# Entropy for Gaussian states

The aim of this problem is to obtain the formulas for entropy of Gaussian states that are used in the problems "Area law for entanglement entropy" and "Entanglement entropy and c-function for a massive scalar field in two dimensions". It is better you attack first these later problems with Mathematica (using this problem as a sheet of formulas). When you have the time you can explore the justification of the formulas for Gaussian states following this guided exercise.

a) Let the Hermitian operators $\phi_i$ and $\pi_j$ (coordinate and conjugate momentum) obey the canonical commutation relations

$$[\phi_i, \pi_j] = i\delta_{ij}\,, \qquad [\phi_i, \phi_j] = [\pi_i, \pi_j] = 0\,. \tag{1}$$

This forms a canonical commutation algebra. A Gaussian state is a state such that all non-zero correlators are obtained from the two point correlators by the prescription

$$\langle \mathcal{O} f_{i_1} f_{i_2} ... f_{i_{2k}} \rangle = \frac{1}{2^k k!} \sum_\sigma \langle \mathcal{O} f_{i_{\sigma(1)}} f_{i_{\sigma(2)}} \rangle ... \langle \mathcal{O} f_{i_{\sigma(2k-1)}} f_{i_{\sigma(2k)}} \rangle\,, \tag{2}$$

where the sum is over all the permutations $\sigma$ of the indices, the $f_i$ can be any of the field or momentum variables, and $\mathcal{O}$ is an ordering prescription, for example, ordering the products inside the expectation values with the field variables at the left and the momentum variables on the right. Once this equation holds for a specific ordering automatically holds for any other ordering. A Gaussian state is also called "free" or "quasifree", and the property (2) is also called Wick's theorem. As we will see a Gaussian state can be pure or mixed.

Show that the two point functions can be written as

$$\langle \phi_i \phi_j \rangle = X_{ij}\,, \qquad \langle \pi_i \pi_j \rangle = P_{ij}\,, \tag{3}$$

$$\langle \phi_i \pi_j \rangle = \langle \pi_j \phi_i \rangle^* = \frac{i}{2}\delta_{ij} + D_{ij}\,, \tag{4}$$

with $X$ and $P$ real, Hermitian, positive definite, and $D$ real.

b) The fundamental state (vacuum) of a quadratic Hamiltonian of the form

$$H = \frac{1}{2}\sum \pi_i^2 + \frac{1}{2}\sum_{ij} \phi_i K_{ij} \phi_j\,, \tag{5}$$

is a Gaussian state (we are not proving this here, the proof is Wick's theorem). Show the two point correlators are given by

$$X_{ij} = \langle \phi_i \phi_j \rangle = \frac{1}{2}(K^{-\frac{1}{2}})_{ij}\,, \tag{6}$$

$$P_{ij} = \langle \pi_i \pi_j \rangle = \frac{1}{2}(K^{\frac{1}{2}})_{ij}\,, \tag{7}$$

$$D_{ij} = 0\,. \tag{8}$$

From now on we will restrict attention to this case with $D_{ij} = 0$.

c) Now we want to study the reduced state to a subset $V$ (a "region") of $N$ degrees of freedom given by some pairs $\phi_i$, $\pi_i$, $i \in V$, of the original variables. By definition the reduced density matrix satisfies

$$\langle O_V \rangle = \mathrm{tr}(\rho_V O_V)\,, \tag{9}$$

for any operator $O_V$ localized inside a $V$, that is, any polynomial of $\phi_i$, $\pi_i$, $i \in V$. Hence, the reduced density matrix must be such that expectation values give the right two point functions and Wick's theorem for the variables in $V$.

Propose the following anzats for the reduced density matrix

$$\rho_V = C e^{-\mathcal{H}} = C\, e^{-\Sigma \epsilon_l a_l^\dagger a_l}\,, \tag{10}$$

in terms of independent creation and annihilation operators

$$[a_i, a_j^\dagger] = \delta_{ij} \,, \tag{11}$$

which are expressed as (at this moment unknown) linear combinations of the $\phi_i$ and $\pi_j$, $i, j \in V$,

$$\begin{aligned} \phi_i &= \alpha_{ij} a_j^\dagger + \alpha_{ij} a_j \,, & (12) \\ \pi_i &= -i\beta_{ij} a_j^\dagger + i\beta_{ij} a_j \,. & (13) \end{aligned}$$

Here $\alpha$ and $\beta$ are real matrices.

Note that (10) gives the reduced density matrices as a product of independent density matrices for oscillators with mode annihilation operators $a_i$, and that the state on each of these independent modes is a thermal state for a harmonic oscillator. Compute the normalization constant $C$.

As a first step in showing the anzats (10) gives the correct reduced state, argue that this state satisfies Wick's theorem for the operators $a_i$, $a_j^\dagger$, and that if Wick's theorem holds for certain variables it will hold for linear combinations of the variables. In this way we know that the state (10) satisfies Wick's theorem for the original $\phi_i$, $\pi_j$ variables.

d) Show that in order that Bogoliubov transformations (12), (13) satisfy the canonical commutation relations we have

$$\alpha \beta^T = -\frac{1}{2} \,. \tag{14}$$

e) Compute the two point correlation functions from (9), $\mathrm{tr}(\rho_V \phi_i \phi_j) = X_{ij}^V$, $\mathrm{tr}(\rho_V \pi_i \pi_j) = P_{ij}^V$, to obtain the matrix equations

$$\begin{aligned} \alpha(2n+1)\alpha^T &= X^V \,, & (15) \\ \beta(2n+1)\beta^T &= P^V \,, & (16) \end{aligned}$$

where $n$ is the diagonal matrix of the expectation value of the occupation number

$$n_{kk} = \left\langle a_k^\dagger a_k \right\rangle = (e^{\epsilon_k} - 1)^{-1} \,. \tag{17}$$

f) These equations give

$$\alpha \frac{1}{4}(2n+1)^2 \alpha^{-1} = X^V P^V \,. \tag{18}$$

This last equation gives the spectra of the density matrix in terms of the spectrum of $XP$,

$$(1/2)\coth(\epsilon_k/2) = \nu_k \,, \tag{19}$$

where $\nu_k$ are the (positive) eigenvalues of

$$C^V = \sqrt{X^V P^V} \,. \tag{20}$$

Using this in the density matrix compute the entropy:

$$S(V) = \mathrm{tr}\left( (C^V + 1/2)\log(C^V + 1/2) - (C^V - 1/2)\log(C^V - 1/2) \right) \,. \tag{21}$$

g) The expression (21) for the entropy requires that $C^V \geq 1/2$ or

$$X^V . P^V \geq \frac{1}{4} \,, \tag{22}$$

in matrix sense, that means the eigenvalues of $X^V.P^V$ are greater than $1/4$. Can you explain why this inequality for the correlators is always true? What is the entropy for the global vacuum with correlators (6), (7)?

Notice (21) gives the entropy in $V$ just from diagonalizing a $N \times N$ matrix $C_V$ ($N$ is the number of sites in $V$). This is a big simplification that holds for Gaussian states. Compare with an arbitrary state in a set of $N$ qubits.