```
<< Local`QFTToolKit2`
"Local notational definitions";
rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := |a|;
it[a_] := Style[a, Italic]
iD := it[D]
iA := it[A]
iI := it["I"]
C∞ := C"∞"
Bx_ := T[B, "d", {x}]
("∇"S)n_ := T["∇S", "d", {n}]

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
    ""];
selectDef[heads_, with_: {}, all_: Null] := tuRuleSelect[$defall][Flatten[{heads}]] //
     Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;

Clear[expandDC];
expandDC[sub_: {}, scalar_: {}] :=
  tuRepeat[{sub, tuOpDistribute[Dot], tuOpSimplify[Dot, scalar],
     tuOpDistribute[CircleTimes], tuOpSimplify[CircleTimes, scalar]}];
Clear[expandCom]

expandCom[subs_: {}][exp_] := Block[{tmp = exp},
    tmp = tmp //. tuCommutatorExpand // expandDC[];
    tmp = tmp /. toxDot //. Flatten[{subs}];
    tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
    tmp
  ];

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] :→ CommutatorP[b, a] /; OrderedQ[{b, a}],
    CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
    tt : T[γ, "u", {μ}].T[γ, "d", {5}] :> -Reverse[tt]
  };
$symmetries // ColumnBar

tt : g^μ_ν_ :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : F^μ_ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
[a_, b_]_ :→ -[b, a]_ /; OrderedQ[{b, a}]
{a_, b_}+ :→ {b, a}+ /; OrderedQ[{b, a}]
{γ5, γ^μ}+ → 0
tt : γ^μ.γ5 :→ -Reverse[tt]
```

```
$defall = {};(*accumulator for all definitions*)
PR[CO["We use equivalence symbol ≃
    for isomorphism, and Mod[] symbol for quotient group?"]
]
```

We use equivalence symbol ≃ for isomorphism, and Mod[] symbol for quotient group?

1204.0328: Particle Physics From Almost Commutative Spacetime

2. Almost Commutative Manifolds and Gauge Theories -- Canonical Triple

2.1 Spin manifolds in noncommutative geometry

```
PR["● M is 4-dim Reimannian spin manifold with canonical triple ",
 $ = {𝒜 → C∞[M], ℋ -> L²[M, S[CG["spinor"]]], 𝒟 -> slash[iD],
   (f ∈ 𝒜)[ψ ∈ ℋ] → {f[x].ψ[x], x ∈ M}
  }; $ // ColumnBar, accumDef[$];
 NL, "Using the spin connection: ", $connection = "∇ˢ"[CG["S[bundles]"]],
 Imply, "Dirac operator: ", $ = {slash[iD][ψ] → -I T[γ, "u", {μ}].tuDDown["∇ˢ"][ψ, μ]
   (*T["∇"ˢ,"d",{μ}][ψ]*), ψ ∈ Γ[M, S][CG["spinor"]],
   tuDDown["∇ˢ"][f . ψ, μ] → f .tuDDown["∇ˢ"][ψ, μ] + tuPartialD[f, μ] ψ,
   CommutatorM[slash[iD], f].ψ → -I T[γ, "u", {μ}]. tuPartialD[f, μ].ψ,
   f[CG["scalar"]]
  }; accumDef[$]; $ // ColumnBar,
 NL, "• Define a ℤ₂-grading(chirality): ",
 $s = {T[γ, "d", {5}][CG["ℤ₂-grading(chirality)"]] → Product[T[γ, "u", {μ}], {μ, 4}],
   T[γ, "d", {5}].T[γ, "d", {5}] → 1,
   ct[T[γ, "d", {5}]] -> T[γ, "d", {5}],
   CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
   T[γ, "d", {5}][L²[M, S]] -> L²[M, S]⁺ ⊕ L²[M, S]⁻}; $s // ColumnBar,
 accumDef[$s];
 NL, "• Define Charge conjugation: ", $ = {J_M[CG["Charge conjugation"]], J_M.J_M → -1,
   CommutatorM[J_M, slash[iD]] → 0, CommutatorM[J_M, T[γ, "d", {5}]] → 0,
   CommutatorM[J_M, a[CG[ℋ]]] → 0,
   CommutatorP[J_M, T[γ, "u", {μ}]] → 0
  };
 $ // ColumnForms, accumDef[$]
]
```

● M is 4-dim Reimannian spin manifold with canonical triple

$\mathcal{A} \to C^\infty[M]$
$\mathcal{H} \to L^2[M, S[\text{spinor}]]$
$\mathcal{D} \to \not{D}$
$(f \in \mathcal{A})[\psi \in \mathcal{H}] \to \{f[x].\psi[x], x \in M\}$

Using the spin connection: $\nabla^S[S[\text{bundles}]]$

⇒ Dirac operator:
$(\not{D})[\psi] \to -i\, \gamma^\mu . \nabla^S_{-\mu}[\psi]$
$\psi \in \Gamma[M, S][\text{spinor}]$
$\nabla^S_{-\mu}[f.\psi] \to f.\nabla^S_{-\mu}[\psi] + \psi\, \partial_{-\mu}[f]$
$[\not{D}, f]_- . \psi \to -i\, \gamma^\mu . \partial_{-\mu}[f].\psi$
$f[\text{scalar}]$

• Define a $\mathbb{Z}_2$-grading(chirality):
$\gamma_5[\mathbb{Z}_2\text{-grading(chirality)}] \to \gamma^1 \gamma^2 \gamma^3 \gamma^4$
$\gamma_5 . \gamma_5 \to 1$
$(\gamma_5)^\dagger \to \gamma_5$
$\{\gamma_5, \gamma^\mu\}_+ \to 0$
$\gamma_5[L^2[M, S]] \to L^2[M, S]^+ \oplus L^2[M, S]^-$

• Define Charge conjugation:
$J_M[\text{Charge conjugation}]$
$J_M . J_M \to -1$
$[J_M, \not{D}]_- \to 0$
$[J_M, \text{Tensor}[\gamma, |\text{Void}, |5]]_- \to 0$
$[J_M, a[\mathcal{H}]]_- \to 0$
$\{J_M, \text{Tensor}[\gamma, |\mu, |\text{Void}]\}_+ \to 0$

2.2 Almost-commutative manifolds

```
PR["● Almost commutative manifolds: ",
  M[CG["spin manifold"]]×F[CG["finite space internal degrees of freedom"]],
  imply, "gauge theory on M",
  NL,
  "• F→finite space triple: ", F → {𝒜F, ℋF, iDF},
  " where ", {𝒜F[CG[MN[ℂ]]], ℋF[CG["N-dim complex Hilbert space"]],
    iDF[CG["hermitian MN[ℂ]"]], MN[ℂ][CG["NxN matrix"]]} // ColumnBar,
  NL, "• ℋF is ℤ2 graded (even) if ∃ a grading operator: ",
  $ = {γF[CG["grading operator"]], ct[γF] → γF, γF.γF → 1F, γF[ℋF] -> ℋF⁺ ⊕ℋF⁻,
    {γF[ψ ∈ ℋF] → ±ψ},
    CommutatorM[γF, a ∈ AF] → 0,
    CommutatorP[γF, iDF] → 0
   }; $ // ColumnForms, accumDef[$]
 ];
```

● Almost commutative manifolds:
$M[\text{spin manifold}] \times F[\text{finite space internal degrees of freedom}]$ ⇒ gauge theory on M

• F→finite space triple: $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F\}$ where
$\mathcal{A}_F[M_N[\mathbb{C}]]$
$\mathcal{H}_F[\text{N-dim complex Hilbert space}]$
$\mathcal{D}_F[\text{hermitian } M_N[\mathbb{C}]]$
$M_N[\mathbb{C}][\text{NxN matrix}]$

• $\mathcal{H}_F$ is $\mathbb{Z}_2$ graded (even) if ∃ a grading operator:
$\gamma_F[\text{grading operator}]$
$(\gamma_F)^\dagger \to \gamma_F$
$\gamma_F . \gamma_F \to 1_F$
$\gamma_F[\mathcal{H}_F] \to (\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^-$
$|\gamma_F[\psi \in \mathcal{H}_F] \to \pm\psi$
$[\gamma_F, a \in A_F]_- \to 0$
$\{\gamma_F, \mathcal{D}_F\}_+ \to 0$

```
εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1,}}},
   {ε → table[[1, n+1]], ε' → table[[2, n+1]], ε'' → table[[3, n+1]]}
  ];
PR["● Def.2.4: Almost-commutative spin manifold: ",
 $ = M×F → {C"∞"[M, 𝒜F], L²[M, S]⊗ℋF, 𝒟 -> slash[iD]⊗1N + T[γ, "d", {5}]⊗iDF};
 accumDef[$];
 ColumnForms[$],
 NL, "with grading: ", γ -> T[γ, "d", {5}]⊗γF,
 NL, "•Distance: ", {d𝒟[x, y] → sup[‖a[x] - a[y]‖], a ∈ 𝒜 && ‖CommutatorM[iD, a]‖ ≤ 1},
 NL, "●Charge conjugation for F: even space F is real if ∃ ",
 $J = JF[ℋF] -> { JF. JF -> ε, JF.iDF -> ε'. iDF.JF, JF.γF -> ε''.γF.JF};
 ColumnForms[$J], accumDef[$J];
 NL, "where the routine εRule[KOdim_] is provided ",
 CR[" What is the meaning of ε's?"],
 NL, "•", $ = ForAll[{a, b}, a | b ∈ 𝒜F, {CommutatorM[a, rghtA[b]] → 0,
     rghtA[b] → JF.ct[b].ct[JF]}}][CG["Order-0 condition"]],
 accumDef[$];
 NL, "•", $ = ForAll[{a, b}, a | b ∈ 𝒜F, {CommutatorM[CommutatorM[iDF, a], rghtA[b]] → 0,
     rghtA[b] → JF.ct[b].ct[JF]}}][CG["Order-1 condition"]],
 accumDef[$]
]
```

● Def.2.4: Almost-commutative spin manifold:

$$M\times F\to \left|\begin{array}{l} C^{\infty}[M,\mathcal{A}_F] \\ L^2[M,S]\otimes\mathcal{H}_F \\ \mathcal{D}\to(\not{D})\otimes 1_N+\mathrm{Tensor}[\gamma,\ |\mathrm{Void}\ ,\ |5]\otimes D_F \end{array}\right.$$

with grading: $\gamma\to\gamma_5\otimes\gamma_F$

•Distance: $\{d_{\mathcal{D}}[x,y]\to \sup[\|a[x]-a[y]\|],\ a\in\mathcal{A}\ \&\&\ \|[D,a]_-\|\leq 1\}$

●Charge conjugation for F: even space F is real if ∃ $J_F[\mathcal{H}_F]\to\left|\begin{array}{l} J_F.J_F\to\varepsilon \\ J_F.D_F\to\varepsilon'.D_F.J_F \\ J_F.\gamma_F\to\varepsilon''.\gamma_F.J_F\end{array}\right.$

where the routine εRule[KOdim_] is provided What is the meaning of ε's?

•$(\forall_{\{a,b\},a|b\in\mathcal{A}_F}\{[a,b^o]_-\to 0,\ b^o\to J_F.b^\dagger.(J_F)^\dagger\})$[Order-0 condition]

•$(\forall_{\{a,b\},a|b\in\mathcal{A}_F}\{[[D_F,a]_-,b^o]_-\to 0,\ b^o\to J_F.b^\dagger.(J_F)^\dagger\})$[Order-1 condition]

```
PR["●Lemma2.7. Definition 2.5: ", $J[[2]],
  NL, "Where γF decomposes ", $h = ℋ → Table[ℋi,j, {i, 2}, {j, 2}];
  MatrixForms[$h],
  " into ", ℋ → ℋ⁺ ⊕ ℋ⁻ , " i.e. ", $gh = γF.ℋ → {{ℋ⁺, 0}, {0, ℋ⁻}};
  MatrixForms[$gh], accumDef[$gh];
  $gh0 = $gh /. {ℋ⁺ -> ℋ1,1, ℋ⁻ -> ℋ2,2};
  Yield, $gh1 = γF.{{a_, b_}, {c_, d_}} -> DiagonalMatrix[{a, d}];
  MatrixForms[$gh1], accumDef[$gh1];
  NL, "Represent ", $j = JF → Table[ji,j, {i, 2}, {j, 2}];
  MatrixForms[$j], " of the same dimensions.",
  NL, "•For: ",
  $JF = {JF → U.cc, U.ct[U] → 1N, U ∈ U[ℋ"±"], cc → Conjugate, CommutatorP[JF, I] → 0},
  accumDef[$JF];
  NL, "where: ",
  $cc = {ct[cc] → cc, Conjugate[cc] → cc, cc . cc → 1, cc.a_ :→ Conjugate[a].cc},
  accumDef[$cc];

  Imply, $0 = $ = JF.ct[JF],
  yield, $ = $0 → ($ // tuRepeat[{tuRule[$JF[[1 ;; 3]]], $cc}, tuOpSimplifyF[Dot]]);
```

```
    Framed[$],
    Yield, $ = $ /. ConjugateTranspose → SuperDagger /. toxDot /. $j /.
      SuperDagger[a_] :→ Map[Thread[SuperDagger[#]] &, Transpose[a]] /; MatrixQ[a];
    MatrixForms[$],
    Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
    Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
    Yield, $JJ = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
    FramedColumn[$JJ], CK
   ];
 PR[
  line, "•For ", $s = n → 0; Framed[$s],
  yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
  Column[$1],
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = # . ℋ & /@ $, "POFF",
  Yield, $ = $ /. $gh0;
  Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] → γF.xDot[a];
  Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$];
  Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$];
  Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
  Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
  Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
  NL, "•For ", $ = $1[[1]] /. 1 → 1N; Framed[$],
  Yield, $ = $ /. Dot → xDot /. $j,
  Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
  Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
  Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
  FramedColumn[$JJ1], CK,
  NL, "•Then we have: ", $ = {$JJ1, $JJ, $Jg}; ColumnForms[$],
  Yield, $ = $ /. j1,2 | j2,1 → 0 // ConjugateCTSimplify1[{}]; ColumnForms[$],
  Imply, {ct[j1,1] -> j1,1, ct[j2,2] -> j2,2} // FramedColumn
 ]
 PR[
  line, "•For ", $s = n → 2; Framed[$s],
  yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
  Column[$1],
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = # . ℋ & /@ $, "POFF",
  Yield, $ = $ /. $gh0 // tuDotSimplify[],
  Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
     γF.xDot[a], CK,
  Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
  Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
  Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
  Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
  Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
  NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
  Yield, $ = $ /. Dot → xDot /. $j,
  Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
  Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
  Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
  FramedColumn[$JJ1], CK,
  NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
  NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
  ColumnForms[$],
  Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
  Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
```

```
 Imply, {ct[j1,2] → -j2,1, ct[j2,1] → -j1,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 4; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
     γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,2 | j2,1 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply, {ct[j1,1] → -j1,1, ct[j2,2] → -j2,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 6; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
     γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. 1 → 1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply, {ct[j1,2] → j2,1, ct[j2,1] → j1,2} // FramedColumn
]
```

●Lemma2.7. Definition 2.5: $\{J_F.J_F \to \varepsilon,\ J_F.D_F \to \varepsilon'.D_F.J_F,\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F\}$

Where γ_F decomposes $\mathcal{H} \to \begin{pmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} \end{pmatrix}$ into $\mathcal{H} \to \mathcal{H}^+ \oplus \mathcal{H}^-$ i.e. $\gamma_F.\mathcal{H} \to \begin{pmatrix} \mathcal{H}^+ & 0 \\ 0 & \mathcal{H}^- \end{pmatrix}$

→ $\gamma_F.\begin{pmatrix} a\_ & b\_ \\ c\_ & d\_ \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Represent $J_F \to \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}$ of the same dimensions.

•For: $\{J_F \to U.cc,\ U.U^\dagger \to 1_N,\ U \in U[\mathcal{H}^\pm],\ cc \to \text{Conjugate},\ \{J_F,\ i\}_+ \to 0\}$
where: $\{cc^\dagger \to cc,\ cc^* \to cc,\ cc.cc \to 1,\ cc.(a\_) :\to a^*.cc\}$

⇒ $J_F.(J_F)^\dagger \longrightarrow \boxed{J_F.(J_F)^\dagger \to 1_N}$

→ $\text{xDot}[\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}, \begin{pmatrix} (j_{1,1})^\dagger & (j_{2,1})^\dagger \\ (j_{1,2})^\dagger & (j_{2,2})^\dagger \end{pmatrix}] \to 1_N$

→ $\begin{pmatrix} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger & j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger & j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \end{pmatrix} \to 1_N$

→ $\{\{j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger,\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger\},$
$\{j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger,\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger\}\} \to \{\{1_{N^+},\ 0\},\ \{0,\ 1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \to 0 \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger \to 0 \\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \to 1_{N^-} \end{array}}$ ←CHECK

•For $\boxed{n \to 0}$ ⟶ $J_F \cdot J_F \to 1$
$J_F \cdot \gamma_F \to \gamma_F \cdot J_F$

→ $\boxed{J_F \cdot \gamma_F \to \gamma_F \cdot J_F}$ ⟶ $J_F \cdot \gamma_F \cdot \mathcal{H} \to \gamma_F \cdot J_F \cdot \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} + j_{1,2} \cdot \mathcal{H}_{2,1} & j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 & j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,1} \cdot \mathcal{H}_{1,2} + j_{2,2} \cdot \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} + j_{1,2} \cdot \mathcal{H}_{2,1} \\ j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,1} \cdot \mathcal{H}_{1,2} + j_{2,2} \cdot \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F \cdot J_F \to 1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → 1_N

→ $\begin{pmatrix} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} & j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} & j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \end{pmatrix} \to 1_N$

→ {{$j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1}$, $j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2}$}, {$j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1}$, $j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2}$}} →
{{1_{N^+}, 0}, {0, 1_{N^-}}}

→ $\boxed{\begin{array}{l} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} \to 1_{N^+} \\ j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \to 0 \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} \to 0 \\ j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \to 1_{N^-} \end{array}}$ ⟵CHECK

•Then we have:
$$\begin{array}{l} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} \to 1_{N^+} \\ j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \to 0 \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} \to 0 \\ j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \to 1_{N^-} \\ j_{1,1} \cdot (j_{1,1})^\dagger + j_{1,2} \cdot (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} \cdot (j_{2,1})^\dagger + j_{1,2} \cdot (j_{2,2})^\dagger \to 0 \\ j_{2,1} \cdot (j_{1,1})^\dagger + j_{2,2} \cdot (j_{1,2})^\dagger \to 0 \\ j_{2,1} \cdot (j_{2,1})^\dagger + j_{2,2} \cdot (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} + j_{1,2} \cdot \mathcal{H}_{2,1} \\ j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,1} \cdot \mathcal{H}_{1,2} + j_{2,2} \cdot \mathcal{H}_{2,2} \end{array}$$

→
$$\begin{array}{l} j_{1,1} \cdot j_{1,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} \cdot j_{2,2} \to 1_{N^-} \\ j_{1,1} \cdot (j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} \cdot (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,2} \cdot \mathcal{H}_{2,2} \end{array}$$

⇒ $\boxed{\begin{array}{l} (j_{1,1})^\dagger \to j_{1,1} \\ (j_{2,2})^\dagger \to j_{2,2} \end{array}}$

•For $\boxed{n \to 2}$ ⟶ $J_F . J_F \to -1$
$J_F . \gamma_F \to -\gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to -\gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to (-\gamma_F . J_F) . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to -1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → -1_N

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to -1_N$

→ $\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1},\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\},\ \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1},\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to \{\{-1_{N^+},\ 0\},\ \{0,\ -1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}}$ ⟵CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions:
$$\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,2} . \mathcal{H}_{2,2} \end{array}$$

⇒ $\boxed{j_{1,1} \mid j_{2,2} \to 0}$

→
$$\begin{array}{l} j_{1,2} . j_{2,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . j_{1,2} \to -1_{N^-} \\ j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . (j_{2,1})^\dagger \to 1_{N^-} \\ 0 \to 0 \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{array}$$

⇒ $\boxed{\begin{array}{l} (j_{1,2})^\dagger \to -j_{2,1} \\ (j_{2,1})^\dagger \to -j_{1,2} \end{array}}$

•For $\boxed{n \to 4}$ ⟶ $J_F . J_F \to -1$
$J_F . \gamma_F \to \gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to \gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to \gamma_F . J_F . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to -1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → -1_N

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to -1_N$

→ {{$j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1}$, $j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}$}, {$j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1}$, $j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}$}} →
{{-1_{N^+}, 0}, {0, -1_{N^-}}}

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}}$ ⟵CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions:
$$\left\|\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{array}\right.$$

⇒ $\boxed{j_{1,2} \mid j_{2,1} \to 0}$

→
$$\left\|\begin{array}{l} j_{1,1} . j_{1,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . j_{2,2} \to -1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{array}\right.$$

⇒ $\boxed{\begin{array}{l} (j_{1,1})^\dagger \to -j_{1,1} \\ (j_{2,2})^\dagger \to -j_{2,2} \end{array}}$

•For $\boxed{n \to 6}$ ⟶ $J_F . J_F \to 1$
$J_F . \gamma_F \to -\gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to -\gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to (-\gamma_F . J_F) . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to 1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → 1_N

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to 1_N$

→ $\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1},\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\},\ \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1},\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to$
$\{\{1_{N^+},\ 0\},\ \{0,\ 1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \end{array}}$ ⟵CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions:
$$\left\| \begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \\ j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \\ j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,2} . \mathcal{H}_{2,2} \end{array} \right.$$

⇒ $\boxed{j_{1,1} \mid j_{2,2} \to 0}$

→
$$\left\| \begin{array}{l} j_{1,2} . j_{2,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . j_{1,2} \to 1_{N^-} \\ j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . (j_{2,1})^\dagger \to 1_{N^-} \\ 0 \to 0 \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{array} \right.$$

⇒ $\boxed{\begin{array}{l} (j_{1,2})^\dagger \to j_{2,1} \\ (j_{2,1})^\dagger \to j_{1,2} \end{array}}$

2.3 Subgroups and subalgebras

2.3.1 Commutative Subalgebras

```
PR["● Define subalgebra of 𝒜: ",
 $sAt = 𝒜̃J → {a ∈ 𝒜, a.J → J.ct[a], rghtA[a] → a}, accumDef[$sAt];
 NL, "•Unitary group: ", U[𝒜] → {u ∈ 𝒜, u.ct[u] | ct[u].u → 1N},
 Imply, ForAll[x ∈ M, u[x].ct[u[x]] | ct[u[x]].u[x] → 1N],
 Imply, u ∈ U[𝒜] ⇔ u[x] ∈ U[𝒜F],
 NL, "•Lie algebra: ", u[𝒜] → {X ∈ 𝒜, ct[X] → -X} → C"∞"[M, u[𝒜F]],
 NL, "•Special unitary group: ", SU[𝒜F] → {υ ∈ U[𝒜F], Det[u] → 1},
 NL, "•Lie algebra SU[𝒜F]: ", su[𝒜F] → {X ∈ 𝒜F, ct[X] → -X, Tr[X] → 0}
]
PR["● 2.3.3 Adjoint action. space: ",
 $F = F → Table[Subscript[i, F], {i, {𝒜, ℋ, iD, γ, J}}],
 NL, "Define: for ", ξ ∈ $F[[2, 2]],
 Yield, $ = {Ad[U[𝒜F]] → Endo[$F[[2, 2]]], ad[u[$F[[2, 1]]]] → Endo[$F[[2, 2]]]};
 Column[$],
 yield, $ = {Ad[u][ξ] → u.ξ.ct[u] → u.rghtA[ct[u]].ξ,
   ad[A][ξ] → A.ξ - ξ.A → (A - rghtA[A]).ξ}; accumDef[$]; $ // ColumnBar,
 NL, "Since ", $s = selectDef[rghtA[b]] // tuAddPatternVariable[b],
 Yield, $ = $ /. $s;
 $ = {$[[1, 1]] -> $[[1, 2, 2]], $[[2, 1]] -> $[[2, 2, 2]]};
 $0 = $ = $ /. ξ → 1 /. fn_[1] → fn /. tuOpSimplify[Dot];
 accumDef[$]; $ // ColumnBar,
 NL, "For ", $s = ct[A] → -A,
 Yield, $ = $[[2]] /. $s /. tuOpSimplify[Dot],
 NL, "For ",
 $s = {A → B.ii, CO["Track ii→I since ", selectDef[CommutatorP[JF, _]]], ct[B] → B,
   ct[ii] → -ii, (tuCommutatorSolve[1][selectDef[CommutatorP[JF, _]]] /.  I → ii)}, CK,
 Yield, $ = $0[[2]] //. tuRule[$s] //. tuOpSimplify[Dot],
 Yield,
 $ = $ /. tuRule[$s] //. tuOpSimplify[Dot, {ii}] /. tuOpSimplify[ad, {ii}] /. ii → I;
 Yield, $ = {tuRuleSolve[$, ad[B]] // First, B[CG["Hermitian"]]};
 $ // Framed, accumDef[$]
]
```

● Define subalgebra of $\mathcal{A}$: $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A},\ a.J \to J.a^\dagger,\ a^o \to a\}$
•Unitary group: $U[\mathcal{A}] \to \{u \in \mathcal{A},\ u.u^\dagger \mid u^\dagger.u \to 1_N\}$
⇒ $\forall_{x\in M} (u[x].ct[u[x]] \mid ct[u[x]].u[x] \to 1_N)$
⇒ $u \in U[\mathcal{A}] \Leftrightarrow u[x] \in U[\mathcal{A}_F]$
•Lie algebra: $u[\mathcal{A}] \to \{X \in \mathcal{A},\ X^\dagger \to -X\} \to C^\infty[M, u[\mathcal{A}_F]]$
•Special unitary group: $SU[\mathcal{A}_F] \to \{\upsilon \in U[\mathcal{A}_F],\ Det[u] \to 1\}$
•Lie algebra SU[$\mathcal{A}_F$]: $su[\mathcal{A}_F] \to \{X \in \mathcal{A}_F,\ X^\dagger \to -X,\ Tr[X] \to 0\}$

● 2.3.3 Adjoint action. space: $F \to \{\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F\}$
Define: for $\xi \in \mathcal{H}_F$

$\to \begin{array}{l} Ad[U[\mathcal{A}_F]] \to Endo[\mathcal{H}_F] \\ ad[u[\mathcal{A}_F]] \to Endo[\mathcal{H}_F] \end{array} \longrightarrow \left| \begin{array}{l} Ad[u][\xi] \to u.\xi.u^\dagger \to u.u^{\dagger o}.\xi \\ ad[A][\xi] \to A.\xi - \xi.A \to (A - A^o).\xi \end{array} \right.$

Since $b\_^o \to J_F.b^\dagger.(J_F)^\dagger$

$\to \left| \begin{array}{l} Ad[u] \to u.J_F.u.(J_F)^\dagger \\ ad[A] \to A - J_F.A^\dagger.(J_F)^\dagger \end{array} \right.$

For $A^\dagger \to -A$

$\to ad[A] \to A + J_F.A.(J_F)^\dagger$

For $\{A \to B.ii$, Track ii→I since , $\{J_F, i\}_+ \to 0$, $B^\dagger \to B$, $ii^\dagger \to -ii$, $J_F.ii \to -ii.J_F\}$⟵CHECK

$\to ad[B.ii] \to B.ii + J_F.ii.B.(J_F)^\dagger$

→

$\to \boxed{\{ad[B] \to B - J_F.B.(J_F)^\dagger, B[\text{Hermitian}]\}}$

2.4 Gauge Symmetry

```
PR["●2.4.1 Diffeomorphisms and automorphisms. ",
  {ϕ[M] → M, CG["diffeomorphism of C∞[M]"]},
  NL, "• define automorphism: ",
  {α_ϕ[f] → f.inv[ϕ], α_ϕ[CG["algebra"]], f ∈ (C^"∞") [M]},
  NL, "• define diffeomorphism: ", Diff[M×F] → Aut[(C^"∞") [M, 𝒜F]],
  Imply, $ = {a ∈ (C^"∞") [M, 𝒜F], α_ϕ[a] → a∘inv[ϕ], α_ϕ[a][x] → a[inv[ϕ][x]]
   }; accumDef[$]; $ // ColumnBar,
  NL, "• Inner automorphism, Inn[], is such a case: ",
  $ = Inn[𝒜] -> {u ∈ (C∞) [M, U[𝒜F]], α_u[a] → u.a.ct[u], α_u[a][x] → u[x].a[x].ct[u][x]
    }; accumDef[$]; $ // ColumnForms,

  NL, "• Define outer automorphism: ", Out[𝒜] → Mod[Aut[𝒜], Inn[𝒜]],
  NL, "• Define kernel: ", $ = Ker[ϕ] → {ForAll[a ∈ 𝒜, u.a.ct[u] → a],
     ϕ[U[𝒜]][CG["surjective"]] → Inn[𝒜],
     ϕ[u] → α_u, u ∈ U[𝒜]
    }; accumDef[$]; $ // ColumnForms,
  NL, "For ", {(Z ⊂ U[𝒜] && CommutatorM[Z, 𝒜] → 0) ⇒ (Ker[ϕ] → Z),
    Inn[𝒜] ≈ Mod[U[𝒜], Z]
   } // ColumnBar
 ];
```

●2.4.1 Diffeomorphisms and automorphisms. $\{\phi[M] \to M$, diffeomorphism of $C^\infty[M]\}$
• define automorphism: $\{\alpha_\phi[f] \to f.\phi^{-1}, \alpha_\phi[\text{algebra}], f \in C^\infty[M]\}$
• define diffeomorphism: $Diff[M \times F] \to Aut[C^\infty[M, \mathcal{A}_F]]$

$\Rightarrow \left| \begin{array}{l} a \in C^\infty[M, \mathcal{A}_F] \\ \alpha_\phi[a] \to a \circ \phi^{-1} \\ \alpha_\phi[a][x] \to a[\phi^{-1}[x]] \end{array} \right.$

• Inner automorphism, Inn[], is such a case: $Inn[\mathcal{A}] \to \left| \begin{array}{l} u \in C^\infty[M, U[\mathcal{A}_F]] \\ \alpha_u[a] \to u.a.u^\dagger \\ \alpha_u[a][x] \to u[x].a[x].u^\dagger[x] \end{array} \right.$

• Define outer automorphism: $Out[\mathcal{A}] \to Mod[Aut[\mathcal{A}], Inn[\mathcal{A}]]$

• Define kernel: $Ker[\phi] \to \left| \begin{array}{l} \forall_{a \in \mathcal{A}} (u.a.ct[u] \to a) \\ \phi[U[\mathcal{A}]][\text{surjective}] \to Inn[\mathcal{A}] \\ \phi[u] \to \alpha_u \\ u \in U[\mathcal{A}] \end{array} \right.$

For $\left| \begin{array}{l} (Z \subset U[\mathcal{A}] \&\& [Z, \mathcal{A}]_- \to 0) \Rightarrow (Ker[\phi] \to Z) \\ Inn[\mathcal{A}] \approx Mod[U[\mathcal{A}], Z] \end{array} \right.$

```
PR["● 2.4.2: Unitary transform. Given a triple: ", {𝒜, ℋ, 𝒟},
  " the representation(π) of 𝒜 on ℋ: ", π[a][ℋ] ,
  NL, "•Define unitary transform: ",
  $0 = U -> {U[ℋ] → ℋ, {𝒜, ℋ, 𝒟} -> {𝒜, ℋ, U. 𝒟.ct[U]}, (a ∈ 𝒜) -> U. π[a].ct[U],
     γ -> U. γ.ct[U][CG["ACM even"]], J -> U. J.ct[U][CG["ACM real"]]};
  ColumnForms[$0],
  NL, "•EG1. ", $s = {U → π[u], u ∈ U[𝒜][CG["ACM real even"]]},
  Yield, $ = U.π[a].ct[U],
  Yield, $ = $ /. tuRule[$s],
  Yield, $ = $ /. ct[π[a_]] → π[ct[a]] //. π[a_].π[b_] → π[a.b],
  Yield, $ = $ /. (selectDef[αu[_]] // Reverse // tuAddPatternVariable[{a, u}]),
  CG[back, "Inn"],

  NL, "•EG2. (adjoint action) ", $s = {U → Ad[u], U → u.J.u.ct[J]},
  NL, "Charge conjugation rule: ", selectDef[JF.γF],
  imply, CommutatorM[U, γ] → 0, imply, "γ unchanged",
  Yield, $ = U.π[a].ct[U],
  Yield, $ = $ /. ($s[[2]] /. u → π[u]) // tuConjugateTransposeSimplify[],
  NL, "Order-0 condition: ", $s = selectDef[CommutatorM[a, _]] /. selectDef[rghtA[b]],
  Imply, $ = $ /. aa_ . bb_ . π[a] → aa . π[a].bb, (*could be more specific*)
  Yield, $ = $ // tuRepeat[{ct[J_] . J_ → 1, J_ .ct[J_] → 1}, tuDotSimplify[]],
  Yield, $ = $ /. π[a_].π[b_]. ct[π[c_]] → π[a. b. ct[c]],
  Yield, $ = $ /. u_ .a_ .ct[u_] → αu[a]
 ];
```

● 2.4.2: Unitary transform. Given a triple:
{𝒜, ℋ, 𝒟} the representation(π) of 𝒜 on ℋ: π[a][ℋ]

•Define unitary transform: $U \to \left|\begin{array}{l} U[\mathcal{H}] \to \mathcal{H} \\ \left|\begin{array}{l}\mathcal{A}\\ \mathcal{H}\\ \mathcal{D}\end{array}\right. \to \left|\begin{array}{l}\mathcal{A}\\ \mathcal{H}\\ U.\mathcal{D}.U^\dagger\end{array}\right. \\ a \in \mathcal{A} \to U.\pi[a].U^\dagger \\ \gamma \to U.\gamma.U^\dagger[\text{ACM even}] \\ J \to U.J.U^\dagger[\text{ACM real}] \end{array}\right.$

•EG1. $\{U \to \pi[u],\ u \in U[\mathcal{A}][\text{ACM real even}]\}$
→ $U.\pi[a].U^\dagger$
→ $\pi[u].\pi[a].\pi[u]^\dagger$
→ $\pi[u.a.u^\dagger]$
→ $\pi[\alpha_u[a]]$ ⟵Inn
•EG2. (adjoint action) $\{U \to Ad[u],\ U \to u.J.u.J^\dagger\}$
Charge conjugation rule: $J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F$ ⇒ $[U, \gamma]_- \to 0$ ⇒ γ unchanged
→ $U.\pi[a].U^\dagger$
→ $\pi[u].J.\pi[u].J^\dagger.\pi[a].J.\pi[u]^\dagger.J^\dagger.\pi[u]^\dagger$
Order-0 condition: $[a, J_F.b^\dagger.(J_F)^\dagger]_- \to 0$
⇒ $\pi[u].\pi[a].J.\pi[u].J^\dagger.J.\pi[u]^\dagger.J^\dagger.\pi[u]^\dagger$
→ $\pi[u].\pi[a].\pi[u]^\dagger$
→ $\pi[u.a.u^\dagger]$
→ $\pi[\alpha_u[a]]$

```
PR["•2.4.3: Define Gauge group: ", 𝒢[M×F] → {u.J.u.ct[J], u ∈ U[𝒜]},
 NL, "Consider: ", {Ad[U[𝒜]] → 𝒢[M×F], Ad[u] → u.rghtA[ct[u]]} // ColumnBar,
 Imply, $ = {Ker[Ad] → {u ∈ U[𝒜], (u.J.u.ct[J] → 1) ⇒ (u.J → J.ct[u])}, Ker[Ad] → U[𝒜̃J]
  }; $ // ColumnBar,
 NL, "•Define finite gauge group for finite space F: ",
 𝒢[F] → {ℋF → U[(𝒜̃F)JF], hF → u[(𝒜̃F)JF]} // ColumnForms,
 NL, "• Proposition 2.12.  The ACM gauge group 𝒢[M×F] is ", yield,
 $ = {C∞[M, 𝒢[F]], 𝒢[F] → Mod[U[𝒜F], HF][CG["local group of the finite space"]]};
 $ // ColumnBar,
 NL, "• Unimodularity. ", "Two possibilities ",
 $ = {{𝒜F[CG["Complex algebra with identity 𝕀"]] → (ℂ𝕀 ⊂ 𝒜̃FJF),
    imply, U[1] ⊂ HF, U[1][CG["normal subgroup"]]},
   {𝒜F[CG["Real algebra with identity 𝕀"]] → (ℝ𝕀 ⊂ 𝒜̃FJF), imply,
    {1, -1} ⊂ HF, {1, -1}[CG["normal subgroup"]]}};
 $ // ColumnBar,
 NL, "• Proposition 2.13. ",
 e213 = $ = {𝒢[F][CG["gauge group"]] ≃ Mod[SU[𝒜F[CG["complex algebra"]]], SHF],
    SHF → {g ∈ HF, Det[g] → 1}};
 $ // ColumnBar ,
 NL, "●Proof 2.13: ",
 NL, "• define UH-equivalence: ",
 $su = u_ ⇔ u_ . h_ -> ForAll[h, h ∈ HF, (u | u.h ∈ U[𝒜F])],
 Yield, $G = {𝒢[F] ≃ Mod[U[𝒜F], HF]} → {u ⇔ u.h},
 Yield, $ = $G /. $su,
 NL, "• define SUSH equivalence: ",
 $su = su_ ⇔ su_ . g_ -> ForAll[g, g ∈ SHF, (su | su.g ∈ SU[𝒜F])],
 Yield, $SU = {Mod[SU[𝒜F], SHF]} → {su ⇔ su.g},
 Yield, $0 = $SU /. $su,
 NL, "(1)• Is SHF a normal subgroup of SU[𝒜F]?: ",
 $ = ForAll[{g, v}, g ∈ SHF && v ∈ SU[𝒜F], (v.g. inv[v]) ∈ SHF],

 NL, "•Evaluate: ", $ = Det[$0 = v.g. inv[v] ∈ HF],
 yield, $ = $ /. a_ ∈ b_ → a,
 yield, $ = Thread[$, Dot] /. Det[inv[a_]] → 1 / Det[a] /. Dot → Times,
 NL, "Since: ", g ∈ SHF,
 imply, $s = Det[g] → 1,
 imply, $0 ∈ SHF,
 imply, "SHF Normal Subgroup of SU[𝒜F]" // Framed
]
```

•2.4.3: Define Gauge group: $\mathcal{G}[M\times F]\to\{u.J.u.J^{\dagger},\ u\in U[\mathcal{A}]\}$

Consider: $\left|\begin{array}{l}\mathrm{Ad}[U[\mathcal{A}]]\to\mathcal{G}[M\times F]\\ \mathrm{Ad}[u]\to u.u^{\dagger\circ}\end{array}\right.$

$\Rightarrow\left|\begin{array}{l}\mathrm{Ker}[\mathrm{Ad}]\to\{u\in U[\mathcal{A}],\ (u.J.u.J^{\dagger}\to 1)\Rightarrow(u.J\to J.u^{\dagger})\}\\ \mathrm{Ker}[\mathrm{Ad}]\to U[\tilde{\mathcal{A}}_J]\end{array}\right.$

•Define finite gauge group for finite space F: $\mathcal{G}[F]\to\left|\begin{array}{l}\mathcal{H}_F\to U[\tilde{\mathcal{A}}_{F\,J_F}]\\ h_F\to u[\tilde{\mathcal{A}}_{F\,J_F}]\end{array}\right.$

• Proposition 2.12. The ACM gauge group $\mathcal{G}[M\times F]$ is

$\longrightarrow\left|\begin{array}{l}C^{\infty}[M,\ \mathcal{G}[F]]\\ \mathcal{G}[F]\to\mathrm{Mod}[U[\mathcal{A}_F],\ H_F][\text{local group of the finite space}]\end{array}\right.$

• Unimodularity. Two possibilities $\left|\begin{array}{l}\{\mathcal{A}_F[\text{Complex algebra with identity }\mathbb{I}]\to\mathbb{C}\mathbb{I}\subset\tilde{\mathcal{A}}_{F\,J_F},\ \Rightarrow\\ \{\mathcal{A}_F[\text{Real algebra with identity }\mathbb{I}]\to\mathbb{R}\mathbb{I}\subset\tilde{\mathcal{A}}_{F\,J_F},\ \Rightarrow\ ,\ \{\end{array}\right.$

• Proposition 2.13. $\left|\begin{array}{l}\mathcal{G}[F][\text{gauge group}]\simeq\mathrm{Mod}[SU[\mathcal{A}_F[\text{complex algebra}]],\ SH_F]\\ SH_F\to\{g\in H_F,\ \mathrm{Det}[g]\to 1\}\end{array}\right.$

●Proof 2.13:

• define UH-equivalence: $(u\_).(h\_)\Leftrightarrow u\_\to\forall_{h,h\in H_F}(u\mid u.h\in U[\mathcal{A}_F])$

→ $\{\mathcal{G}[F]\simeq\mathrm{Mod}[U[\mathcal{A}_F],\ H_F]\}\to\{u\Leftrightarrow u.h\}$

→ $\{\mathcal{G}[F]\simeq\mathrm{Mod}[U[\mathcal{A}_F],\ H_F]\}\to\{\forall_{h,h\in H_F}(u\mid u.h\in U[\mathcal{A}_F])\}$

• define SUSH equivalence: $(su\_).(g\_)\Leftrightarrow su\_\to\forall_{g,g\in SH_F}(su\mid su.g\in SU[\mathcal{A}_F])$

→ $\{\mathrm{Mod}[SU[\mathcal{A}_F],\ SH_F]\}\to\{su\Leftrightarrow su.g\}$

→ $\{\mathrm{Mod}[SU[\mathcal{A}_F],\ SH_F]\}\to\{\forall_{g,g\in SH_F}(su\mid su.g\in SU[\mathcal{A}_F])\}$

(1)• Is SH_F a normal subgroup of $SU[\mathcal{A}_F]$?: $\forall_{\{g,v\},g\in SH_F\&\&v\in SU[\mathcal{A}_F]}\ v.g.v^{-1}\in SH_F$

•Evaluate: $\mathrm{Det}[v.g.v^{-1}\in H_F]\longrightarrow\mathrm{Det}[v.g.v^{-1}]\longrightarrow\mathrm{Det}[g]$

Since: $g\in SH_F\Rightarrow\mathrm{Det}[g]\to 1\Rightarrow(v.g.v^{-1}\in H_F)\in SH_F\Rightarrow\boxed{SH_F\ \text{Normal Subgroup of}\ SU[\mathcal{A}_F]}$

```
PR["•Property of unitary matrix u: ",
 {Abs[Det[u]] → 1,
   {"Eigenvalues of u", λu ∈ U[1],
    Exists[{u, u'}, u ∈ U[𝒜F] && u' ∈ U[N], u'.u.ct[u'] -> λu 1N]}} // FramedColumn,
 Imply, Exists[λu, λu ∈ U[1] && λu^N → Det[u] && N → dim[ℋF] && U[1] ≤ U[𝒜F]],
 Imply, $ = ($0 = inv[λu].u ∈ SU[𝒜F]) ⟸ {$ = Det[$0[[1]]], $ = Thread[$, Dot],
     $ = $ /. Det[inv[λu]] → λu^(-N), $ = $ /. Det[u] → λu^N, SU[𝒜F]} // ColumnForms,

 NL, "■ define group homomorphism from UH->SUSH: ",
 $ph = {φ[$G[[1, 1]]] → Mod[SU[𝒜F], SHF], φ[{u}] → {inv[λu].u}};
 Column[$ph],
 NL, "□ Check if φ is independent of representative ", λu,
 NL, "•suppose: ", Implies[Exists[λu', (λu')^N → Det[u]],
  inv[λu].λu' ∈ μN[CG["multiplicative group Nth root of unity"]]],
 NL, "•", Implies[Implies[Implies[U[1] ≤ HF, μN ≤ SHF], {inv[λu].u} == {inv[λu'].u}],
  Framed[φ[CG["independent of λu"]]]],
 NL, "□ Check if φ is independent of representative ", u ∈ U[𝒜F],
 NL, "?: ", $0 = $ = ForAll[u, u ∈ HF, φ[{u}]],
 Yield, $ = $ /. $ph, "POFF",
 NL, "For ", $s = (g -> inv[λh].h) ∈ SHF,
 Yield, $ = $ /. dd : HoldPattern[Dot[a_]] → dd .g,
 Yield, $ = $ /. $s[[1]],
 Yield, $ = $ /. dd : HoldPattern[Dot[_]] :→ tuDotTermLeft[inv[_], {inv[λu]}][dd],
 Yield, $ = $ /. inv[a_]. inv[b_] → inv[b.a],
 Yield, $[[3]] = φ[{u.h}]; $, "PONdd",
 yield, $[[3]] == $0[[3]] // Framed,
 NL, "•Suppose ", $ = ForAll[{u1, u2}, {u1 | u2 ∈ U[𝒜F]}, φ[{u1}] == φ[{u2}]],
 Yield, $ = $ /. φ[{a_}] → {inv[λa].a.ga} /. gu1 → 1 /. gu2 → (g ∈ SHF),
 Yield, $ = $ /. HoldPattern[Dot[a_]] → Dot[λu1, a],
 Yield, $ = $ /. a_ . inv[a_] → 1 /. g ∈ SHF → g // tuDotSimplify[],
 Yield, $3 = $ = $[[3]],
 " for some: ", $ = $[[2]] // First // DeleteCases[#, u2] & // tuDotSimplify[];
 $ ∈ SHF,
 imply, "φ is injective.",
 Imply, $ = $3 /. Thread[Apply[List, $] → 1] // tuDotSimplify[]; Framed[$]
]
```

•Property of unitary matrix u:

$\boxed{\begin{array}{l}\text{Abs[Det[u]]} \to 1 \\ \{\text{Eigenvalues of u, } \lambda_u \in \text{U[1]}, \exists_{\{u,u'\},u\in U[\mathcal{A}_F]\&\&u'\in U[N]} (u'.u.(u')^\dagger \to 1_N \lambda_u)\}\end{array}}$

⇒ $\exists_{\lambda_u} (\lambda_u \in \text{U[1]} \;\&\&\; \lambda_u^N \to \text{Det[u]} \;\&\&\; N \to \dim[\mathcal{H}_F] \;\&\&\; \text{U[1]} \leq \text{U}[\mathcal{A}_F])$

⇒ $(\lambda_u^{-1}.u \in \text{SU}[\mathcal{A}_F]) \Longleftarrow \left| \begin{array}{l}\text{Det}[\lambda_u^{-1}.u] \\ \text{Det}[\lambda_u^{-1}].\text{Det[u]} \\ \lambda_u^{-N}.\text{Det[u]} \\ \lambda_u^{-N}.\lambda_u^N \\ \text{SU}[\mathcal{A}_F]\end{array}\right.$

■ define group homomorphism from UH->SUSH:

$\varphi[\mathcal{G}[F] \simeq \text{Mod}[\text{U}[\mathcal{A}_F], H_F]] \to \text{Mod}[\text{SU}[\mathcal{A}_F], \text{SH}_F]$

$\varphi[\{u\}] \to \{\lambda_u^{-1}.u\}$

□ Check if φ is independent of representative λ_u

•suppose: $\exists_{\lambda_u'} ((\lambda_u')^N \to \text{Det[u]}) \Rightarrow \lambda_u^{-1}.\lambda_u' \in \mu_N$[multiplicative group Nth root of unity]

•$((\text{U[1]} \leq H_F \Rightarrow \mu_N \leq \text{SH}_F) \Rightarrow \{\lambda_u^{-1}.u\} == \{(\lambda_u')^{-1}.u\}) \Rightarrow$ $\boxed{\varphi[\text{independent of } \lambda_u]}$

□ Check if φ is independent of representative $u \in \text{U}[\mathcal{A}_F]$

?: $\forall_{u,u\in H_F} \varphi[\{u\}]$

→ $\forall_{u,u\in H_F} \{\lambda_u^{-1}.u\}$

.... ⟶ $\boxed{\varphi[\{u.h\}] == \varphi[\{u\}]}$

•Suppose $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \varphi[\{u_1\}] == \varphi[\{u_2\}]$

→ $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \{\lambda_{u_1}^{-1}.u_1.1\} == \{\lambda_{u_2}^{-1}.u_2.(g \in \text{SH}_F)\}$

→ $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \{\lambda_{u_1}.\lambda_{u_1}^{-1}.u_1.1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.(g \in \text{SH}_F)\}$

→ $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \{u_1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$

→ $\{u_1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$ for some: $\lambda_{u_1}.\lambda_{u_2}^{-1}.g \in \text{SH}_F$ ⇒ φ is injective.

⇒ $\boxed{\{u_1\} == \{u_2\}}$

```
PR["●2.4.4 Full symmetry group. ",
 NL, "• For two groups {H,N} the action ",
 H[N], " is given by a homomorphism θ: ", θ[H] → Aut[N],
 NL, "• Define semi-direct product ", $ = N ⊳ H → {{n, h}, n ∈ N && h ∈ H,
    $sdg = {
      {n_, h_ } · {n1_, h1_} → {n . θ[h]. n1, h . h1},
      1sdp -> {1N, 1H}[CG["unit"]],
      invSDG[{n_, h_}][CG["inverse"]] → {θ[inv[h]].inv[n], inv[h]
       }}}; $ // ColumnForms,
 "POFF",
 NL, "•Check inverse: ",
 NL, "Let: ", $n = {n, h},
 and, "inverse: ", $i = invSDG[$n],
 NL, "For ", $ = $n · $i,
 Yield, $ = $ //. tuRule[$sdg],
 NL, "If ", $s = {inv[a_] .a_ → 1, a_ .inv[a_] → 1, θ[a_]. θ[inv[a_]] → 1,
   θ[a_] . n1_ . θ[a_] . n2_ → θ[a].n1.n2, (*homomorphic property*)
   {θ[a_], b_} -> {1, b}(* Is θ[h].1→1? *)
  }; $s // ColumnBar,
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], CK,
 NL, "For ", $ = $i · $n,
 Yield, $ = $ //. tuRule[$sdg],
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], OK,
 "PONdd",

 NL, "•Invariance under Diff[M]: ",
 NL, $ = xExists [θ[CG["homomorphism"]],
   {θ[Diff[M]] → Aut[𝒢[M×F]] → {θ[ϕ].U → U∘inv[ϕ], ϕ ∈ Diff[M], U ∈ 𝒢[M×F]}}];
 $ // ColumnForms,
 Yield, "Full symmetry group: ", 𝒢[M×F] ⊳ Diff[M]
]
```

●2.4.4 Full symmetry group.

• For two groups {H,N} the action
H[N] is given by a homomorphism θ: $\theta[H] \to \mathrm{Aut}[N]$

• Define semi-direct product $N \rhd H \to \left|\begin{array}{l} \left|\begin{array}{l} n \\ h \end{array}\right. \\ n \in N \,\&\&\, h \in H \\ \left|\begin{array}{l} \left|\begin{array}{l} n\_ \\ h\_ \end{array}\right. \cdot \left|\begin{array}{l} n1\_ \\ h1\_ \end{array}\right. \to \left|\begin{array}{l} n.\theta[h].n1 \\ h.h1 \end{array}\right. \\ 1_{sdp} \to \left|\begin{array}{l} 1_N \\ 1_H \end{array}\right. [\text{unit}] \\ \text{invSDG}[\left|\begin{array}{l} n\_ \\ h\_ \end{array}\right.][\text{inverse}] \to \left|\begin{array}{l} \theta[h^{-1}].n^{-1} \\ h^{-1} \end{array}\right. \end{array}\right. \end{array}\right.$

....

•Invariance under Diff[M]:

xExists[θ[homomorphism], $\left|\ \theta[\mathrm{Diff}[M]] \to \mathrm{Aut}[\mathcal{G}[M\times F]] \to \left|\begin{array}{l} \theta[\phi].U \to U\circ\phi^{-1} \\ \phi \in \mathrm{Diff}[M] \\ U \in \mathcal{G}[M\times F] \end{array}\right.\right.$]

→ Full symmetry group: $\mathcal{G}[M\times F] \rhd \mathrm{Diff}[M]$

```
PR["● Principal bundles. ",
  NL, "Let ", $ = {G[CG["Lie group"]], {P[CG["principal G-bundle"]] ↦ (π[P] → M)},
    Aut[P] -> {f[P] → P, ForAll[{p, g}, p ∈ P && g ∈ G, f[p.g] → f[p].g]},
    Implies[f, Exists[f̄, {(f̄[M] → M) ↦ (f̄[π[p]] → π[f[p]]), f̄[CG["diffeomorphism"]]}]]
   }; ColumnBar[$],
  NL, "•Gauge transformation of P: ",
  𝒢[P] → ForAll[g, g ∈ Aut[P], {ḡ ⩵ 1_M, π[g[p]] → π[p]}],
  NL, "?Is 𝒢[P] a normal subgroup: ",
  NL, "Since ", $ = f̄[π[p]] → π[f[p]],
  Yield, $ = $ /. f → f∘g∘inv[f],
  NL, "Since: ", $s = {(c__∘a_∘b_)[p_] → (c∘a)[b[p]], (a_ ∘ b_)[p_] → a[b[p]]},
  Yield, $ = MapAt[# //. $s &, $, 2],
  NL, "Using: ", $s = {π[f_[p_]] -> f̄[π[p]], a_[b_[π[p]]] :→ Flatten[a∘b][π[p]]},
  Yield, $ = MapAt[# //. $s &, $, 2]; Framed[Head /@ $],
  NL, "For ", $s = {ḡ → 1_M, f_∘1_M∘f1_ → f∘f1, f_∘inv[f_] → 1_M},
  Yield, $ = $ //. $s; $ = Head /@ $,
  imply, $ = $[[1, 1]] ∈ 𝒢[P]; Framed[$ ≤ Aut[P]],
  NL, "Quotient: ", Quotient[Aut[P], 𝒢[P]] ≃ Diff[M]
 ];
```

● Principal bundles.

Let $\left|\begin{array}{l} G[\text{Lie group}] \\ \{P[\text{principal G-bundle}] \mapsto (\pi[P] \to M)\} \\ \text{Aut}[P] \to \{f[P] \to P,\ \forall_{\{p,g\},p\in P\&\&g\in G}\ (f[p.g] \to f[p].g)\} \\ f \Rightarrow \{(\bar{f}[M] \to M) \mapsto (\bar{f}[\pi[p]] \to \pi[f[p]]),\ \bar{f}[\text{diffeomorphism}]\} \end{array}\right.$

•Gauge transformation of P: $\mathcal{G}[P] \to \forall_{g,g\in\text{Aut}[P]}\ \{\bar{g} == 1_M,\ \pi[g[p]] \to \pi[p]\}$

?Is $\mathcal{G}[P]$ a normal subgroup:

Since $\bar{f}[\pi[p]] \to \pi[f[p]]$

→ $\overline{f\circ g\circ f^{-1}}[\pi[p]] \to \pi[(f\circ g\circ f^{-1})[p]]$

Since: $\{(c\_\_\circ a\_\circ b\_)[p\_] \to (c\circ a)[b[p]],\ (a\_\circ b\_)[p\_] \to a[b[p]]\}$

→ $\overline{f\circ g\circ f^{-1}}[\pi[p]] \to \pi[f[g[f^{-1}[p]]]]$

Using: $\{\pi[f\_[p\_]] \to \bar{f}[\pi[p]],\ a\_[b\_[\pi[p]]] :\to \text{Flatten}[a\circ b][\pi[p]]\}$

→ $\boxed{\overline{f\circ g\circ f^{-1}} \to \bar{f}\circ\bar{g}\circ\overline{f^{-1}}}$

For $\{\bar{g} \to 1_M,\ \bar{f\_}\circ 1_M\circ f1\_ \to f\circ f1,\ \bar{f\_}\circ\overline{f\_^{-1}} \to 1_M\}$

→ $\overline{f\circ g\circ f^{-1}} \to 1_M$ ⇒ $\boxed{(f\circ g\circ f^{-1} \in \mathcal{G}[P]) \le \text{Aut}[P]}$

Quotient: $\text{Quotient}[\text{Aut}[P], \mathcal{G}[P]] \simeq \text{Diff}[M]$

2.5 Inner fluctuations and gauge transformations

```
PR["● Definition 2.15:  Given a Real ACM: ", M×F → {𝒜, ℋ, 𝒟, J},
  NL, "and a set: ", $O = Ω𝒟"1" → {xSum[aj. CommutatorM[𝒟, bj], {j}], aj | bj ∈ 𝒜},
  CR["What is this set with Sum? What is index j?"],
  NL, "• Define the inner\!\(\*
StyleBox[\" \",\nFontVariations->{\"Underline\"->True}]\)fluctuations: ",
  $iA = iAf[CG["inner fluctuations"]] → {iA, iA ∈ $O[[1]], ct[iA] -> iA},
  NL, "• Define fluctuated Dirac operator: ",
  $DA = 𝒟𝒜[CG["fluctuated Dirac operator"]] → 𝒟 + iAf + ε'.J.iAf.ct[J],
  accumDef[{$O, $iA, $DA}]
 ];
PR["• Inner fluctuation on M-space where: ",
  $A = $0 = {iA → a.CommutatorM[slash[iD], b], a | b ∈ C"∞"[M],
     iD → slash[iD],
     slash[iD] → -I T[γ, "u", {μ}] tuDs["∇"][_, μ]}; $0 // ColumnBar,
  Yield,
  $ = $0[[1]] /. $0[[-1]] /. tuCommutatorExpand // tuDotSimplify[{T[γ, "u", {μ}]}],
  NL, "Expand operator: ",
  $s = tuDs["∇"][_, μ].b -> tuDs["∇"][b, μ] + b.tuDs["∇"][_, μ],
  Yield, $0 = $ = $ /. $s // tuDotSimplify[{T[γ, "u", {μ}]}],
  NL, "Define ", $Am = $ = I T[iA, "d", {μ}] -> $[[2]] /. T[γ, "u", {μ}] → I;
  $ = -I # & /@ $;
  Framed[$ ∈ Real[C"∞"[M]]],
  and, $Am0 = $0 /. Reverse[$],
  NL, "A hermitian",
  imply, a.T[γ, "u", {μ}] -> T[γ, "u", {μ}].a
 ];
PR["• Proof:",
  "POFF",
  NL, $0;
  $1 = ct /@ $0 // ConjugateCTSimplify1[{}, {}, {T[γ, "u", {μ}]}];
  $2 = 𝒜 → ct[𝒜];
  $ = {$0, $1, $2}; $ // ColumnBar,
  Yield, $ = tuEliminate[$, {iA}],
  yield, $ = Implies[$[[-1]], $[[-1, 2]] ∈ Reals] /. T[γ, "u", {μ}] → 1;
  Framed[$],
  "PONdd",
  NL, "For ", $ = slash[iD]iA -> slash[iD] + iA + JM.iA.ct[JM],
  NL, "Since: ", $s = {jj : JM.iA :> -Reverse[jj], JM.ct[JM] → 1},
  imply, $ = slash[iD]iA -> slash[iD] + iA + JM.iA.ct[JM]
    // tuRepeat[$s, tuDotSimplify[]]
 ];
```

● Definition 2.15: Given a Real ACM: $M\times F\to\{\mathcal{A},\mathcal{H},\mathcal{D},J\}$
and a set: $\Omega^1_{\mathcal{D}}\to\{\sum_{\{j\}}[a_j.[\mathcal{D},b_j]_-],\ a_j\mid b_j\in\mathcal{A}\}$

What is this set with Sum? What is index j?
• Define the inner fluctuations: $A_f[\text{inner fluctuations}]\to\{A,\ A\in\Omega^1_{\mathcal{D}},\ A^\dagger\to A\}$
• Define fluctuated Dirac operator: $\mathcal{D}_{\mathcal{A}}[\text{fluctuated Dirac operator}]\to\mathcal{D}+\varepsilon'.J.A_f.J^\dagger+A_f$

• Inner fluctuation on M-space where: $A \to a.[\not{D}, b]_-$; $a \mid b \in C^\infty[M]$; $D \to \not{D}$; $\not{D} \to -i\,\gamma^\mu \underset{-\mu}{\nabla}[\_]$

→ $A \to i\, a.b.\underline{\nabla}_\mu[\_]\,\gamma^\mu - i\, a.\underline{\nabla}_\mu[\_].b\,\gamma^\mu$

Expand operator: $\underline{\nabla}_\mu[\_].b \to b.\underline{\nabla}_\mu[\_] + \underline{\nabla}_\mu[b]$

→ $A \to -i\, a.\underline{\nabla}_\mu[b]\,\gamma^\mu$

Define $\boxed{(A_\mu \to -i\, a.\underset{-\mu}{\nabla}[b]) \in \mathrm{Real}[C^\infty[M]]}$ and $A \to \gamma^\mu A_\mu$

A hermitian ⇒ $a.\gamma^\mu \to \gamma^\mu.a$

• Proof:

....

For $\not{D}_A \to J_M.A.(J_M)^\dagger + \not{D} + A$

Since: $\{jj : J_M.A :\to -\mathrm{Reverse}[jj],\ J_M.(J_M)^\dagger \to 1\}$ ⇒ $\not{D}_A \to \not{D}$

```
PR["● The Inner fluctuation on M×F: ", $ = $A[[1]],
 NL, "• For Dirac operator: ",
 $d = slash[iD] → slash[iD] ⊗ 1_F + T[γ, "d", {5}] ⊗ iD_F, accumDef[$d];
 Yield, $ = $0 = $ /. $d /. tuCommutatorExpand // tuDotSimplify[],
 NL, "Explicitly label a,b: ", $s = ab : a | b → ab_M ⊗ ab_F,
 Yield, $ = $ /. $s // tuCircleTimesExpand,

 NL, "Simiplying Rule ", $s = {T[γ, "d", {5}].b_M -> b_M.T[γ, "d", {5}], (a_F_ . 1_F) → a_F};
 $s // ColumnBar,
 Yield, $ = $ /. $s /. tuOpDistribute[Dot] /. tuOpDistribute[CircleTimes];
 $ // ColumnSumExp,
 Yield, $ = $ //. tuOpCollect[CircleTimes] //. tuOpCollect[Dot];
 $ // ColumnSumExp,

 NL, "Use relationships: ",
 $s = {(CommutatorM[iD_F, b] /. tuCommutatorExpand) -> CommutatorM[iD_F, b],
    (CommutatorM[slash[iD], b] /. tuCommutatorExpand) -> CommutatorM[slash[iD], b]
   } // tuAddPatternVariable[b],
 Yield, $ = $ /. $s; $ // ColumnSumExp,
 NL, "Use: ",
 $s =
  {tuRuleSelect[$A][iA][[1]] /. iA → iA_M // Reverse // tuAddPatternVariable[{a, b}]},
 Yield, $ = $ /. $s,
 NL, "Use: ",
 $s = {$Am0 /. iA → iA_M, a_F.CommutatorM[iD_F, b_F] → φ},
 Yield, $ = $ /. $s /. tuOpSimplify[CircleTimes, {Tensor[iA_M, _, _]}];
 $ // ColumnSumExp // Framed, accumDef[$];
 NL, "Hence, identify to general algebra ",
 $ = $ /. a_n_ | b_n_ → 1_n //. tuOpSimplify[Dot, {1_M}] /. 1_M → 1 /. 1_n_ . 1_n_ → 1_n;
 $ // Framed, accumDef[$],
 NL, CR["(2.13) defines ", T[iA, "d", {μ}],
  " into F-space whereas the origin is from the
    Dirac operator (defined on M-space) on the algebra 𝒜."]
]
```

- The Inner fluctuation on M×F: $A \to a.[\not{D}, b]_-$
- For Dirac operator: $\not{D} \to (\not{D}) \otimes 1_F + \gamma_5 \otimes D_F$

→ $A \to -a.b.((\not{D}) \otimes 1_F) - a.b.(\gamma_5 \otimes D_F) + a.((\not{D}) \otimes 1_F).b + a.(\gamma_5 \otimes D_F).b$

Explicitly label a,b: $ab : a \mid b \to ab_M \otimes ab_F$

→ $A \to a_M.(\not{D}).b_M \otimes a_F.1_F.b_F - a_M.b_M.(\not{D}) \otimes a_F.b_F.1_F - a_M.b_M.\gamma_5 \otimes a_F.b_F.D_F + a_M.\gamma_5.b_M \otimes a_F.D_F.b_F$

Simiplying Rule $\left|\begin{array}{l} \gamma_5.b_M \to b_M.\gamma_5 \\ a\_{}_F.1_F \to a_F \end{array}\right.$

→ $A \to \sum\left[\left|\begin{array}{l} a_M.(\not{D}).b_M \otimes a_F.b_F \\ -(a_M.b_M.(\not{D}) \otimes a_F.b_F) \\ -(a_M.b_M.\gamma_5 \otimes a_F.b_F.D_F) \\ a_M.b_M.\gamma_5 \otimes a_F.D_F.b_F \end{array}\right.\right]$

→ $A \to \sum\left[\left|\begin{array}{l} a_M.((\not{D}).b_M - b_M.(\not{D})) \otimes a_F.b_F \\ a_M.b_M.\gamma_5 \otimes a_F.(-b_F.D_F + D_F.b_F) \end{array}\right.\right]$

Use relationships: $\{-(b\_).D_F + D_F.(b\_) \to [D_F, b]_-, -(b\_).(\not{D}) + (\not{D}).(b\_) \to [\not{D}, b]_-\}$

→ $A \to \sum\left[\left|\begin{array}{l} a_M.[\not{D}, b_M]_- \otimes a_F.b_F \\ a_M.b_M.\gamma_5 \otimes a_F.[D_F, b_F]_- \end{array}\right.\right]$

Use: $\{(a\_).[\not{D}, b\_]_- \to A_M\}$

→ $A \to a_M.b_M.\gamma_5 \otimes a_F.[D_F, b_F]_- + A_M \otimes a_F.b_F$

Use: $\{A_M \to \gamma^\mu A_{M_\mu}, a_F.[D_F, b_F]_- \to \phi\}$

→ $\boxed{A \to \sum\left[\left|\begin{array}{l} a_M.b_M.\gamma_5 \otimes \phi \\ \gamma^\mu \otimes a_F.b_F\, A_{M_\mu} \end{array}\right.\right]}$

Hence, identify to general algebra $\boxed{A \to \gamma_5 \otimes \phi + \gamma^\mu \otimes 1_F\, A_{M_\mu}}$

(2.13) defines A_μ into F-space whereas the origin is from the Dirac operator (defined on M-space) on the algebra $\mathcal{A}$.

```
PR["● The Fluctuate the Dirac operator: ", $ = tuRule[$DA],
 Yield, $ = a.CommutatorM[𝒟𝒜, b],
 Yield, $ = $ /. tuRule[$DA] // expandCom[] // tuOpSimplifyF[Dot, {ε'}],
 NL, "Use: ", $s =
  {selectDef[𝒟, {γ}] /. 1N → 1F, iAf → iA, J → JM ⊗ JF, selectDef[iA], ab : a | b → abM ⊗ abF};
 $s // ColumnBar,
 Yield, $ = $ //. $s[[1 ;; 2]] /. $s; $ // ColumnSumExp,
 NL, "Distribute ct[] and Expand ⊗ ",
 $ = $ //. tuOpDistribute[ConjugateTranspose, CircleTimes] //. tuOpDistribute[Dot] /.
      a_ ⊗ 1F b_ → a ⊗ b //. tuOpSimplify[Dot] // Expand // tuCircleTimesExpand;
 $ // ColumnSumExp,
 Yield,
 NL, "Use ", $s = {T[γ, "d", {5}].b_M -> bM.T[γ, "d", {5}],
   (a_F . 1N|F) → aF,
   tuCommutatorSolve[1][selectDef[CommutatorM[JM, _], {γ}]],
   tuCommutatorSolve[1][selectDef[CommutatorP[JM, _], {γ}]],
   JM_.ct[JM_] → 1, tt : T[γ, "u", {μ}].bM → Reverse[tt]},
 NL, CR["Not sure of last Rule(see above)."],
 Yield, $ = $ // tuRepeat[{$s, tuOpSimplify[Dot]}, {tuCircleTimesSimplify}, 1];
 $ // ColumnSumExp;
 NL, "Gather into commutator with: ",
 $s = a_ . f_ . b_ ff_: 1 - a_ . b_ . f_ ff_: 1 → a.CommutatorM[f, b] ff,
 $ = $ //. tuOpCollect[CircleTimes]; $ // ColumnSumExp;
 $ //. $s // Simplify // tuCircleTimesSimplify; $ // ColumnSumExp;
 Yield, $ = $ //. $s /. tt : CommutatorM[bF, a_] → -Reverse[tt] //. tuOpSimplify[Dot];
 $ // ColumnSumExp,
 NL, "In generalized form ",
 $pass = $ = $ /. a_ . CommutatorM[f_, b_] → f /. an_.bn_ → 1n /. 1n_.a_ | a_.1n_ → a;
 $ // ColumnSumExp
]
PR[$0 = $pass;
 "Defining ",
 $s = {(tuTermSelect[T[γ, "u", {μ}] ⊗ a_][$0] /. a_ ⊗ b_ :> a ⊗ Apply[Plus, b] // First) ->
     T[γ, "u", {μ}] ⊗ T[B, "d", {μ}],
    (tuTermExtract[T[γ, "d", {5}] ⊗ a_][$0] /. a_ ⊗ b_ :> a ⊗ Apply[Plus, b]) ->
     T[γ, "d", {5}] ⊗ Φ
   }; $s // ColumnBar,
 Yield, $ = $0 //. $s; $1 = tuRule[$DA][[1, 1]] → $, accumDef[{$, $1, $s}]
]
```

● The Fluctuate the Dirac operator: $\{\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \varepsilon'.J.\mathcal{A}_f.J^\dagger + \mathcal{A}_f\}$

→ $a.[\mathcal{D}_{\mathcal{A}}, b]_-$

→ $-a.b.\mathcal{D} - a.b.\mathcal{A}_f + a.\mathcal{D}.b + a.\mathcal{A}_f.b - a.b.J.\mathcal{A}_f.J^\dagger\,\varepsilon' + a.J.\mathcal{A}_f.J^\dagger.b\,\varepsilon'$

Use: $\begin{array}{|l} \mathcal{D} \to (\not{D})\otimes 1_F + \gamma_5\otimes D_F \\ \mathcal{A}_f \to A \\ J \to J_M\otimes J_F \\ A \to \gamma_5\otimes\phi + \gamma^\mu\otimes 1_F\, A_{M_\mu} \\ ab : a \mid b \to ab_M\otimes ab_F \end{array}$

→ $\sum[\begin{array}{|l} -(a_M\otimes a_F).(b_M\otimes b_F).((\not{D})\otimes 1_F + \gamma_5\otimes D_F) \\ -(a_M\otimes a_F).(b_M\otimes b_F).(\gamma_5\otimes\phi + \gamma^\mu\otimes 1_F\, A_{M_\mu}) \\ (a_M\otimes a_F).((\not{D})\otimes 1_F + \gamma_5\otimes D_F).(b_M\otimes b_F) \\ (a_M\otimes a_F).(\gamma_5\otimes\phi + \gamma^\mu\otimes 1_F\, A_{M_\mu}).(b_M\otimes b_F) \\ -(a_M\otimes a_F).(b_M\otimes b_F).(J_M\otimes J_F).(\gamma_5\otimes\phi + \gamma^\mu\otimes 1_F\, A_{M_\mu}).(J_M\otimes J_F)^\dagger\,\varepsilon' \\ (a_M\otimes a_F).(J_M\otimes J_F).(\gamma_5\otimes\phi + \gamma^\mu\otimes 1_F\, A_{M_\mu}).(J_M\otimes J_F)^\dagger.(b_M\otimes b_F)\,\varepsilon' \end{array}]$

Distribute ct[] and Expand ⊗ $\sum[\begin{array}{|l} a_M.(\not{D}).b_M\otimes a_F.1_F.b_F \\ -(a_M.b_M.(\not{D})\otimes a_F.b_F.1_F) \\ -(a_M.b_M.\gamma_5\otimes a_F.b_F.\phi) \\ -(a_M.b_M.\gamma_5\otimes a_F.b_F.D_F) \\ -(a_M.b_M.\gamma^\mu\otimes a_F.b_F.A_{M_\mu}) \\ a_M.\gamma_5.b_M\otimes a_F.\phi.b_F \\ a_M.\gamma_5.b_M\otimes a_F.D_F.b_F \\ a_M.\gamma^\mu.b_M\otimes a_F.A_{M_\mu}.b_F \\ -(a_M.b_M.J_M.\gamma_5.(J_M)^\dagger\otimes a_F.b_F.J_F.\phi.(J_F)^\dagger)\,\varepsilon' \\ -(a_M.b_M.J_M.\gamma^\mu.(J_M)^\dagger\otimes a_F.b_F.J_F.A_{M_\mu}.(J_F)^\dagger)\,\varepsilon' \\ a_M.J_M.\gamma_5.(J_M)^\dagger.b_M\otimes a_F.J_F.\phi.(J_F)^\dagger.b_F\,\varepsilon' \\ a_M.J_M.\gamma^\mu.(J_M)^\dagger.b_M\otimes a_F.J_F.A_{M_\mu}.(J_F)^\dagger.b_F\,\varepsilon' \end{array}]$

→

Use $\{\gamma_5.b\_{}_M \to b_M.\gamma_5,\ a\_{}_F.1_{N|F} \to a_F,$
$J_M.\gamma_5 \to \gamma_5.J_M,\ J_M.\gamma^\mu \to -\gamma^\mu.J_M,\ J_M\_.(J_M\_)^\dagger \to 1,\ tt : \gamma^\mu.b_M \to \text{Reverse}[tt]\}$

Not sure of last Rule(see above).

→

Gather into commutator with:

$-(a\_).(b\_).(f\_)\,(ff\_ : 1) + (a\_).(f\_).(b\_)\,(ff\_ : 1) \to ff\, a.[f, b]_-$

→ $\sum[\begin{array}{|l} a_M.[\not{D}, b_M]_-\otimes a_F.b_F \\ a_M.b_M.\gamma_5\otimes(a_F.[\phi, b_F]_- + a_F.[D_F, b_F]_- + a_F.[J_F.\phi.(J_F)^\dagger, b_F]_-\,\varepsilon') \\ a_M.b_M.\gamma^\mu\otimes(a_F.[A_{M_\mu}, b_F]_- - a_F.[J_F.A_{M_\mu}.(J_F)^\dagger, b_F]_-\,\varepsilon') \end{array}]$

In generalized form $\sum[\begin{array}{|l} (\not{D})\otimes 1_F \\ \gamma_5\otimes(\phi + D_F + J_F.\phi.(J_F)^\dagger\,\varepsilon') \\ \gamma^\mu\otimes(A_{M_\mu} - J_F.A_{M_\mu}.(J_F)^\dagger\,\varepsilon') \end{array}]$

Defining $\begin{array}{|l} \gamma^\mu\otimes(A_{M_\mu} - J_F.A_{M_\mu}.(J_F)^\dagger\,\varepsilon') \to \gamma^\mu\otimes B_\mu \\ \gamma_5\otimes(\phi + D_F + J_F.\phi.(J_F)^\dagger\,\varepsilon') \to \gamma_5\otimes\Phi \end{array}$

→ $\mathcal{D}_{\mathcal{A}} \to (\not{D})\otimes 1_F + \gamma_5\otimes\Phi + \gamma^\mu\otimes B_\mu$

```
$hermitian = {𝒜μ};
PR["Since: ", $ = Implies[Inactive[tuMemberQ[𝒜μ, $hermitian]], ct[𝒜μ] == 𝒜μ] ,
 imply, $ = -I # & /@ Activate[$] /. -I ct[a_] → SuperDagger[I a],
 imply, Framed[I $[[2]] ∈ I u],
 NL, "For ", I 𝔤[F] → I Mod[𝔲[F], 𝔥[F]],
 imply, $ = $e219 = 𝒜μ ∈ C"∞"[M, I 𝔤[F]]; $ // Framed
]
```

```
Since: Inactive[tuMemberQ[𝒜μ, $hermitian]] ⇒ (𝒜μ)† == 𝒜μ ⇒ (ⅈ 𝒜μ)† == -ⅈ 𝒜μ ⇒ [𝒜μ ∈ ⅈ u]

For ⅈ 𝔤[F] → ⅈ Mod[𝔲[F], 𝔥[F]] ⇒ [𝒜μ ∈ C∞[M, ⅈ 𝔤[F]]]
```

```
PR["● Gauge transformation on fluctuating Dirac operator. ",
  Yield, $00 = $0 = 𝒟𝒜 → 𝒟 + iA + ε'.J.iA.ct[J],
  NL, "Expanding Rules: ", $s0 = {U → u.J.u.ct[J], CommutatorM[a, rghtA[b]] → 0,
    CommutatorM[iA, J.u.ct[J]] → 0,
    CommutatorM[CommutatorM[𝒟, a], rghtA[b]] → 0, J.𝒟 → ε'.𝒟.J,
    rghtA[b] → J.ct[b].ct[J], JJ_ .ct[JJ_] :> 1 /; MemberQ[{J, u}, JJ],
    ct[JJ_].JJ_ :> 1 /; MemberQ[{J, u}, JJ],
    ε^2 → 1};
  Yield, $s0x =
   $s0 /. tuCommutatorExpand // tuDotSimplify[{ε'}] // tuRuleEliminate[{rghtA[b]}];
  FramedColumn[$s0x],
  NL, "Evaluate: ",
  $0a = $ = U.#.ct[U] & /@ $0 // tuDotSimplify[{ε', ε}],

  Yield,
  $1 = $ = $[[2]] // tuRepeat[$s0x, tuDotSimplify[]] // ConjugateCTSimplify1[{ε', ε}];
  $1 = $1 // tuRepeat[$s0x, tuDotSimplify[]], (*Need to repeat for some reason*)
  NL, "From commutation rules: ",
  $s = tuRuleSolve[$s0x[[5]], Dot[𝒟, J]],

  NL, "■Simplify the term: ",
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  yield, $ = $ /. $s0x[[7]] // tuDotSimplify[{ε', ε}],
  NL, "From ", $s = u.CommutatorM[𝒟, ct[u]] -> u.MCommutator[𝒟, ct[u]],
  $s = $s // tuDotSimplify[];
  yield, $s = $s /. $s0 // tuDotSimplify[],
  yield, $s = tuRuleEliminate[{u.𝒟.ct[u]}][{$s}]; Framed[$s],
  Imply, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  Yield, $ = $ //. $s0 // tuDotSimplify[{ε', ε}],
  yield, $1a = $ = $ /. $s; Framed[$], CK
 ];
PR[
  "■Simplify the term: ",
  Yield, $0 = $ = $1[[1]]; Framed[$],
  NL, "Use: ", $s = tuRuleSolve[$s0x /. u → ct[u], iA ._],
  Yield, $ = $ /. $s // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1b = $]
 ];
$s0x /. xu → ct[u];
PR[
 "■Simplify the term: ",
 Yield, $0 = $ = $1[[3]]; Framed[$],
```

```
  NL, "Append 1→ ", $s = J.ct[J],
  imply, $ = $. $s // tuDotSimplify[{ε', ε}],
  NL, "Use ",
  $s = tuRuleSolve[$s0x /. u → ct[u], iA ._],
  " with ConjugateTranspose: ", $sa = aa : a | J → ct[aa],
  Yield, $s = $s /. ConditionalExpression[a_, b_] → a /. $sa //
    tuAddPatternVariable[{a, b}],
  NL, "The Rule applies to: ", $sa = iA → u.iA.ct[u],
  yield, $s = $s /. $sa,
  Imply, $ = $ /. $s,
  yield, $ = $ // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1c = $]
 ]
PR["■Check if equal to (2.20). Our calculation: ",
  $ = $0a[[1]] -> $1a + $1b + $1c; Framed[$],
  NL, "Evaluate (2.20) with ", $ = $00 /. iA → iAᵘ, CK,
  Yield,
  $[[2]] = $[[2]] /. iAᵘ → u.iA.ct[u] + u.CommutatorM[𝒟, ct[u]] // tuDotSimplify[{ε'}];
  Framed[$],
  NL, CR["Almost equal."]
 ];
```

● Gauge transformation on fluctuating Dirac operator.

→ $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \varepsilon'.J.A.J^{\dagger} + A$

Expanding Rules:

→ $\boxed{\begin{array}{l} U \to u.J.u.J^{\dagger} \\ a.J.b^{\dagger}.J^{\dagger} - J.b^{\dagger}.J^{\dagger}.a \to 0 \\ -J.u.J^{\dagger}.A + A.J.u.J^{\dagger} \to 0 \\ -a.\mathcal{D}.J.b^{\dagger}.J^{\dagger} + J.b^{\dagger}.J^{\dagger}.a.\mathcal{D} - J.b^{\dagger}.J^{\dagger}.\mathcal{D}.a + \mathcal{D}.a.J.b^{\dagger}.J^{\dagger} \to 0 \\ J.\mathcal{D} \to \mathcal{D}.J\,\varepsilon' \\ (JJ\_).JJ\_^{\dagger} :\to 1 /; \text{MemberQ}[\{J, u\}, JJ] \\ JJ\_^{\dagger}.(JJ\_) :\to 1 /; \text{MemberQ}[\{J, u\}, JJ] \\ \varepsilon^2 \to 1 \end{array}}$

Evaluate: $U.\mathcal{D}_{\mathcal{A}}.U^{\dagger} \to U.\mathcal{D}.U^{\dagger} + U.A.U^{\dagger} + U.J.A.J^{\dagger}.U^{\dagger}\,\varepsilon'$

→ $u.J.u.J^{\dagger}.\mathcal{D}.J.u^{\dagger}.J^{\dagger}.u^{\dagger} + u.J.u.J^{\dagger}.A.J.u^{\dagger}.J^{\dagger}.u^{\dagger} + u.J.u.A.u^{\dagger}.J^{\dagger}.u^{\dagger}\,\varepsilon'$

From commutation rules: $\{\mathcal{D}.J \to \frac{J.\mathcal{D}}{\varepsilon'}\}$

■Simplify the term:

→ $\boxed{u.J.u.J^{\dagger}.A.J.u^{\dagger}.J^{\dagger}.u^{\dagger}} \longrightarrow u.J.u.J^{\dagger}.A.J.u^{\dagger}.J^{\dagger}.u^{\dagger} \longrightarrow u.J.u.J^{\dagger}.A.J.u^{\dagger}.J^{\dagger}.u^{\dagger}$

From $u.[\mathcal{D}, u^{\dagger}]_- \to u.(\mathcal{D}.u^{\dagger} - u^{\dagger}.\mathcal{D}) \longrightarrow u.[\mathcal{D}, u^{\dagger}]_- \to -\mathcal{D} + u.\mathcal{D}.u^{\dagger} \longrightarrow \boxed{\{u.\mathcal{D}.u^{\dagger} \to \mathcal{D} + u.[\mathcal{D}, u^{\dagger}]_-\}}$

⇒ $u.J.u.J^{\dagger}.A.J.u^{\dagger}.J^{\dagger}.u^{\dagger}$

→ $u.J.u.J^{\dagger}.A.J.u^{\dagger}.J^{\dagger}.u^{\dagger} \longrightarrow \boxed{u.J.u.J^{\dagger}.A.J.u^{\dagger}.J^{\dagger}.u^{\dagger}}$ ←CHECK

■Simplify the term:

→ $\boxed{u.J.u.J^{\dagger}.\mathcal{D}.J.u^{\dagger}.J^{\dagger}.u^{\dagger}}$

Use: $\{A.J.u^{\dagger}.J^{\dagger} \to J.u^{\dagger}.J^{\dagger}.A\}$

→ $\boxed{u.J.u.J^{\dagger}.\mathcal{D}.J.u^{\dagger}.J^{\dagger}.u^{\dagger}}$

■Simplify the term:

→ $\boxed{u.J.u.\mathcal{A}.u^\dagger.J^\dagger.u^\dagger\ \varepsilon'}$

Append 1→ $J.J^\dagger$ ⇒ $u.J.u.\mathcal{A}.u^\dagger.J^\dagger.u^\dagger.J.J^\dagger\ \varepsilon'$

Use $\{\mathcal{A}.J.u^\dagger.J^\dagger \to J.u^\dagger.J^\dagger.\mathcal{A}\}$ with ConjugateTranspose: $aa : a \mid J \to aa^\dagger$

→ $\{\mathcal{A}.J^\dagger.u^\dagger.J \to J^\dagger.u^\dagger.J.\mathcal{A}\}$

The Rule applies to: $\mathcal{A} \to u.\mathcal{A}.u^\dagger \longrightarrow \{u.\mathcal{A}.u^\dagger.J^\dagger.u^\dagger.J \to J^\dagger.u^\dagger.J.u.\mathcal{A}.u^\dagger\}$

⇒ $u.J.J^\dagger.u^\dagger.J.u.\mathcal{A}.u^\dagger.J^\dagger\ \varepsilon' \longrightarrow \boxed{J.u.\mathcal{A}.u^\dagger.J^\dagger\ \varepsilon'}$

■Check if equal to (2.20). Our calculation:

$\boxed{U.\mathcal{D}_{\mathcal{A}}.U^\dagger \to u.J.u.J^\dagger.\mathcal{D}.J.u^\dagger.J^\dagger.u^\dagger + u.J.u.J^\dagger.\mathcal{A}.J.u^\dagger.J^\dagger.u^\dagger + J.u.\mathcal{A}.u^\dagger.J^\dagger\ \varepsilon'}$

Evaluate (2.20) with $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \varepsilon'.J.\mathcal{A}^u.J^\dagger + \mathcal{A}^u$⟵CHECK

→ $\boxed{\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + u.[\mathcal{D}, u^\dagger]_- + u.\mathcal{A}.u^\dagger + J.u.[\mathcal{D}, u^\dagger]_-.J^\dagger\ \varepsilon' + J.u.\mathcal{A}.u^\dagger.J^\dagger\ \varepsilon'}$

Almost equal.

```
PR["●Define bilinear form: ", $0 = $ = 𝔘𝒟[ξ, ξp] → BraKet[J.ξ, 𝒟.ξp](*⟨J.ξ,𝒟.ξp⟩*),
  Yield, $ = $ /. dd : 𝒟.ξp → -J.J.dd //. simpleBraKet[],
  Yield, $ = $ /. BraKet[J.a_, J.b_] → BraKet[b, a] /. J.𝒟 → 𝒟.J,
  Yield, $ = $ /. BraKet[𝒟.a_, b_] → BraKet[a, 𝒟.b](*𝒟 is Hermitian*),
  Yield, $s = Reverse[$0] // tuAddPatternVariable[{ξp, ξ}],
  Yield, $ = $ /. $s; Framed[$]
 ];
```

●Define bilinear form: $\mathfrak{U}_{\mathcal{D}}[\xi, \xi p] \to \langle J.\xi \mid \mathcal{D}.\xi p\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi, \xi p] \to -\langle J.\xi \mid J.J.\mathcal{D}.\xi p\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi, \xi p] \to -\langle \mathcal{D}.J.\xi p \mid \xi\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi, \xi p] \to -\langle J.\xi p \mid \mathcal{D}.\xi\rangle$

→ $\langle J.(\xi\_) \mid \mathcal{D}.(\xi p\_)\rangle \to \mathfrak{U}_{\mathcal{D}}[\xi, \xi p]$

→ $\boxed{\mathfrak{U}_{\mathcal{D}}[\xi, \xi p] \to -\mathfrak{U}_{\mathcal{D}}[\xi p, \xi]}$

```
PR["●Define classical fermions: ", (ℋ⁺)cl → {ξ̃ → Grassmann, ξ ∈ ℋ⁺},
  NL, "●Define action functional: ", $S = S → Sb + Sf → Tr[f[𝒟𝒜 / Λ]] + BraKet[J.ξ̃, 𝒟𝒜.ξ̃] / 2
 ];
PR["●Invariance of action functional under ", $s = {𝒟𝒜 → U.𝒟𝒜.ct[U], xx : ξ̃ → U.xx},
  NL, "■Boson ", $0 = $ = tuExtractPattern[Tr[_]][$S] // First,
  yield, $ = $ /. $s,
  yield, xSum[f[λn / Λ], n], CG[" Invariant"],
  NL, "■Fermion ", $0 = $ = tuExtractPattern[BraKet[_, _]][$S] // First,
  Yield, $ = $ /. $s,
  NL, "Apply ", $s = {J.U → U.J, ct[u_].u_ → 1, BraKet[U . a_, U.b_] -> BraKet[a, b]},
  Yield, $ = $ //. $s // tuDotSimplify[], CG[" Invariant"]
 ];
```

●Define classical fermions: $\mathcal{H}^{+}{}_{cl} \to \{\tilde{\xi} \to \text{Grassmann},\ \xi \in \mathcal{H}^{+}\}$

●Define action functional: $S \to S_b + S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle + \text{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

●Invariance of action functional under $\{\mathcal{D}_{\mathcal{A}} \to U.\mathcal{D}_{\mathcal{A}}.U^{\dagger},\ xx : \tilde{\xi} \to U.xx\}$

■Boson $\text{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \longrightarrow \text{Tr}[f[\frac{U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}}{\Lambda}]] \longrightarrow \sum_n[f[\frac{\lambda_n}{\Lambda}]]$ Invariant

■Fermion $\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$

→ $\langle J.U.\tilde{\xi} \mid U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}.U.\tilde{\xi}\rangle$

Apply $\{J.U \to U.J,\ u\_^{\dagger}.(u\_) \to 1,\ \langle U.(a\_) \mid U.(b\_)\rangle \to \langle a \mid b\rangle\}$

→ $\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$ Invariant

```
Clear[i];
PR["●Theorem 2.19. A real even almost-commutative manifold M×F describes
    a gauge theory on M with gauge group 𝒢[M×F]->C∞[M,𝒢[F]]. ",
  NL, "•Sketch of Proof: ",
  $t219 = $ = {{"(2.19)" → {I 𝒜μ[x] ∈ 𝔤[F] → Mod[𝓊[𝒜F], 𝒽F]},
      𝒜[CG["Total algebra"]] → C"∞"[M, 𝒜F] → xSum[section[ii, Γ[M×𝒜F]], {ii}],
      {ω → I T[𝒜, "d", {μ}] . DifForm[T[x, "u", {μ}]], ω[CG["𝔤[F]-valued 1-form"]]},
      P[CG["Principal bundle"]] → M×𝒢[F],
      "(2.22)" → ω[CG["connection form on P"]],
      "group of gauge transform"[P] -> C"∞"[M, 𝒢[F]],
      "(2.12)" ⇒ 𝒢[M×F][CG["group of gauge transform"]][P],
      "(2.11)" ⇒ 𝒢[M×F] -> {U → u.J.u.ct[J], u ∈ U[𝒜]},
      (rep[𝒜F[ℋF]] ⇒ rep[𝒢[F][ℋF]])
       ⇒ (M×ℋF ↔ "vector bundle of"[P → M×𝒢[F]])
     }}; Grid[Transpose[$], Frame → All],
  NL, "Note: ", {("E" → M×ℋF) ↔
     (P[CG["Principal bundle"]] → M×𝒢[F])⟹CG["action of gauge group on fermions"],
    ℋ["ACM"] → L²[M, S]⊗ℋF → L²[M, S⊗"E"],
    "⇒ particle fields"⟶section[S⊗"E"]} // ColumnBar
 ];
tuSaveAllVariables[];
```

●Theorem 2.19. A real even almost-commutative manifold $M\times F$ describes a gauge theory on M with gauge group $\mathcal{G}[M\times F]\text{->}C^\infty[M,\mathcal{G}[F]]$.

•Sketch of Proof:

$(2.19)\to\{\mathbb{i}\,\mathcal{A}_\mu[x]\in\mathfrak{g}[F]\to \mathrm{Mod}[u[\mathcal{A}_F],\,h_F]\}$
$\mathcal{A}[\text{Total algebra}]\to C^\infty[M,\,\mathcal{A}_F]\to \sum_{\{ii\}}[\mathrm{section}[ii,\,\Gamma[M\times\mathcal{A}_F]]]$
$\{\omega\to\mathbb{i}\,\mathcal{A}_\mu.d[x^\mu],\ \omega[\mathfrak{g}[F]\text{-valued 1-form}]\}$
$P[\text{Principal bundle}]\to M\times\mathcal{G}[F]$
$(2.22)\to\omega[\text{connection form on P}]$
group of gauge transform$[P]\to C^\infty[M,\,\mathcal{G}[F]]$
$(2.12)\Rightarrow\mathcal{G}[M\times F][\text{group of gauge transform}][P]$
$(2.11)\Rightarrow\mathcal{G}[M\times F]\to\{U\to u.J.u.J^\dagger,\ u\in U[\mathcal{A}]\}$
$(\mathrm{rep}[\mathcal{A}_F[\mathcal{H}_F]]\Rightarrow\mathrm{rep}[\mathcal{G}[F][\mathcal{H}_F]])\Rightarrow M\times\mathcal{H}_F\leftrightarrow$ vector bundle of$[P\to M\times\mathcal{G}[F]]$

Note:
- $(E\to M\times\mathcal{H}_F)\leftrightarrow(P[\text{Principal bundle}]\to M\times\mathcal{G}[F])\Longrightarrow$ action of gauge group on fermions
- $\mathcal{H}[ACM]\to L^2[M,\,S]\otimes\mathcal{H}_F\to L^2[M,\,S\otimes E]$
- $\Rightarrow$ particle fields $\longrightarrow$ section$[S\otimes E]$