Twisted spectral triple for the Standard Model and spontaneous breaking of the Grand Symmetry

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Abstract

Grand symmetry models in noncommutative geometry have been introduced to explain how to generate minimally (i.e. without adding new fermions) an extra scalar field beyond the standard model, which both stabilizes the electroweak vacuum and makes the computation of the mass of the Higgs compatible with its experimental value. In this paper, we use Connes-Moscovici twisted spectral triples to cure a technical problem of the grand symmetry, that is the appearance together with the extra scalar field of unbounded vectorial terms. The twist makes these terms bounded and - thanks to a twisted version of the first-order condition that we introduce here - also permits to understand the breaking to the standard model as a dynamical process induced by the spectral action, as conjectured in [23]. This is a spontaneous breaking from a pre-geometric Pati-Salam model to the almost-commutative geometry of the standard model, with two Higgs-like fields: scalar and vector.

1 Introduction

Noncommutative geometry [NCG] provides a description of the standard model of elementary particles [SM] in which the mass of the Higgs − at unification scale Λ − is a function of the other parameters of the theory, especially the Yukawa coupling of fermions [9]. Assuming there is no new physics between the electroweak and the unification scales (the "big desert hypothesis"), the flow of this mass under the renormalization group yields a prediction for the Higgs observable mass m_H. It is well known that in the absence of new physics the three constants of interaction fail to meet at a single unification scale, but form a triangle which lays between 10^{13} and 10^{17} GeV. The situation can be improved by taking into account higher order term in the NCG action [22], or gravitational effects [21]. Nevertheless, the prediction of m_H is not much sensible on the precise choice of the unification scale. Since the beginning of the model in the early 90' [14, 15], for Λ between 10^{13} and 10^{17}GeV this prediction had been around 170 GeV, a value ruled out by Tevratron in 2008. Consequently, either the model should be abandoned, or the big desert hypothesis questioned.

The recent discovery of the Higgs boson with a mass $m_H \simeq 126\,$Gev suggests the big desert hypothesis should be questioned. There is indeed an instability in the electroweak vacuum which is meta-stable rather than stable (see [5] for the most recent update). There does not seem to be a consensus in the community whether this is an important problem or not: on the one hand the mean time of this meta-stable state is longer than the age of the universe, on the other hand in some cosmological scenario the meta-stabililty may be problematic [26, 27]. Still,

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the fact that m_H is almost at the boundary value between the stable and meta-stable phases of the electroweak vacuum suggests that "something may be going on". In particular, particle physicists have shown how a new scalar field suitably coupled to the Higgs - usually denoted σ - can cure the instability (e.g. [13, 25]).

Taking into account this extra field in the NCG description of the SM induces a modification of the flow of the Higgs mass, governed by the parameter $r = \frac{k_\nu}{k_t}$ which is the ratio of the Dirac mass of the neutrino and of the Yukawa coupling of the quark top. Remarkably, for any value of Λ between 10^{12} and 10^{17} Gev, there exists a realistic value $r \simeq 1$ which brings back the computed value of m_H to 126 Gev [8].

The question is then to generate the extra field σ in agreement with the tools of noncommutative geometry. Early attempts in this direction have been done in [34], but they require the adjunction of new fermions (see [35] for a recent state of the art). In [8], a scalar σ correctly coupled to the Higgs is obtained without touching the fermionic content of the model, simply by turning the Majorana mass k_R of the neutrino into a field

$$k_R \to k_R \, \sigma. \tag{1.1}$$

Usually the bosonic fields in NCG are generated by inner fluctuations of the geometry. However this does not work for the field σ because of the first-order condition

$$[[D, a], Jb^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A} \tag{1.2}$$

where $\mathcal{A}$ and D are the algebra and the Dirac operator of the spectral triple of the standard model, and J the real structure.

In [11, 12] it was shown how to obtain σ by an inner fluctuation that does not satisfy the first-order condition, but in such a way that the latter is retrieved dynamically, as a minimum of the spectral action. The field σ is then interpreted as an excitation around this minimum. Previously in [23] another way had been investigated to generate σ in agreement with the first-order condition, taking advantage of the fermion doubling in the Hilbert space $\mathcal{H}$ of the spectral triple of the SM [31–33].

More specifically, under natural assumptions on the representation of the algebra and an ad-hoc symplectic hypothesis, it is shown in [7] that the algebra in the spectral triple of the SM should be a sub-algebra of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$, where $\mathcal{M}$ is a Riemannian compact spin manifold (usually of dimension 4) while

$$\mathcal{A}_F = M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C}) \quad a \in \mathbb{N}. \tag{1.3}$$

The algebra of the standard model

$$\mathcal{A}_{sm} := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{1.4}$$

is obtained from $\mathcal{A}_F$ for $a = 2$, by the grading and the first-order conditions. Starting instead with the "grand algebra" $(a = 4)$

$$\mathcal{A}_G := M_4(\mathbb{H}) \oplus M_8(\mathbb{C}), \tag{1.5}$$

one generates the field σ by a inner fluctuation which respects the first-order condition imposed by the part D_M of the Dirac operator that contains the Majorana mass k_R [23]. The breaking to $\mathcal{A}_{sm}$ is then obtained by the first-order condition imposed by the free Dirac operator

$$\not{D} := \not{\partial} \otimes \mathbb{I}_F \tag{1.6}$$

where $\mathbb{I}_F$ is the identity operator on the finite dimensional Hilbert space $\mathcal{H}_F$ on which acts $\mathcal{A}_G$.

Unfortunately, before this breaking not only is the first-order condition not satisfied, but the commutator

$$[\partial\!\!\!/, A] \quad A \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_G \tag{1.7}$$

is never bounded. This is problematic both for physics, because the connection 1-form describing the gauge bosons is unbounded; and from a mathematical point of view, because the construction of a Fredholm module over $\mathcal{A}$ and Hochschild character cocycle depends on the boundedness of the commutator (1.7).

In this paper, we solve this problem by using instead a *twisted spectral triple* $(\mathcal{A}, \mathcal{H}, D, \rho)$ [17]*. Rather than requiring the boundedness of the commutator, one asks that there exists a automorphism ρ of $\mathcal{A}$ such that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D \tag{1.8}$$

is bounded for any $a \in \mathcal{A}$. Accordingly, we introduce in Def. 3.1 a *twisted first-order condition*

$$[[D, a]_\rho, Jb^*J^{-1}]_\rho := [D, a]_\rho Jb^*J^{-1} - J\rho(b^*)J^{-1}[D, a]_\rho = 0 \quad \forall a, b \in \mathcal{A}. \tag{1.9}$$

We then show that for a suitable choice of a subalgebra $\mathcal{B}$ of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$, a *twisted fluctuation* of $\partial\!\!\!/ + D_M$ that satisfies (1.9) generates a field $\boldsymbol{\sigma}$ - slightly different from the one of [8] - together with an additional vector field X_μ.

Furthermore, the breaking to the standard model is now spontaneous, as conjectured by Lizzi in [23]. Namely the reduction of the grand algebra $\mathcal{A}_G$ to $\mathcal{A}_{sm}$ is obtained dynamically, as a minimum of the spectral action. The scalar field σ then play a role similar as the one of the Higgs in the electroweak symmetry breaking.

Mathematically, twists make sense as explained in [17], for the Chern character of finitely summable spectral triples extends to the twisted case, and lands in ordinary (untwisted) cyclic cohomology. Twisted spectral triples have been introduced to deal with type III examples, such as those arising from transverse geometry of codimension one foliation, and have been used in various context since, like quantum statistical dynamical systems [30]. It is quite surprising that the same tool gives a possibility to implement in NCG of the idea of a "bigger symmetry beyond the SM". The main results of the paper are summarized in the following theorem.

Theorem 1.1. *Let $\mathcal{H}$ be the Hilbert space of the standard model described in §2.1. There exists a sub-algebra $\mathcal{B}$ of the grand algebra $\mathcal{A}_G$ containing $\mathcal{A}_{sm}$ together with an automorphism ρ of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ such that*

i) $T := (C^\infty(\mathcal{M}) \otimes \mathcal{B}, \mathcal{H}, \partial\!\!\!/ + D_M; \rho)$ is a twisted spectral triple satisfying the twisted 1^{st}-order condition (1.9)*;*

ii) twisted fluctuations of $\partial\!\!\!/ + D_M$ by $\mathcal{B}$ are parametrized by a scalar field $\boldsymbol{\sigma}$ and a vector field X_μ;

iii) the spectral triple of the standard model is obtained from T by minimizing the potential of the vector field X_μ induced by the spectral action coming from a twisted fluctuation of $\partial\!\!\!/$;

iv) the spectral triple of the standard model is also obtained by minimizing the potential induced by the spectral action of a twisted fluctuation of the whole Dirac operator $\partial\!\!\!/ + D_M$. Such a fluctuation provides a potential for the scalar field $\boldsymbol{\sigma}$, which is minimum when $\partial\!\!\!/ + D_M$ is fluctuated by $\mathcal{A}_{SM}$, that is when $\boldsymbol{\sigma}$ is the constant field k_R.

*Also called σ-triple, but to avoid confusion with the field σ, we denote by ρ the automorphism called σ in [17].

Explicitly, $\mathcal{B}$ is a sub-algebra $\mathbb{H}^2 \oplus \mathbb{C}^2 \oplus M_3(\mathbb{C})$ of $\mathcal{A}_G$. Labeling the two copies of the quaternions and complex algebras by the left/right spinorial indices l, r and the left/right internal indices L/R, that is

$$\mathcal{B} = \mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^l \oplus \mathbb{C}_R^r \oplus M_3(\mathbb{C}), \tag{1.10}$$

the automorphism ρ is the exchange of the left/right spinorial indices:

$$\rho\,(\,q_L^l,\, q_L^r,\, c_R^l,\, c_R^r,\, m) \to (q_L^r,\, q_L^l,\, c_R^r,\, c_R^l,\, m) \tag{1.11}$$

where $m \in M_3(\mathbb{C})$ while the q's and c's are quaternions and complex numbers belonging to their respective copy of $\mathbb{H}$ and $\mathbb{C}$.

The paper is organized as follows. In section 2 we recall briefly the spectral triple of the standard model (§2.1), the tensorial notation used all along the paper (§2.2), and the results of [23] on the grand algebra (§2.3). We discuss the unboundedness of the commutator (1.7) in §2.4. Section 3 deals with the twist. It begins with the definition of the twisted first-order condition in Def. 3.1. In §3.1 we fix the representation of the grand algebra, which differs from the one used in [23]. It is used in §3.2 to build a twisted spectral triple with the free Dirac operator (Prop. 3.4). In §3.3 the twisted first-order condition for D_M yields the reduction to the algebra $\mathcal{B}$ and the construction of the spectral triple T (Prop. 3.5). This proves the first point of theorem 1.1. In section 4 we compute the twisted fluctuations D_X of the free Dirac operator $\not{\partial}$ (§4.2), and D_σ of the Majorana-Dirac operator D_M (§4.3). This yields the additional vector field in Prop. 4.1, and the extra scalar field $\boldsymbol{\sigma}$ in Prop. 4.4, proving the second point of theorem 1.1. In section 5, after some generalities on the spectral action in §5.1, we compute the generalized Lichnerowicz formula for the twisted-fluctuated Dirac operator in §5.2. The comparison with the non-twisted case is made in § 5.3. The dynamical reduction of $\mathcal{B}$ to the standard model by minimizing the potential of the additional vector field is obtained in §5.4. The potential of the scalar field is treated in §5.5, and the potential of interaction between the vector and the scalar field in §5.6. These results are discussed in section 6. In §6.1 we stress how twisting the almost commutative geometry of the SM may open the way to models where the algebra is not the tensor product of a manifold by a finite dimensional geometry. This justifies the choice of the representation of $\mathcal{A}_G$ made in the present paper, but we show in §6.2 that the results are the same with the representation used in [23].

2 Standard model and the grand algebra

2.1 The spectral triple of the standard model

The main tools of NCG [18] are encoded within a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra acting on a Hilbert space $\mathcal{H}$, and D is a selfadjoint operator on $\mathcal{H}$. These three elements come with two more operators, a real structure J [19] and a graduation Γ that are generalizations to the noncommutative setting of the charge conjugation and the chirality operators of quantum field theory. These five objects satisfy a set of properties guaranteeing that given any spectral triple with $\mathcal{A}$ unital and commutative, then there exists a closed Riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$ [20]. These conditions still make sense in the noncommutative case [14], hence the definition of a *noncommutative geometry* as a spectral triple where the algebra $\mathcal{A}$ is non necessarily commutative.

Among these conditions, the ones that play an important role in this work are the first-order condition (1.2), the boundedness and the grading conditions

$$[D, a] \in \mathcal{B}(\mathcal{H}), \quad [\Gamma, a] = 0 \quad \forall a \in \mathcal{A}, \tag{2.1}$$

as well as the order-zero condition

$$[a, Jb^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}. \tag{2.2}$$

A gauge theory is described by an *almost commutative geometry*

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \ \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \ D = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma^5 \otimes D_F, \tag{2.3}$$

which is the product of the canonical spectral triple $\left(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/\right)$ associated to a oriented closed spin manifold $\mathcal{M}$ of (even) dimension m, by a finite dimensional spectral triple

$$(\mathcal{A}_F, \mathcal{H}_F, D_F). \tag{2.4}$$

Here $L^2(\mathcal{M}, S)$ is the space of square integrable spinors on $\mathcal{M}$, and

$$\partial\!\!\!/ = -i \sum_{\mu=1}^{m} \gamma^\mu \nabla_\mu^S \quad \text{ with } \quad \nabla_\mu^S = \partial_\mu + \omega_\mu^S \tag{2.5}$$

is the Dirac operator with $\gamma^\mu = \gamma^{\mu\dagger}$ the selfadjoint Dirac matrices and ω_μ^S the spin connection. The chirality operator γ^5 is a graduation of $L^2(\mathcal{M}, S)$ which commutes with $C^\infty(\mathcal{M})$ and anticommutes with $\partial\!\!\!/$. The notation is justified assuming $\mathcal{M}$ has dimension 4 (what we do from now on): γ^5 is then the product of the four Dirac matrices.

The choice of the finite dimensional spectral triple (2.4) is dictated by the physical contains of the theory. For the SM, the algebra is $\mathcal{A}_{sm}$ given in (1.4), whose group of unitary elements yields the gauge group of the standard model. The finite dimensional Hilbert space $\mathcal{H}_F$ is spanned by the particle content of the theory. The standard model has 96 such degrees of freedom: 8 fermions (electron, neutrino, up and down quarks with three colors each) for N=3 generations and two chiralities L, R, plus antiparticles. Therefore one takes

$$\mathcal{H}_F = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c = \mathbb{C}^{96}. \tag{2.6}$$

The finite dimensional Dirac operator $D_F = D_0 + D_R$ is a 96×96 matrix where

$$D_0 := \begin{pmatrix} 0_{8N} & \mathcal{M}_0 & 0_{8N} & 0_{8N} \\ \mathcal{M}_0^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & \bar{\mathcal{M}}_0 \\ 0_{8N} & 0_{8N} & \mathcal{M}_0^T & 0_{8N} \end{pmatrix} \ \text{ and } \ D_R := \begin{pmatrix} 0_{8N} & 0_{8N} & \mathcal{M}_R & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & 0_{8N} \\ \mathcal{M}_R^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & 0_{8N} \end{pmatrix}. \tag{2.7}$$

The matrix $\mathcal{M}_0$ contains the Yukawa couplings of fermions, the Dirac mass of neutrinos, the Cabibbo matrix and the mixing matrix for neutrinos. The matrix $\mathcal{M}_R$ contains the Majorana mass of neutrinos. Explicitly

$$\mathcal{M}_0 = \begin{pmatrix} M_u & 0_4 \\ 0_4 & M_d \end{pmatrix} \otimes \mathbb{I}_N \qquad \mathcal{M}_R = \begin{pmatrix} M_R & 0_4 \\ 0_4 & 0_4 \end{pmatrix} \otimes \mathbb{I}_N \tag{2.8}$$

where, for the first generation, M_u contains the Yukawa coupling of the up quark and the Dirac mass of ν_e, M_d contains the down quark and the electron masses, and M_R the Majorana mass of ν_e. The structure is repeated for the other two generations.

The real structure

$$J = \mathcal{J} \otimes J_F \tag{2.9}$$

acts as the charge conjugation operator $\mathcal{J} = i\gamma^0\gamma^2 cc$ on $L^2(\mathcal{M}, S)$, and as

$$J_F := \begin{pmatrix} 0 & \mathbb{I}_{16N} \\ \mathbb{I}_{16N} & 0 \end{pmatrix} cc \tag{2.10}$$

on $\mathcal{H}_F$, where it exchanges the blocks $\mathcal{H}_R \oplus \mathcal{H}_L$ of particles with the block $\mathcal{H}_R^c \oplus \mathcal{H}_L^c$ of antiparticles. The graduation is

$$\Gamma = \gamma^5 \otimes \gamma_F \quad \text{ where } \quad \gamma_F := \begin{pmatrix} \mathbb{I}_{8N} & & & \\ & -\mathbb{I}_{8N} & & \\ & & -\mathbb{I}_{8N} & \\ & & & \mathbb{I}_{8N} \end{pmatrix}. \tag{2.11}$$

The operators γ_F, J_F and D_F are such that $J_F^2 = \mathbb{I}$, $J\gamma_F = -\gamma_F J_F$, $J_F D_F = D_F J_F$, meaning that the finite part of the spectral triple of the standard model has KO-dimension 6 [3, 9]. Meanwhile the continuous part of the spectral triple has KO-dimension 4, that is $\mathcal{J}^2 = -\mathbb{I}$, $\mathcal{J}\gamma^5 = \gamma^5\mathcal{J}$ and $\mathcal{J}\partial\!\!\!/ = \partial\!\!\!/\mathcal{J}$.

Gauge fields are obtained by fluctuating the operator D by $\mathcal{A}$, that is substituting it with the *covariant Dirac operator*

$$D_A := D + A + JAJ^{-1} \tag{2.12}$$

where

$$A = \sum_i a_i[D, b_i] \quad a_i, b_i \in \mathcal{A} \tag{2.13}$$

is a selfadjoint 1-form of the almost commutative manifold.

As stressed in the introduction, the field σ cannot be generated by a fluctuation of the Majorana part

$$D_M := \gamma^5 \otimes D_R \tag{2.14}$$

of the Dirac operator, because $[D_R, a] = 0$ for any $a \in \mathcal{A}_{sm}$. The obstruction has its origin in the first-order condition. Indeed one easily checks [23] that for any $a, b \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$

$$[[D_M, A],\ Jb^*J^{-1}] = 0 \text{ if and only if } [D_M, A] = 0. \tag{2.15}$$

Hence the necessity to make the first-order condition more flexible [12], or to enlarge the algebra one is starting with, in order to have enough space to generate the field σ without violating the first-order condition. This enlargement is made possible by mixing the internal degrees of freedom of $\mathcal{H}_F$ with the spinorial degrees of freedom of $L^2(\mathcal{M}, S)$. This has been done in [23] and is recalled in the next two paragraphs.

2.2 Mixing of spinorial and internal degrees of freedom

The total Hilbert space $\mathcal{H}$ of the almost commutative geometry (2.3) is the tensor product of four dimensional spinors by the 96-dimensional elements of $\mathcal{H}_F$. Any of its element is a $\mathbb{C}^{384}$-vector valued function on $\mathcal{M}$. From now on, we work with $N = 1$ generation only, and consider instead $384/3 = 128$ components vector. The total Hilbert space can thus be written - at least in a local trivialization - in two ways:

$$\mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F \ \text{ or } \ \mathcal{H} = L^2(\mathcal{M}) \otimes \mathsf{H}_F \tag{2.16}$$

where $\mathsf{H_F} \simeq \mathbb{C}^{128}$ takes into account both external (i.e. spin) and internal (i.e. particle) degrees of freedom. We label the basis of $\mathsf{H_F}$ with a multi-index $s\dot{s}\mathsf{C}\mathrm{I}\alpha$ where:

- $s, \dot{s}$ are the four spinor indices: $s = r, l$ runs over the right, left parts and $\dot{s} = \dot{0}, \dot{1}$ over the particle, antiparticle parts of the spinors.
- C indicates wether we are considering "particles" ($\mathsf{C} = 0$) or "antiparticles" ($\mathsf{C} = 1$).
- I is a "lepto-color" index: $\mathrm{I} = 0$ identifies leptons while $\mathrm{I} = 1, 2, 3$ are the three colors of QCD.
- α is the flavor index. It runs over the set u_R, d_R, u_L, d_L when $\mathrm{I} = 1, 2, 3$, and ν_R, e_R, ν_L, e_L when $\mathrm{I} = 0$.

On this basis, an element Ψ of $\mathcal{H}$ has components $\Psi^{\mathsf{C}\mathrm{I}}_{s\dot{s}\alpha} \in L^2(\mathcal{M})$. The position of the indices is arbitrary: Ψ evaluated at $x \in \mathcal{M}$ is a column vector, so all the indices are raw indices. An element A in $\mathcal{B}(\mathcal{H})$ is a 128×128 matrix whose coefficients are function of M, and carries the indices

$$A = A^{\mathsf{C}t\mathrm{I}\dot{t}\beta}_{\mathsf{D}s\mathrm{J}\dot{s}\alpha} \tag{2.17}$$

where $\mathsf{D}, t, \mathrm{J}, \dot{t}, \beta$ are column indices with the same range as $\mathsf{C}, s, \mathrm{I}, \dot{s}, \alpha$.

This choice of indices yields the chiral basis for the Euclidean Dirac matrices:[†]

$$\gamma^\mu = \begin{pmatrix} 0_2 & \sigma^\mu \\ \tilde{\sigma} & 0_2 \end{pmatrix}_{st}, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix}_{st}, \tag{2.18}$$

where for $\mu = 0, 1, 2, 3$ one defines

$$\sigma^\mu = \{\mathbb{I}_2, -i\sigma_i, \}, \quad \tilde{\sigma}^\mu = \{\mathbb{I}_2, i\sigma_i\} \tag{2.19}$$

with σ_i, $i = 1, 2, 3$ the Pauli matrices. Explicitly,

$$\sigma^0 = \mathbb{I}_2,\ \sigma^1 = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}_{\dot{s}\dot{t}},\ \sigma^2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\dot{s}\dot{t}},\ \sigma^3 = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}_{\dot{s}\dot{t}}.$$

The free Dirac operator $\partial\!\!\!/$ extended to $\mathcal{H}$ according to (1.6) acts as [‡]

$$D\!\!\!\!/ := \delta^{\mathsf{C}\mathrm{I}\beta}_{\mathsf{D}\mathrm{J}\alpha}\partial\!\!\!/ = -i\begin{pmatrix} \delta^{\mathrm{I}\beta}_{\mathrm{J}\alpha}\,\gamma^\mu\nabla^S_\mu & 0_{64} \\ 0_{64} & \delta^{\mathrm{I}\beta}_{\mathrm{J}\alpha}\,\gamma^\mu\nabla^S_\mu \end{pmatrix}_{\mathsf{CD}}. \tag{2.20}$$

In tensorial notation, the charge conjugation operator is

$$\mathcal{J} = i\gamma^0\gamma^2 cc = i\begin{pmatrix} \tilde{\sigma}^2 & 0_2 \\ 0_2 & \sigma^2 \end{pmatrix}_{st} cc = -i\eta^t_s\tau^{\dot{t}}_{\dot{s}}\,cc, \tag{2.21}$$

while

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_{\mathsf{CD}} cc, \tag{2.22}$$

[†] The multi-index st after the closing parenthesis is to recall that the block-entries of the γ's matrices are labelled by indices s, t taking values in the set $\{l, r\}$. For instance the l-raw, l-column block of γ^5 is $\mathbb{I}_2$. Similarly the entries of the σ's matrices are labelled by $\dot{s}, \dot{t}$ indices taking value in the set $\{\dot{0}, \dot{1}\}$: for instance ${\sigma^2}^{\dot{0}}_{\dot{0}} = {\sigma^2}^{\dot{1}}_{\dot{1}} = 0$.

[‡] We use Einstein summation on alternated up/down indices. For any n pairs of indices (x_1, y_1), (x_2, y_2), ... (x_n, y_n), we write $\delta^{y_1 y_2 \dots y_n}_{x_1 x_2 \dots x_n}$ instead of $\delta^{y_1}_{x_1}\delta^{y_2}_{x_2}\dots\delta^{y_n}_{x_n}$. For the tensorial notation to be coherent, $\partial\!\!\!/$ and γ^μ should carry lower $s\dot{s}$ and upper $t\dot{t}$ indices. We systematically omit them to facilitate the reading.

hence

$$(J\Psi)^{\mathsf{CI}}_{s\dot{s}\alpha} = -i\eta^t_s\,\tau^{\dot{t}}_{\dot{s}}\,\xi^{\mathsf{C}}_{\mathsf{D}}\,\delta^{\mathrm{I}\beta}_{\mathrm{J}\alpha}\,\bar{\Psi}^{\mathsf{DJ}}_{t\dot{t}\beta} \tag{2.23}$$

where for any pair of indices $x, y \in [1, ..., n]$ one defines

$$\xi^x_y = \begin{pmatrix} 0_n & \mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix}, \quad \eta^x_y = \begin{pmatrix} \mathbb{I}_n & 0_n \\ 0_n & -\mathbb{I}_n \end{pmatrix}, \quad \tau^x_y = \begin{pmatrix} 0_n & -\mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix}. \tag{2.24}$$

The chirality acts as $\gamma^5 = \eta^t_s\delta^{\dot{t}}_{\dot{s}}$ on the spin indices, and as $\gamma_F = \eta^{\mathsf{C}}_{\mathsf{D}}\,\delta^{\mathrm{I}}_{\mathrm{J}}\,\eta^{\beta}_{\alpha}$ on the internal indices:

$$(\Gamma\Psi)^{\mathsf{CI}}_{s\dot{s}\alpha} = \eta^t_s\delta^{\dot{t}}_{\dot{s}}\;\eta^{\mathsf{C}}_{\mathsf{D}}\,\delta^{\mathrm{I}}_{\mathrm{J}}\,\eta^{\beta}_{\alpha}\;\Psi^{\mathsf{DJ}}_{t\dot{t}\beta}. \tag{2.25}$$

2.3 The grand algebra

Under natural assumptions (irreducibility of the representation, existence of a separating vector), a "symplectic hypothesis" and the requirement that the KO-dimension is 6, the most general finite algebra that satisfies the conditions for the real structure is [7]

$$\mathcal{A}_F = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \quad a \in \mathbb{N}, \tag{2.26}$$

acting on a Hilbert space of dimension $2(2a)^2$. To have a non-trivial grading on $\mathbb{M}_a(\mathbb{H})$ the integer a must be at least 2, meaning the simplest possibility is $\mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})$. The dimension of the Hilbert space is thus $2(2 \cdot 2)^2 = 32$, which is precisely the dimension of $\mathcal{H}_{\mathcal{F}}$ for one generation. The grading condition $[a, \Gamma] = 0$ imposes the reduction to the left-right algebra,

$$\mathcal{A}_{LR} := \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C}), \tag{2.27}$$

and the order one condition $[[D_F, a], Jb^*J^{-1}] = 0$ reduces further the algebra to $\mathcal{A}_{sm}$ in (1.4).

The case $a = 3$ requires an Hilbert space of dimension $2(2 \cdot 3)^2 = 72$, which has no obvious physical interpretation so far.

For $a = 4$, the dimension is $2(2 \cdot 4)^2 = 128$, which turns out to be precisely the dimension of the "fermion doubled" space H_F. In other terms, the mixing of the internal and the spin degrees of freedom provides exactly the space required to represent the "grand algebra"

$$\mathcal{A}_G = \mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C}). \tag{2.28}$$

Any elements of $\mathcal{A}_G$ is seen as a pair of 8×8 complex matrices $Q \in \mathbb{M}_4(\mathbb{H}), M \in \mathbb{M}_8(\mathbb{C})$, each having a block structure of four 4×4 matrices

$$Q = \begin{pmatrix} Q^1_1 & Q^2_1 \\ Q^1_2 & Q^2_2 \end{pmatrix}, \quad M = \begin{pmatrix} M^1_1 & M^2_1 \\ M^1_2 & M^2_2 \end{pmatrix} \tag{2.29}$$

where $Q^j_i \in M_2(\mathbb{H})$ and $M^j_i \in M_4(\mathbb{C})$ for any $i, j = 1, 2$. By further imposing all the conditions defining a spectral triple, one intends to find back the algebra $\mathcal{A}_{sm}$ of the standard model acting suitably on $\mathcal{H}_F$. This imposes that Q acts on the particle subspace $\mathsf{C} = 0$, trivially on the lepto-color index I, meaning the complex components of each of the four 4×4 matrices Q^j_i are labelled by the flavor index α. Similarly, one asks that M acts on antiparticles $\mathsf{C} = 1$, trivially on the flavor index, meaning the components of each of the four M^j_i are labelled by the lepto-color index I. Thus any element $(Q, M) \in \mathcal{A}_G$ acts on H_F as

$$\delta^{0\mathrm{J}}_{\mathsf{C}\mathrm{I}}\,Q^{j\beta}_{i\alpha} + \delta^{1\beta}_{\mathsf{C}\alpha}\,M^{j\mathrm{J}}_{i\mathrm{I}}. \tag{2.30}$$

The representation of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ is obtained viewing $Q^{j\beta}_{i\alpha}$, $M^{j\mathrm{J}}_{i\mathrm{I}}$ no longer as constants but as L^2 functions on $\mathcal{M}$.

There is still some freedom on how to label the blocks of the matrices Q and M. One simply needs indices i, j that live on the $s\dot{s}$ spinorial space, take two values each and are compatible with the order-zero condition (2.2). The natural choice is to label the blocks of either Q or M by the chiral index $s = r, l$ and the other blocks by the (anti)-particle index $\dot{s} = \dot{0}, \dot{1}$ (although in principle one could also consider combinations of them). In [23] we chose to label the quaternions by the anti-(particle) index and the complex matrices by the chiral index,

$$Q = Q^{\dot{t}\beta}_{\dot{s}\alpha}, \quad M = M^{t\mathrm{J}}_{s\mathrm{I}}. \tag{2.31}$$

The reduction of $\mathcal{A}_G$ to the algebra of the standard model is then obtained as follows

$$\begin{array}{rcll}
\mathcal{A}_G & = & M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) & (2.32)\\
& \Downarrow & \text{grading condition} & \\
\mathcal{A}'_G & = & \mathbb{M}_2(\mathbb{H})_L \oplus \mathbb{M}_2(\mathbb{H})_R \oplus M^l_4(\mathbb{C}) \oplus M^r_4(\mathbb{C}) & \\
& \Downarrow & 1^{\text{st}}\text{-order for the Majorana-Dirac operator } D_M & \\
\mathcal{A}''_G & = & (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}^l \oplus M^l_3(\mathbb{C}) \oplus \mathbb{C}^r \oplus M^r_3(\mathbb{C}))\text{with } \mathbb{C}_R = \mathbb{C}^r = \mathbb{C}^l & \\
& \Downarrow & 1^{\text{st}}\text{-order for the free Dirac operator } \not{D} & (2.33)\\
\mathcal{A}_{sm} & = & \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) &
\end{array}$$

The interest of the grand algebra is the possibility to generate the field σ thanks to a fluctuation of the Majorana mass term D_M (2.14) which respects the first-order condition imposed by this same Majorana mass term. Namely [23], and this has to be put in contrast with (2.15):

$$\text{for } A \in \mathcal{A}''_G,\ [D_M, A] \text{ is not necessarily zero.} \tag{2.34}$$

2.4 Unboundedness of the commutator

As explained in [24], there is no spectral triple for the grand algebra because the commutator $[\not{D}, A]$ of any of its element with the free Dirac operator is never bounded. This can be seen from eq. (5.3) in [23] and has been pointed out to us by W. v. Suijlekom. In order to have bounded commutators, the action of $\mathcal{A}_G$ on spinors has to be trivial.

Proposition 2.1. *Let A_F be a finite dimensional algebra acting on the Hilbert space H_F in (2.16). For any $A \in C^\infty(\mathcal{M}) \otimes \mathsf{A}_F$, the commutator $[\not{D}, A]$ is bounded iff A_F acts as the identity operator on the spinors indices $s\dot{s}$. In particular, the biggest sub-algebra of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ acting as in (2.30) and whose commutator with $\not{D}$ is bounded is $C^\infty(\mathcal{M}) \otimes (M_2(\mathbb{H}) \oplus M_4(\mathbb{C}))$.*

Proof. In tensorial notation, a generic element of A_F is $A = A^{\mathsf{C}t\mathrm{I}\dot{t}\beta}_{\mathsf{D}s\mathrm{J}\dot{s}\alpha}$. For any such A, by (2.20) and omitting the indices $st\dot{s}\dot{t}$ for the Dirac matrices, one gets

$$[\not{D}, A] = [\delta^{\mathsf{C}\mathrm{I}\beta}_{\mathsf{D}\mathrm{J}\alpha}\not{\partial}, A^{\mathsf{C}t\mathrm{I}\dot{t}\beta}_{\mathsf{D}s\mathrm{J}\dot{s}\alpha}] = -i[\delta^{\mathsf{C}\mathrm{I}\beta}_{\mathsf{D}\mathrm{J}\alpha}\gamma^\mu, A^{\mathsf{C}t\mathrm{I}\dot{t}\beta}_{\mathsf{D}s\mathrm{J}\dot{s}\alpha}]\nabla^S_\mu - i\gamma^\mu[\nabla^S_\mu, A^{\mathsf{C}t\mathrm{I}\dot{t}\beta}_{\mathsf{D}s\mathrm{J}\dot{s}\alpha}]. \tag{2.35}$$

This is bounded iff the first term in the r.h.s. is zero. The only matrices that commute with all the Dirac matrices are the multiple of the identity, hence $[\not{D}, A]$ is bounded iff $A = \lambda\delta^{t\dot{t}}_{s\dot{s}}A^{\mathsf{C}\mathrm{I}\beta}_{\mathsf{D}\mathrm{J}\alpha}$ for some scalar λ. This means $Q^{j\beta}_{i\alpha} = \lambda\delta^{t\dot{t}}_{s\dot{s}}Q^\alpha_\beta \in M_2(\mathbb{H})$ and $M^{j\mathrm{J}}_{i\mathrm{I}} = \lambda\delta^{t\dot{t}}_{s\dot{s}}M^{\mathrm{J}}_{\mathrm{I}} \in M_4(\mathbb{C})$ in (2.30). ∎

In other term, to build a spectral triple with the grand algebra ($a = 4$ in (2.26)), one has to consider its subalgebra given by $a = 2$, that acts without mixing spinorial and internal indices. This of course is not interesting from our perspective, since the aim of the grand algebra is precisely to mix spinorial with internal degrees of freedom. A solution is to consider instead *twisted spectral triples*. They have been introduced in [17] precisely to solve the problem of the unboundedness of the commutator, which may occur in very elementary situations such as the lift to spinors of a conformal transformation. Using twists to make $[\not{D}, A]$ bounded has been suggested independently to the second author by J.-C. Wallet, and to the first author by W. v. Suijlekom, who also brought our attention on ref. [17].

3 Twisting the standard model

A *twisted spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra acting on a Hilbert space $\mathcal{H}$ and D a selfadjoint operator on $\mathcal{H}$ with compact resolvent, together with an automorphism ρ of $\mathcal{A}$ such that

$$[D, a]_\rho = Da - \rho(a)D \tag{3.1}$$

is bounded for any $a \in \mathcal{A}$. It is graded if, in addition, there is a selfadjoint operator Γ of square $\mathbb{I}$ which commutes the algebra and anticommutes with D.

As far as we know, the other conditions satisfied by a spectral triple have not been adapted to the twisted case yet. As long as the commutator between the algebra and the Dirac operator is not involved, one can keep the definitions of an ordinary spectral triple, for instance the order-zero condition. In the 1^{st}-order condition (1.2) it is natural to substitute $[D, a]$ with the twisted commutator $[D, a]_\rho$. The question is whether to twist the commutator with Jb^*J^{-1} as well. As explained in [17, Prop. 3.4], the set Ω^1_D of twisted 1-forms, that is all the operators of the form

$$\mathbb{A} = \sum_i b^i [D, a_i]_\rho, \tag{3.2}$$

is a $\mathcal{A}$-bimodule for the left and right actions

$$a \cdot \omega \cdot b := \rho(a)\omega b \quad \forall a, b \in \mathcal{A}, \omega \in \Omega^1_D. \tag{3.3}$$

Therefore it is natural to twist the commutator with JbJ^{-1}. As pointed out below propositions 3.4 and 3.5, this choice is also the one which is efficient for our purposes. Furthermore we assume that ρ is a $*$-automorphism that commutes with the real structure J, which permits us to define the twisted version of the 1^{st}-order condition as follows.

Definition 3.1. *A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D, \rho)$ with real structure J satisfies the twisted 1^{st}-order condition if and only if*

$$[[D, a]_\rho,\, JbJ^{-1}]_\rho = [D, a]_\rho\, JbJ^{-1} - J\rho(b)J^{-1}[D, a]_\rho = 0 \qquad \forall a, b \in \mathcal{A}. \tag{3.4}$$

3.1 Representation

For reasons explained in § 6.1, it is convenient to work with the other natural representation of the grand algebra than the one used in [23]. Namely instead of (2.31) one asks that quaternions carry the chiral index s of spinors while the complex matrices carry the (anti)-particle index:

$$Q = Q^{t\beta}_{s\alpha}, \quad M = M^{\dot{t}\mathrm{J}}_{\dot{s}\mathrm{I}}. \tag{3.5}$$

Explicitly, the representation of the grand algebra $\mathcal{A}_G$ is

$$Q = \begin{pmatrix} Q_r^r & Q_r^l \\ Q_l^r & Q_l^l \end{pmatrix}_{st} \in M_4(\mathbb{H}), \quad M = \begin{pmatrix} M_{\dot 0}^{\dot 0} & M_{\dot 0}^{\dot 1} \\ M_{\dot 1}^{\dot 0} & M_{\dot 1}^{\dot 1} \end{pmatrix}_{\dot s\dot t} \in M_8(\mathbb{C}), \tag{3.6}$$

where for any $s,t \in \{l,r\}$ and $\dot s, \dot t \in \{\dot 0, \dot 1\}$ one defines

$$Q_s^t = \begin{pmatrix} Q_{sa}^{ta} & Q_{sa}^{tb} & Q_{sa}^{tc} & Q_{sa}^{td} \\ Q_{sb}^{ta} & Q_{sb}^{tb} & Q_{sb}^{tc} & Q_{sb}^{td} \\ Q_{sc}^{ta} & Q_{sc}^{tb} & Q_{sc}^{tc} & Q_{sc}^{td} \\ Q_{sd}^{ta} & Q_{sd}^{tb} & Q_{sd}^{tc} & Q_{sd}^{td} \end{pmatrix}_{\alpha\beta} \in M_2(\mathbb{H}), \quad M_{\dot s}^{\dot t} = \begin{pmatrix} M_{\dot s0}^{\dot t0} & M_{\dot s0}^{\dot t1} & M_{\dot s0}^{\dot t2} & M_{\dot s0}^{\dot t3} \\ M_{\dot s1}^{\dot t0} & M_{\dot s1}^{\dot t1} & M_{\dot s1}^{\dot t2} & M_{\dot s1}^{\dot t3} \\ M_{\dot s2}^{\dot t0} & M_{\dot s2}^{\dot t1} & M_{\dot s2}^{\dot t2} & M_{\dot s2}^{\dot t3} \\ M_{\dot s3}^{\dot t0} & M_{\dot s3}^{\dot t1} & M_{\dot s3}^{\dot t2} & M_{\dot s3}^{\dot t3} \end{pmatrix}_{\mathrm{IJ}} \in M_4(\mathbb{C}).$$

Here we use a,b,c,d to denote the value of the flavor index α. On the remaining indices, Q and M act trivially, that is as the identity operator. The representation of $A = (Q,M) \in \mathcal{A}_G$ on H_F is thus

$$A_{\mathsf{D}s\mathrm{J}\dot s\alpha}^{\mathsf{C}t\mathrm{I}\dot t\beta} = \left(\delta_{0\dot s\mathrm{J}}^{\mathsf{C}\dot t\mathrm{I}}\, Q_{s\alpha}^{t\beta} + \delta_1^{\mathsf{C}} M_{\dot s\mathrm{J}}^{\dot t\mathrm{I}} \delta_{s\alpha}^{t\beta}\right) = \begin{pmatrix} \delta_{\dot s\mathrm{J}}^{\dot t\mathrm{I}}\, Q_{s\alpha}^{t\beta} & 0_{64} \\ 0_{64} & M_{\dot s\mathrm{J}}^{\dot t\mathrm{I}}\, \delta_{s\alpha}^{t\beta} \end{pmatrix}_{\mathsf{CD}}. \tag{3.7}$$

One easily checks the order-zero condition (2.2): with $A = (R,N) \in \mathcal{A}_G$, a generic element of the opposite algebra is

$$JAJ^{-1} = -JAJ = \begin{pmatrix} -\delta_{s\alpha}^{t\beta}\,(\tau\bar N\tau)_{\dot s\mathrm{J}}^{\dot t\mathrm{I}} & 0_{64} \\ 0_{64} & \delta_{\dot s\mathrm{J}}^{\dot t\mathrm{I}}\,(\eta\bar R\eta)_{s\alpha}^{t\beta} \end{pmatrix}_{\mathsf{CD}} \tag{3.8}$$

where the bar denotes the complex conjugate and we used

$$\mathcal{J}R\mathcal{J} := (\tau^2)_{\dot s}^{\dot t}\,(\eta\bar R\eta)_{s\alpha}^{t\beta} = -\delta_{\dot s}^{\dot t}\,(\eta\bar R\eta)_{s\alpha}^{t\beta}, \quad \mathcal{J}N\mathcal{J} := (\eta^2)_s^t\,(\tau\bar N\tau)_{\dot s\mathrm{J}}^{\dot t\mathrm{I}} = \delta_s^t(\tau\bar N\tau)_{\dot s\mathrm{J}}^{\dot t\mathrm{I}}. \tag{3.9}$$

Obviously (3.7) commutes with (3.8).

Lemma 3.2. *The biggest subalgebra of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ that satisfies the grading condition* (2.1) *and has bounded commutator with $\not{D}$ is the left-right algebra $\mathcal{A}_{LR}$ given in* (2.27).

Proof. By (2.25), for the quaternion sector $[\Gamma, A] = 0$ amounts to asking $[\eta_s^t \eta_\alpha^\beta, Q_{s\alpha}^{t\beta}] = 0$. This imposes

$$Q = \begin{pmatrix} Q_r^r & 0_4 \\ 0_4 & Q_l^l \end{pmatrix}_{st} \tag{3.10}$$

where

$$Q_r^r = \begin{pmatrix} q_R^r & 0_2 \\ 0_2 & q_L^r \end{pmatrix}_{\alpha\beta}, \; Q_l^l = \begin{pmatrix} q_R^l & 0_2 \\ 0_2 & q_L^l \end{pmatrix}_{\alpha\beta} \quad \text{with } q_R^r, q_L^r, q_R^l, q_L^l \in \mathbb{H}. \tag{3.11}$$

For matrices, one asks $[\delta_{\dot s\mathrm{J}}^{\dot t\mathrm{I}}, M_{\dot s\mathrm{J}}^{\dot t\mathrm{I}}] = 0$ which is trivially satisfied. So the grading condition $[\Gamma, A] = 0$ imposes the reduction of $\mathcal{A}_G$ to

$$\mathcal{B}_{LR} := (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus M_8(\mathbb{C}). \tag{3.12}$$

For $A = (Q,M) \in C^\infty(\mathcal{M}) \otimes \mathcal{B}_{LR}$, the boundedness of the commutator §

$$[\not{D}, A] = \begin{pmatrix} \delta_J^I\,[\not\partial, Q] & 0_{64} \\ 0_{64} & \delta_\alpha^\beta\,[\not\partial, M] \end{pmatrix}_{\mathsf{CD}} \tag{3.13}$$

§To lighten notation, we omit the trivial indices in the product (hence in the commutators) of operators. From (3.5) one knows that Q carries the indices $s\alpha$ while γ^μ carries $s\dot s$, hence $[\not\partial, Q]$ carries indices $s\dot s\alpha$ and should be written $[\delta_\alpha^\beta \not\partial, \delta_{\dot s}^{\dot t} Q]$. As well, $[\not\partial, M]$ carries indices $s\dot s I$ and holds for $[\delta_I^J \not\partial, \delta_s^t M]$.

means that

$$[\partial\!\!\!/, Q] = -i\gamma^\mu(\nabla^S_\mu Q) - i[\gamma^\mu, Q]\nabla^S_\mu \quad \text{and} \quad [\partial\!\!\!/, M] = -i\gamma^\mu(\nabla^S_\mu M) - i[\gamma^\mu, M]\nabla^S_\mu \tag{3.14}$$

are bounded. This is obtained if and only if Q and M commute with all the Dirac matrices, i.e. are proportional to $\delta^{t\dot{t}}_{s\dot{s}}$. For Q this means $Q^r_r = Q^l_l$ in (3.10), hence the reductions

$$\mathbb{H}^r_R \oplus \mathbb{H}^l_R \to \mathbb{H}_R, \quad \mathbb{H}^r_L \oplus \mathbb{H}^l_L \to \mathbb{H}_L. \tag{3.15}$$

For M, this means that all the components $M^{\dot{t}}_{\dot{s}}$ in (3.6) are equal, that is the reduction

$$M_8(\mathbb{C}) \to M_4(\mathbb{C}). \tag{3.16}$$

Therefore $\mathcal{B}_{LR}$ is reduced to $\mathcal{A}_{LR}$, acting diagonally on spinors. ■

This lemma is nothing but a restatement of Prop. 2.1 in the peculiar representation (3.7) and taking into account the grading condition. Nevertheless, it is useful to have it explicitly, in order to understand how to get rid of the unboundedness of the commutator. It is also worth stressing the difference with the representation (2.31), for which the grading breaks both matrices and quaternions and reduces $\mathcal{A}_G$ to $\mathcal{A}'_G$. Here only quaternions are broken by the grading.

To cure the unboundedness of the commutator, the idea we propose is the following: impose the reduction (3.16) by hand, and deal with the unboundedness of $[\partial\!\!\!/, Q]$ thanks to a twist. This is a "middle term solution": imposing by hand both reductions (3.16) and (3.15) is not interesting from the grand algebra point of view, since it brings us back to an almost commutative geometry where spinorial and internal indices are not mixed; solving both the unboundedness of $[\partial\!\!\!/, Q]$ and $[\partial\!\!\!/, M]$ by a twist yields some complications discussed in §6.1. The remarkable point is that this middle term solution is sufficient to obtain the σ-field by a fluctuation that respects the twisted first-order condition of definition 3.1.

3.2 The twist and the first-order condition for the free Dirac

Imposing (3.16) on the grand algebra $\mathcal{A}_G$ reduced by the grading to $\mathcal{B}_{LR}$ yields

$$\mathcal{B}' := (\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{H}^l_R \oplus \mathbb{H}^r_R) \oplus M_4(\mathbb{C}). \tag{3.17}$$

An element $A = (Q, M)$ of $\mathcal{B}'$ is given by (3.7) where Q is as in (3.10) while M in (3.5) is proportional to $\delta^{\dot{t}}_{\dot{s}}$:

$$M = \delta^{\dot{t}}_{\dot{s}} M^{\mathrm{I}}_{\mathrm{J}} \in M_4(\mathbb{C}). \tag{3.18}$$

The algebra $\mathcal{B}'$ contains the algebra of the standard model $\mathcal{A}_{sm}$, and still has a part (the quaternion) that acts in a non-trivial way on the spin degrees of freedom. In this sense $\mathcal{B}'$ is still from the grand algebra side, even if it is "not so grand".

Let ρ be the automorphism of $(\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{H}^l_R \oplus \mathbb{H}^r_R)$ that exchanges Q^r_r and Q^l_l in (3.10), that is the exchange

$$\mathbb{H}^r_R \leftrightarrow \mathbb{H}^l_R, \quad \mathbb{H}^r_L \leftrightarrow \mathbb{H}^l_L. \tag{3.19}$$

This means in components

$$\rho\left(\begin{pmatrix} Q^r_r & 0_4 \\ 0_4 & Q^l_l \end{pmatrix}_{st}\right) = \begin{pmatrix} Q^l_l & 0_4 \\ 0_4 & Q^r_r \end{pmatrix}_{st}. \tag{3.20}$$

Lemma 3.3. *Denote by the same letter the extension of* ρ *to* $C^\infty(\mathcal{M})\otimes(\mathbb{H}_L^l\oplus\mathbb{H}_L^r\oplus\mathbb{H}_R^l\oplus\mathbb{H}_R^r)$. *For any* μ *one has*

$$\gamma^\mu Q = \rho(Q)\gamma^\mu, \quad \gamma^\mu\rho(Q) = Q\gamma^\mu. \tag{3.21}$$

Thus

$$[\partial\!\!\!/, Q]_\rho = -i\gamma^\mu(\nabla_\mu^S Q). \tag{3.22}$$

Proof. Writing explicitly the δ's, one gets

$$\gamma^\mu Q = \left(\delta_\alpha^\beta \begin{pmatrix} 0_2 & \sigma^\mu \\ \tilde\sigma^\mu & 0_2 \end{pmatrix}_{st}\right)\left(\begin{pmatrix} Q_r^r & 0_4 \\ 0_4 & Q_l^l \end{pmatrix}_{st} \delta_{\dot s}^{\dot t}\right) = \begin{pmatrix} 0_8 & \sigma^\mu Q_l^l \\ \bar\sigma^\mu Q_r^r & 0_8 \end{pmatrix}_{st} \tag{3.23}$$

$$= \left(\begin{pmatrix} Q_l^l & 0_4 \\ 0_4 & Q_r^r \end{pmatrix}_{st} \delta_{\dot s}^{\dot t}\right)\left(\delta_\alpha^\beta \begin{pmatrix} 0_2 & \sigma^\mu \\ \tilde\sigma^\mu & 0_2 \end{pmatrix}_{st}\right) = \rho(Q)\gamma^\mu. \tag{3.24}$$

The second part of (3.21) follows because $\rho^2 = \mathbb{I}$. Eq. (3.22) comes from

$$[\partial\!\!\!/, Q]_\rho = -i\gamma^\mu(\nabla_\mu^S Q) - i[\gamma^\mu, Q]_\rho \nabla_\mu^S,$$

where the second term is zero by (3.21). ■

We still denote by the same letter the extension of ρ to $C^\infty(\mathcal{M})\otimes\mathcal{B}'$:

$$\rho((Q,M)) := ((\rho(Q), M). \tag{3.25}$$

Proposition 3.4. $(C^\infty(\mathcal{M})\otimes\mathcal{B}', \mathcal{H}, D\!\!\!\!/, \rho)$ *together with the graduation* Γ *in* (2.11) *and the real structure* J *in* (2.9) *is a graded twisted spectral triple which satisfies the twisted first-order condition of definition 3.1.*

Proof. Let $A = (Q,M) \in C^\infty(\mathcal{M})\otimes\mathcal{B}'$. The twisted version of (3.13) is

$$[D\!\!\!\!/, A]_\rho = \begin{pmatrix} \delta_J^I\,[\partial\!\!\!/, Q]_\rho & 0_{64} \\ 0_{64} & \delta_\alpha^\beta\,[\partial\!\!\!/, M] \end{pmatrix}_{\mathsf{CD}}. \tag{3.26}$$

From (3.18) and (3.7) M commutes with γ^μ, so that the second equation in (3.14) reduces to

$$[\partial\!\!\!/, M] = -i\gamma^\mu(\nabla_\mu^S M), \tag{3.27}$$

which is a bounded operator. By lemma 3.3, $[\partial\!\!\!/, Q]_\rho = -i\gamma^\mu(\partial_\mu Q)$ is bounded as well. Hence $(C^\infty(\mathcal{M})\otimes\mathcal{B})', \mathcal{H}, D\!\!\!\!/, \rho)$ together with Γ form a graded twisted spectral triple.

We now examine the twisted first-order condition (3.1). Let $B = (R,N) \in C^\infty(\mathcal{M})\otimes\mathcal{B}'$. A generic element of the algebra opposite to $C^\infty(\mathcal{M})\otimes\mathcal{B}'$ is

$$JBJ^{-1} = -JBJ = \begin{pmatrix} \delta_{s\alpha}^{t\beta}\bar N & 0_{64} \\ 0_{64} & \delta_{\dot s \mathrm{I}}^{\dot t \mathrm{J}}\bar R \end{pmatrix}_{\mathsf{CD}} \tag{3.28}$$

where we used (3.8) and noticed that for R as in (3.10) and N as in (3.18) one has

$$(\eta\bar R\eta)_{s\alpha}^{t\beta} = \bar R_{s\alpha}^{t\beta}, \quad (\tau\bar N\tau)_{\dot s\mathrm{J}}^{\dot t\mathrm{I}} = -\bar N_{\dot s\mathrm{J}}^{\dot t\mathrm{I}}. \tag{3.29}$$

As well, one has

$$J\rho(B)J^{-1} = -J\rho(B)J = \begin{pmatrix} \delta_{s\alpha}^{t\beta}\bar N & 0_{64} \\ 0_{64} & \delta_{\dot s\mathrm{I}}^{\dot t\mathrm{J}}\rho(\bar R) \end{pmatrix}_{\mathsf{CD}}. \tag{3.30}$$

Thus $[\not{D}, A]_\rho JBJ^{-1} - J\rho(B)J^{-1}[\not{D}, A]_\rho$ is a diagonal matrix with components

$$[\delta^I_J\,[\not{\partial}, Q]_\rho,\ \delta^{t\beta}_{s\alpha}\bar{N}], \qquad \delta^\beta_\alpha\,[\not{\partial}, M]\,\delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\bar{R} - \delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\rho(\bar{R})\,\delta^\beta_\alpha\,[\not{\partial}, M]. \tag{3.31}$$

The first term vanishes because the only non-trivial index carries by $\bar{N}$ is IJ. The second term is (omitting the deltas and a global $-i$ factor)

$$\begin{aligned}&\begin{pmatrix} 0_8 & \sigma^\mu(\partial_\mu M) \\ \bar{\sigma}^\mu(\partial_\mu M) & 0_8 \end{pmatrix}_{st}\begin{pmatrix} \bar{R}^r_r & 0_8 \\ 0_8 & \bar{R}^l_l \end{pmatrix}_{st} - \begin{pmatrix} \bar{R}^l_l & 0_8 \\ 0_8 & \bar{R}^r_r \end{pmatrix}_{st}\begin{pmatrix} 0_8 & \sigma^\mu(\partial_\mu M) \\ \bar{\sigma}^\mu(\partial_\mu M) & 0_8 \end{pmatrix}_{st} \\ &= \begin{pmatrix} 0_8 & [\sigma^\mu(\partial_\mu M), \bar{R}^l_l] \\ \left[\bar{\sigma}^\mu(\partial_\mu M), \bar{R}^r_r\right] & 0_8 \end{pmatrix}_{st}\end{aligned} \tag{3.32}$$

which vanishes because R only non-trivial index is $\alpha\beta$ while $\left[\bar{\sigma}^\mu(\partial_\mu M), \bar{R}^r_r\right]$ is proportional to δ^β_α. ■

3.3 Twisted first-order condition for the Majorana-Dirac operator

We individuate a subalgebra $\mathcal{B}$ of $\mathcal{B}'$ such that a twisted fluctuation of the *Majorana-Dirac operator* D_M in (2.14) by $\mathcal{B}$ satisfies the twisted first-order condition. Since we are working with one generation of fermions only, in (2.7) the Majorana mass matrix $\mathcal{M}_R$ in D_R is $\Xi^\beta_\alpha k_R$, where

$$\Xi = \begin{pmatrix} 1 & 0 \\ 0 & 0_3 \end{pmatrix} \tag{3.33}$$

denotes the projection on the first component. Therefore

$$D_M = \gamma^5 D_R = \eta^t_s\,\delta^{\dot{t}}_{\dot{s}}\Xi^{\mathrm{I}\beta}_{\mathrm{J}\alpha}\begin{pmatrix} 0 & k_R \\ \bar{k}_R & 0 \end{pmatrix}_{\mathsf{CD}}. \tag{3.34}$$

In this equation the product $\gamma^5 D_R$ is intended with the convention of the footnote p.11, namely this is the tensorial notation $\gamma^{5\,t\dot{t}}_{\ \,s\dot{s}} D_{R\,\mathrm{J}\alpha\mathsf{C}}^{\ \ \beta\mathrm{I}\mathsf{D}}$ in which we omit the indices. In practical, this amounts to omit the tensor product symbol in $\gamma^5 \otimes D_R$, which makes sense because of our choice of viewing the total Hilbert space no longer as the tensor product of spinors by $\mathcal{H}_\mathcal{F}$. These distinctions may seem pedantic here, but they will be important later on, when writing the product $\gamma^\mu X_\mu$ for a vector field X_μ that no longer commutes with the Dirac matrices: $\gamma^\mu X_\mu$ will holds for $\gamma^{\mu\,t\dot{t}}_{\ \ s\dot{s}} X_{\mu\,\mathrm{J}\alpha\mathsf{C}}^{\ \ \beta\mathrm{I}\mathsf{D}}$, while $\gamma^\mu \otimes X_\mu$ no longer makes sense.

Proposition 3.5. *A subalgebra of $\mathcal{B}'$ which satisfies the twisted first-order condition*

$$[[D_M, A]_\rho, JBJ^{-1}]_\rho = 0 \tag{3.35}$$

is

$$\mathcal{B} := \mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{C}^l_R \oplus \mathbb{C}^r_R \oplus M_3(\mathbb{C}). \tag{3.36}$$

Proof. Consider first the subalgebra

$$\tilde{\mathcal{B}} := (\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{C}^l_R \oplus \mathbb{C}^r_R) \oplus (M_3(\mathbb{C}) \oplus \mathbb{C}) \tag{3.37}$$

of $\mathcal{B}'$ obtained by asking that q^l_R, q^r_R in (3.11) are diagonal quaternions, namely

$$q^l_R = \begin{pmatrix} c^l_R & 0 \\ 0 & \bar{c}^l_R \end{pmatrix},\ q^r_R = \begin{pmatrix} c^r_R & 0 \\ 0 & \bar{c}^r_R \end{pmatrix} \text{ with } c^l_R, c^r_R \in \mathbb{C}; \tag{3.38}$$

while M in (3.18) is of the form

$$M = \delta_{\dot s}^{\dot t} \begin{pmatrix} m & 0 \\ 0 & \mathbf{M} \end{pmatrix}_{\mathrm{IJ}} \quad \text{with } m \in \mathbb{C}, \mathbf{M} \in M_3(\mathbb{C}). \tag{3.39}$$

This means that while Q carries non-trivial indices $\dot s, \alpha$, the action of M is non-trivial only in the I index. Define similarly $B = (R, N) \in \tilde{\mathcal{B}}$ with components $d_R^l, d_R^r \in \mathbb{C}$, $n \in \mathbb{C}$, $\mathbf{N} \in M_3(\mathbb{C})$. For any $A, B \in \tilde{\mathcal{B}}$, one has from (3.34) where we write

$$\mathsf{D}_M := \eta_s^t \, \delta_{\dot s}^{\dot t} \, \Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta}, \tag{3.40}$$

and (3.7) (omitting the deltas)

$$[D_M, A]_\rho = \begin{pmatrix} 0_{64} & k_R(\mathsf{D}_M M - \rho(Q)\mathsf{D}_M) \\ \bar k_R(\mathsf{D}_M Q - M\mathsf{D}_M) & 0_{64} \end{pmatrix}_{\mathsf{CD}}. \tag{3.41}$$

By (3.28), (3.30) one obtains

$$[[D_M, A]_\rho, JBJ^{-1}]_\rho = \begin{pmatrix} 0_{64} & k_R\left((\mathsf{D}_M M - \rho(Q)\mathsf{D}_M)\bar R - \bar N(\mathsf{D}_M M - \rho(Q)\mathsf{D}_M)\right) \\ \bar k_R\left((\mathsf{D}_M Q - M\mathsf{D}_M)\bar N - \rho(\bar R)(\mathsf{D}_M Q - M\mathsf{D}_M)\right) & 0_{64} \end{pmatrix}_{\mathsf{CD}}.$$

The terms entering the upper-right components of this matrix are (omitting a global k_R factor)

$$\bar N \mathsf{D}_{\mathsf{M}} M = (\bar N \Xi M)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} (\eta \Xi)_{s\alpha}^{t\beta} = \begin{pmatrix} \bar{\mathsf{n}}\mathsf{m} & 0_4 \\ 0_4 & \bar{\mathsf{n}}\mathsf{m} \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st}, \tag{3.42}$$

$$\bar N \rho(Q) \mathsf{D}_{\mathsf{M}} = (\bar N \Xi)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} \, (\rho(Q)\eta \Xi)_{s\alpha}^{t\beta} = \begin{pmatrix} \bar{\mathsf{n}} & 0_4 \\ 0_4 & \bar{\mathsf{n}} \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \mathsf{c}_R^l & 0_4 \\ 0_4 & -\,\mathsf{c}_R^r \end{pmatrix}_{st}, \tag{3.43}$$

$$\mathsf{D}_{\mathsf{M}} M \bar R = (\Xi M)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} \, (\eta \Xi \bar R)_{s\alpha}^{t\beta} = \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R^r & 0_4 \\ 0_4 & -\bar{\mathsf{d}}_R^l \end{pmatrix}_{st}, \tag{3.44}$$

$$\rho(Q) \mathsf{D}_M \bar R = (\Xi \delta)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} \, (\rho(Q)\eta \Xi R)_{s\alpha}^{t\beta} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \mathsf{c}_R^l \, \bar{\mathsf{d}}_R^r & 0_4 \\ 0_4 & -\mathsf{c}_R^r \, \bar{\mathsf{d}}_R^l \end{pmatrix}_{st}, \tag{3.45}$$

where we defined

$$\mathsf{m} := \begin{pmatrix} m & 0 \\ 0 & 0_3 \end{pmatrix}_{\mathrm{IJ}}, \quad \mathsf{c}_R^r = \begin{pmatrix} c_R^r & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{c}_R^l = \begin{pmatrix} c_R^l & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta} \tag{3.46}$$

and similarly for d_R^r, d_R^l and $\mathbf{n}$. Collecting the various terms, one finds that the upper-right component of $[[D_M, A]_\rho, JBJ^{-1}]_\rho$ vanishes if and only if

$$(c_R^l - m)(\bar d_R^r - \bar n) = 0, \quad (c_R^r - m)(\bar d_R^l - \bar n) = 0. \tag{3.47}$$

Similarly, for the lower-left component of $[[D_M, A]_\rho, JBJ^{-1}]_\rho$ one has

$$\rho(\bar R) M \mathsf{D}_{\mathsf{M}} = (\Xi M)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} (\rho(\bar R)\eta \Xi)_{s\alpha}^{t\beta} = \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R^l & 0_4 \\ 0_4 & -\bar{\mathsf{d}}_R^r \end{pmatrix}_{st}, \tag{3.48}$$

$$\rho(\bar R) \mathsf{D}_M Q = (\Xi \delta)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} \, (\rho(\bar R) \, \eta \Xi \, Q)_{s\alpha}^{t\beta} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \mathsf{c}_R^r \bar{\mathsf{d}}_R^l & 0_4 \\ 0_4 & -\mathsf{c}_R^l \bar{\mathsf{d}}_R^r \end{pmatrix}_{st}, \tag{3.49}$$

$$M \mathsf{D}_M \bar N = (M \Xi \bar N)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} \, (\eta \Xi)_{s\alpha}^{t\beta} = \begin{pmatrix} \bar{\mathsf{n}}\mathsf{m} & 0_4 \\ 0_4 & \bar{\mathsf{n}}\mathsf{m} \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st}, \tag{3.50}$$

$$\mathsf{D}_M Q \bar N = (\Xi \bar N)_{\dot s \mathrm{J}}^{\dot t \mathrm{I}} \, (\eta \Xi Q)_{s\alpha}^{t\beta} = \begin{pmatrix} \bar{\mathsf{n}} & 0_4 \\ 0_4 & \bar{\mathsf{n}} \end{pmatrix}_{\dot s \dot t} \otimes \begin{pmatrix} \mathsf{c}_R^r & 0_4 \\ 0_4 & -\mathsf{c}_R^l \end{pmatrix}_{st}, \tag{3.51}$$

yielding the same condition (3.47). Hence the twisted first-order condition is satisfied as soon as

$$c_R^r = m, \; d_R^r = n, \tag{3.52}$$

which amounts to identify $\mathbb{C}_R^r$ with $\mathbb{C}$. Hence the reduction of $\mathcal{B}'$ to $\mathcal{B}$ as defined in (3.36). ■

One could identify $\mathbb{C}_R^l$ with $\mathbb{C}$, instead of $\mathbb{C}_R^r$, without changing the result. As discussed before definition 3.1, one might also consider a first-order condition where only the commutator with D is twisted, that is

$$[[D_M, A]_\rho, JBJ^{-1}] = 0. \tag{3.53}$$

This is not pertinent in our case however, for this amounts to permuting $\bar{R}_l^l$ with $\bar{R}_r^r$ in - and only in - the second term in (3.32), which then no longer vanishes as soon as $R_r^r \neq R_l^l$.

Proposition 3.5 deals only with the finite dimensional part of the spectral triple. However (3.35) is still satisfied with $A, B \in C^\infty(\mathcal{M}) \otimes \mathcal{B}$ (though, strictly speaking, one can no longer talk of "twisted first-order condition for D_M", for on $L^2(\mathcal{M}) \otimes \mathbb{C}^{128}$ the operator D_M does not have a compact resolvent). Proposition 3.4 is true for the subalgebra $C^\infty(\mathcal{M}) \otimes \mathcal{B}$. Therefore the twisted first-order condition (3.4) with $\mathcal{B}$ is true for $\not{\partial} + D_M$ since it is true for $\not{\partial}$ and D_M independently. This proves the first statement of theorem 1.1.

4 Twisted-covariant Dirac operators

The twisted spectral triple

$$(C^\infty(\mathcal{M}) \otimes \mathcal{B}, \, L^2(\mathcal{M}) \otimes \mathbb{C}^{128}, \, \not{\partial} + D_M; \, \rho) \tag{4.1}$$

of theorem 1.1 solves the problem of the non-boundedness of the commutators $[\not{\partial}, A]$ raised by the non-trivial action of the grand algebra on spinors. But to be of interest, this spectral triple should preserve the property the grand algebra has been invented for, that is generating the field σ by a fluctuation of D_M, or a twisted version of it. As shown in this section this is indeed the case, because although $\mathcal{B}$ is not so grand (it is smaller than $\mathcal{A}_G$), it is neither too small ($C^\infty(\mathcal{M}) \otimes \mathcal{B}$ still has non trivial action on spinors).

4.1 Twisted fluctuation

In analogy with gauge fluctuation of almost commutative geometries described in §2.1, we call *twisted fluctuation of D by $C^\infty(\mathcal{M}) \otimes \mathcal{B}$* the substitution of $D = \not{\partial} + D_M$ with

$$D_{\mathbb{A}} = D + \mathbb{A} + J \, \mathbb{A} \, J^{-1} \tag{4.2}$$

where $\mathbb{A}$ is twisted 1-form

$$\mathbb{A} = B^i [D, A_i]_\rho \quad A_i, B^i \in C^\infty(\mathcal{M}) \otimes \mathcal{B}. \tag{4.3}$$

We do not require $\mathbb{A}$ to be selfadjoint, we only ask that $D_{\mathbb{A}}$ is selfadjoint and called it *twisted-covariant Dirac operator*. It is the sum $D_{\mathbb{A}} = D_X + D_\sigma$ of the twisted-covariant free Dirac operator

$$D_X := \not{\partial} + \not{\mathbb{A}} + J\not{\mathbb{A}}J^{-1} \qquad \not{\mathbb{A}} := B^i[\not{\partial}, A_i]_\rho \tag{4.4}$$

with the twisted-covariant Majorana-Dirac operator

$$D_\sigma := D_M + \mathbb{A}_M + J\mathbb{A}_M J^{-1} \qquad \mathbb{A}_M := B^i[D_M, A_i]_\rho. \tag{4.5}$$

In this section, we compute explicitly D_X and D_σ, and show that they are parametrized by a vector field X_μ and a scalar field $\boldsymbol{\sigma}$.

In the following, $A_i = (Q_i, M_i)$ and $B^i = (R^i, N^i)$ are arbitrary elements of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$, where i a summation index and

$$Q_i = \begin{pmatrix} Q^r_{ri} & 0_4 \\ 0_4 & Q^l_{li} \end{pmatrix}_{st}, \quad M_i = \delta^{\dot t}_{\dot s} \begin{pmatrix} c^r_i & 0 \\ 0 & \mathsf{M}_i \end{pmatrix}_{\mathrm{IJ}} \tag{4.6}$$

with¶ $\mathsf{M}_i \in M_3(\mathbb{C})$ and

$$Q^r_{ri} = \begin{pmatrix} q^r_{Ri} & 0_2 \\ 0_2 & q^r_{Li} \end{pmatrix}_{\alpha\beta}, \; Q^l_{li} = \begin{pmatrix} q^l_{Ri} & 0_2 \\ 0_2 & q^l_{Li} \end{pmatrix}_{\alpha\beta} \tag{4.7}$$

with $q^l_{Li} \in \mathbb{H}^l_L$, $q^r_{Li} \in \mathbb{H}^r_L$ and

$$q^r_{Ri} = \operatorname{diag}(c^r_i, \bar c^r_i), \quad q^l_{Ri} = \operatorname{diag}(c^l_i, \bar c^l_i) \quad \text{with} \quad c^r_i \in \mathbb{C}^r_R,\; c^l_i \in \mathbb{C}^l_R. \tag{4.8}$$

The components R^i, N^i of B^i are defined similarly, with

$$d^{ri} \in \mathbb{C}^r_R,\; d^{li} \in \mathbb{C}^l_R, \quad r^{ri}_L \in \mathbb{H}^r_L,\; r^{ri}_L \in \mathbb{H}^l_L \quad \text{and} \quad \mathsf{N}_i \in M_3(\mathbb{C}). \tag{4.9}$$

4.2 Twisted-covariant free Dirac operator D_X

The twisted fluctuations (4.4) of the free Dirac operator $\partial\!\!\!/$ in (2.20) by $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ are parametrized by a vector field.

Proposition 4.1. *One has*

$$D_X = \partial\!\!\!/ + \mathbb{X}\!\!\!/ \tag{4.10}$$

with

$$\mathbb{X}\!\!\!/ := -i\gamma^\mu \mathbb{X}_\mu, \qquad \mathbb{X}_\mu := \begin{pmatrix} X_\mu & 0_{64} \\ 0_{64} & -\bar X_\mu \end{pmatrix}_{\mathsf{CD}}, \tag{4.11}$$

where we define the bounded-operator valued vector field ‖

$$X_\mu := \delta^I_J\, \rho(R^i)\, \nabla^S_\mu Q_i - \delta^\beta_\alpha\, \bar N^i \nabla^S_\mu \bar M_i \tag{4.12}$$

which commutes with γ^5 and twisted-commutes with γ^ν, that is for all μ, ν one has

$$\gamma^\mu X_\nu = \rho(X_\nu)\gamma^\mu, \quad \gamma^\mu \rho(X_\nu) = X_\nu \gamma^\mu. \tag{4.13}$$

Proof. Given $A_i = (Q_i, M_i)$ and $B^i = (R^i, N^i)$ in $\mathcal{B}$, one gets from (3.26), (3.27) and (3.21)

$$\mathbb{A}\!\!\!/ = -iB^i[\partial\!\!\!/, A_i]_\rho = -i \begin{pmatrix} \delta^I_J\, \gamma^\mu \rho(R^i) \nabla^S_\mu Q_i & 0_{64} \\ 0_{64} & \delta^\beta_\alpha\, \gamma^\mu N^i \nabla^S_\mu M_i \end{pmatrix}_{\mathsf{CD}} \tag{4.14}$$

¶In all this section, the components of the matrices are functions on $\mathcal{M}$. To lighten notation we write $M_3(\mathbb{C})$ instead of $C^\infty(\mathcal{M}) \otimes M_3(\mathbb{C})$. The same is true for the various copies of $\mathbb{H}$ and $\mathbb{C}$.

‖To lighten notations we omit the parenthesis around $\partial_\mu Q_i$ and $\partial_\mu \bar M_i$: the latter are bounded operators and act as matrices, not as differential operators.

where we used that N^i commutes with γ^μ and $R^i\gamma^\mu = \gamma^\mu\rho(R^i)$ (lemma 3.3). By (2.23) one gets

$$J\mathbb{A}J^{-1} = -J\mathbb{A}J = i\begin{pmatrix} \delta_\alpha^\beta\,\gamma^\mu\bar{N}^i\nabla_\mu^S\bar{M}_i & 0_{64} \\ 0_{64} & \delta_J^I\gamma^\mu\rho(\bar{R}^i)\nabla_\mu^S\bar{Q}_i \end{pmatrix}_{\mathsf{CD}} \tag{4.15}$$

where we used that $\mathcal{J}$ anti-commutes with the γ's matrices and commute with ∇_μ^S** so that, inserting $\mathcal{J}^2 = -\mathbb{I}$ before ∇_μ^S, one obtains

$$\mathcal{J}\left(\gamma^\mu N^i\nabla_\mu^S M_i\right)\mathcal{J} = \gamma^\mu(\mathcal{J}N^i\mathcal{J})\,\nabla_\mu^S(\mathcal{J}M_i\mathcal{J}) = \gamma^\mu\bar{N}^i\nabla_\mu^S\bar{M}_i, \tag{4.16}$$

$$\mathcal{J}(\gamma^\mu\rho(R^i)\partial_\mu Q_i)\mathcal{J} = \gamma^\mu(\mathcal{J}\rho(R^i)\mathcal{J})\nabla_\mu^S(\mathcal{J}Q_i\mathcal{J}) = \gamma^\mu\rho(\bar{R}^i)\nabla_\mu^S\bar{Q}_i. \tag{4.17}$$

In both equations above the last term comes from (3.9), noticing that $\rho(R_i)$ and Q_i are now diagonal in the st index and so commute with η, while N_i, M_i are proportional to $\delta_{\dot{s}}^{\dot{t}}$, hence commute with τ. Summing up (4.14) and (4.15), one obtains

$$\mathbb{A} + J\mathbb{A}J^{-1} = -i\gamma^\mu\mathbb{X}_\mu \tag{4.18}$$

with $\mathbb{X}_\mu$ as in (4.12).

X_μ commuting with γ^5 is a consequence of the breaking of $\mathcal{A}_G$ by the grading condition and can be checked explicitly using (4.7) and (3.7). Eq. (4.13) follows from lemma 3.3 and the definition (4.12) of X_μ. ∎

Lemma 4.2. *D_X is selfadjoint, and called twisted-covariant free Dirac operator, if and only if for any $\mu = 0, 1, 2, 3$ one has*

$$\rho(X_\mu) = -X_\mu^\dagger. \tag{4.19}$$

Proof. In the st indices, X_μ is a block diagonal matrix which is proportional to $\delta_{\dot{s}}^{\dot{t}}$,

$$X_\mu = \delta_{\mathrm{J}\dot{s}}^{\mathrm{I}\dot{t}}\begin{pmatrix} R_{\;l}^{il}\nabla_\mu^S Q_{\;ir}^{\,r} & 0_4 \\ 0_4 & R_{\;r}^{ir}\nabla_\mu^S Q_{\;il}^{\,l} \end{pmatrix}_{st} - \delta_{\alpha s\dot{s}}^{\beta t\dot{t}}\bar{N}^i\nabla_\mu^S\bar{M}_i =: \delta_{\dot{s}}^{\dot{t}}\begin{pmatrix} X_\mu^r & 0_{32} \\ 0_{32} & X_\mu^l \end{pmatrix}_{st}, \tag{4.20}$$

thus

$$\gamma^\mu X_\mu = \begin{pmatrix} 0_{32} & \sigma^\mu X_\mu^l \\ \tilde{\sigma}^\mu X_\mu^r & 0_{32} \end{pmatrix}_{st}, \quad (\gamma^\mu X_\mu)^\dagger = \begin{pmatrix} 0_{32} & \sigma^\mu(X_\mu^r)^\dagger \\ \tilde{\sigma}^\mu(X_\mu^l)^\dagger & 0_{32} \end{pmatrix}_{st} = \gamma^\mu\rho(X_\mu^\dagger), \tag{4.21}$$

where we used that X_μ commutes with the σ's matrices and $(\sigma^\mu)^\dagger = \tilde{\sigma}^\mu$. Therefore $\gamma^\mu X_\mu$ is selfadjoint iff

$$\sigma^\mu(X_\mu^r)^\dagger = \sigma^\mu X_\mu^l. \tag{4.22}$$

Since $\mathrm{Tr}\,\bar{\sigma}^\nu\sigma^\mu = 2\delta_\nu^\mu$ and both X_μ^r and X_μ^l are proportional to $\delta_{\dot{s}}^{\dot{t}}$, the partial trace on the $\dot{s}\dot{t}$ indices of the above equation, where both side have been multiplied by $\bar{\sigma}^\lambda$, yields $(X_\mu^r)^\dagger = X_\mu^l$ for any μ, that is

$$X_\mu^\dagger = \rho(X_\mu). \tag{4.23}$$

The lemma is obtained noticing that D_X is selfadjoint if and only if $i\gamma^\mu X^\mu$ is selfadjoint, that is $\gamma^\mu X^\mu$ is anti-selfadjoint. ∎

** $\{\mathcal{J}, \gamma^\mu\} = i(\gamma^0\gamma^2\bar{\gamma}^\mu + \gamma^\mu\gamma^0\gamma^2)cc = 0$ because $\bar{\gamma}^\mu = -\gamma^\mu$ for $\mu = 1, 3$, $\bar{\gamma}^\mu = \gamma^\mu$ for $\mu = 0, 2$. That $\mathcal{J}$ commutes with the spin covariant derivative ∇_μ^S is a classical result, see e.g. [36, Prop. 4.18].

4.3 Twisted-covariant Majorana-Dirac operator D_σ

Twisted fluctuations of the Majorana-Dirac operator D_M are parametrized by a scalar field $\boldsymbol{\sigma}$. To show that, we begin by a short calculation in tensorial notations.

Lemma 4.3. *For $A = (Q, M) \in \mathcal{B}$ with components $c^r, c^l \in \mathbb{C}$ as in (4.8), one has*

$$[D_M, A]_\rho = \begin{pmatrix} 0_2 & k_R(c^r - c^l)\mathcal{S} \\ \bar{k}_R(c^r - c^l)\mathcal{S}' & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \tag{4.24}$$

where

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{st}, \quad \mathcal{S}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{st}. \tag{4.25}$$

Proof. Computing explicitly (3.41) with notations (3.46) and omitting k_R and $\bar{k}_R$ yields

$$\begin{aligned}
\mathsf{D_M} M - \rho(Q)\mathsf{D_M} &= (\Xi M)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} (\eta\Xi)^{t\beta}_{s\alpha} - (\Xi\delta)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\, (\rho(Q)\eta\Xi)^{t\beta}_{s\alpha} \\
&= \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \Xi^{\mathrm{I}}_{\mathrm{J}} & 0_4 \\ 0_4 & -\Xi^{\mathrm{I}}_{\mathrm{J}} \end{pmatrix}_{st} - \begin{pmatrix} \Xi^{\beta}_{\alpha} & 0_4 \\ 0_4 & \Xi^{\beta}_{\alpha} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^l_R & 0_4 \\ 0_4 & -\mathsf{c}^r_R \end{pmatrix}_{st} \\
&= \begin{pmatrix} \begin{pmatrix} (m - c^l_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & (m - c^l_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} & 0_{32} \\ 0_{32} & \begin{pmatrix} -(m - c^r_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & -(m - c^r_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} \end{pmatrix}_{st}
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
\mathsf{D_M}^\dagger Q - M \mathsf{D}^\dagger_\nu &= (\Xi\delta)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\, (\eta\Xi Q)^{t\beta}_{s\alpha} - (\Xi M)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} (\eta\Xi)^{t\beta}_{s\alpha} \\
&= \begin{pmatrix} \Xi^{\beta}_{\alpha} & 0_4 \\ 0_4 & \Xi^{\beta}_{\alpha} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\mathsf{c}}^r_R & 0_4 \\ 0_4 & -\bar{k}_R \mathsf{c}^l_R \end{pmatrix}_{st} - \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\Xi}^{\mathrm{I}}_{\mathrm{J}} & 0_4 \\ 0_4 & -\bar{\Xi}^{\mathrm{I}}_{\mathrm{J}} \end{pmatrix}_{st} \\
&= \begin{pmatrix} \begin{pmatrix} \bar{(}c^r_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & \bar{R}(c^r_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} & 0_{32} \\ 0_{32} & \begin{pmatrix} -\bar{(}c^l_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & -\bar{(}c^l_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}\dot{t}} \end{pmatrix}_{st}.
\end{aligned} \tag{4.27}$$

Identifying c^r_R with m following (3.52) yields the result, where we drop the index R to match notation (4.8). ■

Proposition 4.4. *The selfadjoint twisted fluctuation (4.5) of the Majorana-Dirac operator $D_M = \gamma^5 D_R$ by $C^\infty(\mathcal{M}) \otimes \mathcal{B}$, called twisted-covariant Majorana-Dirac operator, is*

$$D_\sigma = \boldsymbol{\sigma}\gamma^5 D_R \tag{4.28}$$

where

$$\boldsymbol{\sigma} = (\mathbb{I} + \gamma^5\phi) \tag{4.29}$$

with ϕ a real scalar field.

Proof. Let $B^i = (R^i, N^i)$ as in (4.9). From lemma 4.3 one gets

$$\mathbb{A}_M = B^i[D_M, A_i]_\rho = \phi \begin{pmatrix} 0_2 & k_R\,\mathcal{S} \\ \bar{k}_R\,\mathcal{S}' & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\, \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha} \tag{4.30}$$

where

$$\phi := d^{ir}(c^r_i - c^l_i). \tag{4.31}$$

One has $\mathcal{J}(\mathcal{S}\delta^{\dot{t}}_{\dot{s}})\mathcal{J} = -\mathcal{S}\delta^{\dot{t}}_{\dot{s}}$ and $\mathcal{J}(\mathcal{S}'\delta^{\dot{t}}_{\dot{s}})\mathcal{J} = -\mathcal{S}'\delta^{\dot{t}}_{\dot{s}}$. Hence

$$J\mathbb{A}_M J^{-1} = -J\mathbb{A}_M J = \bar{\phi}\begin{pmatrix} 0_2 & k_R\mathcal{S}' \\ \bar{k}_R\mathcal{S} & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\,\Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha} \tag{4.32}$$

so that

$$D_M + \mathbb{A}_M + J\mathbb{A}_M J^{-1} = \begin{pmatrix} 0_2 & k_R\left(\eta^t_s + \phi\,\mathcal{S} + \bar{\phi}\,\mathcal{S}'\right) \\ \bar{k}_R\left(\eta^t_s + \phi\,\mathcal{S}' + \bar{\phi}\mathcal{S}\right) & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\,\Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha}. \tag{4.33}$$

It is selfadjoint if and only if $\phi = \bar{\phi}$. Then

$$D_\sigma := D_M + \mathbb{A}_M + J\mathbb{A}_M J^{-1} = \begin{pmatrix} 0_4 & k_R(\gamma^5 + \phi\mathbb{I}_4) \\ \bar{k}_R(\gamma^5 + \phi\mathbb{I}_4) & 0_4 \end{pmatrix}_{\mathsf{CD}} \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha}, \tag{4.34}$$

$$= (\gamma^5 + \phi\mathbb{I})D_R. \tag{4.35}$$

Factorizing by γ^5, one gets the result. ■

Propositions 4.1 and 4.4 prove the second statement of theorem 1.1. The field $\boldsymbol{\sigma}$ in (4.29) is slightly different from the one obtained in [23] by a non-twisted fluctuation of D_M by $\mathcal{A}_{sm} \otimes\ C^\infty(\mathcal{M})$, namely

$$\sigma = (1+\phi)\mathbb{I}. \tag{4.36}$$

We comment on that in the conclusion.

5 Breaking of the grand symmetry to the standard model

We prove the third and fourth point of theorem 1.1 by computing the spectral action for the twisted-covariant Dirac operator

$$D_{\mathbb{A}} = D_X + D_{\boldsymbol{\sigma}}, \tag{5.1}$$

where D_X and $D_{\boldsymbol{\sigma}}$ have been obtained by twisted fluctuation of $\not{\partial}$ and D_M in (4.10) and (4.28). More precisely, we show that the potential part of this action is minimum when the Dirac operator $\not{\partial} + D_M$ of the twisted spectral triple is fluctuated by a subalgebra of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ which is invariant under the automorphism ρ. The maximal such sub-algebra is precisely the algebra $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}$ of the standard model. Indeed by (3.25) an element (Q, M) of $\mathcal{B}$ is invariant by the automorphism ρ if and only if

$$\rho(Q) = Q, \tag{5.2}$$

which means $\mathbb{H}^r_R = \mathbb{H}^l_R$ and $\mathbb{C}^r_L = \mathbb{C}^l_L$, that is $(Q, M) \in \mathcal{A}_{sm}$.

We begin by some recalls on the spectral action, then we establish the generalized Lichnerowicz formula for $D_{\mathbb{A}}$ and finally we study the potential for the vector field, the scalar field, and their interaction.

5.1 Spectral action

A striking application of noncommutative geometry to physics is to give a gravitational interpretation of the standard model [14]. By this, one intends that the bosonic part of the SM Lagrangian is deduced from an action which is purely geometric, that is which depends only the spectrum of the covariant Dirac operator D_A (2.12) of the almost commutative geometry of the standard model. The most obvious way to define such an action consists in counting the eigenvalues lower than a given energy scale Λ. This is the spectral action [6]

$$S = \operatorname{Tr} f\left(\frac{D_A^2}{\Lambda^2}\right) \tag{5.3}$$

where f is a positive cutoff function, usually the (smoothened) characteristic function on the interval $[0,1]$. It has an asymptotic expansion in power series of Λ,

$$\sum_{n\geq 0} f_{4-n}\, \Lambda^{4-n}\, a_n(D_A^2/\Lambda^2) \tag{5.4}$$

where the f_n are the momenta of f and the a_n the Seeley-de Witt coefficients which are nonzero only for n even. To compute these coefficients, one usually starts with D_A^2 written as an elliptic operator of Laplacian type,

$$D_A^2 = -(g^{\mu\nu}\, \nabla^S_\mu\, \nabla^S_\nu + \alpha^\mu\, \nabla^S_\mu + \beta), \tag{5.5}$$

and introduces the covariant derivative

$$\nabla_\mu := \nabla^S_\mu + \omega_\mu \tag{5.6}$$

associated with the connection 1-form

$$\omega_\mu := \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\right). \tag{5.7}$$

This yields the generalized Lichnerowicz formula

$$D_A^2 = -\nabla_\mu \nabla^\mu - E \tag{5.8}$$

where

$$E := \beta - g^{\mu\nu}\left(\nabla^S_\mu\, \omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right). \tag{5.9}$$

The coefficients a_n are then computed by usual technics of heat kernel. The first ones are [29,37]

$$\begin{aligned}
a_0 &= \frac{1}{16\pi^2}\int \mathrm{d}x^4 \sqrt{g}\, \operatorname{Tr}(Id), && (5.10)\\
a_2 &= \frac{1}{16\pi^2}\int \mathrm{d}x^4 \sqrt{g}\, \operatorname{Tr}\left(-\frac{R}{6} + E\right)\\
a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4 \sqrt{g}\, \operatorname{Tr}(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} && (5.11)\\
&\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu}) && (5.12)
\end{aligned}$$

where $R_{\mu\nu}$ is the Ricci tensor, $-R$ the scalar curvature and $\Omega_{\mu\nu}$ the curvature of the connection ω_μ. Applied to the spectral triple (2.3) of the standard model, fluctuated according to (2.12), the expansion (5.4) yields the bosonic part of Lagrangian of the standard model - including the Higgs - minimally coupled with gravity [9, Sect. 4.1]. For the fermionic action and how it is related to the spectral action see [1], [2] and for a complete and pedagogical treatment of the subject, see the recent book [36].

Here we compute the asymptotic expansion (5.4) for the twisted covariant Dirac operator $D_{\mathbb{A}}$ (5.1). For simplicity we restrict to the flat case $g^{\mu\nu} = \delta^{\mu\nu}$, so that (5.6), (5.7) and (5.9) reduce to

$$\nabla_\mu = \partial_\mu + \omega_\mu\,, \quad \omega_\mu = \frac{1}{2} g_{\mu\nu}\alpha^\nu\,, \quad E = \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu\right), \tag{5.13}$$

that is

$$\nabla_\mu = \partial_\mu + \frac{1}{2}\alpha_\mu, \quad E = \beta - \frac{1}{4}\alpha\cdot\alpha - \frac{1}{2}\partial_\mu\alpha^\mu \tag{5.14}$$

where $\alpha\cdot\alpha := g_{\mu\nu}\alpha^\mu\alpha^\nu$ denotes the inner product defined by the Riemannian metric. Furthermore, in all this section, we consider fluctuations such that D_X and $D_{\boldsymbol{\sigma}}$ are selfadjoint, meaning X_μ satisfies lemma 4.2 and ϕ is a real field.

5.2 Lichnerowicz formula for the twisted-covariant Dirac operator

We define

$$\not{X} := -i\gamma^\mu X_\mu, \quad \not{\rho}(X) := -i\gamma^\mu\rho(X_\mu). \tag{5.15}$$

These are selfadjoint operators since by (4.13) and lemma 4.2 one has

$$\not{X}^\dagger = i\, X_\mu^\dagger\gamma^\mu = -i\rho(X_\mu)\gamma^\mu = -i\gamma^\mu X_\mu = \not{X}, \tag{5.16}$$

and similarly for $\not{\rho}(X)$. The same is true for

$$\not{\bar{X}} := -i\gamma^\mu \bar{X}_\mu, \quad \not{\rho}(\bar{X}) := -i\gamma^\mu\rho(\bar{X}_\mu). \tag{5.17}$$

Similar equations hold for the field $\boldsymbol{\sigma}$, by extending the automorphism ρ to $\mathcal{B}(\mathcal{H})$ as the conjugate action of the unitary operator that exchanges the indices l and r in the basis of H. Doing so, one gets $\rho(\gamma^5) = -\gamma^5$, that is

$$\rho(\boldsymbol{\sigma}) = \mathbb{I} - \gamma^5\phi. \tag{5.18}$$

Thus $\boldsymbol{\sigma}$ twisted-commutes with γ^μ - as X_μ in (4.13) - for the anti-commutativity of γ^μ and γ^5 yields

$$\gamma^\mu\boldsymbol{\sigma} = \rho(\boldsymbol{\sigma})\gamma^\mu, \qquad \gamma^\mu\rho(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\gamma^\mu. \tag{5.19}$$

The standard model algebra $\mathcal{A}_{sm}$ is the subalgebra of $\mathcal{B}$ invariant under the twist. To measure how far the grand symmetry is from the SM, we introduce as physical degrees of freedom the fields

$$\Delta(X)_\mu := X_\mu - \rho(X_\mu), \quad \Delta(\boldsymbol{\sigma}) := (\boldsymbol{\sigma} - \rho(\boldsymbol{\sigma}))D_R. \tag{5.20}$$

Both are selfadjoint, $\Delta(X)_\mu$ by lemma 4.2, $\Delta(\boldsymbol{\sigma})$ because $\boldsymbol{\sigma}$ and D_R are selfadjoint and commute. Moreover, by (4.13) and (5.19) one has

$$\{\gamma^\mu, \Delta(X)_\nu\} = \{\gamma^\mu, \Delta(\boldsymbol{\sigma})\} = 0, \tag{5.21}$$

while γ^5 commuting with X_μ and $\boldsymbol{\sigma}$ guarantee that

$$\left[\gamma^5, \Delta(X)_\nu\right] = \left[\gamma^5, \Delta(\boldsymbol{\sigma})\right] = 0. \tag{5.22}$$

We write

$$\rho(\mathbb{X}_\mu) := \begin{pmatrix} \not{\rho}(X) & 0_{64} \\ 0_{64} & -\not{\rho}(\bar{X}) \end{pmatrix}_{\mathsf{CD}}, \qquad \Delta(\mathbb{X})_\mu := \mathbb{X}_\mu - \rho(\mathbb{X}_\mu), \tag{5.23}$$

and in agreement with (4.10) and (4.11) written as

$$\not{\mathbb{X}} = -i\gamma^\mu \mathbb{X}_\mu = \begin{pmatrix} \not{X} & 0_{64} \\ 0_{64} & -\bar{\not{X}} \end{pmatrix}_{\mathsf{CD}}, \tag{5.24}$$

we also define the selfadjont operators

$$\not{\rho}(\mathbb{X}) := -i\gamma^\mu \rho(\mathbb{X}_\mu), \quad \not{\Delta}(\mathbb{X}) := \not{\mathbb{X}} - \not{\rho}(\mathbb{X}). \tag{5.25}$$

Finally, we let

$$D_\mu := \partial_\mu + \mathrm{ad}\, \mathbb{X}_\mu \tag{5.26}$$

denotes the covariant derivative associated with the connection $\mathbb{X}_\mu$.

Proposition 5.1. *The square of the twisted-covariant Dirac operator* (5.1) *is*

$$D_{\mathbb{A}}^2 = -\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + (\alpha_X^\mu + \alpha_{\boldsymbol{\sigma}}^\mu)\partial_\mu + \beta_X + \beta_{X\boldsymbol{\sigma}} + \beta_{\boldsymbol{\sigma}}\right) \tag{5.27}$$

where

$$\alpha_X^\mu \quad := i\left\{\not{\mathbb{X}}, \gamma^\mu\right\}, \quad \beta_X = i\gamma^\mu(\partial_\mu \not{\mathbb{X}}) - \not{\mathbb{X}}\not{\mathbb{X}}, \tag{5.28}$$

while

$$\alpha_{\boldsymbol{\sigma}}^\mu := i\gamma^\mu\gamma^5\Delta(\boldsymbol{\sigma}), \qquad \beta_{\boldsymbol{\sigma}} := -\boldsymbol{\sigma}^2 D_R^2, \tag{5.29}$$

and

$$\beta_{X\boldsymbol{\sigma}} := i\gamma^\mu\gamma^5\left(D_\mu(\boldsymbol{\sigma} D_R) + \Delta(\boldsymbol{\sigma})\,\mathbb{X}_\mu\right). \tag{5.30}$$

Proof. One has $D_{\mathbb{A}}^2 = D_X^2 + D_\sigma^2 + \{D_X, D_\sigma\}$. By (4.10), the first term is

$$\begin{aligned} D_X^2 &= -\gamma^\mu(\partial_\mu + \mathbb{X}_\mu)\,\gamma^\nu(\partial_\nu + \mathbb{X}_\nu) && (5.31)\\ &= -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - i\left\{\not{\mathbb{X}}, \gamma^\mu\right\}\partial_\mu - i\gamma^\mu(\partial_\mu\not{\mathbb{X}}) + \not{\mathbb{X}}\not{\mathbb{X}} && (5.32)\\ &= -\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \alpha_X^\mu\partial_\mu + \beta_X\right). && (5.33) \end{aligned}$$

By propositions 4.1 one has

$$\{D_X, D_{\boldsymbol{\sigma}}\} = -i\left\{\gamma^\mu\partial_\mu, D_{\boldsymbol{\sigma}}\right\} - i\left\{\gamma^\mu\mathbb{X}_\mu, D_{\boldsymbol{\sigma}}\right\}. \tag{5.34}$$

From proposition 4.4, using (5.19) and $\left\{\gamma^5, \gamma^\mu\right\} = [\gamma^5, \boldsymbol{\sigma}] = 0$, one gets

$$\begin{aligned} \{\gamma^\mu\partial_\mu, D_{\boldsymbol{\sigma}}\} = \left\{\gamma^\mu\partial_\mu, \gamma^5\boldsymbol{\sigma} D_R\right\} &= \gamma^\mu\gamma^5\partial_\mu\boldsymbol{\sigma} D_R - \gamma^\mu\gamma^5\rho(\boldsymbol{\sigma})D_R\,\partial_\mu, && (5.35)\\ &= \gamma^\mu\gamma^5\left(\partial_\mu\boldsymbol{\sigma} D_R\right) + \gamma^\mu\gamma^5\Delta(\boldsymbol{\sigma})D_R\,\partial_\mu. && (5.36) \end{aligned}$$

Similarly, using that γ^5 commutes with X_μ, hence with $\mathbb{X}_\mu$, one has

$$\begin{aligned} \{\gamma^\mu\mathbb{X}_\mu, D_{\boldsymbol{\sigma}}\} = \left\{\gamma^\mu\mathbb{X}_\mu, \boldsymbol{\sigma}\gamma^5 D_R\right\} &= \gamma^\mu\mathbb{X}_\mu\boldsymbol{\sigma}\gamma^5 D_R + \boldsymbol{\sigma}\gamma^5 D_R\gamma^\mu\mathbb{X}_\mu, && (5.37)\\ &= \gamma^\mu\gamma^5[\mathbb{X}_\mu, \boldsymbol{\sigma} D_R]_\rho = \gamma^\mu\gamma^5\left([\mathbb{X}_\mu, \boldsymbol{\sigma} D_R] + \Delta(\boldsymbol{\sigma})\,\mathbb{X}_\mu\right). && (5.38) \end{aligned}$$

Summing (5.38) and (5.36), and using the definition (5.26) of D_μ, one rewrites (5.34) as $\{D_X, D_{\boldsymbol{\sigma}}\} = -(\alpha_{\boldsymbol{\sigma}}^\mu\partial_\mu + \beta_{X\boldsymbol{\sigma}})$. Finally from (4.28) one has $D_{\boldsymbol{\sigma}}^2 = -\beta_{\boldsymbol{\sigma}}$. ■

Remarkably the contributions $\alpha^\mu_{\boldsymbol{\sigma}}$ of the anti-commutator of D_X and $D_{\boldsymbol{\sigma}}$ to the order one part of $D^2_{\mathbb{A}}$ depends on $\boldsymbol{\sigma}$ only, and not on X. The same is true for $\beta_{\boldsymbol{\sigma}}$. The contributions α^μ_X and β_X of D_X depend on X only, and not on $\boldsymbol{\sigma}$. Thus in the Lichnerowicz formula for $D^2_{\mathbb{A}}$, that is

$$D^2_{\mathbb{A}} = -\nabla_\mu \nabla^\mu - E \tag{5.39}$$

with

$$\nabla_\mu = \partial_\mu + \frac{1}{2} g_{\mu\nu}(\alpha^\nu_X + \alpha^\nu_{\boldsymbol{\sigma}}), \tag{5.40}$$

the bounded endormorphism E is the sum

$$E = E_X + E_{\boldsymbol{\sigma}} + E_{X\boldsymbol{\sigma}} \tag{5.41}$$

of three terms:

$$E_X := \beta_X - \frac{1}{4}\alpha_X \cdot \alpha_X - \frac{1}{2}\partial_\mu \alpha^\mu_X, \tag{5.42}$$

which depends only on X,

$$E_{\boldsymbol{\sigma}} := \beta_{\boldsymbol{\sigma}} - \frac{1}{4}\alpha_{\boldsymbol{\sigma}} \cdot \alpha_{\boldsymbol{\sigma}} - \frac{1}{2}\partial_\mu \alpha_{\boldsymbol{\sigma}}, \tag{5.43}$$

that depends only on $\boldsymbol{\sigma}$, and an interaction term

$$E_{X\boldsymbol{\sigma}} := \beta_{X\boldsymbol{\sigma}} - \frac{1}{4}\left(\alpha_X \cdot \alpha_{\boldsymbol{\sigma}} + \alpha_{\boldsymbol{\sigma}} \cdot \alpha_X\right). \tag{5.44}$$

5.3 Deviation from the non-twisted case

We write the endomorphisms $E_X, E_{\boldsymbol{\sigma}}$ and $E_{X\boldsymbol{\sigma}}$ that appear in the Lichnerowicz formula (5.39) of the twisted-covariant Dirac operator $D_{\mathbb{A}}$ in terms of the physical degrees of freedom $\Delta(\boldsymbol{\sigma})$, $\Delta(X)_\mu$ defined in (5.20). This will permit to measure how far the twisted spectral triple (4.1) is from the spectral triple of the SM, basing our measure on the spectral action. Let us start with a technical lemma.

Lemma 5.2. *One has*

$$\alpha_X \cdot \alpha_X = 2\left\{\rho\!\!\!/(\mathbb{X}), \mathbb{X}\!\!\!/\right\} - 4\mathbb{X}\!\!\!/\mathbb{X}\!\!\!/ + 4\mathbb{X}\cdot\rho(\mathbb{X}), \tag{5.45}$$

$$\alpha_X \cdot \alpha_{\boldsymbol{\sigma}} + \alpha_{\boldsymbol{\sigma}} \cdot \alpha_X = -2i\gamma^\mu\gamma^5\left\{2\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu, \Delta(\boldsymbol{\sigma})\right\}. \tag{5.46}$$

Proof. One has

$$\begin{aligned}\alpha^\mu_X \cdot \alpha^\nu_X &= -(\mathbb{X}\!\!\!/\gamma^\mu + \gamma^\mu\mathbb{X}\!\!\!/)(\mathbb{X}\!\!\!/\gamma^\nu + \gamma^\nu\mathbb{X}\!\!\!/) &(5.47)\\ &= -\left(\mathbb{X}\!\!\!/\gamma^\mu\mathbb{X}\!\!\!/\gamma^\nu + \mathbb{X}\!\!\!/\gamma^\mu\gamma^\nu\mathbb{X}\!\!\!/ + \gamma^\mu\mathbb{X}\!\!\!/\mathbb{X}\!\!\!/\gamma^\nu + \gamma^\mu\mathbb{X}\!\!\!/\gamma^\nu\mathbb{X}\!\!\!/\right). &(5.48)\end{aligned}$$

The contraction of the second term with the metric is easily computed using $g_{\mu\nu}\gamma^\mu\gamma^\nu = 4\mathbb{I}$:

$$g_{\mu\nu}\,\mathbb{X}\!\!\!/\gamma^\mu\gamma^\nu\mathbb{X}\!\!\!/ = 4\mathbb{X}\!\!\!/\mathbb{X}\!\!\!/. \tag{5.49}$$

For the remaining terms, (4.13) written as

$$\mathbb{X}_\mu\gamma^\nu = \gamma^\nu\rho(\mathbb{X}_\mu) \tag{5.50}$$

together with $\gamma^\mu\gamma^\nu = 2g^{\mu\nu}\mathbb{I} - \gamma^\nu\gamma^\mu$ yields

$$\mathbb{X}\!\!\!/\gamma^\mu = -2i\rho(\mathbb{X}^\mu) - \gamma^\mu\rho\!\!\!/(\mathbb{X}), \quad \gamma^\mu\mathbb{X}\!\!\!/ = -2i\mathbb{X}^\mu - \rho\!\!\!/(\mathbb{X})\gamma^\mu. \tag{5.51}$$

Therefore

$$
\begin{aligned}
g_{\mu\nu}\, \not{\mathbb{X}}\gamma^\mu\not{\mathbb{X}}\gamma^\nu &= -g_{\mu\nu}\, \not{\mathbb{X}}\gamma^\mu \left(2i\rho(\mathbb{X}^\nu) + \gamma^\nu \not{\rho}(\mathbb{X})\right) = -2\not{\mathbb{X}}\, \not{\rho}(\mathbb{X});\\
g_{\mu\nu}\gamma^\mu\not{\mathbb{X}}\gamma^\nu\not{\mathbb{X}} &= -g_{\mu\nu}\left(2i\mathbb{X}^\mu + \not{\rho}(\mathbb{X})\gamma^\mu\right)\gamma^\nu\not{\mathbb{X}} = -2\not{\rho}(\mathbb{X})\not{\mathbb{X}};\\
g_{\mu\nu}\gamma^\mu\not{\mathbb{X}}\not{\mathbb{X}}\gamma^\nu &= g_{\mu\nu}\left(2i\mathbb{X}^\mu + \not{\rho}(\mathbb{X})\gamma^\mu\right)\left(2i\rho(\mathbb{X}^\nu) + \gamma^\nu\not{\rho}(\mathbb{X})\right),\\
&= -4\mathbb{X}\cdot\rho(\mathbb{X}) - 2\not{\rho}(\mathbb{X})\not{\rho}(\mathbb{X}) - 2\not{\rho}(\mathbb{X})\not{\rho}(\mathbb{X}) + 4\not{\rho}(\mathbb{X})\not{\rho}(\mathbb{X}) = -4\mathbb{X}\cdot\rho(\mathbb{X}).
\end{aligned}
\tag{5.52}
$$

Hence the contraction of (5.48) by $g_{\mu\nu}$ yields (5.45).

To obtain (5.46), one starts with (5.28) and (5.29) together with (5.21). This gives

$$
\alpha^\mu_X\alpha^\nu_{X\boldsymbol{\sigma}} + \alpha^\mu_{X\boldsymbol{\sigma}}\alpha^\nu_X := -\left\{\not{\mathbb{X}},\gamma^\mu\right\}\gamma^\nu\gamma^5\Delta(\boldsymbol{\sigma}) - \gamma^\nu\gamma^5\Delta(\boldsymbol{\sigma})\left\{\not{\mathbb{X}},\gamma^\mu\right\}, \tag{5.53}
$$

$$
= -\left\{\not{\mathbb{X}},\gamma^\mu\right\}\gamma^\nu\gamma^5\Delta(\boldsymbol{\sigma}) - \gamma^5\Delta(\boldsymbol{\sigma})\gamma^\nu\left\{\not{\mathbb{X}},\gamma^\mu\right\}. \tag{5.54}
$$

By (5.51) one has

$$
g_{\mu\nu}\left\{\not{\mathbb{X}},\gamma^\mu\right\}\gamma^\nu = -g_{\mu\nu}\left(2i\rho(\mathbb{X}^\mu) + \gamma^\mu\not{\rho}(\mathbb{X}) + 2i\mathbb{X}^\mu + \not{\rho}(\mathbb{X})\gamma^\mu\right)\gamma^\nu, \tag{5.55}
$$

$$
= 2\not{\mathbb{X}} - 4\not{\mathbb{X}} + 2\not{\rho}(\mathbb{X}) - 4\not{\rho}(\mathbb{X}) = -4\not{\mathbb{X}} + 2\not{\Delta}(\mathbb{X}), \tag{5.56}
$$

and similarly $g_{\mu\nu}\,\gamma^\mu\left\{\not{\mathbb{X}},\gamma^\nu\right\} = -4\not{\mathbb{X}} + 2\not{\Delta}(\mathbb{X})$. Therefore (5.54) gives

$$
\begin{aligned}
\alpha_X\cdot\alpha_{\boldsymbol{\sigma}} + \alpha_{\boldsymbol{\sigma}}\cdot\alpha_X = 2\left\{2\not{\mathbb{X}} - \not{\Delta}(\mathbb{X}),\gamma^5\Delta(\boldsymbol{\sigma})\right\} &= 2\left\{-i\gamma^\mu(2\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu),\gamma^5\Delta(\boldsymbol{\sigma})\right\},\\
&= -2i\gamma^\mu\gamma^5\left\{2\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu,\Delta(\boldsymbol{\sigma})\right\}
\end{aligned}
\tag{5.57}
$$

where in the last line we use that γ^μ anticommutes with both γ^5 and $\Delta(\boldsymbol{\sigma})$, while γ^5 commutes with both $\mathbb{X}_\mu$ and $\Delta(\mathbb{X})_\mu$. ■

Proposition 5.3. *One has*

$$
E_X = \frac{1}{2}\gamma^\mu\gamma^\nu\left(\mathbb{F}_{\mu\nu} + D_\nu\,\Delta(\mathbb{X}_\mu) + \Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu\right), \tag{5.58}
$$

$$
E_{X\boldsymbol{\sigma}} = i\gamma^\mu\gamma^5\left(D_\mu(\boldsymbol{\sigma}D_R) - \frac{1}{2}[\mathbb{X}_\mu,\Delta(\boldsymbol{\sigma})] + \frac{1}{2}\left\{3\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu,\Delta(\boldsymbol{\sigma})\right\}\right) \tag{5.59}
$$

$$
E_{\boldsymbol{\sigma}} = \Delta(\boldsymbol{\sigma})^2 - \boldsymbol{\sigma}^2 D_R^2 - \frac{i}{2}\gamma^\mu\gamma^5\partial_\mu\Delta(\boldsymbol{\sigma}). \tag{5.60}
$$

where

$$
\mathbb{F}_{\mu\nu} := (\partial_\mu\mathbb{X}_\nu) - (\partial_\nu\mathbb{X}_\mu) + [\mathbb{X}_\mu,\mathbb{X}_\nu] \tag{5.61}
$$

is the field strength of $\mathbb{X}_\mu$

Proof. By (5.42), (5.28) and lemma 5.2 one gets

$$
E_X = \frac{i}{2}\left[\gamma^\mu,(\partial_\mu\not{\mathbb{X}})\right] - \frac{1}{2}\left\{\not{\rho}(\mathbb{X}),\not{\mathbb{X}}\right\} - \mathbb{X}\cdot\rho(\mathbb{X}). \tag{5.62}
$$

One further computes, writing Δ_μ for $\Delta(\mathbb{X})_\mu$,

$$
\begin{aligned}
-\frac{1}{2}\left\{\not{\rho}(\mathbb{X}),\not{\mathbb{X}}\right\} - \mathbb{X}\cdot\rho(\mathbb{X}) &= \frac{1}{2}\left(\gamma^\mu\rho\left(\mathbb{X}_\mu\right)\gamma^\nu\mathbb{X}_\nu + \gamma^\mu\mathbb{X}_\mu\gamma^\nu\rho\left(\mathbb{X}_\nu\right) - \left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\mathbb{X}_\mu\rho\left(\mathbb{X}_\nu\right)\right),\\
&= \frac{1}{2}\gamma^\mu\gamma^\nu\left(\mathbb{X}_\mu\Delta_\nu + \rho(\mathbb{X}_\mu)\rho(\mathbb{X}_\nu) - \mathbb{X}_\nu\rho(\mathbb{X}_\mu)\right),\\
&= \frac{1}{2}\gamma^\mu\gamma^\nu\left(\mathbb{X}_\mu\Delta_\nu + (\mathbb{X}_\mu - \Delta_\mu)(\mathbb{X}_\nu - \Delta_\nu) - \mathbb{X}_\nu(\mathbb{X}_\mu - \Delta_\mu)\right),\\
&= \frac{1}{2}\gamma^\mu\gamma^\nu\left([\mathbb{X}_\mu,\mathbb{X}_\nu] + \Delta_\mu\Delta_\nu + [\mathbb{X}_\nu,\Delta_\mu]\right).
\end{aligned}
\tag{5.63}
$$

As well,

$$\frac{i}{2}\left[\gamma^\mu, (\partial_\mu \not{\mathbb{X}})\right] = \gamma^\mu(\partial_\mu \gamma^\nu \mathbb{X}_\nu) - (\partial_\mu \gamma^\nu \mathbb{X}_\nu)\gamma^\mu,$$
$$= \gamma^\mu\gamma^\nu\left(\partial_\mu X_\nu - \partial_\nu \rho(X_\mu)\right) = \gamma^\mu\gamma^\nu\left(\partial_\mu X_\nu - \partial_\nu X_\mu + \partial_\nu \Delta_\mu\right). \tag{5.64}$$

The sum of (5.64) and (5.63) gives (5.58).

From (5.44), (5.29) and lemma 5.2 one obtains

$$E_{X\boldsymbol{\sigma}} = i\gamma^\mu\gamma^5\left(D_\mu(\boldsymbol{\sigma} D_R) + \Delta(\boldsymbol{\sigma})\,\mathbb{X}_\mu\right) + i\gamma^\mu\gamma^5\left\{\mathbb{X}_\mu - \frac{1}{2}\Delta(\mathbb{X})_\mu, \Delta(\boldsymbol{\sigma})\right\}. \tag{5.65}$$

Eq. (5.59) then follows writing

$$\Delta(\boldsymbol{\sigma})\mathbb{X}_\mu = \frac{1}{2}\left\{\mathbb{X}_\mu, \Delta(\boldsymbol{\sigma})\right\} - \frac{1}{2}\left[\mathbb{X}_\mu, \Delta(\boldsymbol{\sigma})\right] \tag{5.66}$$

To prove (5.60) one uses $\{\Delta(\boldsymbol{\sigma}), \gamma^\nu\} = \left[\Delta(\boldsymbol{\sigma}), \gamma^5\right] = 0$ to compute

$$g_{\mu\nu}\alpha^\mu_{\boldsymbol{\sigma}}\alpha^\nu_{\boldsymbol{\sigma}} = -g_{\mu\nu}\gamma^\mu\gamma^5\Delta(\boldsymbol{\sigma})\gamma^\nu\gamma^5\Delta(\boldsymbol{\sigma}) = -g_{\mu\nu}\gamma^\mu\gamma^\nu\Delta^2(\boldsymbol{\sigma}) = -4\Delta^2(\boldsymbol{\sigma}). \tag{5.67}$$

Thus $\beta_{\boldsymbol{\sigma}} - \frac{1}{4}\alpha_{\boldsymbol{\sigma}}\cdot\alpha_{\boldsymbol{\sigma}} = \Delta^2(\boldsymbol{\sigma}) - \boldsymbol{\sigma}^2 D_R^2$ and (5.60) follows from (5.43) and (5.29). ∎

In order to interprete proposition 5.3, it is instructive to confront with the non-twisted case. When the finite dimensional algebra $\mathcal{A}_F$ of an almost-commutative geometry acts trivially on spinors, the full covariant Dirac operator is

$$D_A = D_Y + \gamma^5 \otimes D_R, \tag{5.68}$$

where $D_Y := -i\gamma^\mu\nabla^Y_\mu$ is the covariant Dirac operator of a $U(\mathcal{A}_F)$-bundle over the spin bundle of $\mathcal{M}$, associated with the covariant derivative $\nabla^Y_\mu := \nabla^S_\mu + \mathbb{Y}_\mu$ defined by a connection one-form $\mathbb{Y}_\mu = \delta^{t\dot{t}}_{s\dot{s}}\, Y^{I\beta\mathsf{D}}_{\mathsf{J}\alpha\mathsf{C}}$ whose action on spinors indices is trivial. One gets

$$D_A^2 = D_Y^2 + D_R^2 + \left\{\not{D}_Y, \gamma^5 \otimes D_R\right\}. \tag{5.69}$$

Because

$$[\gamma^\mu, \mathbb{Y}_\nu] = 0, \tag{5.70}$$

one has $D_Y^2 = -\gamma^\mu\gamma^\nu\nabla^Y_\mu\nabla^Y_\nu$. Using $\gamma^\mu\gamma^\nu = g^{\mu\nu}\mathbb{I} + \frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\gamma^\nu\gamma^\mu$, the square of D_Y is rewritten as the sum of the Laplacian and the field strength $F_{\mu\nu} = [\mathbb{Y}_\mu, \mathbb{Y}_\nu]$, namely

$$D_Y^2 = -g^{\mu\nu}\nabla^Y_\mu\nabla^Y_\nu - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}. \tag{5.71}$$

The second term in (5.69) is $D_R^2 = |k_R|^2\mathbb{I}$ and the third is $-i\gamma^\mu\gamma^5[\mathbb{Y}_\mu, D_R]$. Therefore the Lichnerowicz formula for the covariant *non-twisted* Dirac operator is

$$D_A^2 = -g^{\mu\nu}\nabla^Y_\mu\nabla^Y_\nu - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + |k_R|^2\mathbb{I} - i\gamma^\mu\gamma^5[\mathbb{Y}_\mu, D_R]. \tag{5.72}$$

In the twisted case, summing up the terms in Prop. 5.3 one obtains from (5.39)

$$D_{\mathbb{A}}^2 = -g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{1}{2}\gamma^\mu\gamma^\nu\left(\mathbb{F}_{\mu\nu} + D_\nu\,\Delta(\mathbb{X})_\mu + \Delta(\mathbb{X})_\mu\Delta(\mathbb{X}_\nu)\right) \tag{5.73}$$
$$+\boldsymbol{\sigma}^2 D_R^2 - \Delta(\boldsymbol{\sigma})^2 - i\gamma^\mu\gamma^5\left(D_\mu(\boldsymbol{\sigma} D_R) - \frac{1}{2}D_\mu\Delta(\boldsymbol{\sigma})\right) \tag{5.74}$$
$$-\frac{i}{2}\gamma^\mu\gamma^5\left\{3\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu, \Delta(\boldsymbol{\sigma})\right\}. \tag{5.75}$$

There are several important differences with the non-twisted case:

- In (5.73), the covariant derivative of $\Delta(\mathbb{X})_\nu$ and its potential $\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu$ can be traced back to the Lichnerowicz formula for the twisted-covariant free Dirac operator,

$$D_X^2 = -g^{\mu\nu}\nabla_\mu^X\nabla_\nu^X - E_X \text{ where } \quad \nabla_\mu^X := \nabla_\mu^S + \frac{1}{2}\alpha_\mu^X. \tag{5.76}$$

These new terms arise because in the twisted case (5.70) no longer holds, instead one has (4.13), that is

$$[\gamma^\mu, X_\nu]_\rho = [X_\nu, \gamma^\mu]_\rho = 0. \tag{5.77}$$

- The appearance of the covariant derivative of $\boldsymbol{\sigma} D_R$ in (5.74) is not surprising. It is already there in (5.72), where the last term is nothing but the covariant derivative of $\boldsymbol{\sigma} D_R$ for $\boldsymbol{\sigma}$ the constant field 1. Similarly $|k_R|^2$ in (5.72) is the potential term $\boldsymbol{\sigma}^2 D_R^2$ in (5.74) for $\boldsymbol{\sigma} = 1$.

- In (5.74) the scalar $\Delta(\boldsymbol{\sigma})$ is described by a dynamical term $-D_\mu\Delta(\boldsymbol{\sigma})$ and a potential $-\Delta(\boldsymbol{\sigma})^2$, whose sign are opposite to the similar terms for $\boldsymbol{\sigma}$.

- The interaction between $\mathbb{X}_\mu$ and $\Delta(\boldsymbol{\sigma})$ is not totally absorbed in the covariant derivative D_μ. There remains in (5.75) an potential of interaction $\{3\mathbb{X}_\mu, \Delta(\boldsymbol{\sigma})\}$. As well, there is a potential of interaction $\{\Delta(\mathbb{X})_\mu, \Delta(\boldsymbol{\sigma})\}$ between the extra scalar field and the additional vector field.

One may be puzzled by the presence of two distincts covariant derivatives in the Lichnerowicz formula for $D_{\mathbb{A}}$: ∇_μ in the Laplacian and D_μ that encodes the dynamics of $\Delta(\mathbb{X})_\mu$ and $\Delta(\boldsymbol{\sigma})$. In the non-twisted case this is the same covariant derivative ∇_μ^Y which play both role. However, because we switch gravitation off[††] and consider the flat case, in the heat kernel expansion of the spectral action the covariant derivative ∇_μ only appears through the term $\nabla^\mu\nabla_\mu E$ (in a_4). The latter is interpreted as a boundary term (see [16, Rem.1.155]) and we shall not take it into account in this paper. Doing so, only one covariant derivative remains, D_μ. This makes sense from our perspective: the fields $\Delta(\mathbb{X})_\mu$ and $\Delta(\boldsymbol{\sigma})$ are viewed as "excitations" generated by the twist, living on a background gauge theory with connection 1-form X_μ; so their dynamics is encoded by D_μ, not by ∇_μ.

The remaining Seeley-de Witt coefficients are a_0, which is not affected by the twist and is interpreted as the cosmological constant (which recently turns out to be quantized, see [10]) and the integral of the trace of E (in a_2) and E^2 (in a_4) for E given in (5.41). In other terms the potential is the part of

$$V := \Lambda^2 f_2 \operatorname{Tr} E \;+\; \frac{1}{2} f_0 \operatorname{Tr} E^2 \tag{5.78}$$

that does not depend on the covariant derivative D_μ. We analyze it below, dividing it into three pieces: the potential $V(X)$ of the vector field, $V(\boldsymbol{\sigma})$ of the scalar field, and a potential of interaction $V(X, \boldsymbol{\sigma})$.

5.4 The vector field and the breaking to the standard model

The potential $V(X)$ is the part of V that depends on $\Delta(\mathbb{X})_\mu$ and no on its derivative, that is

$$V(X) = \Lambda^2 f_2 \operatorname{Tr} E_X^0 + \frac{1}{2} f_0 \operatorname{Tr} (E_X^0)^2, \tag{5.79}$$

[††] Our aim in this paper is to understand how the twist allows to generate the field $\boldsymbol{\sigma}$. That is why for simplicity we consider the flat case. The curved case, which should be similar, will be studied elsewhere.

where $E^0_X := \frac{1}{2}\gamma^\mu\gamma^\nu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu$ is read in (5.58). One rewrites it as

$$E^0_X = \frac{1}{2}\not{\Delta}^2(\mathbb{X}), \tag{5.80}$$

thanks to (5.21) which guarantees that γ^ν anti-commutes with $\Delta(\mathbb{X})_\mu$ for all μ.

Proposition 5.4. *The potential $V(X)$ is never negative and vanishes iff $\Delta(\mathbb{X})_\mu = 0$ for any μ.*

Proof. Since $\not{\Delta}(\mathbb{X})$ is selfadjoint, E^0_X and $(E^0_X)^2$ are positive. Thus their trace is never negative, and vanishes if and only if $E^0_X = (E^0_X)^2 = 0$. This condition is equivalent to

$$\Delta(\mathbb{X})_\mu = 0 \quad \forall\mu. \tag{5.81}$$

Indeed, since $\{\gamma^\nu, \Delta(\mathbb{X})_\mu\} = 0$ one has

$$\text{Tr}\,(\gamma^\mu\gamma^\nu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) = \text{Tr}\,(\gamma^\mu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu\gamma^\nu) = \text{Tr}\,(\gamma^\nu\gamma^\mu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) \tag{5.82}$$

where the last equality comes from the tracial property. Therefore

$$\text{Tr}\,E^0_X = \frac{1}{4}(\text{Tr}\,(\gamma^\mu\gamma^\nu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) + \text{Tr}\,(\gamma^\nu\gamma^\mu\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu)), \tag{5.83}$$

$$= \frac{1}{2}\,g^{\mu\nu}\,\text{Tr}(\Delta(\mathbb{X})_\mu\Delta(\mathbb{X})_\nu) = \frac{1}{2}\sum_\mu \text{Tr}\left(\Delta^2(\mathbb{X})_\mu\right). \tag{5.84}$$

Since $\Delta(\mathbb{X})_\mu$ is selfadjoint, $\Delta^2(\mathbb{X})_\mu$ is positive. Its trace is never negative and vanishes if and only if $\Delta(\mathbb{X})_\mu$ is zero. The same is true for the sum in (5.84), meaning that Tr E^0_X - hence E^0_X - vanishes if and only if $\Delta(\mathbb{X})_\mu = 0$ for all μ.

The proposition is obtained noticing that f_0 and f_2 are positive numbers. ■

Condition (5.81) is equivalent to $\Delta(X)_\mu = 0$ for any μ. To obtain the breaking to the standard model, one needs to check that the vanishing of $\Delta(X)_\mu$, that is the invariance of X_μ under the twist, implies the invariance of its components R^i, Q_i.

Lemma 5.5. *The biggest unital subalgebra of $\mathcal{B}\otimes\mathcal{C}^\infty(\mathcal{M})$ for wich any combination*

$$X_\mu = \delta^I_J\,\rho(R^i)\,\partial_\mu Q_i - \delta^\beta_\alpha\,\bar{N}^i\partial_\mu\bar{M}_i \tag{5.85}$$

is invariant under the twist is $\mathcal{A}_{SM}\otimes C^\infty(\mathcal{M})$.

Proof. Let $\mathcal{G}$ be any subalgebra of $\mathcal{B}\otimes\mathcal{C}^\infty(\mathcal{M})$ such that any linear combinations X_μ with (R^i, N^i) and (Q_i, M_i) in $\mathcal{G}$ is invariant under the automorphism ρ. This means in particular that for $X = R\partial_\mu Q - Q\partial_\mu R$ with R, Q arbitrary elements in $\mathcal{G}$, one has

$$\rho(X_\mu) - X_\mu = \rho(R)\partial_\mu Q - R\partial_\mu\rho(Q) = 0. \tag{5.86}$$

Taking $R = \mathbb{I}$, this implies

$$\partial_\mu(Q - \rho(Q)) = 0. \tag{5.87}$$

So any element of $\mathcal{G}$ is (Q, M) where

$$Q = \begin{pmatrix} Q^r_r & 0 \\ 0 & Q^r_r + c \end{pmatrix}_{st} \tag{5.88}$$

with c a constant. For $\mathcal{G}$ to be an algebra, (5.88) must be true also for Q^2, that is there must exists a constant c' such that

$$Q^2 = \begin{pmatrix} (Q_r^r)^2 & 0 \\ 0 & (Q_r^r)^2 + c^2 + 2cQ_r^r \end{pmatrix}_{st} = \begin{pmatrix} (Q_r^r)^2 & 0 \\ 0 & (Q_r^r)^2 + c'^2 \end{pmatrix}_{st}. \tag{5.89}$$

This is possible if and only if $c = c' = 0$. Thus $\rho(Q) = Q$ for any $(Q, M) \in \mathcal{G}$. The proposition follows from the identification of $\mathcal{A}_{sm}$ as the biggest ρ-invariant sub-algebra of $\mathcal{B}$. ■

This proves the third statement of theorem 1.1, namely the breaking of grand symmetry to the standard model is dynamical, and induced by the minimal of the spectral action of the twisted-covariant free Dirac operator D_X.

5.5 The scalar field

The part of the potential containing only the extra scalar field and not the vector field is

$$V(\boldsymbol{\sigma}) := \Lambda^2 f_2 \operatorname{Tr} E^0_{\boldsymbol{\sigma}} + \frac{1}{2} f_0 \operatorname{Tr} (E^0_{\boldsymbol{\sigma}})^2, \tag{5.90}$$

where

$$E^0_{\boldsymbol{\sigma}} := \Delta^2(\boldsymbol{\sigma}) - \sigma^2 D_R^2 \tag{5.91}$$

is read in (5.60). Compared to $V(X)$ which contains only $\Delta(\mathbb{X})_\mu$ and not $\mathbb{X}_\mu$, the potential $V(\boldsymbol{\sigma})$ contains both $\boldsymbol{\sigma}$ and $\Delta(\boldsymbol{\sigma})$. This gives two possibilites for minimizing:

Either one considers only $\Delta(\boldsymbol{\sigma})$ as degree of freedom. The potential then reduces to

$$V(\Delta(\boldsymbol{\sigma})) := \Lambda^2 f_2 \operatorname{Tr}(\Delta^2(\boldsymbol{\sigma})) + \frac{1}{2} f_0 \operatorname{Tr}(\Delta^4(\boldsymbol{\sigma})). \tag{5.92}$$

Since $\Delta(\boldsymbol{\sigma})$ is selfadjoint, this potential is positive and vanishes if and only if $\Delta(\boldsymbol{\sigma}) = 0$. Going back to the the definition (5.20) of $\Delta(\boldsymbol{\sigma})$, this means

$$\boldsymbol{\sigma} = \rho(\boldsymbol{\sigma}) = \mathbb{I}. \tag{5.93}$$

Or one may prefer to take into account the whole potential (5.90). In this case it is easier to take as degree of freedom the field ϕ.

Lemma 5.6. *The potential of the scalar field is*

$$V(\boldsymbol{\sigma}) = C_4\, \phi^4 + C_2\, \phi^2 + C_0 \tag{5.94}$$

where $C_4 := 36|k_R|^4 f_0,\ C_2 := 8|k_R|^2(3\Lambda^2 f_2 - |k_R|^2 f_0),\ C_0 := 8|k_R|^2 \left(\frac{|k_R|^2}{2} f_0 - \Lambda^2 f_2\right)$.

Proof. By (4.29), (5.18) and (5.20) one has

$$E^0_{\boldsymbol{\sigma}} = \left((3\phi^2 - 1)\mathbb{I}_4 - 2\gamma^5 \phi\right)\, D_R^2. \tag{5.95}$$

From (3.34),

$$D_R^2 = |k_R|^2\, \delta^{\mathsf{D}}_{\mathsf{C}}\, \Xi^{\mathrm{I}\beta}_{\mathrm{J}\alpha} \tag{5.96}$$

so that $\gamma^5 D_R^2$ has zero trace. Hence

$$\operatorname{Tr} E^0_{\boldsymbol{\sigma}} = (3\phi^2 - 1)\, \operatorname{Tr}\left(\, \mathbb{I}_4 \otimes D_R^2\right) = 8|k_R|^2\, (3\phi^2 - 1). \tag{5.97}$$

Squaring (5.95) one gets

$$(E^0_{\boldsymbol\sigma})^2 = \left((3\phi^2-1)^2\mathbb{I}_4 + 4\phi^2\mathbb{I}_4 - 2\gamma^5\phi(3\phi^2-1)\right)\, D_R^4 \tag{5.98}$$

whose trace is

$$\mathrm{Tr}\,(E^0_{\boldsymbol\sigma})^2 = 8|k_R|^4\left((3\phi^2-1)^2+4\phi^2\right). \tag{5.99}$$

The result follows from (5.90). ■

At a large unification scale Λ it is reasonable to assume that (see e.g. [36, §11.3.2])

$$3\Lambda^2 f_2 \geq f_0|k_R|^2. \tag{5.100}$$

Together with the positivity of f_0 this shows that $V(\boldsymbol\sigma)$ is minimum when $\phi=0$, and one is led to the same conclusion (5.93) obtained by minimizing $V(\Delta(\boldsymbol\sigma))$.

The invariance (5.93) of $\boldsymbol\sigma$ under the twist implies that $D_{\boldsymbol\sigma}=D_M$, so that one is back to the Dirac operator of the standard model. However this does not imply the reduction of the algebra to the one of the standard model. Indeed, from (4.31) the vanishing of ϕ means $c_R^r = c_R^l$, so that the bigger subalgebra of $C^\infty(\mathcal{M})\otimes\mathcal{B}$ for which any fluctuation yields a ρ-invariant $\boldsymbol\sigma$ is

$$C^\infty(\mathcal{M})\otimes(\mathbb{H}_L^l\oplus\mathbb{H}_L^r\oplus\mathbb{C}\oplus M_3(\mathbb{C})), \tag{5.101}$$

which contains, but is different from $C^\infty(\mathcal{M})\otimes\mathcal{A}_{sm}$.

5.6 Potential of interaction

We now consider the remaining part of the potential, that is the interaction term $V(X,\boldsymbol\sigma)$ between the scalar and the vector fields. Writing $V(X,\boldsymbol\sigma)$ explicitly in (5.106) below, it becomes clear it is easier to minimize it together with the potential $V(X)$, which is done in prop. 5.9.

Let

$$E^0_{X\boldsymbol\sigma} := \frac{i}{2}\gamma^\mu\gamma^5\left\{\mathbb{H}_\mu,\Delta(\boldsymbol\sigma)\right\} \quad \text{with} \quad \mathbb{H}_\mu := 3\mathbb{X}_\mu - \Delta(\mathbb{X})_\mu \tag{5.102}$$

denote the part of $E_{X\boldsymbol\sigma}$ in (5.59) that does not depend on the covariant derivative of the fields. The potential of interaction is made of all the terms in the trace of $E^0_X + E^0_{\boldsymbol\sigma} + E^0_{X\boldsymbol\sigma}$ and its square that depend on both X and $\boldsymbol\sigma$.

Lemma 5.7. *$E^0_{X\boldsymbol\sigma}$ is selfadjoint and traceless.*

Proof. Since γ^μ anti-commutes with $\Delta(\boldsymbol\sigma)$ and γ^5, one has

$$E^0_{X\boldsymbol\sigma} = \frac{1}{2}\gamma^5\left[\not{\mathbb{H}},\Delta(\boldsymbol\sigma)\right]. \tag{5.103}$$

Therefore

$$\mathrm{Tr}\,E^0_{X\boldsymbol\sigma} = \frac{1}{2}\mathrm{Tr}\,\gamma^5\left[\not{\mathbb{H}},\Delta(\boldsymbol\sigma)\right] = -\frac{1}{2}\mathrm{Tr}\left[\not{\mathbb{H}},\Delta(\boldsymbol\sigma)\right]\gamma^5 = -\frac{1}{2}\mathrm{Tr}\,\gamma^5\left[\not{\mathbb{H}},\Delta(\boldsymbol\sigma)\right] \tag{5.104}$$

where the first equality come from $\left\{\gamma^5,\not{\mathbb{H}}\right\} = \left[\gamma^5,\Delta(\boldsymbol\sigma)\right] = 0$ and the second from the tracial property. Thus $\mathrm{Tr}\,E^0_{X\boldsymbol\sigma} = -\mathrm{Tr}\,E^0_{X\boldsymbol\sigma}$, and so vanishes. The selfadjointness follows from the commutation properties of γ^5 and the selfadjointness of $\not{\mathbb{H}}$ and $\Delta(\boldsymbol\sigma)$. ■

Furthermore E_X and $E_{\boldsymbol{\sigma}}$ depend solely on X and $\boldsymbol{\sigma}$, so the potential of interaction reduces to the trace of the part of $(E^0_X + E^0_{\boldsymbol{\sigma}} + E^0_{X\boldsymbol{\sigma}})^2$ that contains products of X and $\boldsymbol{\sigma}$, namely

$$E^2_{X\boldsymbol{\sigma}} + \left\{E^0_X, E^0_{\boldsymbol{\sigma}}\right\} + \left\{E^0_X, E^0_{X\boldsymbol{\sigma}}\right\} + \left\{E^0_{\boldsymbol{\sigma}}, E^0_{X\boldsymbol{\sigma}}\right\}. \tag{5.105}$$

The last two terms are traceless because E^0_X and $E^0_{\boldsymbol{\sigma}}$ are diagonal in the CD indices (see (5.80, 5.24, 5.25) and (5.95, 5.96)) while $E^0_{X\boldsymbol{\sigma}}$ is off-diagonal (being the commutator of a diagonal and an off-diagonal matrice). By the tracial property $\mathrm{Tr}\left\{E^0_X, E^0_{\boldsymbol{\sigma}}\right\} = 2\,\mathrm{Tr}\, E^0_X E^0_{\boldsymbol{\sigma}}$, hence the potential of interaction is

$$V(X,\boldsymbol{\sigma}) := \frac{1}{2} f_0\, \mathrm{Tr}\, (E^0_{X\boldsymbol{\sigma}})^2 + f_0\, \mathrm{Tr}\, E^0_X E^0_{\boldsymbol{\sigma}}. \tag{5.106}$$

Lemma 5.8. *One has*

$$Tr\, E^0_X E^0_{\boldsymbol{\sigma}} = \frac{1}{2}(3\phi^2 - 1)\, Tr\left(\not{\Delta}^2(\mathbb{X}) D^2_R\right). \tag{5.107}$$

Proof. By (5.80) and (5.95) one has

$$E^0_X E^0_{\boldsymbol{\sigma}} = \frac{1}{2}(3\phi^2 - 1)\not{\Delta}^2(\mathbb{X}) D^2_R - \phi\not{\Delta}^2(\mathbb{X})\gamma^5 D^2_R. \tag{5.108}$$

The result amounts to show that the second term has vanishing trace. To see it, let use (5.23), (5.24) and (5.96) to write

$$\not{\Delta}^2(\mathbb{X})\gamma^5 D^2_R = |k_R|^2 \begin{pmatrix} \not{\Delta}^2(X)\,\gamma^5\,\Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0_{64} \\ 0_{64} & -\not{\Delta}^2(\bar{X})\,\gamma^5\,\Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\mathsf{CD}}. \tag{5.109}$$

One has

$$\Delta(X)_\mu = \delta^i_{\dot{s}} \begin{pmatrix} X^l_\mu - X^r_\mu & 0_{32} \\ 0_{32} & X^r_\mu - X^l_\mu \end{pmatrix}_{st} =: \delta^i_{\dot{s}} \begin{pmatrix} \Delta_\mu & 0_{32} \\ 0_{32} & -\Delta_\mu \end{pmatrix}_{st}, \tag{5.110}$$

so that

$$\not{\Delta}(X) = -i \begin{pmatrix} 0_{32} & -\sigma^\mu \Delta_\mu \\ \bar{\sigma}^\mu \Delta_\mu & 0_{32} \end{pmatrix}_{st}, \quad \not{\Delta}^2(X) = \begin{pmatrix} \sigma^\mu \tilde{\sigma}^\nu \Delta_\nu \Delta_\mu & 0_{32} \\ 0_{32} & \tilde{\sigma}^\mu \sigma^\nu \Delta_\mu \Delta_\nu \end{pmatrix}_{st}. \tag{5.111}$$

Hence

$$\begin{aligned} \mathrm{Tr}\left(\not{\Delta}^2(X)\,\gamma^5\,\Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}}\right) &= \mathrm{Tr}\begin{pmatrix} \sigma^\mu \tilde{\sigma}^\nu X^l_\mu X^r_\nu & 0_{32} \\ 0_{32} & -\tilde{\sigma}^\mu \sigma^\nu X^r_\mu X^l_\nu \end{pmatrix}_{st} \\ &= \mathrm{Tr}\left(\sigma^\mu \tilde{\sigma}^\nu X^l_\mu X^r_\nu\right) - \mathrm{Tr}\left(\tilde{\sigma}^\mu \sigma^\nu X^r_\mu X^l_\nu\right) = \mathrm{Tr}\left(\sigma^\mu \tilde{\sigma}^\nu X^l_\mu X^r_\nu\right) - \mathrm{Tr}\left(\tilde{\sigma}^\nu \sigma^\mu X^r_\nu X^l_\mu\right) \end{aligned} \tag{5.112}$$

which vanishes by the trace property and the commutation of $[X^l_\mu, \sigma^\mu] = [X^l_\mu, \sigma^\nu] = 0$. The same is true for $\mathrm{Tr}\left(\not{\Delta}^2(\bar{X})\,\gamma^5\,\Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}}\right)$, so that (5.109) has zero trace. ∎

By lemma 5.7, $(E^0_{X\boldsymbol{\sigma}})^2$ is positive, hence its trace is never negative and minimal when $E^0_{X\boldsymbol{\sigma}}$ is zero. However $\mathrm{Tr}\, E^0_X E^0_{X\boldsymbol{\sigma}}$ is not necessarily bounded from below, which makes difficult to minimize the potential of interaction alone. In fact it is easier to minimize it together with the potential $V(X)$ of the vector field.

Proposition 5.9. *The potential*

$$V'(X,\boldsymbol{\sigma}) := V(X) + V(X,\boldsymbol{\sigma}) \tag{5.113}$$

is never negative and vanishes if and only if $\Delta(\mathbb{X})_\mu = 0$ *for all* μ.

Proof. We write $\not{\Delta}$ for $\not{\Delta}(\mathbb{X})$. By summing up (5.79) and (5.106) one gets

$$
\begin{aligned}
V'(X,\boldsymbol{\sigma}) &= \frac{\Lambda^2 f_2}{2}\mathrm{Tr}\,\not{\Delta}^2 + \frac{f_0}{8}\mathrm{Tr}\,\not{\Delta}^4 + \frac{f_0}{2}\mathrm{Tr}\,(E_{X\boldsymbol{\sigma}})^2 + \frac{f_0}{2}(3\phi^2-1)\mathrm{Tr}\,(\not{\Delta}^2 D_R^2), && (5.114)\\
&= \mathrm{Tr}\left(\frac{\Lambda^2 f_2}{2}\,\not{\Delta}^2 - \frac{f_0}{2}\,\not{\Delta}^2 D_R^2\right) + \frac{f_0}{8}\mathrm{Tr}\,\not{\Delta}^4 + \frac{f_0}{2}\mathrm{Tr}\,(E_{X\boldsymbol{\sigma}})^2 + \frac{3f_0}{4}\phi^2\mathrm{Tr}\,(\not{\Delta}^2 D_R^2). && (5.115)
\end{aligned}
$$

Let $p := \delta_{st\mathsf{C}}^{\dot{t}\dot{s}\mathsf{D}}\,\Xi_{\mathrm{J}\alpha}^{\mathrm{I}\beta}$ denote the projection on the non-zero entries of D_R^2, so that $\delta_{st}^{\dot{s}\dot{t}}D_R^2 = |k_R|^2 p$. The first term in (5.115) is

$$
W(X,\boldsymbol{\sigma}) := \left(\frac{\Lambda^2 f_2}{2} - \frac{1}{2}f_0|k_R|^2\right)\mathrm{Tr}\,(\not{\Delta}^2\,p) + \frac{\Lambda^2 f_2}{2}\mathrm{Tr}\,(\not{\Delta}^2\,(1-p)) \tag{5.116}
$$

Because $\not{\Delta}$ is selfadjoint,

$$
\mathrm{Tr}\,(\not{\Delta}^2 p) = \mathrm{Tr}(p\not{\Delta}^2 p) = \mathrm{Tr}\,\left((p\not{\Delta})(p\not{\Delta})^\dagger\right) \tag{5.117}
$$

is positive. The same is true for $\mathrm{Tr}\,(\not{\Delta}^2(1-p))$. Assuming as in (5.100) that at high energy

$$
\Lambda^2 f_2 \geq f_0|k_R|^2, \tag{5.118}
$$

one gets that $W(X,\boldsymbol{\sigma})$ is never negative, and vanishes if and only if $\not{\Delta}^2 p$ and $\not{\Delta}^2(1-p)=0$, that is if and only if $\not{\Delta}^2 = 0$, which is equivalent to $\mathrm{Tr}\,\Delta^2 = 0$ since Δ^2 is positive. By (5.84) this is equivalent to $\Delta(\mathbb{X})_\mu = 0$ for any μ.

The second term in (5.115) is never negative, and vanishes when $\Delta(X)_\mu = 0$ for any μ. The same is true for the third term by lemma 5.8, and for the last term since

$$
\mathrm{Tr}\,(\not{\Delta}^2 D_R^2) = \mathrm{Tr}\,\left((D_R\not{\Delta})(D_R\not{\Delta})^\dagger\right). \tag{5.119}
$$

Hence $V'(X,\boldsymbol{\sigma})$ is never negative, and vanishes if and only if $\Delta(\mathbb{X})_\mu = 0$ for any μ. ■

Combining propositions 5.9 and 5.6 one gets that the whole potential $V(X)+V(\boldsymbol{\sigma})+V(X,\boldsymbol{\sigma})$ is zero if and only if both the scalar field $\boldsymbol{\sigma}$ and the vector field $\Delta(X)_\mu$ are zero. This proves the first statement of point iii) in theorem 1.1. The second statement has been proven below lemma 5.6.

6 Twist and representations

We discuss the choices made in the construction of the twisted spectral triple of the standard model: the middle-term solution consisting in imposing by hand the reduction $M_8(\mathbb{C}) \to M_4(\mathbb{C})$, and the representation of $\mathcal{A}_G$.

6.1 Global twist

Instead of reducing by hand $\mathcal{B}_{LR}$ to $\mathcal{B}'$ by imposing the reduction $M_8(\mathbb{C}) \to M_4(\mathbb{C})$, one could twist $\mathcal{B}_{LR}$ as well. This means finding an automorphism ρ of $M_8(\mathbb{C})$ such that

$$
\sigma^\mu M\,\partial_\mu - \sigma(M)\sigma^\mu\partial_\mu = 0, \qquad \bar{\sigma}^\mu M\,\partial_\mu - \bar{\sigma}(M)\bar{\sigma}^\mu\partial_\mu = 0. \tag{6.1}
$$

Using $\sigma^\mu \bar{\sigma}^\nu \partial_\mu \partial_\nu = \nabla^2$, the first expression yields

$$\sigma(M) = \sigma^\mu M \bar{\sigma}^\nu \frac{1}{\nabla^2} \partial_\mu \partial_\nu. \tag{6.2}$$

This does not define an automorphism of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$. Indeed, writing $T_{\mu\nu} \equiv \frac{1}{\nabla^2}\partial_\mu\partial_\nu$ and $M_1^{\mu\nu} \equiv \sigma^\mu M_1 \bar{\sigma}^\nu$, one gets

$$\begin{aligned}
\sigma(M_1)\sigma(M_2) &= (M_1^{\mu\nu} T_{\mu\nu}) \left(M_2^{\alpha\beta} T_{\alpha\beta} \right) && (6.3)\\
&= M_1^{\mu\nu} \left[T_{\mu\nu}, M_2^{\alpha\beta} \right] T_{\alpha\beta} + M_1^{\mu\nu} M_2^{\alpha\beta} T_{\mu\nu} T_{\alpha\beta}, && (6.4)\\
&= \sigma(M_1 M_2) + M_1^{\mu\nu} \left[T_{\mu\nu}, M_2^{\alpha\beta} \right] T_{\alpha\beta} && (6.5)
\end{aligned}$$

where we compute

$$\begin{aligned}
M_1^{\mu\nu} M_2^{\alpha\beta} T_{\mu\nu} T_{\alpha\beta} &= \sigma^\mu M_1 \bar{\sigma}^\nu \sigma^\alpha M_2 \bar{\sigma}^\beta \frac{1}{\nabla^2} \frac{1}{\nabla^2} \partial_\mu \partial_\nu \partial_\alpha \partial_\beta \\
&= \sigma^\mu M_1 M_2 \bar{\sigma}^\beta \frac{1}{\nabla^2} \partial_\mu \partial_\beta \\
&= \sigma(M_1 M_2). && (6.6)
\end{aligned}$$

A possible solution is to look for a $\star$ product such that

$$\sigma(M_1) \star \sigma(M_2) = \sigma(M_1 \star M_2), \tag{6.7}$$

that would encode the intrinsic mixing between the manifold (space-time) and the matrix part (gauge sector) that is the core of the Grand Symmetry. This would also force us to consider an algebra $\mathcal{A}_0$ of pseudo-differential operators bigger than $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$. This point is particularly interesting if one believes that almost commutative geometries are an effective low energy description of a more fundamental theory, based on a "truly" non-commutative algebra (that is with a finite dimensional center). This idea has been often advertised by D. Kastler, and it could be that $\mathcal{A}_0$ is not so far from the "noncommutative salmon" he aims at fishing. All this will be investigated in future works.

The reason why we choose the representation (3.5) instead of (2.31) as in [23] is that while it is right that (6.2) is still in $\mathbb{M}_4(\mathbb{C})$, it would not be true for an element $Q = Q_{\dot{s}\alpha}^{\dot{t}\beta} \in M_2(\mathbb{H})$ that $\sigma^\mu Q \bar{\sigma}^\nu$ is still in $M_2(\mathbb{H})$. However, all the results presented in this paper would also be true with the representation (2.31), as explained in the next paragraph.

6.2 Invariance of the constraints

The grand algebra in the representation (3.5) is broken by the grading to [23, eq. (3.17)]

$$\mathcal{A}'_G = M_2(\mathbb{H})_L \oplus M_2(\mathbb{H})_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}). \tag{6.8}$$

To have bounded commutators with $\not{D}$, we impose by hand that quaternions act trivially on the $\dot{s}$ index, yielding the reduction to

$$\mathcal{A}' := \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}) \tag{6.9}$$

whose elements are (Q, M) where

$$Q = \delta_{s\dot{s}}^{t\dot{t}} \begin{pmatrix} q_R & 0_2 \\ 0_2 & q_L \end{pmatrix}_{\alpha\beta}, \quad M = \begin{pmatrix} M_l^l & 0_4 \\ 0_4 & M_r^r \end{pmatrix}_{st} \quad \text{with } q_r \in \mathbb{H}, M_l^l, M_r^r \in M_4(\mathbb{C}). \tag{6.10}$$

The twist ρ is still defined as the exchange of the left and right part of spinors, but it now acts on the matrix part

$$\rho(M) = \begin{pmatrix} M_r^r & 0_4 \\ 0_4 & M_l^l \end{pmatrix}_{st}. \tag{6.11}$$

This guarantees that

$$[\not{D}, M]_\rho = (\not\partial M) + [\gamma^\mu, M]_\rho = (\not\partial M) \tag{6.12}$$

is bounded, so that $(C^\infty(\mathcal{M}) \otimes \mathcal{A}', \mathcal{H}, \not{D} + D_M; \rho)$ is a twisted spectral triple. The twisted first-order condition for $\not{D}$ is checked as in proposition 3.4.

For the twisted first-order condition imposed by D_M, one first consider the subalgebra of $\mathcal{A}'$

$$\tilde{\mathcal{A}} := \mathbb{H}_L \oplus \mathbb{C}_R \oplus M_3^l(\mathbb{C}) \oplus \mathbb{C}^l \oplus M_3^r(\mathbb{C}) \oplus \mathbb{C}^r \tag{6.13}$$

obtained by asking

$$q_R = \begin{pmatrix} c_R & 0 \\ 0 & \bar{c}_R \end{pmatrix} \quad \text{with } c_R \in \mathbb{C} \tag{6.14}$$

in (6.10) and

$$M_r^r = \begin{pmatrix} m^r & 0_2 \\ 0_2 & \mathbf{M}^r \end{pmatrix}_{\mathrm{IJ}}, \quad M_l^l = \begin{pmatrix} m^l & 0_2 \\ 0_2 & \mathbf{M}^l \end{pmatrix}_{\mathrm{IJ}} \text{ with } \mathbf{M}^r, \mathbf{M}^l \in M_3(\mathbb{C}),\ m^r, m^l \in \mathbb{C}. \tag{6.15}$$

Let $B = (R, N) \in \tilde{\mathcal{B}}$ be another element of $\tilde{\mathcal{A}}$, with components $d_r, n^r, n^l \in \mathbb{C}$ and $\mathbf{N}^r, \mathbf{N}^l \in M_3(\mathbb{C})$. The double twisted commutator $[[D_M, A]_\rho,\ JBJ^{-1}]_\rho$ is an off-diagonal matrix with components

$$(\mathsf{D}_M M - Q\mathsf{D}_M)\bar{R} - \rho(\bar{N})(\mathsf{D}_M M - Q\mathsf{D}_M), \tag{6.16}$$
$$(\mathsf{D}_M Q - \rho(M)\mathsf{D}_M)\bar{N} - \bar{R}(\mathsf{D}_M Q - \rho(M)\mathsf{D}_M). \tag{6.17}$$

One has

$$\rho(\bar{N})\mathsf{D}_\nu M = (\rho(\bar{N})\eta\Xi M)_{s\mathrm{J}}^{t\mathrm{I}}(\Xi\delta)_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \bar{\mathsf{n}}^l\mathsf{m}^\mathsf{r} & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^r\mathsf{m}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.18}$$

$$\rho(\bar{N})Q\mathsf{D}_\nu = (\rho(\bar{N})\eta\Xi)_{s\mathrm{J}}^{t\mathrm{I}}\,(Q\Xi)_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \bar{\mathsf{n}}^l & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^r \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4 \\ 0_4 & \mathsf{c}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.19}$$

$$\mathsf{D}_\nu M\bar{R} = (\eta\Xi M)_{s\mathrm{J}}^{t\mathrm{I}}\,(\Xi\bar{R})_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \mathsf{m}^r & 0_4 \\ 0_4 & -\mathsf{m}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.20}$$

$$Q\mathsf{D}_\nu\bar{R} = (\eta\Xi)_{s\mathrm{J}}^{t\mathrm{I}}\,(Q\Xi\bar{R})_{\dot{s}\alpha}^{\dot{t}\beta} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R\bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \mathsf{c}_R\bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.21}$$

where we defined

$$\mathsf{m}^\mathsf{r} := \begin{pmatrix} m^r & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{m}^\mathsf{l} := \begin{pmatrix} m^l & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{c}_R = \begin{pmatrix} c_R & 0 \\ 0 & 0_3 \end{pmatrix}_{\mathrm{IJ}} \tag{6.22}$$

and similarly for n^r, n^l and d_R. Collecting the various terms, one finds that (6.16) is zero if and only if

$$(c_R - m^r)(\bar{d}_R - \bar{n}^l) = 0, \quad (c_R - m^l)(\bar{d}_R - \bar{n}^r) = 0 \tag{6.23}$$

which are the same constraints (3.47) coming from the other representation. The same is true for (6.17), using

$$\bar{R}\rho(M)\mathsf{D}_\nu = (\rho(M)\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi\bar{R})^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \mathsf{m}^l & 0_4 \\ 0_4 & -\mathsf{m}^r \end{pmatrix}_{st} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.24}$$

$$\bar{R}\mathsf{D}_\nu Q = (\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(\bar{R}\Xi Q)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R\bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \mathsf{c}_R\bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.25}$$

$$\rho(M)\mathsf{D}_\nu\bar{N} = (\rho(M)\eta\Xi\bar{N})^{t\mathrm{I}}_{s\mathrm{J}}(\Xi)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \mathsf{m}^l\,\bar{\mathsf{n}}^r & 0_4 \\ 0_4 & -\mathsf{m}^r\,\bar{\mathsf{n}}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.26}$$

$$\mathsf{D}_\nu Q\bar{N} = (\eta\Xi\bar{N})^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi Q)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \bar{\mathsf{n}}^r & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4 \\ 0_4 & \mathsf{c}_R \end{pmatrix}_{\dot{s}\dot{t}}. \tag{6.27}$$

Solving (3.47) by asking $m^r = c_R$, that is identifying $\mathbb{C}^r$ and $\mathbb{C}_R$ with a single copy $\mathbb{C}^r_R$ of the complex numbers, one reduces $\tilde{A}$ to

$$\mathcal{A} := \mathbb{H}_L \oplus \mathbb{C}^r_R \oplus \mathbb{C}^l \oplus M^l_3(\mathbb{C}) \oplus M^r_3(\mathbb{C}). \tag{6.28}$$

This algebra plays for the representation (2.31) the same role as the algebra $\mathcal{B}$ for the representation (3.5). Repeating the computation of §4.3, one finds a scalar field similar to $\boldsymbol{\sigma}$. Thus, except for the hope of a global twist described in §6.1, there is at the moment no motivation to prefer one or the other of the two natural representations of the grand algebra.

7 Conclusion

Let us summarize our results by the following chain of breaking, to be compared with (2.32):

$$\begin{array}{rcl}
\mathcal{A}_G & = & M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \\
& \Downarrow & \text{grading condition} \\
\mathcal{B}_{LR} & = & (\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{H}^l_R \oplus \mathbb{H}^r_R) \oplus M_8(\mathbb{C}) \\
& \Downarrow & \text{bounded commutator for } M_8(\mathbb{C}) \\
\mathcal{B}' & = & (\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{H}^l_R \oplus \mathbb{H}^r_R) \oplus M_4(\mathbb{C}) \\
& \Downarrow & 1^{\text{st}}\text{-order for the Majorana-Dirac operator} D_M \\
\mathcal{B} & = & (\mathbb{H}^l_L \oplus \mathbb{C}^r_L \oplus \mathbb{H}^l_R \oplus \mathbb{C}^r_R) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \text{ with } \mathbb{C} = \mathbb{C}^r_R \\
& \Downarrow & \text{minimum of the spectral action} \\
\mathcal{A}_{sm} & = & \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
\end{array}$$

Starting with the "not so grand algebra" $\mathcal{B}$, one builds a twisted spectral triple whose fluctuations generate both an extra scalar field $\boldsymbol{\sigma}$ and an additional vector field X_μ. This is a Pati-Salam like model - the unitary of $\mathcal{B}$ yields both an $SU(2)_R$ and an $SU(2)_L$, together with an extra $U(1)$ - but in a pre-geometric phase since the Lorentz symmetry (in our case: the Euclidean $SO(n)$ symmetry) is not explicit. The spectral action spontaneously breaks this model to the standard model, in which the Lorentz symmetry is explicit, with the scalar and the vector fields playing a role similar as the one of Higgs field. We thus have a dynamical model of emergent geometry.

The idea that the scalar field σ is associated to the spontaneous breaking of a bigger symmetry to the standard model had been formulated in [23], but, was not fully implemented, because the fluctuation of the free Dirac operator by the grand algebra $\mathcal{A}_G$ yields an operator whose square is a non-minimal Laplacian. The heat kernel expansion of such operators is notably difficult to compute. Almost simultaneously, a similar idea has been implemented in [12], where the Pati-Salam like symmetry does not come from a bigger algebra, but follows from relaxing the first-order condition. It would be interesting to understand to what extend the twisted fluctuations presented here are a particular case of those inner fluctuation without first oder condition. More generally, the structure of the set of twisted fluctuations and of the associated twisted-gauge transformations of $\mathbb{A}$ needs to be worked out. Let us mention a possibly relevant notion of twisted connections, explained for instance in [28].

The twist ρ is remarkably simple, and its mathematical significance should be studied more in details, in particular how it should be incorporated in the axioms of noncommutative geometry, like the orientability condition where the commutator with the Dirac operator plays a crucial role. Also, the physical meaning of the twist is intriguing: the un-twisting of $\mathcal{B}$ forces the action of the algebra to be the same on the left and right components of spinors. In this sense the breaking of the grand algebra to the standard model is a sort of “primordial” chiral symmetry breaking.

Full phenomenology and comparison with [11] require to take into account all fermions, not only the right neutrino. This means to compute the spectral action of $\not{\partial} + D_M + \gamma^5 \otimes D_0$. This would also allow to check that our $\boldsymbol{\sigma}$ couples to the Higgs as σ does in [8]. The simultaneous occurrence of both a scalar and a vector field offers interesting perspective for physics beyond the standard model. A similar phenomena appears in a recent proposal on how to generate the field $\boldsymbol{\sigma}$ in NCG [4], where it comes together with an additional bosonic field with $B - L$ charge.

Finally, let us mention a very recent work of Chamseddine, Connes and Mukhanov [10] where the algebra $\mathcal{A}_F$ for $a = 2$ is obtained without the ad-hoc symplectic hypothesis, but from an higher degree Heisenberg relation for the space-time coordinates. It would be interesting to understand whether the case $a = 4$ enters this framework.

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