```
<< Local`QFTToolKit2`
$defWhy = {};

rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := |a|;
it[a_] := Style[a, Italic]
iD := it[D]
iA := it[A]
iI := it["I"]
C∞ := C^"∞"
(*
Bx_:=T[B,"d",{x}]
   ("▽"^S)n_:=T["▽"^S,"d",{n}]*)

accumWhy[item_] := Block[{}, $defWhy = tuAppendUniq[item][$defWhy];
   ""];
selectWhy[heads_, with_:{}, all_:Null] := tuRuleSelect[$defWhy][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
Clear[expandDC];
expandDC[sub_:{}, scalar_:{}] :=
 tuRepeat[{sub, tuOpDistribute[Dot], tuOpSimplify[Dot, scalar],
   tuOpDistribute[CircleTimes]}, tuCircleTimesSimplify]
Clear[expandCom]
expandCom[subs_:{}][exp_] := Block[{tmp = exp},
   tmp = tmp //. tuCommutatorExpand // expandDC[];
   tmp = tmp /. toxDot //. Flatten[{subs}];
   tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
   tmp
  ];
(**)
$sgeneral := {
  T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
  T[γ, "d", {5}].T[γ, "d", {5}] → 1,
  ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
  CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  T["▽", "d", {_}][1n_] → 0, a_ . 1n_ → a, 1n_ . a_ → a}
$sgeneral // ColumnBar;

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
   tt : T[F, "uu", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
   tt : T[F, "dd", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
   CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}],
   CommutatorP[a_, b_] :→ CommutatorP[b, a] /; OrderedQ[{b, a}],
   tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
  };
$symmetries // ColumnBar

εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8]},
   table =
    {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1,}}},
```

```
  {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
 ]
εRule[6]
```

```
tt : g^μ_ ^ν_ :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : F^μ_ ^ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : F_μ_ ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
[a_, b_]_- :→ -[b, a]_- /; OrderedQ[{b, a}]
{a_, b_}_+ :→ {b, a}_+ /; OrderedQ[{b, a}]
tt : γ^μ.γ5 :→ Reverse[tt]
```

```
{ε → 1, ε′ → 1, ε″ → -1}
```

Why the Standard Model

1 Introduction

```
PR["Motive: Show that the space: M[gravity]×F[noncommutative geometries
   of K-dimension Mod[n,8]→6]⇒ Standard Model+Einstein gravity.",
 line,
 New, "The data: ",
 $ = {
   ℋ[CG["finite dimensional Hilbert space"]],
   {J[CG["antilinear isometry of ℋ"]], J J → ε},
   {𝒜[CG["involutive ℝ-algebra on ℋ"]], CommutatorM[a, rghtA[b]] → 0,
    CG["order zero condition"], {a, b} ∈ 𝒜, rghtA[b] → J.ct[b].inv[J]},
   {γ[CG["ℤ/2 grading of ℋ"]], J.γ → ε'' γ.J
   },
   {iD[CG["self-adjoint operator in ℋ"], J.iD → ε' iD.J]
   },
   {CG["K-dimension[F]→Mod[6,8] ⇒ K-dimension[M×F]→Mod[10,8]"],
    εRule[6],
    BraKet[J.ξ, iD.η][CG["antisymmetric"]],
    {ξ, η} ∈ ℋ, γ.ξ → ξ, γ.η → η,
    CG["Functional integral over fermions ⇒Pfaffian"]
   }
  };
 $ // ColumnForms, accumWhy[$],
 New, "To determine F: ",
 $ = {"(1)classify irreducible triplets {𝒜,ℋ,J}",
   "(2)study γ",
   {"(3)classify ", $c3 = {𝒜F ⊂ 𝒜, CommutatorM[𝒜F, Z[𝒜]][CG["Center of 𝒜"]]] ≠ 0,
      CommutatorM[CommutatorM[iD, a], rghtA[b]] → 0, {a, b} ∈ 𝒜F}
   }
  }; $ // ColumnForms, accumWhy[$]
]
```

Motive: Show that the space: M[gravity]×F[noncommutative geometries of K-dimension Mod[n,8]→6]⇒ Standard Model+Einstein gravity.

- The data:
  - $\mathcal{H}$[finite dimensional Hilbert space]
  - J[antilinear isometry of $\mathcal{H}$]
    - $J^2 \to \varepsilon$
  - $\mathcal{A}$[involutive $\mathbb{R}$-algebra on $\mathcal{H}$]
    - $[a, b^o]_- \to 0$
    - order zero condition
    - $a, b \in \mathcal{A}$
    - $b^o \to J.b^\dagger.J^{-1}$
  - γ[$\mathbb{Z}$/2 grading of $\mathcal{H}$]
    - $J.\gamma \to \gamma.J\,\varepsilon''$
  - D[self-adjoint operator in $\mathcal{H}$, $J.D \to D.J\,\varepsilon'$]
  - K-dimension[F]→Mod[6,8] ⇒ K-dimension[M×F]→Mod[10,8]
    - $\varepsilon \to 1$
    - $\varepsilon' \to 1$
    - $\varepsilon'' \to -1$
  - $\langle J.\xi \mid D.\eta \rangle$[antisymmetric]
    - $\xi, \eta \in \mathcal{H}$
    - $\gamma.\xi \to \xi$
    - $\gamma.\eta \to \eta$
  - Functional integral over fermions ⇒Pfaffian

- To determine F:
  - (1)classify irreducible triplets $\{\mathcal{A},\mathcal{H},J\}$
  - (2)study γ
  - (3)classify
    - $\mathcal{A}_F \subset \mathcal{A}$
    - $[\mathcal{A}_F, Z[\mathcal{A}]]$[Center of $\mathcal{A}$]$]_- \neq 0$
    - $[[D, a]_-, b^o]_- \to 0$
    - $a, b \in \mathcal{A}_F$

```
PR["(1)classify irreducible triplets {𝒜,ℋ,J} "
  Imply, $ = {x ∈ M_{k∈ℕ}[ℂ], ℋ -> M_{k∈ℕ}[ℂ], J[x] → ct[x],
   {M_k[ℂ][CG["unitary"]], M_k[ℝ][CG["orthogonal"]], M_{a→k/2}[ℍ][CG["sympletic"]]}};
 $ // ColumnBar,
 NL, "Or",
 Imply, $ = {M_k[ℂ] ⊕ M_k[ℂ], ℋ -> M_k[ℂ] ⊕ M_k[ℂ], J[BraKet[x, y]] → BraKet[ct[y], ct[x]]};
 $ // ColumnBar,
 NL, "• The ℤ/2 grading where ", J.γ → -γ.J, imply, "the 2nd case.",
 Imply,
 $ = {𝒜 → M_2[ℍ] ⊕ M_4[ℂ], ℋ → Hom_ℂ[V, W] ⊕ Hom_ℂ[W, V], V[CG["4-dimensional ℂ vector space"]],
   W[CG["2-dimensional graded right vector space[ℍ]"]],
   𝒜[CG["left action on ℋ"]] → End_ℍ[W] ⊕ End_ℂ[V],
   grading[𝒜 | ℋ][CG["⟵grading of W"]]}; $ // ColumnBar,
 NL, "• Hence, ", Exists[𝒜_F, 𝒜_F ⊂ 𝒜^even[CG["even part of 𝒜"]], $c3],
 NL, "This defines a NonCommutativeGeometry ", {𝒜_F ≃ ℂ ⊕ ℍ ⊕ M_3[ℂ], {ℋ, J, γ}}
]
```

(1)classify irreducible triplets $\{\mathcal{A},\mathcal{H},J\}$

$\Rightarrow$
$x \in M_{k\in\mathbb{N}}[\mathbb{C}]$
$\mathcal{H} \to M_{k\in\mathbb{N}}[\mathbb{C}]$
$J[x] \to x^\dagger$
$\{M_k[\mathbb{C}]$[unitary], $M_k[\mathbb{R}]$[orthogonal], $M_{a\to\frac{k}{2}}[\mathbb{H}]$[sympletic]$\}$

Or

$\Rightarrow$
$M_k[\mathbb{C}] \oplus M_k[\mathbb{C}]$
$\mathcal{H} \to M_k[\mathbb{C}] \oplus M_k[\mathbb{C}]$
$J[\langle x \mid y \rangle] \to \langle y^\dagger \mid x^\dagger \rangle$

• The $\mathbb{Z}/2$ grading where $J.\gamma \to -\gamma.J$ $\Rightarrow$ the 2nd case.

$\Rightarrow$
$\mathcal{A} \to M_2[\mathbb{H}] \oplus M_4[\mathbb{C}]$
$\mathcal{H} \to \mathrm{Hom}_\mathbb{C}[V, W] \oplus \mathrm{Hom}_\mathbb{C}[W, V]$
V[4-dimensional $\mathbb{C}$ vector space]
W[2-dimensional graded right vector space[$\mathbb{H}$]]
$\mathcal{A}$[left action on $\mathcal{H}$] $\to \mathrm{End}_\mathbb{H}[W] \oplus \mathrm{End}_\mathbb{C}[V]$
grading[$\mathcal{A} \mid \mathcal{H}$][⟵grading of W]

• Hence,

$\exists_{\mathcal{A}_F,\mathcal{A}_F\subset\mathcal{A}^{\text{even}}[\text{even part of }\mathcal{A}]} \{\mathcal{A}_F \subset \mathcal{A},\ [\mathcal{A}_F, Z[\mathcal{A}][\text{Center of }\mathcal{A}]]_- \neq 0,\ [[D, a]_-, b^o]_- \to 0,\ \{a, b\} \in \mathcal{A}_F\}$

This defines a NonCommutativeGeometry $\{\mathcal{A}_F \simeq \mathbb{C}\oplus\mathbb{H}\oplus M_3[\mathbb{C}], \{\mathcal{H}, J, \gamma\}\}$

■ 2 The order zero condition and Irreducible pairs: 𝒜, J

```
PR["● From ", ℋ[CG["finite dimensional Hilbert space"]],
  ", find an 𝒜 ∍ ", $ = selectWhy[J^2];
  $ = {selectWhy[J^2] /. selectWhy[ε], selectWhy[rghtA[_]],
    selectWhy[CommutatorM[_, _], {}, all] // First,
    {{𝒜[CG["has separating vector, i.e.",
         Exists[ξ, ξ ∈ ℋ, 𝒜'.ξ → ℋ && 𝒜'["commutant of 𝒜"]]]]}[CG["(1)"]],
     {"representation of 𝒜 and J is irreducible, i.e.",
      ! Exists[e, e[CG["projection"]] ∈ ℒ[ℋ] && CommutatorM[e, 𝒜 | J] → 0]}[CG["(2)"]]
    }
   }; $ // ColumnForms
 ];

PR["Lemma 2.1: Assume conditions (2.2) and (1), (2), then, ",
  Imply, $L21 = $ = {ForAll[e ≠ 1, e ∈ Z[𝒜], e.J. e . inv[J] → 0],
     ForAll[eⱼ ∈ "projection in Z[𝒜]", e₁.e₂ → 0, e₁.J.e₂.inv[J] + e₂.J.e₁.inv[J] ∈ {0, 1}]};
  $ // ColumnBar, accumWhy[$L21];

  line,
  next, "Proof:  Given: ", $ = selectWhy[CommutatorM[_, _], {}, all] // First,
  NL, "Letting ", $s = {a → 𝒜, 𝒜 → e, b → e, ct[e] → e},
  note, "i.e., for {a,b} projections",

  Imply, $ = $ /. $s,
  yield, $ = $ /. (selectWhy[rghtA[_]] // tuAddPatternVariable[b]),
  yield, $ = $ /. $s,
  NL, "Apply J on LHS: ", $ = J.# & /@ $ // expandDC[],
  yield, $ = $ /. tuCommutatorExpand // expandDC[],
  NL, "Apply different forms of ",
  $s = {J.J → 1, J.inv[J] → 1, e.inv[J].e → e.J.J.inv[J].e.inv[J].J},
  Imply, $ = $ /. $s // Inactivate[#, Plus] & // expandDC[],
  Yield, $[[1]] = $[[1]] /. e.inv[J].e → e.J.J.inv[J].e.inv[J].J,
  yield, $ = $ /. $s[[2]] // expandDC[] // Activate,
  Yield, $ = $ /. a_ .b_ - b_ .a_ → CommutatorM[a, b]; $ // Framed,
  imply, $1 = $[[1, 2]] ∈ {0, 1};
  CR["?by irreducibility of "], $1 // Framed,
  NL, CR["Why does this imply that ", $1, "?" ],
  NL, "The condition ", e ≠ 1, imply, $[[1, 2]] → 0,
  NL, CR[" Clarify the idea of projection.  Is the idea that since ",
   e ∈ Z[𝒜][CG["Center of 𝒜"]],
   imply, e.𝒜 → 𝒜.e , imply, "if ", e ≠ {0, 1}, imply, e.𝒜 ⊂ 𝒜,
   " possibly contradictory situation?"
  ]
 ];
```

● From ℋ[finite dimensional Hilbert space], find an 𝒜 ∍

$J^2 \to 1$

$b^o \to J.b^\dagger.J^{-1}$

$[a, b^o]_- \to 0$

𝒜[has separating vector, i.e., $\exists_{\xi,\xi\in\mathcal{H}}\,(\mathcal{A}'.\xi \to \mathcal{H}$ && 𝒜′[commutant of 𝒜])] [(1)]

representation of 𝒜 and J is irreducible, i.e. [(2)]

$\forall_e$! (e[projection] $\in \mathcal{L}[\mathcal{H}]$ && $[e, \mathcal{A} \mid J]_- \to 0$)

Lemma 2.1: Assume conditions (2.2) and (1), (2), then,

$$\Rightarrow \left| \begin{array}{l} \forall_{e\neq 1, e\in Z[\mathcal{A}]}\ (e.J.e.\mathrm{inv}[J] \to 0) \\ \forall_{e_j \in \text{projection in } Z[\mathcal{A}], e_1.e_2\to 0}\ e_1.J.e_2.\mathrm{inv}[J] + e_2.J.e_1.\mathrm{inv}[J] \in \{0, 1\} \end{array} \right.$$

◆Proof: Given: $[a, b^o]_- \to 0$

Letting $\{a \to \mathcal{A}, \mathcal{A} \to e, b \to e, e^\dagger \to e\}$

⌘i.e., for {a,b} projections

$\Rightarrow [\mathcal{A}, e^o]_- \to 0 \longrightarrow [\mathcal{A}, J.e^\dagger.J^{-1}]_- \to 0 \longrightarrow [e, J.e.J^{-1}]_- \to 0$

Apply J on LHS: $J.[e, J.e.J^{-1}]_- \to 0 \longrightarrow -J.J.e.J^{-1}.e + J.e.J.e.J^{-1} \to 0$

Apply different forms of $\{J.J \to 1, J.J^{-1} \to 1, e.J^{-1}.e \to e.J.J.J^{-1}.e.J^{-1}.J\}$

$\Rightarrow -e.J^{-1}.e + J.e.J.e.J^{-1} \to 0$

$\to -e.J.J.J^{-1}.e.J^{-1}.J + J.e.J.e.J^{-1} \longrightarrow J.e.J.e.J^{-1} - e.J.e.J^{-1}.J \to 0$

$\to \boxed{[J, e.J.e.J^{-1}]_- \to 0}$ ⇒ ?by irreducibility of $\boxed{e.J.e.J^{-1} \in \{0, 1\}}$

Why does this imply that $e.J.e.J^{-1} \in \{0, 1\}$?

The condition $e \neq 1 \Rightarrow e.J.e.J^{-1} \to 0$

Clarify the idea of projection. Is the idea that since $e \in Z[\mathcal{A}]$[Center of $\mathcal{A}$]

$\Rightarrow e.\mathcal{A} \to \mathcal{A}.e \Rightarrow$ if $e \neq \{0, 1\} \Rightarrow e.\mathcal{A} \subset \mathcal{A}$ possibly contradictory situation?

```
PR["Define ",
 {𝒜ℂ[CG["complex linear space generated by 𝒜 in ℒ[ℋ][algebra of operators in ℋ]"]],
  𝒜ℂ[CG["involutive complex subalgebra of ℒ[ℋ]"]]},
 line,
 "Lemma 2.2: Assume conditions (2.2) and (1), (2), then, ",
 $ = {Z[𝒜ℂ] → ℂ, or,
   {Z[𝒜ℂ] → ℂ ⊕ ℂ, J.e1.inv[J] → e2, ej ∈ Z[𝒜ℂ], ej[CG["minimal projections"]]
   }}; $ // ColumnBar,
 line,
 next, "Proof by contradiction:
Assume ", $ = ! Z[𝒜ℂ] → ℂ, CO[" (i.e. second case)"],
 imply, $ = {xSum[ej, j] → 1[CG["minimal projections"]],
   xSum[ei.J.ej.inv[J], i ≠ j] → 1[CG["i⩵j is 0(L.2.1)"]],
   ei.J.ej.inv[J][CG["pairwise orthogonal projections"]],
   e1.J.e2.inv[J] + e2.J.e1.inv[J] → 1,
   xSum[ei.J.ej.inv[J], i ∉ {1, 2}, j] → 0
  }; $ // ColumnBar
]
```

Define
{𝒜ℂ[complex linear space generated by 𝒜 in ℒ[ℋ][algebra of operators in ℋ]],
𝒜ℂ[involutive complex subalgebra of ℒ[ℋ]]}

Lemma 2.2: Assume conditions (2.2) and (1), (2), then,

$Z[\mathcal{A}_{\mathbb{C}}] \to \mathbb{C}$
or
$\{Z[\mathcal{A}_{\mathbb{C}}] \to \mathbb{C}\oplus\mathbb{C},\ J.e_1.J^{-1} \to e_2,\ e_j \in Z[\mathcal{A}_{\mathbb{C}}],\ e_j$[minimal projections]$\}$

◆Proof by contradiction:

Assume $!\,Z[\mathcal{A}_{\mathbb{C}}] \to \mathbb{C}$ (i.e. second case) ⇒

$\sum_{j}[e_j] \to 1$[minimal projections]
$\sum_{i\neq j}[e_i.J.e_j.J^{-1}] \to 1$[i⩵j is 0(L.2.1)]
$e_i.J.e_j.J^{-1}$[pairwise orthogonal projections]
$e_1.J.e_2.J^{-1} + e_2.J.e_1.J^{-1} \to 1$
$\sum_{i\notin\{1,2\},\ j}[e_i.J.e_j.J^{-1}] \to 0$

2.1 The case $Z[\mathcal{A}_{\mathbb{C}}]\to\mathbb{C}$

```
PR["Assume ", Z[𝒜ℂ] → ℂ,
 imply, Exists[k, {k ∈ ℕ, 𝒜ℂ[CG["involutive"]] → Mk[ℂ],
   𝒜ℂ ⊗ rghtA[𝒜ℂ][CG["~ Mk² [ℂ]"]] → ℒ[ℋ], ForAll[{x, y}, {x, y} ∈ 𝒜ℂ,
    β[CG["homomorphism"]][x ⊗ y] → x.rghtA[y][CG["an injection"]]]
  }],
 line,
 NL, "Lemma 2.4: The representation ",
 $l24 = {β → π[𝒜ℂ ⊗ rghtA[𝒜ℂ]][ℋ], β[x ⊗ y] → x.rghtA[y]},
 " is irreducible.", accumWhy[$l24];
 line,
 NL, "Proof: ",
 $ = {𝒜ℂ ⊗ rghtA[𝒜ℂ] ~ Mk² [ℂ],
   imply, $r = {β → π[𝒜ℂ] π[rghtA[𝒜ℂ]], 𝒜ℂ → Mk[ℂ]},
   "What is the multiplicity?",
   {"Let ",
    e[CG["minimal projection of ", $r[[-1]]]], it["E"] →
     e.J.e.inv[J][CG["a minimal projection of ", ℬ → $l24[[1]], "dimension m"]],
    {CommutatorM[it["E"], J] → 0, imply,
     "J restrict to an antilinear isometric involution of square 1 on Eℋ"},
    CR["??"]
   }
  };
 $ // ColumnForms
]
```

Assume $Z[\mathcal{A}_{\mathbb{C}}]\to\mathbb{C}$ ⇒ $\exists_k\,\{k\in\mathbb{N},\ \mathcal{A}_{\mathbb{C}}[\text{involutive}]\to M_k[\mathbb{C}],$
$\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}[\sim M_{k^2}[\mathbb{C}]]\to\mathcal{L}[\mathcal{H}],\ \forall_{\{x,y\},\{x,y\}\in\mathcal{A}_{\mathbb{C}}}\,(\beta[\text{homomorphism}][x\otimes y]\to x.y^{o}[\text{an injection}])\}$

Lemma 2.4: The representation $\{\beta\to\pi[\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}][\mathcal{H}],\ \beta[x\otimes y]\to x.y^{o}\}$ is irreducible.

Proof:
$\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}\sim M_{k^2}[\mathbb{C}]$
⇒
$\beta\to\pi[\mathcal{A}_{\mathbb{C}}]\,\pi[\mathcal{A}_{\mathbb{C}}{}^{o}]$
$\mathcal{A}_{\mathbb{C}}\to M_k[\mathbb{C}]$
What is the multiplicity?
Let
e[minimal projection of , $\mathcal{A}_{\mathbb{C}}\to M_k[\mathbb{C}]$]
$E\to e.J.e.J^{-1}$[a minimal projection of , $\mathcal{B}\to\beta\to\pi[\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}][\mathcal{H}]$, dimension m]
$[E,\ J]_{-}\to 0$
⇒
J restrict to an antilinear isometric involution of square 1 on $E\mathcal{H}$
??

```
PR["Proposition 2.5: Let ℋ be a Hilbert space of dimension
   n.  Then an irreducible solution with Z[𝒜ℂ]→ℂ exists iff n→k² is
   a square.  It is given by 𝒜ℂ→Mk[ℂ] acting by left multiplication
   on itself and antilinear involution ", {J[x] → ct[x], x ∈ Mk[ℂ]},
 line,
 NL, "Proof: ", $ = {𝒜ℂ ⊗ rghtA[𝒜ℂ] ~ Mk²[ℂ], selectWhy[β][CG["irreducible"]]},
 Imply, n → k²[CG["square"]][CR["?"]],
 NL, "The action ", $ = 𝒜ℂ ⊗ rghtA[𝒜ℂ][𝒜ℂ -> Mk[ℂ]],
 " is realized by representation ", selectWhy[β],
 Yield, σ[a ⊗ rghtA[b]] → b ⊗ rghtA[a][CG["canonical antiautomorphism"]],
 NL, "Implemented by involution ",
 {J0[x] → ct[x], σ[x] → J0.ct[x].inv[J0], x ∈ 𝒜ℂ ⊗ rghtA[𝒜ℂ]},
 NL, CR["? do not follow rest where they claim that the same process holds for ",
  {𝒜, J}, " and conclude " J0 → J]
]
```

Proposition 2.5: Let $\mathcal{H}$ be a Hilbert space of dimension n.
Then an irreducible solution with $Z[\mathcal{A}_{\mathbb{C}}]\to\mathbb{C}$ exists iff $n\to k^2$ is a square. It is given by $\mathcal{A}_{\mathbb{C}}\to M_k[\mathbb{C}]$ acting by left multiplication on itself and antilinear involution $\{J[x]\to x^\dagger,\ x\in M_k[\mathbb{C}]\}$

Proof: $\{\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}\sim M_{k^2}[\mathbb{C}],\ (\beta\to\pi[\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}][\mathcal{H}])[\text{irreducible}]\}$

$\Rightarrow$ $n\to k^2[\text{square}][?]$

The action $\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}[\mathcal{A}_{\mathbb{C}}\to M_k[\mathbb{C}]]$ is realized by representation $\beta\to\pi[\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}][\mathcal{H}]$

$\to$ $\sigma[a\otimes b^o]\to b\otimes a^o[\text{canonical antiautomorphism}]$

Implemented by involution $\{J_0[x]\to x^\dagger,\ \sigma[x]\to J_0.x^\dagger.J_0^{-1},\ x\in\mathcal{A}_{\mathbb{C}}\otimes\mathcal{A}_{\mathbb{C}}{}^{o}\}$

? do not follow rest where they claim that the same process holds for $\{\mathcal{A},J\}$ and conclude $J_0\to J$

```
PR["Possibilities for ",
 $ = {𝒜[CG["involutive algebra of Mk[ℂ]"]][Inactivate[𝒜 + I 𝒜, Plus] -> Mk[ℂ]],
   Z[𝒜] ⊂ (Z[Mk[ℂ]] → ℂ),
   {(Z[𝒜] → ℂ) ⇒ {I ∈ 𝒜, 𝒜 -> Mk[ℂ]},
    (Z[𝒜] → ℝ) ⇒
     {𝒜 -> Mk[ℝ], 𝒜[CG["fixed point algebra of antilinear automorphism α[Mk[ℂ]]",
        α[a + I b] → a - I b]],
      Exists[it["I"], it["I"][CG["antilinear isometry"]][ℂ^k],
       {α[x] -> it["I"].x.inv[it["I"]], x ∈ Mk[ℂ]
       }
      ],
      α.α → 1,
      it["I"].it["I"] → ±1[CG["scalar λ∈ℂ of modulus 1"]],
      𝒜[CG["commutant of ", it["I"]]] ⇒ Or[{it["I"].it["I"] → 1, Mk[ℝ]},
        {it["I"].it["I"] → -1,
         it["I"][ℂ^k] → {"right vector space"[ℍ], 𝒜 → Ma[ℍ], k → 2 a}}],
      Or[𝒜 → Mk[ℂ][CG["unitary"]],
       𝒜 → Mk[ℝ][CG["orthogonal"]],
       𝒜 → Mk[ℍ][CG["symplectic k→2a"]] ],

      {𝒜[CG["left action"]][Mk[ℂ]],
       J[x] → ct[x][CG["antilinear involution"]]
      }
     }
   }
  }; $ // ColumnForms
]
```

```
Possibilities for
 𝒜[involutive algebra of Mk[ℂ]][𝒜 + ⅈ 𝒜 → Mk[ℂ]]
 Z[𝒜] ⊂ (Z[Mk[ℂ]] → ℂ)
   (Z[𝒜] → ℂ) ⇒ | ⅈ ∈ 𝒜
                 | 𝒜 → Mk[ℂ]
   (Z[𝒜] → ℝ) ⇒
     𝒜 → Mk[ℝ]
     𝒜[fixed point algebra of antilinear automorphism α[Mk[ℂ]], α[a + ⅈ b] → a - ⅈ b]
     ∃ it[I],I[antilinear isometry][ℂ^k] | α[x] → it[I].x.it[I]^-1
                                        | x ∈ Mk[ℂ]
     α.α → 1
     I.I → ±1[scalar λ∈ℂ of modulus 1]
                                                   | I.I → -1
     𝒜[commutant of , I] ⇒ | I.I → 1   ||          |            | right vector space[ℍ]
                           | Mk[ℝ]                 | I[ℂ^k] →   | 𝒜 → Ma[ℍ]
                                                   |            | k → 2 a
     (𝒜 → Mk[ℂ][unitary]) || (𝒜 → Mk[ℝ][orthogonal]) || (𝒜 → Mk[ℍ][symplectic k→2a])
      | 𝒜[left action][Mk[ℂ]]
      | J[x] → x^†[antilinear involution]
```

2.2 The case $Z[\mathcal{A}_{\mathbb{C}}]\to\mathbb{C}\otimes\mathbb{C}$

```
PR["", $ = Z[𝒜_ℂ] → ℂ ⊗ ℂ,
 imply, Exists[k_j, k_j ∈ ℕ, {𝒜_ℂ → M_{k_1}[ℂ] ⊕ M_{k_2}[ℂ][CG["involutive algebra"[ℂ]]]}],
 NL, "Let ", {e_j[CG["minimal projections" ∈ Z[𝒜_ℂ]]], e_j ~ M_{k_j}},
 NL,
 "● Lemma 2.7: ",
 NL, "(1) The representation ", selectWhy[β],
 " is the direct sum of two irreducible representations in the decomposition ",
 $ = {ℋ → e_1.ℋ ⊕ e_2.ℋ, ℋ → ℋ_1 ⊕ ℋ_2, β → β_1 ⊕ β_2}; $ // ColumnBar,
 NL, "(2) The representation β_1(resp. β_2) is the
   only irreducible representation of the reduced algebra of ",
 ℬ -> e_1 ⊕ rghtA[e_2], "(resp.", e_2 ⊕ rghtA[e_1], ")",
 NL, "(3) The dimension of ℋ_j", yield, k_1 k_2,
 line,
 NL, "Proof: ",
 NL, "(1)",
 $ = {{{ℋ_j → e_j.ℋ,
        e_j ∈ Z[𝒜]} -> 𝒜[ℋ][CG["diagonal"]],
      {J.e_j.inv[J] → e_k, k ≠ j} → rghtA[𝒜][ℋ][CG["diagonal"]]
     } → {β -> β_1 ⊕ β_2},
    {J[ℋ_1 ⊕ ℋ_2] -> ℋ_2 ⊕ ℋ_1,
      F_1[CG["invariant subspace for β_1[ℋ]"]] ∋
       F_1 ⊕ J.F_1[CG["invariant under ℬ,J"]] ⊂ ℋ,
      ℋ[CG["irreducible"]]
     } → {F_1 ⊕ J.F_1 → ℋ,
      F_1 → ℋ_1
     }
  };
 $ // ColumnForms,
 NL, "(2) Reduction of ℬ by ",
 {e_i ⊗ rghtA[e_j], i ≠ j} → "irreducible rep. isomorphic to ", M_{k_i}[ℂ] ⊗ M_{k_j}[ℂ] ~ M_{k_1 k_2}[ℂ]


]
```

$Z[\mathcal{A}_{\mathbb{C}}] \to \mathbb{C}\otimes\mathbb{C} \Rightarrow \exists_{k_j, k_j \in \mathbb{N}} \{\mathcal{A}_{\mathbb{C}} \to M_{k_1}[\mathbb{C}] \oplus M_{k_2}[\mathbb{C}][\text{involutive algebra}[\mathbb{C}]]\}$

Let $\{e_j[\text{minimal projections} \in Z[\mathcal{A}_{\mathbb{C}}]], e_j \sim M_{k_j}\}$

● Lemma 2.7:

(1) The representation $\mathcal{B} \to \pi[\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}{}^{o}][\mathcal{H}]$
is the direct sum of two irreducible representations in the decomposition

$$\left| \begin{array}{l} \mathcal{H} \to e_1.\mathcal{H} \oplus e_2.\mathcal{H} \\ \mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2 \\ \mathcal{B} \to \mathcal{B}_1 \oplus \mathcal{B}_2 \end{array} \right.$$

(2) The representation $\mathcal{B}_1$(resp. $\mathcal{B}_2$) is the only irreducible
representation of the reduced algebra of $\mathcal{B} \to e_1 \oplus e_2{}^{o}$(resp.$e_2 \oplus e_1{}^{o}$)

(3) The dimension of $\mathcal{H}_j \longrightarrow k_1\, k_2$

Proof:

(1)

$$\left| \begin{array}{l} \left| \begin{array}{l} \left| \begin{array}{l} \mathcal{H}_j \to e_j.\mathcal{H} \\ e_j \in Z[\mathcal{A}] \end{array} \right. \to \mathcal{A}[\mathcal{H}][\text{diagonal}] \\ \left| \begin{array}{l} J.e_j.J^{-1} \to e_k \\ k \neq j \end{array} \right. \to \mathcal{A}^{o}[\mathcal{H}][\text{diagonal}] \end{array} \right. \to \left| \mathcal{B} \to \mathcal{B}_1 \oplus \mathcal{B}_2 \right. \\ \left| \begin{array}{l} J[\mathcal{H}_1 \oplus \mathcal{H}_2] \to \mathcal{H}_2 \oplus \mathcal{H}_1 \\ F_1[\text{invariant subspace for } \mathcal{B}_1[\mathcal{H}]] \ni F_1 \oplus J.F_1[\text{invariant under } \mathcal{B}, J] \subset \mathcal{H} \\ \mathcal{H}[\text{irreducible}] \end{array} \right. \to \left| \begin{array}{l} F_1 \oplus J.F_1 \to \mathcal{H} \\ F_1 \to \mathcal{H}_1 \end{array} \right. \end{array} \right.$$

(2) Reduction of $\mathcal{B}$ by $\{e_i \otimes e_j{}^{o},\ i \neq j\} \to$ irreducible rep. isomorphic to $M_{k_i}[\mathbb{C}] \otimes M_{k_j}[\mathbb{C}] \sim M_{k_1\, k_2}[\mathbb{C}]$

```
PR["● Proposition 2.8: Let ℋ be a Hilbert space of dimension n.
   Then an irreducible solution with ", Z[𝒜ℂ] → ℂ⊕ℂ, " exists iff ",
 n → 2 k², " is twice a square.  It is given by ", 𝒜ℂ -> Mk[ℂ] ⊕ Mk[ℂ],
 " acting by left multiplication on itself and antilinear involution ",
 {J[{x, y}] → {ct[y], ct[x]}, {x | y} ∈ Mk[ℂ]},
 line,
 NL, "Proof: ", $ = {{dim[𝒜ℂ] → k1² + k2², dim[ℋ] → 2 k1 k2,
      "separating condition" ⇒ dim[𝒜ℂ] ≤ dim[ℋ],
      a ∈ 𝒜ℂ → {a.ξ ∈ ℋ, 𝒜'[CR["?"]].ξ → ℋ}
     } → {k1² + k2² ≤ 2 k1 k2, k1 == k2, n → 2 k²,
      β[CG["π[ℬ]"]] → {e1 ⊗ rghtA[e2] ⊕ e2 ⊗ rghtA[e1]}[ℬ]},
    {J0[{x, y}] → {ct[y], ct[x]},
      CommutatorM[inv[J0].J, ℬ] → 0
     } → {inv[J0].J[CG[ℋ → ℋ1 ⊕ℋ2]] → DiagonalMatrix[{λ1, λ2}],
      (J.J → 1) ⇒ (λ1 → λ2),
      J → λ J0}

  }; $ // ColumnForms
]
```

● Proposition 2.8: Let $\mathcal{H}$ be a Hilbert space of dimension n. Then an irreducible solution with $Z[\mathcal{A}_{\mathbb{C}}] \to \mathbb{C}\oplus\mathbb{C}$ exists iff $n \to 2\,k^2$ is twice a square. It is given by $\mathcal{A}_{\mathbb{C}} \to M_k[\mathbb{C}] \oplus M_k[\mathbb{C}]$ acting by left multiplication on itself and antilinear involution $\{J[\{x, y\}] \to \{y^\dagger, x^\dagger\}, \{x \mid y\} \in M_k[\mathbb{C}]\}$

Proof:

$$\left|\begin{array}{l} \left|\begin{array}{l} \dim[\mathcal{A}_{\mathbb{C}}] \to k_1^2 + k_2^2 \\ \dim[\mathcal{H}] \to 2\,k_1\,k_2 \\ \text{separating condition} \Rightarrow \dim[\mathcal{A}_{\mathbb{C}}] \le \dim[\mathcal{H}] \\ a \in \mathcal{A}_{\mathbb{C}} \to \left|\begin{array}{l} a.\xi \in \mathcal{H} \\ \mathcal{A}'[?].\xi \to \mathcal{H} \end{array}\right. \end{array}\right. \to \left|\begin{array}{l} k_1^2 + k_2^2 \le 2\,k_1\,k_2 \\ k_1 == k_2 \\ n \to 2\,k^2 \\ \beta[\pi[\mathcal{B}]] \to |\, e_1 \otimes e_2{}^\circ \oplus e_2 \otimes e_1{}^\circ\ [\mathcal{B}] \end{array}\right. \\ \left|\begin{array}{l} J_0\left[\left|\begin{array}{l} x \\ y \end{array}\right.\right] \to \left|\begin{array}{l} y^\dagger \\ x^\dagger \end{array}\right. \\ [J_0^{-1}.J, \mathcal{B}]_- \to 0 \end{array}\right. \to \left|\begin{array}{l} J_0^{-1}.J[\mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2] \to \left|\left|\begin{array}{l} \lambda_1 \\ 0 \\ 0 \\ \lambda_2 \end{array}\right.\right. \\ (J.J \to 1) \Rightarrow (\lambda_1 \to \lambda_2) \\ J \to \lambda\,J_0 \end{array}\right. \end{array}\right.$$

```
PR["● Remark 2.9. A more intrinsic solutions ",
 NL, $ = {{V, W}[CG["k-dim ℂ-Hilbert"]],
   𝒜ℂ → Endℂ[W] ⊕ Endℂ[V],
   ℋ[CG["bimodule over 𝒜ℂ"]] → ℰ ⊕ ct[ℰ], J[{ξ, η}] → {ct[η], ct[ξ]},
   ℰ → Homℂ[V, W], ct[ℰ] → Homℂ[W, V],
   {w, v}[{g, h}] → {w∘g, v∘h}
  }; $ // ColumnForms,
 $ = remark29 → $; accumWhy[$]
]
```

● Remark 2.9. A more intrinsic solutions

$\begin{array}{l} V \\ W \end{array}$ [k-dim ℂ-Hilbert]

$\mathcal{A}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}[W] \oplus \mathrm{End}_{\mathbb{C}}[V]$

$\mathcal{H}$[bimodule over $\mathcal{A}_{\mathbb{C}}$] $\to \mathcal{E} \oplus \mathcal{E}^{\dagger}$

$J\left[\begin{array}{l} \xi \\ \eta \end{array}\right] \to \begin{array}{l} \eta^{\dagger} \\ \xi^{\dagger} \end{array}$

$\mathcal{E} \to \mathrm{Hom}_{\mathbb{C}}[V, W]$

$\mathcal{E}^{\dagger} \to \mathrm{Hom}_{\mathbb{C}}[W, V]$

$\begin{array}{l} w \\ v \end{array}\left[\begin{array}{l} g \\ h \end{array}\right] \to \begin{array}{l} w \circ g \\ v \circ h \end{array}$

■ 3. ℤ/2-grading

```
zer[a_] := a"0"
PR["Assume: ", $ = ℋ[CG["even, ℤ/2-graded"]] → {γ[CG["grading operator"]],
    γ.γ → 1, γ → ct[γ], γ.𝒜.inv[γ] → 𝒜, CommutatorM[γ, a ∈ 𝒜^even] → 0};
 $ // ColumnForms,
 line,
 NL, "● Lemma 3.1. In the case ", Z[𝒜ℂ] → ℂ[CG["Prop.2.5"]], " let ",
 $ = {(γ[CG["ℤ/2-grading"]][ℋ] ∋ {γ.𝒜.inv[γ] → 𝒜, J.γ → ϵ'' γ.J, ϵ'' → ±1}) ⇒ {ϵ'' → 1}};
 $ // ColumnForms,
 line,
 NL, "Proof: ", $ = {"Assume Prop.2.5.",
   {δ[CG[Aut[𝒜ℂ]]][a] → γ.a.inv[γ],
    δ.δ → 1
   },
   {rghtA[δ][CG[Aut[rghtA[𝒜ℂ]]]][rghtA[b]] → γ.rghtA[b].inv[γ],
    γ.rghtA[𝒜ℂ].inv[γ] -> rghtA[𝒜ℂ]
   },
   {δ⊗rghtA[δ][CG[Aut[𝒜ℂ⊗rghtA[𝒜ℂ]]]] → γ.β[x].inv[γ] → β[δ⊗inv[δ][x]],
     x ∈ ℬ -> 𝒜ℂ⊗rghtA[𝒜ℂ]
    } ⇒
    {γ[CG[Aut[Mk[ℂ]]⊗Aut[Mk[ℂ]]]][a] → u.a.ct[v], a ∈ 𝒜ℂ,
     Exists[{u, v}, {u, v}[CG["unitary"]] ∈ Mk[ℂ]]
    },
   imply,
   {J.γ.inv[J][a] → ct[u.ct[a].ct[v]] → v.a.ct[u],
    (J.γ.inv[J] → -γ) ⇒
     {(v.a.ct[u] → -u.a.ct[v]), ct[u].v → z, z.a.z → -a, z.z → -1, z.a → a.z, z → η. iI,
      η → {±1}, v → η . iI.u, γ[a] → -η.iI.u.a.ct[u], γ.γ[a] → -u.u.a.inv[u].inv[u],
      γ.γ → 1,
      a → -u.u.a.inv[u].inv[u][CG["contradiction for a→1"]]
     }
   }
  }; $ // ColumnForms,
 line,
 NL, "Hence, ", J.γ → -γ.J, " is not possible for ", Z[𝒜ℂ] → ℂ,
 NL, "So the case ", Z[𝒜ℂ] → ℂ⊕ℂ, " is only possibility."
]
```

Assume: $\mathcal{H}$[even, $\mathbb{Z}/2$-graded] $\to$
- γ[grading operator]
- $\gamma.\gamma \to 1$
- $\gamma \to \gamma^\dagger$
- $\gamma.\mathcal{A}.\gamma^{-1} \to \mathcal{A}$
- $[\gamma, a \in \mathcal{A}^{even}]_- \to 0$

● <u>Lemma</u> <u>3.1.</u> In the case $Z[\mathcal{A}_\mathbb{C}] \to \mathbb{C}$[Prop.2.5]

let (γ[$\mathbb{Z}/2$-grading][$\mathcal{H}$] $\ni$ {$\gamma.\mathcal{A}.\gamma^{-1} \to \mathcal{A}$; $J.\gamma \to \gamma.J\,\epsilon''$; $\epsilon'' \to \pm 1$}) $\Rightarrow$ $\epsilon'' \to 1$

Proof:

Assume Prop.2.5.

$\delta[\text{Aut}[\mathcal{A}_\mathbb{C}]][a] \to \gamma.a.\gamma^{-1}$

$\delta.\delta \to 1$

$\delta^o[\text{Aut}[\mathcal{A}_\mathbb{C}{}^o]][b^o] \to \gamma.b^o.\gamma^{-1}$

$\gamma.\mathcal{A}_\mathbb{C}{}^o.\gamma^{-1} \to \mathcal{A}_\mathbb{C}{}^o$

{$\delta\otimes\delta^o[\text{Aut}[\mathcal{A}_\mathbb{C}\otimes\mathcal{A}_\mathbb{C}{}^o]] \to \gamma.\beta[x].\gamma^{-1} \to \beta[\delta\otimes\delta^{-1}[x]]$; $x \in \mathcal{B} \to \mathcal{A}_\mathbb{C}\otimes\mathcal{A}_\mathbb{C}{}^o$} $\Rightarrow$ {$\gamma[\text{Aut}[M_k[\mathbb{C}]]\otimes\text{Aut}[M_k[\mathbb{C}]]][a] \to u.a.v^\dagger$; $a \in \mathcal{A}_\mathbb{C}$; u, v [unitary] $\in M_k[\mathbb{C}]$}

$\Rightarrow$

$J.\gamma.J^{-1}[a] \to (u.a^\dagger.v^\dagger)^\dagger \to v.a.u^\dagger$

$(J.\gamma.J^{-1} \to -\gamma) \Rightarrow$
- $v.a.u^\dagger \to -u.a.v^\dagger$
- $u^\dagger.v \to z$
- $z.a.z \to -a$
- $z.z \to -1$
- $z.a \to a.z$
- $z \to \eta.I$
- $\eta \to \pm 1$
- $v \to \eta.I.u$
- $\gamma[a] \to -\eta.I.u.a.u^\dagger$
- $\gamma.\gamma[a] \to -u.u.a.u^{-1}.u^{-1}$
- $\gamma.\gamma \to 1$
- $a \to -u.u.a.u^{-1}.u^{-1}$[contradiction for a$\to$1]

Hence, $J.\gamma \to -\gamma.J$ is not possible for $Z[\mathcal{A}_\mathbb{C}] \to \mathbb{C}$

So the case $Z[\mathcal{A}_\mathbb{C}] \to \mathbb{C}\oplus\mathbb{C}$ is only possibility.

```
PR["• What is ", {𝒜ℂ[CG["real or of 𝒜"]] → Mk ⊕ Mk}, "?",
 line,
 NL, "Assume W in ", selectWhy[remark29],
 " is right vector space over ℍ with nontrivial ℤ/2
   grading. The right action of ℍ ⇒ antilinear isometry ",
 $ = {iI[W] , iI.iI → -1, "nontrival ℤ/2-grading" ⇒ dim[W] ≥ 2 };
 $ // ColumnForms,
 NL, "Let ", $ = {W[CG["2-dim over ℍ"], V[CG["4-dim ℂ vector space"]]],
   𝒜 → (Endℍ[W] ⊕ Endℂ[V] ~ M2[ℍ] ⊕ M4[ℂ]),
   𝒜[CG["ℤ/2 graded from W, nontrivial grading only on M2[ℍ] part"]]
  }; $ // ColumnForms,
 line,
 NL, "● Proposition.3.2: There exists up to equivalence a unique
   ℤ/2-grading of ℋ compatible with the graded representation of 𝒜 and ",
 $ = {J.γ → -γ.J, ℋ → ℰ⊖ct[ℰ], γ[{ξ, η}] → {γ.ξ, -γ.η}};
 $ // ColumnForms,
 line,
 NL, "• Remark.3.3: ", ℰ → Homℂ[V, W],
 " is related to the classification of instantons."
]
```

• What is $\{\mathcal{A}_{\mathbb{C}}[\text{real or of } \mathcal{A}] \to M_k \oplus M_k\}$?

Assume W in

remark29 → {{V, W}[k-dim ℂ-Hilbert], $\mathcal{A}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}[W] \oplus \mathrm{End}_{\mathbb{C}}[V]$, $\mathcal{H}$[bimodule over $\mathcal{A}_{\mathbb{C}}$] → $\mathcal{E} \oplus \mathcal{E}^\dagger$, J[{ξ, η}] → {$\eta^\dagger$, $\xi^\dagger$}, $\mathcal{E} \to \mathrm{Hom}_{\mathbb{C}}[V, W]$, $\mathcal{E}^\dagger \to \mathrm{Hom}_{\mathbb{C}}[W, V]$, {w, v}[{g, h}] → {w ∘ g, v ∘ h}}

is right vector space over ℍ with nontrivial ℤ/2 grading. The right action of ℍ ⇒ antilinear isometry

$$\left|\begin{array}{l} I[W] \\ I.I \to -1 \\ \text{nontrival } \mathbb{Z}/2\text{-grading} \Rightarrow \dim[W] \geq 2 \end{array}\right.$$

Let
$$\left|\begin{array}{l} W[\text{2-dim over } \mathbb{H}, V[\text{4-dim } \mathbb{C} \text{ vector space}]] \\ \mathcal{A} \to \mathrm{End}_{\mathbb{H}}[W] \oplus \mathrm{End}_{\mathbb{C}}[V] \sim M_2[\mathbb{H}] \oplus M_4[\mathbb{C}] \\ \mathcal{A}[\mathbb{Z}/2 \text{ graded from W, nontrivial grading only on } M_2[\mathbb{H}] \text{ part}] \end{array}\right.$$

● Proposition.3.2: There exists up to equivalence a unique ℤ/2-grading of $\mathcal{H}$ compatible with the graded representation of $\mathcal{A}$ and

$$\left|\begin{array}{l} J.\gamma \to -\gamma.J \\ \mathcal{H} \to \mathcal{E} \ominus \mathcal{E}^\dagger \\ \gamma\left[\left|\begin{array}{l}\xi \\ \eta\end{array}\right.\right] \to \left|\begin{array}{l}\gamma.\xi \\ -\gamma.\eta\end{array}\right. \end{array}\right.$$

• Remark.3.3: $\mathcal{E} \to \mathrm{Hom}_{\mathbb{C}}[V, W]$ is related to the classification of instantons.

■ 4. The subalgebra and the order one condition

```
PR["● For ", {𝒜, ℋ, J, γ},
 Imply, {Z[𝒜][CG["center of 𝒜 is non-trivial and 𝒜 is not connected"]]},
 NL, "Look for ", iD[CG["Dirac operator"]],
 " that connects the two spaces via off-diagonal: ",
 CommutatorM[iD, Z[𝒜]] ≠ {0},
 NL, "Require order-one condition ", selectWhy[CommutatorM[__]],
 NL, "Find ", {𝒜F ⊂ 𝒜even, CommutatorM[iD, Z[𝒜]] ≠ {0}},
 line,
 NL, "● Theorem 4.1. Up to an automorphism of ", 𝒜even,
 ", there exists a unique involutive subalgebra ", 𝒜F ⊂ 𝒜even,
 " of maximal dimension admitting off-diagonl Dirac operators.  It is given by: ",
 𝒜F → {{λ⊕q, λ⊕m}, λ ∈ ℂ, q ∈ ℍ, m ∈ M3[ℂ]} ⊂ (ℍ⊕ℍ⊕M4[ℂ]),
 " using the field morphism ", ℂ → ℍ, ".  The involutive algebra ", 𝒜F,
 " is isomorphic to ", ℂ⊕ℍ⊕M3[ℂ], " and together with its representation in ",
 {ℋ, J, γ}, " it give the noncommutative geometry F "
]
```

● For $\{\mathcal{A}, \mathcal{H}, J, \gamma\}$
⇒ $\{Z[\mathcal{A}]$[center of $\mathcal{A}$ is non-trivial and $\mathcal{A}$ is not connected]$\}$
Look for D[Dirac operator]
that connects the two spaces via off-diagonal: $[D, Z[\mathcal{A}]]_- \neq \{0\}$
Require order-one condition $[[D, a]_-, b^o]_- \to 0$
Find $\{\mathcal{A}_F \subset \mathcal{A}^{even}, [D, Z[\mathcal{A}]]_- \neq \{0\}\}$

● Theorem 4.1. Up to an automorphism of $\mathcal{A}^{even}$, there exists a unique involutive subalgebra $\mathcal{A}_F \subset \mathcal{A}^{even}$ of maximal dimension admitting off-diagonl Dirac operators. It is given by: $\mathcal{A}_F \to \{\{\lambda\oplus q, \lambda\oplus m\}, \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3[\mathbb{C}]\} \subset \mathbb{H}\oplus\mathbb{H}\oplus M_4[\mathbb{C}]$ using the field morphism $\mathbb{C} \to \mathbb{H}$. The involutive algebra $\mathcal{A}_F$ is isomorphic to $\mathbb{C}\oplus\mathbb{H}\oplus M_3[\mathbb{C}]$ and together with its representation in $\{\mathcal{H}, J, \gamma\}$ it give the noncommutative geometry F

```
PR["● Consider ", {ℋ → e1.ℋ⊕e2.ℋ, ℋ → ℋ1 ⊕ℋ2, β → β1 ⊕β2, 𝒜F ⊂ 𝒜^even, πj[𝒜F] → 𝒜F ⊂ ℋj},
 line,
 NL, "● Lemma.4.2. If the representations πj are
   disjoint, then there is no off diagonal Dirac operators for 𝒜F.",
 line,
 NL, "Proof: ", $ = {{ej[CG["minimal projection in ", Z[𝒜]]], J.e1.inv[J] → e2} ⇒
     {ej[CG["minimal projection in ", Z[rghtA[𝒜]]]]},
   {Inactive[ForAll][a ∈ 𝒜F, {πj[a] → a.ej, πj[a] → ej.a, πj[CG["disjoint"]],

        T[CG["operator in ℋ"]], CommutatorM[T, a] → 0,
        CommutatorM[e1.T.e2, a] → 0
       }]
      ⇒
      {πj[CG["disjoint"]] ⇒ "intertwining operator→0", e1.T.e2 → 0, e2.T.e1 → 0,
       CommutatorM[T, rghtA[a]] → 0
      },
     Inactive[ForAll][{a, b} ∈ 𝒜F, CommutatorM[CommutatorM[iD, a], rghtA[b]] → 0]
      ⇒
      {e2.CommutatorM[iD, a].e1 → 0,
       CommutatorM[e2.iD.e1, a] → 0,
       e2.iD.e1 → 0[CG["no off-diagonal elements."]]
      }
   }
  }; $ // ColumnForms,
 NL, "For ",
 $ = {T[ℋ1] → ℋ2, 𝒜[T] → {b ∈ 𝒜^even, π2[b].T → T.π1[b], π2[ct[b]].T → T.π1[ct[b]]},
   CG["involutive unital subalgebra of 𝒜^even"]};
 $ // ColumnForms
]
```

● Consider $\{\mathcal{H}\to e_1.\mathcal{H}\oplus e_2.\mathcal{H},\ \mathcal{H}\to\mathcal{H}_1\oplus\mathcal{H}_2,\ \beta\to\beta_1\oplus\beta_2,\ \mathcal{A}_F\subset\mathcal{A}^{even},\ \pi_j[\mathcal{A}_F]\to\mathcal{A}_F\subset\mathcal{H}_j\}$

● Lemma.4.2. If the representations π_j are disjoint, then there is no off diagonal Dirac operators for $\mathcal{A}_F$.

Proof: $\left|\begin{array}{l} \left|\begin{array}{l} e_j[\text{minimal projection in }, Z[\mathcal{A}]] \\ J.e_1.J^{-1}\to e_2 \end{array}\right. \Rightarrow \left| e_j[\text{minimal projection in }, Z[\mathcal{A}^o]]\right. \\ \text{ForAll}\left[a\in\mathcal{A}_F, \left|\begin{array}{l} \pi_j[a]\to a.e_j \\ \pi_j[a]\to e_j.a \\ \pi_j[\text{disjoint}] \\ T[\text{operator in }\mathcal{H}] \\ [T, a]_-\to 0 \\ [e_1.T.e_2, a]_-\to 0 \end{array}\right.\right] \Rightarrow \left|\begin{array}{l} \pi_j[\text{disjoint}]\Rightarrow\text{intertwining operator}\to 0 \\ e_1.T.e_2\to 0 \\ e_2.T.e_1\to 0 \\ [T, a^o]_-\to 0 \end{array}\right. \\ \text{ForAll}\left[\left|\begin{array}{l} a \\ b \end{array}\right. \in\mathcal{A}_F, [[D, a]_-, b^o]_-\to 0\right] \Rightarrow \left|\begin{array}{l} e_2.[D, a]_-.e_1\to 0 \\ [e_2.D.e_1, a]_-\to 0 \\ e_2.D.e_1\to 0[\text{no off-diagonal elements.}] \end{array}\right. \end{array}\right.$

For $\left|\begin{array}{l} T[\mathcal{H}_1]\to\mathcal{H}_2 \\ \mathcal{A}[T]\to\left|\begin{array}{l} b\in\mathcal{A}^{even} \\ \pi_2[b].T\to T.\pi_1[b] \\ \pi_2[b^\dagger].T\to T.\pi_1[b^\dagger] \end{array}\right. \\ \text{involutive unital subalgebra of }\mathcal{A}^{even} \end{array}\right.$

```
PR["Proof of Theorem.4.1.(contd).  Let ",
 $ = {𝒜F[CG["involutive subalgebra with off-diagonal operator"]] ⊂ 𝒜even,
    "Lemma42"
   } ⇒ {πj[CG["not disjoint"]] ⇒
     Exists[T, {
       (T[ℋ1] → ℋ2[CG["not zero"]]) ∍ 𝒜F ⊂ 𝒜[T],
       {T → c2.T.c1[CG[let]], CommutatorM[cj, 𝒜even] → 0} ∍
        {𝒜F ⊂ 𝒜[c2.T.c1]},
       {CommutatorM[cj, πj[b]] → 0,
         support[T] → 𝒜even[ℋ1],
         range[T] → 𝒜even[ℋ2]}
        ⇒
        {π1 → 2 × (ℍ ⊂ ℂ^2), CR["Why ℍ and not M4[ℂ]"],
         π2 → M4[ℂ] ⊂ ℂ^4,
         C[CG["Projection"]][𝒜even → ℍ ⊕ ℍ ⊕ M4[ℂ]] -> ℍ ⊕ M4[ℂ],
         T[ℂ^2] → ℂ^4,
         C[T] → {b ∈ C, π2[b].T → T.π1[b], π2[ct[b]].T → T.π1[ct[b]]},
         𝒜[T] → {{q, y}, q ∈ ℍ, y ∈ C[T]},
         dim[𝒜[T]] → 4 + dim[C[T]]
        },
       {rank[T] → 2} ⇒
        {range[T] → R[CG["2-dim subspace of ℂ^4"]],
         "invariant under C",
         C[T] ⊂ ℍ ⊕ M2[ℂ] ⊕ M2[ℂ],
         ℍ[CG["π1[support[T]]"]] -> π2[b ∈ C] ⊂ (M2[ℂ][CG["range[T]"]]),
         C[T] ⊂ ℍ ⊕ M2[ℂ],
         dimℝ[C[T]] ≤ 4 + 8
        },

       {rank[T] → 1} ⇒
        {range[T] → R[CG["1-dim subspace of ℂ^4"]],
         "invariant under C",
         (support[T] ⊂ ℂ^2) → S["1-dim subspace,"],
         {SU[2][ℍ], U[4][M4[ℂ]]}[CG["act transitive on {ℂ^2,ℂ^4}"]],
         S → {{a, 0} ∈ ℂ^2, range[T] → {a, 0, 0, 0} ∈ ℂ^4, T[{a, b}] → {a, 0, 0, 0}},
         {ℂ[CG["embedded"]] ⊂ ℍ, λ → {{λ, 0}, {0, cc[λ]}}},
         C[T] → {{λ, λ ⊕ m} ∈ ℍ ⊕ M4[ℂ], λ ∈ ℂ, m ∈ M3[ℂ]},

         dimℝ[C[T]] -> 2 + 18
        }
      }],


   }; $ // ColumnForms
]
```

Proof of Theorem.4.1.(contd). Let

$\mathcal{A}_F$[involutive subalgebra with off-diagonal operator] $\subset \mathcal{A}^{even}$ $\Rightarrow$
Lemma42

π_j[not disjoint] $\Rightarrow \exists_T$

$(T[\mathcal{H}_1] \to \mathcal{H}_2$[not zero]$) \ni \mathcal{A}_F \subset \mathcal{A}[T]$

$T \to c_2.T.c_1$[let]
$[c_j, \mathcal{A}^{even}]_- \to 0$
$\ni$ $\mathcal{A}_F \subset \mathcal{A}[c_2.T.c_1]$

$[c_j, \pi_j[b]]_- \to 0$
support[T] $\to \mathcal{A}^{even}[\mathcal{H}_1]$
range[T] $\to \mathcal{A}^{even}[\mathcal{H}_2]$
$\Rightarrow$

- $\pi_1 \to 2 \times (\mathbb{H} \subset \mathbb{C}^2)$
- Why $\mathbb{H}$ and not $M_4[\mathbb{C}]$
- $\pi_2 \to M_4[\mathbb{C}] \subset \mathbb{C}^4$
- $\mathcal{C}$[Projection]$[\mathcal{A}^{even} \to \mathbb{H} \oplus \mathbb{H} \oplus M_4[\mathbb{C}]] \to \mathbb{H} \oplus M_4[\mathbb{C}]$
- $T[\mathbb{C}^2] \to \mathbb{C}^4$
- $\mathcal{C}[T] \to$ $b \in \mathcal{C}$; $\pi_2[b].T \to T.\pi_1[b]$; $\pi_2[b^\dagger].T \to T.\pi_1[b^\dagger]$
- $\mathcal{A}[T] \to$ $\begin{pmatrix} q \\ y \end{pmatrix}$; $q \in \mathbb{H}$; $y \in \mathcal{C}[T]$
- $\dim[\mathcal{A}[T]] \to 4 + \dim[\mathcal{C}[T]]$

rank[T] $\to 2$ $\Rightarrow$

- range[T] $\to$ R[2-dim subspace of $\mathbb{C}$^4]
- invariant under $\mathcal{C}$
- $\mathcal{C}[T] \subset \mathbb{H} \oplus M_2[\mathbb{C}] \oplus M_2[\mathbb{C}]$
- $\mathbb{H}[\pi_1$[support[T]]] $\to \pi_2[b \in \mathcal{C}] \subset M_2[\mathbb{C}]$[range[T]]
- $\mathcal{C}[T] \subset \mathbb{H} \oplus M_2[\mathbb{C}]$
- $\dim_{\mathbb{R}}[\mathcal{C}[T]] \le 12$

rank[T] $\to 1$ $\Rightarrow$

- range[T] $\to$ R[1-dim subspace of $\mathbb{C}$^4]
- invariant under $\mathcal{C}$
- support[T] $\subset \mathbb{C}^2 \to$ S[1-dim subspace,]
- SU[2][$\mathbb{H}$]; U[4][$M_4[\mathbb{C}]$] [act transitive on $\{\mathbb{C}^2, \mathbb{C}^4\}$]
- S $\to$ $\begin{pmatrix} a \\ 0 \end{pmatrix} \in \mathbb{C}^2$; range[T] $\to \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^4$; $T[\begin{pmatrix} a \\ b \end{pmatrix}] \to \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}$
- $\mathbb{C}$[embedded] $\subset \mathbb{H}$; $\lambda \to \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$
- $\mathcal{C}[T] \to$ $\begin{pmatrix} \lambda \\ \lambda \oplus m \end{pmatrix} \in \mathbb{H} \oplus M_4[\mathbb{C}]$; $\lambda \in \mathbb{C}$; $m \in M_3[\mathbb{C}]$
- $\dim_{\mathbb{R}}[\mathcal{C}[T]] \to 20$

```
PR["● Theorem 4.3.  Let M be a Riemannian spin 4-manifold and F the finite
   noncommutative geometry of K-theoretic dimension 6 described above,
   but with multipliciy 3.  Let M×F be endowed with the product metric.
(1) The unimodular subgroup of the unitary group acting by the adjoint
   representation Ad[u] in ℋ is the group of gauge transformations of SM.
(2) The unimodular inner fluctuations of the metric give the gauge bosons of the SM.
(3) The full standard model (with neutrino mixing
   and seesaw mechanism) minimally coupled to Einsteing granvity
   is given in Euclidean form by the action of functional ",
 NL, {S → Tr[f[iD_A / Λ]] + 1 / 2 BraKet[J.ξ̃, iD_A.ξ̃], ξ̃ ∈ ℋcl^+},
 " where ", iD_A, " is the Dirac operator with the unimodular inner fluctuations."
]
```

● Theorem 4.3. Let M be a Riemannian spin 4-manifold and F the finite noncommutative geometry of K-theoretic dimension 6 described above, but with multipliciy 3. Let M×F be endowed with the product metric.
(1) The unimodular subgroup of the unitary group acting by the adjoint representation Ad[u] in $\mathcal{H}$ is the group of gauge transformations of SM.
(2) The unimodular inner fluctuations of the metric give the gauge bosons of the SM.
(3) The full standard model (with neutrino mixing and seesaw mechanism) minimally coupled to Einsteing granvity is given in Euclidean form by the action of functional

$\{S \to \frac{1}{2}\langle J.\tilde{\xi} \mid D_A.\tilde{\xi}\rangle + \mathrm{Tr}[f[\frac{D_A}{\Lambda}]],\ \tilde{\xi} \in (\mathcal{H}_{cl})^+\}$ where D_A
is the Dirac operator with the unimodular inner fluctuations.