Entanglment from integrability - Ising model example

based on 0706.3384

Form factors of the twist fields

One of the consequences of the integrability is that the scattering of asymptotic states is very simple - the number of particles is conserved, they have the same momenta before and after the scattering and the S-matrix is complitely determined by the 2 particle scattering. In the relativistic models this is just a phase factor $S_{ij}(\theta_1 - \theta_2)$, which depends on the types of particles scattered (denoted my index i and j) and on the difference of the rapidities of the particles θ, defined so that

$$E = m\cosh\theta \quad , \quad p = m\cosh\theta$$

where m is the rest mass of the particle.

Form factors are the matrix elements of some operator $\mathcal{O}(x)$ between the vacuum and some state with particles

$$F^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_k\rangle_{\mu_1,\ldots,\mu_k}$$

we will consider the simples theory with scalar particles. Because we have to take n copies of our theory in order to define the twist operator each particle will come with an index $\mu = 1\ldots n$ determining to which sheet the asymptotic particles belong.

We assume the following axioms

$$F^{\ldots,\mu_l,\mu_{l+1},\ldots}(\ldots,\theta_l,\theta_{l+1},\ldots) = S_{\mu_l,\mu_{l+1}}(\theta_l - \theta_{l+1})F^{\ldots,\mu_{l+1},\mu_l,\ldots}(\ldots,\theta_{l+1},\theta_l,\ldots)$$

$$F^{\mu_1,\mu_2,\ldots,\mu_k}(\theta_1 + 2\pi i,\theta_2,\ldots,\theta_k) = F^{\mu_2,\ldots,\mu_k,\mu_1+1}(\theta_2,\ldots,\theta_k,\theta_1)$$

$$-i\mathrm{Res}_{\bar\theta_0=\theta_0}F^{\mu_0,\mu_0,\mu_1,\ldots,\mu_k}(\bar\theta_0 + \pi i,\theta_0,\theta_1,\ldots,\theta_k) = F^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)$$

$$-i\mathrm{Res}_{\bar\theta_0=\theta_0}F^{\mu_0,\mu_0+1,\mu_1,\ldots,\mu_k}(\bar\theta_0 + \pi i,\theta_0,\theta_1,\ldots,\theta_k) = -\prod_{i=1}^{k} S_{\mu\mu_i}(\theta_0 - \theta_i)F^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)$$

- Define a substitution rules `Rule1[l]` and `Rule2` which will implement the first two transformations

```
F[μ1, μ2, μ3][θ1, θ2, θ3] /. Rule1[1]
```

$F(\mu_2,\mu_1,\mu_3)(\theta_2,\theta_1,\theta_3)\, S(\mu_1,\mu_2)(\theta_1-\theta_2)$

```
F[μ1, μ2, μ3][θ1, θ2, θ3] /. Rule2
```

$F(\mu_2,\mu_3,\mu_1+1)(\theta_2,\theta_3,\theta_1-2\,i\,\pi)$

- Assume there are only two particles. In this case `F[m1,m2][t1,t2]=F[m1,m2][t1-t2]`
Show that `F[i,i+k][t]=F[j,j+k][t]` for any i, j, k
Show that `F[1,j][t]=F[1,1][-t+2Pi I(j-1)]`
which means that only `f[t]=F[1,1][t]` is needed. Find all relations on `f[t]`

$F(1,1)(\theta) \;=\; F(1,2)(-\theta+2\,i\,\pi)$

$F(1,1)(\theta) \;=\; F(1,3)(-\theta+4\,i\,\pi)$

```
F[1, 1][θ] "=" S[θ] F[1, 1][-θ]
```

$F(1,1)(\theta) \;=\; S(\theta)\,F(1,1)(-\theta)$

• For the simples case of the Ising model we simply have $S(\theta) = -1$ so that `F[1,1][t]=-F[1,1][-t]` and `F[1,1][t]=F[1,1][-t+2Pi n i]` as it is a $4\pi n i$ periodic function we can introduce the variable $x = e^{\frac{\theta}{2n}}$

Write an ansatz for a function of x such that
1) it has only two poles at $\theta = \pm\pi i \bmod 2\pi i n$
2) decays at $\theta \to \pm\infty$
You should find:

```
f[θ] /. slA12 // FullSimplify
```

$$-\frac{\text{A3}\sinh\left(\frac{\theta}{2n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cosh\left(\frac{\theta}{n}\right)}$$

Fix the remaining constant by requiring the residue to be $iF0$

```
F211 = f[θ] /. slA12 /. slA3 // FullSimplify
```

$$\frac{\text{F0}\cos\left(\frac{\pi}{2n}\right)\csc\left(\frac{\pi-i\theta}{2n}\right)\csc\left(\frac{\pi+i\theta}{2n}\right)\sinh\left(\frac{\theta}{2n}\right)}{n}$$

Finally, find the full two point form-factor `F2[i,j][θ]` and define the corresponding function.

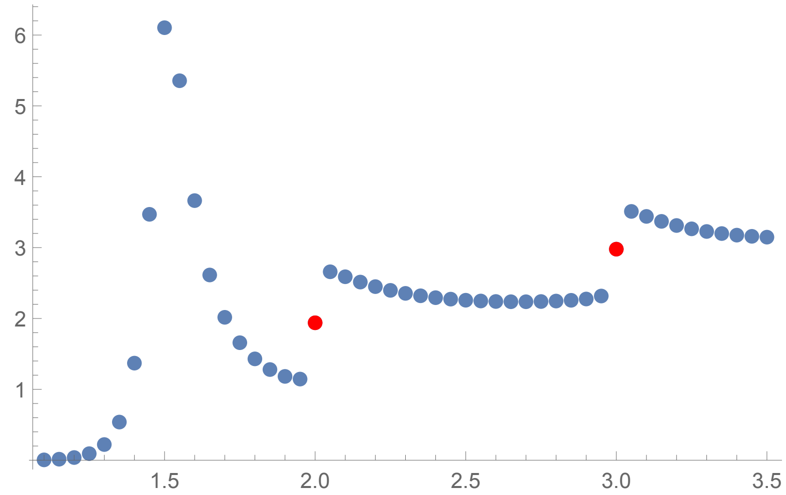
• The two-point function in the two-particle approximation is given by

$$\langle \mathcal{T}(r)\bar{\mathcal{T}}(0)\rangle = F_0^2 + \sum_{i,j=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{2!(2\pi)^2}|F_2^{ij}(\theta_1-\theta_2)|^2 e^{-rm(\cosh\theta_1+\cosh\theta_2)}$$

change the coordinates to $\theta = \theta_1 - \theta_2$ and $\eta = \frac{\theta_1+\theta_2}{2}$ and perform the integration in η

$$\frac{4\,\text{F0}^2\cos^2\left(\frac{\pi}{2n}\right)\left(\cos\left(\frac{2\pi(j-1)}{n}\right)-\cosh\left(\frac{\theta}{n}\right)\right)K_0\left(2\,m\,r\cosh\left(\frac{\theta}{2}\right)\right)}{n\left(\cos\left(\frac{\pi(3-2j)}{n}\right)-\cosh\left(\frac{\theta}{n}\right)\right)\left(\cosh\left(\frac{\theta}{n}\right)-\cos\left(\frac{\pi-2\pi j}{n}\right)\right)}$$

• We have to sum over $j = 1\ldots n$ and then analytically continue in n to the vicinity of $n = 1$ First, use the Poisonn resummation formula $\sum_{j^n} f_j = \sum_{k=-\infty}^{\infty}\int_0^n dj e^{-2\pi ijk} f_j dj$ and plot the result as below



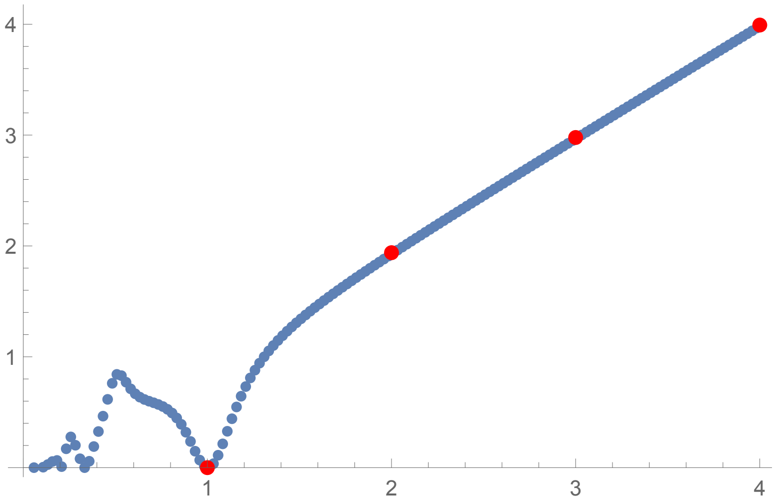
• Now use the pole decomposition in the variable j. You should get the followign representation. Check that the sum over the poles reproduces the initial expression

```
(*representation by poles*)
sing =
 F0^2
  ((4 Sin[π/(2 n)] Cos[π/(2 n)]^2 Csc[π/n] Sin[π/(2 n) + ⅈ θ/n] Csc[π/n + ⅈ θ/n] Csch[θ/n]
      (1/(θ - (2 π I (j - 1) + I π + 2 π I n l)) + 1/(θ + (2 π I (j - 1) - I π + 2 π I n l))) +
     4 Sin[π/(2 n)] Cos[π/(2 n)]^2 Csc[π/n] Sin[π/(2 n) - ⅈ θ/n] Csc[π/n - ⅈ θ/n] Csch[θ/n]
      (1/(θ + (2 π I (j - 1) + I π + 2 π I n l)) + 1/(θ - (2 π I (j - 1) - I π + 2 π I n l)))))
```

```
tosuminn / Sum[sing, {l, -∞, ∞}] /. F0 → 1 // FullSimplify
```

1

• Sum over n analytically. Choose the analytic continuation carefelly depending on the sign of l. In this way you should get the only good analytic continuation. Plot the result



• Check that in the limit $n \to 1$ the analytic continuation is simply proportional to the $\delta(\theta)$, which should lead to the final result

$$S_A = -\frac{1}{6}\log(\epsilon m) - \frac{1}{8}K_0(2rm)$$

(the first simple term comes from the renormalization of the twist field, which we do not consider here)