/ Phys 234A HW 4 John Chae.

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1.1) $S = \int_{\mathbb{C}} d^2z\, b\, \bar{\partial}\, c.$

$z \mapsto f(z) \equiv z'$

$(z)\, b \mapsto b'(z') = \left(\frac{\partial z}{\partial z'}\right)^{\lambda} \cdot b(z)$

$(z)\, c \mapsto c'(z') = \left(\frac{\partial z}{\partial z'}\right)^{1-\lambda} c(z)$

$\frac{\partial}{\partial \bar{z}} \mapsto \frac{\partial}{\partial \bar{z}'} = \frac{\partial \bar{z}}{\partial \bar{z}'} \frac{\partial}{\partial \bar{z}}$

$dz \mapsto dz' = \frac{\partial z'}{\partial z} dz$

$d\bar{z} \mapsto d\bar{z}' = \frac{\partial \bar{z}'}{\partial \bar{z}} d\bar{z}$

$\Rightarrow S \mapsto \int d^2z'\, b'(z')\, \bar{\partial}'\, c'(z') \equiv S'$

$= \int dz d\bar{z}\, \cancel{\frac{\partial z'}{\partial z}} \cancel{\frac{\partial \bar{z}'}{\partial \bar{z}}} \cancel{\left(\frac{\partial z}{\partial z'}\right)^{\lambda}} \cancel{\left(\frac{\partial z}{\partial z'}\right)^{1-\lambda}} \cancel{\frac{\partial \bar{z}}{\partial \bar{z}'}}\, b \bar{\partial} c$

$= S = S'$ $\therefore$ S_{bc} is conformaly Inv.

1.2) $b(z)\cdot c(w) = :b(z)\cdot c(w): + \dfrac{1}{z-w}$?

$$\sim \frac{1}{z-w}, \quad \Bigg\| \quad \{b_n, c_m\} = \delta_{n+m,0}$$

$$b(z)\cdot c(w) = \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{n+\lambda}} \sum_{m\in\mathbb{Z}} \frac{c_m}{w^{m+1-\lambda}}, \quad \begin{array}{l} b_n|0\rangle = 0,\ n>-\lambda \\ c_m|0\rangle = 0,\ n>\lambda- \end{array}$$

$$= \Big(\underbrace{\sum_{-\infty}^{-\lambda}}_{\text{Creat}^{n}} + \underbrace{\sum_{1-\lambda}^{\infty}}_{\text{Annihil}^{n}}\Big) \frac{b_n}{z^{n+\lambda}} \times \Big(\underbrace{\sum_{-\infty}^{\lambda-1}}_{\text{Creat}^{n}} + \underbrace{\sum_{\lambda}^{\infty}}_{\text{Annihil}^{n}}\Big) \frac{c}{w^{m+}}$$

$$= \sum_{n=-\infty}^{-\lambda} \sum_{m=-\infty}^{\infty} \frac{b_n c_m}{z^{n+\lambda} w^{m+1-\lambda}} + \underbrace{\sum_{n=1-\lambda}^{\infty} \sum_{m=-\infty}^{\lambda-1} \frac{b_n c_m}{z^{n+\lambda} w^{m}}}_{(*),\ \text{requires ordering}}$$

$$+ \sum_{n=1-\lambda}^{\infty} \sum_{m=\lambda}^{\infty} \frac{b_n c_m}{z^{n+\lambda} w^{m+1-\lambda}}$$

$$(*) \underset{\{b,c\}\ \text{relation}}{=} \sum_{n=1-\lambda}^{\infty} \sum_{m=-\infty}^{\lambda-1} \frac{1}{z^{n+\lambda} w^{m+1-\lambda}} \left(-c_m b_n + \delta_{n+m,0}\right)$$

$$\sum_{n=-\lambda+1}^{\infty} \sum_{m=-\infty}^{\lambda-1} \frac{\delta_{n,-m}}{z^{n+\lambda} w^{m+1-\lambda}}$$

$m \mapsto -m$

$$= \sum_{n=-\lambda+1}^{\infty} \sum_{m=+\infty}^{-\lambda+1} \frac{\delta_{n,m}}{z^{n+\lambda} w^{-m+1-\lambda}}$$

$\sum_m \delta_{n,m} w^m$

$$= \left[\sum_{n=-\lambda+1}^{\infty} \frac{w^n}{z^n}\right] \frac{1}{w} \left(\frac{w}{z}\right)^{\lambda}$$

geometric series,

$|z| > |w|$

$\Leftrightarrow$ Ratio $= \frac{w}{z}$

$$= \frac{1}{w} \cancel{\left(\frac{w}{z}\right)^{\lambda}} \left(\frac{w}{z}\right)^{1-\cancel{\lambda}} \frac{1}{1 - w/z}$$

$$= \frac{1}{z} \frac{1}{1 - w/z}$$

$$= \frac{1}{z-w}$$

$$\xrightarrow{\text{sub}} (*) = \sum_{n=1-\lambda}^{\infty} \sum_{m=-\infty}^{\lambda-1} \frac{-c_m \cdot b_n}{z^{n+\lambda} w^{m+1-\lambda}} + \frac{1}{z-w}.$$

$$\therefore \underbrace{b(z)\cdot c(w)}_{-c(w)\cdot b(z)} = \sum_{n=-\infty}^{-\lambda}\sum_{m=-\infty}^{\infty}\frac{b_n\cdot c_m}{z^{n+\lambda}w^{m+1-\lambda}} +$$

$$\sum_{n=1-\lambda}^{\infty}\left[\left(\sum_{m=-\infty}^{\lambda-1} - c_m\cdot b_n + \sum_{\lambda}^{\infty} b_n\cdot c_m\right)\frac{1}{z^{n+\lambda}\cdot w^{m+1-\lambda}}\right] + \frac{1}{z-w}$$

$$\propto \frac{1}{z-w} \xRightarrow{\langle\ \rangle} \langle b(z)\cdot c(w)\rangle = \frac{1}{z-w}$$

$$:b(z)\cdot c(w): = \sum_{n=-\infty}^{-\lambda}\sum_{m=-\infty}^{\infty}\frac{b_n\cdot c_m}{z^{n+\lambda}w^{m+1-\lambda}} +$$

$$\sum_{n=-\lambda+1}^{\infty}\left[\sum_{m=-\infty}^{\lambda-1}\frac{-c_m\cdot b_n}{z^{n+\lambda}w^{m+1-\lambda}} + \sum_{m=\lambda}^{\infty}\frac{b_n\cdot c_m}{z^{n+\lambda}w^{m+1-\lambda}}\right]$$

1.3) $T^{(g)}(z) :b(w): = ?$

expect$^{\underline{n}}$; $\text{RHS} \supset \frac{h_b}{(z-w)^2} b$.

<u>Idea -</u>

B/C $T^{(g)}$ is normally ordered, we need to use $:F::G: = :FG: + \underbrace{\sum \text{Cross-contract}^{\underline{n}}\text{s}}_{\substack{\cup \\ \langle \text{fields} \rangle}}$

Then focus on the cross-contractions containing only b to verify h_b.

$$\text{RHS} = \left(:(\partial_z b) \overset{C(z)}{\widetilde{C_z}}:(1-\lambda) - \lambda \cdot :b_z \partial_z C: \right) :b_w:$$

$$= \left[:\partial_z b\, C_z\, b_w: (1-\lambda) \overset{\{C,b\}=0}{-} \langle \partial_z b \cdot b_w \rangle C_z (1-\lambda) \right.$$

$$\left. + \partial_z b \langle C_z b_w \rangle (1-\lambda) \right] - \lambda :b_z \partial_z C\, b_w:$$

$$-\lambda \Big(\underbrace{b_z \langle \partial_z C\, b_w \rangle}_{(*),\ \text{relevant term}} - \langle b_z b_w \rangle \partial_z C \Big)$$

$$(*) = -\lambda \cdot b_z \langle -b_w \partial_z c \rangle$$

$$= +\lambda \cdot b_z \partial_z \langle b_w c_z \rangle$$

$$= \lambda b_z \partial_z \frac{1}{w-z}$$

$$= \lambda b_z \frac{1}{(w-z)^2} \Longrightarrow \underline{\underline{h_b = \lambda}}$$

Similiarly for h_c,

$$T^{(g)}(z) :c(w): \;\propto\; :(\partial_z b) c_z: (1-\lambda) :c_w:$$

$$= (1-\lambda)\Big(:\partial_z b \; c_z \; c_w: +$$

$$\partial_z b \langle c_z c_w \rangle \underbrace{- c_z \langle \partial_z b \; c_w \rangle}_{(*)} \Big)$$

$$(*) = -(1-\lambda) c_z \partial_z \langle b_z c_w \rangle$$

$$= -(1-\lambda) c_z \frac{-1}{(z-w)^2} \Longrightarrow h_c = 1-\lambda$$

4) $T^{(g)}_{(z)}\, T^{(g)}(w) \supset \dfrac{c/2}{(z-w)^4}$.

Applying the same strategy as the previous problem,

$$:T(z)::T(w): \;=\; :T(z)\cdot T(w): + \underbrace{\sum \text{cross-contract}^{n}\text{s}}_{\downarrow\; c}$$

Remark) $\tilde{b}, \tilde{c}$ absent so $\tilde{c}^{(g)} = 0$.

The c-c terms will be

$$\left[(\partial_z b)\, c_z (1-\lambda) - \lambda\, b_z \partial_z c\right]\cdot\left[(\partial_w b)\, c_w (1-\lambda) - \lambda\cdot b_w \partial_w c\right]$$

|| $\langle\ \rangle$ are suppressed for later steps.

$$= (\partial_z b)\, c_z (\partial_w b)\, c_w (1-\lambda)^2 + \lambda^2\, b_z \partial_z c\, b_w \partial_w c$$

$$- \lambda\cdot(1-\lambda)\cdot\left((\partial_z b)\, c_z\, b_w \partial_w c + b_z (\partial_z c)(\partial_w b)\, c_w\right).$$

Cross-contractions applied

$$\rightarrow \underbrace{(-)^3}_{\text{2 con-tr}^{n}\text{s.}} (1-\lambda)^2 \langle \partial_z b\, c_w\rangle\langle \partial_w b\, c_z\rangle + (-)^3 \lambda^2 \langle b_z \partial_w c\rangle\cdot$$

$$\langle b_w \partial_z c\rangle - \lambda(1-\lambda)\cdot(-)^3\left[\langle \partial_z b\, \partial_w c\rangle\langle b_w c_z\rangle + \right.$$

$\langle b_z c_w \rangle \langle \partial_w b \, \partial_z c \rangle \Big]$, where only non-vanishing VEVs are kept.

$$= -(1-\lambda)^2 \left(\partial_z \frac{1}{z-w} \right) \partial_w \frac{1}{w-z} - \lambda^2 \left(\partial_w \frac{1}{z-w} \right)$$

$$\partial_z \frac{1}{w-z} + \lambda (1-\lambda) \left[\left(\partial_z \partial_w \frac{1}{z-w} \right) \frac{1}{w-z} + \frac{1}{z-w} \partial_w \partial_z \frac{1}{w-z} \right]$$

$$= -(1-\lambda)^2 \frac{1}{(z-w)^2} \frac{-1}{(w-z)^2} - \lambda^2 \frac{1}{(z-w)^2} \frac{1}{(w-z)^2} + \lambda(1-\lambda) \left[\frac{-2}{(z-w)^3} \frac{-1}{z-w} + \frac{+1}{z-w} \frac{-2}{(w-z)^3} \right]$$

$$= \frac{1}{(z-w)^4} \left(-1 - \lambda^2 + 2\lambda - \lambda^2 + 4\lambda (1-\lambda) \right)$$ $\Big\|$ $\tilde{C}^{(g)} = 0$ &

$$= \frac{1}{(z-w)^4} \left(-6\lambda^2 + 6\lambda - 1 \right) \frac{2}{2} \Rightarrow \underline{c^{(g)} = -3(2\lambda-1)^2 + 1}$$

Applying this to the first term, we obtain

$$\begin{aligned}\langle 0|\mathcal{R}\left[:\partial b(z)c(z)::\partial b(w)c(w):\right]|0\rangle &= \left(\partial_z\langle 0|\mathcal{R}\left[b(z)c(w)\right]|0\rangle\right)\left(\partial_w\langle 0|\mathcal{R}\left[c(z)b(w)\right]|0\rangle\right)\\ &= \left(-\frac{1}{(z-w)^2}\right)\left(\frac{1}{(z-w)^2}\right) = -\frac{1}{(z-w)^4}.\end{aligned}$$

Similarly,

$$\langle 0|\mathcal{R}\left[:b(z)\partial c(z)::b(w)\partial c(w):\right]|0\rangle = -\frac{1}{(z-w)^4}.$$

We also have that

$$\langle 0|\mathcal{R}\left[:\partial b(z)c(z)::b(w)\partial c(w):\right]|0\rangle = \partial_z\partial_w\left(\frac{1}{z-w}\right)\frac{1}{z-w} = -\frac{2}{(z-w)^4},$$

and similarly for the final term. Putting this all together, we find

$$\langle 0|\mathcal{R}[T(z)T(w)]|0\rangle = \frac{-6\lambda^2+6\lambda-1}{(z-w)^4}.$$

Comparing this with (6), we see that

$$\boxed{c = 2(-6\lambda^2+6\lambda-1) = -3(2\lambda-1)^2+1}.$$

Problem 2. Define

$$S := \frac{1}{2\pi}\int_P \mathrm{d}^2z\,\left[\frac{1}{2}\partial X\bar{\partial}X + b\bar{\partial}c + \tilde{b}\partial\tilde{c}\right],$$

where X is a primary field with conformal weights $(0,0)$, b and c are primary fields with conformal weights $(\lambda,0)$ and $(1-\lambda,0)$ respectively, and $\tilde{b}$ and $\tilde{c}$ are primary fields with conformal weights $(0,\lambda)$ and $(0,1-\lambda)$ respectively. (In string theory, $\lambda=2$, and we will indeed need to specialize to this case later on). Then, the energy-momentum tensor is given by

$$T = T^{(m)} + T^{(g)} \text{ and } \tilde{T} = \tilde{T}^{(m)} + \tilde{T}^{(g)},$$

where

$$T^{(m)} = -\frac{1}{2}:\partial X\partial X \text{ and } \tilde{T}^{(m)} = -\frac{1}{2}:\bar{\partial}X\bar{\partial}X:,$$

and

$$T^{(g)} = (1-\lambda):\bar{\partial}bc:-\lambda:b\bar{\partial}c: \text{ and } \tilde{T}^{(g)} = (1-\lambda):\bar{\partial}\tilde{b}\tilde{c}:-\lambda:\tilde{b}\bar{\partial}\tilde{c}:$$

We define the modes of the fields in the usual way. Then, define

$$Q := \sum_{n\in\mathbb{Z}}\left[c_nL_{-n}^{(m)} + \tilde{c}_n\tilde{L}_{-n}^{(m)}\right] + \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[:c_nL_{-n}^{(g)}:+:\tilde{c}_n\tilde{L}_{-n}^{(g)}:\right] - (c_0+\tilde{c}_0).$$

(2.1). We have that

$$\boxed{\begin{aligned}[L_m,b_n] = [L_m^{(g)},b_n] &= \int_0\frac{\mathrm{d}w}{2\pi\mathrm{i}}\,w^{n+\lambda-1}\int_w\frac{\mathrm{d}z}{2\pi\mathrm{i}}\,z^{m+1}\mathcal{R}\left[T^{(g)}(z)b(w)\right]\\ &= \int_0\frac{\mathrm{d}w}{2\pi\mathrm{i}}\,w^{n+\lambda-1}\left[\lambda b(w)\left(\int_w\frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\frac{z^{m+1}}{(z-w)^2}\right) + \partial b(w)\left(\int_w\frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\frac{z^{m+1}}{z-w}\right)\right]\\ &= \int_0\frac{\mathrm{d}w}{2\pi\mathrm{i}}\,w^{n+\lambda-1}\left(\lambda b(w)(m+1)w^m + \partial b(w)w^{m+1}\right)\\ &= \lambda(m+1)\int_0\frac{\mathrm{d}w}{2\pi\mathrm{i}}\sum_{k\in\mathbb{Z}}\frac{b_k}{w^{k-n+1-m}} - \int_0\frac{\mathrm{d}w}{2\pi\mathrm{i}}\sum_{k\in\mathbb{Z}}\frac{(k+\lambda)b_k}{w^{k-n+1-m}}\\ &= \lambda(m+1)b_{m+n} - ((m+n)+\lambda)b_{m+n} = ((\lambda-1)m-n)\,b_{m+n}\end{aligned}}$$

The classical BRST transformation of $b(z)$ is $\delta b = \eta T$ (η is the Grassman parameter of the transformation). As Q generates this transformation in the quantum theory, we must have that

$$\{Q, b\} = T,$$

from which it follows that

$$\boxed{\{Q, b_m\} = L_m}$$

(because, for the string, $\lambda = 2$, in which case the modes of b and the modes of T are defined in the same way.).
(2.2).

$$\boxed{\begin{aligned}\{[Q, L_m], b_n\} - [\{b_n, Q\}, L_m] - \{[L_m, b_n], Q\} &= \{QL_m - L_mQ, b_n\} + [L_m, b_nQ + Qb_n] \\ &\quad + \{b_nL_m - L_mb_n, Q\} \\ &= QL_mb_n - L_mQb_n + b_nQL_m - b_nL_mQ \\ &\quad + L_mb_nQ + L_mQb_n - b_nQL_m - Qb_nL_m \\ &\quad + b_nL_mQ - L_mb_nQ + Qb_nL_m - QL_mb_n \\ &= 0.\end{aligned}}$$

(2.3). Using the result of (2.2) and then the results of (2.1), we have

$$\boxed{\begin{aligned}\{[Q, L_m], b_n\} &= [\{b_n, Q\}, L_m] + \{[L_m, b_n], Q\} = [L_n, L_m] + \{(m-n)b_{m+n}, Q\} \\ &= (n-m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m,-n} + (m-n)L_{m+n} \\ &= \frac{c}{12}(n^3 - n)\delta_{m,-n}.\end{aligned}}$$

Thus, $\{[Q, L_m], b_n\}$ vanishes if $c = 0$.
(2.4). Q has ghost number 1 and L_m has ghost number 0. It then follows from the Jacobi identity (compute $[U, [Q, L_m]]$ with U the ghost operator) that $[Q, L_m]$ has ghost number 1. It follows that $[Q, L_m]$ will be a linear combination of terms that involve 1 more ghost than antighost. In particular, each term in $[Q, L_m]$ must contain at least one ghost, say c_k. Then, the anticommutator of this term with b_k will be a sum of terms, all of which vanish, except for the term containing $\{c_k, b_k\} = 1$ (this is because there can only be a single c_k in the term and the anticommutator of b_k with everything else vanishes). However, this anticommutator must vanish if $c = 0$ (from (2.3)), and so the coefficient of this term must vanish, and so $[Q, L_m] = 0$.
(2.5).

$$[b_m, \{Q, Q\}] = [\{b_m, Q\}, Q] - [Q, \{b_m, Q\}] = [L_m, Q] - [Q, L_m] = 2[L_m, Q].$$

By the result of (2.4), if the central charge vanishes, then so does $[L_m, Q]$, and hence $[b_m, Q^2]$ vanishes for every b_m, and so by the same logic as in (2.4), Q^2 itself must vanish.

Problem 3. An arbitrary state at level 0 is given by

$$|\phi\rangle := |p\rangle|-, \tilde{-}\rangle,$$

where $|-, \tilde{-}\rangle$ is the ghost vacuum[2], and $|0\rangle$ is the oscillator vacuum. The condition that $|\phi\rangle$ be physical, i.e. $Q|\phi\rangle = 0$, reads

$$\begin{aligned}0 = Q|\phi\rangle &= \left[c_0L_0^{(m)} + \tilde{c}_0\tilde{L}_0^{(m)} - (c_0 + \tilde{c}_0)\right]|p\rangle|-, \tilde{-}\rangle = \left(L_0^{(m)} - 1\right)|p\rangle|+, \tilde{-}\rangle + \left(\tilde{L_0}^{(m)} - 1\right)|p\rangle|-, \tilde{+}\rangle \\ &= \left(\frac{1}{8\pi\mu_0}p^2 - 1\right)|p\rangle|+, \tilde{-}\rangle + \left(\frac{1}{8\pi\mu}p^2 - 1\right)|p\rangle|-, \tilde{+}\rangle,\end{aligned} \tag{7}$$

[2]There are three other ghost vacuums that are excluded by the physical state conditions $b_0|\psi\rangle = 0 = \tilde{b}_0|\psi\rangle$.

Problem 3

- $N_L = N_R = 0$

We analyse left-movers first, as the hint says.

If $|\phi\rangle$ is physical, $Q|\phi\rangle = 0$.

We have $\begin{cases} b_n|\phi\rangle = 0, & n \geq 0 \\ c_n|\phi\rangle = 0, & n > 0 \end{cases}$

This implies $L_0|\phi\rangle = |\phi\rangle$, since $L_0 - 1 = \{Q, b_0\}$ and both b_0 and Q kill $|\phi\rangle$.

We consider the state $|\phi\rangle = |0,k\rangle$, where $-k^2 = -\frac{4}{\alpha'}$

(note $L_0 = \frac{\alpha'}{4}(p^\mu p_\mu + m^2)$ where $\frac{\alpha' m^2}{4} = \sum_{n=1}^{\infty} n\left(N_{b_n} + N_{c_n} + \sum_{\mu=1}^{\infty} N_{\mu n}\right) - 1$)

From the form of Q in problem 2, we see $Q|0,k\rangle = 0$.

Also, there are no Q-exact states at this level, so each invariant state $|0,k\rangle$ corresponds to a cohomology class.

The right-movers don't change the reasoning, with $\tilde{m}^2 = m^2 = \frac{-4}{\alpha'}$

so $|\phi\rangle = |0,0,k\rangle$, and we found the $\boxed{\text{closed string tachyon.}}$

- $N_L = N_R = 1$

Again, left-movers first:

Write the general $|\phi\rangle = (e\cdot\alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0,k\rangle$ with $26+1+1 = 28$ degrees of freedom.

But $0 = (L_0 - 1)|\phi\rangle$

$$= (L_0^{(m)} - 1)|\phi\rangle + \sum_{n>0} n(c_{-n}b_n + b_{-n}c_n)|\phi\rangle$$

$$= (L_0^{(m)} - 1)|\phi\rangle + (c_{-1}b_1 + b_{-1}c_1)(e\cdot\alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0,k\rangle$$

But $c_{-1}b_1|0,k\rangle = 0$ and $b_{-1}c_1|0,k\rangle = 0$

Also $\{b_1, c_{-1}\} = 1$ and $\{c_1, b_{-1}\} = 1$

So we get $0 = (L_0^{(m)} - 1)|\phi\rangle + (\beta b_{-1} + \gamma c_{-1})|0,k\rangle \qquad (*)$

Also, $0 = Q|\phi\rangle$

$$= \left[\sum_{n\in\mathbb{Z}} \left(c_n L_{-n}^{(m)} - \frac{1}{2}\sum_{m\in\mathbb{Z}} (m-n) : c_{-m}c_{-n}b_{m+n} :\right) - c_0\right]|\phi\rangle$$

The first sum will give only 3 surviving terms, since:

$$\sum_{n\in\mathbb{Z}} c_n L_{-n}^{(m)}(e\cdot\alpha_{-1})|0,k\rangle = \frac{1}{2}\sum_{n,m\in\mathbb{Z}} c_n : \alpha^{\mu}_{-n-m}\alpha_{\mu m} : (e\cdot\alpha_{-1})|0,k\rangle$$

$$= L_0^{(m)}(e\cdot\alpha_{-1})|0,k\rangle + \frac{1}{2}\sum_{m\in\mathbb{Z}} c_{-1} : \alpha^{\mu}_{1-m}\alpha_{\mu m} : e^{\nu}\alpha_{\nu -1}|0,k\rangle$$

$$+ \frac{1}{2}\sum_{m\in\mathbb{Z}} c_1 : \alpha^{\mu}_{-1-m}\alpha_{\mu m} : e^{\nu}\alpha_{\nu -1}|0,k\rangle$$

The above sums will collapse using $[\alpha_{\mu m}, \alpha_{\nu \ell}] = m\delta_{m,\ell}\eta_{\mu\nu}$,

so we get $0 = Q|\phi\rangle = c_0\left((L_0^{(m)} - 1)|\phi\rangle + (\beta b_{-1} + \gamma c_{-1})|0,k\rangle\right)$

$$+ \left(c_1\frac{2}{2}(\alpha_0\cdot\alpha_{-1}) + c_{-1}\frac{2}{2}(\alpha_0\cdot\alpha_1)\right)(e\cdot\alpha_{-1} + \beta b_{-1})|0,k\rangle$$

after similar reasoning on the ghosts for the second term.

Now use (*) to replace $(L_0^{(m)}-1)|\phi\rangle$ with a cancelling expression:

$$0 = c_0\overset{\to 0}{\left(-(\beta b_{-1}+\gamma c_{-1}) + (\beta b_{-1}+\gamma c_{-1})\right)}|\phi\rangle + c_{-1}(\alpha_0\cdot\alpha_1)(e\cdot\alpha_{-1}+\beta b_{-1})|0,k\rangle + c_1(\alpha_0\cdot\alpha_{-1})(e\cdot\alpha_{-1}+\beta b_{-1})|0,k\rangle$$

$$= (c_{-1}\,\alpha_0\cdot e + \alpha_0\cdot\alpha_{-1}\beta)|0,k\rangle$$

where I used $[\alpha_1^\mu, \alpha_{-1}^\nu] = \eta^{\mu\nu}$ in the first term and $\{c_1, b_{-1}\} = 1$ in the second.

But α_0 is proportional to p, so $c_{-1}k\cdot e + \beta k\cdot\alpha_{-1} = 0$

So we want:

$$\boxed{k\cdot e = 0 \text{ and } \beta = 0.}$$

There are 26 states left that are lineary independent. There are then 2 zero norm states, those created by c_{-1} and those created by $k\cdot\alpha_{-1}$.

So our Q-exact state is $Q|\chi\rangle = \left(c_{-1}(\alpha_0\cdot e') + \beta'(\alpha_0\cdot\alpha_{-1})\right)|0,k\rangle$ where e'_μ and β' are new constants, and we should exclude this. Since we found that $c_{-1}|0,k\rangle$ and $p\cdot\alpha_{-1}|0,k\rangle$ are spurious states we get $26-2=24$ states in the end. Including the right-movers, we get $\boxed{24^2 \text{ states for } N_L = N_R = 1}$.