrm{Tr}[f[\frac{D_A}{\Lambda}]] \rightarrow \sum_{\{k\geq 0\}} [\frac{\Lambda^{4-k} \int 2\, u^{3-k}\, f[u]\, du\, a_k[D_A^2]}{\Gamma[2-\frac{k}{2}]}]}$

```
PR["●Proposition 3.5. For canonical triple ", {C"∞"[M], L²[M, S], slash[iD]},
  Yield,
  $p35 = $ = {Tr[f[slash[iD] / Λ]] → xIntegral[ℒM[T[g, "dd", {μ, ν}]] √Det[g], x⁴],
     ℒM[T[g, "dd", {μ, ν}]] → f₄ Λ⁴ / (2 π²) - f₂ Λ²
         / (24 π²) + f[0] / (16 π²) (△[s] / 30 -
          T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] / 20 + 11 / 360 R*.R*)};
  ColumnBar[$],

  line,
  NL, CO["Sketch proof: with ",
   $sdim = $s0 = {m → dim[M], dim[M] → 4, Tr"E"x[1N] → dim[S], dim[S] → 2^(m/2)}],
  NL, "■Evaluate terms in Theorem.3.4. ", $t34s = $t34 /. 𝒟𝒜 → slash[iD],

  next, "For ", $0 = $ = tuExtractPattern[a₀[_]][$t34s[[1, 2]]] // First,
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {M} → {x, x ∈ M} /. g → g[x] /.
    x ∈ M → x,
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[1]]],
  Yield, $a0 = $0 -> $ /. $t32[[3 ;; -1]] //. $sdim // tuIntegralSimplify;
  Framed[$a0],

  next, "For ", $0 = $ = tuExtractPattern[a₂[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = F → -s / 4 1N,
  Yield, $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /.
     {{M} → {x, x ∈ M}, g → g[x]} /. x ∈ M → x,
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[2]]] //. $sF,
  Yield, $ = ($ // tuArgSimplify[Tr"E"x, {s}]) /. s → s[x],
  Yield, $a2 = $0 -> $ /. $t32[[3 ;; -1]] //. $sdim // tuIntegralSimplify;
  Framed[$a2],

  next, "For ", $0 = $ = tuExtractPattern[a₄[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = {s → s . 1N, F → -s / 4 1N, Ω"E" → Ω"S"},
  Yield, $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /.
     {{M} → {x, x ∈ M}, g → g[x]} /. x ∈ M → x,
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[3]]] /. $sF, "xPOFF",
  Yield, $ = $ // tuDotSimplify[{s}] // (# //. $sgeneral[[-2 ;; -1]] &),
  Yield, $ = $ // tuArgSimplify[△, {1N}] // tuArgSimplify[Tr"E"x, {s, △[s]}],
  Yield, $ = $ /. s → s[x] // tuArgSimplify[Tr"E"x, {s, △[s]}] //
     tuIntegralSimplify // (# //. $sdim &),
  "PONdd", Framed[$a4b = $0 -> $ //. $sdim]
 ];
```

●Proposition 3.5. For canonical triple $\{C^\infty[M], L^2[M, S], \not{D}\}$

$$\to \left|\begin{array}{l} \mathrm{Tr}[f[\frac{\not{D}}{\Lambda}]] \to \int \sqrt{\mathrm{Det}[g]}\ \mathcal{L}_M[g_{\mu\nu}]\, dx^4 \\ \mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} + \frac{f[0]\,(\frac{11 R^*.R^*}{360} - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{\Delta[s]}{30})}{16\pi^2} \end{array}\right.$$

Sketch proof: with $\{m \to \dim[M], \dim[M] \to 4, \mathrm{Tr}_{E_x}[1_N] \to \dim[S], \dim[S] \to 2^{m/2}\}$

■Evaluate terms in Theorem.3.4.

$$\{\mathrm{Tr}[f[\frac{\not{D}}{\Lambda}]] \to 2\,(\frac{\Lambda^4 f_4\, a_0[(\not{D})^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2\, a_2[(\not{D})^2]}{\Gamma[1]}) + f[0]\, a_4[(\not{D})^2],\ f_{j\_} \to \int v^{-1+j} f[v]\, dv\}$$

◆For $a_0[(\not{D})^2]$

$$\to \int \sqrt{\mathrm{Det}[g[x]]}\ a_0[x, (\not{D})^2]\, dx$$

$\rightarrow \int 2^{-n}\,\pi^{-n/2}\,\sqrt{\mathrm{Det}[g[x]]}\;\mathrm{Tr}_{E_x}[1_N]\,dx$

$\rightarrow \boxed{a_0[(D\!\!\!/)^2]\rightarrow \dfrac{\int\sqrt{\mathrm{Det}[g[x]]}\,dx}{4\,\pi^2}}$

◆For $a_2[(D\!\!\!/)^2]$ using $F\rightarrow -\dfrac{s\,1_N}{4}$

$\rightarrow \int\sqrt{\mathrm{Det}[g[x]]}\;a_2[x,\,(D\!\!\!/)^2]\,dx$

$\rightarrow \int 2^{-n}\,\pi^{-n/2}\,\sqrt{\mathrm{Det}[g[x]]}\;\mathrm{Tr}_{E_x}[-\dfrac{s\,1_N}{12}]\,dx$

$\rightarrow \int -\dfrac{1}{3}\,2^{-2-n}\,\pi^{-n/2}\,\sqrt{\mathrm{Det}[g[x]]}\;s[x]\,\mathrm{Tr}_{E_x}[1_N]\,dx$

$\rightarrow \boxed{a_2[(D\!\!\!/)^2]\rightarrow -\dfrac{\int\sqrt{\mathrm{Det}[g[x]]}\;s[x]\,dx}{48\,\pi^2}}$

◆For $a_4[(D\!\!\!/)^2]$ using $\{s\rightarrow s.1_N,\;F\rightarrow -\dfrac{s\,1_N}{4},\;\Omega^E\rightarrow\Omega^S\}$

$\rightarrow \int\sqrt{\mathrm{Det}[g[x]]}\;a_4[x,\,(D\!\!\!/)^2]\,dx$

$\rightarrow$

$$\int \frac{1}{45}\,2^{-3-n}\,\pi^{-n/2}\,\sqrt{\mathrm{Det}[g[x]]}\;\mathrm{Tr}_{E_x}[180\,(-\frac{s\,1_N}{4}).(-\frac{s\,1_N}{4}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu} + 60\,s.1_N.(-\frac{s\,1_N}{4}) + 5\,s.1_N.s.1_N - 12\,\triangle[s.1_N] - 60\,\triangle[-\frac{s\,1_N}{4}]]\,dx\text{xPOFF}$$

$$\rightarrow \int \frac{1}{45}\,2^{-3-n}\,\pi^{-n/2}\,\sqrt{\mathrm{Det}[g[x]]}\;\mathrm{Tr}_{E_x}[-2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu} + \frac{5\,s^2\,1_N}{4} - 60\,\triangle[-\frac{s\,1_N}{4}] - 12\,\triangle[s\,1_N]]\,dx$$

$$\rightarrow \int \frac{1}{45}\,2^{-3-n}\,\pi^{-n/2}\,\sqrt{\mathrm{Det}[g[x]]}\;(-2\,\mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\,\mathrm{Tr}_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] + \frac{5}{4}\,s^2\,\mathrm{Tr}_{E_x}[1_N] + 3\,\triangle[s]\,\mathrm{Tr}_{E_x}[1_N])\,dx$$

$$\rightarrow \frac{1}{45}\,2^{-3-n}\,\pi^{-n/2}\int\sqrt{\mathrm{Det}[g[x]]}\;(5\,s[x]^2 + 12\,\triangle[s[x]] - 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\,\mathrm{Tr}_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}])\,dx\text{PONdd}$$

$$\boxed{a_4[(D\!\!\!/)^2]\rightarrow \frac{1}{45}\,2^{-3-n}\,\pi^{-n/2}\int\sqrt{\mathrm{Det}[g[x]]}\;(5\,s[x]^2 + 12\,\triangle[s[x]] - 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\,\mathrm{Tr}_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}])\,dx}$$

```
PR["Using (3.14): ", $s = e314 =
    T[Ω"S", "dd", {μ, ν}] → 1 / 4 T[R, "dddd", {μ, ν, ρ, σ}] T[γ, "u", {ρ}].T[γ, "u", {σ}],
  yield, $s314 = {e314, e314 /. ρ → ρ1 /. σ → σ1 // tuIndicesRaise[{μ, ν}]} //
    tuAddPatternVariable[{μ, ν}], accumDef[$s314];
  NL, "Evaluate: ", $ = $a4b // tuExtractPattern[
      T[Ω"S", "dd", {μ, ν}].T[Ω"S", "uu", {μ, ν}]] // First;
  $t0 = $ = Tr[$],
  Yield, $ = $ /. $s314 // tuDotSimplify[{Tensor[R, __]}],
  NL, "Tr[] scalars: ", $s = {Tensor[R, _, _]},
  Yield, $ = $ // tuTrSimplify[$s],
  Yield, $ = $ /. subTraceGamma0,
  Yield, $ = $ // Expand // ContractUpDn[g],
  NL, "Use: ", $s = {T[R, "ddud", {μ_, ν_, ρ_, ρ_}] → 0, T[R, "dduu", {μ, ν, ρ1_, σ1_}] :>
     -T[R, "dduu", {μ, ν, σ1, ρ1}] /; OrderedQ[{σ1, ρ1}]},
  Yield, $t0 = $t0 -> $ /. $s /. Tr -> Tr"E"x; Framed[$t0], accumDef[$t0];
  Imply, $ = $a4b /. $t0; Framed[$],
  (**)
  NL, "Remaining Dot[] are scalars: ",
  Yield, $ = $ /. dd : HoldPattern[Dot[_]] → 1N dd /.
     tuOpSimplify[Tr"E"x, {HoldPattern[Dot[_]]}] //. $sdim,
  Yield, $ = UpDownIndexSwap[{ρ1, σ1}][$] /. ρ1 → ρ /. σ1 → σ /.
      tt : T[R, "dddd", {_, _, _, _}] ⧴ tuTensorAntiSymmetricOrdered[tt, {3, 4}] /. Dot →
      Times // Simplify;
  Framed[$a4c = $], CG[" (3.16)"],
  (**)
  NL, "■Convert expression in terms of: ",
  NL, "•Weyl tensor: ", T[C, "dddd", {μ, ν, ρ, σ}],
  Yield,
  $ = T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] -> T[R, "dddd", {μ, ν, ρ, σ}]
      T[R, "uuuu", {μ, ν, ρ, σ}] - 2 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] + s[x]^2 / 3,
  NL, "•Pontryagin class ",
  $1 = R*.R* → s[x]^2 - 4 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] +
     T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
  NL, "The ",
  $2 = $a4c // tuExtractIntegrand;
  $2a0 = $2 // tuExtractPositionPattern[Plus[_, __]];
  $2a = integrandTerm → $2a0[[1, 2]],
  $ = {$, $1, $2a}; $ // ColumnBar,
  Imply,
  $ = tuEliminate[$, {T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
     T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}]}], CK,
  Yield, $ = tuRuleSolve[$, integrandTerm],
  Yield, $2a0[[1, 2]] = $[[1, 2]]; $2a0,
  Yield, $2 = tuReplacePart[$2, $2a0],
  Yield, $a4d = $ = tuReplacePart[$a4c, {$2}]; Framed[$], CG[" QED"]
 ];
```

Using (3.14): $\Omega^S{}_{\mu\nu} \to \frac{1}{4}\gamma^\rho.\gamma^\sigma R_{\mu\nu\rho\sigma} \longrightarrow \{\Omega^S{}_{\mu\_\nu\_} \to \frac{1}{4}\gamma^\rho.\gamma^\sigma R_{\mu\nu\rho\sigma},\ \Omega^{S\mu\_\nu\_} \to \frac{1}{4}\gamma^{\rho 1}.\gamma^{\sigma 1} R^{\mu\nu}{}_{\rho 1\sigma 1}\}$

Evaluate: $\mathrm{Tr}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}]$

$\to \mathrm{Tr}[\frac{1}{16}\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}]$

Tr[] scalars: {Tensor[R, _, _]}

$\to \frac{1}{16} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1} \mathrm{Tr}[\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1}]$

$\to \frac{1}{4}(g^{\rho\sigma} g^{\rho 1\sigma 1} + g^{\rho\sigma 1} g^{\sigma\rho 1} - g^{\rho\rho 1} g^{\sigma\sigma 1}) R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}$

$\to -\frac{1}{4} R_{\mu\nu}{}^{\rho 1\sigma 1} R^{\mu\nu}{}_{\rho 1\sigma 1} + \frac{1}{4} R_{\mu\nu}{}^{\sigma 1\rho 1} R^{\mu\nu}{}_{\rho 1\sigma 1} + \frac{1}{4} R_{\mu\nu}{}^{\sigma}{}_{\sigma} R^{\mu\nu\sigma 1}{}_{\sigma 1}$

Use: $\{R_{\mu\_\nu\_}{}^{\rho\_}{}_{\rho\_} \to 0,\ R_{\mu\nu}{}^{\rho 1\_\sigma 1\_} :\to -T[R, dduu, \{\mu, \nu, \sigma 1, \rho 1\}] /; \mathrm{OrderedQ}[\{\sigma 1, \rho 1\}]\}$

$\to \boxed{\mathrm{Tr}_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] \to -\frac{1}{2} R_{\mu\nu}{}^{\rho 1\sigma 1} R^{\mu\nu}{}_{\rho 1\sigma 1}}$

⇒

$$\boxed{a_4[(\not{D})^2] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (5\, s[x]^2 - 15 R_{\mu\nu}{}^{\rho 1\sigma 1} R^{\mu\nu}{}_{\rho 1\sigma 1} + 12 \triangle[s[x]] - 2\, \mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\, \mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}])\, dx}$$

Remaining Dot[] are scalars:

$\to a_4[(\not{D})^2] \to$
$\frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (-8 R_{\mu\nu}.R^{\mu\nu} + 8 R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 5\, s[x]^2 - 15 R_{\mu\nu}{}^{\rho 1\sigma 1} R^{\mu\nu}{}_{\rho 1\sigma 1} + 12 \triangle[s[x]])\, dx$

$$\to \boxed{a_4[(\not{D})^2] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (5\, s[x]^2 - 8 R_{\mu\nu} R^{\mu\nu} - 7 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 12 \triangle[s[x]])\, dx}$$

(3.16)

■Convert expression in terms of:

•Weyl tensor: $C_{\mu\nu\rho\sigma}$

$\to C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \to \frac{s[x]^2}{3} - 2 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

•Pontryagin class $R^*.R^* \to s[x]^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

The integrandTerm $\to 5\, s[x]^2 - 8 R_{\mu\nu} R^{\mu\nu} - 7 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]$

$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \to \frac{s[x]^2}{3} - 2 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$
$R^*.R^* \to s[x]^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$
integrandTerm $\to 5\, s[x]^2 - 8 R_{\mu\nu} R^{\mu\nu} - 7 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]$

⇒ integrandTerm $+ 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - 12 \triangle[s[x]] == 11 R^*.R^* \longleftarrow$ CHECK

$\to$ {integrandTerm $\to 11 R^*.R^* - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]$}

$\to \{\{2, 2\} \to 11 R^*.R^* - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]\}$

$\to \{2, 4, 1\} \to \sqrt{\mathrm{Det}[g[x]]}\ (11 R^*.R^* - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]])$

$$\to \boxed{a_4[(\not{D})^2] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (11 R^*.R^* - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]])\, dx}$$ QED

```
PR["•NOTE: In 4-dim compact orientable manifold M without boundary ",
  Yield,
  {IntegralOp[{{M}}, R*.R* νg] → 8 π^2 χ[M], χ[M] → "Euler Characteristic"} // Column,
  imply, "Topological term",
  yield, "Constant",
  yield, "Ignore",
  NL, "With no boundaries the ", △[s[x]], " term does not contribute."
 ];
```

•NOTE: In 4-dim compact orientable manifold M without boundary

→ $\int_{\{M\}} [R^*.R^* \nu_g] \to 8\pi^2 \chi[M]$, $\chi[M] \to$ Euler Characteristic ⇒ Topological term ⟶ Constant ⟶ Ignore

With no boundaries the △[s[x]] term does not contribute.

```
PR["To derive Proposition 3.5.
• Insert a's into ", $ = $t34s; $ // ColumnSumExp,
  NL, "Using: ",
  $s = {R*.R* → 0, △[s[x]] → 0, tt : Tensor[C, _, _] → tt[x], n → 4, Γ → Gamma},
  Yield, $t34s1 = $ = $[[1]] /. {$a0, $a2, $a4d} /. $s //. tuIntegralGather // Simplify;
  $ // ColumnSumExp,
  NL, "•Comparing with (3.19). The relevant term in integrand: ",
  $ = $t34s1 // tuExtractIntegrand // Last // (# /. √_ → 1 &);
  $ // ColumnSumExp,
  Yield, $LM = ℒM[T[g, "dd", {μ, ν}]] -> $ // Expand, CG[" Agrees with (3.19)."]
 ];
```

To derive Proposition 3.5.

• Insert a's into $\{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \to \sum\left[\begin{array}{l} 2\left(\frac{\Lambda^4 f_4 a_0[(D\!\!\!/)^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2 a_2[(D\!\!\!/)^2]}{\Gamma[1]}\right) \\ f[0]\, a_4[(D\!\!\!/)^2] \end{array}\right],\ f_{j\_} \to \int v^{\sum\left|\begin{smallmatrix}-1\\ j\end{smallmatrix}\right]} f[v]\, dv\}$

Using: {R*.R* → 0, △[s[x]] → 0, tt : Tensor[C, _, _] → tt[x], n → 4, Γ → Gamma}

→ $\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \to \int \frac{\sum\left[\begin{array}{l} -40\,\Lambda^2 s[x]\, f_2 \\ 480\,\Lambda^4 f_4 \\ -3 f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x] \end{array}\right] \sqrt{\mathrm{Det}[g[x]]}}{960\,\pi^2}\, dx$

•Comparing with (3.19). The relevant term in integrand:

$$\frac{\sum\left[\begin{array}{l} -40\,\Lambda^2 s[x]\, f_2 \\ 480\,\Lambda^4 f_4 \\ -3 f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x] \end{array}\right]}{960\,\pi^2}$$

→ $\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 s[x]\, f_2}{24\,\pi^2} + \frac{\Lambda^4 f_4}{2\,\pi^2} - \frac{f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x]}{320\,\pi^2}$ Agrees with (3.19).

p.35

```
PR["●Proposition 3.7. The spectral action of the fluctuated Dirac operator is ",
  $p37 = $ = {Tr[f[𝒟A / Λ]] → xIntegral[√Det[g[x]] ℒ[T[g, "dd", {μ, ν}], Bμ, Φ], x ∈ M],
     ℒ[T[g, "dd", {μ, ν}], Bμ, Φ] →
      N ℒM[T[g, "dd", {μ, ν}]] + ℒB[Bμ] + ℒH[T[g, "dd", {μ, ν}], Bμ, Φ],
     $LM,
     N → dim[ℋF],
     ℒB[Bμ] → f[0] / (24 π^2) Tr[T[F, "dd", {μ, ν}] T[F, "uu", {μ, ν}]],
     ℒB[Bμ] → "Kinetic term gauge fields",
     ℒH[T[g, "dd", {μ, ν}], Bμ, Φ] → -2 f2 Λ^2 / (4 π^2) Tr[Φ.Φ] +
       f[0] / (8 π^2) Tr[Φ.Φ.Φ.Φ] + f[0] / (24 π^2) △[Tr[Φ.Φ]] + f[0] / (48 π^2) s[x] Tr[Φ.Φ] +
       f[0] / (8 π^2) Tr[T[iD, "d", {μ}][Φ].T[iD, "u", {μ}][Φ]],
     ℒH[T[g, "dd", {μ, ν}], Bμ, Φ] → "Higgs lagrangian",
     N → Tr[1ℋF]
    }; FramedColumn[$]
 ];
```

●Proposition 3.7. The spectral action of the fluctuated Dirac operator is

$$\mathrm{Tr}[f[\tfrac{\mathcal{D}_A}{\Lambda}]] \to \int_{x\in M} \sqrt{\mathrm{Det}[g[x]]}\ \mathcal{L}[g_{\mu\nu}, B_\mu, \Phi]$$

$$\mathcal{L}[g_{\mu\nu}, B_\mu, \Phi] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + N\,\mathcal{L}_M[g_{\mu\nu}]$$

$$\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2\, s[x]\, f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} - \frac{f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x]}{320\pi^2}$$

$$N \to \dim[\mathcal{H}_F]$$

$$\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$$

$\mathcal{L}_B[B_\mu] \to$ Kinetic term gauge fields

$$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2\, \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\,\mathrm{Tr}[D_\mu[\Phi].D^\mu[\Phi]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$$

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$ Higgs lagrangian

$$N \to \mathrm{Tr}[1_{\mathcal{H}_F}]$$

```
PR["●Proof: Starting with the formulas from Theorem 3.3 ", $ = $t33[[1 ;; 3]];
  $ // ColumnBar,
  NL, "let ",
  $s = {F → Q, H → 𝒟A}, ". Using explicit tensor notation. ", H → S × ℋF,
  Yield,
  $t33a = {{$ /. $s, $31[[-1]]} /. (tt : Tr_)[1N] :→ tt[1N ⊗ 1ℋF] /. s 1N → s /. s ⊗ 1ℋF → s /.
        s → (s 1N ⊗ 1ℋF) /. 1Nx → 1N ⊗ 1ℋ
         1N → "Identity for spinor field"} // Flatten; FramedColumn[$t33a]
 ];
```

●Proof: Starting with the formulas from Theorem 3.3

$$a_0[x, H] \to 2^{-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N]$$

$$a_2[x, H] \to 2^{-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[F + \tfrac{s\,1_N}{6}]$$

$$a_4[x, H] \to \tfrac{1}{45} 2^{-3-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[180\,F.F + 60\,s.F + 5$$

let $\{F \to Q, H \to \mathcal{D}_A\}$. Using explicit tensor notation. $H \to S \times \mathcal{H}_F$

→

$$a_0[x, \mathcal{D}_A] \to 2^{-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_2[x, \mathcal{D}_A] \to 2^{-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[Q + \tfrac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_4[x, \mathcal{D}_A] \to \tfrac{1}{45} 2^{-3-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[180\,Q.Q + 60\,(s\,1_N\otimes 1_{\mathcal{H}_F}).Q + 5\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s\,1_N\otimes 1_{\mathcal{H}_F}]]$$

$$Q \to -i\,\gamma^\mu.\gamma_5 \otimes D_\mu.\Phi + \tfrac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \tfrac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$$

```
PR["●Compute the aₙ terms of ", $t34[[1, 1]], (*
  " relative to ",$p35[[1,1]],*)
  NL, "for ", $s04 = Join[$sdim, {Tr[1_N] → dim[S], n → dim[M]}],
  Yield, $t33a // FramedColumn
 ];
```

●Compute the a_n terms of $\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

for $\{m \to \dim[M], \dim[M] \to 4, \mathrm{Tr}_{E_x}[1_N] \to \dim[S], \dim[S] \to 2^{m/2}, \mathrm{Tr}[1_N] \to \dim[S], n \to \dim[M]\}$

→

$$a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_2[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[Q + \frac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45} 2^{-3-n}\pi^{-n/2}\,\mathrm{Tr}_{E_x}[180\, Q.Q + 60\,(s\, 1_N \otimes 1_{\mathcal{H}_F}).Q + 5\,(s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[Q] - 12\, \triangle[s\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

$$Q \to -i\,\gamma^\mu.\gamma_5 \otimes D_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$$

```
PR[next, "For ", $ = $t33a[[1]],

  next, "For ", $ = $t33a[[1]] /. Tr_ → Tr /. $t32[[3]] /. $sdim,
  Yield,
  $ = $ /. tuOpDistribute[Tr, CircleTimes] //. tuOpSimplify[CircleTimes, {Tr[_]}],
  " ", "Recall ", $s = $t33[[1]] //. Join[{H → slash[iD], Tr_ → Tr}, $s04[[{2, -1}]]],
  Imply, $a0a = tuRuleEliminate[{Tr[1N]}][{$s, $}] // First; Framed[$a0a],

  next, "For ", $ = $t33a[[2]] /. Tr_ → Tr /. $t32[[3]] /. $sdim,
  Yield,
  $ = $ /. tuRuleSelect[$t33a][Q] //. tuOpDistribute[Tr] // tuArgSimplify[Tr, {s}] //
     tuOpDistributeF[Tr, CircleTimes] // tuOpSimplifyF[CircleTimes, {Tr[_]}],

  NL, "• ", T[F, "dd", {μ, ν}], " is anti-symmetric and ", $symmetries[[-1]], yield,
  $s = {Tr[T[γ, "u", {μ}].T[γ, "u", {ν}]] Tr[T[F, "dd", {μ, ν}]] → 0,
    Tr[T[γ, "u", {μ}].T[γ, "d", {5}]] → 0};
  $s // ColumnBar,
  Imply, $ = $ /. $s,
  NL, "Recall ",
  $s = $t33[[2]] //. Join[{H → slash[iD], Tr_ → Tr, $sF[[2]]}, $s04[[{2, -1}]]] //
    tuArgSimplify[Tr, {s}],
  Imply, $a2a = $ /. tuRuleSolve[$s, {s Tr[1N]}] // Expand; Framed[$a2a]
 ];
```

◆For $a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$

◆For $a_0[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{\mathrm{Tr}[1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

→ $a_0[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{\mathrm{Tr}[1_N]\,\mathrm{Tr}[1_{\mathcal{H}_F}]}{16\,\pi^2}$ Recall $a_0[x, \not{D}] \to \dfrac{\mathrm{Tr}[1_N]}{16\,\pi^2}$

⇒ $\boxed{a_0[x, \mathcal{D}_{\mathcal{A}}] \to \mathrm{Tr}[1_{\mathcal{H}_F}]\,a_0[x, \not{D}]}$

◆For $a_2[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{\mathrm{Tr}[Q + \frac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

→ $a_2[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{-i\,\mathrm{Tr}[\gamma^\mu.\gamma_5]\,\mathrm{Tr}[D_\mu.\Phi] - \mathrm{Tr}[\Phi.\Phi]\,\mathrm{Tr}[1_N] - \frac{1}{12} s\,\mathrm{Tr}[1_N]\,\mathrm{Tr}[1_{\mathcal{H}_F}] + \frac{1}{2} i\,\mathrm{Tr}[\gamma^\mu.\gamma^\nu]\,\mathrm{Tr}[F_{\mu\nu}]}{16\,\pi^2}$

• $F_{\mu\nu}$ is anti-symmetric and tt : $\gamma^\mu.\gamma_5$:→ Reverse[tt] ⟶ $\left| \begin{array}{l} \mathrm{Tr}[\gamma^\mu.\gamma^\nu]\,\mathrm{Tr}[F_{\mu\nu}] \to 0 \\ \mathrm{Tr}[\gamma^\mu.\gamma_5] \to 0 \end{array} \right.$

⇒ $a_2[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{-\mathrm{Tr}[\Phi.\Phi]\,\mathrm{Tr}[1_N] - \frac{1}{12} s\,\mathrm{Tr}[1_N]\,\mathrm{Tr}[1_{\mathcal{H}_F}]}{16\,\pi^2}$

Recall $a_2[x, \not{D}] \to -\dfrac{s\,\mathrm{Tr}[1_N]}{192\,\pi^2}$

⇒ $\boxed{a_2[x, \mathcal{D}_{\mathcal{A}}] \to -\dfrac{\mathrm{Tr}[\Phi.\Phi]\,\mathrm{Tr}[1_N]}{16\,\pi^2} + \mathrm{Tr}[1_{\mathcal{H}_F}]\,a_2[x, \not{D}]}$

```
PR["■For: ", $0 = $ = $t33a[[3]] /. Tr_ → Tr /. $t32[[3]] /. $sdim;
 Framed[$],
 NL, "Add product space explicitly: ",
 $s = {tt : Tensor[R, _, _].Tensor[R, _, _] → tt 1N ⊗ 1HF},
 Yield, $ = $ /. $s,
 NL, "Let scalars: ", $scal = {s, △[s], Tensor[R, _, _]},
 $sQ = {Map[#.(# /. {μ → μ1, ν → ν1}) &, $t33a[[4]]], $t33a[[4]]};
```

```
  next, "Use: ", $s = Join[($sQ), {$s34}, $s314]; FramedColumn[$s],
  Yield, $ = $ //. $s; ColumnSumExp[$],
  Yield, $ = $ // tuDotSimplify[]; ColumnSumExp[$];
  Yield, $ = $ // tuArgSimplify[△] // tudExpand[△, {1_, Tensor[γ, _, _]}] // expandDC[];
  next, "Combine product of operator product: ", $s = {};
  $ = $ //. tuOpSimplify[Dot, {s}] //. tuOpSimplify[CircleTimes] //.
     {(a_ ⊗ b_) .( c_ ⊗ d_) → a.c ⊗ b.d,
      1n_ . a_ → a, a_ . 1n_ → a} // Expand; $ // ColumnSumExp;
  next, "Apply Tr[] over each space: ", $s = {Tr[a_ ⊗ b_] → Tr[a] Tr[b]},
  Yield,
  $ = $ //. tuOpDistribute[Tr] //
    tuArgSimplify[Tr, {s, △[s], Tensor[R, _, _].Tensor[R, _, _]}];
  $ = $ /. $s;
  next, "Reduce Tr[γ's]: ",
  $ = $ //. tuTrGamma // Expand;

  line,
  next, "Evaluate g,F terms using symmetry: ",
  $symg = {T[g, "uu", {μ_, ν_}] :→ T[g, "uu", {ν, μ}] /; OrderedQ[{ν, μ}],
    T[F, "dd", {μ_, ν_}] :→ -T[F, "dd", {ν, μ}] /; OrderedQ[{ν, μ}],
    T[F, "uu", {μ_, ν_}] :→ -T[F, "uu", {ν, μ}] /; OrderedQ[{ν, μ}],
    tt : T[g, "uu", {a_, b_}] A_ :→ 0 /; ! FreeQ[tt, T[F, "dd", {a, b}]]},
  {$s, $} = $ // tuTermApply[{g, F, μ, ν, μ1, ν1}, {}, $symg,
     {tuIndexContractUpDn[g, {μ1, ν1}] /@ # &, tuOpSimplifyF[Dot], tuTrSimplify[]}, 1];
  {$s, $} = $ // tuTermApply[{g, F}, {}, $symg, {tuOpSimplifyF[Dot], tuTrSimplify[]}, 1];
  $ // ColumnSumExp
 ]

 PR["Use ",
  $s = {dim[N] → 4, tt : T[γ, "u", {μ_}].T[γ, "d", {5}] . T[γ, "u", {μ1_}].T[γ, "d", {5}] →
     -T[γ, "u", {μ}].T[γ, "u", {μ1}]},
  Yield, $ = $ /. $s // tuArgSimplify[Tr] // (# /. tuTrGamma &) // Simplify;
  $ // ColumnSumExp,

  next, "Gather △'s and contracting indices: ",
  $s = △[Tr[Φ.Φ]];
  $s = $s → ($s // tudExpand[△] // (# /. tuTrExpand &));
  $s = tuRuleSolve[$s, $s[[2, 1]]];
  {$s, $} = $ // tuTermApply[{△, Φ}, {}, {$s}, {}, 1];
  New, $s,
  Yield, $ = $ // Expand;
  {$s, $} =
   $ // tuTermApply[{T[g, "uu", {_, _}]}, {}, {}, {tuIndexContractUpDn[g, {μ1}]}, 1];
  New, $s,
  Yield, $pass = $ = $ // Simplify; $ // ColumnSumExp,

  NL, "• Comparing to text(p.37)", CR[" there are 2 differences, but evaluate ",
   $oEE = tuTermSelect[Tr[Tensor[Ω"E", _, _].Tensor[Ω"E", _, _]]][$pass // Expand] //
      First // Numerator],
  Yield, $ = 360 (4 π)^2 # & /@ $;
  $p37a4 = $ = Collect[$, dim[_], Simplify];
  $ // ColumnFormOn[Plus] // Framed
 ]
```

For: $a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\mathrm{Tr}[180\,Q.Q + 60\,(s\,1_N \otimes 1_{\mathcal{H}_F}).Q + 5\,(s\,1_N \otimes 1_{\mathcal{H}_F}).(s\,1_N \otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s\,1_N \otimes 1_{\mathcal{H}_F}]]$

Add product space explicitly: {tt : Tensor[R, _, _].Tensor[R, _, _] → tt $1_N \otimes 1_{\mathcal{H}_F}$}

→ $a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}$ Tr[180 Q.Q + 60 (s $1_N\otimes 1_{\mathcal{H}_F}$).Q + 5 (s $1_N\otimes 1_{\mathcal{H}_F}$).(s $1_N\otimes 1_{\mathcal{H}_F}$) − 2 × $1_N\otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu}.R^{\mu\nu}$ + 2 × $1_N\otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}$ + 30 $\Omega^E_{\mu\nu}.\Omega^{E\mu\nu}$ − 60 △[Q] − 12 △[s $1_N\otimes 1_{\mathcal{H}_F}$]]

Let scalars: {s, △[s], Tensor[R, _, _]}

◆Use:

$$Q.Q \to (-i\,\gamma^\mu.\gamma_5\otimes D_\mu.\Phi + \tfrac{1}{2} i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \tfrac{1}{4} s\,1_N\otimes 1_{\mathcal{H}_F}).(-i\,\gamma^{\mu1}.\gamma_5\otimes D_{\mu1}.\Phi + \tfrac{1}{2} i\,\gamma^{\mu1}.\gamma^{\nu1}\otimes F_{\mu1\,\nu1} - 1_N\otimes\Phi.\Phi - \tfrac{1}{4} s\,1_N\otimes 1_{\mathcal{H}_F})$$

$$Q \to -i\,\gamma^\mu.\gamma_5\otimes D_\mu.\Phi + \tfrac{1}{2} i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \tfrac{1}{4} s\,1_N\otimes 1_{\mathcal{H}_F}$$

$$\Omega^E[\mu,\nu] \to 1_N\otimes(i\,F_{\mu\nu}) + (-\underset{-\nu}{\nabla^S}[\underset{-\mu}{\nabla^S}[\_]] + \underset{-\mu}{\nabla^S}[\underset{-\nu}{\nabla^S}[\_]])\otimes 1_{\mathcal{H}_F}$$

$$\Omega^S{}_{\mu\_\,\nu\_} \to \tfrac{1}{4}\gamma^\rho.\gamma^\sigma R_{\mu\nu\rho\sigma}$$

$$\Omega^{S\mu\_\,\nu\_} \to \tfrac{1}{4}\gamma^{\rho1}.\gamma^{\sigma1} R^{\mu\nu}{}_{\rho1\,\sigma1}$$

→ $a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}$ Tr[∑[
5 (s $1_N\otimes 1_{\mathcal{H}_F}$).(s $1_N\otimes 1_{\mathcal{H}_F}$)
60 (s $1_N\otimes 1_{\mathcal{H}_F}$).($-i\,\gamma^\mu.\gamma_5\otimes D_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \frac{1}{4} s\,1_N\otimes 1_{\mathcal{H}_F}$)
180 ($-i\,\gamma^\mu.\gamma_5\otimes D_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \frac{1}{4} s\,1_N\otimes 1_{\mathcal{H}_F}$).($-i\,\gamma^{\mu1}.\gamma_5\otimes D_{\mu1}.\Phi + \frac{1}{2} i\,\gamma^{\mu1}.\gamma^{\nu1}\otimes F_{\mu1\,\nu1} - 1_N\otimes\Phi.\Phi - \frac{1}{4} s\,1_N\otimes 1_{\mathcal{H}_F}$)
−2 $1_N\otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu}.R^{\mu\nu}$
2 × $1_N\otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}$
30 $\Omega^E_{\mu\nu}.\Omega^{E\mu\nu}$
−12 △[s $1_N\otimes 1_{\mathcal{H}_F}$]
−60 △[$-i\,\gamma^\mu.\gamma_5\otimes D_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \frac{1}{4} s\,1_N\otimes 1_{\mathcal{H}_F}$]
]]

→

→

◆Combine product of operator product:

◆Apply Tr[] over each space: {Tr[a_⊗b_] → Tr[a] Tr[b]}

→

◆Reduce Tr[γ's]:

◆Evaluate g,F terms using symmetry:

{$g^{\mu\_\,\nu\_}$:→ T[g, uu, {ν, μ}] /; OrderedQ[{ν, μ}], $F_{\mu\_\,\nu\_}$:→ −T[F, dd, {ν, μ}] /; OrderedQ[{ν, μ}], $F^{\mu\_\,\nu\_}$:→ −T[F, uu, {ν, μ}] /; OrderedQ[{ν, μ}], tt : A_ $g^{a\_\,b\_}$:→ 0 /; ! FreeQ[tt, T[F, dd, {a, b}]]}

$a_4[x, \mathcal{D}_{\mathcal{A}}] \to$ ∑[
$\frac{s^2\,\dim[N]\,\dim[\mathcal{H}_F]}{4608\,\pi^2}$
$-\frac{\dim[N]\,\dim[\mathcal{H}_F]\,R_{\mu\nu}.R^{\mu\nu}}{2880\,\pi^2}$
$\frac{\dim[N]\,\dim[\mathcal{H}_F]\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}}{2880\,\pi^2}$
$\frac{s\,\dim[N]\,\mathrm{Tr}[\Phi.\Phi]}{192\,\pi^2}$
$\frac{\dim[N]\,\mathrm{Tr}[\Phi.\triangle[\Phi]]}{96\,\pi^2}$
$\frac{\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}]}{16\,\pi^2}$
$\frac{\mathrm{Tr}[\Omega^E_{\mu\nu}.\Omega^{E\mu\nu}]}{192\,\pi^2}$
$\frac{\dim[N]\,\mathrm{Tr}[\triangle[\Phi].\Phi]}{96\,\pi^2}$
$\frac{\dim[N]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{32\,\pi^2}$
$-\frac{\mathrm{Tr}[\gamma^\mu.\gamma_5.\gamma^{\mu1}.\gamma_5]\,\mathrm{Tr}[D_\mu.\Phi.D_{\mu1}.\Phi]}{32\,\pi^2}$
$\frac{\dim[N]\,\dim[\mathcal{H}_F]\,\triangle[s]}{1920\,\pi^2}$
]

Use {dim[N] → 4, tt : γ^{μ}–.γ_5.$\gamma^{\mu1}$–.γ_5 → –γ^{μ}.$\gamma^{\mu1}$}

$$\to a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{\sum\left[\begin{array}{l} 30\,(4\,s\,\mathrm{Tr}[\Phi.\Phi] + 8\,\mathrm{Tr}[\Phi.\triangle[\Phi]] + 12\,\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] + \\ \quad \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] + 8\,\mathrm{Tr}[\triangle[\Phi].\Phi] + 24\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] + 24\,g^{\mu\mu1}\,\mathrm{Tr}[D_{\mu}.\Phi.D_{\mu1}.\Phi]) \\ \dim[\mathcal{H}_F]\,(5\,s^2 - 8\,R_{\mu\nu}.R^{\mu\nu} + 8\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 12\,\triangle[s]) \end{array}\right]}{5760\,\pi^2}$$

◆Gather △'s and contracting indices:

- $\frac{\mathrm{Tr}[\Phi.\triangle[\Phi]]}{24\,\pi^2} + \frac{\mathrm{Tr}[\triangle[\Phi].\Phi]}{24\,\pi^2} \to \frac{\mathrm{Tr}[\triangle[\Phi].\Phi]}{24\,\pi^2} + \frac{-\mathrm{Tr}[\triangle[\Phi].\Phi] + \triangle[\mathrm{Tr}[\Phi.\Phi]]}{24\,\pi^2}$

→

- $\frac{g^{\mu\mu1}\,\mathrm{Tr}[D_{\mu}.\Phi.D_{\mu1}.\Phi]}{8\,\pi^2} \to \frac{\mathrm{Tr}[D_{\mu}.\Phi.D^{\mu}.\Phi]}{8\,\pi^2}$

$$\to a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\sum\left[\begin{array}{l} \dim[\mathcal{H}_F]\,(5\,s^2 - 8\,R_{\mu\nu}.R^{\mu\nu} + 8\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 12\,\triangle[s]) \\ 30\,(4\,s\,\mathrm{Tr}[\Phi.\Phi] + 12\,\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] + \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] + 24\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] + 24\,\mathrm{Tr}[D_{\mu}.\Phi.\ldots \end{array}\right.$$

• Comparing to text(p.37) there are 2 differences, but evaluate Tr[$\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}$]

$$\to \boxed{5760\,\pi^2\,a_4[x, \mathcal{D}_{\mathcal{A}}] \to \left|\begin{array}{l} \dim[\mathcal{H}_F]\left|\begin{array}{l} 5\,s^2 \\ -8\,R_{\mu\nu}.R^{\mu\nu} \\ 8\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} \\ 12\,\triangle[s] \end{array}\right. \\ 30\left|\begin{array}{l} 4\,s\,\mathrm{Tr}[\Phi.\Phi] \\ 12\,\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] \\ \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] \\ 24\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] \\ 24\,\mathrm{Tr}[D_{\mu}.\Phi.D^{\mu}.\Phi] \\ 8\,\triangle[\mathrm{Tr}[\Phi.\Phi]] \end{array}\right. \end{array}\right.}$$

```
PR["Compute: ", $0 = $ = $oEE,
 Yield, $ = $ //. selectDef[{Tensor[Ω"E", _, _]}, {Ω"S"}, all] // expandDC[] // Expand;
 $ = $ //. tuOpDistribute[Tr] // tuCircleTimesGather[] // (# //. { a_ . 1n_ | 1n_ . a_ → a} &);
 $ // ColumnSumExp;
 Yield,
 $ = $ // tuArgSimplify[Tr] // tuOpDistributeF[Tr, CircleTimes] // tuIndexDummyOrdered //
    Simplify;
 $ // ColumnSumExp,
 NL, "Apply: ", $s = (selectDef[Tr"E"x[_]]
       /. Trx_ → Tr /. tt : Tensor[R, a_, b_] Tensor[R, a1_, b1_] :→ Apply[Dot, tt] //
     tuIndexSwapUpDown[{ρ1, σ1}] // tuIndexDummyOrdered),
 Yield, $ = $ //. $s,
 $ // ColumnSumExp,
 Yield, $s = $ = $0 → ($ /. CircleTimes → Times), CK,
 NL, "Using ",
 $s1 = {Tr[1N] → 4, Tr[1n_] → dim[n], ρ1 → ρ, σ1 → σ, Tr[Tensor[F, _, _]] → 0},
 Yield, $s = $s //. $s1,
 NL, "● The above: ", $ = $p37a4; $[[1]],
 Yield, $ = $ /. $s;
 Yield, $ = $ // Simplify // ColumnFormOn[Plus] // Framed,
 NL, "Which is the expression on p.37."
]
```

Compute: $\mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}]$

→

→ $\sum\left[\begin{array}{|l}\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}]\otimes\mathrm{Tr}[1_{\mathcal{H}_F}]\\ -(\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}])\\ 2\,i\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}]\otimes\mathrm{Tr}[F^{\mu\nu}]\end{array}\right]$

Apply: $\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}]\to-\frac{1}{2}R_{\mu\nu\rho1\sigma1}.R^{\mu\nu\rho1\sigma1}$

→ $(-\frac{1}{2}R_{\mu\nu\rho1\sigma1}.R^{\mu\nu\rho1\sigma1})\otimes\mathrm{Tr}[1_{\mathcal{H}_F}]-\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}]+2\,i\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}]\otimes\mathrm{Tr}[F^{\mu\nu}]$

$\sum\left[\begin{array}{|l}(-\frac{1}{2}R_{\mu\nu\rho1\sigma1}.R^{\mu\nu\rho1\sigma1})\otimes\mathrm{Tr}[1_{\mathcal{H}_F}]\\ -(\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}])\\ 2\,i\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}]\otimes\mathrm{Tr}[F^{\mu\nu}]\end{array}\right]$

→ $\mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}]\to-\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}]\,\mathrm{Tr}[1_N]-\frac{1}{2}R_{\mu\nu\rho1\sigma1}.R^{\mu\nu\rho1\sigma1}\,\mathrm{Tr}[1_{\mathcal{H}_F}]+2\,i\,\mathrm{Tr}[F^{\mu\nu}]\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}]$ ⟵CHECK

Using $\{\mathrm{Tr}[1_N]\to4,\ \mathrm{Tr}[1_{n\_}]\to\mathrm{dim}[n],\ \rho1\to\rho,\ \sigma1\to\sigma,\ \mathrm{Tr}[\mathrm{Tensor}[F,\_,\_]]\to0\}$

→ $\mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}]\to-\frac{1}{2}\mathrm{dim}[\mathcal{H}_F]\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}-4\,\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}]$

● The above: $5760\,\pi^2\,a_4[x,\mathcal{D}_{\mathcal{A}}]$

→

→ $\boxed{5760\,\pi^2\,a_4[x,\mathcal{D}_{\mathcal{A}}]\to\left|\begin{array}{l}\mathrm{dim}[\mathcal{H}_F]\left|\begin{array}{l}5\,s^2\\ -8\,R_{\mu\nu}.R^{\mu\nu}\\ -7\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}\\ 12\,\triangle[s]\end{array}\right.\\ 120\left|\begin{array}{l}s\,\mathrm{Tr}[\Phi.\Phi]\\ 2\left|\begin{array}{l}\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}]\\ 3\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]\\ 3\,\mathrm{Tr}[D_\mu.\Phi.D^\mu.\Phi]\\ \triangle[\mathrm{Tr}[\Phi.\Phi]]\end{array}\right.\end{array}\right.\end{array}\right.}$

Which is the expression on p.37.

Aside: Compute Q.Q

```
PR["•Evaluate: ", $ = $sQ[[1]],
  Yield, $ = $ // tuDotSimplify[],
  NL, CO["Is there a Logical order to the operations? "],
  $sX = { (a_ ⊗ b_ ) .( c_ ⊗ d_ ) → a.c ⊗ b.d,
    1n_ . a_ → a, a_ . 1n_ → a};
  Yield, $ = $ // tuRepeat[{$sX, tuOpSimplify[Dot, {s}]}] , $ // ColumnSumExp;
  Yield, $ = Tr[#] & /@ $ // tuTrSimplify[{s}]; $ // ColumnSumExp;
  Yield, $ = $ //. tuOpDistribute[Tr, CircleTimes] /. Tr[a_] ⊗ Tr[b_] → Tr[a] Tr[b];
  $ // ColumnSumExp;
  NL, "Use: ",
  $s = {dim[N] → 4, tt : T[γ, "u", {μ_}].T[γ, "d", {5}] . T[γ, "u", {μ1_}].T[γ, "d", {5}] →
     -T[γ, "u", {μ}].T[γ, "u", {μ1}]},
  Yield, $ = $ /. $s //. tuTrGamma // tuTrSimplify[]; $ // ColumnSumExp;
  NL, "Apply symmetries ",
  $ss = tt : T[g, "uu", {a_, b_}] A_ ⧴ 0 /; ! FreeQ[tt, T[F, "dd", {a, b}]],
  Yield, $ = $ /. $ss //. tuTrGamma // Expand;
  Yield, $ = $ // tuIndexContractUpDn[g, {ν1, μ1}]; $ // ColumnSumExp;
  NL, "Apply: ", $s = {aa : Tensor[g, _, _] A_ :> tuIndexContractUpDn[g, {ν1, μ1, ν}][aa],
    μ1 | ν1 → ν, tt : T[F, "du", {a_, b_}].Tensor[F, _, _] ⧴ tuIndexSwapUpDown[μ][tt],
    T[F, "ud", {a_, a_}] → 0},
  Yield, $sQQ = $ = $ //. $s /. $symmetries //. tuOpSimplify[Dot] // tuTrSimplify[];
  $ // ColumnSumExp // Framed
 ];
```

•Evaluate: $Q.Q \to (-i\,\gamma^{\mu}.\gamma_5 \otimes D_{\mu}.\Phi + \frac{1}{2} i\,\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}).$
$(-i\,\gamma^{\mu 1}.\gamma_5 \otimes D_{\mu 1}.\Phi + \frac{1}{2} i\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F})$

$\to$ $Q.Q \to -(\gamma^{\mu}.\gamma_5 \otimes D_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes D_{\mu 1}.\Phi) + \frac{1}{2} (\gamma^{\mu}.\gamma_5 \otimes D_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) +$
$i\,(\gamma^{\mu}.\gamma_5 \otimes D_{\mu}.\Phi).(1_N \otimes \Phi.\Phi) + \frac{1}{4} i\,(\gamma^{\mu}.\gamma_5 \otimes D_{\mu}.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + \frac{1}{2} (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma_5 \otimes D_{\mu 1}.\Phi) -$
$\frac{1}{4} (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) - \frac{1}{2} i\,(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(1_N \otimes \Phi.\Phi) -$
$\frac{1}{8} i\,(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + i\,(1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes D_{\mu 1}.\Phi) - \frac{1}{2} i\,(1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) +$
$(1_N \otimes \Phi.\Phi).(1_N \otimes \Phi.\Phi) + \frac{1}{4} (1_N \otimes \Phi.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + \frac{1}{4} i\,(s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5 \otimes D_{\mu 1}.\Phi) -$
$\frac{1}{8} i\,(s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + \frac{1}{4} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(1_N \otimes \Phi.\Phi) + \frac{1}{16} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F})$

Is there a Logical order to the operations?

$\to$ $Q.Q \to \frac{1}{4} i\, s\, \gamma^{\mu}.\gamma_5 \otimes D_{\mu}.\Phi + i\,\gamma^{\mu}.\gamma_5 \otimes D_{\mu}.\Phi.\Phi.\Phi - \frac{1}{2} i\,\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}.\Phi.\Phi - \frac{1}{8} i\, s\, \gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} +$
$\frac{1}{4} i\, s\, \gamma^{\mu 1}.\gamma_5 \otimes D_{\mu 1}.\Phi + i\,\gamma^{\mu 1}.\gamma_5 \otimes \Phi.\Phi.D_{\mu 1}.\Phi - \frac{1}{2} i\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes \Phi.\Phi.F_{\mu 1\,\nu 1} - \frac{1}{8} i\, s\, \gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} -$
$\gamma^{\mu}.\gamma_5.\gamma^{\mu 1}.\gamma_5 \otimes D_{\mu}.\Phi.D_{\mu 1}.\Phi + \frac{1}{2} \gamma^{\mu}.\gamma_5.\gamma^{\mu 1}.\gamma^{\nu 1} \otimes D_{\mu}.\Phi.F_{\mu 1\,\nu 1} + \frac{1}{2} \gamma^{\mu}.\gamma^{\nu}.\gamma^{\mu 1}.\gamma_5 \otimes F_{\mu\nu}.D_{\mu 1}.\Phi -$
$\frac{1}{4} \gamma^{\mu}.\gamma^{\nu}.\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu\nu}.F_{\mu 1\,\nu 1} + \frac{1}{2} s\, 1_N \otimes \Phi.\Phi + 1_N \otimes \Phi.\Phi.\Phi.\Phi + \frac{1}{16} s^2\, 1_N \otimes 1_{\mathcal{H}_F}$

$\to$

$\to$

Use: {dim[N] → 4, tt : γ^μ_.γ5.γ^μ1_.γ5 → -γ^μ.γ^μ1}

$\to$

Apply symmetries tt : A_ g^a_ ^b_ :→ 0 /; ! FreeQ[tt, T[F, dd, {a, b}]]

$\to$

$\to$

Apply: {aa : A_Tensor[g, _, _] :→ tuIndexContractUpDn[g, {ν1, μ1, ν}][aa],
μ1 | ν1 → ν, tt : F_a_ ^b_.Tensor[F, _, _] :→ tuIndexSwapUpDown[μ][tt], F^a_ _a_ → 0}

$\to$ $\boxed{Tr[Q.Q] \to \sum\left[\begin{array}{l} \frac{1}{16} s^2\, dim[N]\, dim[\mathcal{H}_F] \\ \frac{1}{2} s\, dim[N]\, Tr[\Phi.\Phi] \\ 2\, Tr[F^{\mu\nu}.F_{\mu\nu}] \\ dim[N]\, Tr[\Phi.\Phi.\Phi.\Phi] \\ 4\, Tr[D_{\mu}.\Phi.D^{\mu}.\Phi] \end{array}\right]}$

■ 4. Electrodynamics (p.38)

● 4.1 A two-point space

```
PR["● Take the Two point space. ",
  {X -> {x, y}, C[X] → ℂ^2, C[CG["complex functions"]]},
  NL, "•Construct an even finite space ",
  {F_X → {C[X], ℋ_F, T[γ, "u", {v_}]_F, γ_F}, dim[ℋ_F] ≥ 2, γ_F[CG["ℤ^2-grading"]] },
  Yield, γ_F ⇒ {ℋ_F → ℋ_F^+ ⊕ ℋ_F^- → ℂ ⊕ ℂ, ℋ_F^"±" → {ψ ∈ ℋ_F | γ_F.ψ → ±ψ}},
  imply, $ = γ_F → {{1, 0}, {0, -1}}; MatrixForms[$],
  NL, "• Since ", $sD0 = {CommutatorM[γ_F, a] → 0,
    CommutatorP[iD_F, γ_F] → 0, iD_F[CG["offDiagonal"]], iD_F → {{0, du}, {dl, 0}}},
  Imply, {a.ψ → Inactive[Dot][{{a₊, 0}, {0, a₋}}, {{ψ₊}, {ψ₋}}], a ∈ 𝒜_F, ψ ∈ ℋ_F} //
   MatrixForms,

  Imply, $sFX = F_X → {{𝒜_F, ℋ_F, 𝒟_F, γ_F} → {ℂ^2, ℂ^2, {{0, t}, {t̄, 0}}, {{1, 0}, {0, -1}}}, t ∈ ℂ};
  $sFX // MatrixForms,
  line,
  NL, "■ Prop.4.1. Only a real structure ", $ = J_F ⇒ {iD_F → 0}, " exists on F_X.",
  line,
  NL, "Proof: Determine iD_F for even KO dimensions by requiring: ",
  $def = selectDef[{CommutatorM[_, rghtA[b]], rghtA[b]}, {}, all] // DeleteDuplicates;
  $c = $ = Join[$J[[2]], $def]; ColumnBar[$],

  NL, "■ KOdim→0: ", $sj = {J_F → {{j₊, 0}, {0, j₋}}.cc, j_"±" ∈ U[1]};
  $sj // MatrixForms,
  NL, "for ", $sa = ab : a | b -> {{ab₊, 0}, {0, ab₋}}; MatrixForms[$sa],
  NL, "•Compute ", $0 = $ = tuRuleSelect[$c][{rghtA[b]}] // First,
  yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];
  yield, $[[2]] = $[[2]] // tuRepeat[$cc, tuConjugateTransposeSimplify[]];
  MatrixForms[$],
  yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
  MatrixForms[$sb = $] // Framed, yield, b,
  Yield, "This is diagonal hence satisfies 0-order condition: ",
  tuRuleSelect[$c][CommutatorM[_, _]][[1]] // Framed,

  NL, "• The 1-order condition ",
  $ = tuRuleSelect[$c][{CommutatorM[CommutatorM[_, _], rghtA[b]]}] // First, "POFF",
  $sa = ab : a | xb -> {{ab₊, 0}, {0, ab₋}};
  Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
  yield, $ = $ //. tuCommutatorExpand // expandDC[];
  yield, $ = $ /. tuRule[$sD0][[-1]] // Simplify;
  MatrixForms[$],
  $x = tuExtractPattern[du _][$][[1]] / du; "PONdd",
  yield, $ = $x.(# / $x) & /@ $ /. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
  imply, Framed[iD_F → 0]
 ];
```

● Take the Two point space. $\{X \to \{x, y\}, C[X] \to \mathbb{C}^2, C[\text{complex functions}]\}$

•Construct an even finite space $\{F_X \to \{C[X], \mathcal{H}_F, \gamma^{v}{}_{-F}, \gamma_F\}, \dim[\mathcal{H}_F] \geq 2, \gamma_F[\mathbb{Z}^2\text{-grading}]\}$

→ $\gamma_F \Rightarrow \{\mathcal{H}_F \to (\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^- \to \mathbb{C}\oplus\mathbb{C}, \mathcal{H}_F^{\pm} \to \{\psi \in \mathcal{H}_F \mid \gamma_F.\psi \to \pm\psi\}\}$ ⇒ $\gamma_F \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

• Since $\{[\gamma_F, a]_- \to 0, \{\mathcal{D}_F, \gamma_F\}_+ \to 0, \mathcal{D}_F[\text{offDiagonal}], \mathcal{D}_F \to \{\{0, du\}, \{dl, 0\}\}\}$

⇒ $\{a.\psi \to \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}.\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, a \in \mathcal{A}_F, \psi \in \mathcal{H}_F\}$

⇒ $F_X \to \{\{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F\} \to \{\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}, t \in \mathbb{C}\}$

■ Prop.4.1. Only a real structure $J_F \Rightarrow \{\mathcal{D}_F \to 0\}$ exists on F_X.

Proof: Determine iD_F for even KO dimensions by requiring:
$$\left| \begin{array}{l} J_F.J_F \to \varepsilon \\ J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F \\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F \\ [a, b^o]_- \to 0 \\ [[\mathcal{D}_F, a]_-, b^o]_- \to 0 \\ b^o \to J_F.b^\dagger.(J_F)^\dagger \end{array} \right.$$

■ KOdim→0: $\{J_F \to \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix}.cc, j_\pm \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

•Compute $b^o \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow \;\longrightarrow b^o \to \begin{pmatrix} (j_+)^* b_+ j_+ & 0 \\ 0 & (j_-)^* b_- j_- \end{pmatrix} \longrightarrow \boxed{b^o \to \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}} \longrightarrow b$

→ This is diagonal hence satisfies 0-order condition: $\boxed{[a, b^o]_- \to 0}$

• The 1-order condition $[[\mathcal{D}_F, a]_-, b^o]_- \to 0$

.... $\longrightarrow ((a_- - a_+)(b_- - b_+)).\mathcal{D}_F \to 0$ ⇒ $\boxed{\mathcal{D}_F \to 0}$

```
PR[
  "■ KOdim→2: ", $sj = {J_F → {{0, j}, {-j, 0}}.cc, j ∈ U[1]};
  $sj // MatrixForms,
  NL, "for ", $sa = ab : a | b -> {{ab+, 0}, {0, ab-}}; MatrixForms[$sa],
  NL, "Compute ", $0 = $ = tuRuleSelect[$c][rghtA[b]] // DeleteDuplicates // First,
  yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];
  yield, $[[2]] = $[[2]] // tuRepeat[$cc, tuConjugateTransposeSimplify[]];
  MatrixForms[$sb = $],
  yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
  MatrixForms[$sb = $],
  Yield, "This is diagonal hence satisfies 0-order condition: ",
  tuRuleSelect[$c][CommutatorM[_, _]][[1]] // Framed, (**)

  NL, "• The 1-order condition ",
  $ = tuRuleSelect[$c][{CommutatorM[CommutatorM[_, _], rghtA[b]]}] // First, "xPOFF",
  $sa = ab : a | xb -> {{ab+, 0}, {0, ab-}};
  Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
  Yield, $ = $ //. tuCommutatorExpand //
    tuRepeat[Join[tuOpSimplify[Dot], {tuOpDistribute[Dot]}]];
  Yield, $ = $ /. $sD0[[-1]] // Simplify;
  MatrixForms[$],
  $x = tuExtractPattern[du _][$][[1]] / du;
  yield, $ = $x.(# / $x) & /@ $ //. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
  imply, Framed[iD_F → 0]
 ];
PR["■ KOdim→4:",
  NL, "■ KOdim→6:"
 ];
```

■ KOdim→2: $\{J_F \to \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}.cc,\ j \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

Compute $b^o \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow\ \longrightarrow b^o \to \begin{pmatrix} j\, j^*\, b_- & 0 \\ 0 & j\, j^*\, b_+ \end{pmatrix} \longrightarrow b^o \to \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}$

→ This is diagonal hence satisfies 0-order condition: $\boxed{[a, b^o]_- \to 0}$

• The 1-order condition $[[\mathcal{D}_F, a]_-, b^o]_- \to 0$xPOFF

→ $[[\mathcal{D}_F, \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}]_-, \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}]_- \to 0$

→

→ $\begin{pmatrix} 0 & -du\,(a_- - a_+)\,(b_- - b_+) \\ -dl\,(a_- - a_+)\,(b_- - b_+) & 0 \end{pmatrix} \to 0$

$\longrightarrow -((a_- - a_+)\,(b_- - b_+)).\mathcal{D}_F \to 0 \Rightarrow \boxed{\mathcal{D}_F \to 0}$

■ KOdim→4:
■ KOdim→6:

4.1.2 The product space

```
PR["The product space  ",
 $ = {M × Fx → {𝒜 -> C"∞"[M, ℂ²], ℋ -> L²[M, S] ⊗ ℂ², 𝒟 -> slash[iD] ⊗ 1F, γ -> γ5 ⊗ γF, J -> JM ⊗ JF},
   M[CG["4-dim Riemann spin manifold"]],
   Fx[CG["two-point space"]],
   C∞[M, ℂ²] -> C∞[M] ⊕ C∞[M],
   ℋ -> L²[M, S] ⊕ L²[M, S],
   {(a ⊕ b).(ψ ⊕ ϕ) -> (a.ψ ⊕ b.ϕ), a ⊕ b ∈ C∞[M] ⊕ C∞[M], ψ ⊕ ϕ ∈ ℋ}
  }; $ // ColumnForms,
 accumDef[$]
]
```

```
The product space   M × Fx → | 𝒜 → C∞[M, ℂ²]
                             | ℋ → L²[M, S] ⊗ ℂ²
                             | 𝒟 → (D̸) ⊗ 1F
                             | γ → γ5 ⊗ γF
                             | J → JM ⊗ JF
                    M[4-dim Riemann spin manifold]
                    Fx[two-point space]
                    C∞[M, ℂ²] → C∞[M] ⊕ C∞[M]
                    ℋ → L²[M, S] ⊕ L²[M, S]
                    | (a⊕b).(ψ⊕ϕ) → a.ψ⊕b.ϕ
                    | a⊕b ∈ C∞[M] ⊕ C∞[M]
                    | ψ⊕ϕ ∈ ℋ
```

Distance

```
PR["1•  Restrict distance formula to F_X: ",
 Yield, $0 = {d_iD_F[x, y] → sup[‖a[x] - a[y]‖], a ∈ 𝒜_F, Abs[Det[CommutatorM[iD_F, a]]] ≤ 1},
 NL, "Using: ", $s = $sFX;
 $s = Thread[$s[[2, 1]]]; $s // MatrixForms,
 NL, "Define algebra for the two points {x,y}: ", $s1 = a → {{a[x], 0}, {0, a[y]}};
 $s1 // MatrixForms,

 NL, "• Determine influence of: ", $ = Abs[Det[CommutatorM[iD_F, a]]] ≤ 1,
 Imply, $ = ($ /. $s1 /. $s /. CommutatorM → MCommutator // Simplify),
 Yield, $ = $ /. t̄ → Conjugate[t] /. Abs[t^2 a_] → Abs[t^2] Abs[a],
 Yield, $ = # / Abs[t^2] & /@ $,
 NL, "A real structure J_F (Prop.4.1)",
 imply, iD_F → 0, imply, t → 0, imply, $0[[1, 1]] → ∞,

 line,
 NL, "2•  For the case with points: ", {{p, x}, {p, y}, p ∈ M},
 NL, "Let ", $sa2 = {{a[n] → a_x[p], a_x_[p_] → a[p, x], a_x[CG[C∞[M]]]},
   {d_slash[iD]⊗1_F[n_, m_] → sup[‖a[n] - a[m]‖],
    a ∈ 𝒜, Abs[Det[CommutatorM[slash[iD], a]]] ≤ 1, n_ | m_ ∈ N}
  }; $sa2 // ColumnForms,
 Yield, $ = tuRuleSelect[$sa2][d_[_, _]] // First,
 NL, "Define ", $s =
  tuRuleSelect[$sa2][a[n]] /. {x → x[n], p → p[n]} // tuAddPatternVariable[n] // First,
 Yield, $1 = $ = $ /. $s,
 NL, "• For ", $s = {x[m] | x[n] → g, CG["i.e. the same F-space points "]},
 Yield, $ = $ /. tuRule[$s],
 NL, "This can be identified with normal distance in M.",

 line,
 NL, "• For different F-space points the requirement: ",
 $1 = $ = Select[Flatten[$sa2], MatchQ[#, Abs[_] ≤ 1] &][[1]],
 NL, "implies different requirements depending on
   the definition of the algebra and Dirac operator.",
 next, "For: ", $s = {slash[iD] → iD_M ⊗ iD_F, a → a_M ⊗ a_F},
 Yield, $ = $ /. $s,
 NL, "• If the space is a disjoint product ",
 Yield, $s = Abs[Det[CommutatorM[a_ ⊗ b_, c_ ⊗ d_]]] →
   Abs[Det[CommutatorM[a, c]]] Abs[Det[CommutatorM[b, d]]],
 Yield, $ = $ /. $s, CR[
  " which is the same as the previous example so the distance is ∞. "],

 NL, "• If there is cross talk between the spaces ",
 $s = {slash[iD] → iD_M ⊗ iD_F, CG["only"]},
 Yield, $ = $1 /. tuRule[$s],
 NL, "Let ", $s = Abs[Det[CommutatorM[a_ ⊗ b_, c_]]] →
   Abs[Det[CommutatorM[a, c]]] Abs[Det[CommutatorM[b, c]]],
 Yield, $ /. $s, CO[" possible finite distance."]
]
```

1• Restrict distance formula to F_X:
→ $\{d_{D_F}[x, y] \to \sup[\|a[x] - a[y]\|], a \in \mathcal{A}_F, \text{Abs}[\text{Det}[[D_F, a]_-]] \le 1\}$

Using: $\{\mathcal{A}_F \to \mathbb{C}^2, \mathcal{H}_F \to \mathbb{C}^2, D_F \to \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \gamma_F \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$

Define algebra for the two points {x,y}: $a \to \begin{pmatrix} a[x] & 0 \\ 0 & a[y] \end{pmatrix}$

• Determine influence of: $\text{Abs}[\text{Det}[[D_F, a]_-]] \le 1$
⇒ $\text{Abs}[\text{Det}[-\{\{a[x], 0\}, \{0, a[y]\}\}.D_F + D_F.\{\{a[x], 0\}, \{0, a[y]\}\}]] \le 1$
→ $\text{Abs}[\text{Det}[-\{\{a[x], 0\}, \{0, a[y]\}\}.D_F + D_F.\{\{a[x], 0\}, \{0, a[y]\}\}]] \le 1$
→ $\dfrac{\text{Abs}[\text{Det}[-\{\{a[x], 0\}, \{0, a[y]\}\}.D_F + D_F.\{\{a[x], 0\}, \{0, a[y]\}\}]]}{\text{Abs}[t]^2} \le \dfrac{1}{\text{Abs}[t]^2}$

A real structure J_F (Prop.4.1) ⇒ $D_F \to 0$ ⇒ $t \to 0$ ⇒ $d_{D_F}[x, y] \to \infty$

2• For the case with points: $\{\{p, x\}, \{p, y\}, p \in M\}$

Let
$$\left\| \begin{array}{l} a[n] \to a_x[p] \\ a_{x}\_[p\_] \to a[p, x] \\ a_x[C^\infty[M]] \\ d_{(D)\otimes 1_F}[n\_, m\_] \to \sup[\|-a[m] + a[n]\|] \\ a \in \mathcal{A} \\ \text{Abs}[\text{Det}[[D\!\!\!/, a]_-]] \le 1 \\ n\_ \mid m\_ \in N \end{array} \right.$$

→ $d_{(D)\otimes 1_F}[n\_, m\_] \to \sup[\|-a[m] + a[n]\|]$
Define $a[n\_] \to a_{x[n]}[p[n]]$
→ $d_{(D)\otimes 1_F}[n\_, m\_] \to \sup[\|-a_{x[m]}[p[m]] + a_{x[n]}[p[n]]\|]$
• For $\{x[m] \mid x[n] \to g$, i.e. the same F-space points $\}$
→ $d_{(D)\otimes 1_F}[n\_, m\_] \to \sup[\|-a_g[p[m]] + a_g[p[n]]\|]$
This can be identified with normal distance in M.

• For different F-space points the requirement: $\text{Abs}[\text{Det}[[D\!\!\!/, a]_-]] \le 1$
implies different requirements depending
on the definition of the algebra and Dirac operator.
◆For: $\{D\!\!\!/ \to D_M \otimes D_F, a \to a_M \otimes a_F\}$
→ $\text{Abs}[\text{Det}[[D_M \otimes D_F, a_M \otimes a_F]_-]] \le 1$
• If the space is a disjoint product
→ $\text{Abs}[\text{Det}[[a\_\otimes b\_, c\_\otimes d\_]_-]] \to \text{Abs}[\text{Det}[[a, c]_-]]\,\text{Abs}[\text{Det}[[b, d]_-]]$
→ $\text{Abs}[\text{Det}[[D_F, a_F]_-]]\,\text{Abs}[\text{Det}[[D_M, a_M]_-]] \le 1$
which is the same as the previous example so the distance is ∞.
• If there is cross talk between the spaces $\{D\!\!\!/ \to D_M \otimes D_F$, only$\}$
→ $\text{Abs}[\text{Det}[[D_M \otimes D_F, a]_-]] \le 1$
Let $\text{Abs}[\text{Det}[[a\_\otimes b\_, c\_]_-]] \to \text{Abs}[\text{Det}[[a, c]_-]]\,\text{Abs}[\text{Det}[[b, c]_-]]$
→ $\text{Abs}[\text{Det}[[D_F, a]_-]]\,\text{Abs}[\text{Det}[[D_M, a]_-]] \le 1$ possible finite distance.

4.1.3 U[1] gauge theory

```
PR["U[1] gauge theory for: ", tuRuleSelect[$defall][M×FX] // First,
 NL, "gauge group: ", 𝒢[𝒜] → Mod[U[𝒜], U[$sAt[[1]]]], U[𝒜] ≠ U[$sAt[[1]]],
 NL, "where ",
 {$t219[[1, -2]], U[𝒜] ≠ U[$sAt[[1]]][CG["non-trivial"]], $sAt} // ColumnBar,
 NL, "non-triviality", imply, "KOdim[JF]" → {2, 6},
 ", i.e., off diagonal.

Only KOdim→6 for Standard Model so used in this case. ",
 Imply, "Can use Def.2.17 for action functional ",
 $d217 = $ = {S → Sb + Sf, Sb → Tr[f[𝒟𝒜 / Λ]], Sf → 1 / 2 BraKet[J.ξ̃, 𝒟𝒜.ξ̃],
    ξ̃ ∈ ℋcl⁺, ℋcl⁺ → {ξ̃ | ξ ∈ ℋ⁺}, ξ̃[CG["GrassmannVariable"]]};
 $ // ColumnBar,
 NL, "•Consider ", $Fx = FX → {ℂ², ℂ², 0, γF → {{1, 0}, {0, -1}}, JF → {{0, C}, {C, 0}}};
 MatrixForms[$Fx]
]
```

U[1] gauge theory for:

$M\times F_X\to\{\mathcal{A}\to C^\infty[M,\mathbb{C}^2],\ \mathcal{H}\to L^2[M,S]\otimes\mathbb{C}^2,\ \mathcal{D}\to(\not{D})\otimes 1_F,\ \gamma\to\gamma_5\otimes\gamma_F,\ J\to J_M\otimes J_F\}$

gauge group: $\mathcal{G}[\mathcal{A}]\to \mathrm{Mod}[U[\mathcal{A}],U[\tilde{\mathcal{A}}_J]]U[\mathcal{A}]\neq U[\tilde{\mathcal{A}}_J]$

where
$$\left|\begin{array}{l}(2.11)\Rightarrow\mathcal{G}[M\times F]\to\{U\to u.J.u.J^\dagger,\ u\in U[\mathcal{A}]\}\\ U[\mathcal{A}]\neq U[\tilde{\mathcal{A}}_J][\text{non-trivial}]\\ \tilde{\mathcal{A}}_J\to\{a\in\mathcal{A},\ a.J\to J.a^\dagger,\ a^o\to a\}\end{array}\right.$$

non-triviality ⇒ $\mathrm{KOdim}[J_F]\to\{2,6\}$, i.e., off diagonal.

Only KOdim→6 for Standard Model so used in this case.

⇒ Can use Def.2.17 for action functional
$$\left|\begin{array}{l}S\to S_b+S_f\\ S_b\to\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]\\ S_f\to\frac{1}{2}\langle J.\tilde{\xi}\mid\mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle\\ \tilde{\xi}\in(\mathcal{H}_{cl})^+\\ (\mathcal{H}_{cl})^+\to\{\tilde{\xi}\mid\xi\in\mathcal{H}^+\}\\ \tilde{\xi}[\text{GrassmannVariable}]\end{array}\right.$$

•Consider $F_X\to\{\mathbb{C}^2,\mathbb{C}^2,0,\gamma_F\to\begin{pmatrix}1&0\\0&-1\end{pmatrix},J_F\to\begin{pmatrix}0&C\\C&0\end{pmatrix}\}$

```
PR["Prop.4.2. The gauge group of ", {𝒢[𝒜F] → U[1], 𝒜F[CG["2-point space"]]},
  line,
  NL, "Proof: Note: ", U[𝒜F] → U[1] × U[1],
  NL, "The subspace: ", $sAt // ColumnForms,
  yield, $ = ForAll[a,
    a ∈ ℂ^2 && a ∈ ($sAtj = ($sAt[[1]] /. J → F)_JF), (JF.ConjugateTranspose[a].JF → a)],
  NL, "Compute ", $0 = $ = tuExtractPattern[Rule[__]][$][[1]],
  yield, $ = $ /. $Fx[[2, -2 ;; -1]]; MatrixForms[$],
  NL, "The 2-point algebra ", $sCC = $s = {a -> DiagonalMatrix[{a1, a2}],
     C.a_ :> Conjugate[a].C /; FreeQ[a, C], Conjugate[C] → C, C.C → 1};
  $s // MatrixForms,
  Yield, $ = $ /. Dot → xDot /. $s // OrderedxDotMultiplyAll[];
  MatrixForms[$],
  yield, $ = $ // tuRepeat[$s, ConjugateCTSimplify1[{}]];
  MatrixForms[$] // Framed,
  imply, a1 → a2, imply, a ∝ "identity",
  imply, $pass4 = $ = $sAtj ≃ ℂ,
  imply, (U[$[[1]]] → U[1]) ⊂ U[𝒜F], CG[" QED"]
 ];
```

Prop.4.2. The gauge group of $\{\mathcal{G}[\mathcal{A}_F] \to U[1], \mathcal{A}_F[\text{2-point space}]\}$

Proof: Note: $U[\mathcal{A}_F] \to U[1] \times U[1]$

The subspace: $\tilde{\mathcal{A}}_J \to \left| \begin{array}{l} a \in \mathcal{A} \\ a.J \to J.a^\dagger \\ a^o \to a \end{array} \right. \longrightarrow \forall_{a, a\in\mathbb{C}^2 \&\& a\in\tilde{\mathcal{A}}_{F_{J_F}}} (J_F.a^\dagger.J_F \to a)$

Compute $J_F.a^\dagger.J_F \to a \longrightarrow \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}.a^\dagger.\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \to a$

The 2-point algebra $\{a \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix}, C.(a\_) :\to a^*.C /; \text{FreeQ}[a, C], C^* \to C, C.C \to 1\}$

$\to \begin{pmatrix} C.a2^*.C & 0 \\ 0 & C.a1^*.C \end{pmatrix} \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix} a2 & 0 \\ 0 & a1 \end{pmatrix} \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix}}$

$\Rightarrow a1 \to a2 \Rightarrow a \propto \text{identity} \Rightarrow \tilde{\mathcal{A}}_{F_{J_F}} \simeq \mathbb{C} \Rightarrow (U[\tilde{\mathcal{A}}_{F_{J_F}}] \to U[1]) \subset U[\mathcal{A}_F]$ QED

```
PR["■Determine Bμ of Prop.3.7: Since ", $pass4,
  yield, (hF → u[$sAtj]) ≃ I ℝ,
  NL, "Gauge field: ",
  Aμ[x] ∈ (I gF -> I Mod[u[($a = $sAt[[1]] /. J → F)], I ℝ]) → (I𝔰𝔲[$a] ≃ ℝ),
  NL, "Arbitrary hermitian field ",
  $sA = {Aμ → -I a tuDPartial[b, μ], Aμ → {{T[X"1", "d", {μ}], 0}, {0, T[X2, "d", {μ}]}},
    {T[X"1", "d", {μ}], T[X2, "d", {μ}]} ∈ C"∞"[M, ℝ], C.tt : T[X"1"|2, "d", {μ}] → tt.C};
  $sA // MatrixForms,
  NL, "Since ", Aμ , " is always in form ", $ = Bμ -> Aμ - JF.Aμ.inv[JF],
  Yield, $ = $ /. $Fx[[2, -1]] /. inv[cc : 0 | C] → cc /. Dot → xDot /.
       dd : xDot[__] :→ (dd /. $sA[[2]] //. $sA[[-1]]) /.
      Plus → xPlus /. $sA[[2]] // OrderedxDotMultiplyAll[];
  Yield, $ = $ /. xPlus → Plus /. $sA[[-1]] /. $sCC /. tuOpSimplify[Dot];
  MatrixForms[$B = $],
  " define ", $ = $ -> {{T[Y, "d", {μ}], 0}, {0, -T[Y, "d", {μ}]}};
  $ = Flatten /@ ($[[1, 2]] -> $[[-1]]);
  $sb = Thread[$] // DeleteCases[#, 0 → 0] & // First,
  imply, $B = $B /. {$sb, -1 # & /@ $sb};
  MatrixForms[$B -> T[Y, "d", {μ}] ⊗ γF] // Framed, CG[" (4.3)"]
 ];
```

■Determine B_μ of Prop.3.7: Since $\mathcal{A}_{F J_F} \simeq \mathbb{C}$ ⟶ $(h_F \to u[\mathcal{A}_{F J_F}]) \simeq i\,\mathbb{R}$
Gauge field: $A_\mu[x] \in (i\, g_F \to i\, \mathrm{Mod}[u[\mathcal{A}_F],\, i\,\mathbb{R}]) \to I\mathfrak{su}[\mathcal{A}_F] \simeq \mathbb{R}$
Arbitrary hermitian field
$\{A_\mu \to -i\, a\, \underline{\partial}_\mu[b],\ A_\mu \to \begin{pmatrix} X^1{}_\mu & 0 \\ 0 & X^2{}_\mu \end{pmatrix}\},\ \{X^1{}_\mu,\ X^2{}_\mu\} \in C^\infty[M, \mathbb{R}],\ C.(tt : X^{1|2}{}_\mu) \to tt.C\}$
Since A_μ is always in form $B_\mu \to -J_F.A_\mu.J_F^{-1} + A_\mu$
→
→ $B_\mu \to \begin{pmatrix} -X^2{}_\mu + X^1{}_\mu & 0 \\ 0 & X^2{}_\mu - X^1{}_\mu \end{pmatrix}$ define $-X^2{}_\mu + X^1{}_\mu \to Y_\mu$ ⇒ $\boxed{(B_\mu \to \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix}) \to Y_\mu \otimes \gamma_F}$ (4.3)

```
PR["●Prop.4.3. The inner fluctuations
    for ACM M×FX are parameterized by a U[1]-gauge field Yμ ",
  Yield, 𝒟 ↦ (𝒟' → 𝒟 + T[γ, "u", {μ}].T[Y, "d", {μ}] ⊗ γF),
  NL, "The action of gauge group ", 𝒢[𝒜] ≃ C"∞"[M, U[1]][𝒟'],
  Yield,
  {T[Y, "d", {μ}] -> T[Y, "d", {μ}] - I u.tuDPartial[ConjugateTranspose[u], μ], u ∈ 𝒢[𝒜]}
 ];
```

●Prop.4.3. The inner fluctuations
for ACM $M{\times}F_X$ are parameterized by a U[1]-gauge field Y_μ
→ $\mathcal{D} \mapsto (\mathcal{D}' \to \mathcal{D} + \gamma^\mu.Y_\mu \otimes \gamma_F)$
The action of gauge group $\mathcal{G}[\mathcal{A}] \simeq C^\infty[M, U[1]][\mathcal{D}']$
→ $\{Y_\mu \to -i\, u.\underline{\partial}_\mu[u^\dagger] + Y_\mu,\ u \in \mathcal{G}[\mathcal{A}]\}$

4.2 Electrodynamics

```
$defEM = {};
accumEM[item_] := Block[{}, $defEM = tuAppendUniq[item][$defEM]; ""];
selectEM[heads_, with_ : {}, all_ : Null] := tuRuleSelect[$defEM][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] // If[all === Null, Last[#], #] &;
```

```
PR["■Two modifications of ACM M×FX needed for E-M: ",
 $ = {iDF[CG["non-zero"]], Sfermion[CG["action"]] ⇒ "2 independent spinors",
   S[CG["action"]] → xIntegral[-I ψ̄ .(T[γ, "u", {μ}].tuDPartial[_, μ] - m).ψ, x^4]};
 $ // ColumnBar,
 NL, "•Let ", $ = {{e, ē}[CG["basis of ℋF, OverBar→charge conjugate"]],
   e[CG["basis of ℋF+"]],
   ē[CG["basis of ℋF-"]],
   JF.e -> ē,
   JF.ē → e,
   γF.e → e,
   γF.ē → -ē
  }; $ // ColumnBar, accumEM[$];
 imply,
 $H = {ℋ → L2[M, S] ⊗ ℋF, L2[M, S] -> L2[M, S]+ ⊕ L2[M, S]-,
   ℋ+[CG["positiveEigenSpace of γ→γ5⊗γF"]] -> L2[M, S]+ ⊗ℋF+ ⊕ L2[M, S]- ⊗ℋF-,
   {ξ[CG["arbitrary"]] ∈ ℋ+,
    ξ → ψL ⊗ e + ψR ⊗ ē,
    ψL ∈ L2[M, S]+,
    ψR ∈ L2[M, S]-,
    ψ -> ψL + ψR,
    CG["⇒one Dirac spinor⇒too restrictive"]}
  }; accumEM[$H]; $H // ColumnForms,
 line,
 NL, "• Solution is to double the space ",
 C"∞"[M, ℂ2] ↔ (N → M × X ≃ M ⊔ M), CR["⊔ disjoint union, labels {L,R}"],
 NL, "Let ", $se = {{eR, eL, ēR, ēL} → basis [ℋF → ℂ4],
   γF.eL → eL, γF.eR → -eR, γF.ēL → -ēL,
   γF.ēR → ēR, γF[CG["decompose ℋ->ℋL[eL,ēR]⊕ℋR[eR,ēL]"]], (*
   JF.eR→-ēL,JF.eL→-ēR,JF.ēL→-eR,JF.ēR→-eL,*)
   JF.eR → ēR, JF.eL → ēL, JF.ēL → eL,
   JF.ēR → eR, JF[CG["interchanges particle-antiparticle"]],
   KOdim → 6, JF.JF → 1F, JF.γF → -γF.JF}; $se // ColumnBar, accumEM[$se]
  NL, "Chirality ", $ = {JF.γF.eL → -γF.JF.eL, JF.γF.eR → -γF.JF.eR} //
   tuRepeat[tuRule[{$se, tuOpSimplify[Dot]}]];
 $ // ColumnBar, accumEM[$];
 Imply, $sgj = {γF → DiagonalMatrix[{-1, 1, 1, -1}],
    JF → SparseArray[{Band[{1, 3}] → C, Band[{3, 1}] → C}, {4, 4}],
    C[CG["charge conjugation"]]} // Normal;
 $sgj // MatrixForms,
 NL, "•The elements ",
 $sa = {a ∈ (𝒜F → ℂ2), a[{eR, eL, ēR, ēL}] → DiagonalMatrix[{a1, a1, a2, a2}]};
 accumEM[{$sa, $sgj}]; MatrixForms[$sa]
]
PR["■Prop.4.5. ", FED → {ℂ2, ℂ4, 0, γF, JF}, " is a real even finite space of KOdim→6."
]
```

■Two modifications of ACM $M\times F_X$ needed for E-M:

D_F[non-zero]
$S_{fermion}$[action] ⇒ 2 independent spinors
S[action] → $\int -i\,\bar{\psi}.(-m+\gamma^{\mu}.\partial_{-\mu}[\_]).\psi\, dx^4$

•Let
{e, $\bar{e}$}[basis of $\mathcal{H}_F$, OverBar→charge conjugate]
e[basis of $\mathcal{H}_F{}^+$]
$\bar{e}$[basis of $\mathcal{H}_F{}^-$]
$J_F.e \to \bar{e}$
$J_F.\bar{e} \to e$
$\gamma_F.e \to e$
$\gamma_F.\bar{e} \to -\bar{e}$
⇒

$\mathcal{H} \to L^2[M, S]\otimes\mathcal{H}_F$
$L^2[M, S] \to L^2[M, S]^+ \oplus L^2[M, S]^-$
$\mathcal{H}^+$[positiveEigenSpace of $\gamma\to\gamma_5\otimes\gamma_F$] $\to L^2[M, S]^+\otimes(\mathcal{H}_F)^+ \oplus L^2[M, S]^-\otimes(\mathcal{H}_F)^-$
ξ[arbitrary] $\in \mathcal{H}^+$
$\xi \to \psi_L\otimes e + \psi_R\otimes\bar{e}$
$\psi_L \in L^2[M, S]^+$
$\psi_R \in L^2[M, S]^-$
$\psi \to \psi_L + \psi_R$
⇒one Dirac spinor⇒too restrictive

• Solution is to double the space
$C^\infty[M, \mathbb{C}^2] \leftrightarrow (N \to M\times X \simeq M\sqcup M)$ $\sqcup$ disjoint union, labels {L,R}

Let
$\{e_R, e_L, \bar{e}_R, \bar{e}_L\} \to$ basis$[\mathcal{H}_F \to \mathbb{C}^4]$
$\gamma_F.e_L \to e_L$
$\gamma_F.e_R \to -e_R$
$\gamma_F.\bar{e}_L \to -\bar{e}_L$
$\gamma_F.\bar{e}_R \to \bar{e}_R$
γ_F[decompose $\mathcal{H}$->$\mathcal{H}_L[e_L,\bar{e}_R]\oplus\mathcal{H}_R[e_R,\bar{e}_L]$]
$J_F.e_R \to \bar{e}_R$
$J_F.e_L \to \bar{e}_L$
$J_F.\bar{e}_L \to e_L$
$J_F.\bar{e}_R \to e_R$
J_F[interchanges particle-antiparticle]
KOdim → 6
$J_F.J_F \to 1_F$
$J_F.\gamma_F \to -\gamma_F.J_F$

Chirality
$\bar{e}_L \to \bar{e}_L$
$-\bar{e}_R \to -\bar{e}_R$

$$\Rightarrow \left\{\gamma_F \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ J_F \to \begin{pmatrix} 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \\ C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \end{pmatrix},\ C[\text{charge conjugation}]\right\}$$

•The elements $\left\{a \in (\mathcal{A}_F \to \mathbb{C}^2),\ a[\{e_R, e_L, \bar{e}_R, \bar{e}_L\}] \to \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}\right\}$

■Prop.4.5. $F_{ED} \to \{\mathbb{C}^2, \mathbb{C}^4, 0, \gamma_F, J_F\}$ is a real even finite space of KOdim→6.

4.2.2 A non-trivial finite Dirac operator

```
$accum = {};
PR["■Determine non-trivial Dirac operator iD_F from constraints.",
 next, "Hermitian condition: ", iD_F → ct[iD_F],
 "POFF",
 NL, "General iD_F: ", $d = Table[d_{i,j}, {i, 4}, {j, 4}]; MatrixForms[$d],
```

```
NL, "ConjugateTranspose: ", $ct = ct[$d]; MatrixForms[$ct],
Yield, $ct = $d → $ct //. rr : Rule[_, _] ⧴ Thread[rr] // Flatten // DeleteDuplicates;
$ct, CK, AppendTo[$accum, $ct]; "PONdd",
Yield, $ct = Select[$ct, ! OrderedQ[Apply[List, #[[1, 2 ;; 3]]]] &],

next,
$ = iDF.γF → -γF.iDF, "POFF",
Yield, $d = iDF → $d,
Yield, $ = $ /. $d /. tuRule[$sgj]; MatrixForms[$],
Yield, $ = $ //. rr : Rule[_, _] ⧴ Thread[rr] // Flatten // DeleteDuplicates,
AppendTo[$accum, $];
Yield, $s = tuRuleSolve[$, Flatten[$d[[2]]]],
"PONdd",
Imply, $d0 = $d = $d /. $s /. $ct; MatrixForms[$d] // Framed,

next,
$ = iDF.JF → JF.iDF, "POFF",
Yield, $ = $ /. Dot → xDot /. $d /. tuRule [$sgj] // tuMatrixOrderedMultiply //
    tuOpSimplifyF[dotOps] // (# /. xDot → Dot &);
MatrixForms[$],
Yield, $ = $ /. C . d_ → Conjugate[d].C; MatrixForms[$],
Yield, $ = $ //. rr : Rule[_, _] ⧴ Thread[rr] // Flatten // DeleteDuplicates;
Yield, $ = $ /.  a_ .C → a // DeleteCases[#, a_ → a_] &, AppendTo[$accum, $];
Yield, $ =
 $ /. Rule → xRule /.  aa : xRule[a_, b_] ⧴ Reverse[aa] /; FreeQ[a, 3 | 4] /. xRule → Rule //
   DeleteDuplicates,
"PONdd",
Imply, $d = $d /. $; MatrixForms[$d] // Framed,

next, "Order one condition: ",
$ord1 = selectDef[{CommutatorM[CommutatorM[_, _], _]}],
NL, "• First compute: ",
$Da = $ = CommutatorM[iDF, a],
Yield, $ =
 $ /. $d /. (tuRuleSelect[$defEM][a[_]] /. a[_] → a) /. tuCommutatorExpand // Simplify;
$1 = $Da -> $; $1 // MatrixForms,
NL, "• Let: ", $s = {selectDef[rghtA[b]],
   b -> DiagonalMatrix[{b1, b1, b2, b2}]}, "POFF",
Yield, $ = $ord1 /. $1 /. $s /. $s /. Dot → xDot /. tuRule[$sgj], CK,
Yield, $ = $ /. tuCommutatorExpand /. Dot → xDot,
$ = $ // tuMatrixOrderedMultiply // tuOpSimplifyF[dotOps] // (# /. xDot → Dot &);
"PONdd",
NL, "Let ", $s = { Dot[C , Shortest[e_]] ⧴ Dot[Conjugate[e], C] /; e =!= C,
   Conjugate[C] → C, C.C → 1};
$s // ColumnBar,
"POFF",
Yield, $ = $ // tuRepeat[$s, tuConjugateSimplify[]];
$ // MatrixForms,
"PONdd",
NL, "Determine dn,m for arbitrary a,b: ",
$ = $[[1]] // Flatten // DeleteCases[#, 0] &;
$ = # → 0 & /@ $,
NL, "Let ", $s = a2 → a12 + a1,
Yield, $ = $ /. $s //. tuOpSimplify[dotOps]; $ // Column,
Yield, $ = $ /. Dot → Times // Simplify; $ // ColumnBar,
NL, "Since the a,b's are arbitrary ",
Yield, $ = $ /. {a12 → 1, b1 - b2 → 1},
```

```
  Imply, $e46 = $d = $d /. $;
  accumEM[$d];
  MatrixForms[$d] // Framed, CG[" (4.6)"]
]
```

■Determine non-trivial Dirac operator iD_F from constraints.

◆Hermitian condition: $D_F \to (D_F)^\dagger$

....

→ $\{d_{2,1} \to (d_{1,2})^*, d_{3,1} \to (d_{1,3})^*, d_{3,2} \to (d_{2,3})^*, d_{4,1} \to (d_{1,4})^*, d_{4,2} \to (d_{2,4})^*, d_{4,3} \to (d_{3,4})^*\}$

◆$D_F.\gamma_F \to -\gamma_F.D_F$

....

$$\Rightarrow \boxed{D_F \to \begin{pmatrix} 0 & d_{1,2} & d_{1,3} & 0 \\ (d_{1,2})^* & 0 & 0 & d_{2,4} \\ (d_{1,3})^* & 0 & 0 & d_{3,4} \\ 0 & (d_{2,4})^* & (d_{3,4})^* & 0 \end{pmatrix}}$$

◆$D_F.J_F \to J_F.D_F$

....

$$\Rightarrow \boxed{D_F \to \begin{pmatrix} 0 & d_{1,2} & d_{1,3} & 0 \\ (d_{1,2})^* & 0 & 0 & d_{2,4} \\ (d_{1,3})^* & 0 & 0 & (d_{1,2})^* \\ 0 & (d_{2,4})^* & d_{1,2} & 0 \end{pmatrix}}$$

◆Order one condition: $[[D_F, a]_-, b^o]_- \to 0$

• First compute: $[D_F, a]_-$

$$\to [D_F, a]_- \to \begin{pmatrix} 0 & 0 & (-a_1 + a_2)\, d_{1,3} & 0 \\ 0 & 0 & 0 & (-a_1 + a_2)\, d_{2,4} \\ (d_{1,3})^* (a_1 - a_2) & 0 & 0 & 0 \\ 0 & (d_{2,4})^* (a_1 - a_2) & 0 & 0 \end{pmatrix}$$

• Let: $\{b^o \to J_F.b^\dagger.(J_F)^\dagger, b \to \{\{b_1, 0, 0, 0\}, \{0, b_1, 0, 0\}, \{0, 0, b_2, 0\}, \{0, 0, 0, b_2\}\}\}$

....

Let
$$\left| \begin{array}{l} C.\text{Shortest}[e\_] :\to e^*.C\ /;\ e =!= C \\ C^* \to C \\ C.C \to 1 \end{array} \right.$$

....

Determine $d_{n,m}$ for arbitrary a,b:

$\{(-a_1 + a_2).d_{1,3}.b_1 - b_2.(-a_1 + a_2).d_{1,3} \to 0, (-a_1 + a_2).d_{2,4}.b_1 - b_2.(-a_1 + a_2).d_{2,4} \to 0,$
$(d_{1,3})^*.(a_1 - a_2).b_2 - b_1.(d_{1,3})^*.(a_1 - a_2) \to 0, (d_{2,4})^*.(a_1 - a_2).b_2 - b_1.(d_{2,4})^*.(a_1 - a_2) \to 0\}$

Let $a_2 \to a12 + a_1$

$$\to \begin{array}{l} a12.d_{1,3}.b_1 - b_2.a12.d_{1,3} \to 0 \\ a12.d_{2,4}.b_1 - b_2.a12.d_{2,4} \to 0 \\ -(d_{1,3})^*.a12.b_2 + b_1.(d_{1,3})^*.a12 \to 0 \\ -(d_{2,4})^*.a12.b_2 + b_1.(d_{2,4})^*.a12 \to 0 \end{array}$$

$$\to \left| \begin{array}{l} a12\,(b_1 - b_2)\, d_{1,3} \to 0 \\ a12\,(b_1 - b_2)\, d_{2,4} \to 0 \\ a12\,(d_{1,3})^*\,(b_1 - b_2) \to 0 \\ a12\,(d_{2,4})^*\,(b_1 - b_2) \to 0 \end{array} \right.$$

Since the a,b's are arbitrary

→ $\{d_{1,3} \to 0, d_{2,4} \to 0, (d_{1,3})^* \to 0, (d_{2,4})^* \to 0\}$

$$\Rightarrow \boxed{D_F \to \begin{pmatrix} 0 & d_{1,2} & 0 & 0 \\ (d_{1,2})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & (d_{1,2})^* \\ 0 & 0 & d_{1,2} & 0 \end{pmatrix}} \quad (4.6)$$

4.2.3 The almost commutative manifold

```
PR["●Then ", $ = selectDef[M×FX]; $ // ColumnForms,
 " becomes ",
 $ = M×FED → {𝒜 -> C"∞"[M, ℂ^2], ℋ -> L^2[M, S]⊗ℂ^4,
    𝒟 -> slash[iD]⊗1F + T[γ, "d", {5}]⊗iDF, γ -> γ5⊗γF, J -> JM⊗JF};
 $ // ColumnForms, accumEM[$];
 NL, "Decompose ", $ = {𝒜 -> C"∞"[M, ℂ^2] -> C"∞"[M, ℂ]⊕C"∞"[M, ℂ],
   (ℋ -> L^2[M, S]⊗ℂ^4) -> L^2[M, S]⊗ℋe⊕L^2[M, S]⊗ℋe,
   a ∈ 𝒜 → $sa[[2]]
  }; $ // MatrixForms // ColumnBar, accumEM[$];
 NL, "Gauge group for 2-point space 𝒜F (Prop.4.2): ", 𝒢[𝒜F] ≃ U[1],
 Yield, $B = {T[B, "d", {μ}] → T[A, "d", {μ}] - JF.T[A, "d", {μ}].ct[JF], T[B, "d", {μ}] →
    DiagonalMatrix[{T[Y, "d", {μ}], T[Y, "d", {μ}], -T[Y, "d", {μ}], -T[Y, "d", {μ}]}],
   T[Y, "d", {μ}][x] ∈ ℝ};
 MatrixForms[$B] // ColumnBar, CG["(4.7)"], accumEM[$B]
]
```

●Then $M\times F_X \to$ $\left|\begin{array}{l} \mathcal{A}\to C^\infty[M,\ \mathbb{C}^2] \\ \mathcal{H}\to L^2[M,\ S]\otimes\mathbb{C}^2 \\ \mathcal{D}\to (\not{D})\otimes 1_F \\ \gamma\to\gamma_5\otimes\gamma_F \\ J\to J_M\otimes J_F \end{array}\right.$ becomes $M\times F_{ED}\to$ $\left|\begin{array}{l} \mathcal{A}\to C^\infty[M,\ \mathbb{C}^2] \\ \mathcal{H}\to L^2[M,\ S]\otimes\mathbb{C}^4 \\ \mathcal{D}\to (\not{D})\otimes 1_F + \text{Tensor}[\gamma,\ |\text{Void}\ ,\ |5\,]\otimes D_F \\ \gamma\to\gamma_5\otimes\gamma_F \\ J\to J_M\otimes J_F \end{array}\right.$

Decompose $\left|\begin{array}{l} \mathcal{A}\to C^\infty[M,\ \mathbb{C}^2]\to C^\infty[M,\ \mathbb{C}]\oplus C^\infty[M,\ \mathbb{C}] \\ (\mathcal{H}\to L^2[M,\ S]\otimes\mathbb{C}^4)\to L^2[M,\ S]\otimes\mathcal{H}_e\oplus L^2[M,\ S]\otimes\mathcal{H}_{\bar{e}} \\ a\in\mathcal{A}\to a[\{e_R,\ e_L,\ \bar{e}_R,\ \bar{e}_L\}]\to\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \end{array}\right.$

Gauge group for 2-point space $\mathcal{A}_F$ (Prop.4.2): $\mathcal{G}[\mathcal{A}_F]\simeq U[1]$

$\to$ $\left|\begin{array}{l} B_\mu\to -J_F.A_\mu.(J_F)^\dagger + A_\mu \\ B_\mu\to\begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & -Y_\mu & 0 \\ 0 & 0 & 0 & -Y_\mu \end{pmatrix}\ (4.7) \\ Y_\mu[x]\in\mathbb{R} \end{array}\right.$

4.2.4 The Lagrangian

● The spectral action

```
PR["●Prop. 4.6: The spectral action of ", $ = selectEM[M × F_ED];
  $ // ColumnForms,
  Yield,
  $p46 = $ = {Tr[f[𝒟_A / Λ]] → xIntegral[ℒ[T[g, "dd", {μ, ν}], T[Y, "d", {μ}]] √Det[g], x^4],
     ℒ[T[g, "dd", {μ, ν}], T[Y, "d", {μ}]] →
      4 ℒ_M[T[g, "dd", {μ, ν}]] + ℒ_Y[T[Y, "d", {μ}]] + ℒ_H[T[g, "dd", {μ, ν}], d],
     $p35[[2]],
     ℒ_Y[T[Y, "d", {μ}]] → f[0] / (6 π^2) T[ℱ, "dd", {μ, ν}] T[ℱ, "uu", {μ, ν}],
     T[ℱ, "dd", {μ, ν}] → tuDPartial[T[Y, "d", {ν}], μ] - tuDPartial[T[Y, "d", {μ}], ν],
     ℒ_H[T[g, "dd", {μ, ν}], d] →
      2 f_2 Λ^2 / π^2 Abs[d]^2 + f[0] / (2 π^2) Abs[d]^4 + f[0] / (12 π^2) s Abs[d]^2
    }; $ // ColumnSumExp // ColumnBar, accumEM[$]; ""
 ];
PR["● Proof: From Prop.3.7: ", $ = $p37; $ // ColumnBar,
 line,
 "Evaluate each part letting: ",
 $sPhi = { Φ → iD_F, N → dim[ℋ_F], dim[ℋ_F] → 4, Tr[1_ℋ_F] → N, $B[[1]], $e46};
 MatrixForms[$sPhi],
 Yield, $ = #[[1]] → (#[[2]] /. $sPhi) & /@ $p37[[{2, 3, 5, 7}]];
 ColumnBar[$0 = $];

 line,
 next, "The term ", tuRuleSelect[$p37][ℒ_M[_]][[1]] // Framed, " is (3.19).",

 next, "Evaluate the term ", $0 = tuRuleSelect[$p37][ℒ_B[B_μ]] // First,
 NL, "where ", $ = $F,
 NL, "Using ",
 $s = (tuRuleSelect[$B][T[B, "d", {μ}]][[2]] // tuAddPatternVariable[μ]), "POFF",
 Yield, $ = $ /. $s /. Plus → Inactive[Plus] //. tt : tuDPartial[a_, b_] ⧴ Thread[tt] //.
     tuDExpand[DerivOps] /. tuCommutatorExpand // Activate,
 $u = $ // tuIndicesRaise[{μ, ν}], "PONdd",
 $ = Thread[$ . $u, Rule] // Simplify; accumEM[$];
 Yield, $ = Tr[#] & /@ $; $, accumEM[$];
 NL, "Defining ",
 $s = {$$ = tuRuleSelect[$p46][T[ℱ, "dd", {μ, ν}]][[1]], tuIndicesRaise[{μ, ν}][$$]},
 Imply, $s = $ /. Reverse /@ (-# & /@ # & /@ $s) /. Dot → Times,
 Imply, $ = $0 /. $s; Framed[$],

 next,
 "Evaluate term ", $ = $0 = tuRuleSelect[$p37][ℒ_H[__]] // First,
 Yield, $[[2]] = $[[2]] /. $sPhi; MatrixForms[$],
 NL, "Evaluate Tr[]'s (switch )", $s = d_1,2 → d, accumEM[$s];
 $1 = $ // tuExtractPositionPattern[Tr[_]];
 $1 = $1 /. $e46 /. $s //.
    tt : T[iD, "d", {μ}][_] | T[iD, "u", {μ}][_] ⧴ Thread[tt] /. a_[0] → 0,
 Yield, $ = tuReplacePart[$, $1]; Framed[$], CR["Compare p.47"]
]
```

●Prop. 4.6: The spectral action of $M \times F_{ED} \rightarrow \begin{cases} \mathcal{A} \rightarrow C^{\infty}[M, \mathbb{C}^2] \\ \mathcal{H} \rightarrow L^2[M, S] \otimes \mathbb{C}^4 \\ \mathcal{D} \rightarrow (\not{D}) \otimes 1_F + \text{Tensor}[\gamma, \text{Void}, 5] \otimes D_F \\ \gamma \rightarrow \gamma_5 \otimes \gamma_F \\ J \rightarrow J_M \otimes J_F \end{cases}$

$$\rightarrow \begin{cases} \text{Tr}[f[\frac{\mathcal{D}_A}{\Lambda}]] \rightarrow \int \sqrt{\text{Det}[g]}\, \mathcal{L}[g_{\mu\nu}, Y_\mu]\, dx^4 \\ \mathcal{L}[g_{\mu\nu}, Y_\mu] \rightarrow \sum[\begin{cases} \mathcal{L}_H[g_{\mu\nu}, d] \\ 4\, \mathcal{L}_M[g_{\mu\nu}] \\ \mathcal{L}_Y[Y_\mu] \end{cases}] \\ \mathcal{L}_M[g_{\mu\nu}] \rightarrow \sum[\begin{cases} -\frac{\Lambda^2 f_2}{24\pi^2} \\ \frac{\Lambda^4 f_4}{2\pi^2} \\ \frac{f[0]\,(\frac{11 R^*.R^*}{360} - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{\Delta[s]}{30})}{16\pi^2} \end{cases}] \\ \mathcal{L}_Y[Y_\mu] \rightarrow \frac{f[0]\, \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}}{6\pi^2} \\ \mathcal{F}_{\mu\nu} \rightarrow \sum[\begin{cases} -\underset{-\nu}{\partial}[Y_\mu] \\ \underset{-\mu}{\partial}[Y_\nu] \end{cases}] \\ \mathcal{L}_H[g_{\mu\nu}, d] \rightarrow \sum[\begin{cases} \frac{s\, \text{Abs}[d]^2 f[0]}{12\pi^2} \\ \frac{\text{Abs}[d]^4 f[0]}{2\pi^2} \\ \frac{2\Lambda^2 \text{Abs}[d]^2 f_2}{\pi^2} \end{cases}] \end{cases}$$

● Proof: From Prop.3.7:

$\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \to \int_{x\in M} \sqrt{\mathrm{Det}[g[x]]}\ \mathcal{L}[g_{\mu\nu}, B_\mu, \Phi]$

$\mathcal{L}[g_{\mu\nu}, B_\mu, \Phi] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + N\,\mathcal{L}_M[g_{\mu\nu}]$

$\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 s[x] f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} - \frac{f[0] C_{\mu\nu\rho\sigma}[x] C^{\mu\nu\rho\sigma}[x]}{320\pi^2}$

$N \to \dim[\mathcal{H}_F]$

$\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$

$\mathcal{L}_B[B_\mu] \to$ Kinetic term gauge fields

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\,\mathrm{Tr}[D_\mu[\Phi].D^\mu[\Phi]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$ Higgs lagrangian

$N \to \mathrm{Tr}[1_{\mathcal{H}_F}]$

Evaluate each part letting: $\{\Phi \to D_F,\ N \to \dim[\mathcal{H}_F],\ \dim[\mathcal{H}_F] \to 4,$
$\mathrm{Tr}[1_{\mathcal{H}_F}] \to N,\ B_\mu \to -J_F.A_\mu.(J_F)^\dagger + A_\mu,\ D_F \to \begin{pmatrix} 0 & d_{1,2} & 0 & 0 \\ (d_{1,2})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & (d_{1,2})^* \\ 0 & 0 & d_{1,2} & 0 \end{pmatrix}\}$

→

◆The term $\boxed{\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 s[x] f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} - \frac{f[0] C_{\mu\nu\rho\sigma}[x] C^{\mu\nu\rho\sigma}[x]}{320\pi^2}}$ is (3.19).

◆Evaluate the term $\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$

where $F_{\mu\nu} \to i\,[B_\mu, B_\nu]_- - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

Using $B_{\mu\_} \to \{\{Y_\mu, 0, 0, 0\}, \{0, Y_\mu, 0, 0\}, \{0, 0, -Y_\mu, 0\}, \{0, 0, 0, -Y_\mu\}\}$

....

→ $\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] \to 4\,(\underline{\partial}_\nu[Y_\mu] - \underline{\partial}_\mu[Y_\nu])\,(\underline{\partial}^\nu[Y^\mu] - \underline{\partial}^\mu[Y^\nu])$

Defining $\{\mathcal{F}_{\mu\nu} \to -\underline{\partial}_\nu[Y_\mu] + \underline{\partial}_\mu[Y_\nu],\ \mathcal{F}^{\mu\nu} \to -\underline{\partial}^\nu[Y^\mu] + \underline{\partial}^\mu[Y^\nu]\}$

⇒ $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 4\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$

⇒ $\boxed{\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}}{6\pi^2}}$

◆Evaluate term $\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$
$\frac{f[0]\, s[x]\, \mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\,\mathrm{Tr}[D_\mu[\Phi].D^\mu[\Phi]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$

→ $\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[D_F.D_F]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[D_F.D_F]}{2\pi^2} +$
$\frac{f[0]\,\mathrm{Tr}[D_\mu[D_F].D^\mu[D_F]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[D_F.D_F.D_F.D_F]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[D_F.D_F]]}{24\pi^2}$

Evaluate Tr[]'s (switch)$d_{1,2} \to d\{\{2, 1, 5\} \to 4\, d\, d^*, \{2, 2, 5\} \to 4\, d\, d^*,$
$\{2, 3, 4\} \to 2\, D_\mu[d]\, D^\mu[d] + 2\, D_\mu[d^*]\, D^\mu[d^*], \{2, 4, 4\} \to 4\, d^2\, d^{*2}, \{2, 5, 4, 1\} \to 4\, d\, d^*\}$

→

$$\boxed{\begin{array}{l}\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \\ \frac{d^2 d^{*2} f[0]}{2\pi^2} + \frac{d\, d^*\, f[0]\, s[x]}{12\pi^2} - \frac{2\, d\, \Lambda^2 d^* f_2}{\pi^2} + \frac{f[0]\,\triangle[4\, d\, d^*]}{24\pi^2} + \frac{f[0]\,(2\, D_\mu[d]\, D^\mu[d] + 2\, D_\mu[d^*]\, D^\mu[d^*])}{8\pi^2}\end{array}}$$

Compare p.47

4.2.5 Fermionic action

```
PR["The basis vectors for ℋF: ",
 $ = Select[$defEM, MatchQ[#, _ -> basis[_]] &][[1]], $basis = $[[1]];
```

```
Yield, $H[[4]],
NL, "Spanning basis ", {𝓗F+[{eL, ēR}], 𝓗F-[{eR, ēL}]},
NL, "Arbitrary vector ",
$sξ = {ξ -> χR ⊗ eR + χL ⊗ eL + ψL ⊗ ēR + ψR ⊗ ēL, {χL, ψL} ∈ L2[M, S]+, {χR, ψR} ∈ L2[M, S]-};

$sξ // ColumnBar,

line,
NL, "● Prop.4.7: The fermionic action for ", tuRuleSelect[$defEM][M × _],
$Sf = $ = Sf → -I BraKet[JM.χ̃, T[γ, "u", {μ}].(T["∇S", "d", {μ}] - I T[Y, "d", {μ}]).ψ̃] +
     BraKet[JM.χ̃L, ct[d].ψ̃L] - BraKet[JM.χ̃R, d.ψ̃R];
Framed[$], CO["Prop.4.7"],
NL, "where the ~ means ", $sAt,
(****)
line,
NL, "■Proof: Compute: ", $00 = $d217[[3]] /. 𝒟𝒜 → iDiA, CG[" Definition 2.17"],
next, "Determine: The fluctuated Dirac operator ",
Yield, $sDA1 = $ = $sDA[[1]] /. ((tuRule[$sDA[[2]]] // First) /. a_ . b_ → a)
      /. N → M /. tuRuleSelect[$sPhi][Φ] // expandDC[], "POFF", (*M?*)
Yield, $ = $ /. tuOpDistribute[Dot] //. tuOpSimplify[Dot] // Expand,
Yield, $sDA1 = $ = $ //. {a_ . (b_ ⊗ c_) → (a.b) ⊗ c, a_ . 1_ → a}, "PON",

NL, "Since ",
$s = selectDef[slash[iD][_]] /. ψ → Blank[];
$s = tuRuleSolve[$s, Dot[_, _]] /. (dd : slash[_])[_] → dd,
(**)
yield, $ = $ /. $s // expandDC[];
Framed[ColumnSumExp[$sDA0 = $]], CO["p.48"],

line,
NL, "▪Using ",
$sem = tuRuleSelect[$defEM][{iDF, d1,2, T[B, "d", {μ}]}] // Select[#, FreeQ[#, A] &] &,
Yield, $s1 = iDF . # & /@ $basis;
$s2 = iDF.Transpose[{$basis}] /. $sem // Transpose // First;
$sd = Thread[$s1 → $s2];
Yield, $s1 = T[B, "d", {μ}].# & /@ $basis;
$s2 = T[B, "d", {μ}].Transpose[{$basis}] /. $sem // Transpose // First;
$sb = Thread[$s1 → $s2];
NL, "Get Combined Rule[]s: ",
$s0J = {tuRuleSelect[$defEM][JF.(eL | eR | ēL | ēR)], $sd, $sb} // Flatten;
$s0J // ColumnBar,
(**)
$accum = {};
NL, "Compute ", $ = J.ξ;
$ = $ → ($ /. $sξ[[1]] /. selectEM[J] //. tuOpDistribute[Dot] //
      tuCircleTimesGather[] // expandDC[$s0J]);
Framed[$], AppendTo[$accum, $];

$0 = $ = $sDA0[[2, 1]].ξ;
$ = $ → ($ /. $sξ[[1]] /. selectEM[J] //. tuOpDistribute[Dot] // tuCircleTimesGather[] //
     expandDC[{$s0J, $sgeneral}]);
$ // Framed,
AppendTo[$accum, $];

$ = $sDA0[[2, 2]].ξ;
$ =
  $ → ($ /. $sξ[[1]] /. selectEM[J] //. tuOpDistribute[Dot] //. $sX /. $s0J // expandDC[]);
```

```
 $ // Framed,
 AppendTo[$accum, $];
 $ = $sDA0[[2, 3]].ξ;
 $ = $ →
   ($ /. $sξ[[1]] /. selectEM[J] //. tuOpDistribute[Dot] //. $sX /. $s0J // expandDC[]);
 AppendTo[$accum, $]; Framed[$]
]
PR["Substitute these terms into: ", $ = $00,
 Yield, $s = #.ξ̃ & /@ $sDA0 // expandDC[]; $s // ColumnSumExp,
 Yield, $ = $ /. $s /. ξ̃ → ξ /. $accum; $ // ColumnSumExp,
 NL, "Expand BraKet: ",
 Yield, $ = $ //. tuBraKetSimplify[];
 Yield, $ = $ //. BraKet[a_ ⊗ b_, c_ ⊗ d_] -> BraKet[a, c] ⊗ BraKet[b, d];
 Yield,
 $ = $ //. tuBraKetSimplify[{d1,2, Conjugate[d1,2], T[Y, "d", {_}]}] /. $noArg // Expand;

 NL, "Impose orthogonality on F-space Using ",
 $s = {BraKet[a_ , a_] b_ : 1 → b, bb__ ⊗ (BraKet[a_, b_] y_ : 1) ⧴ 0 /; ! a === b},
 Yield, $pass1 = $ = $ /. $s /. CircleTimes → Times; $ // ColumnSumExp
]
PR[CO["NOTE: Chirality changing and Weyl basis. Let "],
 $s = {γ0, γi} -> {{{0, 1}, {1, 0}}, {{0, σ}, {-σ, 0}} } // Thread;
 $s // MatrixForms,
 and, $v = ψ -> {{ψL}, {ψR}}; $v // MatrixForms,
 NL, "Let ", $ = $0 = ψL → {{1, 0}, {0, 0}}.ψ; $ // MatrixForms,
 imply,
 $ = γi.# & /@ $; $ = MapAt[# /. $s /. $v &, $, 2]; $ // MatrixForms,
 and,
 $ = γ0.# & /@ $0; $ = MapAt[# /. $s /. $v &, $, 2]; $ // MatrixForms,
 NL, "Let ",
 $0 = $ = ψR → {{0, 0}, {0, 1}}.ψ; $ // MatrixForms,
 imply, $ = γi.# & /@ $; $ = MapAt[# /. $s /. $v &, $, 2]; $ // MatrixForms,
 and,
 $ = γ0.# & /@ $0; $ = MapAt[# /. $s /. $v &, $, 2]; $ // MatrixForms,
 NL, "i.e., The γ matrices exchange chirality ⇒ ",
 T[γ, "u", {μ}].tuDPartial[_, μ], " change chirality."
]
PR[$ = $pass1;
 "The ", slash[iD], " terms are symmetrizable, ", slash[iD],
 " change chirality.  Combine into single expression ",
 $s = {HoldPattern[AA : (cc_ : 1) BraKet[aa_ . ψa_, bb_ . χb_] ] ⧴
    BraKet[aa .χb , bb . ψa] cc /; (! FreeQ[AA, slash[iD]]),
   BraKet[JM .χL , slash[iD] . ψR] + BraKet[JM .χR , slash[iD] . ψL] ->
    BraKet[JM .χ , slash[iD] . ψ]
  }; $s // ColumnBar,
 Imply, {$s, $} = $ // tuTermApply[{slash[iD]}, {}, $s, {}, 1];
 $s // ColumnSumExp,
 Imply, ($pass2 = $) // ColumnSumExp
]
PR[$ = $pass2;
 "The γ's in the ", T[Y, "d", {μ}], " terms change spinor chirality ",
 CR["and these terms are anti-symmetric wrt spinors. "],
 $s = {HoldPattern[(Times[cc___, BraKet[aa_ . ψa_, bb_ . χb_]] )] ⧴
    -BraKet[aa .χb , bb . ψa] cc /; (! FreeQ[cc , Y]),
   (cc_ : 1) BraKet[JM .χL , T[γ, "u", {μ}] . ψR]
     + (cc_) BraKet[JM .χR , T[γ, "u", {μ}] . ψL] → cc BraKet[JM .χ , T[γ, "u", {μ}] . ψ]
  }; $s // ColumnBar,
```

```
 Imply, {$s, $} = $ // tuTermApply[{Y}, {}, $s, {}, 1];
 $s // ColumnSumExp,
 Imply, ($pass3 = $) // ColumnSumExp
]
PR[$ = $pass3;
 "The d terms are symmetric wrt spinors ",
 $s = {HoldPattern[(Times[cc_: 1, BraKet[aa_ . ψa_, bb_ . χb_]])] :→
     BraKet[aa .χb , bb . ψa] cc /; (! FreeQ[cc , d]),
   (cc_: 1) BraKet[JM .χL , T[γ, "u", {μ}] . ψR]
     + (cc_) BraKet[JM .χR , T[γ, "u", {μ}] . ψL] → cc BraKet[JM .χ , T[γ, "u", {μ}] . ψ]
  }; $s // ColumnBar,
 Imply, {$s, $} = $ // tuTermApply[{d}, {}, $s, {}, 1];
 Yield, $s // ColumnSumExp,
 NL, "With Weyl basis ", $s = {T[γ, "d", {5}] → {{-1, 0}, {0, 1}}},
 imply, $s = {T[γ, "d", {5}].ψL → -ψL, T[γ, "d", {5}].ψR → ψR},
 Imply, {$s, $} = $ // tuTermApply[{d}, {}, {$s, tuBraKetSimplify[]}, {}, 1];
 $s // ColumnSumExp,
 Yield, ($pass4 = $) // ColumnSumExp // Framed
]
```

The basis vectors for $\mathcal{H}_F$: $\{e_R, e_L, \bar{e}_R, \bar{e}_L\} \to \text{basis}[\mathcal{H}_F \to \mathbb{C}^4]$

→ $\{\xi$[arbitrary]$\in \mathcal{H}^+, \xi \to \psi_L \otimes e + \psi_R \otimes \bar{e}, \psi_L \in L^2[M, S]^+,$
$\psi_R \in L^2[M, S]^-, \psi \to \psi_L + \psi_R$, ⇒one Dirac spinor⇒too restrictive}

Spanning basis $\{(\mathcal{H}_F)^+[\{e_L, \bar{e}_R\}], (\mathcal{H}_F)^-[\{e_R, \bar{e}_L\}]\}$

Arbitrary vector
$$\xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \bar{e}_R + \psi_R \otimes \bar{e}_L$$
$$\{\chi_L, \psi_L\} \in L^2[M, S]^+$$
$$\{\chi_R, \psi_R\} \in L^2[M, S]^-$$

● Prop.4.7: The fermionic action for
$\{M \times F_{ED} \to \{\mathcal{A} \to C^\infty[M, \mathbb{C}^2], \mathcal{H} \to L^2[M, S] \otimes \mathbb{C}^4, \mathcal{D} \to (\not{D}) \otimes 1_F + \gamma_5 \otimes D_F, \gamma \to \gamma_5 \otimes \gamma_F, J \to J_M \otimes J_F\}\}$

$$\boxed{S_f \to -i \left\langle J_M.\tilde{\chi} \mid \gamma^\mu.(\nabla^S{}_\mu - i\, Y_\mu).\tilde{\psi} \right\rangle + \left\langle J_M.\tilde{\chi}_L \mid d^\dagger.\tilde{\psi}_L \right\rangle - \left\langle J_M.\tilde{\chi}_R \mid d.\tilde{\psi}_R \right\rangle}$$ Prop.4.7

where the ~ means $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A}, a.J \to J.a^\dagger, a^o \to a\}$

■Proof: Compute: $S_f \to \frac{1}{2} \left\langle J.\tilde{\xi} \mid D_A.\tilde{\xi} \right\rangle$ Definition 2.17

◆Determine: The fluctuated Dirac operator

→ $D_A \to \gamma_5 \otimes D_F - i\,(i\, \gamma^\mu.(1_M \otimes B_\mu) + \gamma^\mu.(\underline{\nabla}^S{}_\mu[\_] \otimes 1_{\mathcal{H}_F}))$

Since $\{\gamma^\mu.\underline{\nabla}^S{}_\mu[\_] \to i\,(\not{D})\}$ ⟶ $\boxed{D_A \to \sum\left[\begin{array}{l} (\not{D}) \otimes 1_{\mathcal{H}_F} \\ \gamma_5 \otimes D_F \\ \gamma^\mu \otimes B_\mu \end{array}\right]}$ p.48

■Using $\{D_F \to \{\{0, d_{1,2}, 0, 0\}, \{(d_{1,2})^*, 0, 0, 0\}, \{0, 0, 0, (d_{1,2})^*\}, \{0, 0, d_{1,2}, 0\}\},$
$d_{1,2} \to d, B_\mu \to \{\{Y_\mu, 0, 0, 0\}, \{0, Y_\mu, 0, 0\}, \{0, 0, -Y_\mu, 0\}, \{0, 0, 0, -Y_\mu\}\}\}$

→

→

Get Combined Rule[]s:
$$\begin{array}{l}
J_F.e_R \to \bar{e}_R \\
J_F.e_L \to \bar{e}_L \\
J_F.\bar{e}_L \to e_L \\
J_F.\bar{e}_R \to e_R \\
D_F.e_R \to e_L\, d_{1,2} \\
D_F.e_L \to (d_{1,2})^*\, e_R \\
D_F.\bar{e}_R \to (d_{1,2})^*\, \bar{e}_L \\
D_F.\bar{e}_L \to \bar{e}_R\, d_{1,2} \\
B_\mu.e_R \to e_R\, Y_\mu \\
B_\mu.e_L \to e_L\, Y_\mu \\
B_\mu.\bar{e}_R \to -\bar{e}_R\, Y_\mu \\
B_\mu.\bar{e}_L \to -\bar{e}_L\, Y_\mu
\end{array}$$

Compute $\boxed{J.\xi \to J_M.\chi_L \otimes \bar{e}_L + J_M.\chi_R \otimes \bar{e}_R + J_M.\psi_L \otimes e_R + J_M.\psi_R \otimes e_L}$

$$\boxed{((\not{D}) \otimes 1_{\mathcal{H}_F}).\xi \to (\not{D}).\chi_L \otimes e_L + (\not{D}).\chi_R \otimes e_R + (\not{D}).\psi_L \otimes \bar{e}_R + (\not{D}).\psi_R \otimes \bar{e}_L}$$

$$\boxed{(\gamma_5 \otimes D_F).\xi \to \gamma_5.\chi_L \otimes ((d_{1,2})^*\, e_R) + \gamma_5.\chi_R \otimes (e_L\, d_{1,2}) + \gamma_5.\psi_L \otimes ((d_{1,2})^*\, \bar{e}_L) + \gamma_5.\psi_R \otimes (\bar{e}_R\, d_{1,2})}$$

$$\boxed{(\gamma^\mu \otimes B_\mu).\xi \to \gamma^\mu.\chi_L \otimes (e_L\, Y_\mu) + \gamma^\mu.\chi_R \otimes (e_R\, Y_\mu) - \gamma^\mu.\psi_L \otimes (\bar{e}_R\, Y_\mu) - \gamma^\mu.\psi_R \otimes (\bar{e}_L\, Y_\mu)}$$

Substitute these terms into: $S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid D_A.\tilde{\xi}\rangle$

$$\to D_A.\tilde{\xi} \to \sum\left[\begin{array}{|l}((\not{D})\otimes 1_{\mathcal{H}_F}).\tilde{\xi} \\ (\gamma_5\otimes D_F).\tilde{\xi} \\ (\gamma^\mu\otimes B_\mu).\tilde{\xi}\end{array}\right]$$

$$\to S_f \to \frac{1}{2}\left\langle \sum\left[\begin{array}{|l} J_M.\chi_L\otimes\overline{e_L} \\ J_M.\chi_R\otimes\overline{e_R} \\ J_M.\psi_L\otimes e_R \\ J_M.\psi_R\otimes e_L\end{array}\right] \;\middle|\; \sum\left[\begin{array}{|l} (\not{D}).\chi_L\otimes e_L \\ (\not{D}).\chi_R\otimes e_R \\ (\not{D}).\psi_L\otimes\overline{e_R} \\ (\not{D}).\psi_R\otimes\overline{e_L} \\ \gamma_5.\chi_L\otimes((d_{1,2})^* e_R) \\ \gamma_5.\chi_R\otimes(e_L\, d_{1,2}) \\ \gamma_5.\psi_L\otimes((d_{1,2})^*\,\overline{e_L}) \\ \gamma_5.\psi_R\otimes(\overline{e_R}\, d_{1,2}) \\ \gamma^\mu.\chi_L\otimes(e_L\, Y_\mu) \\ \gamma^\mu.\chi_R\otimes(e_R\, Y_\mu) \\ -(\gamma^\mu.\psi_L\otimes(\overline{e_R}\, Y_\mu)) \\ -(\gamma^\mu.\psi_R\otimes(\overline{e_L}\, Y_\mu))\end{array}\right]\right\rangle$$

Expand BraKet:

→

→

→

Impose orthogonality on F-space Using

{⟨a_ | a_⟩ (b_ : 1) → b, bb__ ⊗ (⟨a_ | b_⟩ (y_ : 1)) :→ 0 /; ! a === b}

$$\to S_f \to \sum\left[\begin{array}{|l} \frac{1}{2}\langle J_M.\chi_L \mid (\not{D}).\psi_R\rangle \\ \frac{1}{2}\langle J_M.\chi_R \mid (\not{D}).\psi_L\rangle \\ \frac{1}{2}\langle J_M.\psi_L \mid (\not{D}).\chi_R\rangle \\ \frac{1}{2}\langle J_M.\psi_R \mid (\not{D}).\chi_L\rangle \\ \frac{1}{2}\langle J_M.\chi_L \mid \gamma_5.\psi_L\rangle (d_{1,2})^* \\ \frac{1}{2}\langle J_M.\psi_L \mid \gamma_5.\chi_L\rangle (d_{1,2})^* \\ \frac{1}{2}\langle J_M.\chi_R \mid \gamma_5.\psi_R\rangle d_{1,2} \\ \frac{1}{2}\langle J_M.\psi_R \mid \gamma_5.\chi_R\rangle d_{1,2} \\ -\frac{1}{2}\langle J_M.\chi_L \mid \gamma^\mu.\psi_R\rangle Y_\mu \\ -\frac{1}{2}\langle J_M.\chi_R \mid \gamma^\mu.\psi_L\rangle Y_\mu \\ \frac{1}{2}\langle J_M.\psi_L \mid \gamma^\mu.\chi_R\rangle Y_\mu \\ \frac{1}{2}\langle J_M.\psi_R \mid \gamma^\mu.\chi_L\rangle Y_\mu \end{array}\right]$$

NOTE: Chirality changing and Weyl basis. Let $\{\gamma 0 \to \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \gamma i \to \begin{pmatrix}0 & \sigma\\ -\sigma & 0\end{pmatrix}\}$ and $\psi \to \begin{pmatrix}\psi L\\ \psi R\end{pmatrix}$

Let $\psi L \to \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.\psi \Rightarrow \gamma i.\psi L \to \begin{pmatrix}0\\ -\sigma\,\psi L\end{pmatrix}$ and $\gamma 0.\psi L \to \begin{pmatrix}0\\ \psi L\end{pmatrix}$

Let $\psi R \to \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.\psi \Rightarrow \gamma i.\psi R \to \begin{pmatrix}\sigma\,\psi R\\ 0\end{pmatrix}$ and $\gamma 0.\psi R \to \begin{pmatrix}\psi R\\ 0\end{pmatrix}$

i.e., The γ matrices exchange chirality $\Rightarrow \gamma^\mu.\underline{\partial}_\mu[\_]$ change chirality.

The $\not{D}$ terms are symmetrizable, $\not{D}$ change chirality. Combine into single expression

$$\text{HoldPattern}[AA:(cc\_:1)\,\langle(aa\_).\psi_{a\_}\mid(bb\_).\chi_{b\_}\rangle] \mapsto \langle aa.\chi_b \mid bb.\psi_a\rangle\, cc\ /;\ !\,\text{FreeQ}[AA,\not{D}]$$
$$\langle J_M.\chi_L \mid (\not{D}).\psi_R\rangle + \langle J_M.\chi_R \mid (\not{D}).\psi_L\rangle \to \langle J_M.\chi \mid (\not{D}).\psi\rangle$$

$$\Rightarrow\ \sum\Big[\ \frac{1}{2}\langle J_M.\chi_L \mid (\not{D}).\psi_R\rangle\ \ \frac{1}{2}\langle J_M.\chi_R \mid (\not{D}).\psi_L\rangle\ \ \frac{1}{2}\langle J_M.\psi_L \mid (\not{D}).\chi_R\rangle\ \ \frac{1}{2}\langle J_M.\psi_R \mid (\not{D}).\chi_L\rangle\ \Big] \to \langle J_M.\chi \mid (\not{D}).\psi\rangle$$

$$\Rightarrow\ S_f \to \sum\Big[\ \langle J_M.\chi \mid (\not{D}).\psi\rangle\ \ \frac{1}{2}\langle J_M.\chi_L \mid \gamma_5.\psi_L\rangle (d_{1,2})^*\ \ \frac{1}{2}\langle J_M.\psi_L \mid \gamma_5.\chi_L\rangle (d_{1,2})^*\ \ \frac{1}{2}\langle J_M.\chi_R \mid \gamma_5.\psi_R\rangle d_{1,2}\ \ \frac{1}{2}\langle J_M.\psi_R \mid \gamma_5.\chi_R\rangle d_{1,2}\ \ -\frac{1}{2}\langle J_M.\chi_L \mid \gamma^\mu.\psi_R\rangle Y_\mu\ \ -\frac{1}{2}\langle J_M.\chi_R \mid \gamma^\mu.\psi_L\rangle Y_\mu\ \ \frac{1}{2}\langle J_M.\psi_L \mid \gamma^\mu.\chi_R\rangle Y_\mu\ \ \frac{1}{2}\langle J_M.\psi_R \mid \gamma^\mu.\chi_L\rangle Y_\mu\ \Big]$$

The γ's in the Y_μ terms change spinor chirality
and these terms are anti-symmetric wrt spinors.

$$\text{HoldPattern}[cc\_\_\_\ \langle(aa\_).\psi_{a\_}\mid(bb\_).\chi_{b\_}\rangle] \mapsto -\langle aa.\chi_b \mid bb.\psi_a\rangle\, cc\ /;\ !\,\text{FreeQ}[cc, Y]$$
$$\langle J_M.\chi_L \mid \gamma^\mu.\psi_R\rangle (cc\_:1) + \langle J_M.\chi_R \mid \gamma^\mu.\psi_L\rangle\, cc\_ \to cc\,\langle J_M.\chi \mid \gamma^\mu.\psi\rangle$$

$$\Rightarrow\ \sum\Big[\ -\frac{1}{2}\langle J_M.\chi_L \mid \gamma^\mu.\psi_R\rangle Y_\mu\ \ -\frac{1}{2}\langle J_M.\chi_R \mid \gamma^\mu.\psi_L\rangle Y_\mu\ \ \frac{1}{2}\langle J_M.\psi_L \mid \gamma^\mu.\chi_R\rangle Y_\mu\ \ \frac{1}{2}\langle J_M.\psi_R \mid \gamma^\mu.\chi_L\rangle Y_\mu\ \Big] \to -\langle J_M.\chi \mid \gamma^\mu.\psi\rangle Y_\mu$$

$$\Rightarrow\ S_f \to \sum\Big[\ \langle J_M.\chi \mid (\not{D}).\psi\rangle\ \ \frac{1}{2}\langle J_M.\chi_L \mid \gamma_5.\psi_L\rangle (d_{1,2})^*\ \ \frac{1}{2}\langle J_M.\psi_L \mid \gamma_5.\chi_L\rangle (d_{1,2})^*\ \ \frac{1}{2}\langle J_M.\chi_R \mid \gamma_5.\psi_R\rangle d_{1,2}\ \ \frac{1}{2}\langle J_M.\psi_R \mid \gamma_5.\chi_R\rangle d_{1,2}\ \ -\langle J_M.\chi \mid \gamma^\mu.\psi\rangle Y_\mu\ \Big]$$

The d terms are symmetric wrt spinors

```
HoldPattern[(cc_ : 1) ⟨(aa_).ψa_ | (bb_).χb_⟩] ⧴ ⟨aa.χb | bb.ψa⟩ cc /; ! FreeQ[cc, d]
⟨J_M.χ_L | γ^μ.ψ_R⟩ (cc_ : 1) + ⟨J_M.χ_R | γ^μ.ψ_L⟩ cc_ → cc ⟨J_M.χ | γ^μ.ψ⟩
```

⇒

$$\to \sum\left[\begin{array}{l} \frac{1}{2}\langle J_M.\chi_L \mid \gamma_5.\psi_L\rangle (d_{1,2})^* \\ \frac{1}{2}\langle J_M.\psi_L \mid \gamma_5.\chi_L\rangle (d_{1,2})^* \\ \frac{1}{2}\langle J_M.\chi_R \mid \gamma_5.\psi_R\rangle d_{1,2} \\ \frac{1}{2}\langle J_M.\psi_R \mid \gamma_5.\chi_R\rangle d_{1,2} \end{array}\right] \to \sum\left[\begin{array}{l} \langle J_M.\chi_L \mid \gamma_5.\psi_L\rangle (d_{1,2})^* \\ \langle J_M.\chi_R \mid \gamma_5.\psi_R\rangle d_{1,2} \end{array}\right]$$

With Weyl basis $\{\gamma_5 \to \{\{-1, 0\}, \{0, 1\}\}\} \Rightarrow \{\gamma_5.\psi_L \to -\psi_L, \gamma_5.\psi_R \to \psi_R\}$

$$\Rightarrow \sum\left[\begin{array}{l} \langle J_M.\chi_L \mid \gamma_5.\psi_L\rangle (d_{1,2})^* \\ \langle J_M.\chi_R \mid \gamma_5.\psi_R\rangle d_{1,2} \end{array}\right] \to \sum\left[\begin{array}{l} -\langle J_M.\chi_L \mid \psi_L\rangle (d_{1,2})^* \\ \langle J_M.\chi_R \mid \psi_R\rangle d_{1,2} \end{array}\right]$$

$$\to \boxed{S_f \to \sum\left[\begin{array}{l} \langle J_M.\chi \mid (\not{D}).\psi\rangle \\ -\langle J_M.\chi_L \mid \psi_L\rangle (d_{1,2})^* \\ \langle J_M.\chi_R \mid \psi_R\rangle d_{1,2} \\ -\langle J_M.\chi \mid \gamma^\mu.\psi\rangle Y_\mu \end{array}\right]}$$

Theorem 4.9

```
PR["●Theorem 4.9.  For ", $ = selectEM[M×F_ED]; $ // ColumnForms,
 NL, "the full Lagrangian is: ",
 $1 = $ = ℒ_grav[T[g, "dd", {μ, ν}]] → 4 ℒ_M[T[g, "dd", {μ, ν}]] + ℒ_H[T[g, "dd", {μ, ν}]];
 $ // ColumnSumExp, CG[" Prop.4.6"],
 NL, "plus the E-M Lagrangian ",
 $2 = $ = ℒ_EM[T[g, "dd", {μ, ν}]] →
    -I BraKet[J_M.χ̃, (T[γ, "u", {μ}].(("∇")_μ - I T[Y, "d", {μ}]) - m).ψ̃]_ℒ +
     f[0]/(6 π^2) T[F, "dd", {μ, ν}] T[F, "uu", {μ, ν}];
 $ // ColumnSumExp, CG[" Prop.4.7"],
 NL, "define ", $3 = BraKet[ξ, ψ] → xIntegral[√Abs[Det[g]] BraKet[ξ, ψ]_ℒ, x ∈ M],
 " to get theorem.",
 NL, "where ",
 $s = {BraKet[ξ, ψ]_ℒ[CG["hermitian pointwise inner product on fibres "]], m → I d_{1,2}};
 $s // ColumnBar

]
```

●Theorem 4.9. For $M\times F_{ED}\to \left|\begin{array}{l} \mathcal{A}\to C^{\infty}[M,\ \mathbb{C}^2] \\ \mathcal{H}\to L^2[M,\ S]\otimes\mathbb{C}^4 \\ \mathcal{D}\to (\not{D})\otimes 1_F+\text{Tensor}[\gamma,\ |\text{Void},\ |5]\otimes\mathcal{D}_F \\ \gamma\to\gamma_5\otimes\gamma_F \\ J\to J_M\otimes J_F \end{array}\right.$

the full Lagrangian is: $\mathcal{L}_{grav}[g_{\mu\nu}]\to\sum\left[\left|\begin{array}{l}\mathcal{L}_H[g_{\mu\nu}] \\ 4\ \mathcal{L}_M[g_{\mu\nu}]\end{array}\right.\right]$ Prop.4.6

plus the E-M Lagrangian $\mathcal{L}_{EM}[g_{\mu\nu}]\to\sum\left[\left|\begin{array}{l}-i\left\langle J_M.\tilde{\chi}\ \middle|\ (-m+\gamma^{\mu}.(\nabla_{\mu}-i\,Y_{\mu})).\tilde{\psi}\right\rangle_{\mathcal{L}} \\ \frac{f[0]\,F_{\mu\nu}\,F^{\mu\nu}}{6\pi^2}\end{array}\right.\right]$ Prop.4.7

define $\langle\xi\mid\psi\rangle\to\int_{x\in M}\sqrt{\text{Abs}[\text{Det}[g]]}\ \langle\xi\mid\psi\rangle_{\mathcal{L}}$ to get theorem.

where $\left|\begin{array}{l}\langle\xi\mid\psi\rangle_{\mathcal{L}}[\text{hermitian pointwise inner product on fibres }] \\ m\to i\,d_{1,2}\end{array}\right.$

4.2.6 Fermionic degrees of freedom

```
PR["Grassmann variable definition: ",
 $grassmann = $ = {θ[CG["Grassman vector"]] → {T[θ, "d", {i}], {i, N}},
    T[θ, "d", {i}] .T[θ, "d", {j}] → -T[θ, "d", {j}].T[θ, "d", {i}],
    xIntegral[1, T[θ, "d", {i}]] → 0,
    xIntegral[T[θ, "d", {i}], T[θ, "d", {i}]] → 1,
    (iD[θ] -> xProduct[d[T[θ, "d", {i}]], {i, N}])[CG["for integrals"]],
    (iD[η, θ] -> xProduct[d[T[η, "d", {i}]] · d[T[θ, "d", {i}]], {i, N}])[
     CG["for two vectors"]],
    xIntegral[Exp[Transpose[θ].A.η], iD[η, θ]] → Det[A],
    Det[A] → 1 / N! xSum[(-1)^(Abs[σ]+Abs[τ])
        T[A, "dd", {σ[1], τ[1]}]. … .T[A, "dd", {σ[N], τ[N]}], {σ, τ} ∈ Π_N],
    Π_dim[F][CG["permutations of {1,N}"]],
    {N → 2 n, θ → η, xIntegral[Exp[Transpose[η].A.η / 2], iD[η]] → Pf[A],
     Pf[A] → (-1)^n / (2^n n!)
       xSum[(-1)^Abs[σ] T[A, "dd", {σ[1], σ[2]}].….T[A, "dd", {σ[2 n - 1], σ[2 n]}]],
     {A[CG["skewsymmetric"]],
      Det[A] → Pf[A]^2
     }
    }

   }; $ // ColumnBar
]
```

Grassmann variable definition:

θ[Grassman vector] $\rightarrow \{\theta_i, \{i, N\}\}$

$\theta_i . \theta_j \rightarrow -\theta_j . \theta_i$

$\int 1 \, d\theta_i \rightarrow 0$

$\int \theta_i \, d\theta_i \rightarrow 1$

$(D[\theta] \rightarrow \prod_{\overline{\{i,N\}}} [d[\theta_i]])$[for integrals]

$(D[\eta, \theta] \rightarrow \prod_{\overline{\{i,N\}}} [d[\eta_i] \cdot d[\theta_i]])$[for two vectors]

$\int e^{\theta^T . A . \eta} \, dD[\eta, \theta] \rightarrow \mathrm{Det}[A]$

$\mathrm{Det}[A] \rightarrow \frac{\sum_{\overline{\{\sigma,\tau\}\in\Pi_N}} [(-1)^{\mathrm{Abs}[\sigma]+\mathrm{Abs}[\tau]} A_{\sigma[1]\,\tau[1]} . \ldots . A_{\sigma[N]\,\tau[N]}]}{N!}$

$\Pi_{\mathrm{dim}[F]}$[permutations of {1,N}]

$\{N \rightarrow 2n, \theta \rightarrow \eta, \int e^{\frac{1}{2}\eta^T . A . \eta} \, dD[\eta] \rightarrow \mathrm{Pf}[A],$

$\mathrm{Pf}[A] \rightarrow \frac{(-\frac{1}{2})^n \,\mathrm{xSum}[(-1)^{\mathrm{Abs}[\sigma]} A_{\sigma[1]\,\sigma[2]} . \ldots . A_{\sigma[-1+2n]\,\sigma[2n]}]}{n!}$, {A[skewsymmetric], $\mathrm{Det}[A] \rightarrow \mathrm{Pf}[A]^2$}}

```
PR["Consider ", xIntegral[Exp[S], ϕ],
 NL, "with ", $ = {U[ξ, ζ][CG["antisymmetric bilinear form on ℋ⁺", {ξ, ζ} ∈ ℋ⁺]] →
     BraKet[J.ξ, iD_A.ζ],
   B[χ, ψ][CG["bilinear form on L²[M,S] ", {χ, ψ} ∈ L²[M, S]]] →
    -I BraKet[J_M.χ, (T[γ, "u", {μ}].(("∇S")_μ - I T[Y, "d", {μ}] ) - m).ψ],
   $sξ, χ → χ_L + χ_R, ψ → ψ_L + ψ_R,
   $sDA1
  }; $ // ColumnBar,
 line,
 NL, "Show ", ($ = U[ξ, ξ]) → 2 B[χ, ψ],
 Yield, $,
 line,
 NL, "They get ",
 Yield, $ = {Pf[U] → (xIntegral[Exp[1 / 2 U[ξ̃, ξ̃]], iD[ξ̃]] →
      (xIntegral[Exp[B[ξ̃, ψ̃]], iD[ξ̃], iD[ψ̃]] →
        Det[B]))}; $ // ColumnBar
]
```

Consider $\int e^{S}\, d\phi$

with

$U[\xi, \zeta][\text{antisymmetric bilinear form on } \mathcal{H}^+, \{\xi, \zeta\} \in \mathcal{H}^+] \to \langle J.\xi \mid D_A.\zeta \rangle$

$B[\chi, \psi][\text{bilinear form on } L^2[M,S] \ , \{\chi, \psi\} \in L^2[M, S]] \to -i \langle J_M.\chi \mid (-m + \gamma^\mu.(\nabla^S{}_\mu - i\, Y_\mu)).\psi \rangle$

$\{\xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \overline{e_R} + \psi_R \otimes \overline{e_L},\ \{\chi_L, \psi_L\} \in L^2[M, S]^+,\ \{\chi_R, \psi_R\} \in L^2[M, S]^-\}$

$\chi \to \chi_L + \chi_R$

$\psi \to \psi_L + \psi_R$

$D_A \to -i\, \gamma^\mu.\nabla^S_{-\mu}\ [\_] \otimes 1_{\mathcal{H}_F} + \gamma_5 \otimes D_F + \gamma^\mu \otimes B_\mu$

Show $U[\xi, \xi] \to 2\, B[\chi, \psi]$

→ $U[\xi, \xi]$

They get

→ $Pf[U] \to \int e^{\frac{1}{2} U[\tilde{\xi}, \tilde{\xi}]}\, dD[\tilde{\xi}] \to \int\int e^{B[\tilde{\xi}, \tilde{\psi}]}\, dD[\tilde{\psi}]\, dD[\tilde{\xi}] \to Det[B]$

```
tuSaveAllVariables[]
```