Perimeter Institute, August 24-29 2015
Mathematica Summer School

Lectures on Tensor Networks, Guifre Vidal (Perimeter Institute)

1- Tensor networks and many-body entanglement
Matrix product state (MPS)
2- Multi-scale entanglement renormalization ansatz (MERA)
3- Tensor network renormalization (TNR)

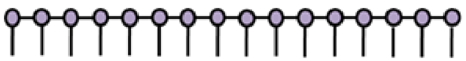
Slides used during the lectures
(Tuesday 25th - Thursday 27th 2015)

LECTURE 3

Summary MPS / MERA

- Important aspects of a Tensor Network?
    - ➢ efficient representation and *efficient computation*
    - ➢ structural properties (correlations and entanglement)

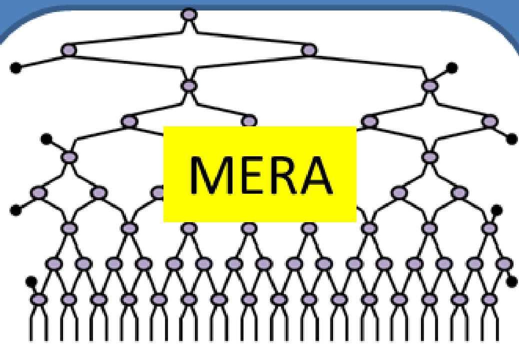
MPS



$$\langle \hat{o}(0)\hat{o}(L)\rangle \approx e^{-L/\xi}$$

$$S(A) \leq const$$

ground state
gapped Hamiltonian



$$\langle \hat{o}(0)\hat{o}(L)\rangle \approx \frac{1}{L^p}$$

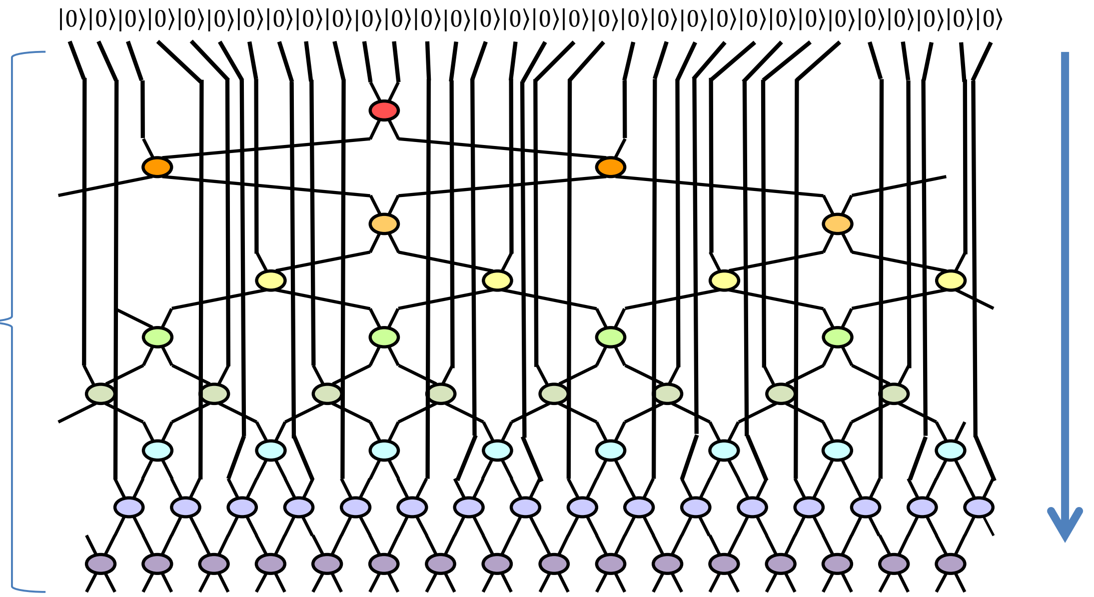
$$S(A) \approx \log L$$

ground state
gapless Hamiltonian

Important: In *practice,* MPS can also be used for critical systems!
and MERA can also be used for gapped systems!

MERA as a quantum circuit

quantum circuit U



"time"

ground state ansatz $|\Psi\rangle = U\,|0\rangle^{\otimes N}$

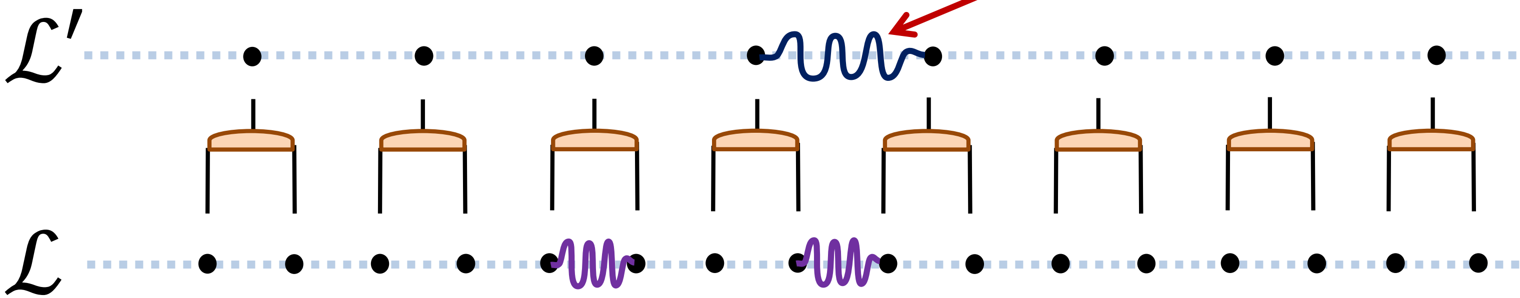
Entanglement introduced by gates at different "times" (= length scales)

MERA as a (real space) Renormalizatin Group transformation

Kadanoff (1966) + White (1992)

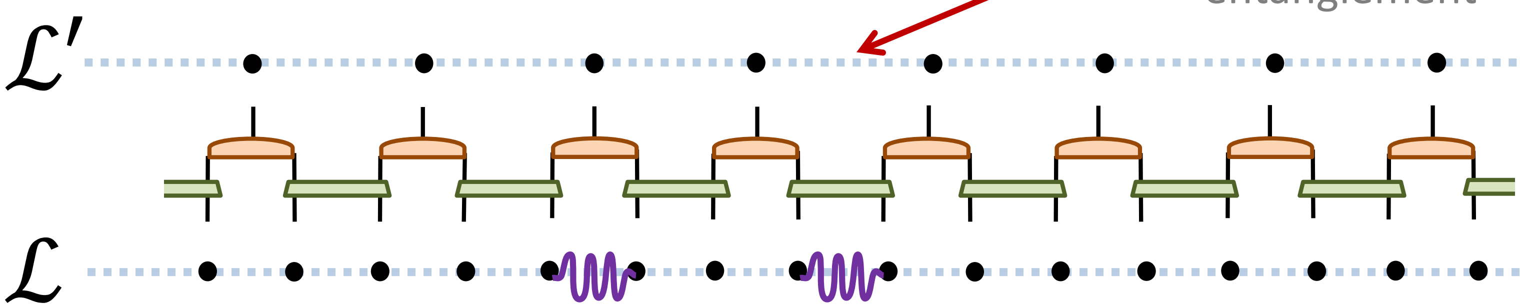
blocking   variational optimization

failure to remove ***some*** short-range entanglement !



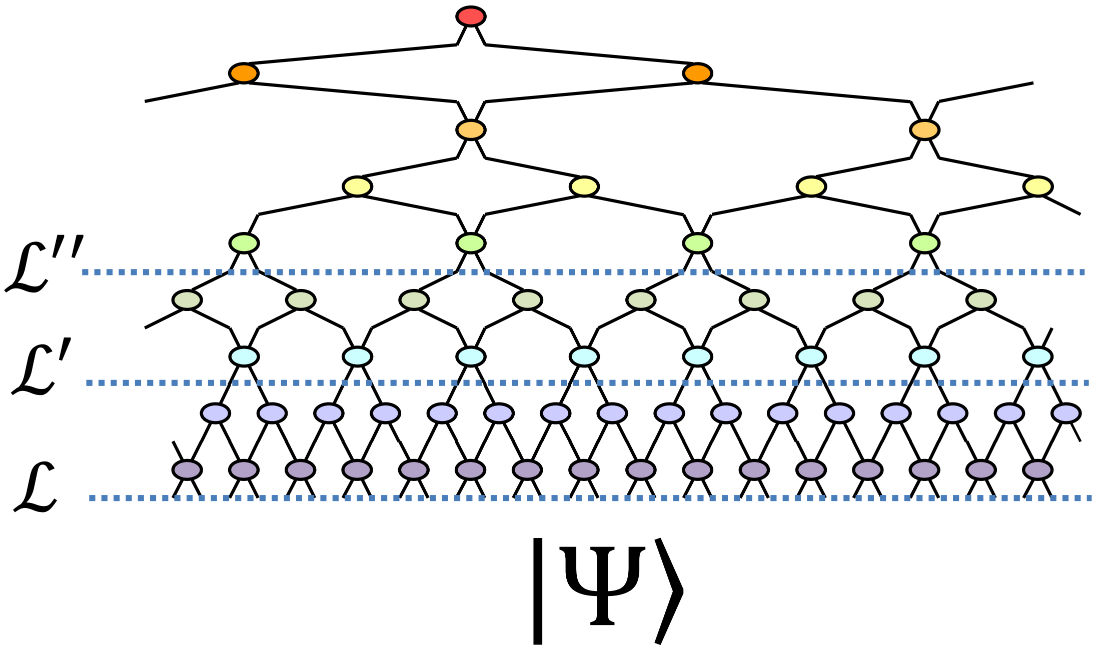
Entanglement renormalization (2005)

removal of ***all*** short-range entanglement



MERA as a (real space) Renormalizatin Group transformation

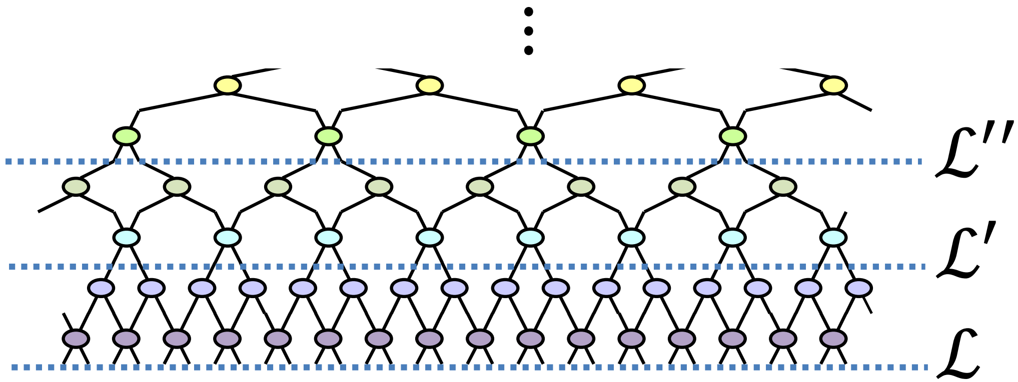
sequence of ground state wave-functions

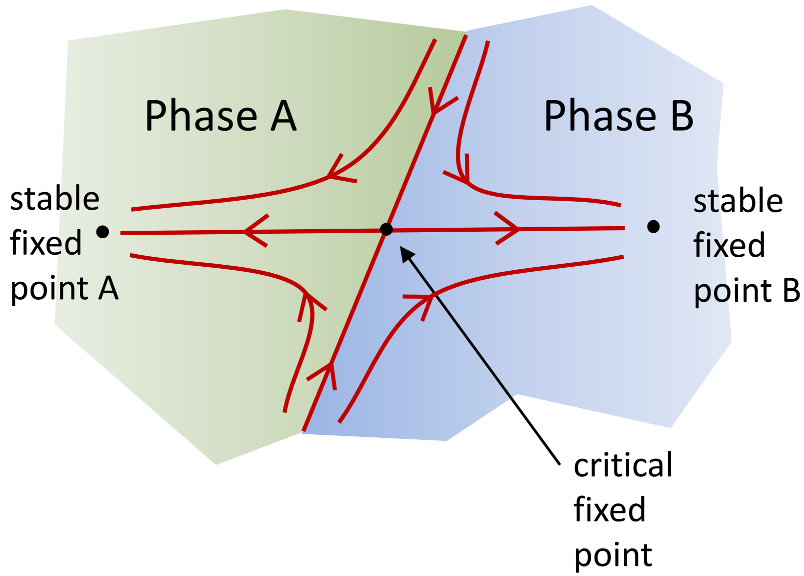


$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$

MERA defines an RG flow in the space of wave-functions

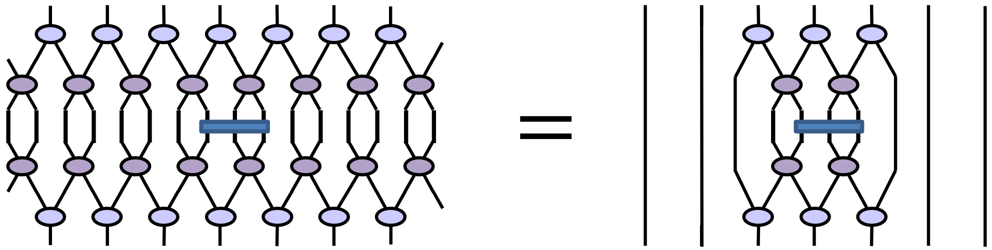
$$|\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow \cdots$$





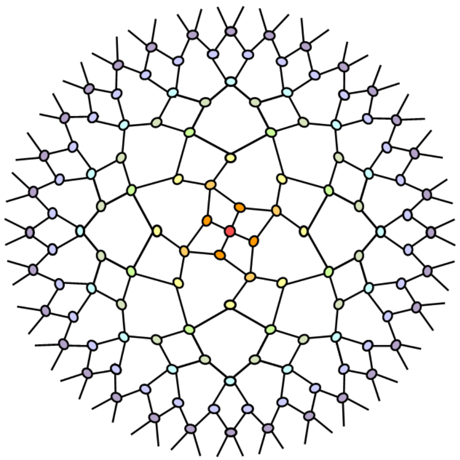
… and in the space of Hamiltonians

$$H \rightarrow H' \rightarrow H'' \rightarrow \cdots$$



local operators are mapped into local operators !

MERA and CFT



MERA

input

1D quantum Hamiltonian

- **on the lattice**
- **at a critical point**

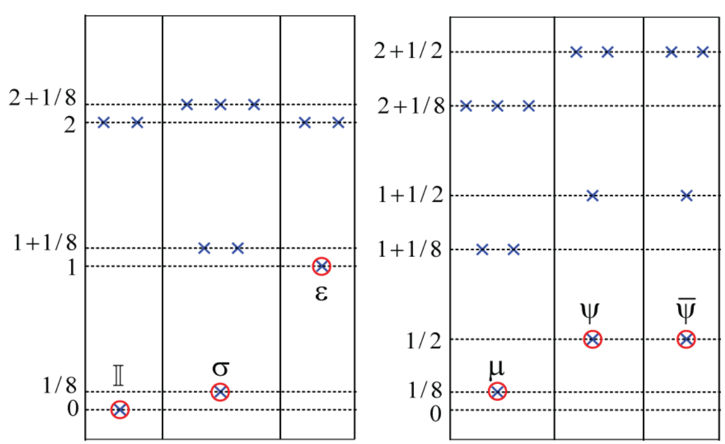
output

Numerical determination of conformal data:

- **central charge** c
- **scaling dimensions** $\Delta_\alpha \equiv h_\alpha + \bar{h}_\alpha$ **and conformal spins** $s_\alpha \equiv h_\alpha - \bar{h}_\alpha$
- **OPE coefficients** $C_{\alpha\beta\gamma}$

e.g. critical Ising model (approx. an hour on your laptop)

Pfeifer, Evenbly, Vidal 08



$(\Delta_{\mathbb{I}} = 0)$

$\Delta_\sigma \approx 0.124997$

$\Delta_\varepsilon \approx 0.99993$

$\Delta_\mu \approx 0.125002$

$\Delta_\psi \approx 0.500001$

$\Delta_{\bar{\psi}} \approx 0.500001$

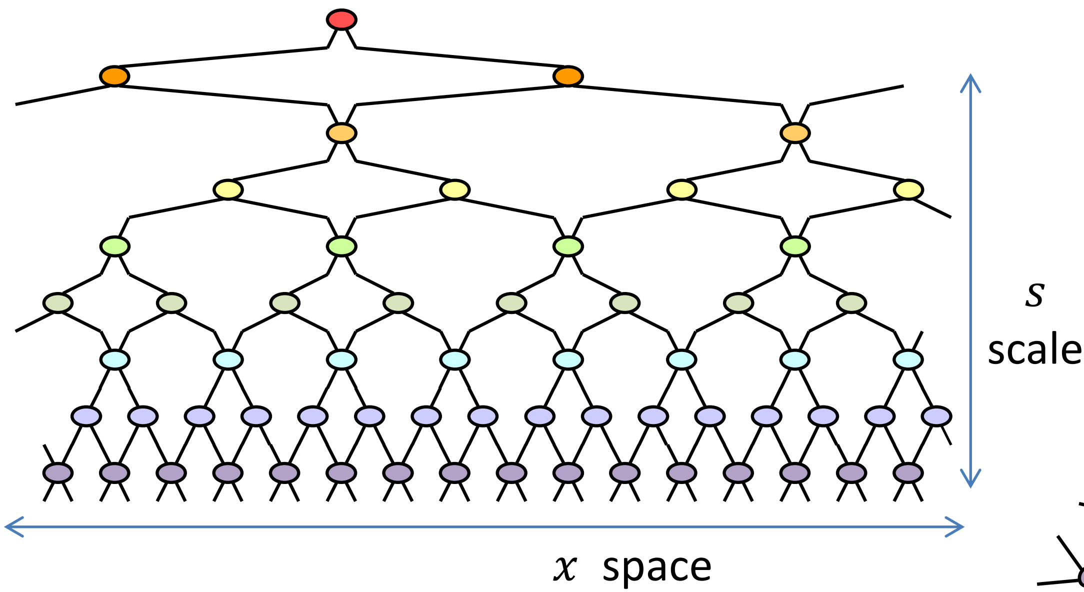
$C_{\epsilon\sigma\sigma} = \frac{1}{2}$ $\quad C_{\epsilon\mu\mu} = -\frac{1}{2}$

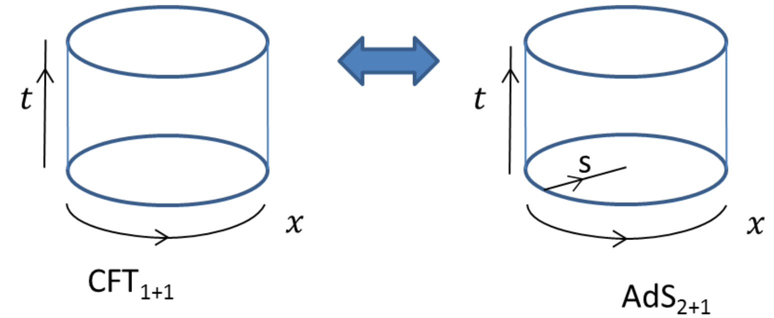
$C_{\epsilon\psi\bar{\psi}} = i$ $\quad C_{\epsilon\bar{\psi}\psi} = -i$

$C_{\psi\mu\sigma} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}$ $\quad C_{\bar{\psi}\mu\sigma} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}$

$(\pm 6 \times 10^{-4})$

MERA and holography?





- entanglement entropy

$$S_L \approx \log(L)$$

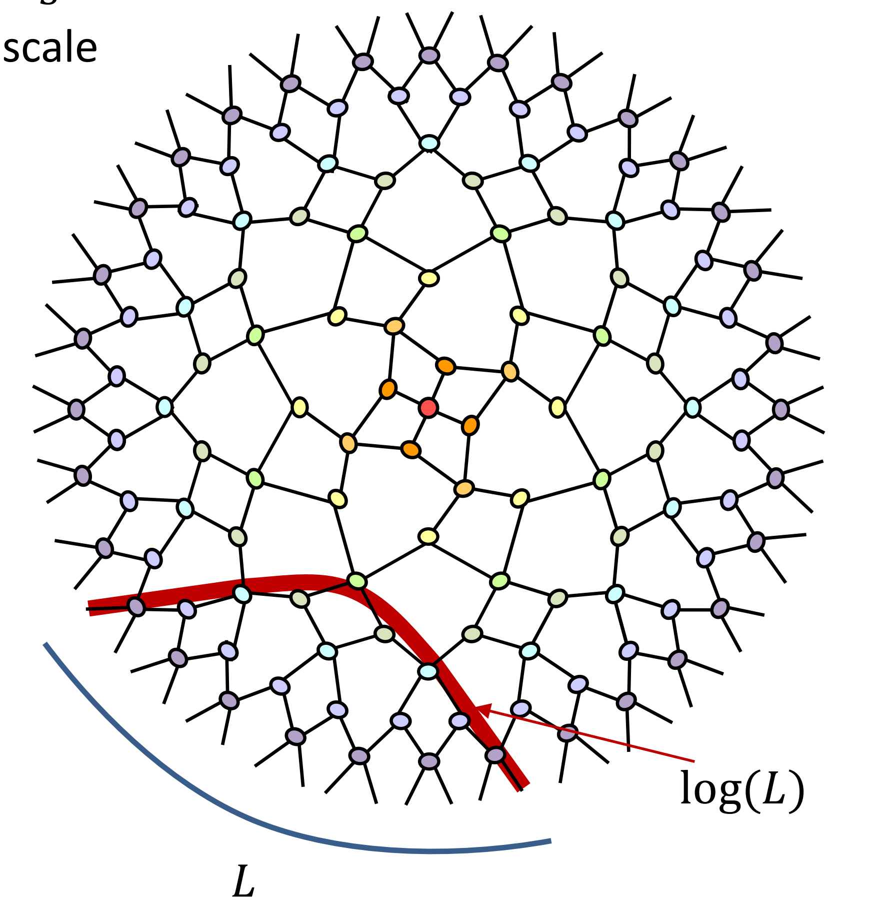
parallel to area of minimal surface in Ryu-Takayanagi

- two-point correlations

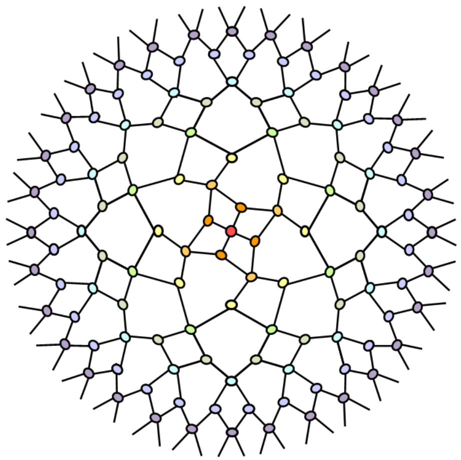
$$C(L) \approx L^{-2\Delta}$$

geodesic distance $D \approx \log(L)$ as in a hyperbolic geometry

$$C(L) \approx e^{-D} = e^{-2\Delta \log(L)} = L^{-2\Delta}$$



MERA and holography?



MERA
(2005)

MERA ↔ AdS/CFT

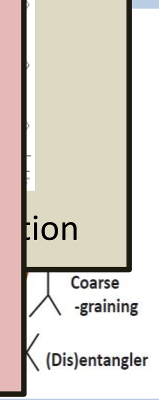
Swingle, 2009

“Entanglement renormalization for quantum fields”
Haegeman, Osborne, Verschelde, Verstraete, 2011

“Holographic Geometry of Entanglement Renormalization in Quantum Field Theories”
Nozaki, Ryu, Takayanagi, 2012

“Time Evolution of Entanglement Entropy from Black Hole Interiors”
Hartman, Maldacena, 2013

“Exact holographic mapping and emergent space-time geometry”
Xiaoliang Qi, 2013

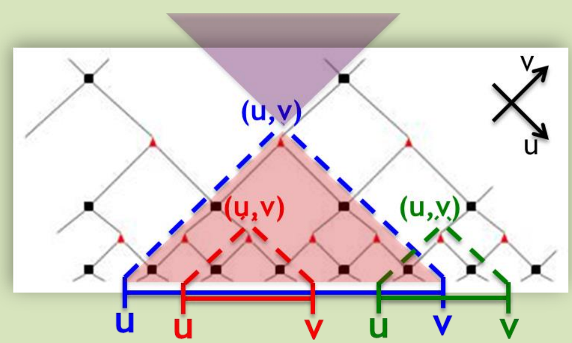


tion

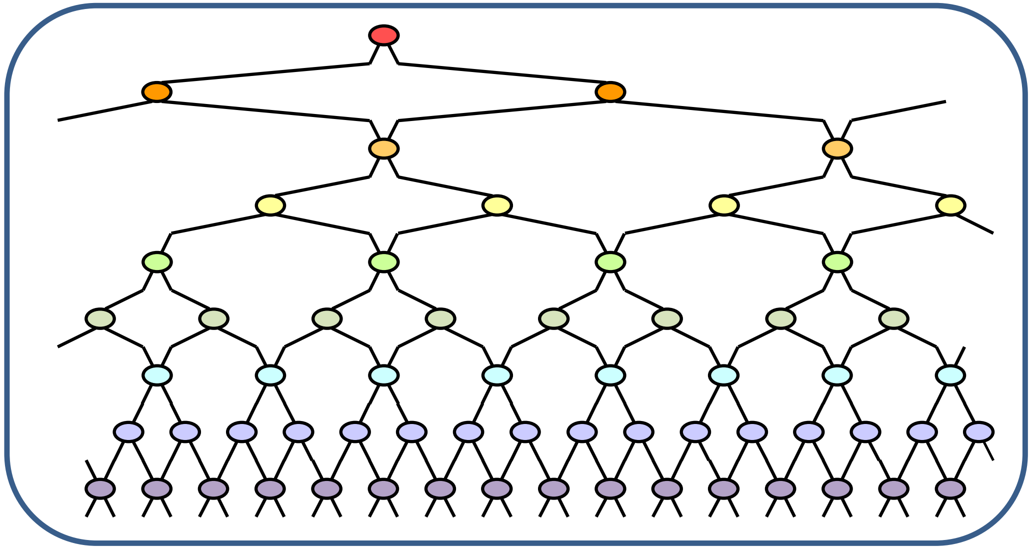
“Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence”
Pastawki, Yoshida, Harlow, Preskill, 2015

code

“Integral Geometry and Holography”
Czech, Lamprou, McCandlish, Sully, 2015

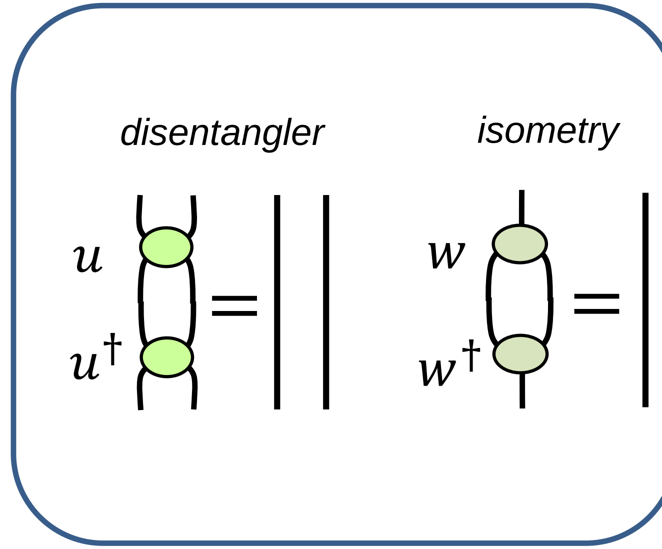


MERA = tensor network + isometric/unitary constraints



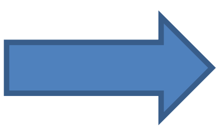
~ hyperbolic plane?

(Swingle 2009)



~ de Sitter space?

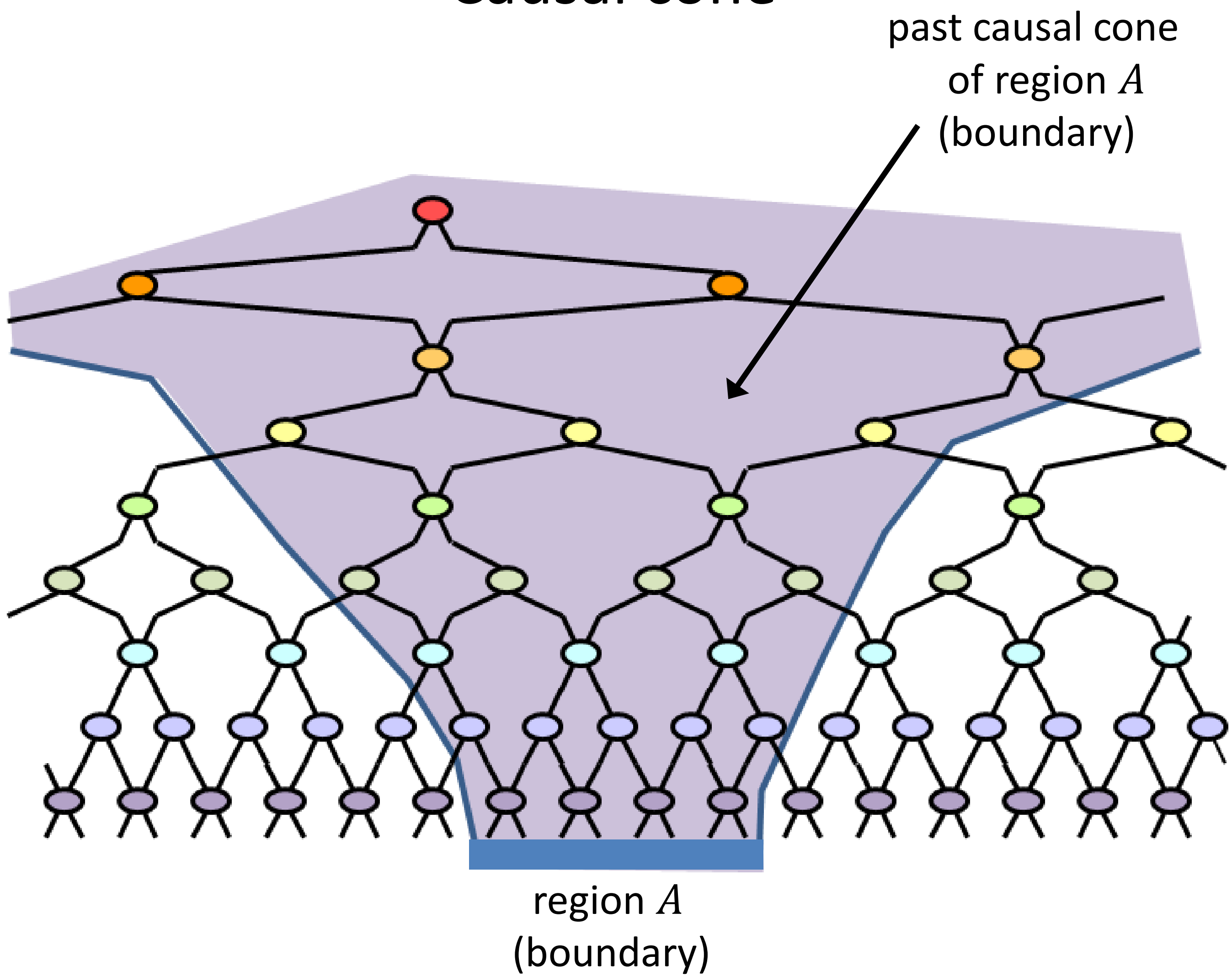
(Beny 2011, Czech 2015)



Causal structure

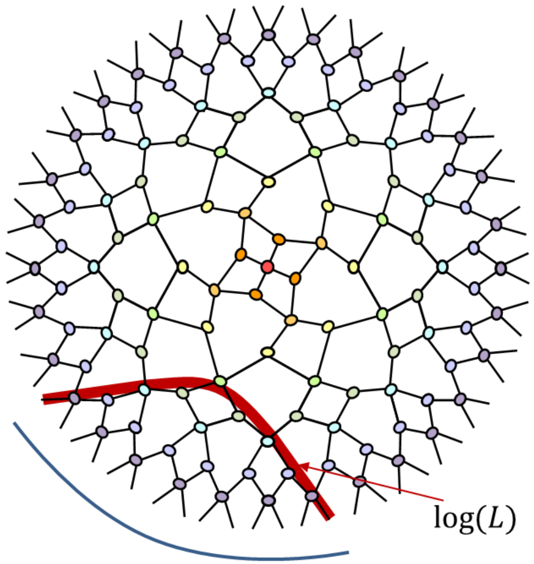
essential for many MERA properties
and computational efficiency

Causal cone



MERA = RG

Tensor network for ground state/Hilbert space of CFT, organized in extra dimension corresponding to scale



MERA represents a generic CFT *(no large N or strong interactions)*

e.g. for Ising model

MERA/CFT dictionary

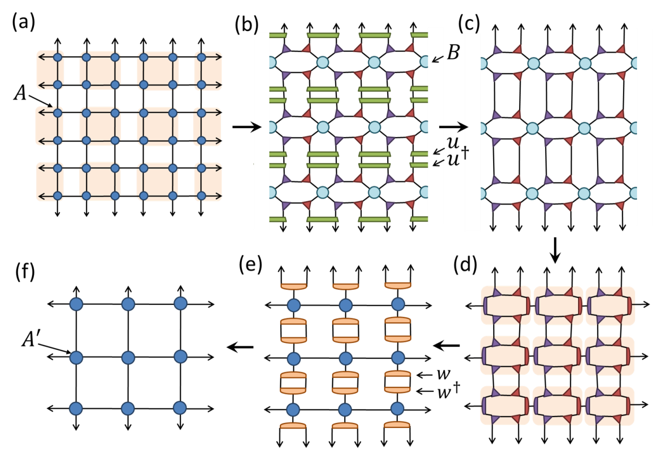
boundary	bulk
state of CFT (or Hilbert space of CFT)	tensor network (?) (or unitary map, EHM)
scaling dimension Δ	mass $\sim\Delta$
entanglement entropy	"minimal connecting surface"
global on-site symmetry (e.g. Z_2)	local/gauge symmetry (e.g. Z_2)

MERA operates at scale of AdS radius
For smaller scale? → cMERA

Useful testing ground / nice drawings

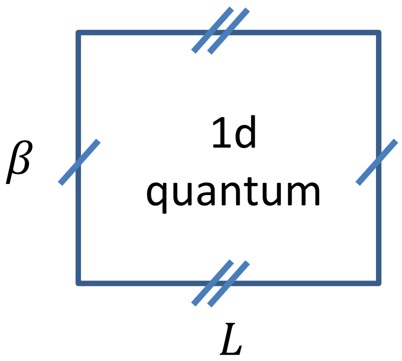
Generalized notion of *holographic* description?

“Lecture 3”- Tensor network renormalization (TNR)



Euclidean path integral

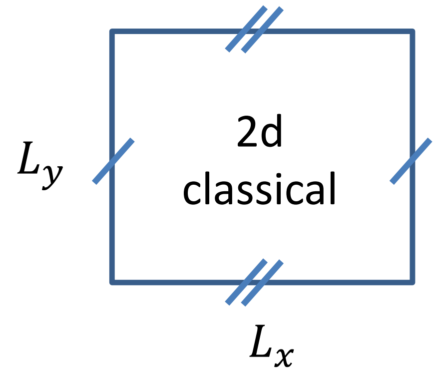
$$Z(\lambda) = tr\, e^{-\beta H_q^{1d}}$$



$\sim$

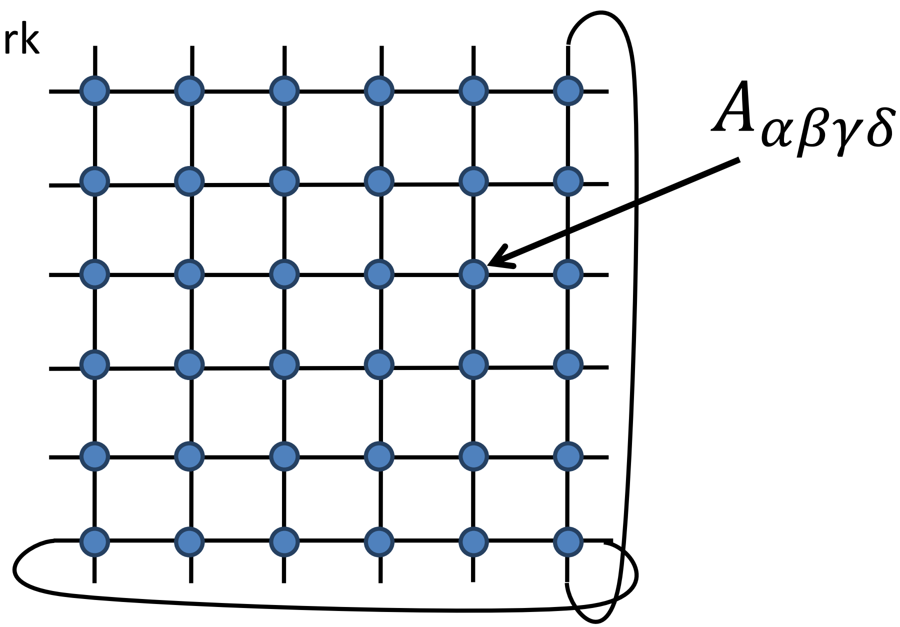
Statistical partition function

$$Z(T) = \sum_{\{s\}} e^{-\frac{1}{T} H_{cl}^{2d}}$$



as a tensor network

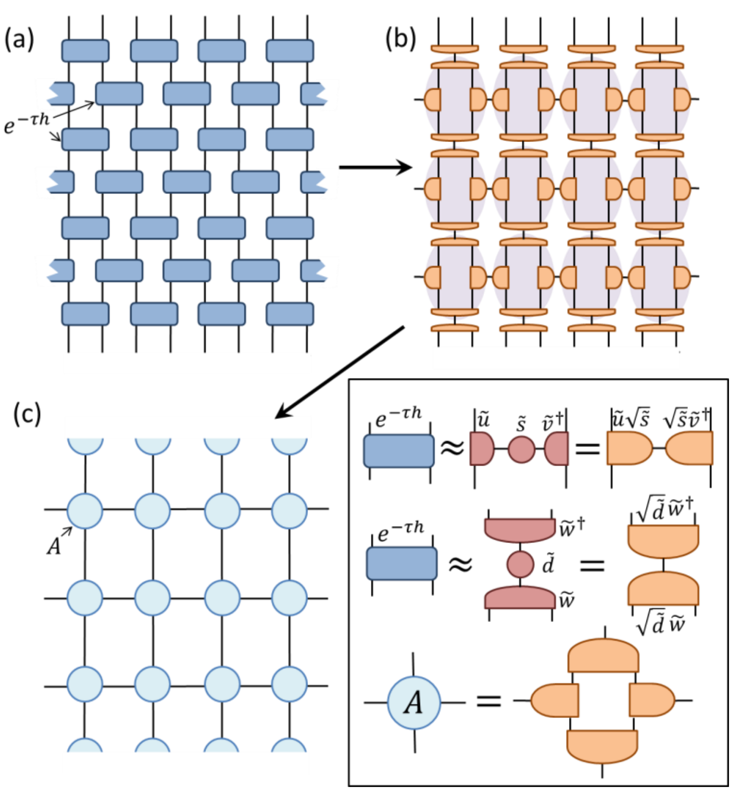
$$Z =$$



Euclidean path integral

$$Z(\lambda) = tr\ e^{-\beta H_q^{1d}}$$

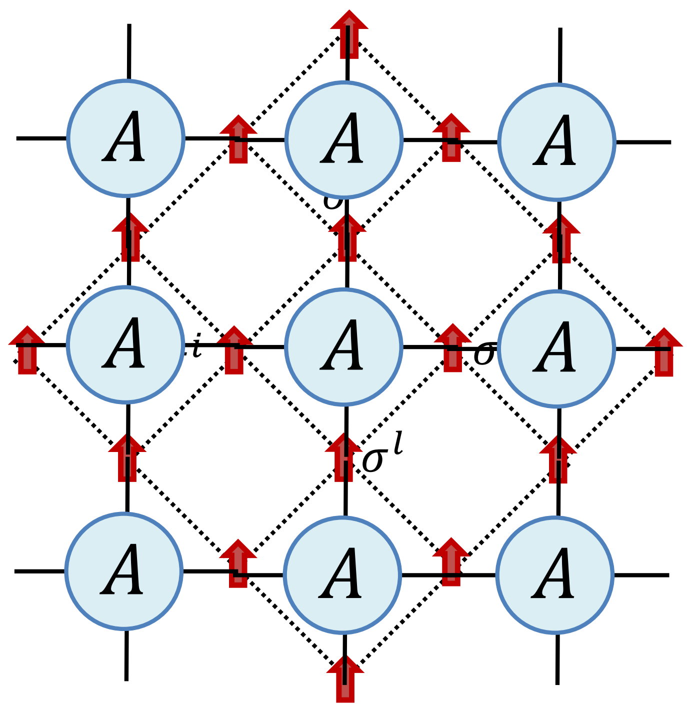
$$H_q^{1d} = \sum_i (\sigma_z^i + \sigma_x^i \sigma_x^{i+1})$$



Statistical partition function

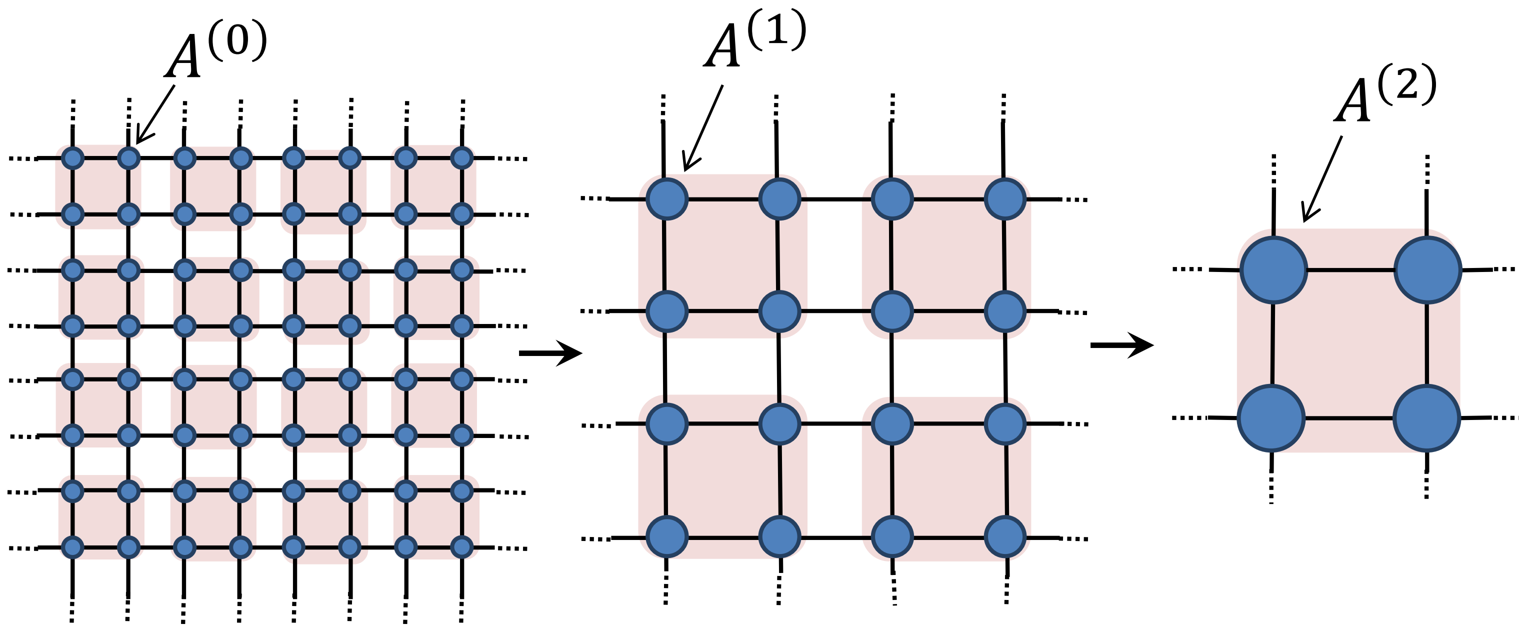
$$Z(T) = \sum_{\{s\}} e^{-\frac{1}{T} H_{cl}^{2d}}$$

$$H_{cl}^{2d} = \sum_{<i,j>} \sigma^i \sigma^j$$



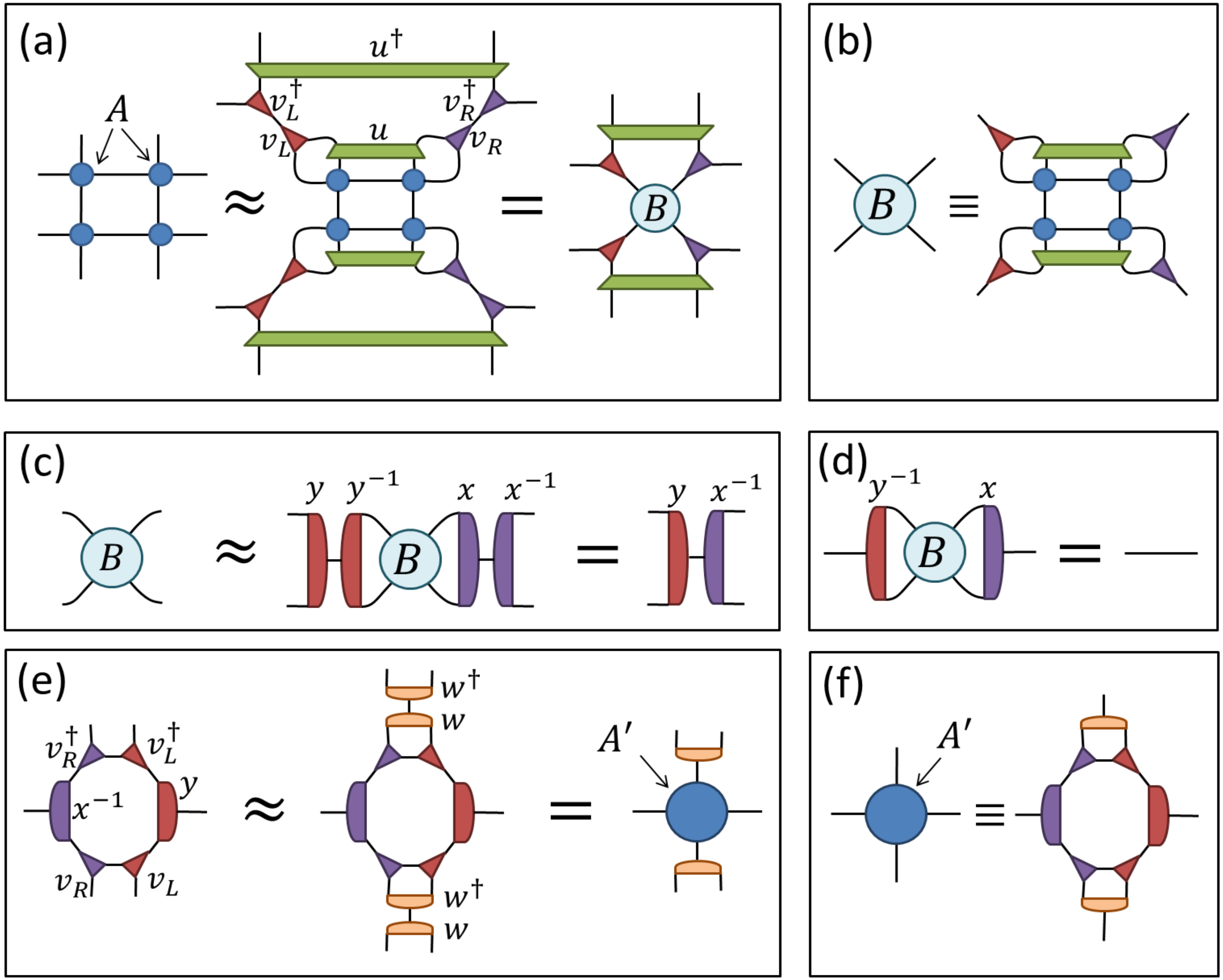
$$A_{ijkl} = e^{-(\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_l + \sigma_l\sigma_i)/T}$$

Goal: define an RG flow in the space of tensor networks



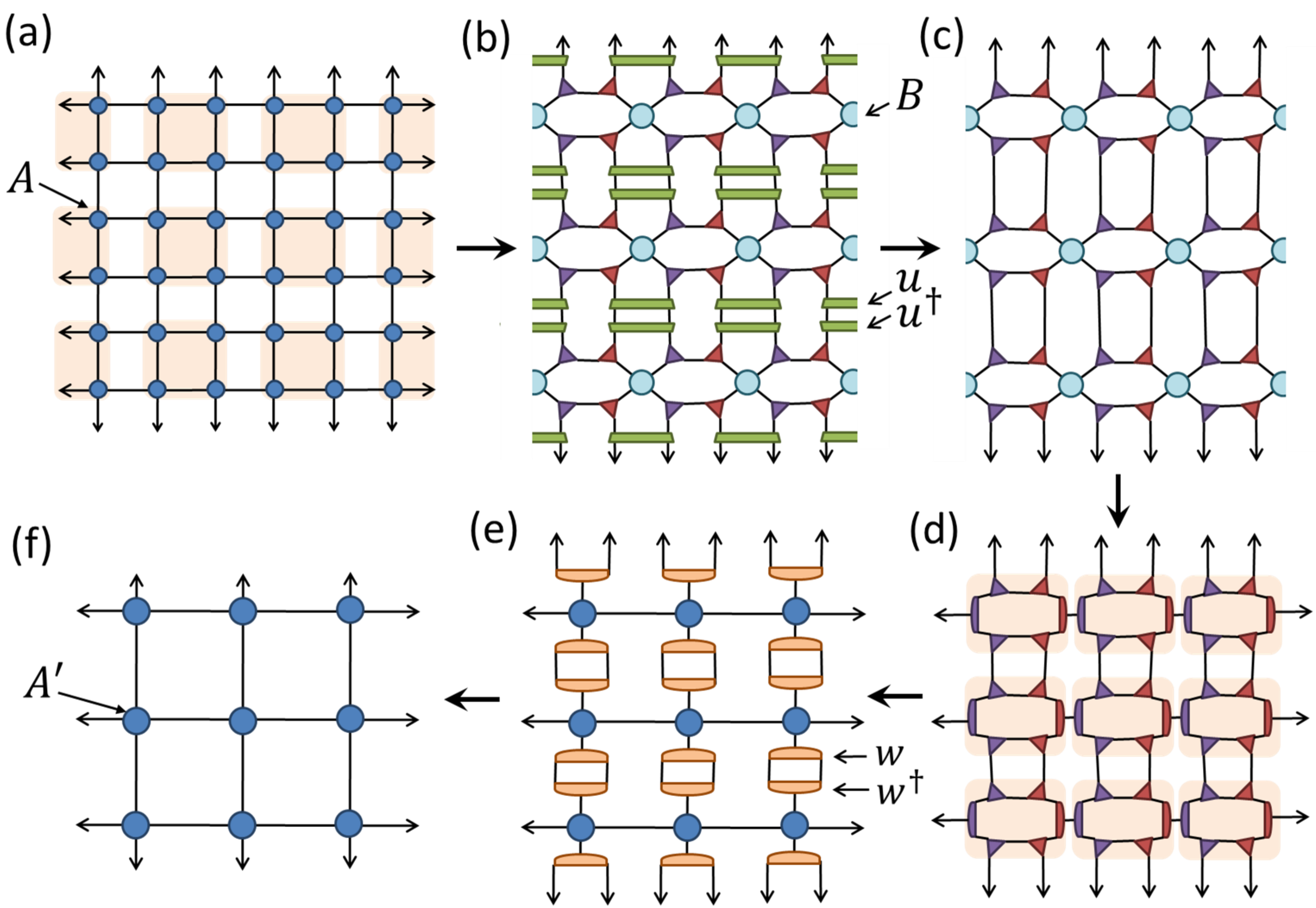
Tensor Network Renormalization (TNR)

[Evenbly, Vidal 14-15]

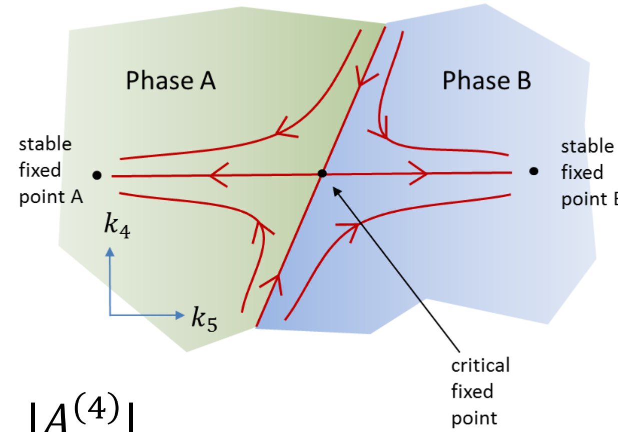


Tensor Network Renormalization (TNR)

[Evenbly, Vidal 14-15]



TNR -> proper RG flow
Example: 2D classical Ising



$$A \to A' \to A'' \to \cdots \to A^{fp}$$

$|A^{(1)}|$ $|A^{(2)}|$ $|A^{(3)}|$ $|A^{(4)}|$

below critical

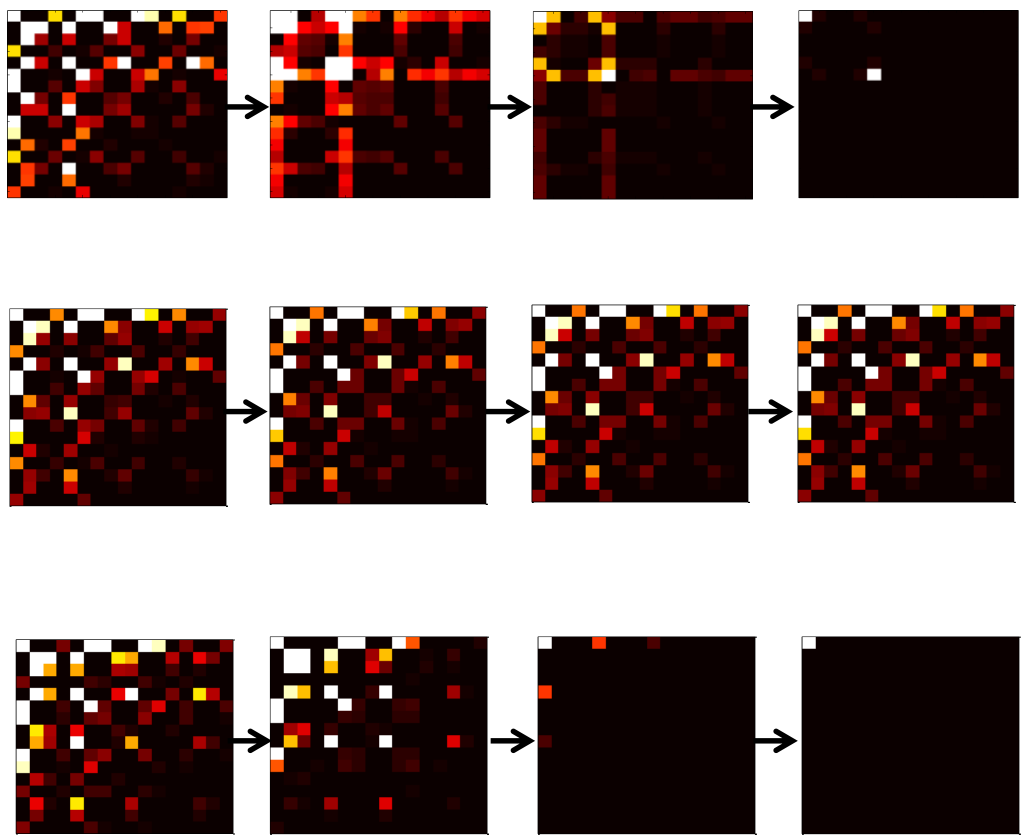
$T = 0.9\, T_c$

critical

$T = T_c$

above critical

$T = 1.1\, T_c$



ordered (Z2) fixed point

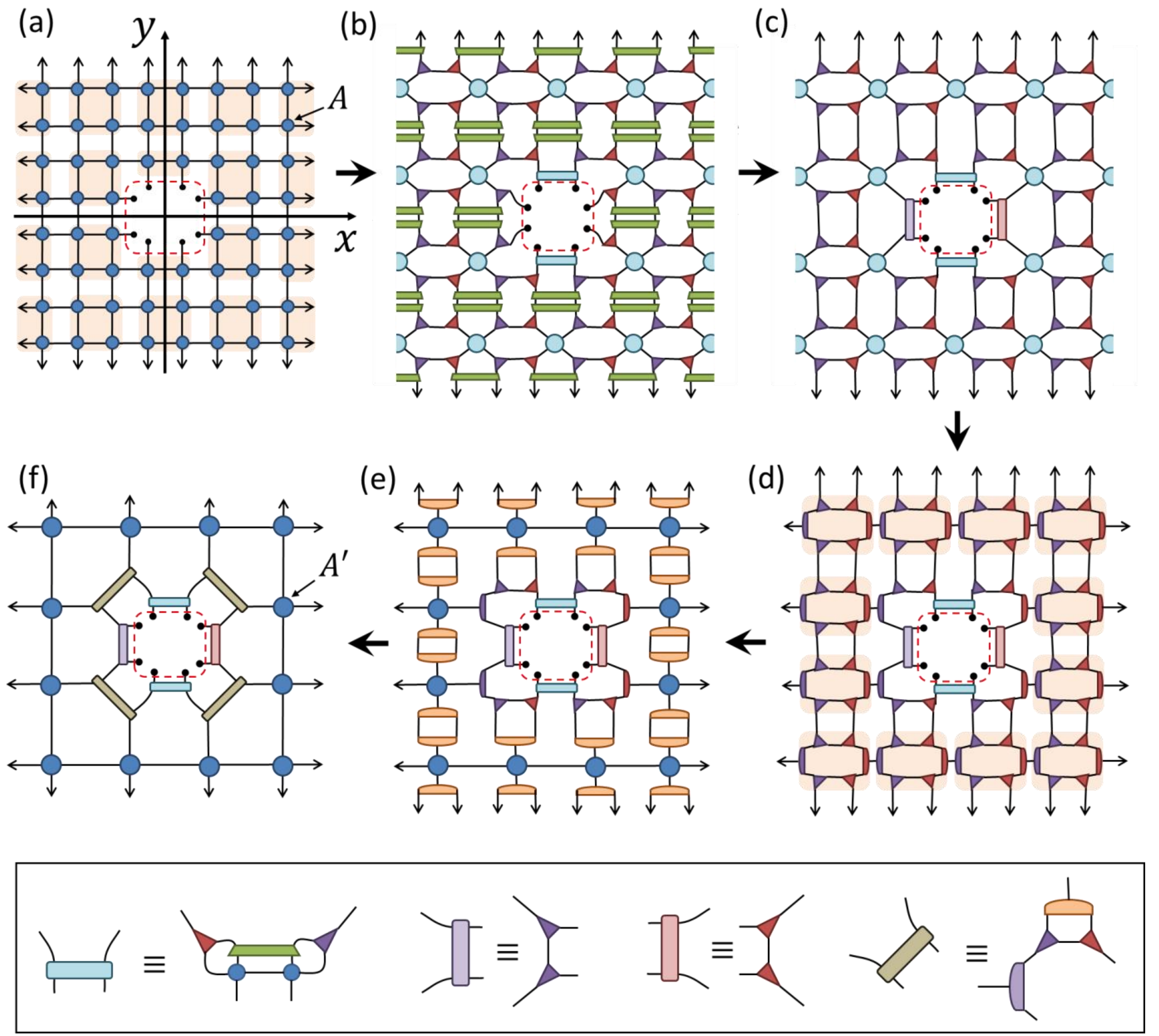
critical fixed point

disordered (trivial) fixed point

local scale transformations

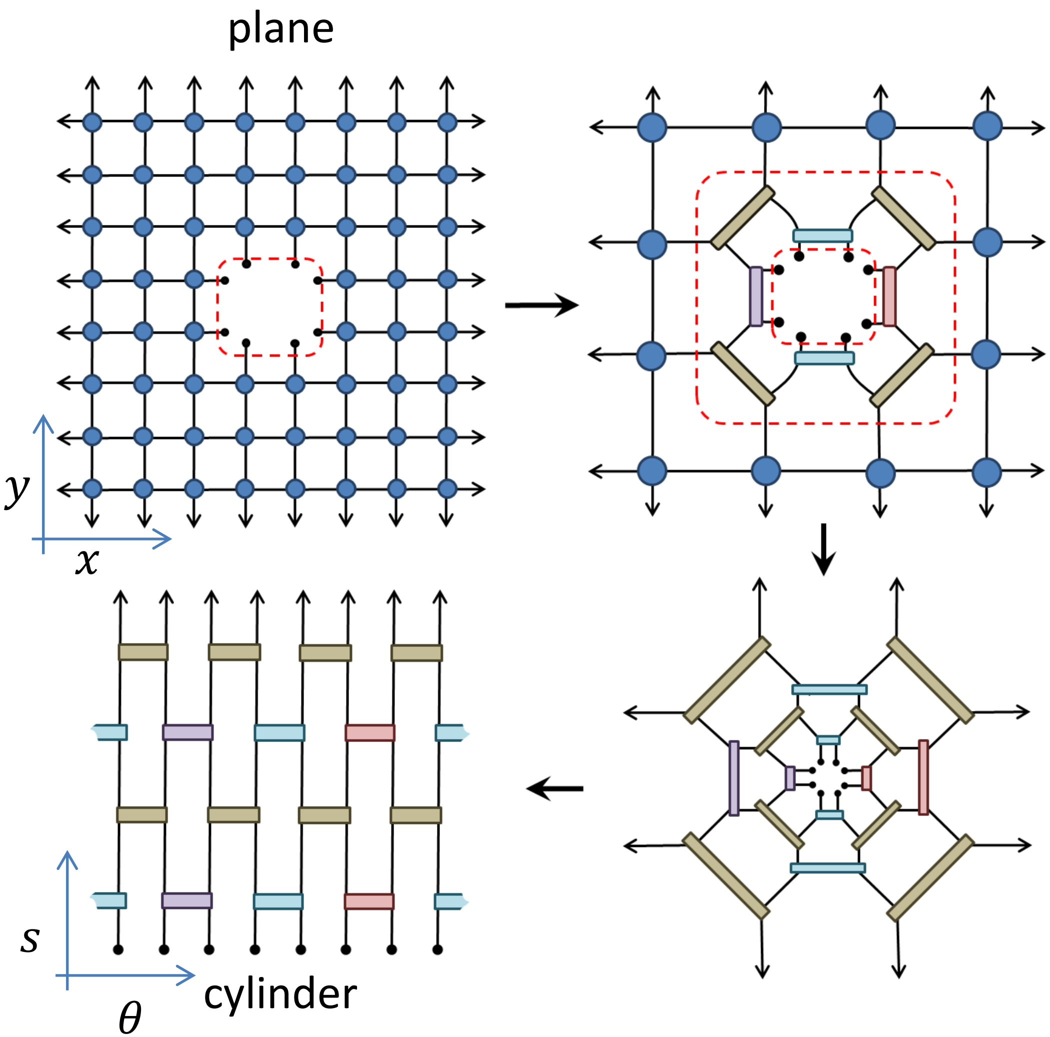
[Evenbly et al, in preparation]

example 1: Plane to cylinder



local scale transformations

[Evenbly et al, in preparation]



example 1: **Plane to cylinder**

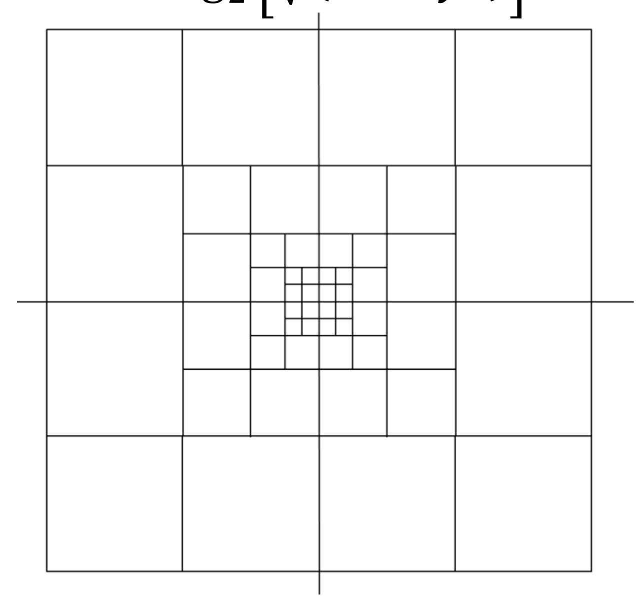
(radial quantization in CFT)

$$z \equiv x + iy$$

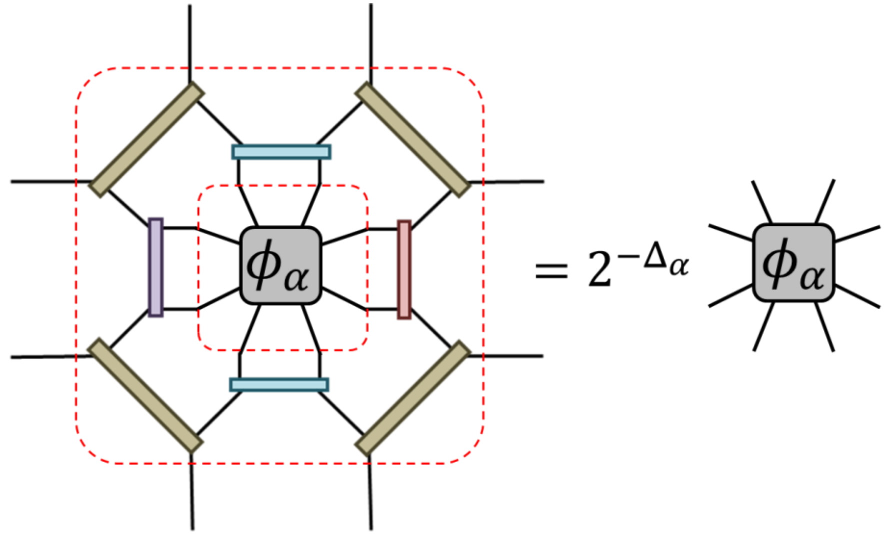
$$\boxed{z = 2^{w}}$$

$$w \equiv s + i\theta$$

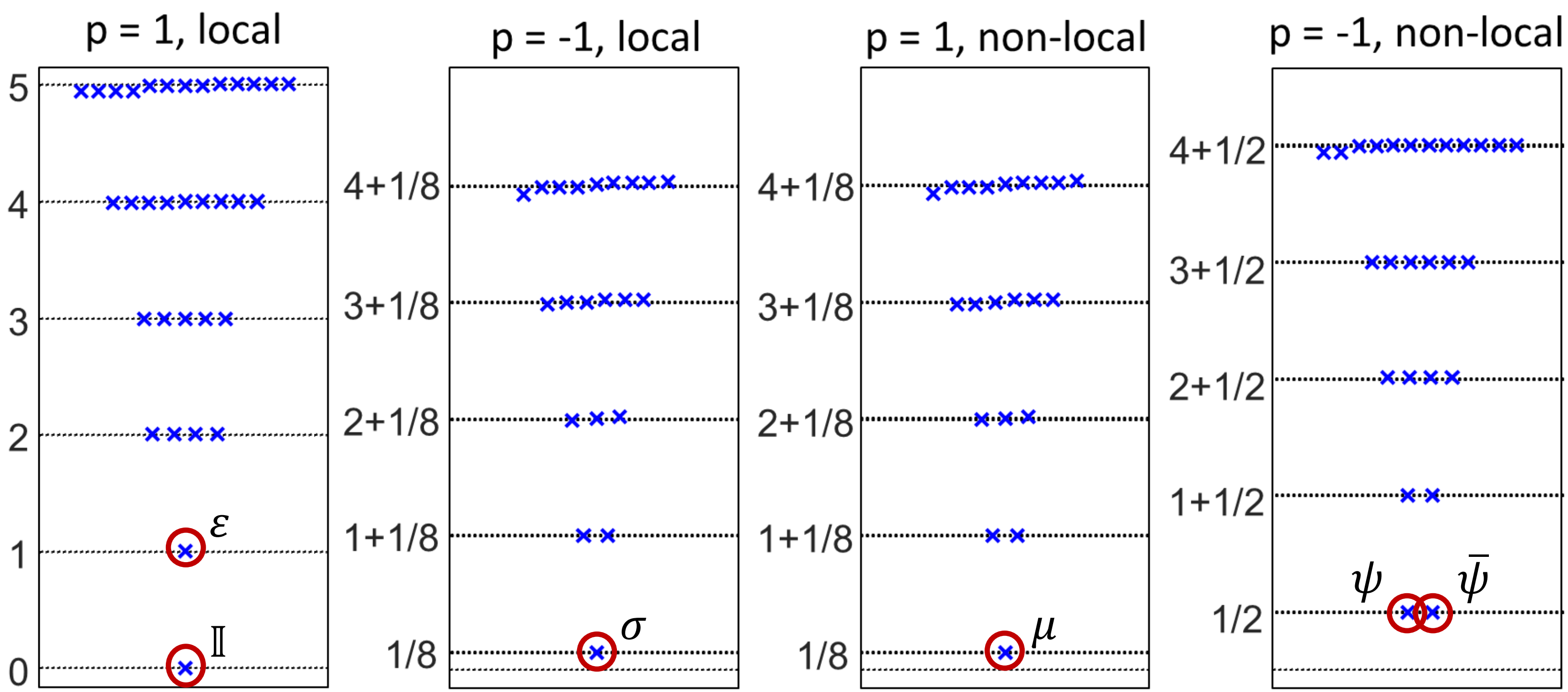
$$s \equiv \log_2 \left[\sqrt{(x^2 + y^2)} \right]$$

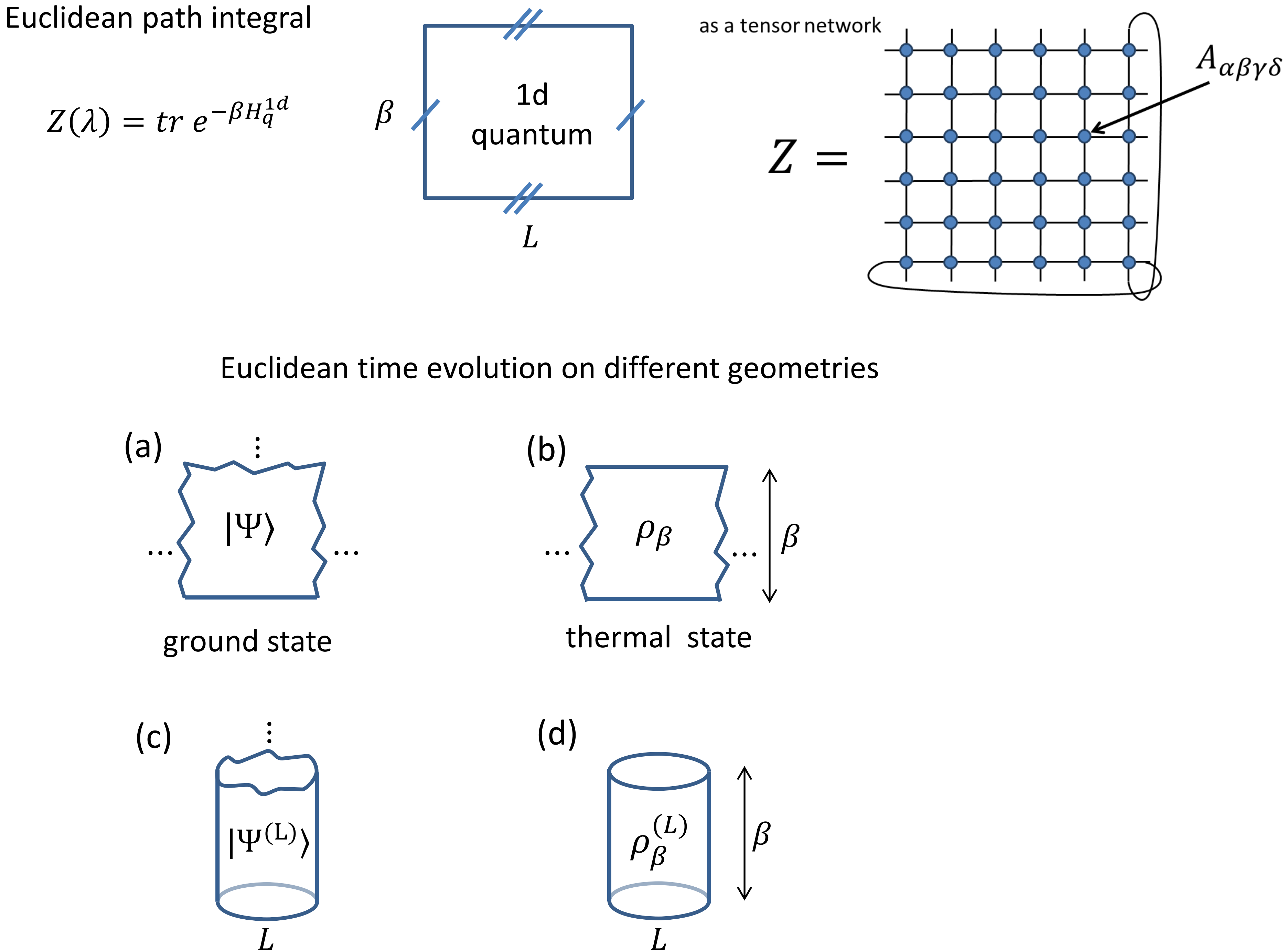


- Extraction of scaling dimensions, OPE



Example: 2D classical Ising





Euclidean path integral
$Z(\lambda) = tr\ e^{-\beta H_q^{1d}}$
β
1d
quantum
L
as a tensor network
$Z =$
$A_{\alpha\beta\gamma\delta}$
Euclidean time evolution on different geometries
(a)
$|\Psi\rangle$
ground state
(b)
ρ_β
β
thermal state
(c)
$|\Psi^{(L)}\rangle$
L
(d)
$\rho_\beta^{(L)}$
β
L

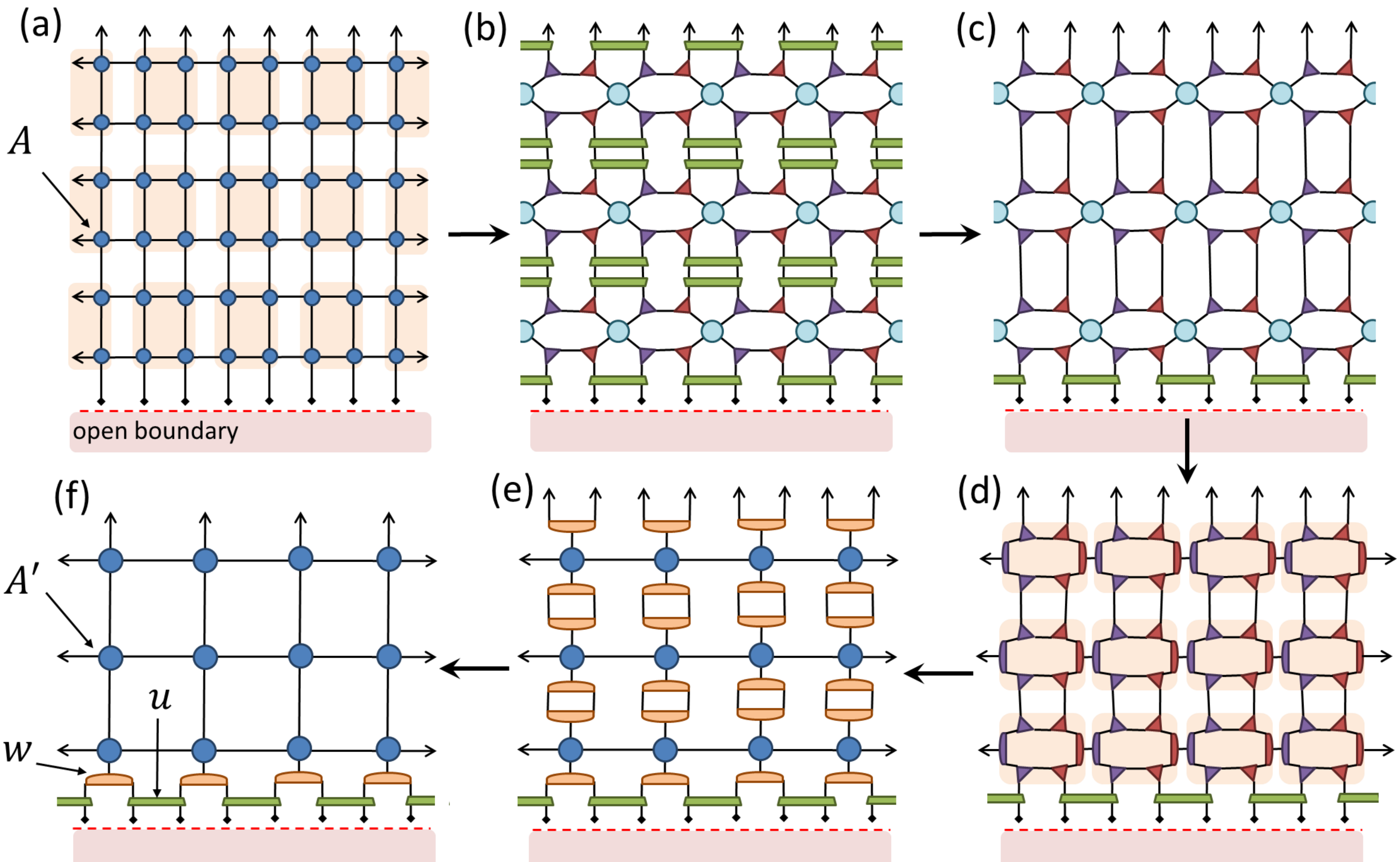
local scale transformations

example 2:

[Evenbly, Vidal, 15]

Upper half plane to hyperbolic plane

$|\Psi\rangle \sim e^{-\tau H}|\phi_0\rangle$



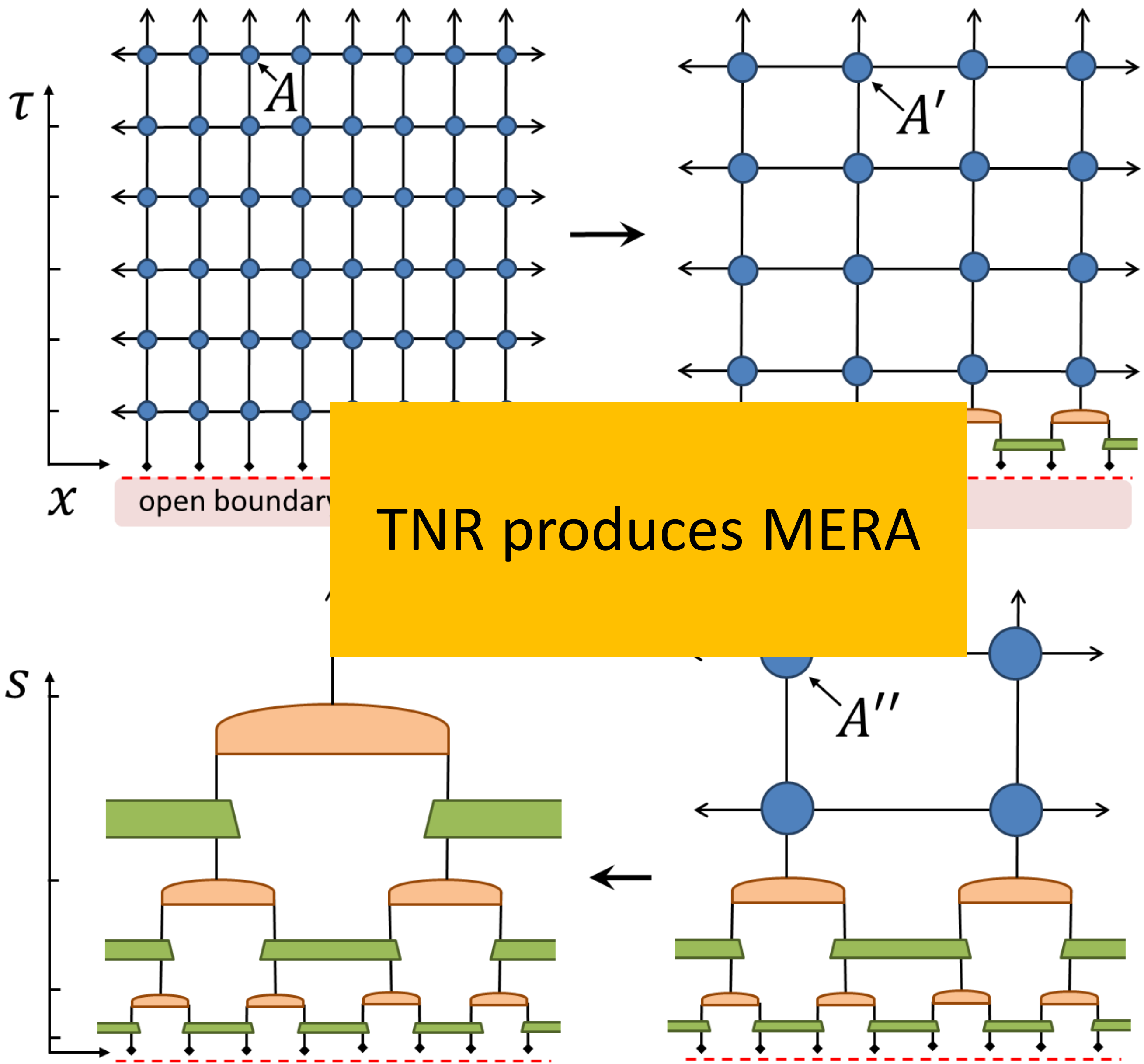
local scale transformations

example 2:

[Evenbly, Vidal, 15]

Upper half plane to hyperbolic plane

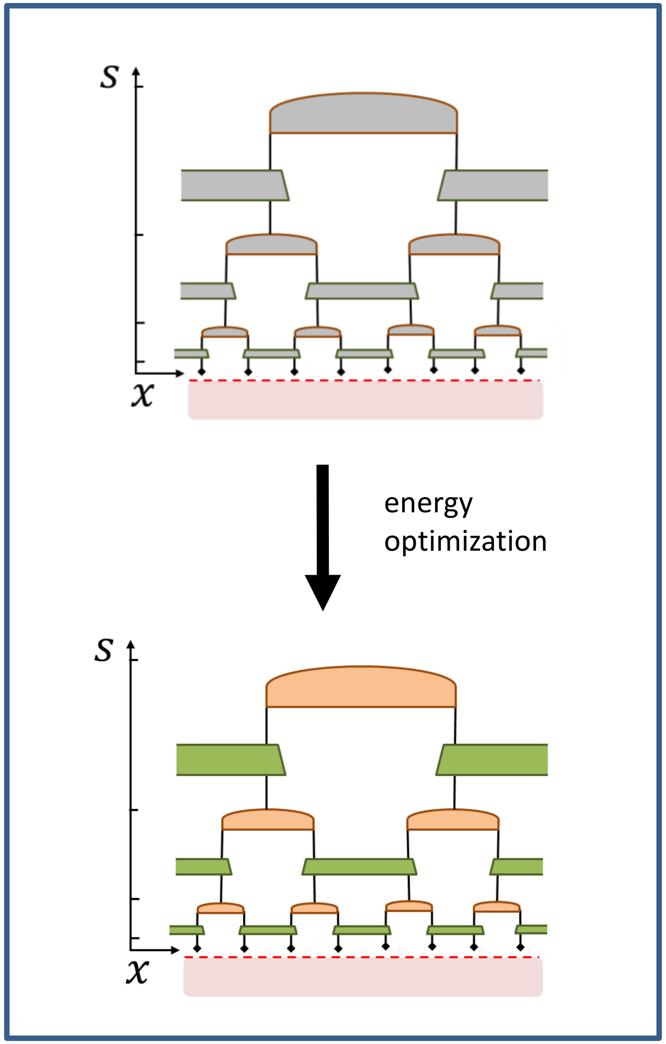
$|\Psi\rangle \sim e^{-\tau H}|\phi_0\rangle$



TNR produces MERA

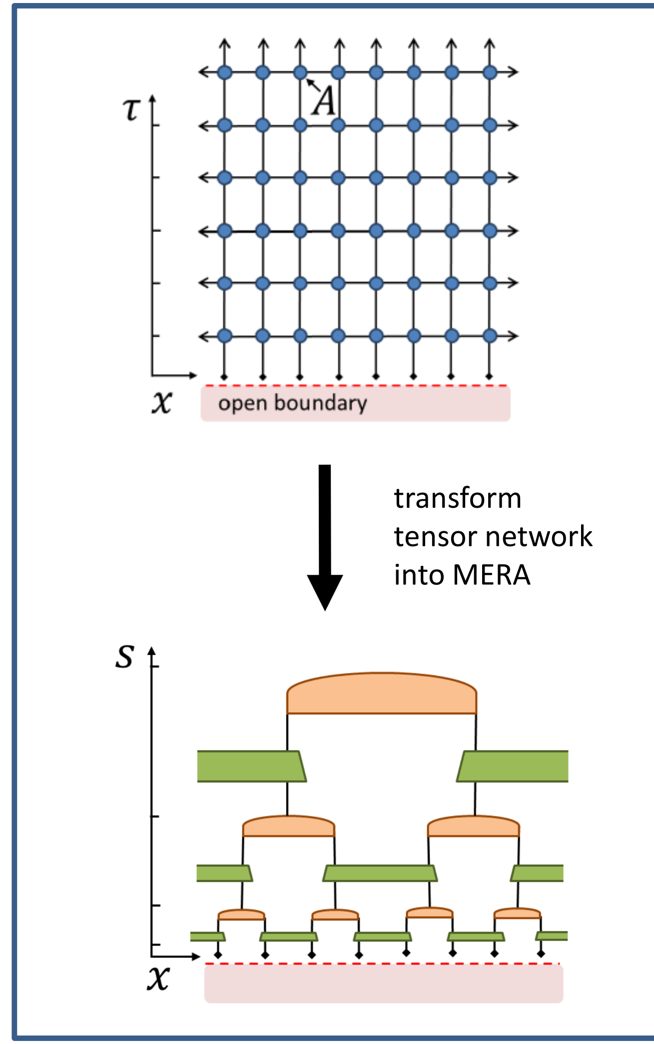
MERA = variational ansatz

MERA = by-product of TNR



energy minimization

- 1000s of iterations over scale
- local minima
- correct ground ?

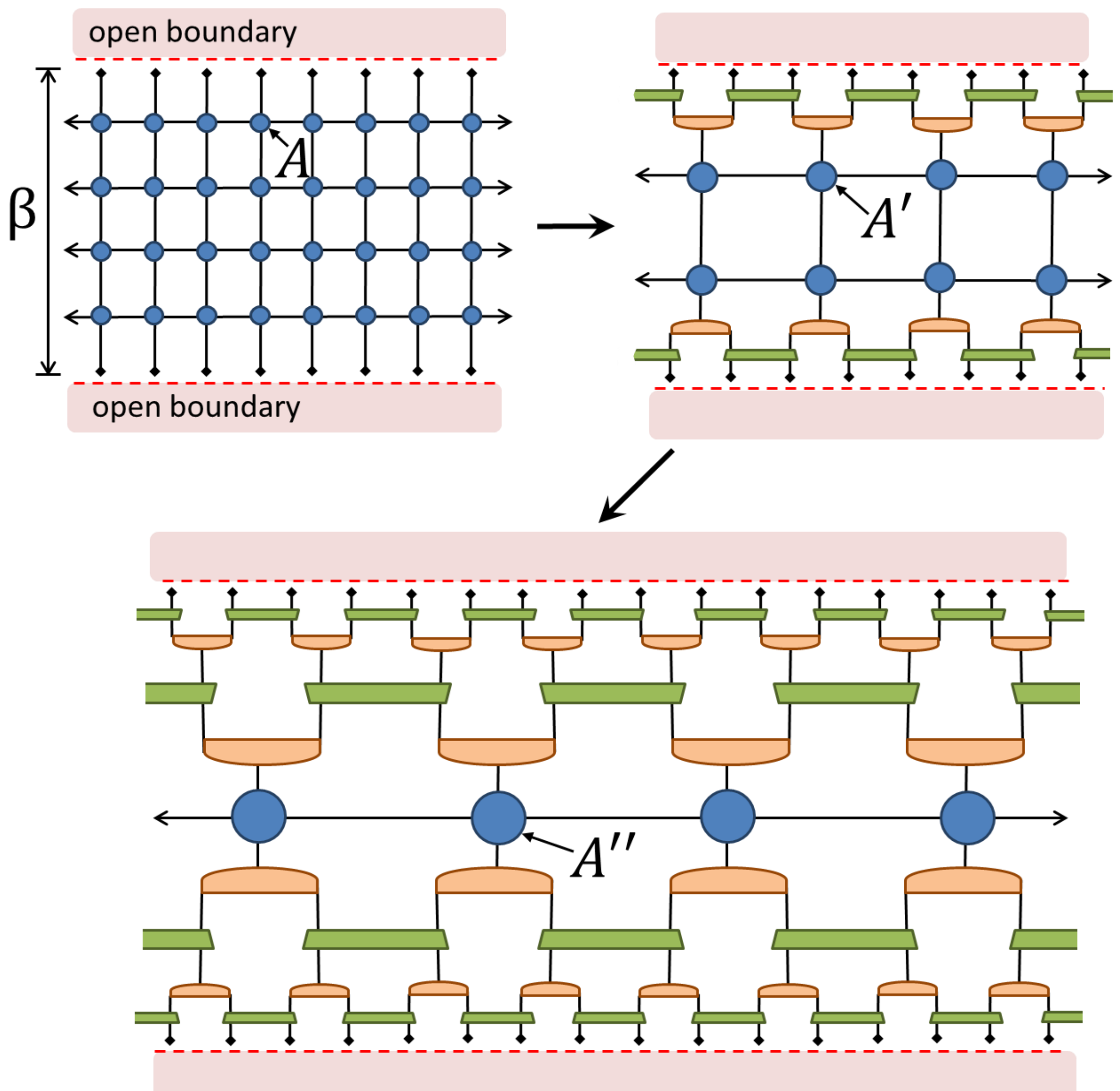


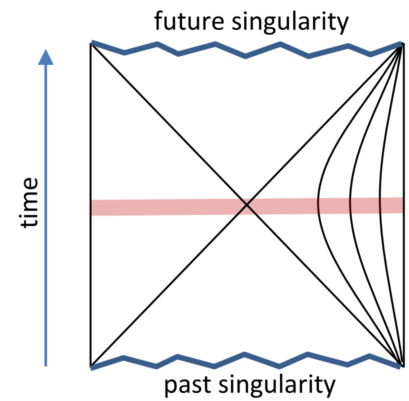
TNR -> MERA

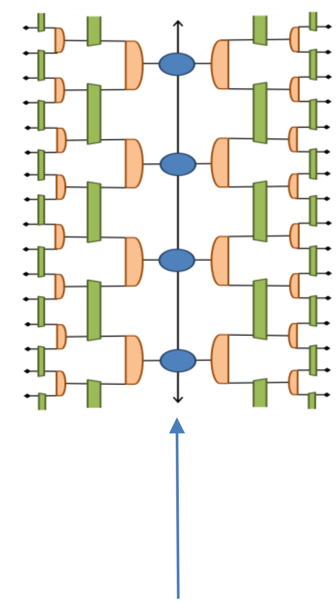
- single iteration over scale
- rewrite tensor network for ground state
- certificate of accuracy

MERA for a thermal state (or black hole in holography)

$$\rho_\beta \sim e^{-\beta H}$$







Einstein-Rosen bridge

Summary: three lectures on Tensor Networks

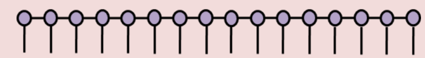
Lecture 1

Tensor networks and many-body entanglement

$S_L^{gapped} = const.$

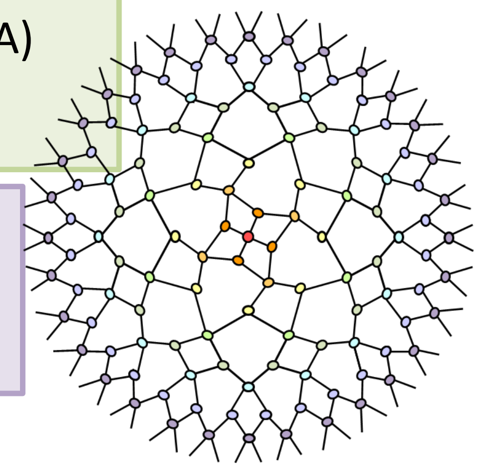
$S_L^{gapless} = \log L$

Matrix product state (MPS)



Lecture 2

Multi-scale entanglement renormalization ansatz (MERA)



Lecture 3

MERA and holography

Tensor network renormalization (TNR)