```
<< Local`QFTToolKit2`
Get[NotebookDirectory[] <>
  "1204.0328 ParticlePhysicsFromAlmostCommutativeSpacetime.2.GWSmodel.out"]
$defStdMdl = {};

{Temporary}

"Notational definitions"
"Note that in the text the symbols may reference
  different Hilbert spaces. This has caused confusion in some of the
  calculations.  To address this problem we will try to label the
  variables by subscripts to designate the applicable Hilbert space.
  NOTE: Need to do notational change for .1,.2 notebooks."

rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := |a|;
it[a_] := Style[a, Italic]
iD := it[D]
iA := it[A]
iI := it["I"]
C∞ := C^"∞"
B_x_ := T[B, "d", {x}]
("▽"^S)_n_ := T["▽"^S, "d", {n}]

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
   ""];
accumStdMdl[item_] := Block[{}, $defStdMdl = tuAppendUniq[item][$defStdMdl];
   ""];
selectStdMdl[heads_, with_: {}, all_: Null] :=
  tuRuleSelect[$defStdMdl // tuExtractPattern[(Rule | RuleDelayed)[_, _]] // tuRule][
     Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectGWS[heads_, with_: {}, all_: Null] :=
  tuRuleSelect[$defGWS // tuExtractPattern[(Rule | RuleDelayed)[_, _]] // tuRule
     ][Flatten[{heads}]] // Select[#, tuHasAllQ[#, Flatten[{with}]] &] & //
   If[all === Null, Last[#], #] &;
selectDef[heads_, with_: {}, all_: Null] :=
  tuRuleSelect[$defall // tuExtractPattern[(Rule | RuleDelayed)[_, _]] // tuRule][
     Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;

Clear[expandDC];
expandDC[sub_: {}, scalar_: {}] :=
 tuRepeat[{sub, tuOpDistribute[Dot], tuOpSimplify[Dot, scalar],
   tuOpDistribute[CircleTimes]}, tuCircleTimesSimplify]
Clear[expandCom]
expandCom[subs_: {}][exp_] := Block[{tmp = exp},
   tmp = tmp //. tuCommutatorExpand // expandDC[];
   tmp = tmp /. toxDot //. Flatten[{subs}];
   tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
   tmp
```

```
  ];
(**)
$sgeneral := {
  T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
  T[γ, "d", {5}].T[γ, "d", {5}] → 1,
  ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
  CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  T["▽", "d", {_}][1n_] → 0, a_ . 1n_ → a, 1n_ . a_ → a}
$sgeneral // ColumnBar

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] ⧴ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
   tt : T[F, "uu", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
   tt : T[F, "dd", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
   CommutatorM[a_, b_] ⧴ -CommutatorM[b, a] /; OrderedQ[{b, a}],
   CommutatorP[a_, b_] ⧴ CommutatorP[b, a] /; OrderedQ[{b, a}],
   tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
  };
$symmetries // ColumnBar

εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1,}}},
  {ε → table[[1, n+1]], ε' → table[[2, n+1]], ε'' → table[[3, n+1]]}
 ]
εRule[6]
```

Notational definitions

Note that in the text the symbols may reference different Hilbert spaces. This has caused confusion in some of the calculations. To address this problem we will try to label the variables by subscripts to designate the applicable Hilbert space. NOTE: Need to do notational change for .1,.2 notebooks.

$\gamma_5 \to \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$
$\gamma_5 . \gamma_5 \to 1$
$(\gamma_5)^\dagger \to \gamma_5$
$\{\gamma_5, \gamma^\mu\}_+ \to 0$
$\nabla\_[1_{n\_}] \to 0$
$(a\_).1_{n\_} \to a$
$1_{n\_}.(a\_) \to a$

$tt : g^{\mu\_\,\nu\_} \mapsto tuIndexSwap[\{\mu, \nu\}][tt] \;/;\; OrderedQ[\{\nu, \mu\}]$
$tt : F^{\mu\_\,\nu\_} \mapsto -tuIndexSwap[\{\mu, \nu\}][tt] \;/;\; OrderedQ[\{\nu, \mu\}]$
$tt : F_{\mu\_\,\nu\_} \mapsto -tuIndexSwap[\{\mu, \nu\}][tt] \;/;\; OrderedQ[\{\nu, \mu\}]$
$[a\_, b\_]_- \mapsto -[b, a]_- \;/;\; OrderedQ[\{b, a\}]$
$\{a\_, b\_\}_+ \mapsto \{b, a\}_+ \;/;\; OrderedQ[\{b, a\}]$
$tt : \gamma^\mu.\gamma_5 \mapsto Reverse[tt]$

$\{\varepsilon \to 1,\ \varepsilon' \to 1,\ \varepsilon'' \to -1\}$

6. The Standard Model

■ 6.1 The Finite space

```
PR["● The algebra: left-right symmetric algebra ",
  iA_LR, " and subalgebra ", iA_F ⊂ iA_LR, " with Dirac operator ", iD_F,
  NL, "The space: ", $sSM = {KOdim -> 6,

iA_F[CG["ℂ⊕ℍ⊕M₃[3x3 ℂ matrices(for 3 generations)]"]],
    ℋ_l[CG[ℂ^4[{ν_R, e_R, ν_L, e_L}]]],
    ℋ_q[CG[ℂ^4[{u_R, d_R, u_L, d_L}]]]⊗ℂ^3[CG["color"]],
    ℋ_F → (ℋ_l ⊕ ℋ_l̄ ⊕ ℋ_q ⊕ ℋ_q̄)^"⊕3[generation]",
    a ∈
iA_F,
    a → {λ, q[CG[M₂[ℂ]]], m[CG[M₃[ℂ]]]},
    a_l → {λ, q, m}_ℋ_l, selectGWS[a_l],
    a_q → {λ, q, m}_ℋ_q,
    ($ = selectGWS[a_l] /. l → q; $[[2]] = $[[2]]⊗1₃[CG["color"]]; $),

    a_l̄ → {λ, q, m}_ℋ_l̄, a_l̄ . l̄ → λ 1₄ . l̄,
    a_q̄ → {λ, q, m}_ℋ_q̄, a_q̄ . q̄ → λ (1₄⊗m).q̄,
    {CG["fermionic{f_L,f_R} grading"],
     γ_F.f_L → f_L,
     γ_F.f_R → -f_R
    },
    {CG["fermionic Charge conjugation(single generation, no color)"], (*
     J_F.f_:→
      If[FreeQ[f,OverBar],f,f[[1]]] /;tuMemberQ[f,selectStdMdl[basisSM][[2]]],*)
     J_F.Tensor[f_, a_, b_] :→ Tensor[If[FreeQ[f, OverBar], f̄, f[[1]]], a, b]
    },

    iD_F → {{S, ct[T]}, {T, Conjugate[S]}},
    S_l ->
     Normal[SparseArray[{{1, 3} -> Y_ν, {2, 4} -> Y_e, {3, 1} → ct[Y_ν], {4, 2} → ct[Y_e]}]],
    S_q⊗1₃ → Normal[SparseArray[{{1, 3} -> Y_u, {2, 4} -> Y_d,
          {3, 1} → ct[Y_u], {4, 2} → ct[Y_d]}]]⊗1₃,
    {Y_ν, Y_e, Y_u, Y_d} ∈ M₃[CG["3 generation mass matrix, symmetric"]],
    T . ν_R → Y_R[CG["3×3 symmetric Majorana generation mass matrix"]].ν̄_R,
    T.f :→ 0 /; f =!= ν_R,
    ν_R → Table[{T[ν_R, "d", {i}]}, {i, 3}][CG["with generations"]]
   }; $sSM // MatrixForms // ColumnBar, accumStdMdl[$sSM],
  NL,
  CO["Note: ", {a_l, a_l̄, a_q, a_q̄}, " only operate on their respective ℋilbert spaces."]
 ];
```

● The algebra: left-right symmetric algebra $\mathcal{A}_{LR}$ and subalgebra $\mathcal{A}_F \subset \mathcal{A}_{LR}$ with Dirac operator $\mathcal{D}_F$

The space:

KOdim → 6

$\mathcal{A}_F$[ℂ⊕ℍ⊕M_3[3x3 ℂ matrices(for 3 generations)]]

$\mathcal{H}_l[\mathbb{C}^4[\{\nu_R, e_R, \nu_L, e_L\}]]$

$\mathcal{H}_q[\mathbb{C}^4[\{u_R, d_R, u_L, d_L\}]]\otimes\mathbb{C}^3$[color]

$\mathcal{H}_F \to (\mathcal{H}_l\oplus\mathcal{H}_{\bar{l}}\oplus\mathcal{H}_q\oplus\mathcal{H}_{\bar{q}})^{\oplus 3[\text{generation}]}$

$a \in \mathcal{A}_F$

$a \to \{\lambda,\ q[M_2[\mathbb{C}]],\ m[M_3[\mathbb{C}]]\}$

$a_l \to \{\lambda,\ q,\ m\}_{\mathcal{H}_l}$

$$a_l \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$$

$a_q \to \{\lambda,\ q,\ m\}_{\mathcal{H}_q}$

$$a_q \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}\otimes 1_3[\text{color}]$$

$a_{\bar{l}} \to \{\lambda,\ q,\ m\}_{\mathcal{H}_{\bar{l}}}$

$a_{\bar{l}}.\bar{l} \to \lambda\, 1_4.\bar{l}$

$a_{\bar{q}} \to \{\lambda,\ q,\ m\}_{\mathcal{H}_{\bar{q}}}$

$a_{\bar{q}}.\bar{q} \to \lambda\,(1_4\otimes m).\bar{q}$

{fermionic{f_L,f_R} grading, $\gamma_F.f_L \to f_L$, $\gamma_F.f_R \to -f_R$}

{fermionic Charge conjugation(single generation, no color),
 J_F.Tensor[f_, a_, b_] :→ Tensor[If[FreeQ[f, OverBar], $\bar{f}$, f⟦1⟧], a, b]}

$$\mathcal{D}_F \to \begin{pmatrix} S & T^\dagger \\ T & S^* \end{pmatrix}$$

$$S_l \to \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ (Y_\nu)^\dagger & 0 & 0 & 0 \\ 0 & (Y_e)^\dagger & 0 & 0 \end{pmatrix}$$

$$S_q\otimes 1_3 \to \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ (Y_u)^\dagger & 0 & 0 & 0 \\ 0 & (Y_d)^\dagger & 0 & 0 \end{pmatrix}\otimes 1_3$$

$\{Y_\nu,\ Y_e,\ Y_u,\ Y_d\} \in M_3$[3 generation mass matrix, symmetric]

$T.\nu_R \to Y_R$[3×3 symmetric Majorana generation mass matrix]$.\bar{\nu}_R$

$T.f :\to 0$ /; $f =!= \nu_R$

$$\nu_R \to \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix}[\text{with generations}]$$

Note: $\{a_l, a_{\bar{l}}, a_q, a_{\bar{q}}\}$ only operate on their respective $\mathcal{H}$ilbert spaces.

```
(*PR["Hilbert space basis: ",
 $={$smbasis={l→{νR,eR,νL,eL},l̄->{ν̄R,ēR,ν̄L,ēL},q->{uR,dR,uL,dL},q̄->{ūR,d̄R,ūL,d̄L}},
   {q,q̄}→{qcolor,q̄color̄},color→{1,2,3},generations→{1,2,3}};
 $//ColumnForms[#,1]&,accumStdMdl[$],
 NL,"(8[l,l̄]+3[color]*8[q,q̄])*3[generations]->96 dimensions",
 NL,"Dirac operator: ",selectStdMdl[iDF]//MatrixForms,
 " is a 96 x 96 matrix operator.",
 NL,"Use as basis: ",
 $basisSM=basisSM[CG["without generations(3) and color(3 for u,d) indices"]]->
   Flatten[#[[2]]&/@selectStdMdl/@{l,q,l̄,q̄}],
 accumStdMdl[$basisSM]
]*)
```

```
PR["Hilbert space basis from basic fermions: ", $fermion = $ = {ν, e, u, d},
  NL, "Leptons: ", $fermion1 = $ = {l -> $fermion[[1 ;; 2]], l̄ → (#̄ & /@ $fermion[[1 ;; 2]]),
     q -> $fermion[[3 ;; 4]], q̄ → (#̄ & /@ $fermion[[3 ;; 4]])},
  NL, "Chirality added ",
  $ = $fermion1;
  $fermion2 =
   $ = $ /. Rule[a_, b_] :→ Rule[a, Flatten[{T[#, "d", {R}] & /@ b, T[#, "d", {L}] & /@ b}]];
  $ = {$smbasis = $fermion2, {q, q̄} → {q_color, q_colōr},
    color → {1, 2, 3}, generations → {1, 2, 3}};
  NL, $ // ColumnForms[#, 1] &,

  accumStdMdl[$];
  NL, "(8[l,l̄]+3[color]*8[q,q̄])*3[generations]->96 dimensions",
  NL, "Dirac operator: ",
  selectStdMdl[iD_F] // MatrixForms, " is a 96 x 96 matrix operator.",
  NL, "Use as basis: ", $basisSM =
   basisSM[CG["without generations(3) and color(3 for u,d) indices"]] ->
    Flatten[#[[2]] & /@ selectStdMdl /@ {l, q, l̄, q̄}],
  accumStdMdl[$basisSM]
 ];
```

Hilbert space basis from basic fermions: $\{\nu, e, u, d\}$

Leptons: $\{l \to \{\nu, e\}, \bar{l} \to \{\bar{\nu}, \bar{e}\}, q \to \{u, d\}, \bar{q} \to \{\bar{u}, \bar{d}\}\}$

Chirality added

$$l \to \begin{array}{|l} \nu_R \\ e_R \\ \nu_L \\ e_L \end{array} \quad \bar{l} \to \begin{array}{|l} \bar{\nu}_R \\ \bar{e}_R \\ \bar{\nu}_L \\ \bar{e}_L \end{array} \quad q \to \begin{array}{|l} u_R \\ d_R \\ u_L \\ d_L \end{array} \quad \bar{q} \to \begin{array}{|l} \bar{u}_R \\ \bar{d}_R \\ \bar{u}_L \\ \bar{d}_L \end{array}$$

$$\begin{array}{|l} q \\ \bar{q} \end{array} \to \begin{array}{|l} q_{\text{color}} \\ q_{\overline{\text{color}}} \end{array}$$

$$\text{color} \to \begin{array}{|l} 1 \\ 2 \\ 3 \end{array} \qquad \text{generations} \to \begin{array}{|l} 1 \\ 2 \\ 3 \end{array}$$

(8[l,l̄]+3[color]*8[q,q̄])*3[generations]->96 dimensions

Dirac operator: $D_F \to \begin{pmatrix} S & T^\dagger \\ T & S^* \end{pmatrix}$ is a 96 x 96 matrix operator.

Use as basis: basisSM[without generations(3) and color(3 for u,d) indices] → $\{\nu_R, e_R, \nu_L, e_L, u_R, d_R, u_L, d_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L, \bar{u}_R, \bar{d}_R, \bar{u}_L, \bar{d}_L\}$

Proposition 6.1

```
PR["Proposition 6.1.  The data ", $ = F_SM → (#_F & /@ {
iA, ℋ, iD, γ, J}),
 " define a real even finite space of KO-dimension 6.", accumStdMdl[$]
]
```

Proposition 6.1. The data $F_{SM} \to \{A_F, \mathcal{H}_F, D_F, \gamma_F, J_F\}$
 define a real even finite space of KO-dimension 6.

6.2 The gauge theory

The gauge group

```
PR["Manifold(p.69) ", M×F_SM,
  NL, "Define sub-algebra: ", $subalg = $ = { iÃ_{F J_F} ⊂ $sSM[[2]],
     a ∈ iÃ_{F J_F}, a.J_F → J_F.ct[a], {λ → cc[λ], α → λ, β → 0, m → λ 1_3}, a ≃ λ[CG[ℝ]]};
  ColumnBar[$], accumStdMdl[$],
  Imply, $subalg[[1, 1]] ≃ ℝ,
  Imply, "LieAlgebra", yield, {h_F → u[ $[[1, 1]] ], u[ $[[1, 1]] ] → {0}},
  line,
  NL, "●Examine the statement that ", $subalg[[2 ;; 3]],
  imply, tuRuleSelect[$subalg] /@ {λ, α, β, m} // Flatten // ColumnBar,
  NL, "We have: ",
  NL, "Algebra form: ", $a = b → (selectStdMdl[a_1] // Last);
  $a // MatrixForms,
  NL, "Real form: ",
  $s = selectGWS[J_{F_4}]; $s // MatrixForms,
  NL, "Subalgebra relationship: ",
  $ = selectDef[rghtA[b]] /. {rghtA[b] → b, F → F_4},
  Yield, $ = $ /. toxDot /. $s /. $a;
  Yield, $ =
   OrderedxDotMultiplyAll[][$] /. {cc .a_ → Conjugate[a].cc} // tuConjugateSimplify[] //
     (# /. cc.cc → 1) & // tuOpSimplifyF[Dot];
  $ = $ //. rr : Rule[__] :> Thread[rr] // Flatten // DeleteDuplicates //
    (# /. Rule → Equal &), CK,
  Imply, $ = tuRuleSolve[$, {λ, β, Conjugate[α], Conjugate[λ]}];
  Framed[$],
  CR[" ", λ^* → λ, " not indicated."]
 ];
```

Manifold(p.69) $M \times F_{SM}$

Define sub-algebra:

$\tilde{A}_{F J_F} \subset A_F[\mathbb{C} \oplus \mathbb{H} \oplus M_3$[3x3 $\mathbb{C}$ matrices(for 3 generations)]]
$a \in \tilde{A}_{F J_F}$
$a.J_F \to J_F.a^\dagger$
$\{\lambda \to \lambda^*, \alpha \to \lambda, \beta \to 0, m \to \lambda\, 1_3\}$
$a \approx \lambda[\mathbb{R}]$

⇒ $\tilde{A}_{F J_F} \simeq \mathbb{R}$

⇒ LieAlgebra ⟶ $\{\mathfrak{h}_F \to \mathfrak{u}[\tilde{A}_{F J_F}], \mathfrak{u}[\tilde{A}_{F J_F}] \to \{0\}\}$

●Examine the statement that $\{a \in \tilde{A}_{F J_F}, a.J_F \to J_F.a^\dagger\}$ ⇒
$\lambda \to \lambda^*$
$\alpha \to \lambda$
$\beta \to 0$
$m \to \lambda\, 1_3$

We have:

Algebra form: $b \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$

Real form: $J_{F_4} \to \begin{pmatrix} 0 & 0 & cc & 0 \\ 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 \\ 0 & cc & 0 & 0 \end{pmatrix}$

Subalgebra relationship: $b \to J_{F_4}.b^\dagger.(J_{F_4})^\dagger$

→

→ $\{\lambda == \alpha, 0 == -\beta^*, \text{True}, 0 == \beta, \lambda^* == \alpha^*, \alpha == \lambda, \beta == 0, -\beta^* == 0, \alpha^* == \lambda^*\}$ ⟵CHECK

⇒ $\boxed{\{\lambda \to \alpha, \beta \to 0, \alpha^* \to \alpha^*\}}$ $\lambda^* \to \lambda$ not indicated.

Proposition 6.2

```
PR[
 NL, "Prop.6.2:  The local gauge group: ", {𝒢[F_SM] ≃ mod[(U[1]×SU[2]×U[3]), {1, -1}]},
 NL, "Demand unimodularity: ", Det[u]_ℋF → 1, imply, (λ Det[m])^12 → 1, " for ",
 u ∈ U[1]×SU[2]×U[3],
 NL, CR["Why 12? Possible rational: "], Det[u]_ℋF → Det[λ] Det[q] Det[m] → 1,
 and, {Det[q] → 1, Det[λ] → λ},
 and, "there is 2 x 2 x 3 possible phases freedoms in Det[u]_ℋF.",
 NL, "Let ", U → u.J.u.ct[J] ↔ 𝒢[F_SM],
 NL, "The subgroup: ",
 $ = S𝒢[F_SM] → {U → u.J.u.ct[J] ∈ 𝒢[F_SM], u → {λ, q, m}, (λ Det[m])^12 → 1};
 $ // ColumnForms,
 NL, "The condition ", $[[2, 3]] ⇒ mod[Det[m] ≃ cc[λ], μ12] ,
 NL, "True gauge group of the SM: ",
 𝒢SM → mod[U[1]×SU[2]×SU[3], μ6]
]
```

Prop.6.2: The local gauge group: $\{\mathcal{G}[F_{SM}] \simeq \mathrm{mod}[U[1]\times SU[2]\times U[3], \{1, -1\}]\}$
Demand unimodularity: $\mathrm{Det}[u]_{\mathcal{H}_F} \to 1 \Rightarrow \lambda^{12}\,\mathrm{Det}[m]^{12} \to 1$ for $u \in U[1]\times SU[2]\times U[3]$
Why 12? Possible rational: $\mathrm{Det}[u]_{\mathcal{H}_F} \to \mathrm{Det}[m]\,\mathrm{Det}[q]\,\mathrm{Det}[\lambda] \to 1$ and $\{\mathrm{Det}[q] \to 1, \mathrm{Det}[\lambda] \to \lambda\}$
  and there is 2 x 2 x 3 possible phases freedoms in $\mathrm{Det}[u]_{\mathcal{H}_F}$.
Let $U \to u.J.u.J^{\dagger} \leftrightarrow \mathcal{G}[F_{SM}]$

The subgroup: $S\mathcal{G}[F_{SM}] \to \left|\begin{array}{l} U \to u.J.u.J^{\dagger} \in \mathcal{G}[F_{SM}] \\ u \to \left|\begin{array}{l}\lambda \\ q \\ m\end{array}\right. \\ \lambda^{12}\,\mathrm{Det}[m]^{12} \to 1 \end{array}\right.$

The condition $(\lambda^{12}\,\mathrm{Det}[m]^{12} \to 1) \Rightarrow \mathrm{mod}[\mathrm{Det}[m] \simeq \lambda^{*}, \mu_{12}]$
True gauge group of the SM: $\mathcal{G}_{SM} \to \mathrm{mod}[U[1]\times SU[2]\times SU[3], \mu_6]$

Proposition 6.3

```
PR["Prop 6.3: The unimodular gauge group ", S𝒢[F_SM] ≈ 𝒢SM × μ12,
 line,
 CR["Did not understand proof."]
]
```

Prop 6.3: The unimodular gauge group $S\mathcal{G}[F_{SM}] \approx \mathcal{G}_{SM}\times\mu_{12}$

Did not understand proof.

6.2.2 The gauge fields and the Higgs field

```
PR["Calculate ", {T[A, "d", {μ}], ϕ},
 " From 2.13 and 2.14 ", (*define in and get from $defall*)
 {$e213 = T[γ, "u", {μ}] ⊗ T[A, "d", {μ}] →
     a CommutatorM[slash[iD] ⊗ 1_F, b] → -I T[γ, "u", {μ}] ⊗ (a tuDDown["∂"][b, μ]),
   $e214 = T[γ, "d", {5}] ⊗ ϕ -> a CommutatorM[T[γ, "d", {5}] ⊗ iD_F, b] ->
      T[γ, "d", {5}] ⊗ (a CommutatorM[iD_F, b])
  } // ColumnBar,

 Imply, "Higgs field ", $e61 = $ = {
    ϕ_H1 → {{0, ct[Y]}, {Y, 0}},
    ϕ_HI → 0,
    ϕ_Hq → {{0, ct[X]}, {X, 0}} ⊗ 1_3[CG["color"]],
    ϕ_Hq → 0 ,
    {ϕ1, ϕ2} ∈ CG[ℂ],
    Y → {{Y_ν ϕ1, -Y_e Conjugate[ϕ2]}, {Y_ν ϕ2, Y_e Conjugate[ϕ1]}},
    X → {{Y_u ϕ1, -Y_d Conjugate[ϕ2]}, {Y_u ϕ2, Y_d Conjugate[ϕ1]}},
    Φ → Inactivate[iD_F2 + {{ϕ, 0}, {0, 0}} + J_F.{{ϕ, 0}, {0, 0}}.ct[J_F], Plus] →
      {{S + ϕ, ct[T]}, {T, Conjugate[S + ϕ]}}
   }; $ // Column // MatrixForms // Framed, accumStdMdl[$], CG[" (6.1,6.2)"],
 NL, "same as GWS: ", selectGWS[Tensor[iA, _, _], {Λ, Q}]
 (*Symbol for A inconsistent.*)
]
```

Calculate $\{A_\mu, \phi\}$ From 2.13 and 2.14 $\left|\begin{array}{l}\gamma^\mu\otimes A_\mu \to a\,[(D\!\!\!/)\otimes 1_F,\, b]_- \to -i\,\gamma^\mu\otimes(a\,\partial_{-\mu}\,[b]) \\ \gamma_5\otimes\phi \to a\,[\gamma_5\otimes D_F,\, b]_- \to \gamma_5\otimes(a\,[D_F,\, b]_-)\end{array}\right.$

⇒ Higgs field

$$\boxed{\begin{array}{l}
\phi_{\mathcal{H}_1} \to \begin{pmatrix} 0 & Y^\dagger \\ Y & 0 \end{pmatrix} \\
\phi_{\mathcal{H}_I} \to 0 \\
\phi_{\mathcal{H}_q} \to \begin{pmatrix} 0 & X^\dagger \\ X & 0 \end{pmatrix} \otimes 1_3[\text{color}] \\
\phi_{\mathcal{H}_q} \to 0 \\
\{\phi_1,\, \phi_2\} \in \mathbb{C} \\
Y \to \begin{pmatrix} Y_\nu\, \phi_1 & -(\phi_2)^*\, Y_e \\ Y_\nu\, \phi_2 & (\phi_1)^*\, Y_e \end{pmatrix} \\
X \to \begin{pmatrix} Y_u\, \phi_1 & -(\phi_2)^*\, Y_d \\ Y_u\, \phi_2 & (\phi_1)^*\, Y_d \end{pmatrix} \\
\Phi \to D_{F_2} + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F.\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}.(J_F)^\dagger \to \begin{pmatrix} S+\phi & T^\dagger \\ T & (S+\phi)^* \end{pmatrix}
\end{array}}$$

(6.1,6.2)

same as GWS: $A_\mu \to \{\{\Lambda_\mu,\, 0,\, 0\},\, \{0,\, -\Lambda_\mu,\, 0\},\, \{0,\, 0,\, Q_\mu\}\}$

```
PR["•The field from the term ", $ = -I (a tuDDown["∂"][b, μ]), " on ", ℋq,
  NL, "Let ", $s = {a → m, b → m', {m, m'} ∈ ℋq, {m, m'} ∈ M3[ℂ]},
  Yield, $ = T[V', "d", {μ}] -> $ /. tuRule[$s], accumStdMdl[$];
  NL, "If ", $[[1]], " hermitian ⇒ ", $[[1, 1]] ∈ I u[3], imply, $[[1, 1]] ∈ U[3],
  NL, "Impose unimodularity condition to get SU[3] gauge field. ",
  CR["Why is I included?"],
  NL, "Since ", $ = Tr_ℋF[T[A, "d", {μ}]] → 0,
  Yield, $ = $ /. Tr_[a_] :→ Sum[Tr_h[a], {h, {l, q, l̄, q̄}}],
  Yield, $ = $ /. Tr_l|q[_] → 0,
  Yield, Tr_ℋl̄[T[Λ, "d", {μ}] 1_4] + Tr_ℋq̄[1_4 ⊗ T[V', "d", {μ}]] → 0,
  imply, Tr[T[V', "d", {μ}]] → -T[Λ, "d", {μ}],
  NL, "For a traceless SU[3] gauge field define: ",
  $ = cc[T[V, "d", {μ}]] → -T[V', "d", {μ}] - 1_3 T[Λ, "d", {μ}] / 3;
  $ // Framed, accumStdMdl[$];

  NL, "Then the gauge field becomes: ", T[A, "d", {μ}],
  Yield, $e63a = $ = {T[A_ℋl, "d", {μ}] →
      DiagonalMatrix[{T[Λ, "d", {μ}], -T[Λ, "d", {μ}], T[Q, "d", {μ}]}],
     T[A_ℋl̄, "d", {μ}] → 1_4 T[Λ, "d", {μ}],
     T[A_ℋq, "d", {μ}] →
      DiagonalMatrix[{T[Λ, "d", {μ}], -T[Λ, "d", {μ}], T[Q, "d", {μ}]}] ⊗ 1_3,
     T[A_ℋq̄, "d", {μ}] → -1_4 ⊗ (cc[T[V, "d", {μ}]] + 1_3 T[Λ, "d", {μ}] / 3),
     T[Λ, "d", {μ}] ∈ U[1],
     T[Q, "d", {μ}] ∈ SU[2]
    }; $ // Column // MatrixForms // Framed,
  line,
  "The action on fermions of the field: ",
  $e63b = T[B, "d", {μ}] -> T[A, "d", {μ}] - J_F.T[A, "d", {μ}].inv[J_F],
  Yield, $e63 = $ = {T[B_ℋl, "d", {μ}] →
      DiagonalMatrix[{0, -2 T[Λ, "d", {μ}], T[Q, "d", {μ}] - T[Λ, "d", {μ}] 1_2}],
     T[B_ℋq, "d", {μ}] → DiagonalMatrix[{4 / 3 T[Λ, "d", {μ}] 1_3 + T[V, "d", {μ}],
        -2 / 3 T[Λ, "d", {μ}] 1_3 + T[V, "d", {μ}],

        (T[Q, "d", {μ}] + 1 / 3 T[Λ, "d", {μ}] 1_2) ⊗ 1_3 + 1_2 ⊗ T[V, "d", {μ}]}]};
  $ // Column // MatrixForms // Framed, accumStdMdl[{$e63, $e63a, $e63b}]
 ];
PR["Hypercharge assignments(coefficient of Λ's): ", $hypercharge = Association[
   {ν_R → 0, e_R → -2, ν_L → -1, e_L → 2, u_R → 4 / 3, d_R → -2 / 3, u_L → 1 / 3, d_L → 1 / 3}],
 NL, CR["How are ", T[Λ, "d", {μ}], " coefficient determined?"]
]
```

•The field from the term $-i\, a\, \underline{\partial}_\mu[b]$ on $\mathcal{H}_q$

Let $\{a \to m,\ b \to m',\ \{m, m'\} \in \mathcal{H}_q,\ \{m, m'\} \in M_3[\mathbb{C}]\}$

$\to\ V'_\mu \to -i\, m\, \underline{\partial}_\mu[m']$

If V'_μ hermitian $\Rightarrow\ V' \in i\, \mathfrak{u}[3]\ \Rightarrow\ V' \in U[3]$

Impose unimodularity condition to get SU[3] gauge field. Why is I included?

Since $\mathrm{Tr}_{\mathcal{H}_F}[A_\mu] \to 0$

$\to\ \mathrm{Tr}_1[A_\mu] + \mathrm{Tr}_{\bar{q}}[A_\mu] + \mathrm{Tr}_{\bar{I}}[A_\mu] + \mathrm{Tr}_q[A_\mu] \to 0$

$\to\ \mathrm{Tr}_{\bar{I}}[A_\mu] + \mathrm{Tr}_q[A_\mu] \to 0$

$\to\ \mathrm{Tr}_{\mathcal{H}_{\bar{I}}}[1_4\, \Lambda_\mu] + \mathrm{Tr}_{\mathcal{H}_q}[1_4 \otimes V'_\mu] \to 0\ \Rightarrow\ \mathrm{Tr}[V'_\mu] \to -\Lambda_\mu$

For a traceless SU[3] gauge field define: $\boxed{(V_\mu)^* \to -\frac{1}{3} 1_3\, \Lambda_\mu - V'_\mu}$

Then the gauge field becomes: A_μ

$$\to\ \boxed{\begin{array}{l} A_{\mathcal{H}_1\,\mu} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix} \\ A_{\mathcal{H}_{\bar{I}}\,\mu} \to 1_4\, \Lambda_\mu \\ A_{\mathcal{H}_{\bar{q}}\,\mu} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix} \otimes 1_3 \\ A_{\mathcal{H}_q\,\mu} \to -1_4 \otimes \left((V_\mu)^* + \frac{1}{3} 1_3\, \Lambda_\mu\right) \\ \Lambda_\mu \in U[1] \\ Q_\mu \in SU[2] \end{array}}$$

The action on fermions of the field: $B_\mu \to -J_F . A_\mu . J_F^{-1} + A_\mu$

$$\to\ \boxed{\begin{array}{l} B_{\mathcal{H}_1\,\mu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu - 1_2\,\Lambda_\mu \end{pmatrix} \\ B_{\mathcal{H}_q\,\mu} \to \begin{pmatrix} V_\mu + \frac{4}{3} 1_3\,\Lambda_\mu & 0 & 0 \\ 0 & V_\mu - \frac{2}{3} 1_3\,\Lambda_\mu & 0 \\ 0 & 0 & 1_2 \otimes V_\mu + \left(Q_\mu + \frac{1}{3} 1_2\,\Lambda_\mu\right) \otimes 1_3 \end{pmatrix} \end{array}}$$

Hypercharge assignments(coefficient of Λ's):

$$\left\langle \left| \nu_R \to 0,\ e_R \to -2,\ \nu_L \to -1,\ e_L \to 2,\ u_R \to \frac{4}{3},\ d_R \to -\frac{2}{3},\ u_L \to \frac{1}{3},\ d_L \to \frac{1}{3} \right| \right\rangle$$

How are Λ_μ coefficient determined?

```
PR["We compute: ",
 $ = T[B, "d", {μ}] -> T[A, "d", {μ}] - JF.T[A, "d", {μ}].Inverse[JF] //
    Inactivate[#, Plus] &;
 $ = $ /. T[aa : A | B, "d", {μ}] → T[aa_ℋ1, "d", {μ}] ⊕ T[aa_ℋ1̄, "d", {μ}],
 NL, "Using ",
 $s = selectStdMdl[T[A_, "d", {μ}], {}, all]; $s // ColumnBar,
 $ = $ /. Reverse[$s]; $ // MatrixForms,
 NL, "Expanding to 8×8 matrices ",
 Yield, $ = $ // tuCirclePlus2Matrix;
 $ = $ /. 1_4 → DiagonalMatrix[{1, 1, 1, 1}] // tuArrayFlatten;
 $ // MatrixForms;
 $ = $ /. {F → F_8, qq : T[Q, "d", {μ}] → Table[qq_i,j, {i, 2}, {j, 2}]} // tuArrayFlatten;
 $ // MatrixForms;
 Yield, $ = $ /. toxDot /. selectGWS[J_F_, {}, all]; $ // MatrixForms,
 $ = $ // tuMatrixOrderedMultiply; $ // MatrixForms;
 $ = $ // tuOpSimplifyF[xDot];
 Yield, $ = $ /. toDot // Activate; $ // MatrixForms;
 NL, CO["The standard result of Λ∈U[1]: ", $s = cc.Λ_ .(1 / cc) → cc[Λ], Imply,
  $1 = $ /. $s; $1 // MatrixForms,
  NL, " which is different from the result on p.71.",
  NL, "We would get their results if ",
  $s = {T[Λ, "d", {μ}] ∈ ℝ["as used on p.56"], cc[T[Λ, "d", {μ}]] -> T[Λ, "d", {μ}]},
  Yield, $1 = $1 /. tuRule[$s]; $1 // MatrixForms,
  NL,
  "Similar result may be obtained if the antiparticle A-elements were Conjugate: ",
  $s = selectStdMdl[T[A_, "d", {μ}], {}, all] /. 1_n_ T[Λ, "d", {μ}] -> 1_n cc[T[Λ, "d", {μ}]]
 ]
]
```

We compute: $B_{\mathcal{H}_{1\mu}}\oplus B_{\mathcal{H}_{\bar{1}\mu}} \to -J_F.(A_{\mathcal{H}_{1\mu}}\oplus A_{\mathcal{H}_{\bar{1}\mu}}).\text{Inverse}[J_F] + A_{\mathcal{H}_{1\mu}}\oplus A_{\mathcal{H}_{\bar{1}\mu}}$

Using
$$\left|\begin{array}{l} A_{\mathcal{H}_{1\mu}} \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\} \\ A_{\mathcal{H}_{\bar{1}\mu}} \to 1_4\, \Lambda_\mu \\ A_{\mathcal{H}_{q\mu}} \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\}\otimes 1_3 \\ A_{\mathcal{H}_{\bar{q}\mu}} \to -1_4\otimes((V_\mu)^* + \frac{1}{3} 1_3\, \Lambda_\mu) \end{array}\right.$$

$$B_{\mathcal{H}_{1\mu}}\oplus B_{\mathcal{H}_{\bar{1}\mu}} \to -J_F.(\begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix}\oplus 1_4\, \Lambda_\mu).\text{Inverse}[J_F] + \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix}\oplus 1_4\, \Lambda_\mu$$

Expanding to 8×8 matrices

→

$$\to \begin{pmatrix} B_{\mathcal{H}_{1\mu}} & 0 \\ 0 & B_{\mathcal{H}_{\bar{1}\mu}} \end{pmatrix} \to -\text{xDot}[\begin{pmatrix} 0 & 0 & 0 & 0 & cc & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & cc & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & cc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & cc & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & cc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cc & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{\mu 1,1} & Q_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{\mu 2,1} & Q_{\mu 2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu \end{pmatrix},$$

$$\left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & \frac{1}{cc} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{cc} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{cc} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{cc} \\ \frac{1}{cc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{cc} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{cc} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{cc} & 0 & 0 & 0 & 0 \end{array}\right)] + \left(\begin{array}{cccccccc} \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{\mu 1,1} & Q_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{\mu 2,1} & Q_{\mu 2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu \end{array}\right)$$

→

The standard result of $\Lambda \in U[1]$: $cc.(\Lambda\_).\frac{1}{cc} \to \Lambda^*$

$$\Rightarrow \left(\begin{array}{cc} B_{\mathcal{H}_{1\,\mu}} & 0 \\ 0 & B_{\mathcal{H}_{\mathrm{I}\,\mu}} \end{array}\right) \to \left(\begin{array}{cccccc} -(\Lambda_\mu)^* + \Lambda_\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\Lambda_\mu)^* - \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\Lambda_\mu)^* + Q_{\mu 1,1} & Q_{\mu 1,2} & 0 & 0 \\ 0 & 0 & Q_{\mu 2,1} & -(\Lambda_\mu)^* + Q_{\mu 2,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\Lambda_\mu)^* + \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & (\Lambda_\mu)^* + \Lambda_\mu \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

which is different from the result on p.71.
We would get their results if $\{\Lambda_\mu \in \mathbb{R}$[as used on p.56], $(\Lambda_\mu)^* \to \Lambda_\mu\}$

$$\to \left(\begin{array}{cc} B_{\mathcal{H}_{1\,\mu}} & 0 \\ 0 & B_{\mathcal{H}_{\mathrm{I}\,\mu}} \end{array}\right) \to \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{\mu 1,1} - \Lambda_\mu & Q_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{\mu 2,1} & Q_{\mu 2,2} - \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(Q_{\mu 1,1})^* + \Lambda_\mu & -(Q_{\mu 1,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(Q_{\mu 2,1})^* & -(Q_{\mu 2,2})^* + \Lambda_\mu \end{array}\right)$$

Similar result may be obtained if the antiparticle A-elements were Conjugate:
$\{A_{\mathcal{H}_{1\,\mu}} \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\}, A_{\mathcal{H}_{\mathrm{I}\,\mu}} \to (\Lambda_\mu)^* 1_4,$

$A_{\mathcal{H}_{q\,\mu}} \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\} \otimes 1_3, A_{\mathcal{H}_{\bar{q}\,\mu}} \to -1_4 \otimes ((V_\mu)^* + \frac{1}{3}(\Lambda_\mu)^* 1_3)\}$

```
PR["■Check Calculation of B's. For: ", #[[1]] & /@ $e63 // First,
 NL, "Using ", {$e63b, $sJ =
   {tuRuleSelect[$sr][J_F8][[1]] /. F8 → F, inv[J_F] → J_F, inv[aa : cc | 0] → aa, cc^2 → 1}},
 NL, "•For form 8x8 : ", $s = {$e63a[[1, 1]], $e63a[[2, 1]]},

 NL, "Expand elements of: ",
 $sQ = T[Q, "d", {μ}] -> Table[q_i,j, {i, 2}, {j, 2}],
 $sQ = T[Q, "d", {μ}] -> Table[T[q, "ddd", {μ, i, j}], {i, 2}, {j, 2}], (*TEST*)
 $s1 = $e63a[[1]] /. $sQ // MapAt[ArrayFlatten[#] &, #, 2] &;
 $s2 = $e63a[[2]] /. 1_4 → DiagonalMatrix[Table[1, {4}]];
 $sA8 =
  $e63a[[1, 1]] → ({{$e63a[[1, 1]], 0}, {0, $e63a[[2, 1]]}} /. $s1 /. $s2 // ArrayFlatten);
 $sA8 // MatrixForms,
 NL, "Compute ",
 $ = $e63b /. Plus → Inactive[Plus] /. Tensor[a_, i_, j_] :→ Tensor[a_H1, i, j],
 Yield, $ = $ // expandCom[{$sJ, $sA8}];
 Yield, $ = $ /. cc . a_ :→ cc[a].cc /; FreeQ[a, cc] /. cc.cc → 1;
 $ // MatrixForms,
 Yield, $Bhl = $ = $ /. 1 → 1 ⊕ I // tuConjugateSimplify[{cc, T[Λ, "d", {μ}]}] // Activate;
 $ // MatrixForms // Framed, accumStdMdl[$], OK,
 NL, CR["Assumes Λ∈ℝ. Note previous block."]
]
```

■Check Calculation of B's. For: $B_{\mathcal{H}_{1\,\mu}}$

Using $\{B_\mu \to -J_F.A_\mu.J_F^{-1} + A_\mu,$

$\{J_F \to \{\{0, 0, 0, 0, cc, 0, 0, 0\}, \{0, 0, 0, 0, 0, cc, 0, 0\}, \{0, 0, 0, 0, 0, 0, cc, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, cc\}, \{cc, 0, 0, 0, 0, 0, 0, 0\}, \{0, cc, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, cc, 0, 0, 0, 0, 0\}, \{0, 0, 0, cc, 0, 0, 0, 0\}\}, J_F^{-1} \to J_F, (aa : cc \mid 0)^{-1} \to aa, cc^2 \to 1\}\}$

•For form 8x8 : $\{A_{\mathcal{H}_{1\,\mu}}, A_{\mathcal{H}_{\mathrm{I}\,\mu}}\}$

Expand elements of: $Q_\mu \to \{\{q_{1,1}, q_{1,2}\}, \{q_{2,1}, q_{2,2}\}\}$

$$Q_\mu \to \{\{q_{\mu\,1\,1}, q_{\mu\,1\,2}\}, \{q_{\mu\,2\,1}, q_{\mu\,2\,2}\}\} A_{\mathcal{H}_{1\,\mu}} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,1\,1} & q_{\mu\,1\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,2\,1} & q_{\mu\,2\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu \end{pmatrix}$$

Compute $B_{\mathcal{H}_{1\,\mu}} \to -J_F.A_{\mathcal{H}_{1\,\mu}}.J_F^{-1} + A_{\mathcal{H}_{1\,\mu}}$

→

$$\to B_{\mathcal{H}_{1\,\mu}} \to \begin{pmatrix} -(\Lambda_\mu)^*.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\Lambda_\mu)^*.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\Lambda_\mu)^*.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\Lambda_\mu)^*.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\Lambda_\mu)^*.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\Lambda_\mu)^*.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,1\,1})^*.1 & -(q_{\mu\,1\,2})^*.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,2\,1})^*.1 & -(q_{\mu\,2\,2})^*.1 \end{pmatrix} +$$

$$\begin{pmatrix} \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,1\,1} & q_{\mu\,1\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,2\,1} & q_{\mu\,2\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu \end{pmatrix}$$

$$\to \boxed{B_{\mathcal{H}_{1\oplus\mathrm{I}\,\mu}} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,1\,1} - \Lambda_\mu & q_{\mu\,1\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,2\,1} & q_{\mu\,2\,2} - \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,1\,1})^* + \Lambda_\mu & -(q_{\mu\,1\,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,2\,1})^* & -(q_{\mu\,2\,2})^* + \Lambda_\mu \end{pmatrix}} \quad \text{OK}$$

Assumes $\Lambda \in \mathbb{R}$. Note previous block.

```
PR["■Check Calculation of B's. For: ", #[[1]] & /@ $e63 // Last,
  "With ", $0 = $ = {$e63b, $sJ =
      {tuRuleSelect[$sr][J_F8][[1]] /. F8 → F, inv[J_F] → J_F, inv[cc : cc | 0] → cc, cc^2 → 1}};
  $ // MatrixForms // ColumnBar,
  NL, "•Select one copy of the finite portions: ",
  $s = selectStdMdl[Tensor[A_, _, _], #] & /@ {Q, q̄};
  $s // ColumnBar,
  NL, "Expand so q,q̄ versions of A_H are 4x4 so action of J_F matrix is unambiguous: ",
  $a1 = T[A, "d", {μ}] → $e63a[[3, 1]] ⊕ $e63a[[4, 1]],
  $sQ = T[Q, "d", {μ}] -> Table[q_i,j, {i, 2}, {j, 2}];
  $sQ = T[Q, "d", {μ}] -> Table[T[q, "ddd", {μ, i, j}], {i, 2}, {j, 2}];
  (*TEST*)
  NL, $s1 = $e63a[[3]] /. $sQ /. ll : List[List[__], __] ⧴ ArrayFlatten[ll];
  NL, $s2 = $e63a[[4]] /. -1_4 ⊗ a_ :> DiagonalMatrix[Table[-a, {4}]];
  NL,
  $sA8 =
   $a1[[1]] → ({{$e63a[[3, 1]], 0}, {0, $e63a[[4, 1]]}} /. Plus → Inactive[Plus] /. $s1 /.
         $s2 /. a_ ⊗ 1_3 → a 1_3 /. Plus → Inactive[Plus]) /.
    ll : List[List[__], __] ⧴ ArrayFlatten[ll];
  $sA8 // MatrixForms, "POFF",
  Yield, $ = $0[[1]] /. Plus → Inactive[Plus] /. $sA8; $ // MatrixForms,
  Yield, $ = $ // expandCom[{$sJ, $sA8}] // Activate; $ // MatrixForms,

  Yield, $ = $ /. cc . a_ ⧴ Conjugate[a].cc /; FreeQ[a, cc] /. cc.cc → 1 // expandDC[];
  $ // MatrixForms, CK,
  "PONdd",
  Yield, $BHq = $ = $ /. B → B_(H_q⊕H_q̄) // tuConjugateSimplify[{cc, T[Λ, "d", {μ}], 1_3}] //
     tuOpSimplifyF[Dot, {1_3}];
  $ // MatrixForms // Framed, CG["(6.3)"], accumStdMdl[$],
  NL, CR["Note: ", T[V, "d", {μ}] ∈ M_3[ℂ], " so the notation is ambiguous."]
 ];
```

■Check Calculation of B's. For: $B_{\mathcal{H}_{q\,\mu}}$ With

$B_\mu \to -J_F . A_\mu . J_F^{-1} + A_\mu$

$$\left\{J_F \to \begin{pmatrix} 0&0&0&0&cc&0&0&0\\ 0&0&0&0&0&cc&0&0\\ 0&0&0&0&0&0&cc&0\\ 0&0&0&0&0&0&0&cc\\ cc&0&0&0&0&0&0&0\\ 0&cc&0&0&0&0&0&0\\ 0&0&cc&0&0&0&0&0\\ 0&0&0&cc&0&0&0&0 \end{pmatrix},\ J_F^{-1} \to J_F,\ (cc : cc \mid 0)^{-1} \to cc,\ cc^2 \to 1\right\}$$

•Select one copy of the finite portions:

$A_{\mathcal{H}_{q\,\mu}} \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\} \otimes 1_3$

$A_{\mathcal{H}_{\bar{q}\,\mu}} \to -1_4 \otimes ((V_\mu)^* + \frac{1}{3} 1_3 \Lambda_\mu)$

Expand so q,$\bar{q}$ versions of $A_{\mathcal{H}}$ are 4x4 so action of J_F matrix is unambiguous:

$A_\mu \to A_{\mathcal{H}_{q\,\mu}} \oplus A_{\mathcal{H}_{\bar{q}\,\mu}}$

$$A_\mu \to \begin{pmatrix} 1_3\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1_3\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1_3 q_{\mu\,1\,1} & 1_3 q_{\mu\,1\,2} & 0 & 0 & 0 & 0\\ 0 & 0 & 1_3 q_{\mu\,2\,1} & 1_3 q_{\mu\,2\,2} & 0 & 0 & 0 & 0\\ 0&0&0&0& -(V_\mu)^* + -\frac{1}{3}1_3\Lambda_\mu & 0&0&0\\ 0&0&0&0&0& -(V_\mu)^* + -\frac{1}{3}1_3\Lambda_\mu & 0&0\\ 0&0&0&0&0&0& -(V_\mu)^* + -\frac{1}{3}1_3\Lambda_\mu & 0\\ 0&0&0&0&0&0&0& -(V_\mu)^* + -\frac{1}{3}1_3\Lambda_\mu \end{pmatrix}$$

....

→

$$B_{\mathcal{H}_q \oplus \mathcal{H}_{\bar{q}\,\mu}} \to \begin{pmatrix} V_\mu + \frac{4}{3}1_3\Lambda_\mu & 0 & 0 & 0 & 0 & 0\\ 0 & V_\mu - \frac{2}{3}1_3\Lambda_\mu & 0 & 0 & 0 & 0\\ 0 & 0 & 1_3 q_{\mu\,1\,1} + V_\mu + \frac{1}{3}1_3\Lambda_\mu & 1_3 q_{\mu\,1\,2} & 0 & 0\\ 0 & 0 & 1_3 q_{\mu\,2\,1} & 1_3 q_{\mu\,2\,2} + V_\mu + \frac{1}{3}1_3\Lambda_\mu & 0 & 0\\ 0&0&0&0& -(V_\mu)^* - \frac{4}{3}1_3\Lambda_\mu & 0\\ 0&0&0&0&0& -(V_\mu)^* + \frac{2}{3}\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

(6.3)

Note: $V_\mu \in M_3[\mathbb{C}]$ so the notation is ambiguous.

Proposition 6.4

```
PR["●Prop.6.4. The action of the gauge
   group SG[M×FSM] on the fluctuated Dirac operator: ",
 iDA → slash[iD] ⊗ 1F + T[γ, "u", {μ}] ⊗ T[B, "d", {μ}] + T[γ, "d", {5}] ⊗ Φ,
 NL, "is implemented by: ",
 $p64 = $ = {T[Λ, "d", {μ}] -> T[Λ, "d", {μ}] - I λ.tuDDown["∂"][Conjugate[λ], μ],
    T[Q, "d", {μ}] → q. T[Q, "d", {μ}].ct[q] - I q.tuDDown["∂"][ct[q], μ],
    Conjugate[T[V, "d", {μ}]] →
     m. Conjugate[T[V, "d", {μ}]] .ct[m] - I m.tuDDown["∂"][ct[m], μ],
    {{ϕ1 + 1}, {ϕ2}} → Conjugate[λ] q .{{ϕ1 + 1}, {ϕ2}},
    λ ∈ C"∞"[M, U[1]],
    q ∈ C"∞"[M, SU[2]],
    m ∈ C"∞"[M, SU[3]]
   }; $ // Column // MatrixForms // Framed,
 line,
 NL, "The proof examines the action of ",
 u → {λ, q, m} ∈ C"∞"[M, U[1] × SU[2] × SU[3]],
 NL, "as in Proposition 5.3 for ", selectGWS[Tensor[it[A], _, _]],
 Yield, $ = {T[Q, "d", {μ}] → q. T[Q, "d", {μ}].ct[q] ,
   Conjugate[T[V, "d", {μ}]] → m. Conjugate[T[V, "d", {μ}]] .ct[m] ,
   -I u.tuDDown["∂"][ct[u], μ][{νR, uR, ℋI}] → -I λ.tuDDown["∂"][Conjugate[λ], μ],
   -I u.tuDDown["∂"][ct[u], μ][{eR, dR}] → I λ.tuDDown["∂"][Conjugate[λ], μ],
   -I u.tuDDown["∂"][ct[u], μ][{νL, eL, uL, dL}] → -I q.tuDDown["∂"][ct[q], μ],
   -I u.tuDDown["∂"][ct[u], μ][{ℋq}] → -I m.tuDDown["∂"][ct[m], μ]
  }; $ // Column
]
```

●Prop.6.4. The action of the gauge group $S\mathcal{G}[M\times F_{SM}]$ on the fluctuated Dirac operator: $D_A \to (\not{D})\otimes 1_F + \gamma_5 \otimes \Phi + \gamma^\mu \otimes B_\mu$

is implemented by:

$$\boxed{\begin{array}{l}
\Lambda_\mu \to -i\, \lambda.\underset{-\mu}{\partial}[\lambda^*] + \Lambda_\mu \\
Q_\mu \to -i\, q.\underset{-\mu}{\partial}[q^\dagger] + q.Q_\mu.q^\dagger \\
(V_\mu)^* \to -i\, m.\underset{-\mu}{\partial}[m^\dagger] + m.(V_\mu)^*.m^\dagger \\
\begin{pmatrix} 1+\phi_1 \\ \phi_2 \end{pmatrix} \to \lambda^*\, q.\begin{pmatrix} 1+\phi_1 \\ \phi_2 \end{pmatrix} \\
\lambda \in C^\infty[M, U[1]] \\
q \in C^\infty[M, SU[2]] \\
m \in C^\infty[M, SU[3]]
\end{array}}$$

The proof examines the action of $u \to \{\lambda, q, m\} \in C^\infty[M, U[1]\times SU[2]\times SU[3]]$
as in Proposition 5.3 for $A_\mu \to -i\, u.\underline{\partial}_\mu[u^\dagger] + u.A_\mu.u^\dagger$

$$\begin{array}{l}
Q_\mu \to q.Q_\mu.q^\dagger \\
(V_\mu)^* \to m.(V_\mu)^*.m^\dagger \\
-i\, u.\underset{-\mu}{\partial}[u^\dagger][\{\nu_R, u_R, \mathcal{H}_I\}] \to -i\, \lambda.\underset{-\mu}{\partial}[\lambda^*] \\
\to\ -i\, u.\underset{-\mu}{\partial}[u^\dagger][\{e_R, d_R\}] \to i\, \lambda.\underset{-\mu}{\partial}[\lambda^*] \\
-i\, u.\underset{-\mu}{\partial}[u^\dagger][\{\nu_L, e_L, u_L, d_L\}] \to -i\, q.\underset{-\mu}{\partial}[q^\dagger] \\
-i\, u.\underset{-\mu}{\partial}[u^\dagger][\{\mathcal{H}_q\}] \to -i\, m.\underset{-\mu}{\partial}[m^\dagger]
\end{array}$$

6.3 The spectral action - bosonic part of $\mathcal{L}_{SM}$

Lemma 6.5

```
PR["●Lemma 6.5. ",
 $165 = Tr[T[FHq, "dd", {μ, ν}]. T[FHq, "uu", {μ, ν}]] →
```

```
  24 (10 / 3 T[Λ, "dd", {μ, ν}] T[Λ, "uu", {μ, ν}] + Tr[
      T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]] + Tr[T[V, "dd", {μ, ν}] T[V, "uu", {μ, ν}]]),
line,
NL, "Proof:",
next, "The leptonic sector is as in
  Lemma 5.4 multiplied by 3 for the number of generations.",
next, "The quark sector: ",

next, "Calculate F's. Using: ",
NL, "Using ", $e63[[2]] // MatrixForms,
NL, "• Canonical form: ",
$ = selectDef[Tensor[F, _, _]] /. Tensor[F_, i_, j_] → Tensor[F_ℋq, i, j], "POFF",

$ = $ /. Plus → Inactive[Plus],
$ = $ // expandCom[($e63 // tuAddPatternVariable[μ])];
$ = $ // tuDerivativeExpand[{1_}] // Activate // Expand;
Yield, $ // MatrixForms,

$ = $ // tuCircleTimesGather[] // tuOpSimplifyF[Dot, {Tensor[Λ, _, _]}] // Simplify;
Yield, $ // MatrixForms,
"PONdd",
NL, "Using ",
$s = (selectDef[Tensor[F, _, _]] /. tuCommutatorExpand // Reverse //
    (# /. F | B → V &) // Expand);
$s = tuRuleSolve[$s, T[V, "d", {μ}].T[V, "d", {ν}]] // First;
$s = {$s, $s /. V → Q};
$sx = selectGWS[Tensor[B, _, _]] /. tuCommutatorExpand // Reverse // (# /. B → Λ &);
$s = Append[$s, tuRuleSolve[$sx, $sx[[1, 2]]] ] // Flatten;
$s // ColumnBar,
Yield, $ = $ /. $s // Expand // Simplify;
NL, "Simplifying with: ", $simple = {a_ . 1_n_ → 1_n.a,
  1_n_ . 1_n_ a_ :→ a 1_n, 1_2. qq : Tensor[Q, _, _] → qq, 1_3. qq : Tensor[V, _, _] → qq},
Yield, $ = $ //. $simple // Simplify // tuCircleTimesSimplify;
$ // ColumnSumExp,
$sF = {$, $ // tuIndicesRaise[{μ, ν}]};

line,
next, "Compute ", $ = $165[[1]],
Yield, $ = $ /. Tr → xTr /. toxDot /. $sF // tuMatrixOrderedMultiply //
  tuOpSimplifyF[xDot, {Tensor[Λ, _, _]}];
Yield, $ = $ /. toDot // expandDC[] // tuOpSimplifyF[Dot, {Tensor[Λ, _, _]}] //
  tuCircleTimesGather[];
$ // MatrixForms // ColumnSumExp;
$ = $ //. $simple // tuIndexDummyOrdered //
   tuOpSimplifyF[Dot, {Tensor[Λ, _, _]}] // (# //. $simple &);
$ = $ //. tuOpDistribute[CircleTimes] // tuIndexDummyOrdered //
  (# //. tuOpCollect[CircleTimes] &);
$ // ColumnSumExp,

NL, "Compute xTr[] ",
$ = $ //. xx : xTr[a_] :> Thread[xx] /. xTr[0] → 0 /. aa : CircleTimes[a_, 1_n_] :→
     tuOpDistributeF[CircleTimes][aa] //. tuOpDistribute[xTr] // Tr[#] &;
$ // ColumnSumExp,
NL, "• The Q and V are members of SU[2] and SU[3],
  respectively; hence their Tr[]'s are zero, as well as their
  products. The Tr[] of single Q,V's and Λ will be zero as well.",
NL, "• Use Rule: ", $s = xTr[a_] :→ 0 /; (tuExtractPattern[Tensor[Q | V | Λ, _, _]][{a}] //
```

```
      tuHasAllQ[#, {V, Q}] || tuHasAllQ[#, {V, Λ}] || tuHasAllQ[#, {Λ, Q}] &),
  $ = $ /. $s;
  Yield, $ // ColumnSumExp;
  NL, "Use Rules: ",
  $s = {xTr[1n_ ⊗ a_] → n xTr[a], xTr[a_ ⊗ 1n_] → n xTr[a], xTr[a_ 1n_] → n xTr[a],
    Tr[ll : Tensor[Λ, a_, b_] Tensor[Λ, c_, d_]] → ll, xTr → Tr, Dot → Times};
  $s // ColumnBar,
  Yield, $ = $ //. $s // tuTrSimplify[] // tuIndexDummyOrdered // (# /. $s &);
  $165[[1]] -> $ // ColumnSumExp // Framed,
  NL, CR["Need to add contribution from I,l,q̄  to get complete result."]
]
```

●Lemma 6.5. $\mathrm{Tr}[F_{\mathcal{H}_q\,\mu\nu}.F_{\mathcal{H}_q}{}^{\mu\nu}] \to 24\,(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}] + \mathrm{Tr}[V_{\mu\nu}V^{\mu\nu}])$

Proof:

◆The leptonic sector is as in
   Lemma 5.4 multiplied by 3 for the number of generations.

◆The quark sector:

◆Calculate F's. Using:

Using $B_{\mathcal{H}_q\,\mu} \to \begin{pmatrix} V_\mu + \frac{4}{3}1_3\Lambda_\mu & 0 & 0 \\ 0 & V_\mu - \frac{2}{3}1_3\Lambda_\mu & 0 \\ 0 & 0 & 1_2\otimes V_\mu + (Q_\mu + \frac{1}{3}1_2\Lambda_\mu)\otimes 1_3 \end{pmatrix}$

• Canonical form: $F_{\mathcal{H}_q\,\mu\nu} \to i\,[B_{\mathcal{H}_q\,\mu}, B_{\mathcal{H}_q\,\nu}]_- - \underline{\partial}_\nu[B_{\mathcal{H}_q\,\mu}] + \underline{\partial}_\mu[B_{\mathcal{H}_q\,\nu}]$

....

Using
$$\left|\begin{array}{l} V_\mu.V_\nu \to V_\nu.V_\mu + i\,(-V_{\mu\nu} - \underset{-\nu}{\partial}[V_\mu] + \underset{-\mu}{\partial}[V_\nu]) \\ Q_\mu.Q_\nu \to Q_\nu.Q_\mu + i\,(-Q_{\mu\nu} - \underset{-\nu}{\partial}[Q_\mu] + \underset{-\mu}{\partial}[Q_\nu]) \\ \underset{-\mu}{\partial}[\Lambda_\nu] \to \Lambda_{\mu\nu} + \underset{-\nu}{\partial}[\Lambda_\mu] \end{array}\right.$$

→

Simplifying with:

{(a_).1n_ → 1n.a, 1n_.1n_ a_ :→ a 1n, 12.(qq : Tensor[Q, _, _]) → qq, 13.(qq : Tensor[V, _, _]) → qq}

→ $F_{\mathcal{H}_q\,\mu\nu} \to \{\{\sum[\left|\begin{array}{l} V_{\mu\nu} \\ \frac{4}{3}1_3\Lambda_{\mu\nu}\end{array}\right.], 0, 0\}, \{0, \sum[\left|\begin{array}{l} V_{\mu\nu} \\ -\frac{2}{3}1_3\Lambda_{\mu\nu}\end{array}\right.], 0\}, \{0, 0, \sum[\left|\begin{array}{l} 1_2\otimes V_{\mu\nu} \\ (Q_{\mu\nu} + \frac{1}{3}1_2\Lambda_{\mu\nu})\otimes 1_3\end{array}\right.]\}\}$

◆Compute $\mathrm{Tr}[F_{\mathcal{H}_q\,\mu\nu}.F_{\mathcal{H}_q}{}^{\mu\nu}]$

→

→ $\mathrm{xTr}[\{\{\sum[\left|\begin{array}{l} V_{\mu\nu}.V^{\mu\nu} \\ \frac{8}{3}V^{\mu\nu}\Lambda_{\mu\nu} \\ \frac{16}{9}1_3\Lambda_\mu{}^\nu\Lambda^\mu{}_\nu \end{array}\right.], 0, 0\},$

$\{0, \sum[\left|\begin{array}{l} V_{\mu\nu}.V^{\mu\nu} \\ -\frac{4}{3}V^{\mu\nu}\Lambda_{\mu\nu} \\ \frac{4}{9}1_3\Lambda_\mu{}^\nu\Lambda^\mu{}_\nu \end{array}\right.], 0\}, \{0, 0, \sum[\left|\begin{array}{l} 1_2\otimes V_{\mu\nu}.V^{\mu\nu} \\ 2\,(Q_{\mu\nu} + \frac{1}{3}1_2\Lambda_{\mu\nu})\otimes V^{\mu\nu} \\ (Q_{\mu\nu}.Q^{\mu\nu} + \frac{2}{3}Q^{\mu\nu}\Lambda_{\mu\nu} + \frac{1}{9}1_2\Lambda_\mu{}^\nu\Lambda^\mu{}_\nu)\otimes 1_3 \end{array}\right.]\}\}]$

Compute xTr[] ∑[
```
xTr[Q_{μ ν}.Q^{μ ν}⊗1_3]
xTr[1_2⊗V_{μ ν}.V^{μ ν}]
xTr[(2/3 Q^{μ ν} Λ_{μ ν})⊗1_3]
xTr[2 (Q_{μ ν} + 1/3 1_2 Λ_{μ ν})⊗V^{μ ν}]
xTr[(1/9 1_2 Λ_μ^ν Λ^μ_ν)⊗1_3]
2 xTr[V_{μ ν}.V^{μ ν}]
xTr[-4/3 V^{μ ν} Λ_{μ ν}]
xTr[8/3 V^{μ ν} Λ_{μ ν}]
xTr[4/9 1_3 Λ_μ^ν Λ^μ_ν]
xTr[16/9 1_3 Λ_μ^ν Λ^μ_ν]
```
]

- The Q and V are members of SU[2] and SU[3], respectively; hence their Tr[]'s are zero, as well as their products. The Tr[] of single Q,V's and Λ will be zero as well.
- Use Rule: xTr[a_] :→ 0 /; (tuHasAllQ[#1, {V, Q}] || tuHasAllQ[#1, {V, Λ}] || tuHasAllQ[#1, {Λ, Q}] &)[tuExtractPattern[Tensor[Q | V | Λ, _, _]][{a}]]

→

Use Rules:
```
xTr[1_n_ ⊗a_] → n xTr[a]
xTr[a_⊗1_n_] → n xTr[a]
xTr[a_ 1_n_] → n xTr[a]
Tr[ll : Tensor[Λ, a_, b_] Tensor[Λ, c_, d_]] → ll
xTr → Tr
Dot → Times
```

→ $\boxed{\mathrm{Tr}[F_{\mathcal{H}_q\,\mu\nu}.F_{\mathcal{H}_q}{}^{\mu\nu}] \to \sum\left[\begin{array}{l} \frac{22}{3}\Lambda_\mu{}^\nu \Lambda^\mu{}_\nu \\ 3\,\mathrm{Tr}[Q_\mu{}^\nu Q^\mu{}_\nu] \\ 4\,\mathrm{Tr}[V_\mu{}^\nu V^\mu{}_\nu] \end{array}\right]}$

Need to add contribution from I,l,q̄ to get complete result.

Lemma 6.6

```
PR["●Lemma 6.6 ",
 $166 = $ = {Tr[Φ^2] → 4 a Abs[H']^2 + 2 c,
    Tr[Φ^4] → 4 b Abs[H']^4 + 8 e Abs[H']^2 + 2 d,
    H' → {ϕ1 + 1, ϕ2},
    a → Tr[ct[Yν].Yν + ct[Ye].Ye + 3 ct[Yu].Yu + 3 ct[Yd].Yd],
    b → Tr[(ct[Yν].Yν)^2 + (ct[Ye].Ye)^2 + 3 (ct[Yu].Yu)^2 + 3 (ct[Yd].Yd)^2],
    c → Tr[ct[YR].YR],
    d -> Tr[(ct[YR].YR)^2],
    e -> Tr[ct[YR].YR.ct[Yν].Yν]
   };
 $ // ColumnBar,
 line,
 NL, "Proof: Compute: ", $0 = $ = $166[[1]],

 NL, "Given ", $sΦ = $e61; $sΦ // MatrixForms // ColumnBar,
 Yield, $ = tuRuleSelect[$sΦ][Φ][[1]]; $ // MatrixForms;
 Yield, $0 = $[[1]] → $[[2, 2]];
 (*****)
 line,
 NL, "What does this look like for basis (without generations and color): ",
 $basisSM = selectStdMdl[basisSM],
 line,
 NL, "Determine S: ", $Slq = $ = S1q -> S1 ⊕ Sq,
 NL, "where ",
 $sS = {selectStdMdl[S1], First[#] & /@ selectStdMdl[Sq ⊗ _]},
 Imply, $ = $ /. $sS // tuCirclePlus2Matrix; $ // MatrixForms, CK,
 accumStdMdl[{$}]
]
selectStdMdl[T.νR]
selectStdMdl[T.f]
```

●Lemma 6.6

$Tr[\Phi^2] \to 2\,c + 4\,a\,Abs[H']^2$
$Tr[\Phi^4] \to 2\,d + 8\,e\,Abs[H']^2 + 4\,b\,Abs[H']^4$
$H' \to \{1 + \phi_1, \phi_2\}$
$a \to Tr[3\,(Y_d)^\dagger.Y_d + (Y_e)^\dagger.Y_e + 3\,(Y_u)^\dagger.Y_u + (Y_\nu)^\dagger.Y_\nu]$
$b \to Tr[3\,((Y_d)^\dagger.Y_d)^2 + ((Y_e)^\dagger.Y_e)^2 + 3\,((Y_u)^\dagger.Y_u)^2 + ((Y_\nu)^\dagger.Y_\nu)^2]$
$c \to Tr[(Y_R)^\dagger.Y_R]$
$d \to Tr[((Y_R)^\dagger.Y_R)^2]$
$e \to Tr[(Y_R)^\dagger.Y_R.(Y_\nu)^\dagger.Y_\nu]$

Proof: Compute: $Tr[\Phi^2] \to 2\,c + 4\,a\,Abs[H']^2$

Given

$\phi_{\mathcal{H}_1} \to \begin{pmatrix} 0 & Y^\dagger \\ Y & 0 \end{pmatrix}$
$\phi_{\mathcal{H}_{\bar{1}}} \to 0$
$\phi_{\mathcal{H}_q} \to \begin{pmatrix} 0 & X^\dagger \\ X & 0 \end{pmatrix} \otimes 1_3[color]$
$\phi_{\mathcal{H}_{\bar{q}}} \to 0$
$\{\phi_1, \phi_2\} \in \mathbb{C}$
$Y \to \begin{pmatrix} Y_\nu\,\phi_1 & -(\phi_2)^*\,Y_e \\ Y_\nu\,\phi_2 & (\phi_1)^*\,Y_e \end{pmatrix}$
$X \to \begin{pmatrix} Y_u\,\phi_1 & -(\phi_2)^*\,Y_d \\ Y_u\,\phi_2 & (\phi_1)^*\,Y_d \end{pmatrix}$
$\Phi \to D_{F_2} + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F.\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}.(J_F)^\dagger \to \begin{pmatrix} S+\phi & T^\dagger \\ T & (S+\phi)^* \end{pmatrix}$

→
→

What does this look like for basis (without generations and color):

$basisSM \to \{\nu_R, e_R, \nu_L, e_L, u_R, d_R, u_L, d_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L, \bar{u}_R, \bar{d}_R, \bar{u}_L, \bar{d}_L\}$

Determine S: $S_{1q} \to S_1 \oplus S_q$

where $\{S_1 \to \{\{0, 0, Y_\nu, 0\}, \{0, 0, 0, Y_e\}, \{(Y_\nu)^\dagger, 0, 0, 0\}, \{0, (Y_e)^\dagger, 0, 0\}\},$
$S_q \to \{\{0, 0, Y_u, 0\}, \{0, 0, 0, Y_d\}, \{(Y_u)^\dagger, 0, 0, 0\}, \{0, (Y_d)^\dagger, 0, 0\}\}\}$

$$\Rightarrow S_{1q} \to \begin{pmatrix} 0 & 0 & Y_\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_e & 0 & 0 & 0 & 0 \\ (Y_\nu)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (Y_e)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_d \\ 0 & 0 & 0 & 0 & (Y_u)^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (Y_d)^\dagger & 0 & 0 \end{pmatrix} \longleftarrow \text{CHECK}$$

$T.\nu_R \to Y_R.\bar{\nu}_R$

$T.f :\to 0 \;/;\; f =!= \nu_R$

```
PR["•Construction of SM Φ: ",
  NL, "Using relationships: ",
  $ = $sΦ; $ // MatrixForms,
  next, "Construct {l,q} version of: ",
  Imply, $ = selectStdMdl[ϕℋl]; $ // MatrixForms,
  yield, $l = $ /. selectStdMdl[Y] // MapAt[ArrayFlatten[#] &, #, 2] &;
  $l // MatrixForms,
  $q = selectStdMdl[ϕℋq] /. a_ ⊗ b_ → a; $q // MatrixForms,
  yield, $q = $q /. selectStdMdl[X] // MapAt[ArrayFlatten[#] &, #, 2] &;
  $q // MatrixForms,
  NL, "Taking ", $ϕ = ϕ -> $l[[1]] ⊕ $q[[1]],
  Imply, $0 = $ = {{ϕ, 0}, {0, 0}},
  yield, $ = $ /. $ϕ; $ // MatrixForms,
  Yield, $ = $ /. $l /. $q // tuCirclePlus2Matrix; $ // MatrixForms,
  NL, "Add 0's of dimension ", $d0 = Dimensions[$[[1, 1]]],
  $d0 = Table[Table[0, $d0[[1]]], $d0[[2]]];
  $ = $ // ArrayFlatten;
  Yield, $[[-1, -1]] = $d0;
  $ϕ = $ = $0 → ($ // ArrayFlatten); $ // MatrixForms
 ];
PR["Check calculation of: ", $0 = JF.{{ϕ, 0}, {0, 0}}.ct[JF],
  NL, "Construct: ",
  $ = DiagonalMatrix[Table[cc, {8}], 8] + DiagonalMatrix[Table[cc, {8}], -8];
  $ // MatrixForm;
  $j = $ = JF → $; $ // MatrixForms,
  Imply,
  $ = JF.{{ϕ, 0}, {0, 0}}.ct[JF] /. Dot → xDot /. $j /. $ϕ // OrderedxDotMultiplyAll[];
  $ // MatrixForms;
  Yield,
  $JphJ = $ = $0 -> $ // tuRepeat[{Conjugate[cc] → cc, cc.cc → 1, Conjugate[cc].cc → 1,
       cc . a_ :> Conjugate[a].cc /; a =!= cc}, tuConjugateSimplify[{}]];
  $ // MatrixForms
 ];
```

•Construction of SM Φ:
Using relationships:

$$\left\{\phi_{\mathcal{H}_1}\to\begin{pmatrix}0&Y^\dagger\\Y&0\end{pmatrix},\ \phi_{\mathcal{H}_{\mathrm{T}}}\to 0,\ \phi_{\mathcal{H}_q}\to\begin{pmatrix}0&X^\dagger\\X&0\end{pmatrix}\otimes 1_3[\mathrm{color}],\ \phi_{\mathcal{H}_{\mathrm{q}}}\to 0,\ \{\phi_1,\phi_2\}\in\mathbb{C},\ Y\to\begin{pmatrix}Y_\nu\,\phi_1&-(\phi_2)^*\,Y_e\\Y_\nu\,\phi_2&(\phi_1)^*\,Y_e\end{pmatrix},\right.$$

$$\left.X\to\begin{pmatrix}Y_u\,\phi_1&-(\phi_2)^*\,Y_d\\Y_u\,\phi_2&(\phi_1)^*\,Y_d\end{pmatrix},\ \Phi\to D_{F_2}+\begin{pmatrix}\phi&0\\0&0\end{pmatrix}+J_F.\begin{pmatrix}\phi&0\\0&0\end{pmatrix}.(J_F)^\dagger\to\begin{pmatrix}S+\phi&T^\dagger\\T&(S+\phi)^*\end{pmatrix}\right\}$$

◆Construct {l,q} version of:

$$\Rightarrow\ \phi_{\mathcal{H}_1}\to\begin{pmatrix}0&Y^\dagger\\Y&0\end{pmatrix}\longrightarrow\phi_{\mathcal{H}_1}\to\begin{pmatrix}0&0&(Y_\nu\,\phi_1)^*&(Y_\nu\,\phi_2)^*\\0&0&-(Y_e)^*\,\phi_2&(Y_e)^*\,\phi_1\\Y_\nu\,\phi_1&-(\phi_2)^*\,Y_e&0&0\\Y_\nu\,\phi_2&(\phi_1)^*\,Y_e&0&0\end{pmatrix}$$

$$\phi_{\mathcal{H}_q}\to\begin{pmatrix}0&X^\dagger\\X&0\end{pmatrix}\longrightarrow\phi_{\mathcal{H}_q}\to\begin{pmatrix}0&0&(Y_u\,\phi_1)^*&(Y_u\,\phi_2)^*\\0&0&-(Y_d)^*\,\phi_2&(Y_d)^*\,\phi_1\\Y_u\,\phi_1&-(\phi_2)^*\,Y_d&0&0\\Y_u\,\phi_2&(\phi_1)^*\,Y_d&0&0\end{pmatrix}$$

Taking $\phi\to\phi_{\mathcal{H}_1}\oplus\phi_{\mathcal{H}_q}$

$$\Rightarrow\ \{\{\phi,0\},\{0,0\}\}\longrightarrow\begin{pmatrix}\phi_{\mathcal{H}_1}\oplus\phi_{\mathcal{H}_q}&0\\0&0\end{pmatrix}$$

$$\to\begin{pmatrix}\begin{pmatrix}0&0&(Y_\nu\,\phi_1)^*&(Y_\nu\,\phi_2)^*&0&0&0&0\\0&0&-(Y_e)^*\,\phi_2&(Y_e)^*\,\phi_1&0&0&0&0\\Y_\nu\,\phi_1&-(\phi_2)^*\,Y_e&0&0&0&0&0&0\\Y_\nu\,\phi_2&(\phi_1)^*\,Y_e&0&0&0&0&0&0\\0&0&0&0&0&0&(Y_u\,\phi_1)^*&(Y_u\,\phi_2)^*\\0&0&0&0&0&0&-(Y_d)^*\,\phi_2&(Y_d)^*\,\phi_1\\0&0&0&0&Y_u\,\phi_1&-(\phi_2)^*\,Y_d&0&0\\0&0&0&0&Y_u\,\phi_2&(\phi_1)^*\,Y_d&0&0\end{pmatrix}&0\\0&0\end{pmatrix}$$

Add 0's of dimension {8, 8}

$$\to\begin{pmatrix}\phi&0\\0&0\end{pmatrix}\to$$

$$\left(\begin{array}{cccccccccccccccc}0&0&(Y_\nu\,\phi_1)^*&(Y_\nu\,\phi_2)^*&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&-(Y_e)^*\,\phi_2&(Y_e)^*\,\phi_1&0&0&0&0&0&0&0&0&0&0&0&0\\Y_\nu\,\phi_1&-(\phi_2)^*\,Y_e&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\Y_\nu\,\phi_2&(\phi_1)^*\,Y_e&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&(Y_u\,\phi_1)^*&(Y_u\,\phi_2)^*&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-(Y_d)^*\,\phi_2&(Y_d)^*\,\phi_1&0&0&0&0&0&0&0&0\\0&0&0&0&Y_u\,\phi_1&-(\phi_2)^*\,Y_d&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&Y_u\,\phi_2&(\phi_1)^*\,Y_d&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\end{array}\right)$$

Check calculation of: $J_F.\{\{\phi, 0\}, \{0, 0\}\}.(J_F)^\dagger$

Construct: $J_F \to \begin{pmatrix} 0&0&0&0&0&0&0&0&cc&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&cc&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&cc&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&cc&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&cc&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&cc&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&cc&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&cc \\ cc&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&cc&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&cc&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&cc&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&cc&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&cc&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&cc&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&cc&0&0&0&0&0&0&0&0 \end{pmatrix}$

⇒

→ $J_F.\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}.(J_F)^\dagger \to \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&Y_\nu.\phi_1&Y_\nu.\phi_2&0& \\ 0&0&0&0&0&0&0&0&0&0&-Y_e.(\phi_2)^*&Y_e.(\phi_1)^*&0& \\ 0&0&0&0&0&0&0&0&(Y_\nu)^*.(\phi_1)^*&-\phi_2.(Y_e)^*&0&0&0& \\ 0&0&0&0&0&0&0&0&(Y_\nu)^*.(\phi_2)^*&\phi_1.(Y_e)^*&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&0&(Y_u)^*.(\phi_1)^*&-\phi_2. \\ 0&0&0&0&0&0&0&0&0&0&0&0&(Y_u)^*.(\phi_2)^*&\phi_1. \end{pmatrix}$

```
PR[next, "Construct 16x16: ", $0 = selectStdMdl[{iDF}, {T, S}],
 NL, "Given: ", $ = {$Slq = selectStdMdl[S1q], $sT = selectStdMdl[T.νR]};
 $ // MatrixForms,
 Imply, $sT = T → ({Normal[SparseArray[{{1, 1} -> YR}, {8, 8}]]} // ArrayFlatten // First);
 $ // MatrixForms,
 NL, "Inserting into: ", $ = $0,
 Yield, $DF = $ = $ /. ($Slq /. S1q → S) /. $sT // MapAt[ArrayFlatten[#] &, #, 2] &;
 $ // MatrixForms, accumStdMdl[$]
]
```

◆Construct 16x16: $D_F \to \{\{S, T^\dagger\}, \{T, S^*\}\}$

Given: $\{S_{1q} \to \begin{pmatrix} 0 & 0 & Y_\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_e & 0 & 0 & 0 & 0 \\ (Y_\nu)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (Y_e)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_d \\ 0 & 0 & 0 & 0 & (Y_u)^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (Y_d)^\dagger & 0 & 0 \end{pmatrix}, T.\nu_R \to Y_R.\overline{\nu_R}\}$

⇒ $\{S_{1q} \to \begin{pmatrix} 0 & 0 & Y_\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_e & 0 & 0 & 0 & 0 \\ (Y_\nu)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (Y_e)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_d \\ 0 & 0 & 0 & 0 & (Y_u)^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (Y_d)^\dagger & 0 & 0 \end{pmatrix}, T.\nu_R \to Y_R.\overline{\nu_R}\}$

Inserting into: $D_F \to \{\{S, T^\dagger\}, \{T, S^*\}\}$

→ $D_F \to$

$$\begin{pmatrix}
0 & 0 & Y_\nu & 0 & 0 & 0 & 0 & 0 & (Y_R)^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Y_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(Y_\nu)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (Y_e)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Y_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (Y_u)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (Y_d)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_\nu)^* & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_e)^* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_\nu)^{\dagger *} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_e)^{\dagger *} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_u)^* & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_d)^* \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_u)^{\dagger *} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_d)^{\dagger *} & 0 & 0
\end{pmatrix}$$

```
PR["Substitute into: ", $ = selectStdMdl[Φ, {J_F}],
 $ = $[[1]] → $[[2, 1]]; $ // MatrixForms,
 Yield, $Φ = $ = $ /. ($DF /. F → F2) /. $JphJ /. $ϕ // Activate;
 $ // MatrixForms, accumStdMdl[$]
]
```

Substitute into:

$\Phi \to D_{F_2} + \{\{\phi, 0\}, \{0, 0\}\} + J_F.\{\{\phi, 0\}, \{0, 0\}\}.(J_F)^\dagger \to \{\{S+\phi, T^\dagger\}, \{T, (S+\phi)^*\}\}$

$\Phi \to D_{F_2} + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F.\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}.(J_F)^\dagger$

$$\rightarrow \quad \Phi \to \left(\begin{matrix}
0 & 0 & (Y_\nu\,\phi_1)^* + Y_\nu & (Y_\nu\,\phi_2)^* & 0 & 0 & 0 \\
0 & 0 & -(Y_e)^*\,\phi_2 & Y_e + (Y_e)^*\,\phi_1 & 0 & 0 & 0 \\
(Y_\nu)^\dagger + Y_\nu\,\phi_1 & -(\phi_2)^*\,Y_e & 0 & 0 & 0 & 0 & 0 \\
Y_\nu\,\phi_2 & (Y_e)^\dagger + (\phi_1)^*\,Y_e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (Y_u\,\phi_1)^* + Y \\
0 & 0 & 0 & 0 & 0 & 0 & -(Y_d)^*\,\phi_2 \\
0 & 0 & 0 & 0 & (Y_u)^\dagger + Y_u\,\phi_1 & -(\phi_2)^*\,Y_d & 0 \\
0 & 0 & 0 & 0 & Y_u\,\phi_2 & (Y_d)^\dagger + (\phi_1)^*\,Y_d & 0 \\
Y_R & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} \right.$$

```
PR["Compute ", $0 = Tr[Φ.Φ],
 NL, "• with scalars: ", $scal = {φ1, φ2},
 NL, "• symmetry of Y's: ", $sY = {Transpose[(yy : Yn_)] → yy, ct[(yy : Yn_)] → cc[yy]},
 NL, "• defining ",
 $ = {{1 + φ1}, {φ2}};
 $sH = Abs[H']^2 -> ct[$].$,
 $sH = tuRuleSolve[$sH, cc[φ2] φ2][[1]];
 $sH2 = #^2 & /@ $sH // Expand;
 (**)
 Yield, $TrPP = $ = xTr[ct[Φ].Φ] /. toxDot /. $Φ // tuConjugateTransposeExpand //
      tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
 $ // ColumnSumExp;
 $ = $ // tuTrEvaluate[{}] // (# /. xTr → Tr &); $ // ColumnSumExp;
 $ = $ // tuConjugateTransposeSimplify[{}, $scal]; $ // ColumnSumExp;
 $ = $ /. $sY // Simplify; $ // ColumnSumExp;
 $0a = $ =
   $ /. tt : Tr[a_] :→ tuTrSimplify[{φ1, φ2}][tt] /. tt : Tr[a_] :> tuTrCanonicalOrder[tt] /.
     $sH // Collect[#, Tr[_], Simplify] &;
 $ // ColumnSumExp,
 NL, CR["If ", $s = φ1 → 0],
 Yield, $ = $ /. $s; $ // ColumnSumExp,
 NL, CR["we get the result in the Lemma."],
 NL, "• Similarly, An examination of the φ1 terms ",
 NL, "with ", $ = Im[a] → (a - cc[a]) / 2;
 Yield, $cc = tuRuleSolve[$, cc[a]] /. a → φ1,
 Yield, $ = $0a /. $cc // Collect[#, Tr[_], Simplify] &; $ // ColumnSumExp,
 Yield, $s = Map[#[$] &, (tuTermSelect /@ {cc[Ye].cc[Ye], Ye.Ye})] // Flatten // Column,
 Imply, "Let ", $s = {Im[φ1] → φ1, cc[Ye_].cc[Ye_] -> Ye.Ye},
 Yield, $ = $ /. $s; $ // ColumnSumExp,
 NL,
 CR["If {φ1 pure imaginary, Y's ∈ℝ} the lemma is also satisfied.  This contraint on
    φ1 and Y's seems to be missing in the text. "],

 note, " The u,d terms need factors of 3 to account for the 3-color space."
]
```

Compute $Tr[\Phi.\Phi]$
• with scalars: $\{\phi_1, \phi_2\}$
• symmetry of Y's: $\{yy : Y_{n\_}{}^{T} \to yy,\ (yy : Y_{n\_})^{\dagger} \to yy^{*}\}$
• defining $Abs[H']^2 \to \{\{(1 + (\phi_1)^*)\ (1 + \phi_1) + (\phi_2)^*\ \phi_2\}\}$

$$\to \sum\left[\begin{array}{l} 4\ \phi_1\ Tr[(Y_d)^*.(Y_d)^*] \\ -4\ (-Abs[H']^2 + (\phi_1)^* + \phi_1)\ Tr[(Y_d)^*.Y_d] \\ 4\ \phi_1\ Tr[(Y_e)^*.(Y_e)^*] \\ -4\ (-Abs[H']^2 + (\phi_1)^* + \phi_1)\ Tr[(Y_e)^*.Y_e] \\ 2\ Tr[(Y_R)^*.Y_R] \\ 4\ (\phi_1)^*\ Tr[(Y_u)^*.(Y_u)^*] \\ -4\ (-Abs[H']^2 + (\phi_1)^* + \phi_1)\ Tr[(Y_u)^*.Y_u] \\ 4\ (\phi_1)^*\ Tr[(Y_\nu)^*.(Y_\nu)^*] \\ -4\ (-Abs[H']^2 + (\phi_1)^* + \phi_1)\ Tr[(Y_\nu)^*.Y_\nu] \\ 4\ (\phi_1)^*\ Tr[Y_d.Y_d] \\ 4\ (\phi_1)^*\ Tr[Y_e.Y_e] \\ 4\ \phi_1\ Tr[Y_u.Y_u] \\ 4\ \phi_1\ Tr[Y_\nu.Y_\nu] \end{array}\right]$$

If $\phi_1 \to 0$

$$\to \sum\left[\begin{array}{l} 4\ Abs[H']^2\ Tr[(Y_d)^*.Y_d] \\ 4\ Abs[H']^2\ Tr[(Y_e)^*.Y_e] \\ 2\ Tr[(Y_R)^*.Y_R] \\ 4\ Abs[H']^2\ Tr[(Y_u)^*.Y_u] \\ 4\ Abs[H']^2\ Tr[(Y_\nu)^*.Y_\nu] \end{array}\right]$$

we get the result in the Lemma.
• Similarly, An examination of the ϕ_1 terms with
$\to \{(\phi_1)^* \to -2\ Im[\phi_1] + \phi_1\}$

$$\to \sum\left[\begin{array}{l} 4\ \phi_1\ Tr[(Y_d)^*.(Y_d)^*] \\ 4\ (Abs[H']^2 + 2\ Im[\phi_1] - 2\ \phi_1)\ Tr[(Y_d)^*.Y_d] \\ 4\ \phi_1\ Tr[(Y_e)^*.(Y_e)^*] \\ 4\ (Abs[H']^2 + 2\ Im[\phi_1] - 2\ \phi_1)\ Tr[(Y_e)^*.Y_e] \\ 2\ Tr[(Y_R)^*.Y_R] \\ 4\ (-2\ Im[\phi_1] + \phi_1)\ Tr[(Y_u)^*.(Y_u)^*] \\ 4\ (Abs[H']^2 + 2\ Im[\phi_1] - 2\ \phi_1)\ Tr[(Y_u)^*.Y_u] \\ 4\ (-2\ Im[\phi_1] + \phi_1)\ Tr[(Y_\nu)^*.(Y_\nu)^*] \\ 4\ (Abs[H']^2 + 2\ Im[\phi_1] - 2\ \phi_1)\ Tr[(Y_\nu)^*.Y_\nu] \\ 4\ (-2\ Im[\phi_1] + \phi_1)\ Tr[Y_d.Y_d] \\ 4\ (-2\ Im[\phi_1] + \phi_1)\ Tr[Y_e.Y_e] \\ 4\ \phi_1\ Tr[Y_u.Y_u] \\ 4\ \phi_1\ Tr[Y_\nu.Y_\nu] \end{array}\right]$$

$$\to \begin{array}{l} 4\ \phi_1\ Tr[(Y_e)^*.(Y_e)^*] \\ 4\ (-2\ Im[\phi_1] + \phi_1)\ Tr[Y_e.Y_e] \end{array}$$

⇒ Let $\{Im[\phi_1] \to \phi_1,\ (Y_{e\_})^*.(Y_{e\_})^* \to Y_e.Y_e\}$

$$\to \sum\left[\begin{array}{l} 4\ Abs[H']^2\ Tr[(Y_d)^*.Y_d] \\ 4\ Abs[H']^2\ Tr[(Y_e)^*.Y_e] \\ 2\ Tr[(Y_R)^*.Y_R] \\ 4\ Abs[H']^2\ Tr[(Y_u)^*.Y_u] \\ 4\ Abs[H']^2\ Tr[(Y_\nu)^*.Y_\nu] \end{array}\right]$$

If $\{\phi_1$ pure imaginary, Y's $\in \mathbb{R}\}$ the lemma is also satisfied.
This contraint on ϕ_1 and Y's seems to be missing in the text.
⌘ The u,d terms need factors of 3 to account for the 3-color space.

```
PR[next, "Compute ", $0 = Tr[Φ.Φ.Φ.Φ],
 Yield, $ = xTr[ct[Φ].Φ.ct[Φ].Φ] /. toxDot /. $Φ // tuConjugateTransposeExpand //
     tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
 $ // ColumnSumExp;
 $ = $ // tuTrEvaluate[{}] // (# /. xTr → Tr &); $ // ColumnSumExp;
 $ = $ // tuConjugateTransposeSimplify[{}, $scal]; $ // ColumnSumExp;
 $ = $ /. $sY // Simplify; $ // ColumnSumExp;
 $0a =
  $ = $ /. tt : Tr[a_] ⧴ tuTrSimplify[{ϕ1, ϕ2}][tt] /. tt : Tr[a_] :> tuTrCanonicalOrder[
          tt] //. $sH2 //. $sH // Collect[#, Tr[_], Simplify] &;
 $ // ColumnSumExp,
 NL, CR["With the previous conditions, i.e., ", $s = {cc[ϕ1] → -ϕ1, cc[tt : Y_] → tt}],
 Yield,
 $ = $0a //. $s /. tt : Tr[_] :> tuTrCanonicalOrder[tt] // Collect[#, Tr[_], Simplify] &;
 $ // ColumnSumExp,
 note, " The u,d terms need factors of 3 to account for the 3-color space."
]
```

◆Compute $\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]$

$$\rightarrow \sum[\ \begin{array}{|l}
4\,\phi_1^2\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.(Y_d)^*.(Y_d)^*] \\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.(Y_d)^*.Y_d] \\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.(Y_u)^*.(Y_u)^*] \\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.Y_d.Y_d] \\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_d)^*.(Y_u)^*.Y_u.Y_d] \\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_d)^*.Y_d.(Y_d)^*.Y \\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_d)^*.Y_d.Y_d.Y_d] \\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_d)^*.Y_d.Y_u.(Y_u)^*] \\
4\,\phi_1^2\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.(Y_e)^*.(Y_e)^*] \\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.(Y_e)^*.Y_e] \\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.(Y_\nu)^*.(Y_\nu)^*] \\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.Y_e.Y_e] \\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_e)^*.(Y_\nu)^*.Y_\nu.Y_e] \\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_e)^*.Y_e.(Y_e)^*.Y \\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_e)^*.Y_e.Y_e.Y_e] \\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_e)^*.Y_e.Y_\nu.(Y_\nu)^*] \\
4\,(\phi_1)^*\,\mathrm{Tr}[(Y_R)^*.(Y_\nu)^*.(Y_\nu)^*.Y_R] \\
4\,\mathrm{Tr}[(Y_R)^*.(Y_\nu)^*.Y_\nu.Y_R] \\
2\,\mathrm{Tr}[(Y_R)^*.Y_R.(Y_R)^*.Y_R] \\
4\,(\phi_1)^*\,\mathrm{Tr}[(Y_R)^*.Y_R.(Y_\nu)^*.(Y_\nu)^*] \\
-4\,(1-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_R)^*.Y_R.(Y_\nu)^*.Y_\nu] \\
4\,\mathrm{Tr}[(Y_R)^*.Y_R.Y_\nu.(Y_\nu)^*] \\
4\,\phi_1\,\mathrm{Tr}[(Y_R)^*.Y_R.Y_\nu.Y_\nu] \\
-4\,(1-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_R)^*.Y_\nu.(Y_\nu)^*.Y_R] \\
4\,\phi_1\,\mathrm{Tr}[(Y_R)^*.Y_\nu.Y_\nu.Y_R] \\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*.(Y_u)^*.(Y_u)^*] \\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*.(Y_u)^*.Y_u] \\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*.Y_u.Y_u] \\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_u)^*.Y_u.(Y_u)^*.Y \\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_u)^*.Y_u.Y_u.Y_u] \\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*.(Y_\nu)^*.(Y_\nu)^*] \\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*.(Y_\nu)^*.Y_\nu] \\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*.Y_\nu.Y_\nu] \\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu.(Y_\nu)^*.Y \\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu.Y_\nu.Y_\nu] \\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[Y_d.Y_d.Y_d.Y_d] \\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[Y_d.Y_d.Y_u.Y_u] \\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[Y_e.Y_e.Y_e.Y_e] \\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[Y_e.Y_e.Y_\nu.Y_\nu] \\
4\,\phi_1^2\,\mathrm{Tr}[Y_u.Y_u.Y_u.Y_u] \\
4\,\phi_1^2\,\mathrm{Tr}[Y_\nu.Y_\nu.Y_\nu.Y_\nu]
\end{array}$$

With the previous conditions, i.e., $\{(\phi_1)^* \to -\phi_1,\ (tt:Y\_)^* \to tt\}$

$$\rightarrow \sum[\ \begin{array}{|l}
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_d.Y_d.Y_d.Y_d] \\
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_e.Y_e.Y_e.Y_e] \\
2\,\mathrm{Tr}[Y_R.Y_R.Y_R.Y_R] \\
8\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[Y_R.Y_R.Y_\nu.Y_\nu] \\
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_u.Y_u.Y_u.Y_u] \\
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_\nu.Y_\nu.Y_\nu.Y_\nu]
\end{array}\ ]$$

⌘ The u,d terms need factors of 3 to account for the 3-color space.

Lemma 6.7

```
PR["Lemma 6.7: ",
 $167 = $ = {Tr[tuDDown[iD][Φ, μ] tuDUp[iD][Φ, μ]] → 4 a Abs[tuDDown[iD̃][H', μ]]^2,
    H' → {ϕ1 + 1, ϕ2}, tuDDown[iD̃][H', μ] →
     tuDDown["∂"][H', μ] + I T[Q, "du", {μ, a}] T[σ, "u", {a}] H' - I T[Λ, "d", {μ}] H'};
 accumStdMdl[$];
 $ // Column
]
```

Lemma 6.7:

$\mathrm{Tr}[\underset{-\mu}{D}[\Phi]\,\underset{-}{D^{\mu}}[\Phi]] \to 4\,a\,\mathrm{Abs}[\underset{-\mu}{\tilde{D}}[H']]^2$

$H' \to \{1+\phi_1,\ \phi_2\}$

$\underset{-\mu}{\tilde{D}}[H'] \to -i\,\Lambda_\mu H' + i\,Q_\mu{}^a\,\sigma^a H' + \underset{-\mu}{\partial}[H']$

Proposition 6.8 : The spectral action of AC − manifold $M \times F_{SM}$

```
PR["Proposition 6.8: The spectral action of AC-manifold M×FSM is ",
  $ = {Tr[f[DA / Λ]] → xIntegral[√Abs[g]
        ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}], H'], x ∈ M],
     ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}], H'] →
      96 ℒM[T[g, "dd", {μ, ν}]] + ℒA[T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}]] +
       ℒH[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'],
     ℒA[T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}]] → f[0]/π^2
       (10/3 T[Λ, "dd", {μ, ν}] T[Λ, "uu", {μ, ν}] + Tr[T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]] +
         Tr[T[V, "dd", {μ, ν}] T[V, "uu", {μ, ν}]]),
     ℒA[CG["kinetic terms of the gauge fields"]],
     ℒH[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] →
      b f[0] Λ^2/(2 π^2) Abs[H']^4 + (-2 a f2 Λ^2 + e f[0])/π^2 Abs[H']^2 - c f2 Λ^2 / π^2 + df[0]/(4 π^2) +
       a f[0]/(12 π^2) s Abs[H']^2 + c f[0]/(24 π^2) s + a f[0]/(2 π^2) Abs[tuDDown[iD̃][H', μ]]^2,
     ℒH[CG["Higgs potential"]]
    }; $ // ColumnBar, accumStdMdl[{$}]
 ];
```

Proposition 6.8: The spectral action of AC-manifold $M\times F_{SM}$ is

$\mathrm{Tr}[f[\frac{D_A}{\Lambda}]] \to \int_{x\in M}\sqrt{\mathrm{Abs}[g]}\ \mathcal{L}[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ V_\mu,\ H']$

$\mathcal{L}[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ V_\mu,\ H'] \to \mathcal{L}_A[\Lambda_\mu,\ Q_\mu,\ V_\mu] + \mathcal{L}_H[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H'] + 96\,\mathcal{L}_M[g_{\mu\nu}]$

$\mathcal{L}_A[\Lambda_\mu,\ Q_\mu,\ V_\mu] \to \frac{f[0]\,(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu}+\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}]+\mathrm{Tr}[V_{\mu\nu}V^{\mu\nu}])}{\pi^2}$

$\mathcal{L}_A$[kinetic terms of the gauge fields]

$\mathcal{L}_H[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H'] \to$

$\frac{d f[0]}{4\pi^2} + \frac{c\,s\,f[0]}{24\pi^2} + \frac{a\,s\,\mathrm{Abs}[H']^2 f[0]}{12\pi^2} + \frac{b\,\Lambda^2\,\mathrm{Abs}[H']^4 f[0]}{2\pi^2} + \frac{a\,\mathrm{Abs}[\underset{-\mu}{\tilde{D}}[H']]^2 f[0]}{2\pi^2} - \frac{c\,\Lambda^2 f_2}{\pi^2} + \frac{\mathrm{Abs}[H']^2\,(e\,f[0]-2\,a\,\Lambda^2 f_2)}{\pi^2}$

$\mathcal{L}_H$[Higgs potential]

6.3.1 Coupling constants and unification.

```
PR["6.3.1 Coupling constants and unification. SU[3] gauge field: ",
 $ = {T[V, "d", {μ}] -> T[V, "du", {μ, i}] T[λ, "d", {i}],
   T[λ, "d", {i}][CG["Gell-Mann matrices"]],
   T[V, "du", {μ, i}][CG[ℝ]]
  }; $ // ColumnBar, accumStdMdl[{$}]
  NL, "Coupling constants rescaling: ",
 $e631 = $ = {T[Λ, "d", {μ}] → 1/2 g₁ T[B, "d", {μ}],
     T[Q, "du", {μ, a}] → 1/2 g₂ T[W, "du", {μ, a}],
     T[V, "du", {μ, i}] → 1/2 g₃ T[G, "du", {μ, i}],
     $[[1]]
   }; $ // ColumnBar,
 NL, "With the relations: ", $ = {Tr[T[σ, "u", {a}] T[σ, "u", {b}]] → 2 T[δ, "uu", {a, b}],
   Tr[T[λ, "u", {a}] T[λ, "u", {b}]] → 2 T[δ, "uu", {i, j}]};
 $ // ColumnBar,
 Yield, $ = ℒA[T[B, "d", {μ}], T[W, "d", {μ}], T[G, "d", {μ}]] →
   f[0]/(2 π^2) (5/3 g₁^2 T[B, "dd", {μ, ν}] T[B, "uu", {μ, ν}] + g₂^2 T[W, "dd", {μ, ν}]
      T[W, "uu", {μ, ν}] + g₃^2 T[G, "dd", {μ, ν}] T[G, "uu", {μ, ν}]), accumStdMdl[{$}],
 NL, "Natural normalization: ", $e66 = $ = {f[0]/(2 π^2) g₃^2 → 1/4,
    f[0]/(2 π^2) g₂^2 → 1/4, (5 f[0])/(6 π^2) g₁^2 → 1/4
   }; $ // Column // Framed,
 Yield, $ = tuEliminate[$, {f[0]}] // Simplify; $ // Framed,
 back, "Relationship between coupling constants at unification.",
 accumStdMdl[{$, $e631, $e66}]
]
```

6.3.1 Coupling constants and unification. SU[3] gauge field:

$V_\mu \to V_\mu{}^i \lambda_i$
λ_i[Gell-Mann matrices]
$V_\mu{}^i$[ℝ]

Coupling constants rescaling: $\Lambda_\mu \to \frac{1}{2} g_1 B_\mu$, $Q_\mu{}^a \to \frac{1}{2} g_2 W_\mu{}^a$, $V_\mu{}^i \to \frac{1}{2} g_3 G_\mu{}^i$, $V_\mu \to V_\mu{}^i \lambda_i$

With the relations: $\mathrm{Tr}[\sigma^a \sigma^b] \to 2\,\delta^{a\,b}$, $\mathrm{Tr}[\lambda^a \lambda^b] \to 2\,\delta^{i\,j}$

$$\to \mathcal{L}_A[B_\mu, W_\mu, G_\mu] \to \frac{f[0]\left(\frac{5}{3} g_1^2 B_{\mu\nu} B^{\mu\nu} + g_3^2 G_{\mu\nu} G^{\mu\nu} + g_2^2 W_{\mu\nu} W^{\mu\nu}\right)}{2\pi^2}$$

Natural normalization: $\boxed{\begin{array}{l} \frac{f[0]\, g_3^2}{2\pi^2} \to \frac{1}{4} \\ \frac{f[0]\, g_2^2}{2\pi^2} \to \frac{1}{4} \\ \frac{5\, f[0]\, g_1^2}{6\pi^2} \to \frac{1}{4} \end{array}}$

$\to \boxed{5\, g_1^2 == 3\, g_3^2 \;\&\&\; 5\, g_1^2 == 3\, g_2^2}$ ⟵Relationship between coupling constants at unification.

Theorem 6.9

```
PR["Theorem 6.9. Spectral action on ACM M×FSM: ",
  Yield,
  $t69 =
```

$$S_B \to \mathrm{xIntegral}[(48\, f_4 \frac{\Lambda^4}{\pi^2} - c\, f_2\, \Lambda^2/\pi^2 + df[0]/(4\pi^2) + (c\, f[0]/(24\pi^2) - 4\, f_2\, \Lambda^2/\pi^2)\, s - 3\frac{f[0]}{10\pi^2} T[C, \text{"dddd"}, \{\mu,\nu,\rho,\sigma\}]\, T[C, \text{"uuuu"}, \{\mu,\nu,\rho,\sigma\}] + T[B, \text{"dd"}, \{\mu,\nu\}]\, T[B, \text{"uu"}, \{\mu,\nu\}]/4 + T[W, \text{"udd"}, \{a,\mu,\nu\}]\, T[W, \text{"uuu"}, \{a,\mu,\nu\}]/4 + T[G, \text{"udd"}, \{i,\mu,\nu\}]\, T[G, \text{"uuu"}, \{i,\mu,\nu\}]/4 + b\frac{\pi^2}{2\, a^2\, f[0]} \mathrm{Abs}[H]^4 - (2\, a\, f_2\, \Lambda^2 - e\, f[0])/(a\, f[0])\, \mathrm{Abs}[H]^2 + s\, \mathrm{Abs}[H]^2/12 + \mathrm{Abs}[\mathrm{tuDDown}[\tilde{iD}][H,\mu]]\,\hat{}\,2/2) \sqrt{\mathrm{Abs}[g]},\ x \in M],\ \mathrm{accumStdMdl}[\$t69]$$

```
];
```

Theorem 6.9. Spectral action on ACM M×F_{SM}:

$$\to S_B \to \int_{x\in M} \sqrt{\mathrm{Abs}[g]} \left(\frac{1}{12} s\, \mathrm{Abs}[H]^2 + \frac{1}{2}\mathrm{Abs}[\tilde{D}_{-\mu}[H]]^2 + \frac{df[0]}{4\pi^2} + \frac{b\,\pi^2\, \mathrm{Abs}[H]^4}{2\, a^2\, f[0]} - \frac{c\,\Lambda^2 f_2}{\pi^2} - \frac{\mathrm{Abs}[H]^2\,(-e\, f[0] + 2\, a\,\Lambda^2 f_2)}{a\, f[0]} + s\left(\frac{c\, f[0]}{24\pi^2} - \frac{4\Lambda^2 f_2}{\pi^2}\right) + \frac{48\,\Lambda^4 f_4}{\pi^2} + \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{3\, f[0]\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}}{10\pi^2} + \frac{1}{4} G^i{}_{\mu\nu} G^{i\,\mu\nu} + \frac{1}{4} W^a{}_{\mu\nu} W^{a\,\mu\nu}\right)$$

6.4 Fermionic action

```
PR["Grassmann fermion basis for M⊗F-spaces compose from the basic fermions: ",
  $ = {ν, e, u, d},
  NL, "•add antiparticles and generation index{1,2,3} and color index: ",
  $ = T[#, "u", {λ}] & /@ ({$, OverBar /@ $} // Flatten);
  $ = $ /. tt : Tensor[u | d | ū | d̄, _, _] :→ tuIndexAdd[2, c][tt], CK,
  NL, "•add chiral symbol(Grassmann): ",
  (*
  $=Map[{#/.Tensor[a_,b_,c_]→Tensor[aL,b,c],#/.Tensor[a_,b_,c_]→Tensor[aR,b,c]}&,$]//
     Flatten;*)
  $ = Map[{tuIndexAdd[, , 1, L][#], tuIndexAdd[, , 1, R][#]} &, $] // Flatten;
  $ = Permute[$, Cycles[{{2, 3}, {6, 7}, {10, 11}, {14, 15}}]];
  (*proper order*)
  $1 = {chiralfermion -> $, λ[CG["generation"]] → {1, 2, 3}, c[CG["color"]] → {r, g, b}};
  $1 // ColumnBar, accumStdMdl[$1];
  NL,
  "M-space Weyl fermions have distinct (opposite) chirality, while the F-space Dirac
    fermions do not. Hence, the antiparticle for M⊗F-space the
    M-space and F-space fermions have opposite chirality.",
  Imply, $fermion = gfermionbasis → Join[Map[# ⊗ # &, $[[1 ;; 8]]],
     Thread[CircleTimes[$[[9 ;; -1]], (# /. {R → L, L → R} &) /@ $[[9 ;; -1]]]]];
  $fermion // ColumnForms, accumStdMdl[$fermion];
  NL, $ξ = ξ̃ -> Apply[Plus, $fermion[[2]]];
  NL, "Grassmann basis vector: ",
  $basisG = $ = {ξ̃[CG["Grassman vector"]] ∈ T[ℋ, "du", {cl, "+"}],
     $ξ,
     T[ℋ, "du", {cl, "+"}] ∈ {ℋM × ℋF, γ.ξ̃ -> ξ̃.γ}
    }; $ // ColumnSumExp, accumStdMdl[$];
  CR["The text notation is confusing: The OverBar on the
     F-space basis refers to its anti-particle, not its Conjugate."],
  NL, "The rational for this basis can be inferred from the GWS model in that ",
  Yield, selectGWS[JF._, {}, all] // ColumnBar, " is implemented by Conjugation: ",
  selectGWS[JF4] // MatrixForms,
  NL, "whereas, for the Weyl fermions ", {JM.xL → xR, JM.xR → xL} // ColumnBar

 ];
PR["•A useful division of ξ is: ", $ξMF = ξ̃ → ξ̃M ⊗ ξ̃F δ[pairs[ξ̃]],
  NL, "where ", $ = selectStdMdl[gfermionbasis],
  Yield, $ = {ξ̃M → First /@ $[[2]], ξ̃F → Last /@ $[[2]]};
  $ // ColumnBar, accumStdMdl[{$ξMF, $}]
 ];
```

Grassmann fermion basis for M⊗F-spaces compose from the basic fermions:

$\{\nu, e, u, d\}$

• add antiparticles and generation index{1,2,3} and color index:

$\{\nu^{\lambda}, e^{\lambda}, u^{\lambda c}, d^{\lambda c}, \bar{\nu}^{\lambda}, \bar{e}^{\lambda}, \bar{u}^{\lambda c}, \bar{d}^{\lambda c}\}$ ⟵CHECK

• add chiral symbol(Grassmann):

$$\begin{array}{|l} \text{chiralfermion} \to \\ \quad \{\nu_L{}^{\lambda}, e_L{}^{\lambda}, \nu_R{}^{\lambda}, e_R{}^{\lambda}, u_L{}^{\lambda c}, d_L{}^{\lambda c}, u_R{}^{\lambda c}, d_R{}^{\lambda c}, \bar{\nu}_L{}^{\lambda}, \bar{e}_L{}^{\lambda}, \bar{\nu}_R{}^{\lambda}, \bar{e}_R{}^{\lambda}, \bar{u}_L{}^{\lambda c}, \bar{d}_L{}^{\lambda c}, \bar{u}_R{}^{\lambda c}, \bar{d}_R{}^{\lambda c}\} \\ \lambda[\text{generation}] \to \{1, 2, 3\} \\ c[\text{color}] \to \{r, g, b\} \end{array}$$

M-space Weyl fermions have distinct (opposite) chirality, while the F-space Dirac fermions do not. Hence, the antiparticle for M⊗F-space the M-space and F-space fermions have opposite chirality.

$$\Rightarrow \text{gfermionbasis} \to \left| \begin{array}{l} \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} \\ e_L{}^{\lambda} \otimes e_L{}^{\lambda} \\ \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} \\ e_R{}^{\lambda} \otimes e_R{}^{\lambda} \\ u_L{}^{\lambda c} \otimes u_L{}^{\lambda c} \\ d_L{}^{\lambda c} \otimes d_L{}^{\lambda c} \\ u_R{}^{\lambda c} \otimes u_R{}^{\lambda c} \\ d_R{}^{\lambda c} \otimes d_R{}^{\lambda c} \\ \bar{\nu}_L{}^{\lambda} \otimes \bar{\nu}_R{}^{\lambda} \\ \bar{e}_L{}^{\lambda} \otimes \bar{e}_R{}^{\lambda} \\ \bar{\nu}_R{}^{\lambda} \otimes \bar{\nu}_L{}^{\lambda} \\ \bar{e}_R{}^{\lambda} \otimes \bar{e}_L{}^{\lambda} \\ \bar{u}_L{}^{\lambda c} \otimes \bar{u}_R{}^{\lambda c} \\ \bar{d}_L{}^{\lambda c} \otimes \bar{d}_R{}^{\lambda c} \\ \bar{u}_R{}^{\lambda c} \otimes \bar{u}_L{}^{\lambda c} \\ \bar{d}_R{}^{\lambda c} \otimes \bar{d}_L{}^{\lambda c} \end{array} \right.$$

Grassmann basis vector:

$$\{\tilde{\xi}[\text{Grassman vector}] \in \mathcal{H}_{cl}{}^{+}, \tilde{\xi} \to \sum\left[\left| \begin{array}{l} d_L{}^{\lambda c} \otimes d_L{}^{\lambda c} \\ d_R{}^{\lambda c} \otimes d_R{}^{\lambda c} \\ e_L{}^{\lambda} \otimes e_L{}^{\lambda} \\ e_R{}^{\lambda} \otimes e_R{}^{\lambda} \\ u_L{}^{\lambda c} \otimes u_L{}^{\lambda c} \\ u_R{}^{\lambda c} \otimes u_R{}^{\lambda c} \\ \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} \\ \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} \\ \bar{d}_L{}^{\lambda c} \otimes \bar{d}_R{}^{\lambda c} \\ \bar{d}_R{}^{\lambda c} \otimes \bar{d}_L{}^{\lambda c} \\ \bar{e}_L{}^{\lambda} \otimes \bar{e}_R{}^{\lambda} \\ \bar{e}_R{}^{\lambda} \otimes \bar{e}_L{}^{\lambda} \\ \bar{u}_L{}^{\lambda c} \otimes \bar{u}_R{}^{\lambda c} \\ \bar{u}_R{}^{\lambda c} \otimes \bar{u}_L{}^{\lambda c} \\ \bar{\nu}_L{}^{\lambda} \otimes \bar{\nu}_R{}^{\lambda} \\ \bar{\nu}_R{}^{\lambda} \otimes \bar{\nu}_L{}^{\lambda} \end{array} \right. \right], \mathcal{H}_{cl}{}^{+} \in \{\mathcal{H}_M \times \mathcal{H}_F, \gamma.\tilde{\xi} \to \tilde{\xi}.\gamma\}\}$$

The text notation is confusing: The OverBar on the F-space basis refers to its anti-particle, not its Conjugate.

The rational for this basis can be inferred from the GWS model in that

$$\to \left| \begin{array}{l} J_F.1 \to \bar{1} \\ J_F.\bar{1} \to 1 \end{array} \right. \text{ is implemented by Conjugation: } J_{F_4} \to \begin{pmatrix} 0 & 0 & cc & 0 \\ 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 \\ 0 & cc & 0 & 0 \end{pmatrix}$$

whereas, for the Weyl fermions $\left| \begin{array}{l} J_M.x_L \to x_R \\ J_M.x_R \to x_L \end{array} \right.$

•A useful division of ξ is: $\tilde{\xi} \to \tilde{\xi}_M \otimes \tilde{\xi}_F\, \delta[\text{pairs}[\tilde{\xi}]]$
where gfermionbasis →
$\{\nu_L{}^\lambda \otimes \nu_L{}^\lambda,\ e_L{}^\lambda \otimes e_L{}^\lambda,\ \nu_R{}^\lambda \otimes \nu_R{}^\lambda,\ e_R{}^\lambda \otimes e_R{}^\lambda,\ u_L{}^{\lambda c} \otimes u_L{}^{\lambda c},\ d_L{}^{\lambda c} \otimes d_L{}^{\lambda c},\ u_R{}^{\lambda c} \otimes u_R{}^{\lambda c},\ d_R{}^{\lambda c} \otimes d_R{}^{\lambda c},$
$\bar{\nu}_L{}^\lambda \otimes \bar{\nu}_R{}^\lambda,\ \bar{e}_L{}^\lambda \otimes \bar{e}_R{}^\lambda,\ \bar{\nu}_R{}^\lambda \otimes \bar{\nu}_L{}^\lambda,\ \bar{e}_R{}^\lambda \otimes \bar{e}_L{}^\lambda,\ \bar{u}_L{}^{\lambda c} \otimes \bar{u}_R{}^{\lambda c},\ \bar{d}_L{}^{\lambda c} \otimes \bar{d}_R{}^{\lambda c},\ \bar{u}_R{}^{\lambda c} \otimes \bar{u}_L{}^{\lambda c},\ \bar{d}_R{}^{\lambda c} \otimes \bar{d}_L{}^{\lambda c}\}$
→ $\tilde{\xi}_M \to \{\nu_L{}^\lambda,\ e_L{}^\lambda,\ \nu_R{}^\lambda,\ e_R{}^\lambda,\ u_L{}^{\lambda c},\ d_L{}^{\lambda c},\ u_R{}^{\lambda c},\ d_R{}^{\lambda c},\ \bar{\nu}_L{}^\lambda,\ \bar{e}_L{}^\lambda,\ \bar{\nu}_R{}^\lambda,\ \bar{e}_R{}^\lambda,\ \bar{u}_L{}^{\lambda c},\ \bar{d}_L{}^{\lambda c},\ \bar{u}_R{}^{\lambda c},\ \bar{d}_R{}^{\lambda c}\}$
$\tilde{\xi}_F \to \{\nu_L{}^\lambda,\ e_L{}^\lambda,\ \nu_R{}^\lambda,\ e_R{}^\lambda,\ u_L{}^{\lambda c},\ d_L{}^{\lambda c},\ u_R{}^{\lambda c},\ d_R{}^{\lambda c},\ \bar{\nu}_R{}^\lambda,\ \bar{e}_R{}^\lambda,\ \bar{\nu}_L{}^\lambda,\ \bar{e}_L{}^\lambda,\ \bar{u}_R{}^{\lambda c},\ \bar{d}_R{}^{\lambda c},\ \bar{u}_L{}^{\lambda c},\ \bar{d}_L{}^{\lambda c}\}$

Gauge fields Transformed

```
PR["For physical gauge fields(5.21): ", $e521 // ColumnBar,
 NL, "Define(6.7-10): ",
 $e67 = $ = {T[Q, "du", {μ, 1}] + I T[Q, "du", {μ, 2}] → g₂ / √2 T[W, "d", {μ}],
    T[Q, "du", {μ, 1}] - I T[Q, "du", {μ, 2}] → g₂ / √2 ct[T[W, "d", {μ}]],
    T[Q, "du", {μ, 3}] - T[Λ, "d", {μ}] → g₂ / (2 c_w) T[Z, "d", {μ}],
    T[Λ, "d", {μ}] → s_w g₂ T[A, "d", {μ}] / 2 - s_w^2 g₂ T[Z, "d", {μ}] / (2 c_w),
    -T[Q, "du", {μ, 3}] - T[Λ, "d", {μ}] →
     -s_w g₂ T[A, "d", {μ}] + g₂ / (2 c_w) (1 - 2 c_w^2) T[Z, "d", {μ}],
    T[Q, "du", {μ, 3}] + T[Λ, "d", {μ}] / 3 → (2 / 3) s_w g₂ T[A, "d", {μ}] -
      g₂ / (6 c_w) (1 - 4 c_w^2) T[Z, "d", {μ}],
    -T[Q, "du", {μ, 3}] + T[Λ, "d", {μ}] / 3 → -(1 / 3) s_w g₂ T[A, "d", {μ}] -
      g₂ / (6 c_w) (1 + 2 c_w^2) T[Z, "d", {μ}],
    H → √(a f[0]) / π {φ₁ + 1, φ₂},
    H → {v + h + I T[φ, "u", {0}], I √2 φ⁻},
    T[φ, "u", {0}] ∈ ℝ,
    φ⁻ ∈ ℂ,
    Y_x[CG["anti-hermitian mass matrix of x"]],
    Y_x → -I √(a f[0]) / (π v) m_x,
    m_x[CG["Hermitian matrix"]],
    Y_R → -I m_R,
    m_R[CG["Majorana mass matrix hermitian symmetric"]]
   }; $ // ColumnBar,
 NL, "Derived relationships: ",
 $ = tuRuleSelect[$e67][H];
 $ = tuRuleSubtract[$] // Thread; $ // Column;
 $e67a = $ = tuRuleSolve[$, {φ₁ + 1, φ₂}]; $ // ColumnBar,
 accumStdMdl[{$, $e521, $e67, $e67a}]

]
```

For physical gauge fields(5.21):

$$W_\mu \to \frac{W_\mu{}^1 + i\, W_\mu{}^2}{\sqrt{2}}$$
$$(W_\mu)^* \to \frac{W_\mu{}^1 - i\, W_\mu{}^2}{\sqrt{2}}$$
$$Z_\mu \to -s_w\, B_\mu + c_w\, W_\mu{}^3$$
$$A_\mu \to c_w\, B_\mu + s_w\, W_\mu{}^3$$

Define(6.7-10):

$$Q_\mu{}^1 + i\, Q_\mu{}^2 \to \frac{g_2\, W_\mu}{\sqrt{2}}$$
$$Q_\mu{}^1 - i\, Q_\mu{}^2 \to \frac{(W_\mu)^\dagger\, g_2}{\sqrt{2}}$$
$$Q_\mu{}^3 - \Lambda_\mu \to \frac{g_2\, Z_\mu}{2\, c_w}$$
$$\Lambda_\mu \to \frac{1}{2}\, g_2\, s_w\, A_\mu - \frac{g_2\, s_w^2\, Z_\mu}{2\, c_w}$$
$$-Q_\mu{}^3 - \Lambda_\mu \to -g_2\, s_w\, A_\mu + \frac{(1-2\, c_w^2)\, g_2\, Z_\mu}{2\, c_w}$$
$$Q_\mu{}^3 + \frac{\Lambda_\mu}{3} \to \frac{2}{3}\, g_2\, s_w\, A_\mu - \frac{(1-4\, c_w^2)\, g_2\, Z_\mu}{6\, c_w}$$
$$-Q_\mu{}^3 + \frac{\Lambda_\mu}{3} \to -\frac{1}{3}\, g_2\, s_w\, A_\mu - \frac{(1+2\, c_w^2)\, g_2\, Z_\mu}{6\, c_w}$$
$$H \to \left\{\frac{\sqrt{a\, f[0]}\, (1+\phi_1)}{\pi},\ \frac{\sqrt{a\, f[0]}\, \phi_2}{\pi}\right\}$$
$$H \to \{h + v + i\, \phi^0,\ i\, \sqrt{2}\, \phi^-\}$$
$$\phi^0 \in \mathbb{R}$$
$$\phi^- \in \mathbb{C}$$
Y_x[anti-hermitian mass matrix of x]
$$Y_x \to -\frac{i\, \sqrt{a\, f[0]}\, m_x}{\pi\, v}$$
m_x[Hermitian matrix]
$$Y_R \to -i\, m_R$$
m_R[Majorana mass matrix hermitian symmetric]

Derived relationships:

$$1 + \phi_1 \to \frac{\pi\, (h+v)}{\sqrt{a\, f[0]}} + \frac{i\, \pi\, \phi^0}{\sqrt{a\, f[0]}}$$
$$\phi_2 \to \frac{i\, \sqrt{2}\, \pi\, \phi^-}{\sqrt{a\, f[0]}}$$

Theorem 6.10

```
PR["Theorem 6.10. Fermionic action: ", NL,
  $t610 = $ = {S_F → IntegralOp[{{x ∈ M}}, √Abs[g] (ℒkin + ℒgf + ℒHf + ℒR)],
     ℒkin → ($ = -I BraKet[J_M.ē, T[γ, "u", {μ}] . tuDDown["∇"^S][e, μ] ])
       + ($ /. e → ν)
       + ($ /. e → u)
       + ($ /. e → d),
     ℒgf[CG["gauge-fermion coupling"]] → s_w g2 T[A, "d", {μ}]
        (($ = -BraKet[J_M.ē, T[γ, "u", {μ}].e]) - (2 / 3) ($ /. e → u) + (1 / 3) ($ /. e → d))
       + g2 T[Z, "d", {μ}] / (4 c_w) (
         BraKet[J_M.ν̄, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).ν]
          + BraKet[J_M.ē, T[γ, "u", {μ}].(4 s_w^2 - 1 - T[γ, "d", {5}]).e]
          + BraKet[J_M.ū, T[γ, "u", {μ}].(-8 / 3 s_w^2 + 1 + T[γ, "d", {5}]).u]
          + BraKet[J_M.d̄, T[γ, "u", {μ}].(4 / 3 s_w^2 - 1 - T[γ, "d", {5}]).d]
        )
       + g2 T[W, "d", {μ}] / (2 √2 ) (
         BraKet[J_M.ē, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).ν]
          + BraKet[J_M.d̄, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).u])
       + g2 ct[T[W, "d", {μ}]] / (2 √2 ) (
         BraKet[J_M.ν̄, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).e]
          + BraKet[J_M.ū, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).d]
        )
       + g3 T[G, "du", {μ, i}] / 2 (
         BraKet[J_M.ū, T[γ, "u", {μ}].T[λ, "d", {i}].u]
          + BraKet[J_M.d̄, T[γ, "u", {μ}].T[λ, "d", {i}].d]
        ),
     ℒHf[CG["Yukawa coupling of Higgs-fermion field"]] →
      I (1 + h / v) (($ = BraKet[J_M.ν̄, m_ν.ν]) + ($ /. ν → e) + ($ /. ν → u) + ($ /. ν → d))
       + T[ϕ, "u", {0}] / v
        (($ = BraKet[J_M.ν̄, T[γ, "d", {5}].m_ν.ν]) - ($ /. ν → e) + ($ /. ν → u) - ($ /. ν → d))
       + ϕ^- / (√2 v) (($ = BraKet[J_M.ē, m_e.(1 + T[γ, "d", {5}]).ν]) -
          ($ /. {m_e → m_ν, tt : T[γ, "d", {5}] → -tt}))
       + ϕ^+ / (√2 v) (($ = BraKet[J_M.ν̄, m_ν.(1 + T[γ, "d", {5}]).e]) -
          ($ /. {m_ν → m_e, tt : T[γ, "d", {5}] → -tt}))
       + ϕ^- / (√2 v) (($ = BraKet[J_M.d̄, m_d.(1 + T[γ, "d", {5}]).u]) -
          ($ /. {m_d → m_u, tt : T[γ, "d", {5}] → -tt}))
       + ϕ^+ / (√2 v) (($ = BraKet[J_M.ū, m_u.(1 + T[γ, "d", {5}]).d]) -
          ($ /. {m_u → m_d, tt : T[γ, "d", {5}] → -tt})),
     ℒR[CG["Majorana mass"]] → ($ = I BraKet[J_M.ν_R, m_R.ν_R]) + ($ /. ν_R -> ν̄_L)
    }; $ // ColumnSumExp // ColumnBar, accumStdMdl[{$t610}]
 ];
```

Theorem 6.10. Fermionic action:

$$S_F \to \int_{\{x\in M\}} \left[\sum\left[\begin{array}{l}\mathcal{L}_{gf}\\ \mathcal{L}_{Hf}\\ \mathcal{L}_{kin}\\ \mathcal{L}_R\end{array}\right]\sqrt{\mathrm{Abs}[g]}\right]$$

$$\mathcal{L}_{kin} \to \sum\left[\begin{array}{l}-i\left\langle J_M.\bar{d}\mid \gamma^\mu.\underset{-\,\mu}{\nabla^S}[d]\right\rangle\\ -i\left\langle J_M.\bar{e}\mid \gamma^\mu.\underset{-\,\mu}{\nabla^S}[e]\right\rangle\\ -i\left\langle J_M.\bar{u}\mid \gamma^\mu.\underset{-\,\mu}{\nabla^S}[u]\right\rangle\\ -i\left\langle J_M.\bar{\nu}\mid \gamma^\mu.\underset{-\,\mu}{\nabla^S}[\nu]\right\rangle\end{array}\right]$$

$\mathcal{L}_{gf}$[gauge-fermion coupling] →

$$\sum\left[\begin{array}{l}\frac{(\langle J_M.\bar{u}|\gamma^\mu.(1+\gamma_5).d\rangle+\langle J_M.\bar{\nu}|\gamma^\mu.(1+\gamma_5).e\rangle)\,(W_\mu)^\dagger\, g_2}{2\sqrt{2}}\\ \left(-\frac{1}{3}\langle J_M.\bar{d}\mid\gamma^\mu.d\rangle-\langle J_M.\bar{e}\mid\gamma^\mu.e\rangle+\frac{2}{3}\langle J_M.\bar{u}\mid\gamma^\mu.u\rangle\right) g_2\, s_w\, A_\mu\\ \frac{1}{2}\left(\langle J_M.\bar{d}\mid\gamma^\mu.\lambda_i.d\rangle+\langle J_M.\bar{u}\mid\gamma^\mu.\lambda_i.u\rangle\right) g_3\, G_\mu{}^i\\ \frac{(\langle J_M.\bar{d}|\gamma^\mu.(1+\gamma_5).u\rangle+\langle J_M.\bar{e}|\gamma^\mu.(1+\gamma_5).\nu\rangle)\, g_2\, W_\mu}{2\sqrt{2}}\\ \frac{\left(\left\langle J_M.\bar{d}|\gamma^\mu.\left(-1+\frac{4 s_w^2}{3}-\gamma_5\right).d\right\rangle+\langle J_M.\bar{e}|\gamma^\mu.(-1+4 s_w^2-\gamma_5).e\rangle+\left\langle J_M.\bar{u}|\gamma^\mu.\left(1-\frac{8 s_w^2}{3}+\gamma_5\right).u\right\rangle+\langle J_M.\bar{\nu}|\gamma^\mu.(1+\gamma_5).\nu\rangle\right) g_2\, Z_\mu}{4 c_w}\end{array}\right]$$

$\mathcal{L}_{Hf}$[Yukawa coupling of Higgs-fermion field] →

$$\sum\left[\begin{array}{l}i\left(1+\frac{h}{v}\right)\left(\langle J_M.\bar{d}\mid m_d.d\rangle+\langle J_M.\bar{e}\mid m_e.e\rangle+\langle J_M.\bar{u}\mid m_u.u\rangle+\langle J_M.\bar{\nu}\mid m_\nu.\nu\rangle\right)\\ \frac{(\langle J_M.\bar{d}|m_d.(1+\gamma_5).u\rangle-\langle J_M.\bar{d}|m_u.(1-\gamma_5).u\rangle)\,\phi^-}{\sqrt{2}\,v}\\ \frac{(\langle J_M.\bar{e}|m_e.(1+\gamma_5).\nu\rangle-\langle J_M.\bar{e}|m_\nu.(1-\gamma_5).\nu\rangle)\,\phi^-}{\sqrt{2}\,v}\\ \frac{(-\langle J_M.\bar{u}|m_d.(1-\gamma_5).d\rangle+\langle J_M.\bar{u}|m_u.(1+\gamma_5).d\rangle)\,\phi^+}{\sqrt{2}\,v}\\ \frac{(-\langle J_M.\bar{\nu}|m_e.(1-\gamma_5).e\rangle+\langle J_M.\bar{\nu}|m_\nu.(1+\gamma_5).e\rangle)\,\phi^+}{\sqrt{2}\,v}\\ \frac{(-\langle J_M.\bar{d}|\gamma_5.m_d.d\rangle-\langle J_M.\bar{e}|\gamma_5.m_e.e\rangle+\langle J_M.\bar{u}|\gamma_5.m_u.u\rangle+\langle J_M.\bar{\nu}|\gamma_5.m_\nu.\nu\rangle)\,\phi^0}{v}\end{array}\right]$$

$$\mathcal{L}_R[\text{Majorana mass}] \to \sum\left[\begin{array}{l}i\left\langle J_M.\nu_R\mid m_R.\nu_R\right\rangle\\ i\left\langle J_M.\bar{\nu}_L\mid m_R.\bar{\nu}_L\right\rangle\end{array}\right]$$

```
PR["●Proof(Theorem 6.10): ", " From the definitions: ",
  NL, $aferm = $ = {
     S_F → BraKet[J . ξ̃, 𝒟_A.ξ̃] / 2,
     𝒟_A → slash[iD] ⊗ 1_F + T[γ, "u", {μ}] ⊗ T[B, "d", {μ}] + T[γ, "d", {5}] ⊗ Φ,
     BraKet[ξ, ψ] → xIntegral[√Det[g] BraKet[ξ, ψ], x ∈ M],
     selectStdMdl[S_F, {ℒ}]}; $ // ColumnBar,
  accumStdMdl[{$}];
  NL, $sJ = {J.ξ̃ → (J_M ⊗ J_F).ξ̃, 𝒟_A.ξ̃ -> (slash[iD] ⊗ 1_F).ξ̃}; $sJ // ColumnBar,
  NL, "and the basis: ", $sbasis = selectStdMdl[ξ̃, {λ}]
 ];
```

●Proof(Theorem 6.10): From the definitions:

$S_F \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_A.\tilde{\xi}\rangle$

$\mathcal{D}_A \to (\not{D}) \otimes 1_F + \gamma_5 \otimes \Phi + \gamma^\mu \otimes B_\mu$

$\langle \xi \mid \psi\rangle \to \int_{x\in M} \langle \xi \mid \psi\rangle \sqrt{\mathrm{Det}[g]}$

$S_F \to \int_{\{x\in M\}} [\sqrt{\mathrm{Abs}[g]}\ (\mathcal{L}_{gf} + \mathcal{L}_{Hf} + \mathcal{L}_{kin} + \mathcal{L}_R)]$

$J.\tilde{\xi} \to (J_M \otimes J_F).\tilde{\xi}$

$\mathcal{D}_A.\tilde{\xi} \to ((\not{D}) \otimes 1_F).\tilde{\xi}$

and the basis:

$\tilde{\xi} \to d_L{}^{\lambda c} \otimes d_L{}^{\lambda c} + d_R{}^{\lambda c} \otimes d_R{}^{\lambda c} + e_L{}^{\lambda} \otimes e_L{}^{\lambda} + e_R{}^{\lambda} \otimes e_R{}^{\lambda} + u_L{}^{\lambda c} \otimes u_L{}^{\lambda c} + u_R{}^{\lambda c} \otimes u_R{}^{\lambda c} + \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} + \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} + \bar{d}_L{}^{\lambda c} \otimes \bar{d}_R{}^{\lambda c} + \bar{d}_R{}^{\lambda c} \otimes \bar{d}_L{}^{\lambda c} + \bar{e}_L{}^{\lambda} \otimes \bar{e}_R{}^{\lambda} + \bar{e}_R{}^{\lambda} \otimes \bar{e}_L{}^{\lambda} + \bar{u}_L{}^{\lambda c} \otimes \bar{u}_R{}^{\lambda c} + \bar{u}_R{}^{\lambda c} \otimes \bar{u}_L{}^{\lambda c} + \bar{\nu}_L{}^{\lambda} \otimes \bar{\nu}_R{}^{\lambda} + \bar{\nu}_R{}^{\lambda} \otimes \bar{\nu}_L{}^{\lambda}$

```
PR["●Evaluate ", $ = ℒ_kin, " portion, i.e., terms containing ", $sD = slash[iD], " of ",
  Yield, $ = selectStdMdl[S_F],
  Yield, $00 = $ = $ /. selectStdMdl[𝒟_A] /. $sJ[[1]],
  Yield, $00a = $ = $ //. tuOpDistribute[Dot] //. tuOpDistribute[BraKet] // Expand;
  $[[2]] = $[[2]] // tuTermExtract[$sD]; $ = $ /. S_F → ℒ_kin,
  Yield, $ = $ // expandDC[$sbasis] // tuCircleTimesExpand // tuOpDistributeF[BraKet];
  NL, "Expand CircleTimes, apply definitions for J_F and orthogonality: ",
  $s = {BraKet[ CircleTimes[a_, b_], CircleTimes[c_, d_]] → CircleTimes[BraKet[a, c],
       BraKet[b, d]], CO["Separate {M,F}-spaces"],
    J_F.a_ :→ (a /. Tensor[s_, i_, j_] :→ If[FreeQ[s, OverBar],
         Tensor[s̄, i, j], Tensor[s[[1]], i, j]] ), CO["Charge conjugation"],
    c_ ⊗ BraKet[a_, a_] :→ c, CO["Simplify Identity"],
    c_ ⊗ BraKet[a_, b_] :→ 0 /; FreeQ[{a, b}, Dot] && a =!= b,
    CO["F-basis Orthogonality"],
    1_F .a_ → a, CO["Remove identity symbol"]
   }; $s // ColumnBar,
  Yield, $0 = $ = $ // tuCircleTimesExpand // (# //. tuRule[$s] &);
  $ // ColumnSumExp,

  NL, "Use symmetry ", $symJM = BraKet[J_M.χ̃, slash[iD].ψ̃] -> BraKet[J_M.ψ̃, slash[iD].ξ̃],
  " to order BraKet[]s and specify {R,L} basis with
    projection operators, P_L|R, the sum of these terms: ",
  $s = {BraKet[J_M.a_, slash[iD].b_] :> BraKet[J_M.b, slash[iD].a] /; FreeQ[a, OverBar],
    tt : Tensor[a_, u_, d_] :→ P_L.tuIndexDelete[L][tt] /; ! FreeQ[d, L],
    tt : Tensor[a_, u_, d_] :> P_R.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
   }; $s // ColumnBar,
  Yield, $ = $ //. $s;

  Yield, $2 = $ = $ // Expand; Framed[$]
 ];
```

●Evaluate $\mathcal{L}_{kin}$ portion, i.e., terms containing $\not{D}$ of

→ $S_F \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_A.\tilde{\xi}\rangle$

→ $S_F \to \frac{1}{2}\langle (J_M \otimes J_F).\tilde{\xi} \mid ((\not{D})\otimes 1_F + \gamma_5 \otimes \Phi + \gamma^\mu \otimes B_\mu).\tilde{\xi}\rangle$

→ $\mathcal{L}_{kin} \to \frac{1}{2}\langle (J_M \otimes J_F).\tilde{\xi} \mid ((\not{D})\otimes 1_F).\tilde{\xi}\rangle$

→
Expand CircleTimes, apply definitions for J_F and orthogonality:

$\langle a\_\otimes b\_ \mid c\_\otimes d\_\rangle \to \langle a \mid c\rangle \otimes \langle b \mid d\rangle$

Separate {M,F}-spaces

```
JF.(a_) :→ (a /. Tensor[s_, i_, j_] :→ If[FreeQ[s, OverBar], Tensor[s̄, i, j], Tensor[s⟦1⟧, i, j]])
```

Charge conjugation

$c\_\otimes\langle a\_ \mid a\_\rangle :\to c$

Simplify Identity

$c\_\otimes\langle a\_ \mid b\_\rangle :\to 0$ /; FreeQ[{a, b}, Dot] && a =!= b

F-basis Orthogonality

$1_F.(a\_) \to a$

Remove identity symbol

→ $\mathcal{L}_{kin} \to \frac{1}{2}\sum[\,$
$\langle J_M.{d_L}^{\lambda c} \mid (\not{D}).{\bar{d}_R}^{\lambda c}\rangle$
$\langle J_M.{d_R}^{\lambda c} \mid (\not{D}).{\bar{d}_L}^{\lambda c}\rangle$
$\langle J_M.{e_L}^{\lambda} \mid (\not{D}).{\bar{e}_R}^{\lambda}\rangle$
$\langle J_M.{e_R}^{\lambda} \mid (\not{D}).{\bar{e}_L}^{\lambda}\rangle$
$\langle J_M.{u_L}^{\lambda c} \mid (\not{D}).{\bar{u}_R}^{\lambda c}\rangle$
$\langle J_M.{u_R}^{\lambda c} \mid (\not{D}).{\bar{u}_L}^{\lambda c}\rangle$
$\langle J_M.{\nu_L}^{\lambda} \mid (\not{D}).{\bar{\nu}_R}^{\lambda}\rangle$
$\langle J_M.{\nu_R}^{\lambda} \mid (\not{D}).{\bar{\nu}_L}^{\lambda}\rangle$
$\langle J_M.{\bar{d}_L}^{\lambda c} \mid (\not{D}).{d_R}^{\lambda c}\rangle$
$\langle J_M.{\bar{d}_R}^{\lambda c} \mid (\not{D}).{d_L}^{\lambda c}\rangle$
$\langle J_M.{\bar{e}_L}^{\lambda} \mid (\not{D}).{e_R}^{\lambda}\rangle$
$\langle J_M.{\bar{e}_R}^{\lambda} \mid (\not{D}).{e_L}^{\lambda}\rangle$
$\langle J_M.{\bar{u}_L}^{\lambda c} \mid (\not{D}).{u_R}^{\lambda c}\rangle$
$\langle J_M.{\bar{u}_R}^{\lambda c} \mid (\not{D}).{u_L}^{\lambda c}\rangle$
$\langle J_M.{\bar{\nu}_L}^{\lambda} \mid (\not{D}).{\nu_R}^{\lambda}\rangle$
$\langle J_M.{\bar{\nu}_R}^{\lambda} \mid (\not{D}).{\nu_L}^{\lambda}\rangle$
]

Use symmetry $\langle J_M.\tilde{\chi} \mid (\not{D}).\tilde{\psi}\rangle \to \langle J_M.\tilde{\psi} \mid (\not{D}).\tilde{\xi}\rangle$
to order BraKet[]s and specify {R,L} basis
with projection operators, $P_{L|R}$, the sum of these terms:

$\langle J_M.(a\_) \mid (\not{D}).(b\_)\rangle :\to \langle J_M.b \mid (\not{D}).a\rangle$ /; FreeQ[a, OverBar]

```
tt : Tensor[a_, u_, d_] :→ PL.tuIndexDelete[L][tt] /; ! FreeQ[d, L]
tt : Tensor[a_, u_, d_] :→ PR.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
```

→

→ $\boxed{\mathcal{L}_{kin} \to \langle J_M.P_L.\bar{d}^{\lambda c} \mid (\not{D}).P_R.d^{\lambda c}\rangle + \langle J_M.P_L.\bar{e}^{\lambda} \mid (\not{D}).P_R.e^{\lambda}\rangle + \langle J_M.P_L.\bar{u}^{\lambda c} \mid (\not{D}).P_R.u^{\lambda c}\rangle + \langle J_M.P_L.\bar{\nu}^{\lambda} \mid (\not{D}).P_R.\nu^{\lambda}\rangle + \langle J_M.P_R.\bar{d}^{\lambda c} \mid (\not{D}).P_L.d^{\lambda c}\rangle + \langle J_M.P_R.\bar{e}^{\lambda} \mid (\not{D}).P_L.e^{\lambda}\rangle + \langle J_M.P_R.\bar{u}^{\lambda c} \mid (\not{D}).P_L.u^{\lambda c}\rangle + \langle J_M.P_R.\bar{\nu}^{\lambda} \mid (\not{D}).P_L.\nu^{\lambda}\rangle}$

```
PR["•Using the relationships: ", $s = {
   JM.PL_ → PL.JM, CO["(JM,P's Commute)"],
   slash[iD] . Pl_ .a_ :→ If[l === L, PR.slash[iD] .a, PL.slash[iD] .a],
   CO[slash[iD], "Changes chirality"],
   JM . Tensor[a̅_, b_, c_] :→ Tensor[a, b, c], CO["(Charge Conjugation)"],
   BraKet[Pl_ . a_ , Pl_ .slash[iD] . a_] :→ BraKet[a, Pl.slash[iD] .a],
   CO["(Chiral orthogonal)"], BraKet[a_, PL.slash[iD] . a_] +
     BraKet[a_, PR.slash[iD] .a_] :→ BraKet[a, slash[iD] .a]};
 accumStdMdl[$s], $s // Column, $scc = tuRule[$s];
 Yield, $ = $ //. $scc,
 NL, "Reinsert JM: ", $s = BraKet[Tensor[a_, b_, c_], slash[iD] . Tensor[a_, b_, c_]] :>
   BraKet[JM.Tensor[a̅, b, c], slash[iD] . Tensor[a, b, c]],
 Yield, $ = $ /. $s; $ // ColumnSumExp // Framed
]
```

•Using the relationships:

$J_M.P_{L\_} \to P_L.J_M$

$(J_M$,P's Commute)

$(\not{D}).P_{l\_}.(a\_) :\to \mathrm{If}[l === L,\ P_R.(\not{D}).a,\ P_L.(\not{D}).a]$

$\not{D}$

Changes chirality

$J_M.\mathrm{Tensor}[\bar{a}\_,\ b\_,\ c\_] :\to \mathrm{Tensor}[a, b, c]$

(Charge Conjugation)

$\langle P_{l\_}.(a\_) \mid P_{l\_}.(\not{D}).(a\_)\rangle :\to \langle a \mid P_l.(\not{D}).a\rangle$

(Chiral orthogonal)

$\langle a\_ \mid P_L.(\not{D}).(a\_)\rangle + \langle a\_ \mid P_R.(\not{D}).(a\_)\rangle :\to \langle a \mid (\not{D}).a\rangle$

$\to \mathcal{L}_{kin} \to \langle d^{\lambda c} \mid (\not{D}).d^{\lambda c}\rangle + \langle e^{\lambda} \mid (\not{D}).e^{\lambda}\rangle + \langle u^{\lambda c} \mid (\not{D}).u^{\lambda c}\rangle + \langle \nu^{\lambda} \mid (\not{D}).\nu^{\lambda}\rangle$

Reinsert J_M:

$\langle \mathrm{Tensor}[a\_, b\_, c\_] \mid (\not{D}).\mathrm{Tensor}[a\_, b\_, c\_]\rangle :\to \langle J_M.\mathrm{Tensor}[\bar{a}, b, c] \mid (\not{D}).\mathrm{Tensor}[a, b, c]\rangle$

$$\to \boxed{\mathcal{L}_{kin} \to \sum\left[\begin{array}{l} \langle J_M.\bar{d}^{\lambda c} \mid (\not{D}).d^{\lambda c}\rangle \\ \langle J_M.\bar{e}^{\lambda} \mid (\not{D}).e^{\lambda}\rangle \\ \langle J_M.\bar{u}^{\lambda c} \mid (\not{D}).u^{\lambda c}\rangle \\ \langle J_M.\bar{\nu}^{\lambda} \mid (\not{D}).\nu^{\lambda}\rangle \end{array}\right]}$$

We check these calculations with the standard Peskin-Schroder chirality operations on Dirac spinors

```
PR["■Examine standard spinor and chirality relationship: ",
 $spin = q → ({#} & /@ {ψL, ψL, ψR, ψR}) /. a_i_ :→ T[a, "d", {i}],
 Yield, $ = T[γ, "u", {5}].# & /@ $spin;
 yield, $[[2]] = $[[2]] /. tuGammaExpand; $,
 NL, "Using: ", $s = {U̅_ -> ConjugateTranspose[U].T[γ, "u", {0}],
   PL → (14 + T[γ, "u", {5}]) / 2, PR → (14 - T[γ, "u", {5}]) / 2};
 $s // ColumnBar,
 NL, "Calculate: ", $ = q̅,
 yield, $ = $ /. $spin;
 yield, $qbar = $ = $ /. $s /. tuGammaExpand; $,
 NL, "Calculate: ", $ = q̅.q,
 yield, $ = $ /. $spin;
 yield, $ = $ /. $s /. tuGammaExpand; $,
 NL, "Calculate: ", $ = (PL.q)‾. PL.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,
 NL, "Calculate: ", $ = (PL.q)‾. PR.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,
 NL, "Calculate: ", $ = (PR.q)‾. PR.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,
 NL, "Calculate: ", $ = (PR.q)‾. PL.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,

 NL, "For spinor components: ",
 $spin = q → ({#} & /@ {aL, bL, cR, dR}),
 NL, " ", $ = q̅,
 yield, $ = $ /. $spin;
 yield, $qbar = $ = $ /. $s /. tuGammaExpand; $,
 NL, "Charge conjugation LR-Rule[]: ",
 {T[q̅_, "d", {L}] :→ Conjugate[T[q, "d", L]],
  T[q̅_, "d", {R}] :→ -Conjugate[T[q, "d", R]]
 },
 NL, CO["Note the sign change for R-terms. This sign
    makes a difference in the outcome of the previous calculation."]
]
```

■Examine standard spinor and chirality relationship: $q \to \{\{\psi_L\}, \{\psi_L\}, \{\psi_R\}, \{\psi_R\}\}$
$\to\ \longrightarrow \gamma^5.q \to \{\{\psi_R\}, \{\psi_R\}, \{\psi_L\}, \{\psi_L\}\}$

Using: $\left|\begin{array}{l} \bar{U}\_ \to U^\dagger.\gamma^0 \\ P_L \to \frac{1}{2}(1_4 + \gamma^5) \\ P_R \to \frac{1}{2}(1_4 - \gamma^5) \end{array}\right.$

Calculate: $\bar{q} \longrightarrow\ \longrightarrow \{\{(\psi_L)^*, (\psi_L)^*, -(\psi_R)^*, -(\psi_R)^*\}\}$
Calculate: $\bar{q}.q \longrightarrow\ \longrightarrow \{\{2\,(\psi_L)^*\,\psi_L - 2\,(\psi_R)^*\,\psi_R\}\}$
Calculate: $\overline{P_L.q}.P_L.q \longrightarrow\ \longrightarrow (\ 0\ )$
Calculate: $\overline{P_L.q}.P_R.q \longrightarrow\ \longrightarrow (\ (\psi_L)^*\,\psi_L + (\psi_R)^*\,\psi_L - (\psi_L)^*\,\psi_R - (\psi_R)^*\,\psi_R\ )$
Calculate: $\overline{P_R.q}.P_R.q \longrightarrow\ \longrightarrow (\ 0\ )$
Calculate: $\overline{P_R.q}.P_L.q \longrightarrow\ \longrightarrow (\ (\psi_L)^*\,\psi_L - (\psi_R)^*\,\psi_L + (\psi_L)^*\,\psi_R - (\psi_R)^*\,\psi_R\ )$
For spinor components: $q \to \{\{a_L\}, \{b_L\}, \{c_R\}, \{d_R\}\}$
$\bar{q} \longrightarrow\ \longrightarrow \{\{(a_L)^*, (b_L)^*, -(c_R)^*, -(d_R)^*\}\}$
Charge conjugation LR-Rule[]: $\{\bar{q}\_{}_L :\to T[q, d, L]^*, \bar{q}\_{}_R :\to -T[q, d, R]^*\}$
Note the sign change for R-terms. This sign
makes a difference in the outcome of the previous calculation.

Evaluate terms with $B_\mu ==$ Gauge terms

```
PR["●Generate B.x Rule[]s from matrix definitions:
•For leptons in matrix form: ",
  NL, $sB = selectStdMdl[T[B_{H_{l⊕I}}, "d", {μ}]]; $sB // MatrixForms,
  NL, "Basis: ",
  $basis = $smbasis /. a-_{_i_} → T[a̅, "du", {i, λ}] /. a_{_i_} → T[a, "du", {i, λ}];
  $basis = $smbasis /. tt : Tensor[_, _, _] :> tuIndexAdd[2, λ][tt];
  $basisll = ξ̃_{l I} → ({#} & /@ ({l, I} /. $basis // Flatten));
  $basisll // ColumnFormOn[List]
 ];
PR["•For colorless quarks in matrix form:",
  NL, $sBq = selectStdMdl[T[B_{H_q⊕H_q̅}, "d", {μ}]]; $sBq // MatrixForms,
  NL, "Basis: ",
  Yield, $basisqq = ξ̃_{q q̅} → ({#} & /@ ({q, q̅} /. $basis // Flatten));
  $basisqq // ColumnFormOn[List]
 ];
PR["•Compute B.x for: ",
 $basisV = MapAt[List /@ # &, $basisSM, 2],
 NL, "•Rearrange B for basis: ",

 $bll = selectStdMdl[T[B_{H_{l⊕I}}, "d", {μ}]];
 $bqq = selectStdMdl[T[B_, "d", {μ}], {H_q}];
 $blqlq = B_{lq I q̅} → $bll[[2, 1 ;; 4, 1 ;; 4]] ⊕ $bqq[[2, 1 ;; 4, 1 ;; 4]] ⊕
     $bll[[2, 5 ;; 8, 5 ;; 8]] ⊕ $bqq[[2, 5 ;; 8, 5 ;; 8]] // tuCirclePlus2Matrix;
 $blqlq // MatrixForms;
 NL, "Compute: ",
 $ = Dot[B_{lq I q̅}, basisSM],
 Yield,
 $ = $ /. toxDot /. $blqlq /. $basisV // tuMatrixOrderedMultiply // tuOpSimplifyF[xDot] //
   (# /. toDot &);
 $ // MatrixForms;
 Yield, $sBlq = Thread[T[B, "d", {μ}].# & /@ Flatten[$basisV[[2]]] → Flatten[$]]
]
```

●Generate B.x Rule[]s from matrix definitions:

•For leptons in matrix form:

$$B_{\mathcal{H}_{1\oplus\bar{1}}\,\mu} \to \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,1\,1} - \Lambda_\mu & q_{\mu\,1\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,2\,1} & q_{\mu\,2\,2} - \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,1\,1})^* + \Lambda_\mu & -(q_{\mu\,1\,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,2\,1})^* & -(q_{\mu\,2\,2})^* + \Lambda_\mu \end{array}\right)$$

Basis: $\tilde{\xi}_{1\,\bar{1}} \to \left\| \begin{array}{l} \nu_R{}^\lambda \\ e_R{}^\lambda \\ \nu_L{}^\lambda \\ e_L{}^\lambda \\ \bar{\nu}_R{}^\lambda \\ \bar{e}_R{}^\lambda \\ \bar{\nu}_L{}^\lambda \\ \bar{e}_L{}^\lambda \end{array} \right\|$

•For colorless quarks in matrix form:

$$B_{\mathcal{H}_q\oplus\mathcal{H}_{\bar{q}}\,\mu} \to \left(\begin{array}{cccccc} V_\mu + \frac{4}{3}\,1_3\,\Lambda_\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & V_\mu - \frac{2}{3}\,1_3\,\Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_3\,q_{\mu\,1\,1} + V_\mu + \frac{1}{3}\,1_3\,\Lambda_\mu & 1_3\,q_{\mu\,1\,2} & 0 & 0 \\ 0 & 0 & 1_3\,q_{\mu\,2\,1} & 1_3\,q_{\mu\,2\,2} + V_\mu + \frac{1}{3}\,1_3\,\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & -(V_\mu)^* - \frac{4}{3}\,1_3\,\Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -(V_\mu)^* + \frac{2}{3}\,1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

Basis:

$\to\ \tilde{\xi}_{q\,\bar{q}} \to \left\| \begin{array}{l} u_R{}^\lambda \\ d_R{}^\lambda \\ u_L{}^\lambda \\ d_L{}^\lambda \\ \bar{u}_R{}^\lambda \\ \bar{d}_R{}^\lambda \\ \bar{u}_L{}^\lambda \\ \bar{d}_L{}^\lambda \end{array} \right\|$

•Compute B.x for: basisSM →
$\{\{\nu_R\}, \{e_R\}, \{\nu_L\}, \{e_L\}, \{u_R\}, \{d_R\}, \{u_L\}, \{d_L\}, \{\bar\nu_R\}, \{\bar e_R\}, \{\bar\nu_L\}, \{\bar e_L\}, \{\bar u_R\}, \{\bar d_R\}, \{\bar u_L\}, \{\bar d_L\}\}$

•Rearrange B for basis:

Compute: $B_{1q\,I\,q}$.basisSM

→

→ $\{B_\mu.\nu_R \to 0,\ B_\mu.e_R \to -2\,\Lambda_\mu.e_R,\ B_\mu.\nu_L \to q_{\mu\,1\,2}.e_L + (q_{\mu\,1\,1} - \Lambda_\mu).\nu_L,$
$B_\mu.e_L \to q_{\mu\,2\,1}.\nu_L + (q_{\mu\,2\,2} - \Lambda_\mu).e_L,\ B_\mu.u_R \to (\frac{4}{3} 1_3.\Lambda_\mu + V_\mu).u_R,\ B_\mu.d_R \to (-\frac{2}{3} 1_3.\Lambda_\mu + V_\mu).d_R,$
$B_\mu.u_L \to (1_3.q_{\mu\,1\,1} + \frac{1}{3} 1_3.\Lambda_\mu + V_\mu).u_L + 1_3.q_{\mu\,1\,2}.d_L,\ B_\mu.d_L \to (1_3.q_{\mu\,2\,2} + \frac{1}{3} 1_3.\Lambda_\mu + V_\mu).d_L + 1_3.q_{\mu\,2\,1}.u_L,$
$B_\mu.\bar\nu_R \to 0,\ B_\mu.\bar e_R \to 2\,\Lambda_\mu.\bar e_R,\ B_\mu.\bar\nu_L \to -(q_{\mu\,1\,2})^*.\bar e_L + (-(q_{\mu\,1\,1})^* + \Lambda_\mu).\bar\nu_L,$
$B_\mu.\bar e_L \to -(q_{\mu\,2\,1})^*.\bar\nu_L + (-(q_{\mu\,2\,2})^* + \Lambda_\mu).\bar e_L,\ B_\mu.\bar u_R \to (-(V_\mu)^* - \frac{4}{3} 1_3.\Lambda_\mu).\bar u_R,$
$B_\mu.\bar d_R \to (-(V_\mu)^* + \frac{2}{3} 1_3.\Lambda_\mu).\bar d_R,\ B_\mu.\bar u_L \to (-(V_\mu)^* - (q_{\mu\,1\,1})^*.1_3 - \frac{1}{3} 1_3.\Lambda_\mu).\bar u_L - (q_{\mu\,1\,2})^*.1_3.\bar d_L,$
$B_\mu.\bar d_L \to (-(V_\mu)^* - (q_{\mu\,2\,2})^*.1_3 - \frac{1}{3} 1_3.\Lambda_\mu).\bar d_L - (q_{\mu\,2\,1})^*.1_3.\bar u_L\}$

```
PR["●Show: ", $ = ℒgf;
 $ = tuRuleSelect[$t610][$[_]] // First;
 $ // ColumnSumExp,
 " i.e., terms containing ", $sB = T[B, "d", {μ}], " in ",
 Yield, $ = selectStdMdl[SF],
 line,
 Yield, $ = $00a,
 next, "Extract ", $sB, " terms:",
 Yield, $[[2]] = $[[2]] // tuTermExtract[$sB];
 $ = $ /. SF → ℒgf,
 NL, "To make manipulation more transparent ignore generation and color
   labels, and decompose basis: ", $s = selectStdMdl[ξ̃, {M}] /. δ[_] → 1,
 Yield, $ = $ /. $s // tuCircleTimesExpand;

 $s = {BraKet[a_ ⊗ b_, c_ ⊗ d_] → BraKet[a, c] ⊗ BraKet[b, d]},
 Yield,
 $ = $ /. BraKet[a_ ⊗ b_, c_ ⊗ d_] → BraKet[a, c] ⊗ BraKet[b, d];
 NL,
 CO["Note: the product basis is not a generalized product space. There is a 1-to-1
    correspondence between the M- and F-spaces which needs special handling."],
 next, "For the F-basis: ",
 $sv = $s = selectStdMdl[ξ̃F] // tuIndexDeleteAll[{λ, c}];
 $sv[[2]] = {#} & /@ $sv[[2]]; $sv,
 next, "Expand the F-space part: ",
 Yield, $;
 Yield, $ = $ /. T[B, "d", {μ}] → B1qIq
       /. $sv //. jj : JF._ :→ Thread[jj] /. selectStdMdl[JF._] /.
   a_ ⊗ BraKet[b_, c_] :→ a ⊗ BraKet[Transpose[b], c];
 Yield, $ = $ /. toxDot /. $blqlq // tuMatrixOrderedMultiply // tuOpSimplifyF[xDot];
 $ = $ /. toDot // expandDC[];
 $ = $ /. a_ ⊗ b_ :→ a ⊗ (b /. BraKet[c_, d_] :> xDot[c, d]
           // tuMatrixOrderedMultiply // tuOpSimplifyF[xDot]) /. toDot // expandDC[];
 $pass3 = $ = $ /. a_ ⊗ b_ → a ⊗ Flatten[b];

 $ // ColumnSumExp
]
```

●Show: $\mathcal{L}_{gf}$[gauge-fermion coupling] →

$|\ (\langle J_M.\bar u|\gamma^\mu.(1+\gamma_5).d\rangle + \langle J_M.\bar\nu|$

i.e., terms containing B_μ in

→ $S_F \to \frac{1}{2} \left\langle J.\tilde{\xi} \mid \mathcal{D}_A.\tilde{\xi} \right\rangle$

→ $S_F \to \frac{1}{2} \left\langle (J_M \otimes J_F).\tilde{\xi} \mid ((\not{D}) \otimes 1_F).\tilde{\xi} \right\rangle + \frac{1}{2} \left\langle (J_M \otimes J_F).\tilde{\xi} \mid (\gamma_5 \otimes \Phi).\tilde{\xi} \right\rangle + \frac{1}{2} \left\langle (J_M \otimes J_F).\tilde{\xi} \mid (\gamma^\mu \otimes B_\mu).\tilde{\xi} \right\rangle$

◆Extract B_μ terms:

→ $\frac{1}{2} \left\langle (J_M \otimes J_F).\tilde{\xi} \mid (\gamma^\mu \otimes B_\mu).\tilde{\xi} \right\rangle$

To make manipulation more transparent ignore generation and color labels, and decompose basis: $\tilde{\xi} \to \tilde{\xi}_M \otimes \tilde{\xi}_F$

→ $\{\langle a\_ \otimes b\_ \mid c\_ \otimes d\_ \rangle \to \langle a \mid c \rangle \otimes \langle b \mid d \rangle\}$

→

Note: the product basis is not a generalized product space. There is a 1-to-1 correspondence between the M- and F-spaces which needs special handling.

◆For the F-basis: $\tilde{\xi}_F \to \{\{\nu_L\}, \{e_L\}, \{\nu_R\}, \{e_R\}, \{u_L\}, \{d_L\}, \{u_R\}, \{d_R\}, \{\bar{\nu}_R\}, \{\bar{e}_R\}, \{\bar{\nu}_L\}, \{\bar{e}_L\}, \{\bar{u}_R\}, \{\bar{d}_R\}, \{\bar{u}_L\}, \{\bar{d}_L\}\}$

◆Expand the F-space part:

→

→

$$
\to \mathcal{L}_{gf} \to \frac{1}{2}\left\langle J_M.\tilde{\xi}_M \mid \gamma^\mu.\tilde{\xi}_M \right\rangle \otimes \{\textstyle\sum[\;
\begin{array}{l}
-d_L.(V_\mu)^*.\bar{d}_L \\
-d_R.(V_\mu)^*.\bar{d}_R \\
-e_L.(q_{\mu\,2\,1})^*.\bar{\nu}_L \\
-e_L.(q_{\mu\,2\,2})^*.e_L \\
e_L.\Lambda_\mu.e_L \\
2\,e_R.\Lambda_\mu.e_R \\
-u_L.(V_\mu)^*.u_L \\
-u_R.(V_\mu)^*.u_R \\
-\nu_L.(q_{\mu\,1\,1})^*.\bar{\nu}_L \\
-\nu_L.(q_{\mu\,1\,2})^*.e_L \\
\nu_L.\Lambda_\mu.\bar{\nu}_L \\
\bar{d}_L.V_\mu.d_L \\
\bar{d}_R.V_\mu.d_R \\
-2\,e_L.\Lambda_\mu.e_L \\
e_R.q_{\mu\,2\,1}.\nu_R \\
e_R.q_{\mu\,2\,2}.e_R \\
-e_R.\Lambda_\mu.e_R \\
u_L.V_\mu.u_L \\
u_R.V_\mu.u_R \\
\bar{\nu}_R.q_{\mu\,1\,1}.\nu_R \\
\bar{\nu}_R.q_{\mu\,1\,2}.e_R \\
-\bar{\nu}_R.\Lambda_\mu.\nu_R \\
-d_L.(q_{\mu\,2\,1})^*.1_3.u_L \\
-d_L.(q_{\mu\,2\,2})^*.1_3.\bar{d}_L \\
-\frac{1}{3}\,d_L.1_3.\Lambda_\mu.\bar{d}_L \\
\frac{2}{3}\,d_R.1_3.\Lambda_\mu.\bar{d}_R \\
-u_L.(q_{\mu\,1\,1})^*.1_3.u_L \\
-u_L.(q_{\mu\,1\,2})^*.1_3.\bar{d}_L \\
-\frac{1}{3}\,u_L.1_3.\Lambda_\mu.u_L \\
-\frac{4}{3}\,u_R.1_3.\Lambda_\mu.u_R \\
-\frac{2}{3}\,\bar{d}_L.1_3.\Lambda_\mu.d_L \\
\bar{d}_R.1_3.q_{\mu\,2\,1}.u_R \\
\bar{d}_R.1_3.q_{\mu\,2\,2}.d_R \\
\frac{1}{3}\,\bar{d}_R.1_3.\Lambda_\mu.d_R \\
\frac{4}{3}\,u_L.1_3.\Lambda_\mu.u_L \\
u_R.1_3.q_{\mu\,1\,1}.u_R \\
u_R.1_3.q_{\mu\,1\,2}.d_R \\
\frac{1}{3}\,u_R.1_3.\Lambda_\mu.u_R
\end{array}
\;]\}
$$

```
$x = <| |>;

PR["• Revert q's to SU[2] Q's (ℝ) so we can relate this to
    physical gauge parameters via: ", $ = selectGWS[T[Q, "d", {_}], {σ}];
  $sq = Table[T[q, "ddd", {μ, i, j}], {i, 2}, {j, 2}] -> $[[2]] /. xSum → Sum /.
      tuPauliExpand //. rr : Rule[__] ⧴ Thread[rr] // Flatten;
  $sq // ColumnBar
 ];
PR[$ = $pass3;
 NL, "• BraKet to Dot notation require ConjugateTranspose of the first term: ",
 $0 = $ = $ /. (a_ ⊗ b_ ⧴ a ⊗ (b /. HoldPattern[Shortest[a1_].b1_ .Shortest[c1_]] ⧴
          (J_F.a1 /. selectStdMdl[J_F._]) . b1.c1)
    ); $ // ColumnSumExp;

 NL, "Convert to gauge fields {A,Z,W,G} with Rule[]s for Q's : ",
$sQ = {
    cc /@ # & /@ selectStdMdl[Tensor[Q, _, _] + _, {W}, all],
    selectStdMdl[Tensor[Q, _, _] + _, {W}, all],
    tuRuleSolve[selectStdMdl[T[Q, "du", {μ, 3}] + _, {}, all] // First,
     T[Q, "du", {μ, 3}]],
    tuRuleSolve[selectStdMdl[T[Q, "du", {μ, 3}] + _, {}, all] //
      tuRuleEliminate[{T[Q, "du", {μ, 3}]}]
     , T[Λ, "d", {μ}] ],
    ($s = selectStdMdl[Tensor[V, _, _], {}, all];
     $s[[1]] /. $s[[2]])
   } // Flatten; $sQ // ColumnBar,
 Yield, $ = $ /. $sq;
 Yield, $ = $ //. tuRule[$sQ] // Expand;
 $ = $ //. tuConjugateDistribute // tuConjugateSimplify[] // Expand;
 $ = $ // expandDC[(-1 + c_w^2) → -s_w^2];
 $0 = $ = $ // tuConjugateSimplify[{s_w, c_w, g_, Tensor[A | Z, _, _]}];

 NL, "Extract terms with only: ", $sx = e, $no = {}; (**)
 Yield, $ = $0 // tuTermExtract[$sx, $no]; $ // ColumnSumExp;
 Yield, $x = Append[$x, $sx → $];
 NL, "Extract terms with only: ", $sx = ν, $no = {}; (**)
 Yield, $ = $0 // tuTermExtract[$sx, $no]; $ // ColumnSumExp;
 Yield, $x = Append[$x, $sx → $];
 NL, "Extract terms with only: ", $sx = u, $no = {}; (**)
 Yield, $ = $0 // tuTermExtract[$sx, $no]; $ // ColumnSumExp;
 Yield, $x = Append[$x, $sx → $];
 NL, "Extract terms with only: ", $sx = d, $no = {}; (**)
 Yield, $ = $0 // tuTermExtract[$sx, $no]; $ // ColumnSumExp;
 Yield, $x = Append[$x, $sx → $];
]
```

• Revert q's to SU[2] Q's (ℝ) so we can relate this to physical gauge parameters via:

$$\left|\begin{array}{l} q_{\mu\,1\,1} \to Q_\mu{}^3 \\ q_{\mu\,1\,2} \to Q_\mu{}^1 - i\, Q_\mu{}^2 \\ q_{\mu\,2\,1} \to Q_\mu{}^1 + i\, Q_\mu{}^2 \\ q_{\mu\,2\,2} \to -Q_\mu{}^3 \end{array}\right.$$

• BraKet to Dot notation require ConjugateTranspose of the first term:

Convert to gauge fields {A,Z,W,G} with Rule[]s for Q's :

$$(Q_\mu{}^1)^* - i\,(Q_\mu{}^2)^* \to \frac{(g_2\,W_\mu)^*}{\sqrt{2}}$$
$$(Q_\mu{}^1)^* + i\,(Q_\mu{}^2)^* \to \frac{((W_\mu)^\dagger\,g_2)^*}{\sqrt{2}}$$
$$Q_\mu{}^1 + i\,Q_\mu{}^2 \to \frac{g_2\,W_\mu}{\sqrt{2}}$$
$$Q_\mu{}^1 - i\,Q_\mu{}^2 \to \frac{(W_\mu)^\dagger\,g_2}{\sqrt{2}}$$
$$Q_\mu{}^3 \to \frac{g_2\,Z_\mu + 2\,c_w\,\Lambda_\mu}{2\,c_w}$$
$$\Lambda_\mu \to -\frac{-c_w\,g_2\,s_w\,A_\mu + g_2\,Z_\mu - c_w^2\,g_2\,Z_\mu}{2\,c_w}$$
$$V_\mu \to \frac{1}{2}\,g_3\,G_\mu{}^i\,\lambda_i$$

→
→
Extract terms with only: e
→
→
Extract terms with only: ν
→
→
Extract terms with only: u
→
→
Extract terms with only: d
→
→ Null

```
$fspace = <||>;
$extractCollect :=
 ($ = $x[$s] // tuTermExtract[$sg] // expandDC[{}, {1n_}] // (# //. tuOpCollect[] &) //
     expandDC[{}, {Tensor[Z, _, _], cc[Tensor[Z, _, _]], g_, cw, sw, 1n_}] // Simplify;
  AppendTo[$fspace, {$sg, $s} -> $])

PR[
 "Examine F-space ", $sg = A, " terms ", (***)
 NL, "For ", $s = e,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = ν,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = u,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = d,
 Yield, $extractCollect; $ // ColumnSumExp,

 next, "Examine F-space ", $sg = Z, " terms ", (***)
 NL, "For ", $s = e,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = ν,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = u,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = d,
 Yield, $extractCollect; $ // ColumnSumExp,

 next, "Examine F-space ", $sg = G, " terms ", (***)
 NL, "For ", $s = e,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = ν,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = u,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = d,
 Yield, $extractCollect; $ // ColumnSumExp,

 next, "Examine F-space ", $sg = W, " terms ", (***)
 NL, "For ", $s = e,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = ν,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = u,
 Yield, $extractCollect; $ // ColumnSumExp,
 NL, "For ", $s = d,
 Yield, $extractCollect; $ // ColumnSumExp,


]
```

Examine F-space A terms
For e

$$\rightarrow \sum\left[\begin{array}{|l}-e_L\cdot A_\mu\cdot e_L\\ -e_R\cdot A_\mu\cdot e_R\\ \bar{e}_L\cdot A_\mu\cdot \bar{e}_L\\ \bar{e}_R\cdot A_\mu\cdot \bar{e}_R\end{array}\right] g_2\, s_w$$

For ν

$\to 0$

For u

$$\to \frac{2}{3}\sum\left[\begin{vmatrix} u_L . A_\mu . u_L \\ u_R . A_\mu . u_R \\ -\overline{u}_L . A_\mu . \overline{u}_L \\ -\overline{u}_R . A_\mu . \overline{u}_R \end{vmatrix}\right] 1_3\, g_2\, s_w$$

For d

$$\to \frac{1}{3}\sum\left[\begin{vmatrix} -d_L . A_\mu . d_L \\ -d_R . A_\mu . d_R \\ \overline{d}_L . A_\mu . \overline{d}_L \\ \overline{d}_R . A_\mu . \overline{d}_R \end{vmatrix}\right] 1_3\, g_2\, s_w$$

◆Examine F-space Z terms

For e

$$\to \frac{\sum\left[\begin{vmatrix} -\overline{e}_L . \overline{e}_L \\ -2\, \overline{e}_R . \overline{e}_R \\ 2\, \overline{e}_L . \overline{e}_L\, c_w^2 \\ 2\, \overline{e}_R . \overline{e}_R\, c_w^2 \\ e_R . e_R\, (1 - 2\, c_w^2) \\ -2\, e_L . e_L\, (-1 + c_w^2) \end{vmatrix}\right] g_2\, Z_\mu}{2\, c_w}$$

For ν

$$\to \frac{\sum\left[\begin{vmatrix} \nu_R . \nu_R \\ -\overline{\nu}_L . \overline{\nu}_L \end{vmatrix}\right] g_2\, Z_\mu}{2\, c_w}$$

For u

$$\to \frac{\sum\left[\begin{vmatrix} \overline{u}_L . \overline{u}_L \\ 4\, \overline{u}_R . \overline{u}_R \\ -4\, \overline{u}_L . \overline{u}_L\, c_w^2 \\ -4\, \overline{u}_R . \overline{u}_R\, c_w^2 \\ 4\, u_L . u_L\, (-1 + c_w^2) \\ u_R . u_R\, (-1 + 4\, c_w^2) \end{vmatrix}\right] 1_3\, g_2\, Z_\mu}{6\, c_w}$$

For d

$$\to \frac{\sum\left[\begin{vmatrix} \overline{d}_L . \overline{d}_L \\ -2\, \overline{d}_R . \overline{d}_R \\ 2\, \overline{d}_L . \overline{d}_L\, c_w^2 \\ 2\, \overline{d}_R . \overline{d}_R\, c_w^2 \\ -2\, d_L . d_L\, (-1 + c_w^2) \\ -d_R . d_R\, (1 + 2\, c_w^2) \end{vmatrix}\right] 1_3\, g_2\, Z_\mu}{6\, c_w}$$

◆Examine F-space G terms

For e

$\to 0$

For ν

$\to 0$

For u

$$\to \frac{1}{2}\sum\left[\begin{vmatrix} u_L . (G_\mu{}^i\, \lambda_i) . u_L \\ u_R . (G_\mu{}^i\, \lambda_i) . u_R \\ -\overline{u}_L . ((G_\mu{}^i)^*\, (\lambda_i)^*) . \overline{u}_L \\ -\overline{u}_R . ((G_\mu{}^i)^*\, (\lambda_i)^*) . \overline{u}_R \end{vmatrix}\right] g_3$$

For d

$\rightarrow \frac{1}{2}\sum[\left|\begin{array}{l} d_L.(G_\mu{}^i\,\lambda_i).d_L \\ d_R.(G_\mu{}^i\,\lambda_i).d_R \\ -\overline{d}_L.((G_\mu{}^i)^*\,(\lambda_i)^*).\overline{d}_L \\ -\overline{d}_R.((G_\mu{}^i)^*\,(\lambda_i)^*).\overline{d}_R \end{array}\right.]\,g_3$

◆Examine F-space W terms

For e

$\rightarrow \frac{\sum[\left|\begin{array}{l} e_R.W_\mu.\nu_R \\ \nu_R.(W_\mu)^\dagger.e_R \\ -\overline{e}_L.(W_\mu)^*.\overline{\nu}_L \\ -\overline{\nu}_L.(W_\mu)^{\dagger *}.\overline{e}_L \end{array}\right.]\,g_2}{\sqrt{2}}$

For ν

$\rightarrow \frac{\sum[\left|\begin{array}{l} e_R.W_\mu.\nu_R \\ \nu_R.(W_\mu)^\dagger.e_R \\ -\overline{e}_L.(W_\mu)^*.\overline{\nu}_L \\ -\overline{\nu}_L.(W_\mu)^{\dagger *}.\overline{e}_L \end{array}\right.]\,g_2}{\sqrt{2}}$

For u

$\rightarrow \frac{\sum[\left|\begin{array}{l} d_R.W_\mu.u_R \\ u_R.(W_\mu)^\dagger.d_R \\ -\overline{d}_L.(W_\mu)^*.\overline{u}_L \\ -\overline{u}_L.(W_\mu)^{\dagger *}.\overline{d}_L \end{array}\right.]\,1_3\,g_2}{\sqrt{2}}$

For d

$\rightarrow \frac{\sum[\left|\begin{array}{l} d_R.W_\mu.u_R \\ u_R.(W_\mu)^\dagger.d_R \\ -\overline{d}_L.(W_\mu)^*.\overline{u}_L \\ -\overline{u}_L.(W_\mu)^{\dagger *}.\overline{d}_L \end{array}\right.]\,1_3\,g_2}{\sqrt{2}}$ Null

```
PR["Relationship between bases of ", $ = {ξ̃F, ξ̃M},
 " in context of ℒgh tensor products. ", $ = {$xF = selectStdMdl[$[[1]]],
    $xM = selectStdMdl[$[[2]]]} // tuIndexDeleteAll[{λ, c}];
 $ // ColumnBar,
 NL, "where terms are of form: ",
 BraKet[JM.basism1_, T[γ, "u", {μ}].basism2_] ⊗ BraKet[JF.basisf1_, arb_ .basisf2_],
 NL, "The correspondence of M- to F-bases: ",
 $ = Thread[$, Rule]; $ = Rule @@ # & /@ Thread[$[[2]]];
 $sFM = $
]
```

Relationship between bases of $\{\tilde{\xi}_F, \tilde{\xi}_M\}$ in context of $\mathcal{L}_{gh}$ tensor products.

$\left|\begin{array}{l} \tilde{\xi}_F \rightarrow \{\nu_L, e_L, \nu_R, e_R, u_L, d_L, u_R, d_R, \overline{\nu}_R, \overline{e}_R, \overline{\nu}_L, \overline{e}_L, \overline{u}_R, \overline{d}_R, \overline{u}_L, \overline{d}_L\} \\ \tilde{\xi}_M \rightarrow \{\nu_L, e_L, \nu_R, e_R, u_L, d_L, u_R, d_R, \overline{\nu}_L, \overline{e}_L, \overline{\nu}_R, \overline{e}_R, \overline{u}_L, \overline{d}_L, \overline{u}_R, \overline{d}_R\} \end{array}\right.$

where terms are of form:

$\langle J_M.(basism1\_) \mid \gamma^\mu.(basism2\_)\rangle \otimes \langle J_F.(basisf1\_) \mid (arb\_).(basisf2\_)\rangle$

The correspondence of M- to F-bases: $\{\nu_L \rightarrow \nu_L, e_L \rightarrow e_L, \nu_R \rightarrow \nu_R, e_R \rightarrow e_R, u_L \rightarrow u_L, d_L \rightarrow d_L, u_R \rightarrow u_R, d_R \rightarrow d_R, \overline{\nu}_R \rightarrow \overline{\nu}_L, \overline{e}_R \rightarrow \overline{e}_L, \overline{\nu}_L \rightarrow \overline{\nu}_R, \overline{e}_L \rightarrow \overline{e}_R, \overline{u}_R \rightarrow \overline{u}_L, \overline{d}_R \rightarrow \overline{d}_L, \overline{u}_L \rightarrow \overline{u}_R, \overline{d}_L \rightarrow \overline{d}_R\}$

```
$terms = <||>;
$fspace;
$simplifyFspace :=
 $ = $ /. selectStdMdl[JF._] // tuCircleTimesSimplify // expandDC[{}, $scal] //
    (# //. tuBraKetSimplify[$scal]
         /. BraKet[a_, a_] → 1
        /. {BraKet[a_, b_] ⧴ 1 /; (tuHasAllQ[{a, b}, {e, ν}] ||
              tuHasAllQ[{a, b}, {u, d}]) && FreeQ[{a, b}, OverBar]} /.
       a_ ⊗ aa_ :> a aa /; NumericQ[aa] || tuHasAnyQ[aa, $scal] &) //
   Simplify // (# /. cw^2 → 1 - sw^2 &)
Map[($sFv = #;
   PR["Evaluate terms in ℒgf for the {field,basis}: ", $s = $sFv, (*={A,u}*)(**  *)
    Yield, $ = $fspace[$s],

    NL, "Add back JF to apply basis correspondence: ",
    $s = HoldPattern[Shortest[a_]. c___ .Shortest[b_]] ⧴
      BraKet[JM.(JF.a /. selectStdMdl[JF._]), T[γ, "u", {μ}].(b /. $sFM)] ⊗
       BraKet[JF.(JF.a /. selectStdMdl[JF._]), c.b],
    Yield, $ = $ /. $s,
    NL, "Order JM terms(anti-symmetric): ", $s = HoldPattern[
       BraKet[JM.a_, c_ .Shortest[b_]]] ⧴ -BraKet[JM.b, c.a] /; OrderedQ[{a, b}],
    Yield, $ = $ /. $s;
    NL, "Simplify F-space with Dot[] and ⊗ Scalar: ",
    $scal = {Tensor[A, _, _], Tensor[W, _, _], cc[Tensor[W, _, _]], ct[Tensor[W, _, _]],
      cc[ct[Tensor[W, _, _]]], Tensor[G, _, _], Tensor[λ, _, _]},
    Yield,
    $simplifyFspace,

    NL, "Impose chiral orthogonality ", NL, $s = BraKet[JM.a_, T[γ, "u", {μ}].b_] ⧴
      0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L]),
    Yield, $ = $ /. $s // Expand // tuCircleTimesSimplify,
    NL, "Combine chiral bases: ",
    NL, $s = {(c_: 1 ) BraKet[JM.T[a_, "d", {L}], T[γ, "u", {μ}].T[b_, "d", {L}]] +
        (c_: 1) BraKet[JM.T[a_, "d", {R}], T[γ, "u", {μ}].T[b_, "d", {R}]] →
       c BraKet[JM.T[a, "", {}], T[γ, "u", {μ}].T[b, "", {}]]},

    Yield, $ = $ /. $s; $ // Framed,
    NL, "Add chiral projection operators if possible ",
    $s = {c_ .T[a_, "d", {R}] ⧴ c.PR.T[a, "", {}] /; FreeQ[c, PR],
      BraKet[JM.PR.a_, b_ . PR.c_] -> BraKet[JM.a, b. PR.c]
     },
    Yield, $ = $ //. $s; $ // Framed,
    Yield, $ =
     $ //. tuOpCollect[BraKet] //. tuOpCollect[] // tuCircleTimesSimplify // Simplify;
    $terms = Append[$terms, $sFv -> $];

    $ // Framed
   ];) &, Outer[List, {A, Z, W, G}, {e, ν, u, d}] // Flatten[#, 1] &];
```

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {A, e}

→ $(-e_L.A_\mu.e_L - e_R.A_\mu.e_R + \bar{e}_L.A_\mu.\bar{e}_L + \bar{e}_R.A_\mu.\bar{e}_R)\ g_2\ s_w$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] $:\to \langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

→ $(\langle J_M.e_L \mid \gamma^\mu.\bar{e}_R\rangle \otimes \langle J_F.e_L \mid A_\mu.\bar{e}_L\rangle + \langle J_M.e_R \mid \gamma^\mu.\bar{e}_L\rangle \otimes \langle J_F.e_R \mid A_\mu.\bar{e}_R\rangle -$
$\langle J_M.\bar{e}_L \mid \gamma^\mu.e_L\rangle \otimes \langle J_F.\bar{e}_L \mid A_\mu.e_L\rangle - \langle J_M.\bar{e}_R \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.\bar{e}_R \mid A_\mu.e_R\rangle)\ g_2\ s_w$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] $:\to$
$-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _],
Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

→ $-(\langle J_M.\bar{e}_L \mid \gamma^\mu.e_L\rangle + \langle J_M.\bar{e}_L \mid \gamma^\mu.e_R\rangle + \langle J_M.\bar{e}_R \mid \gamma^\mu.e_L\rangle + \langle J_M.\bar{e}_R \mid \gamma^\mu.e_R\rangle)\ g_2\ s_w\ A_\mu$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle :\to 0$ /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

→ $-\langle J_M.\bar{e}_L \mid \gamma^\mu.e_L\rangle\ g_2\ s_w\ A_\mu - \langle J_M.\bar{e}_R \mid \gamma^\mu.e_R\rangle\ g_2\ s_w\ A_\mu$

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

→ $\boxed{-\langle J_M.\bar{e} \mid \gamma^\mu.e\rangle\ g_2\ s_w\ A_\mu}$

Add chiral projection operators if possible

{(c_).a_$_R$ $:\to$ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle \to \langle J_M.a \mid b.P_R.c\rangle$}

→ $\boxed{-\langle J_M.\bar{e} \mid \gamma^\mu.e\rangle\ g_2\ s_w\ A_\mu}$

→ $\boxed{-\langle J_M.\bar{e} \mid \gamma^\mu.e\rangle\ g_2\ s_w\ A_\mu}$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {A, ν}

→ 0

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] $:\to \langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

→ 0

Order J_M terms(anti-symmetric):

HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] $:\to -\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _],
Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

→ 0

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle :\to 0$ /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

→ 0

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

→ $\boxed{0}$

Add chiral projection operators if possible

{(c_).a_$_R$ $:\to$ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle \to \langle J_M.a \mid b.P_R.c\rangle$}

→ $\boxed{0}$

→ $\boxed{0}$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {A, u}

→ $\frac{2}{3}(u_L.A_\mu.u_L + u_R.A_\mu.u_R - \bar{u}_L.A_\mu.\bar{u}_L - \bar{u}_R.A_\mu.\bar{u}_R)\, 1_3\, g_2\, s_w$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

→ $\frac{2}{3}(-(\langle J_M.u_L \mid \gamma^\mu.\bar{u}_R\rangle \otimes \langle J_F.u_L \mid A_\mu.\bar{u}_L\rangle) - \langle J_M.u_R \mid \gamma^\mu.\bar{u}_L\rangle \otimes \langle J_F.u_R \mid A_\mu.\bar{u}_R\rangle + \langle J_M.\bar{u}_L \mid \gamma^\mu.u_L\rangle \otimes \langle J_F.\bar{u}_L \mid A_\mu.u_L\rangle + \langle J_M.\bar{u}_R \mid \gamma^\mu.u_R\rangle \otimes \langle J_F.\bar{u}_R \mid A_\mu.u_R\rangle)\, 1_3\, g_2\, s_w$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] :→ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

→ $\frac{2}{3}(\langle J_M.\bar{u}_L \mid \gamma^\mu.u_L\rangle + \langle J_M.\bar{u}_L \mid \gamma^\mu.u_R\rangle + \langle J_M.\bar{u}_R \mid \gamma^\mu.u_L\rangle + \langle J_M.\bar{u}_R \mid \gamma^\mu.u_R\rangle)\, 1_3\, g_2\, s_w\, A_\mu$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

→ $\frac{2}{3}\langle J_M.\bar{u}_L \mid \gamma^\mu.u_L\rangle\, 1_3\, g_2\, s_w\, A_\mu + \frac{2}{3}\langle J_M.\bar{u}_R \mid \gamma^\mu.u_R\rangle\, 1_3\, g_2\, s_w\, A_\mu$

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

→ $\boxed{\frac{2}{3}\langle J_M.\bar{u} \mid \gamma^\mu.u\rangle\, 1_3\, g_2\, s_w\, A_\mu}$

Add chiral projection operators if possible

{(c_).a_$_R$:→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle$ → $\langle J_M.a \mid b.P_R.c\rangle$}

→ $\boxed{\frac{2}{3}\langle J_M.\bar{u} \mid \gamma^\mu.u\rangle\, 1_3\, g_2\, s_w\, A_\mu}$

→ $\boxed{\frac{2}{3}\langle J_M.\bar{u} \mid \gamma^\mu.u\rangle\, 1_3\, g_2\, s_w\, A_\mu}$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {A, d}

→ $\frac{1}{3}(-d_L.A_\mu.d_L - d_R.A_\mu.d_R + \overline{d}_L.A_\mu.\overline{d}_L + \overline{d}_R.A_\mu.\overline{d}_R)\, 1_3\, g_2\, s_w$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

→ $\frac{1}{3}(\langle J_M.d_L \mid \gamma^\mu.\overline{d}_R\rangle \otimes \langle J_F.d_L \mid A_\mu.\overline{d}_L\rangle + \langle J_M.d_R \mid \gamma^\mu.\overline{d}_L\rangle \otimes \langle J_F.d_R \mid A_\mu.\overline{d}_R\rangle - \langle J_M.\overline{d}_L \mid \gamma^\mu.d_L\rangle \otimes \langle J_F.\overline{d}_L \mid A_\mu.d_L\rangle - \langle J_M.\overline{d}_R \mid \gamma^\mu.d_R\rangle \otimes \langle J_F.\overline{d}_R \mid A_\mu.d_R\rangle)\, 1_3\, g_2\, s_w$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] :→ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

→ $-\frac{1}{3}(\langle J_M.\overline{d}_L \mid \gamma^\mu.d_L\rangle + \langle J_M.\overline{d}_L \mid \gamma^\mu.d_R\rangle + \langle J_M.\overline{d}_R \mid \gamma^\mu.d_L\rangle + \langle J_M.\overline{d}_R \mid \gamma^\mu.d_R\rangle)\, 1_3\, g_2\, s_w\, A_\mu$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

→ $-\frac{1}{3}\langle J_M.\overline{d}_L \mid \gamma^\mu.d_L\rangle\, 1_3\, g_2\, s_w\, A_\mu - \frac{1}{3}\langle J_M.\overline{d}_R \mid \gamma^\mu.d_R\rangle\, 1_3\, g_2\, s_w\, A_\mu$

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

→ $\boxed{-\frac{1}{3}\langle J_M.\overline{d} \mid \gamma^\mu.d\rangle\, 1_3\, g_2\, s_w\, A_\mu}$

Add chiral projection operators if possible

{(c_).a_R :→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle$ → $\langle J_M.a \mid b.P_R.c\rangle$}

→ $\boxed{-\frac{1}{3}\langle J_M.\overline{d} \mid \gamma^\mu.d\rangle\, 1_3\, g_2\, s_w\, A_\mu}$

→ $\boxed{-\frac{1}{3}\langle J_M.\overline{d} \mid \gamma^\mu.d\rangle\, 1_3\, g_2\, s_w\, A_\mu}$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {Z, e}

$$\to \frac{(-\bar{e}_L.\bar{e}_L - 2\,\bar{e}_R.\bar{e}_R + 2\,\bar{e}_L.\bar{e}_L\,c_w^2 + 2\,\bar{e}_R.\bar{e}_R\,c_w^2 + e_R.e_R\,(1 - 2\,c_w^2) - 2\,e_L.e_L\,(-1 + c_w^2))\,g_2\,Z_\mu}{2\,c_w}$$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$$\to \frac{1}{2\,c_w}(-(\langle J_M.e_L \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.e_L \mid e_L\rangle) - 2\,\langle J_M.e_R \mid \gamma^\mu.e_L\rangle \otimes \langle J_F.e_R \mid e_R\rangle + 2\,\langle J_M.e_L \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.e_L \mid e_L\rangle\,c_w^2 + 2\,\langle J_M.e_R \mid \gamma^\mu.e_L\rangle \otimes \langle J_F.e_R \mid e_R\rangle\,c_w^2 + \langle J_M.e_R \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.e_R \mid e_R\rangle\,(1 - 2\,c_w^2) - 2\,\langle J_M.e_L \mid \gamma^\mu.e_L\rangle \otimes \langle J_F.e_L \mid e_L\rangle\,(-1 + c_w^2))\,g_2\,Z_\mu$$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] :→ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$$\to -\frac{1}{2\,c_w} g_2\,(-2\,\langle J_M.e_L \mid \gamma^\mu.e_L\rangle\,s_w^2 - 2\,\langle J_M.e_L \mid \gamma^\mu.e_R\rangle\,s_w^2 + (\langle J_M.e_R \mid \gamma^\mu.e_L\rangle + \langle J_M.e_R \mid \gamma^\mu.e_R\rangle)\,(-1 + 2\,(1 - s_w^2)))\,Z_\mu$$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$$\to -\frac{\langle J_M.e_R \mid \gamma^\mu.e_R\rangle\,g_2\,Z_\mu}{2\,c_w} + \frac{\langle J_M.e_L \mid \gamma^\mu.e_L\rangle\,g_2\,s_w^2\,Z_\mu}{c_w} + \frac{\langle J_M.e_R \mid \gamma^\mu.e_R\rangle\,g_2\,s_w^2\,Z_\mu}{c_w}$$

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

$$\to \boxed{-\frac{\langle J_M.e_R \mid \gamma^\mu.e_R\rangle\,g_2\,Z_\mu}{2\,c_w} + \frac{\langle J_M.e \mid \gamma^\mu.e\rangle\,g_2\,s_w^2\,Z_\mu}{c_w}}$$

Add chiral projection operators if possible

{(c_).a_$_R$:→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle$ → $\langle J_M.a \mid b.P_R.c\rangle$}

$$\to \boxed{-\frac{\langle J_M.e \mid \gamma^\mu.P_R.e\rangle\,g_2\,Z_\mu}{2\,c_w} + \frac{\langle J_M.e \mid \gamma^\mu.e\rangle\,g_2\,s_w^2\,Z_\mu}{c_w}}$$

$$\to \boxed{\left\langle J_M.e \mid \left(-\frac{g_2\,Z_\mu\,(\gamma^\mu.P_R - 2\,s_w^2\,\gamma^\mu)}{2\,c_w}\right).e\right\rangle}$$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {Z, ν}

$$\rightarrow \frac{(\nu_R \cdot \nu_R - \bar{\nu}_L \cdot \bar{\nu}_L)\, g_2\, Z_\mu}{2\, c_w}$$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M . J_F . a \mid \gamma^\mu . b\rangle \otimes \langle J_F . J_F . a \mid c . b\rangle$

$$\rightarrow \frac{(-(\langle J_M . \nu_L \mid \gamma^\mu . \bar{\nu}_R\rangle \otimes \langle J_F . \nu_L \mid \bar{\nu}_L\rangle) + \langle J_M . \bar{\nu}_R \mid \gamma^\mu . \nu_R\rangle \otimes \langle J_F . \bar{\nu}_R \mid \nu_R\rangle)\, g_2\, Z_\mu}{2\, c_w}$$

Order J_M terms(anti-symmetric):

HoldPattern[$\langle J_M . (a\_) \mid (c\_) . \text{Shortest}[b\_]\rangle$] :→ $-\langle J_M . b \mid c . a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$$\rightarrow \frac{(\langle J_M . \bar{\nu}_R \mid \gamma^\mu . \nu_L\rangle + \langle J_M . \bar{\nu}_R \mid \gamma^\mu . \nu_R\rangle)\, g_2\, Z_\mu}{2\, c_w}$$

Impose chiral orthogonality

$\langle J_M . (a\_) \mid \gamma^\mu . (b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$$\rightarrow \frac{\langle J_M . \bar{\nu}_R \mid \gamma^\mu . \nu_R\rangle\, g_2\, Z_\mu}{2\, c_w}$$

Combine chiral bases:

{$\langle J_M . a\_{}_L \mid \gamma^\mu . b\_{}_L\rangle$ (c_ : 1) + $\langle J_M . a\_{}_R \mid \gamma^\mu . b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M . a \mid \gamma^\mu . b\rangle$}

$$\rightarrow \boxed{\frac{\langle J_M . \bar{\nu}_R \mid \gamma^\mu . \nu_R\rangle\, g_2\, Z_\mu}{2\, c_w}}$$

Add chiral projection operators if possible

{(c_).a_$_R$:→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M . P_R . (a\_) \mid (b\_) . P_R . (c\_)\rangle$ → $\langle J_M . a \mid b . P_R . c\rangle$}

$$\rightarrow \boxed{\frac{\langle J_M . \bar{\nu} \mid \gamma^\mu . P_R . \nu\rangle\, g_2\, Z_\mu}{2\, c_w}}$$

$$\rightarrow \boxed{\frac{\langle J_M . \bar{\nu} \mid \gamma^\mu . P_R . \nu\rangle\, g_2\, Z_\mu}{2\, c_w}}$$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {Z, u}

$$\to \frac{1_3\,(\bar{u}_L.\bar{u}_L + 4\,\bar{u}_R.\bar{u}_R - 4\,\bar{u}_L.\bar{u}_L\,c_w^2 - 4\,\bar{u}_R.\bar{u}_R\,c_w^2 + 4\,u_L.u_L\,(-1+c_w^2) + u_R.u_R\,(-1+4\,c_w^2))\,g_2\,Z_\mu}{6\,c_w}$$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$$\to \frac{1}{6\,c_w} 1_3\,(\langle J_M.u_L \mid \gamma^\mu.u_R\rangle \otimes \langle J_F.u_L \mid u_L\rangle + 4\,\langle J_M.u_R \mid \gamma^\mu.u_L\rangle \otimes \langle J_F.u_R \mid u_R\rangle - 4\,\langle J_M.u_L \mid \gamma^\mu.u_R\rangle \otimes \langle J_F.u_L \mid u_L\rangle\,c_w^2 - 4\,\langle J_M.u_R \mid \gamma^\mu.u_L\rangle \otimes \langle J_F.u_R \mid u_R\rangle\,c_w^2 + 4\,\langle J_M.u_L \mid \gamma^\mu.u_L\rangle \otimes \langle J_F.u_L \mid u_L\rangle\,(-1+c_w^2) + \langle J_M.u_R \mid \gamma^\mu.u_R\rangle \otimes \langle J_F.u_R \mid u_R\rangle\,(-1+4\,c_w^2))\,g_2\,Z_\mu$$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] :→ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$$\to \frac{1}{6\,c_w} 1_3\,g_2\,(-4\,\langle J_M.u_L \mid \gamma^\mu.u_L\rangle\,s_w^2 - 4\,\langle J_M.u_L \mid \gamma^\mu.u_R\rangle\,s_w^2 + (\langle J_M.u_R \mid \gamma^\mu.u_L\rangle + \langle J_M.u_R \mid \gamma^\mu.u_R\rangle)\,(-1+4\,(1-s_w^2)))\,Z_\mu$$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$$\to \frac{\langle J_M.u_R \mid \gamma^\mu.u_R\rangle\,1_3\,g_2\,Z_\mu}{2\,c_w} - \frac{2\,\langle J_M.u_L \mid \gamma^\mu.u_L\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w} - \frac{2\,\langle J_M.u_R \mid \gamma^\mu.u_R\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w}$$

Combine chiral bases:

{$\langle J_M.a\_L \mid \gamma^\mu.b\_L\rangle$ (c_ : 1) + $\langle J_M.a\_R \mid \gamma^\mu.b\_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

$$\to \boxed{\frac{\langle J_M.u_R \mid \gamma^\mu.u_R\rangle\,1_3\,g_2\,Z_\mu}{2\,c_w} - \frac{2\,\langle J_M.u \mid \gamma^\mu.u\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w}}$$

Add chiral projection operators if possible

{(c_).a_R :→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle$ → $\langle J_M.a \mid b.P_R.c\rangle$}

$$\to \boxed{\frac{\langle J_M.u \mid \gamma^\mu.P_R.u\rangle\,1_3\,g_2\,Z_\mu}{2\,c_w} - \frac{2\,\langle J_M.u \mid \gamma^\mu.u\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w}}$$

$$\to \boxed{\left\langle J_M.u \;\middle|\; \frac{1_3\,g_2\,Z_\mu\,(3\,\gamma^\mu.P_R - 4\,s_w^2\,\gamma^\mu)}{6\,c_w}.u \right\rangle}$$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {Z, d}

$$\to \frac{1_3\,(\bar{d}_L.\bar{d}_L - 2\,\bar{d}_R.\bar{d}_R + 2\,\bar{d}_L.\bar{d}_L\,c_w^2 + 2\,\bar{d}_R.\bar{d}_R\,c_w^2 - 2\,d_L.d_L\,(-1 + c_w^2) - d_R.d_R\,(1 + 2\,c_w^2))\,g_2\,Z_\mu}{6\,c_w}$$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] $\mapsto \langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$$\to \frac{1}{6\,c_w} 1_3\,(\langle J_M.d_L \mid \gamma^\mu.\bar{d}_R\rangle \otimes \langle J_F.d_L \mid \bar{d}_L\rangle - 2\,\langle J_M.d_R \mid \gamma^\mu.\bar{d}_L\rangle \otimes \langle J_F.d_R \mid \bar{d}_R\rangle +$$
$$2\,\langle J_M.d_L \mid \gamma^\mu.\bar{d}_R\rangle \otimes \langle J_F.d_L \mid \bar{d}_L\rangle\,c_w^2 + 2\,\langle J_M.d_R \mid \gamma^\mu.\bar{d}_L\rangle \otimes \langle J_F.d_R \mid \bar{d}_R\rangle\,c_w^2 -$$
$$2\,\langle J_M.\bar{d}_L \mid \gamma^\mu.d_L\rangle \otimes \langle J_F.\bar{d}_L \mid d_L\rangle\,(-1 + c_w^2) - \langle J_M.\bar{d}_R \mid \gamma^\mu.d_R\rangle \otimes \langle J_F.\bar{d}_R \mid d_R\rangle\,(1 + 2\,c_w^2))\,g_2\,Z_\mu$$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] $\mapsto$ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$$\to -\frac{1}{6\,c_w} 1_3\,g_2\,(-2\,\langle J_M.\bar{d}_L \mid \gamma^\mu.d_L\rangle\,s_w^2 - 2\,\langle J_M.\bar{d}_L \mid \gamma^\mu.d_R\rangle\,s_w^2 +$$
$$(\langle J_M.\bar{d}_R \mid \gamma^\mu.d_L\rangle + \langle J_M.\bar{d}_R \mid \gamma^\mu.d_R\rangle)\,(1 + 2\,(1 - s_w^2)))\,Z_\mu$$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle \mapsto 0$ /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$$\to -\frac{\langle J_M.\bar{d}_R \mid \gamma^\mu.d_R\rangle\,1_3\,g_2\,Z_\mu}{2\,c_w} + \frac{\langle J_M.\bar{d}_L \mid \gamma^\mu.d_L\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w} + \frac{\langle J_M.\bar{d}_R \mid \gamma^\mu.d_R\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w}$$

Combine chiral bases:

$\{\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle\,(c\_ : 1) + \langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle\,(c\_ : 1) \to c\,\langle J_M.a \mid \gamma^\mu.b\rangle\}$

$$\to \boxed{-\frac{\langle J_M.\bar{d}_R \mid \gamma^\mu.d_R\rangle\,1_3\,g_2\,Z_\mu}{2\,c_w} + \frac{\langle J_M.\bar{d} \mid \gamma^\mu.d\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w}}$$

Add chiral projection operators if possible

$\{(c\_).a\_{}_R \mapsto c.P_R.T[a, , \{\}]$ /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle \to \langle J_M.a \mid b.P_R.c\rangle\}$

$$\to \boxed{-\frac{\langle J_M.\bar{d} \mid \gamma^\mu.P_R.d\rangle\,1_3\,g_2\,Z_\mu}{2\,c_w} + \frac{\langle J_M.\bar{d} \mid \gamma^\mu.d\rangle\,1_3\,g_2\,s_w^2\,Z_\mu}{3\,c_w}}$$

$$\to \boxed{\left\langle J_M.\bar{d} \;\middle|\; \left(-\frac{1_3\,g_2\,Z_\mu\,(3\,\gamma^\mu.P_R - 2\,s_w^2\,\gamma^\mu)}{6\,c_w}\right).d \right\rangle}$$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {W, e}

$$\to \frac{(e_R.W_\mu.\nu_R + \nu_R.(W_\mu)^\dagger.e_R - \bar{e}_L.(W_\mu)^*.\bar{\nu}_L - \bar{\nu}_L.(W_\mu)^{\dagger *}.\bar{e}_L)\, g_2}{\sqrt{2}}$$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$$\to \frac{1}{\sqrt{2}}(-(\langle J_M.e_L \mid \gamma^\mu.\bar{\nu}_R\rangle \otimes \langle J_F.e_L \mid (W_\mu)^*.\bar{\nu}_L\rangle) - \langle J_M.\nu_L \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.\nu_L \mid (W_\mu)^{\dagger *}.e_L\rangle +$$
$$\langle J_M.e_R \mid \gamma^\mu.\nu_R\rangle \otimes \langle J_F.e_R \mid W_\mu.\nu_R\rangle + \langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.\bar{\nu}_R \mid (W_\mu)^\dagger.e_R\rangle)\, g_2$$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] :→ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$$\to \frac{1}{\sqrt{2}} g_2 (\langle J_M.e_R \mid \gamma^\mu.\nu_L\rangle \langle \bar{\nu}_L \mid e_L\rangle (W_\mu)^{\dagger *} + \langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_L\rangle \langle e_L \mid \bar{\nu}_L\rangle (W_\mu)^* +$$
$$\langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle (W_\mu)^\dagger + \langle J_M.e_R \mid \gamma^\mu.\nu_R\rangle W_\mu)$$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$$\to \frac{\langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle (W_\mu)^\dagger g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{e}_R \mid \gamma^\mu.\nu_R\rangle g_2 W_\mu}{\sqrt{2}}$$

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

$$\to \boxed{\frac{\langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle (W_\mu)^\dagger g_2}{\sqrt{2}} + \frac{\langle J_M.e_R \mid \gamma^\mu.\nu_R\rangle g_2 W_\mu}{\sqrt{2}}}$$

Add chiral projection operators if possible

{(c_).a_$_R$:→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle$ → $\langle J_M.a \mid b.P_R.c\rangle$}

$$\to \boxed{\frac{\langle J_M.\bar{\nu} \mid \gamma^\mu.P_R.e\rangle (W_\mu)^\dagger g_2}{\sqrt{2}} + \frac{\langle J_M.e \mid \gamma^\mu.P_R.\nu\rangle g_2 W_\mu}{\sqrt{2}}}$$

$$\to \boxed{\frac{g_2 (\langle J_M.\bar{\nu} \mid \gamma^\mu.P_R.e\rangle (W_\mu)^\dagger + \langle J_M.e \mid \gamma^\mu.P_R.\nu\rangle W_\mu)}{\sqrt{2}}}$$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {W, ν}

$$\to \frac{(e_R.W_\mu.\nu_R + \nu_R.(W_\mu)^\dagger.e_R - \bar{e}_L.(W_\mu)^*.\bar{\nu}_L - \bar{\nu}_L.(W_\mu)^{\dagger *}.\bar{e}_L)\, g_2}{\sqrt{2}}$$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$$\to \frac{1}{\sqrt{2}}(-(\langle J_M.e_L \mid \gamma^\mu.\bar{\nu}_R\rangle \otimes \langle J_F.e_L \mid (W_\mu)^*.\bar{\nu}_L\rangle) - \langle J_M.\nu_L \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.\nu_L \mid (W_\mu)^{\dagger *}.e_L\rangle +$$
$$\langle J_M.e_R \mid \gamma^\mu.\nu_R\rangle \otimes \langle J_F.e_R \mid W_\mu.\nu_R\rangle + \langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle \otimes \langle J_F.\bar{\nu}_R \mid (W_\mu)^\dagger.e_R\rangle)\, g_2$$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] :→ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$$\to \frac{1}{\sqrt{2}} g_2\,(\langle J_M.e_R \mid \gamma^\mu.\nu_L\rangle \langle \bar{\nu}_L \mid e_L\rangle (W_\mu)^{\dagger *} + \langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_L\rangle \langle e_L \mid \bar{\nu}_L\rangle (W_\mu)^* +$$
$$\langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle (W_\mu)^\dagger + \langle J_M.e_R \mid \gamma^\mu.\nu_R\rangle W_\mu)$$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$$\to \frac{\langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle (W_\mu)^\dagger g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{e}_R \mid \gamma^\mu.\nu_R\rangle g_2 W_\mu}{\sqrt{2}}$$

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

$$\to \boxed{\frac{\langle J_M.\bar{\nu}_R \mid \gamma^\mu.e_R\rangle (W_\mu)^\dagger g_2}{\sqrt{2}} + \frac{\langle J_M.e_R \mid \gamma^\mu.\nu_R\rangle g_2 W_\mu}{\sqrt{2}}}$$

Add chiral projection operators if possible

{(c_).a_$_R$:→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle$ → $\langle J_M.a \mid b.P_R.c\rangle$}

$$\to \boxed{\frac{\langle J_M.\bar{\nu} \mid \gamma^\mu.P_R.e\rangle (W_\mu)^\dagger g_2}{\sqrt{2}} + \frac{\langle J_M.e \mid \gamma^\mu.P_R.\nu\rangle g_2 W_\mu}{\sqrt{2}}}$$

$$\to \boxed{\frac{g_2\,(\langle J_M.\bar{\nu} \mid \gamma^\mu.P_R.e\rangle (W_\mu)^\dagger + \langle J_M.e \mid \gamma^\mu.P_R.\nu\rangle W_\mu)}{\sqrt{2}}}$$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {W, u}

$$\rightarrow \frac{(d_R.W_\mu.u_R + u_R.(W_\mu)^\dagger.d_R - \bar{d}_L.(W_\mu)^*.\bar{u}_L - \bar{u}_L.(W_\mu)^{\dagger *}.\bar{d}_L)\, 1_3\, g_2}{\sqrt{2}}$$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ $\langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$$\rightarrow \frac{1}{\sqrt{2}}(-(\langle J_M.d_L \mid \gamma^\mu.u_R\rangle \otimes \langle J_F.d_L \mid (W_\mu)^*.\bar{u}_L\rangle) - \langle J_M.u_L \mid \gamma^\mu.\bar{d}_R\rangle \otimes \langle J_F.u_L \mid (W_\mu)^{\dagger *}.\bar{d}_L\rangle + \langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle \otimes \langle J_F.\bar{d}_R \mid W_\mu.u_R\rangle + \langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle \otimes \langle J_F.\bar{u}_R \mid (W_\mu)^\dagger.d_R\rangle)\, 1_3\, g_2$$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] :→ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

→

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$$\rightarrow \frac{1}{\sqrt{2}} 1_3\, g_2\, (\langle J_M.\bar{d}_R \mid \gamma^\mu.u_L\rangle \langle u_L \mid \bar{d}_L\rangle (W_\mu)^{\dagger *} + \langle J_M.u_R \mid \gamma^\mu.d_L\rangle \langle \bar{d}_L \mid u_L\rangle (W_\mu)^* + \langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle (W_\mu)^\dagger + \langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle W_\mu)$$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle$:→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$$\rightarrow \frac{\langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle (W_\mu)^\dagger\, 1_3\, g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle\, 1_3\, g_2\, W_\mu}{\sqrt{2}}$$

Combine chiral bases:

{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) → c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

$$\rightarrow \boxed{\frac{\langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle (W_\mu)^\dagger\, 1_3\, g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle\, 1_3\, g_2\, W_\mu}{\sqrt{2}}}$$

Add chiral projection operators if possible

{(c_).a_${}_R$:→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle$ → $\langle J_M.a \mid b.P_R.c\rangle$}

$$\rightarrow \boxed{\frac{\langle J_M.\bar{u} \mid \gamma^\mu.P_R.d\rangle (W_\mu)^\dagger\, 1_3\, g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{d} \mid \gamma^\mu.P_R.u\rangle\, 1_3\, g_2\, W_\mu}{\sqrt{2}}}$$

$$\rightarrow \boxed{\frac{1_3\, g_2\, (\langle J_M.\bar{u} \mid \gamma^\mu.P_R.d\rangle (W_\mu)^\dagger + \langle J_M.\bar{d} \mid \gamma^\mu.P_R.u\rangle W_\mu)}{\sqrt{2}}}$$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {W, d}

$$\to \frac{(d_R.W_\mu.u_R + u_R.(W_\mu)^\dagger.d_R - \bar{d}_L.(W_\mu)^*.\bar{u}_L - \bar{u}_L.(W_\mu)^{\dagger *}.\bar{d}_L)\, 1_3\, g_2}{\sqrt{2}}$$

Add back J_F to apply basis correspondence:

```
HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ ⟨J_M.J_F.a | γ^μ.b⟩⊗⟨J_F.J_F.a | c.b⟩
```

$$\to \frac{1}{\sqrt{2}}(-(\langle J_M.d_L \mid \gamma^\mu.u_R\rangle\otimes\langle J_F.d_L \mid (W_\mu)^*.\bar{u}_L\rangle) - \langle J_M.u_L \mid \gamma^\mu.\bar{d}_R\rangle\otimes\langle J_F.u_L \mid (W_\mu)^{\dagger *}.\bar{d}_L\rangle + \langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle\otimes\langle J_F.\bar{d}_R \mid W_\mu.u_R\rangle + \langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle\otimes\langle J_F.\bar{u}_R \mid (W_\mu)^\dagger.d_R\rangle)\, 1_3\, g_2$$

Order J_M terms(anti-symmetric):

```
HoldPattern[⟨J_M.(a_) | (c_).Shortest[b_]⟩] :→ -⟨J_M.b | c.a⟩ /; OrderedQ[{a, b}]
```

→

Simplify F-space with Dot[] and ⊗ Scalar:

```
{Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]^*, Tensor[W, _, _]^†, Tensor[W, _, _]^†*, Tensor[G, _, _], Tensor[λ, _, _]}
```

$$\to \frac{1}{\sqrt{2}} 1_3\, g_2\, (\langle J_M.\bar{d}_R \mid \gamma^\mu.u_L\rangle \langle u_L \mid \bar{d}_L\rangle (W_\mu)^{\dagger *} + \langle J_M.\bar{u}_R \mid \gamma^\mu.d_L\rangle \langle \bar{d}_L \mid u_L\rangle (W_\mu)^* + \langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle (W_\mu)^\dagger + \langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle W_\mu)$$

Impose chiral orthogonality

```
⟨J_M.(a_) | γ^μ.(b_)⟩ :→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])
```

$$\to \frac{\langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle (W_\mu)^\dagger\, 1_3\, g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle\, 1_3\, g_2\, W_\mu}{\sqrt{2}}$$

Combine chiral bases:

```
{⟨J_M.a_L | γ^μ.b_L⟩ (c_ : 1) + ⟨J_M.a_R | γ^μ.b_R⟩ (c_ : 1) → c ⟨J_M.a | γ^μ.b ⟩}
```

$$\to \boxed{\frac{\langle J_M.\bar{u}_R \mid \gamma^\mu.d_R\rangle (W_\mu)^\dagger\, 1_3\, g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{d}_R \mid \gamma^\mu.u_R\rangle\, 1_3\, g_2\, W_\mu}{\sqrt{2}}}$$

Add chiral projection operators if possible

```
{(c_).a_R :→ c.P_R.T[a, , {}] /; FreeQ[c, P_R], ⟨J_M.P_R.(a_) | (b_).P_R.(c_)⟩ → ⟨J_M.a | b.P_R.c⟩}
```

$$\to \boxed{\frac{\langle J_M.\bar{u} \mid \gamma^\mu.P_R.d\rangle (W_\mu)^\dagger\, 1_3\, g_2}{\sqrt{2}} + \frac{\langle J_M.\bar{d} \mid \gamma^\mu.P_R.u\rangle\, 1_3\, g_2\, W_\mu}{\sqrt{2}}}$$

$$\to \boxed{\frac{1_3\, g_2\, (\langle J_M.\bar{u} \mid \gamma^\mu.P_R.d\rangle (W_\mu)^\dagger + \langle J_M.\bar{d} \mid \gamma^\mu.P_R.u\rangle W_\mu)}{\sqrt{2}}}$$

```
Evaluate terms in ℒgf for the {field,basis}: {G, e}
→ 0
Add back JF to apply basis correspondence:
 HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ ⟨JM.JF.a | γ^μ.b⟩⊗⟨JF.JF.a | c.b⟩
→ 0
Order JM terms(anti-symmetric):
 HoldPattern[⟨JM.(a_) | (c_).Shortest[b_]⟩] :→ -⟨JM.b | c.a⟩ /; OrderedQ[{a, b}]
→
Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _],
  Tensor[W, _, _]^*, Tensor[W, _, _]^†, Tensor[W, _, _]^†*, Tensor[G, _, _], Tensor[λ, _, _]}
→ 0
Impose chiral orthogonality
⟨JM.(a_) | γ^μ.(b_)⟩ :→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])
→ 0
Combine chiral bases:
{⟨JM.a_L | γ^μ.b_L⟩ (c_ : 1) + ⟨JM.a_R | γ^μ.b_R⟩ (c_ : 1) → c ⟨JM.a  | γ^μ.b ⟩}
→ [0]
Add chiral projection operators if possible
 {(c_).a_R :→ c.PR.T[a, , {}] /; FreeQ[c, PR], ⟨JM.PR.(a_) | (b_).PR.(c_)⟩ → ⟨JM.a | b.PR.c⟩}
→ [0]
→ [0]
```

```
Evaluate terms in ℒgf for the {field,basis}: {G, ν}
→ 0
Add back JF to apply basis correspondence:
 HoldPattern[Shortest[a_].(c___).Shortest[b_]] :→ ⟨JM.JF.a | γ^μ.b⟩⊗⟨JF.JF.a | c.b⟩
→ 0
Order JM terms(anti-symmetric):
 HoldPattern[⟨JM.(a_) | (c_).Shortest[b_]⟩] :→ -⟨JM.b | c.a⟩ /; OrderedQ[{a, b}]
→
Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _],
  Tensor[W, _, _]^*, Tensor[W, _, _]^†, Tensor[W, _, _]^†*, Tensor[G, _, _], Tensor[λ, _, _]}
→ 0
Impose chiral orthogonality
⟨JM.(a_) | γ^μ.(b_)⟩ :→ 0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])
→ 0
Combine chiral bases:
{⟨JM.a_L | γ^μ.b_L⟩ (c_ : 1) + ⟨JM.a_R | γ^μ.b_R⟩ (c_ : 1) → c ⟨JM.a  | γ^μ.b ⟩}
→ [0]
Add chiral projection operators if possible
 {(c_).a_R :→ c.PR.T[a, , {}] /; FreeQ[c, PR], ⟨JM.PR.(a_) | (b_).PR.(c_)⟩ → ⟨JM.a | b.PR.c⟩}
→ [0]
→ [0]
```

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {G, u}

$\to \frac{1}{2}(u_L.(G_\mu{}^i\,\lambda_i).u_L + u_R.(G_\mu{}^i\,\lambda_i).u_R - \bar{u}_L.((G_\mu{}^i)^*\,(\lambda_i)^*).\bar{u}_L - \bar{u}_R.((G_\mu{}^i)^*\,(\lambda_i)^*).\bar{u}_R)\,g_3$

Add back J_F to apply basis correspondence:
HoldPattern[Shortest[a_].(c___).Shortest[b_]] $\mapsto \langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$\to \frac{1}{2}(-(\langle J_M.u_L \mid \gamma^\mu.\bar{u}_R\rangle \otimes \langle J_F.u_L \mid ((G_\mu{}^i)^*\,(\lambda_i)^*).\bar{u}_L\rangle) - \langle J_M.u_R \mid \gamma^\mu.\bar{u}_L\rangle \otimes \langle J_F.u_R \mid ((G_\mu{}^i)^*\,(\lambda_i)^*).\bar{u}_R\rangle + \langle J_M.\bar{u}_L \mid \gamma^\mu.u_L\rangle \otimes \langle J_F.\bar{u}_L \mid (G_\mu{}^i\,\lambda_i).u_L\rangle + \langle J_M.\bar{u}_R \mid \gamma^\mu.u_R\rangle \otimes \langle J_F.\bar{u}_R \mid (G_\mu{}^i\,\lambda_i).u_R\rangle)\,g_3$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] $\mapsto$ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

$\to$

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$\to \frac{1}{2} g_3\,(\langle J_M.\bar{u}_R \mid \gamma^\mu.u_L\rangle\,\langle \bar{u}_L \mid ((G_\mu{}^i)^*\,(\lambda_i)^*).\bar{u}_L\rangle + \langle J_M.\bar{u}_L \mid \gamma^\mu.u_R\rangle\,\langle \bar{u}_R \mid ((G_\mu{}^i)^*\,(\lambda_i)^*).\bar{u}_R\rangle + (\langle J_M.\bar{u}_L \mid \gamma^\mu.u_L\rangle + \langle J_M.\bar{u}_R \mid \gamma^\mu.u_R\rangle)\,G_\mu{}^i\,\lambda_i)$

Impose chiral orthogonality
$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle \mapsto 0$ /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$\to \frac{1}{2}\langle J_M.\bar{u}_L \mid \gamma^\mu.u_L\rangle\, g_3\, G_\mu{}^i\,\lambda_i + \frac{1}{2}\langle J_M.\bar{u}_R \mid \gamma^\mu.u_R\rangle\, g_3\, G_\mu{}^i\,\lambda_i$

Combine chiral bases:
{$\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle$ (c_ : 1) + $\langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle$ (c_ : 1) $\to$ c $\langle J_M.a \mid \gamma^\mu.b\rangle$}

$\to \boxed{\frac{1}{2}\langle J_M.\bar{u} \mid \gamma^\mu.u\rangle\, g_3\, G_\mu{}^i\,\lambda_i}$

Add chiral projection operators if possible
{(c_).a_$_R$ $\mapsto$ c.P_R.T[a, , {}] /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle \to \langle J_M.a \mid b.P_R.c\rangle$}

$\to \boxed{\frac{1}{2}\langle J_M.\bar{u} \mid \gamma^\mu.u\rangle\, g_3\, G_\mu{}^i\,\lambda_i}$

$\to \boxed{\frac{1}{2}\langle J_M.\bar{u} \mid \gamma^\mu.u\rangle\, g_3\, G_\mu{}^i\,\lambda_i}$

Evaluate terms in $\mathcal{L}_{gf}$ for the {field,basis}: {G, d}

$\rightarrow \frac{1}{2}(d_L.(G_\mu{}^i \lambda_i).d_L + d_R.(G_\mu{}^i \lambda_i).d_R - \overline{d}_L.((G_\mu{}^i)^* (\lambda_i)^*).\overline{d}_L - \overline{d}_R.((G_\mu{}^i)^* (\lambda_i)^*).\overline{d}_R)\, g_3$

Add back J_F to apply basis correspondence:

HoldPattern[Shortest[a_].(c___).Shortest[b_]] $\mapsto \langle J_M.J_F.a \mid \gamma^\mu.b\rangle \otimes \langle J_F.J_F.a \mid c.b\rangle$

$\rightarrow \frac{1}{2}(-(\langle J_M.d_L \mid \gamma^\mu.\overline{d}_R\rangle \otimes \langle J_F.d_L \mid ((G_\mu{}^i)^* (\lambda_i)^*).\overline{d}_L\rangle) - \langle J_M.d_R \mid \gamma^\mu.\overline{d}_L\rangle \otimes \langle J_F.d_R \mid ((G_\mu{}^i)^* (\lambda_i)^*).\overline{d}_R\rangle +$
$\langle J_M.\overline{d}_L \mid \gamma^\mu.d_L\rangle \otimes \langle J_F.\overline{d}_L \mid (G_\mu{}^i \lambda_i).d_L\rangle + \langle J_M.\overline{d}_R \mid \gamma^\mu.d_R\rangle \otimes \langle J_F.\overline{d}_R \mid (G_\mu{}^i \lambda_i).d_R\rangle)\, g_3$

Order J_M terms(anti-symmetric): HoldPattern[$\langle J_M.(a\_) \mid (c\_).\text{Shortest}[b\_]\rangle$] $\mapsto$ $-\langle J_M.b \mid c.a\rangle$ /; OrderedQ[{a, b}]

$\rightarrow$

Simplify F-space with Dot[] and ⊗ Scalar: {Tensor[A, _, _], Tensor[W, _, _], Tensor[W, _, _]*, Tensor[W, _, _]†, Tensor[W, _, _]$^{\dagger *}$, Tensor[G, _, _], Tensor[λ, _, _]}

$\rightarrow \frac{1}{2} g_3 (\langle J_M.\overline{d}_R \mid \gamma^\mu.d_L\rangle \langle \overline{d}_L \mid ((G_\mu{}^i)^* (\lambda_i)^*).\overline{d}_L\rangle +$
$\langle J_M.\overline{d}_L \mid \gamma^\mu.d_R\rangle \langle \overline{d}_R \mid ((G_\mu{}^i)^* (\lambda_i)^*).\overline{d}_R\rangle + (\langle J_M.\overline{d}_L \mid \gamma^\mu.d_L\rangle + \langle J_M.\overline{d}_R \mid \gamma^\mu.d_R\rangle)\, G_\mu{}^i \lambda_i)$

Impose chiral orthogonality

$\langle J_M.(a\_) \mid \gamma^\mu.(b\_)\rangle \mapsto 0$ /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L])

$\rightarrow \frac{1}{2}\langle J_M.\overline{d}_L \mid \gamma^\mu.d_L\rangle g_3 G_\mu{}^i \lambda_i + \frac{1}{2}\langle J_M.\overline{d}_R \mid \gamma^\mu.d_R\rangle g_3 G_\mu{}^i \lambda_i$

Combine chiral bases:

$\{\langle J_M.a\_{}_L \mid \gamma^\mu.b\_{}_L\rangle (c\_ : 1) + \langle J_M.a\_{}_R \mid \gamma^\mu.b\_{}_R\rangle (c\_ : 1) \rightarrow c \langle J_M.a \mid \gamma^\mu.b\rangle\}$

$\rightarrow \boxed{\frac{1}{2}\langle J_M.\overline{d} \mid \gamma^\mu.d\rangle g_3 G_\mu{}^i \lambda_i}$

Add chiral projection operators if possible

$\{(c\_).a\_{}_R \mapsto c.P_R.T[a, , \{\}]$ /; FreeQ[c, P_R], $\langle J_M.P_R.(a\_) \mid (b\_).P_R.(c\_)\rangle \rightarrow \langle J_M.a \mid b.P_R.c\rangle\}$

$\rightarrow \boxed{\frac{1}{2}\langle J_M.\overline{d} \mid \gamma^\mu.d\rangle g_3 G_\mu{}^i \lambda_i}$

$\rightarrow \boxed{\frac{1}{2}\langle J_M.\overline{d} \mid \gamma^\mu.d\rangle g_3 G_\mu{}^i \lambda_i}$

```
PR["Sum terms: ",
 Yield, $ = ℒgf → ($terms // Values // Apply[Plus, #] &); $ // ColumnSumExp
]
```

Sum terms:

$$\to \mathcal{L}_{gf} \to \sum[\ \begin{array}{l}
\langle J_M.\bar{d} \mid (-\frac{1_3\, g_2\, Z_\mu\, (3\,\gamma^\mu.P_R - 2\, s_w^2\, \gamma^\mu)}{6\, c_w}).d \rangle \\
\langle J_M.\bar{e} \mid (-\frac{g_2\, Z_\mu\, (\gamma^\mu.P_R - 2\, s_w^2\, \gamma^\mu)}{2\, c_w}).e \rangle \\
\langle J_M.\bar{u} \mid \frac{1_3\, g_2\, Z_\mu\, (3\,\gamma^\mu.P_R - 4\, s_w^2\, \gamma^\mu)}{6\, c_w}.u \rangle \\
-\langle J_M.\bar{e} \mid \gamma^\mu.e \rangle\, g_2\, s_w\, A_\mu \\
-\frac{1}{3}\langle J_M.\bar{d} \mid \gamma^\mu.d \rangle\, 1_3\, g_2\, s_w\, A_\mu \\
\frac{2}{3}\langle J_M.\bar{u} \mid \gamma^\mu.u \rangle\, 1_3\, g_2\, s_w\, A_\mu \\
\sqrt{2}\ 1_3\, g_2\, (\langle J_M.\bar{u} \mid \gamma^\mu.P_R.d \rangle\, (W_\mu)^\dagger + \langle J_M.\bar{d} \mid \gamma^\mu.P_R.u \rangle\, W_\mu) \\
\sqrt{2}\ g_2\, (\langle J_M.\bar{\nu} \mid \gamma^\mu.P_R.e \rangle\, (W_\mu)^\dagger + \langle J_M.\bar{e} \mid \gamma^\mu.P_R.\nu \rangle\, W_\mu) \\
\frac{\langle J_M.\bar{\nu} \mid \gamma^\mu.P_R.\nu \rangle\, g_2\, Z_\mu}{2\, c_w} \\
\frac{1}{2}\langle J_M.\bar{d} \mid \gamma^\mu.d \rangle\, g_3\, G_\mu{}^i\, \lambda_i \\
\frac{1}{2}\langle J_M.\bar{u} \mid \gamma^\mu.u \rangle\, g_3\, G_\mu{}^i\, \lambda_i
\end{array}\ ]$$

```
PR["●Show: ", $ = ℒHf;
 $ = tuRuleSelect[$t610][$[_]] // First;
 $ // ColumnSumExp,
 line,
 " i.e., terms containing ", $sΦ = Φ, " in ",
 Yield, $ = $00a,
 next, "Extract ", $sΦ, " terms:",
 Yield, $[[2]] = $[[2]] // tuTermExtract[$sΦ];
 $ = $ /. SF → ℒHf,
 next, "Explicit ",
 $s = selectStdMdl[ξ̃, {M}] /. δ[_] → 1,
 $ = $ /. $s // tuCircleTimesExpand;
 next, "Separate M-,F-spaces ",
 $s = {BraKet[a_ ⊗ b_, c_ ⊗ d_] → BraKet[a, c] ⊗ BraKet[b, d]},
 Yield,
 $pass = $ = $ /. BraKet[a_ ⊗ b_, c_ ⊗ d_] → BraKet[a, c] ⊗ BraKet[b, d],
 NL,
 CO["Recall: the product basis is not a generalized product space. There is a 1-to-1
    correspondence between the M- and F-spaces which needs special handling."]
]
```

●Show: $\mathcal{L}_{Hf}$[Yukawa coupling of Higgs-fermion field] →

$$\sum\Big[\begin{array}{l}
i\,(1+\frac{h}{v})\,(\langle J_M.\bar{d}\mid m_d.d\rangle+\langle J_M.\bar{e}\mid m_e.e\rangle+\langle J_M.\bar{u}\mid m_u.u\rangle+\langle J_M.\bar{\nu}\mid m_\nu.\nu\rangle)\\
\frac{(\langle J_M.\bar{d}|m_d.(1+\gamma_5).u\rangle-\langle J_M.\bar{d}|m_u.(1-\gamma_5).u\rangle)\,\phi^-}{\sqrt{2}\,v}\\
\frac{(\langle J_M.\bar{e}|m_e.(1+\gamma_5).\nu\rangle-\langle J_M.\bar{e}|m_\nu.(1-\gamma_5).\nu\rangle)\,\phi^-}{\sqrt{2}\,v}\\
\frac{(-\langle J_M.\bar{u}|m_d.(1-\gamma_5).d\rangle+\langle J_M.\bar{u}|m_u.(1+\gamma_5).d\rangle)\,\phi^+}{\sqrt{2}\,v}\\
\frac{(-\langle J_M.\bar{\nu}|m_e.(1-\gamma_5).e\rangle+\langle J_M.\bar{\nu}|m_\nu.(1+\gamma_5).e\rangle)\,\phi^+}{\sqrt{2}\,v}\\
\frac{(-\langle J_M.\bar{d}|\gamma_5.m_d.d\rangle-\langle J_M.\bar{e}|\gamma_5.m_e.e\rangle+\langle J_M.\bar{u}|\gamma_5.m_u.u\rangle+\langle J_M.\bar{\nu}|\gamma_5.m_\nu.\nu\rangle)\,\phi^0}{v}
\end{array}\Big]$$

---

i.e., terms containing Φ in

$$\rightarrow S_F \rightarrow \frac{1}{2}\langle (J_M\otimes J_F).\tilde{\xi}\mid ((\not{D})\otimes 1_F).\tilde{\xi}\rangle+\frac{1}{2}\langle (J_M\otimes J_F).\tilde{\xi}\mid (\gamma_5\otimes\Phi).\tilde{\xi}\rangle+\frac{1}{2}\langle (J_M\otimes J_F).\tilde{\xi}\mid (\gamma^\mu\otimes B_\mu).\tilde{\xi}\rangle$$

◆Extract Φ terms:

$$\rightarrow \mathcal{L}_{Hf} \rightarrow \frac{1}{2}\langle (J_M\otimes J_F).\tilde{\xi}\mid (\gamma_5\otimes\Phi).\tilde{\xi}\rangle$$

◆Explicit $\tilde{\xi}\rightarrow\tilde{\xi}_M\otimes\tilde{\xi}_F$

◆Separate M-,F-spaces $\{\langle a\_\otimes b\_\mid c\_\otimes d\_\rangle\rightarrow\langle a\mid c\rangle\otimes\langle b\mid d\rangle\}$

$$\rightarrow \mathcal{L}_{Hf} \rightarrow \frac{1}{2}\langle J_M.\tilde{\xi}_M\mid \gamma_5.\tilde{\xi}_M\rangle\otimes\langle J_F.\tilde{\xi}_F\mid \Phi.\tilde{\xi}_F\rangle$$

Recall: the product basis is not a generalized product space. There is a 1-to-1 correspondence between the M- and F-spaces which needs special handling.

```
PR["● Expand the F-space part: ",
 Yield, $ = $pass,
 Yield, $ = $ /. $sv //. jj : JF. _ :→ Thread[jj] /. selectStdMdl[JF._] /.
   a_ ⊗ BraKet[b_, c_] :→ a ⊗ BraKet[Transpose[b], c];
 $ = $ /. toxDot /. selectStdMdl[Φ] // tuMatrixOrderedMultiply // tuOpSimplifyF[xDot];
 $ = $ /. toDot // expandDC[];
 $ = $ /. a_ ⊗ b_ :→ a ⊗ (b /. BraKet[c_, d_] :> xDot[c, d]
          // tuMatrixOrderedMultiply // tuOpSimplifyF[xDot]) /. toDot // expandDC[];
 $ = $ /. a_ ⊗ b_ → a ⊗ Flatten[b];
 NL,
 "• BraKet to Dot notation require ConjugateTranspose(charge conjugation) of the
   first term: ",
 $0 = $ = $ /. (a_ ⊗ b_ :→ a ⊗ (b /. HoldPattern[Shortest[a1_].b1_ .Shortest[c1_]] :→
          (JF.a1 /. selectStdMdl[JF._]) . b1.c1)
    );
 $ = $ // tuConjugateSimplify[];
 NL, "Scalars ", $scalar = {ϕ1, ϕ2, cc[ϕ1], cc[ϕ2], Y_, ct[Y_], cc[Y_], cc[ct[Y_]]},
 Yield, $ = $ //. tuOpSimplify[Dot, $scalar]; $ // ColumnSumExp;
 NL, "Substitute: ",
 $s2 = $s = {selectStdMdl /@ {YR, Yx},
      tuRuleSolve[selectStdMdl[ϕ1 + _], ϕ1],
      selectStdMdl[ϕ2], cc[ϕ⁻] → ϕ⁺} // Flatten // tuAddPatternVariable[{x}];
 $s // ColumnBar, CK,
 NL, "With Reals, Scalars, Hermitian variables: ",
 {$real = { h, √_ , a, f[0], v, T[ϕ, "u", {0}] },
   $scalar = {h, √_ , a, f[0], v, T[ϕ, "u", {0}]}, $hermit = {m_}} // ColumnBar,
 Yield, $pass1 = $ = $ //. tuRule[$s2] // tuConjugateTransposeSimplify[
        $real, $scalar, $hermit] // ExpandAll // (# /. tuRule[$s2] &);
 $ // ColumnSumExp
]
```

● Expand the F-space part:

$\rightarrow \mathcal{L}_{Hf} \rightarrow \frac{1}{2} \langle J_M.\tilde{\xi}_M \mid \gamma_5.\tilde{\xi}_M \rangle \otimes \langle J_F.\tilde{\xi}_F \mid \Phi.\tilde{\xi}_F \rangle$

$\rightarrow$

• BraKet to Dot notation require ConjugateTranspose(charge conjugation) of the first term:

Scalars $\{\phi_1, \phi_2, (\phi_1)^*, (\phi_2)^*, Y\_, (Y\_)^\dagger, (Y\_)^*, (Y\_)^{\dagger *}\}$

$\rightarrow$

Substitute:

$$Y_R \rightarrow -i\, m_R$$
$$Y_{x\_} \rightarrow -\frac{i \sqrt{a\, f[0]}\, m_x}{\pi\, v}$$
$$\phi_1 \rightarrow \frac{h\,\pi + \pi\, v - \sqrt{a\, f[0]}}{\sqrt{a\, f[0]}} + \frac{i\,\pi\,\phi^0}{\sqrt{a\, f[0]}} \longleftarrow \text{CHECK}$$
$$\phi_2 \rightarrow \frac{i\sqrt{2}\,\pi\,\phi^-}{\sqrt{a\, f[0]}}$$
$$(\phi^-)^* \rightarrow \phi^+$$

With Reals, Scalars, Hermitian variables:

$\{h, \sqrt{\_}, a, f[0], v, \phi^0\}$
$\{h, \sqrt{\_}, a, f[0], v, \phi^0\}$
$\{m\_\}$

$$i\,(m_d)^*\, d_L . d_R$$
$$\frac{i\,h\,(m_d)^*\, d_L . d_R}{v}$$
$$i\,(m_e)^*\, e_L . e_R$$
$$\frac{i\,h\,(m_e)^*\, e_L . e_R}{v}$$
$$i\,(m_u)^*\, u_L . u_R$$
$$\frac{i\,h\,(m_u)^*\, u_L . u_R}{v}$$
$$i\,(m_\nu)^*\, \nu_L . \nu_R$$
$$\frac{i\,h\,(m_\nu)^*\, \nu_L . \nu_R}{v}$$
$$i\,(m_R)^*\, \nu_L . \bar{\nu}_R$$
$$i\,(m_d)^*\, \bar{d}_L . \bar{d}_R$$
$$\frac{i\,h\,(m_d)^*\, \bar{d}_L . \bar{d}_R}{v}$$
$$i\,(m_e)^*\, \bar{e}_L . \bar{e}_R$$
$$\frac{i\,h\,(m_e)^*\, \bar{e}_L . \bar{e}_R}{v}$$
$$i\,(m_u)^*\, \bar{u}_L . \bar{u}_R$$
$$\frac{i\,h\,(m_u)^*\, \bar{u}_L . \bar{u}_R}{v}$$
$$i\,(m_\nu)^*\, \bar{\nu}_L . \bar{\nu}_R$$
$$\frac{i\,h\,(m_\nu)^*\, \bar{\nu}_L . \bar{\nu}_R}{v}$$
$$-\frac{i\,(m_d)^*\, d_L . d_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$-\frac{i\,(m_e)^*\, e_L . e_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$-\frac{i\,(m_u)^*\, u_L . u_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$-\frac{i\,(m_\nu)^*\, \nu_L . \nu_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$-\frac{i\,(m_d)^*\, \bar{d}_L . \bar{d}_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$\frac{i\,(m_d)^*\, \bar{d}_R . \bar{d}_L \sqrt{a\,f[0]}}{\pi\, v}$$
$$-\frac{i\,(m_e)^*\, \bar{e}_L . \bar{e}_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$\frac{i\,(m_e)^*\, \bar{e}_R . \bar{e}_L \sqrt{a\,f[0]}}{\pi\, v}$$
$$-\frac{i\,(m_u)^*\, \bar{u}_L . \bar{u}_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$\frac{i\,(m_u)^*\, \bar{u}_R . \bar{u}_L \sqrt{a\,f[0]}}{\pi\, v}$$
$$-\frac{i\,(m_\nu)^*\, \bar{\nu}_L . \bar{\nu}_R \sqrt{a\,f[0]}}{\pi\, v}$$
$$\frac{i\,(m_\nu)^*\, \bar{\nu}_R . \bar{\nu}_L \sqrt{a\,f[0]}}{\pi\, v}$$
$$-i\, d_R . d_L\, m_d$$
$$-\frac{i\,h\, d_R . d_L\, m_d}{v}$$
$$-i\, \bar{d}_R . \bar{d}_L\, m_d$$
$$-\frac{i\,h\, \bar{d}_R . \bar{d}_L\, m_d}{v}$$
$$-\frac{i\, d_L . d_R \sqrt{a\,f[0]}\, m_d}{\pi\, v}$$
$$\frac{i\, d_R . d_L \sqrt{a\,f[0]}\, m_d}{\pi\, v}$$
$$\frac{i\, d_R . d_L \sqrt{a\,f[0]}\, m_d}{\pi\, v^*}$$
$$\frac{i\, \bar{d}_R . \bar{d}_L \sqrt{a\,f[0]}\, m_d}{\pi\, v}$$
$$-i\, e_R . e_L\, m_e$$
$$-\frac{i\,h\, e_R . e_L\, m_e}{v}$$

```
    (# //. tuBraKetSimplify[$scalar]
        /. b_ ⊗ ((c_ : 1) BraKet[a_, a1_]) → b⊗c(*remove F-space BraKet*)
       /. a_⊗aa_ :> a aa /; NumericQ[aa] || tuHasAnyQ[aa, $scalar] &) // Simplify
$termsPh = {};
PR["Do expansion and simplification of the Φ terms:"
]
(**Do over list of symbols**)
$list = {$sym = {mR}, $nosym = {},
   $sym = {a}, $nosym = {},
   $sym = {ν, mν}, $nosym = {a, ϕ},
   $sym = {e, me}, $nosym = {a, ϕ},
   $sym = {u, mu}, $nosym = {a, ϕ},
   $sym = {d, md}, $nosym = {a, ϕ},
   $sym = {ν, T[ϕ, "u", {0}]}, $nosym = {a},
   $sym = {e, T[ϕ, "u", {0}]}, $nosym = {a},
   $sym = {u, T[ϕ, "u", {0}]}, $nosym = {a},
   $sym = {d, T[ϕ, "u", {0}]}, $nosym = {a},
   $sym = {ν, e, ϕ⁻}, $nosym = {a},
   $sym = {d, u, ϕ⁻}, $nosym = {a},
   $sym = {ν, e, ϕ⁺}, $nosym = {a},
   $sym = {d, u, ϕ⁺}, $nosym = {a}
  };
Do[$sym = $list[[i$]]; $nosym = $list[[i$ + 1]];
 PR[$ = $pass1;
  NL, "Extract ", $sym, $nosym,
  (*
  $sym={ν,mν},$nosym={a,ϕ},*)
  $ = tuTermExtract[$sym, $nosym][$pass1]; $ // ColumnSumExp,
  "POFF",
  NL, "Add BraKet[JM._,γ5._] back based on JF basis correspondence: ",
  $s = HoldPattern[Shortest[a_]. c___ .Shortest[b_]] :→
    BraKet[JM.(JF.a /. selectStdMdl[JF._]), T[γ, "d", {5}].(b /. $sFM)] ⊗
     BraKet[JF.(JF.a /. selectStdMdl[JF._]), c.b],
  Yield, $ = $ /. $s, CK,

  NL, "Order JM terms(symmetric with γ5): ",
  $s = HoldPattern[BraKet[JM.a_, c_ .Shortest[b_]]] :→
    BraKet[JM.b, c.a] /; OrderedQ[{a, b}],
  Yield, $ = $ /. $s,

  NL, "Simplify F-space with Dot[] and ⊗ Scalar: ", CR[
   "We remove F-space BraKet since it is the result of off-diagonal elements of Φ and
      its correspondence in the M-space is maintained. "],
  Yield,
  $simplifyFspacePhi;

  NL, "Impose chiral orthogonality ",
  NL, $s = BraKet[JM.a_, T[γ, "d", {5}].b_] :→
    0 /; (! FreeQ[a, L] && ! FreeQ[b, R]) || (! FreeQ[a, R] && ! FreeQ[b, L]),
  Yield, $ = $ /. $s // Expand // tuCircleTimesSimplify // Simplify;
  $ // ColumnSumExp,
  NL, "Apply chiral projection operators and its commutativity with JM: ",
  $s = {T[γ, "d", {5}].T[p_, "d", {R_}] -> T[γ, "d", {5}].PR.T[p, "", {}],
    JM.T[p_, "d", {R_}] → JM.PR.T[p, "", {}],
    BraKet[JM.PR_. a_, T[γ, "d", {5}].PR_. b_] -> BraKet[JM.a, T[γ, "d", {5}].PR.b]
   }; $s // ColumnBar,
  Yield, $ = $ //. $s,
```

```
 NL, "Take as Real: ", $real = {m_},
 Yield, $ = $ // tuConjugateTransposeSimplify[$real, $real] // Simplify,
 NL, "Gather BraKet's ",
 $sgather = $s = {(cR_ : 1) BraKet[JM.a_, T[γ, "d", {5}].PR_.b_] +
       (cL_ : 1) BraKet[JM.a_, T[γ, "d", {5}].PL_.b_] ->
      BraKet[JM.a, T[γ, "d", {5}].(cR PR + cL PL).b]},
 Yield, $ = $ //. $s,
 NL, "Use ", $s = {PL → (1 + T[γ, "d", {5}]) / 2,
   PR → (1 - T[γ, "d", {5}]) / 2, selectDef[Tensor[γ, _, _]._]},
 Yield, $ = $ // expandDC[{$s, $sgather, tuBraKetSimplify[{m_}]}, {m_}] //
   Collect[#, m_] &,
 NL, "Rearrange ", $s = {(cR_ : 1) BraKet[JM.a_, g_.b_] + (cL_ : 1) BraKet[JM.a_, b_] ->
    BraKet[JM.a, (g cR + cL ).b]},
 "PONdd",
 Yield, $ = $ /. $s // Simplify;
 $termsPh = Append[$termsPh, $sym → $];
 $ // Framed
];
, {i$, 1, Length[$list], 2}]
```

Do expansion and simplification of the Φ terms:

Extract $\{m_R\}\{\}\sum[\begin{array}{l} i\,(m_R)^*\,\nu_L.\bar\nu_R \\ -i\,\bar\nu_R.\nu_L\,m_R \end{array}]$

....

→ $\boxed{i \left\langle J_M.\bar\nu \;\middle|\; (\frac{1}{2}(1+\gamma_5)).\bar\nu \right\rangle m_R}$

$$
\begin{array}{l}
-\frac{i\,(m_d)^*\,d_L.d_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,(m_e)^*\,e_L.e_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,(m_u)^*\,u_L.u_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,(m_\nu)^*\,\nu_L.\nu_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,(m_d)^*\,\bar{d}_L.\bar{d}_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
\frac{i\,(m_d)^*\,\bar{d}_R.\bar{d}_L\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,(m_e)^*\,\bar{e}_L.\bar{e}_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
\frac{i\,(m_e)^*\,\bar{e}_R.\bar{e}_L\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,(m_u)^*\,\bar{u}_L.\bar{u}_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
\frac{i\,(m_u)^*\,\bar{u}_R.\bar{u}_L\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,(m_\nu)^*\,\bar\nu_L.\bar\nu_R\,\sqrt{a\,f[0]}}{\pi\,v} \\
\frac{i\,(m_\nu)^*\,\bar\nu_R.\bar\nu_L\,\sqrt{a\,f[0]}}{\pi\,v} \\
-\frac{i\,d_L.d_R\,\sqrt{a\,f[0]}\,m_d}{\pi\,v}
\end{array}
$$

Extract $\{a\}\{\}\sum[$
$$\begin{array}{l}
\frac{i\, d_R.d_L \sqrt{a\, f[0]}\; m_d}{\pi\, v} \\
\frac{i\, d_R.d_L \sqrt{a\, f[0]}\; m_d}{\pi\, v^*} \\
\frac{i\, \bar{d}_R.\bar{d}_L \sqrt{a\, f[0]}\; m_d}{\pi\, v} \\
-\frac{i\, e_L.e_R \sqrt{a\, f[0]}\; m_e}{\pi\, v} \\
\frac{i\, e_R.e_L \sqrt{a\, f[0]}\; m_e}{\pi\, v} \\
\frac{i\, e_R.e_L \sqrt{a\, f[0]}\; m_e}{\pi\, v^*} \\
\frac{i\, \bar{e}_R.\bar{e}_L \sqrt{a\, f[0]}\; m_e}{\pi\, v} \\
-\frac{i\, u_L.u_R \sqrt{a\, f[0]}\; m_u}{\pi\, v} \\
\frac{i\, u_R.u_L \sqrt{a\, f[0]}\; m_u}{\pi\, v} \\
\frac{i\, u_R.u_L \sqrt{a\, f[0]}\; m_u}{\pi\, v^*} \\
\frac{i\, \bar{u}_R.\bar{u}_L \sqrt{a\, f[0]}\; m_u}{\pi\, v} \\
-\frac{i\, \nu_L.\nu_R \sqrt{a\, f[0]}\; m_\nu}{\pi\, v} \\
\frac{i\, \nu_R.\nu_L \sqrt{a\, f[0]}\; m_\nu}{\pi\, v} \\
\frac{i\, \nu_R.\nu_L \sqrt{a\, f[0]}\; m_\nu}{\pi\, v^*} \\
\frac{i\, \bar{\nu}_R.\bar{\nu}_L \sqrt{a\, f[0]}\; m_\nu}{\pi\, v} \\
-\frac{i\, \bar{d}_L.\bar{d}_R \sqrt{a\, f[0]}\; m_d{}^T}{\pi\, v} \\
-\frac{i\, \bar{e}_L.\bar{e}_R \sqrt{a\, f[0]}\; m_e{}^T}{\pi\, v} \\
-\frac{i\, \bar{u}_L.\bar{u}_R \sqrt{a\, f[0]}\; m_u{}^T}{\pi\, v} \\
-\frac{i\, \bar{\nu}_L.\bar{\nu}_R \sqrt{a\, f[0]}\; m_\nu{}^T}{\pi\, v}
\end{array}$$
$]$

....

$$\rightarrow \boxed{-\frac{2\, i \sqrt{a\, f[0]}\; \left(\left\langle J_M.\bar{d}\;\middle|\; d\right\rangle m_d + \left\langle J_M.\bar{e}\;\middle|\; e\right\rangle m_e + \left\langle J_M.\bar{u}\;\middle|\; u\right\rangle m_u + \left\langle J_M.\bar{\nu}\;\middle|\; \nu\right\rangle m_\nu\right)}{\pi\, v}}$$

Extract $\{\nu,\, m_\nu\}\{a,\, \phi\}\sum[$
$$\begin{array}{l}
i\, (m_\nu)^*\, \nu_L.\nu_R \\
\frac{i\, h\, (m_\nu)^*\, \nu_L.\nu_R}{v} \\
i\, (m_\nu)^*\, \bar{\nu}_L.\bar{\nu}_R \\
\frac{i\, h\, (m_\nu)^*\, \bar{\nu}_L.\bar{\nu}_R}{v} \\
-i\, \nu_R.\nu_L\, m_\nu \\
-\frac{i\, h\, \nu_R.\nu_L\, m_\nu}{v} \\
-i\, \bar{\nu}_R.\bar{\nu}_L\, m_\nu \\
-\frac{i\, h\, \bar{\nu}_R.\bar{\nu}_L\, m_\nu}{v}
\end{array}$$
$]$

....

$$\rightarrow \boxed{\frac{i\, (h+v)\, \left\langle J_M.\bar{\nu}\;\middle|\; \nu\right\rangle m_\nu}{v}}$$

$$\text{Extract }\{e, m_e\}\{a, \phi\}\sum\left[\begin{array}{|l} i\,(m_e)^*\, e_L . e_R \\ \frac{i\,h\,(m_e)^*\, e_L . e_R}{v} \\ i\,(m_e)^*\, \bar{e}_L . \bar{e}_R \\ \frac{i\,h\,(m_e)^*\, \bar{e}_L . \bar{e}_R}{v} \\ -i\, e_R . e_L\, m_e \\ -\frac{i\,h\, e_R . e_L\, m_e}{v} \\ -i\, \bar{e}_R . \bar{e}_L\, m_e \\ -\frac{i\,h\, \bar{e}_R . \bar{e}_L\, m_e}{v} \end{array}\right]$$

....

$$\rightarrow \boxed{\frac{i\,(h+v)\,\langle J_M . \bar{e} \mid e \rangle\, m_e}{v}}$$

$$\text{Extract }\{u, m_u\}\{a, \phi\}\sum\left[\begin{array}{|l} i\,(m_u)^*\, u_L . u_R \\ \frac{i\,h\,(m_u)^*\, u_L . u_R}{v} \\ i\,(m_u)^*\, \bar{u}_L . \bar{u}_R \\ \frac{i\,h\,(m_u)^*\, \bar{u}_L . \bar{u}_R}{v} \\ -i\, u_R . u_L\, m_u \\ -\frac{i\,h\, u_R . u_L\, m_u}{v} \\ -i\, \bar{u}_R . \bar{u}_L\, m_u \\ -\frac{i\,h\, \bar{u}_R . \bar{u}_L\, m_u}{v} \end{array}\right]$$

....

$$\rightarrow \boxed{\frac{i\,(h+v)\,\langle J_M . \bar{u} \mid u \rangle\, m_u}{v}}$$

$$\text{Extract }\{d, m_d\}\{a, \phi\}\sum\left[\begin{array}{|l} i\,(m_d)^*\, d_L . d_R \\ \frac{i\,h\,(m_d)^*\, d_L . d_R}{v} \\ i\,(m_d)^*\, \bar{d}_L . \bar{d}_R \\ \frac{i\,h\,(m_d)^*\, \bar{d}_L . \bar{d}_R}{v} \\ -i\, d_R . d_L\, m_d \\ -\frac{i\,h\, d_R . d_L\, m_d}{v} \\ -i\, \bar{d}_R . \bar{d}_L\, m_d \\ -\frac{i\,h\, \bar{d}_R . \bar{d}_L\, m_d}{v} \end{array}\right]$$

....

$$\rightarrow \boxed{\frac{i\,(h+v)\,\langle J_M . \bar{d} \mid d \rangle\, m_d}{v}}$$

Extract $\{\nu, \phi^0\}\{a\}\sum[\frac{(m_\nu)^* \nu_L \cdot \nu_R \phi^0}{v}, \frac{(m_\nu)^* \overline{\nu}_L \cdot \overline{\nu}_R \phi^0}{v}, \frac{\nu_R \cdot \nu_L m_\nu \phi^0}{v}, \frac{\overline{\nu}_R \cdot \overline{\nu}_L m_\nu \phi^0}{v}]$

....

$\to \boxed{\frac{\langle J_M \cdot \overline{\nu} \mid \gamma_5 \cdot \nu \rangle m_\nu \phi^0}{v}}$

Extract $\{e, \phi^0\}\{a\}\sum[-\frac{(m_e)^* e_L \cdot e_R \phi^0}{v}, -\frac{(m_e)^* \overline{e}_L \cdot \overline{e}_R \phi^0}{v}, -\frac{e_R \cdot e_L m_e \phi^0}{v}, -\frac{\overline{e}_R \cdot \overline{e}_L m_e \phi^0}{v}]$

....

$\to \boxed{-\frac{\langle J_M \cdot \overline{e} \mid \gamma_5 \cdot e \rangle m_e \phi^0}{v}}$

Extract $\{u, \phi^0\}\{a\}\sum[\frac{(m_u)^* u_L \cdot u_R \phi^0}{v}, \frac{(m_u)^* \overline{u}_L \cdot \overline{u}_R \phi^0}{v}, \frac{u_R \cdot u_L m_u \phi^0}{v}, \frac{\overline{u}_R \cdot \overline{u}_L m_u \phi^0}{v}]$

....

$\to \boxed{\frac{\langle J_M \cdot \overline{u} \mid \gamma_5 \cdot u \rangle m_u \phi^0}{v}}$

Extract $\{d, \phi^0\}\{a\}\sum[-\frac{(m_d)^* d_L \cdot d_R \phi^0}{v}, -\frac{(m_d)^* \overline{d}_L \cdot \overline{d}_R \phi^0}{v}, -\frac{d_R \cdot d_L m_d \phi^0}{v}, -\frac{\overline{d}_R \cdot \overline{d}_L m_d \phi^0}{v}]$

....

$\to \boxed{-\frac{\langle J_M \cdot \overline{d} \mid \gamma_5 \cdot d \rangle m_d \phi^0}{v}}$

Extract $\{\nu, e, \phi^-\}\{a\}\sum[$ $\frac{\sqrt{2}\,(m_e)^*\, e_L . \nu_R\, \phi^-}{v}$ $\frac{\sqrt{2}\,(m_e)^*\, \bar{\nu}_L . \bar{e}_R\, \phi^-}{v}$ $\frac{\sqrt{2}\, e_R . \nu_L\, m_\nu\, \phi^-}{v}$ $\frac{\sqrt{2}\, \bar{\nu}_R . \bar{e}_L\, m_\nu\, \phi^-}{v}$ $]$

....

$$\rightarrow \boxed{\frac{\sqrt{2}\left(\left\langle J_M . \bar{e} \mid \left(\frac{1}{2}(1+\gamma_5)\right) . \nu\right\rangle m_e + \left\langle J_M . \bar{e} \mid \left(\frac{1}{2}(-1+\gamma_5)\right) . \nu\right\rangle m_\nu\right) \phi^-}{v}}$$

Extract $\{d, u, \phi^-\}\{a\}\sum[$ $\frac{\sqrt{2}\,(m_d)^*\, d_L . u_R\, \phi^-}{v}$ $\frac{\sqrt{2}\,(m_d)^*\, \bar{u}_L . \bar{d}_R\, \phi^-}{v}$ $\frac{\sqrt{2}\, d_R . u_L\, m_u\, \phi^-}{v}$ $\frac{\sqrt{2}\, \bar{u}_R . \bar{d}_L\, m_u\, \phi^-}{v}$ $]$

....

$$\rightarrow \boxed{\frac{\sqrt{2}\left(\left\langle J_M . \bar{d} \mid \left(\frac{1}{2}(1+\gamma_5)\right) . u\right\rangle m_d + \left\langle J_M . \bar{d} \mid \left(\frac{1}{2}(-1+\gamma_5)\right) . u\right\rangle m_u\right) \phi^-}{v}}$$

Extract $\{\nu, e, \phi^+\}\{a\}\sum[$ $\frac{\sqrt{2}\,(m_\nu)^*\, \nu_L . e_R\, \phi^+}{v}$ $\frac{\sqrt{2}\,(m_\nu)^*\, \bar{e}_L . \bar{\nu}_R\, \phi^+}{v}$ $\frac{\sqrt{2}\, \nu_R . e_L\, m_e\, \phi^+}{v}$ $\frac{\sqrt{2}\, \bar{e}_R . \bar{\nu}_L\, m_e\, \phi^+}{v}$ $]$

....

$$\rightarrow \boxed{\frac{\sqrt{2}\left(\left\langle J_M . \bar{\nu} \mid \left(\frac{1}{2}(-1+\gamma_5)\right) . e\right\rangle m_e + \left\langle J_M . \bar{\nu} \mid \left(\frac{1}{2}(1+\gamma_5)\right) . e\right\rangle m_\nu\right) \phi^+}{v}}$$

Extract {d, u, ϕ^+}{a}∑[$\begin{array}{l} \frac{\sqrt{2}\ (m_u)^*\ u_L . d_R\ \phi^+}{v} \\ \frac{\sqrt{2}\ (m_u)^*\ \bar{d}_L . \bar{u}_R\ \phi^+}{v} \\ \frac{\sqrt{2}\ u_R . d_L\ m_d\ \phi^+}{v} \\ \frac{\sqrt{2}\ \bar{d}_R . \bar{u}_L\ m_d\ \phi^+}{v} \end{array}$]

....

→ $\boxed{\frac{\sqrt{2}\ (\langle J_M . \bar{u}\ \mid (\frac{1}{2}(-1+\gamma_5)).d\ \rangle m_d + \langle J_M . \bar{u}\ \mid (\frac{1}{2}(1+\gamma_5)).d\ \rangle m_u)\ \phi^+}{v}}$

```
PR["There are differences: ",
 $ = $termsPh; $ // ColumnBar,

 $notes = {$termsPh[[1, 1]] → "missing other chiral component",
   $termsPh[[2, 1]] → "a f[0] component exists",
   $termsPh[[3, 1]] → "OK",
   $termsPh[[4, 1]] → "OK",
   $termsPh[[5, 1]] → "OK",
   $termsPh[[6, 1]] → "OK",
   $termsPh[[7, 1]] → "OK",
   $termsPh[[8, 1]] → "OK",
   $termsPh[[9, 1]] → "OK",
   $termsPh[[10, 1]] → "OK",
   $termsPh[[11, 1]] → "OK",
   $termsPh[[12, 1]] → "OK",
   $termsPh[[13, 1]] → "OK",
   $termsPh[[14, 1]] → "OK"
  }; $notes // ColumnBar

]
$list;
```

There are differences:

$$\{m_R\} \to i \left\langle J_M . \bar{\nu} \;\middle|\; \left(\frac{1}{2}(1+\gamma_5)\right).\bar{\nu} \right\rangle m_R$$

$$\{a\} \to -\frac{2\,i\,\sqrt{a\,f[0]}\,\left(\langle J_M.\bar{d}\;|d\;\rangle m_d + \langle J_M.\bar{e}\;|e\;\rangle m_e + \langle J_M.\bar{u}\;|u\;\rangle m_u + \langle J_M.\bar{\nu}\;|\nu\;\rangle m_\nu\right)}{\pi\, v}$$

$$\{\nu,\, m_\nu\} \to \frac{i\,(h+v)\,\langle J_M.\bar{\nu}\;|\nu\;\rangle m_\nu}{v}$$

$$\{e,\, m_e\} \to \frac{i\,(h+v)\,\langle J_M.\bar{e}\;|e\;\rangle m_e}{v}$$

$$\{u,\, m_u\} \to \frac{i\,(h+v)\,\langle J_M.\bar{u}\;|u\;\rangle m_u}{v}$$

$$\{d,\, m_d\} \to \frac{i\,(h+v)\,\langle J_M.\bar{d}\;|d\;\rangle m_d}{v}$$

$$\{\nu,\, \phi^0\} \to \frac{\langle J_M.\bar{\nu}\;|\gamma_5.\nu\;\rangle m_\nu\, \phi^0}{v}$$

$$\{e,\, \phi^0\} \to -\frac{\langle J_M.\bar{e}\;|\gamma_5.e\;\rangle m_e\, \phi^0}{v}$$

$$\{u,\, \phi^0\} \to \frac{\langle J_M.\bar{u}\;|\gamma_5.u\;\rangle m_u\, \phi^0}{v}$$

$$\{d,\, \phi^0\} \to -\frac{\langle J_M.\bar{d}\;|\gamma_5.d\;\rangle m_d\, \phi^0}{v}$$

$$\{\nu,\, e,\, \phi^-\} \to \frac{\sqrt{2}\,\left(\left\langle J_M.\bar{e}\;\middle|\left(\frac{1}{2}(1+\gamma_5)\right).\nu\right\rangle m_e + \left\langle J_M.\bar{e}\;\middle|\left(\frac{1}{2}(-1+\gamma_5)\right).\nu\right\rangle m_\nu\right)\phi^-}{v}$$

$$\{d,\, u,\, \phi^-\} \to \frac{\sqrt{2}\,\left(\left\langle J_M.\bar{d}\;\middle|\left(\frac{1}{2}(1+\gamma_5)\right).u\right\rangle m_d + \left\langle J_M.\bar{d}\;\middle|\left(\frac{1}{2}(-1+\gamma_5)\right).u\right\rangle m_u\right)\phi^-}{v}$$

$$\{\nu,\, e,\, \phi^+\} \to \frac{\sqrt{2}\,\left(\left\langle J_M.\bar{\nu}\;\middle|\left(\frac{1}{2}(-1+\gamma_5)\right).e\right\rangle m_e + \left\langle J_M.\bar{\nu}\;\middle|\left(\frac{1}{2}(1+\gamma_5)\right).e\right\rangle m_\nu\right)\phi^+}{v}$$

$$\{d,\, u,\, \phi^+\} \to \frac{\sqrt{2}\,\left(\left\langle J_M.\bar{u}\;\middle|\left(\frac{1}{2}(-1+\gamma_5)\right).d\right\rangle m_d + \left\langle J_M.\bar{u}\;\middle|\left(\frac{1}{2}(1+\gamma_5)\right).d\right\rangle m_u\right)\phi^+}{v}$$

$\{m_R\} \to$ missing other chiral component
$\{a\} \to$ a f[0] component exists
$\{\nu,\, m_\nu\} \to$ OK
$\{e,\, m_e\} \to$ OK
$\{u,\, m_u\} \to$ OK
$\{d,\, m_d\} \to$ OK
$\{\nu,\, \phi^0\} \to$ OK
$\{e,\, \phi^0\} \to$ OK
$\{u,\, \phi^0\} \to$ OK
$\{d,\, \phi^0\} \to$ OK
$\{\nu,\, e,\, \phi^-\} \to$ OK
$\{d,\, u,\, \phi^-\} \to$ OK
$\{\nu,\, e,\, \phi^+\} \to$ OK
$\{d,\, u,\, \phi^+\} \to$ OK

```
(* LAST *)

tuSaveAllVariables[]
```