```
<< Local`QFTToolKit2`;

Get[$HomeDirectory <> "/Mathematica/NonCommutative/1204.0328
      ParticlePhysicsFromAlmostCommutativeSpacetime.1.redo.out"];


"Local notational definitions";
rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := ¦a¦;
it[a_] := Style[a, Italic]
iD := it[D]
iI := it["I"]
C∞ := C^"∞"
B_x_ := T[B, "d", {x}]
("∇"^S)_n_ := T["∇"^S, "d", {n}]

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall]];
Clear[expandDC];
expandDC[sub_: {}] := tuRepeat[{sub, tuOpDistribute[Dot],
    tuOpSimplify[Dot], tuOpDistribute[CircleTimes], tuOpSimplify[CircleTimes]}]
$sgeneral := {T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
  T[γ, "d", {5}].T[γ, "d", {5}] → 1, ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
  CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  T["∇", "d", {_}][1_n_] → 0, a_ . 1_n_ → a, 1_n_ . a_ → a}
$sgeneral // ColumnBar


Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] ⧴ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "dd", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] ⧴ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] ⧴ CommutatorP[b, a] /; OrderedQ[{b, a}],
    tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
  };
$symmetries // ColumnBar
```

$\gamma_5 \to \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$

$\gamma_5 . \gamma_5 \to 1$

$(\gamma_5)^\dagger \to \gamma_5$

$\{\gamma_5, \gamma^\mu\}_+ \to 0$

$\nabla_{\_}[1_{n\_}] \to 0$

$(a\_).1_{n\_} \to a$

$1_{n\_}.(a\_) \to a$

```
tt : g^μ_ ^ν_ :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : F^μ_ ^ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
tt : F_μ_ ν_ :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}]
[a_, b_]_ :→ -[b, a]_ /; OrderedQ[{b, a}]
{a_, b_}+ :→ {b, a}+ /; OrderedQ[{b, a}]
tt : γ^μ.γ5 :→ Reverse[tt]
```

# 1204.0328: Particle Physics From Almost Commutative Spacetime

## 3. The Spectral Action of AC-manifold

### 3.1 The heat expansion of the spectral action

```
PR["●Lichnerowicz formula.",
  NL, "•vector bundle ", "E" → M,
  NL, "•Laplacian ", △^"E"["∇"^"E"[CG["connection on E"]]],
  NL, "•generalized Laplacian ", H → {△^"E" - F, F ∈ Γ[Endo["E"]]},
  NL, "•generalized Dirac operator[ℤ2graded vector bundle E]",
  yield, $ = {iD["E"[CG["ℤ2-graded"]]],
    iD[Γ[M, "E"^"±"]] -> Γ[M, "E"^"∓"],
    iD · iD ∈ H}; $ // ColumnBar,
  NL, CR["Interchange symbols ", 𝒟 ↔ iD]
 ];
```

●Lichnerowicz formula.
•vector bundle E → M
•Laplacian $\triangle^{E}[\nabla^{E}[$connection on E$]]$
•generalized Laplacian $H \to \{-F + \triangle^{E},\ F \in \Gamma[\mathrm{Endo}[E]]\}$
•generalized Dirac operator[$\mathbb{Z}_2$graded vector bundle E] ⟶
$D[E[\mathbb{Z}_2\text{-graded}]]$
$D[\Gamma[M, E^{\pm}]] \to \Gamma[M, E^{\mp}]$
$D \cdot D \in H$

Interchange symbols $\mathcal{D} \leftrightarrow D$

```
PR["■Show ", $ = {𝒟𝒜 -> "generalized Dirac operator", 𝒟𝒜.𝒟𝒜 ∈ H},
  NL, "•compute ", $[[2, 1]],
  " where ",
  $sDA = $s0 = $s = {𝒟𝒜 -> -I T[γ, "u", {μ}].T["∇""E", "d", {μ}] + T[γ, "d", {5}] ⊗ Φ,
      T["∇""E", "d", {μ_}] -> T["∇""S, "d", {μ}] ⊗ 1ℋF + I 1N ⊗ Bμ,
      T["∇""E", "d", {μ}][S ⊗ "E"],
      Φ ∈ Γ[Endo["E"]][CG["Higg's field"]]
     }; $s // Column,
  accumDef[$sDA];
  NL, "•Define ",
  $d = {T[𝒟, "d", {μ}][a_] → ad[T["∇""E", "d", {μ}]][a], ad[aa_][bb_] → aa.bb - bb.aa};
  $d // ColumnBar,
  $defall = $defall // tuAppendUniq[$d];
  "xPOFF",
  Yield, $ = $0 = T[𝒟, "d", {μ}][Φ],
  yield, $ = $ /. $d,
  yield, $ = $ /. $d,
  Yield, $ = $ /. $s[[1 ;; 2]],
  Yield, $ = $ //. tuOpDistribute[Dot] //. tuOpSimplify[Dot], "PONdd",
  NL, "Using ", $s = {(op_ ⊗ 1t_).ph_ → op[ph] ⊗ 1t + ph.(op ⊗ 1t), (1N_ ⊗ op_).ph_ → 1N ⊗ op.ph,
    ph_ .(1N_ ⊗ op_) → 1N ⊗ ph.op, ca_ 1N_ ⊗ a_ + cb_ 1N_ ⊗ b_ -> 1N ⊗ (ca a + cb b),
    a_ .b_ - b_ .a_ → CommutatorM[a, b]
   }; $s // ColumnBar,
  Yield, $ = $0 -> $ // tuRepeat[$s, Simplify]; $ // Framed,
  NL, CR["In the text (3.1) the label S and the 1? got dropped."],
  NL, CR["Use? "], $s = a_ ⊗ 1ℋF → 1N ⊗ a,
  Yield, $ = $ /. $s // expandDC[];
  accumDef[$]; Framed[$D1 = $]
 ];
```

■Show $\{\mathcal{D}_{\mathcal{A}} \to \text{generalized Dirac operator}, \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \in H\}$

•compute $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}}$ where
$\mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \Phi - i\, \gamma^{\mu}.\nabla^{E}{}_{\mu}$
$\nabla^{E}{}_{\mu\_} \to i\, 1_N \otimes B_{\mu} + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}$
$\nabla^{E}{}_{\mu}[S \otimes E]$
$\Phi \in \Gamma[\text{Endo}[E]][\text{Higg's field}]$

•Define $\mathcal{D}_{\mu}[a\_] \to \text{ad}[\nabla^{E}{}_{\mu}][a]$
$\text{ad}[aa\_][bb\_] \to aa.bb - bb.aa$ xPOFF

→ $\mathcal{D}_{\mu}[\Phi] \longrightarrow \text{ad}[\nabla^{E}{}_{\mu}][\Phi] \longrightarrow -\Phi.\nabla^{E}{}_{\mu} + \nabla^{E}{}_{\mu}.\Phi$

→ $-\Phi.(i\, 1_N \otimes B_{\mu} + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}) + (i\, 1_N \otimes B_{\mu} + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).\Phi$

→ $-i\, \Phi.(1_N \otimes B_{\mu}) - \Phi.(\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}) + i\, (1_N \otimes B_{\mu}).\Phi + (\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).\Phi$PONdd

Using
$(op\_ \otimes 1_{t\_}).(ph\_) \to op[ph] \otimes 1_t + ph.(op \otimes 1_t)$
$(1_{N\_} \otimes op\_).(ph\_) \to 1_N \otimes op.ph$
$(ph\_).(1_{N\_} \otimes op\_) \to 1_N \otimes ph.op$
$1_{N\_} \otimes a\_\, ca\_ + 1_{N\_} \otimes b\_\, cb\_ \to 1_N \otimes (a\, ca + b\, cb)$
$(a\_).(b\_) - (b\_).(a\_) \to [a, b]_-$

→ $\boxed{\mathcal{D}_{\mu}[\Phi] \to 1_N \otimes (-i\, [\Phi, B_{\mu}]_-) + \nabla^{S}{}_{\mu}[\Phi] \otimes 1_{\mathcal{H}_F}}$

In the text (3.1) the label S and the 1? got dropped.
Use? $a\_ \otimes 1_{\mathcal{H}_F} \to 1_N \otimes a$

→ $\boxed{\mathcal{D}_{\mu}[\Phi] \to -i\, 1_N \otimes [\Phi, B_{\mu}]_- + 1_N \otimes \nabla^{S}{}_{\mu}[\Phi]}$

```
PR["•Define curvature of Bμ: ",
  $F = T[F, "dd", {μ, ν}] → tuDPartial[Bν, μ] - tuDPartial[Bμ, ν] + I CommutatorM[Bμ, Bν],
  NL, "•Define curvature of ", "∇"^"E", ": ",
  $O = {Ω^"E"[X, Y] -> T["∇"^"E", "d", {X}].T["∇"^"E", "d", {Y}] - T["∇"^"E", "d", {Y}].T["∇"^"E",
        "d", {X}] - T["∇"^"E", "d", {CommutatorM[X, Y]}], {X, Y} → "vector fields"};
  $O // ColumnBar, accumDef[{$F, $O}];
  NL, CO["■For local coordinates(cartesian): "],
  CommutatorM[tuDPartial[_, μ], tuDPartial[_, ν]] → 0,
  NL, "Use: ", {tuDPartial[_, μ] → X, tuDPartial[_, ν] → Y},
  Yield, $s = {CommutatorM[X, Y] → 0, X -> μ, Y → ν, T["∇"^"E", "d", {0}] → 0},
  Imply, e33 = $ = $O[[1]] //. $s,
  Yield, $ = $ /. $sDA[[1 ;; 2]];
  Yield, $ = $ // tuDotSimplify[]; $ // ColumnSumExp,
  NL, "Using: ", $scc = $s = {
     (a_ ⊗ b_).(c_ ⊗ d_) :→ a . c ⊗ b . d +
       If[! FreeQ [a, "∇"] && ! FreeQ [d, B | Φ], c ⊗ a[d] , 0] +
       If[! FreeQ [b, "∇"] && ! FreeQ [d, B | Φ], a ⊗ b[d], 0],
     1N_ . a_ → a, a_ . 1N_ → a , (a_ ⊗ 1HF) - (b_ ⊗ 1HF) → (a - b) ⊗ 1HF,
     (1N_ ⊗ a_) - (1N_ ⊗ b_) → 1N ⊗ (a - b)};
  ColumnBar[$s],

  Yield, $ = $ //. $s // Simplify // Expand;
  accumDef[$];
  $ // ColumnSumExp // Framed,
  NL, "Use ", $s = {I 1N ⊗ a_ - I 1N ⊗ b_ -> 1N ⊗ ( I a - I b),
    1N ⊗ a_ + 1N ⊗ b_ -> 1N ⊗ ( a + b), T["∇"^"S", "d", {a_}][b_] → tuDPartial[b, a]
   }; ColumnBar[$s],
  Yield, $ = $ //. $s,
  NL, "Apply (3.2) ",
  $s = tuRuleSolve[$F, CommutatorM[_, _]] /. CommutatorM → MCommutator // First //
    Map[-# &, #] &,
  NL, "Define ", $s1 = $O[[1]] //. {"E" → S, CommutatorM[X, Y] → 0,
     X -> μ, Y → ν, T["∇"^"S", "d", {0}] → 0},
  Yield, $s34 = e34 = $ = $ /. $s /. Reverse[$s1] // Simplify;
  $s34x = $s34 /. {Ω^n_[a_, b_] → T[Ω^n, "dd", {a, b}]};
  $s34x = {$s34x, tuIndicesRaise[{μ, ν}][$s34x]};
  accumDef[{$s34, $s34x, $s1}]; Framed[$], CG[" (3.4)"]
 ];
```

•Define curvature of $B_\mu$: $F_{\mu\nu} \to i [B_\mu, B_\nu]_- - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

•Define curvature of $\nabla^E$: $\left| \begin{array}{l} \Omega^E[X, Y] \to \nabla^E{}_X.\nabla^E{}_Y - \nabla^E{}_Y.\nabla^E{}_X - \nabla^E{}_{[X,Y]_-} \\ \{X, Y\} \to \text{vector fields} \end{array} \right.$

▪For local coordinates(cartesian): $[\underline{\partial}_\mu[\_], \underline{\partial}_\nu[\_]]_- \to 0$

Use: $\{\underline{\partial}_\mu[\_] \to X, \underline{\partial}_\nu[\_] \to Y\}$

→ $\{[X, Y]_- \to 0, X \to \mu, Y \to \nu, \nabla^E{}_0 \to 0\}$

⇒ $\Omega^E[\mu, \nu] \to \nabla^E{}_\mu.\nabla^E{}_\nu - \nabla^E{}_\nu.\nabla^E{}_\mu$

→

$$\to \Omega^E[\mu, \nu] \to \sum\left[ \left| \begin{array}{l} -(1_N \otimes B_\mu).(1_N \otimes B_\nu) \\ i\,(1_N \otimes B_\mu).(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) \\ (1_N \otimes B_\nu).(1_N \otimes B_\mu) \\ -i\,(1_N \otimes B_\nu).(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}) \\ i\,(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(1_N \otimes B_\nu) \\ (\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) \\ -i\,(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}).(1_N \otimes B_\mu) \\ -(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}) \end{array} \right. \right]$$

Using:
```
(a_⊗b_).(c_⊗d_) :→ a.c⊗b.d + If[! FreeQ[a, ∇] && ! FreeQ[d, B | Φ], c⊗a[d], 0] +
   If[! FreeQ[b, ∇] && ! FreeQ[d, B | Φ], a⊗b[d], 0]
1N_.(a_) → a
(a_).1N_ → a
a_⊗1HF - b_⊗1HF → (a - b)⊗1HF
1N_⊗a_ - 1N_⊗b_ → 1N⊗(a - b)
```

$$\to \boxed{\Omega^E[\mu, \nu] \to \sum\left[ \left| \begin{array}{l} (\nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu) \otimes 1_{\mathcal{H}_F} \\ 1_N \otimes (-B_\mu.B_\nu + B_\nu.B_\mu) \\ i\,1_N \otimes \nabla^S{}_\mu[B_\nu] \\ -i\,1_N \otimes \nabla^S{}_\nu[B_\mu] \end{array} \right. \right]}$$

Use $\left| \begin{array}{l} i\,1_N \otimes a\_ - i\,1_N \otimes b\_ \to 1_N \otimes (i\,a - i\,b) \\ 1_N \otimes a\_ + 1_N \otimes b\_ \to 1_N \otimes (a + b) \\ \nabla^S{}_{a\_}[b\_] \to \underset{-a}{\partial}[b] \end{array} \right.$

→ $\Omega^E[\mu, \nu] \to (\nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu) \otimes 1_{\mathcal{H}_F} + 1_N \otimes (-B_\mu.B_\nu + B_\nu.B_\mu - i\,\underline{\partial}_\nu[B_\mu] + i\,\underline{\partial}_\mu[B_\nu])$

Apply (3.2) $-B_\mu.B_\nu + B_\nu.B_\mu \to -i\,(-F_{\mu\nu} - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu])$

Define $\Omega^S[\mu, \nu] \to \nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu$

$$\to \boxed{\Omega^E[\mu, \nu] \to 1_N \otimes (i\,F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}} \quad (3.4)$$

```
PR["•Calculate ", $0 = $ = CommutatorM[T[𝒟, "d", {μ}], T[𝒟, "d", {ν}]].Φ,
  Yield, $ = $ /. CommutatorM → MCommutator // tuDotSimplify[],
  yield, $ = $ //. a_ . b_ → a[b],
  NL, "From the definition: ", $d,
  Yield, $ = $ //. ($d // tuAddPatternVariable[μ]);
  Yield, $ = $ // tuDotSimplify[],
  NL, "Use ", $s =
   {a_ .Φ - b_ .Φ → (a - b).Φ, Φ.a_ - Φ. b_ → Φ.(a - b), a_ .b_ - b_ .a_ → CommutatorM[a, b],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}]};
  $s // ColumnBar,
  Yield, $ = $ // tuRepeat[$s, tuDotSimplify[]]; Framed[$0 → $],
  NL, "From ", $s1 = e33,
  yield, $s1 = $s1 /. $s // Reverse // tuAddPatternVariable[{μ, ν}],
  Imply, $ = $ /. $s1; Framed[$0 → $],
  yield, $ = $ /. CommutatorM → MCommutator /.
    ($d[[2]] // RemovePatterns // Reverse // tuAddPatternVariable[{aa, bb}]);
  Framed[$0 → $], CG[" (3.4)"],
  NL, "Since ", $ = Flatten[{$s34, CommutatorM[$s34[[1]], Φ] → 0}],
  Yield, $ = CommutatorM[#, Φ] & /@ $[[1]] /. $[[2]],
  Yield, $ = $ // tuCommutatorSimplify[],
  NL, "Using ",
  $s = $d[[2]] /. a_ .b_ - b_ .a_ → CommutatorM[a, b] // Reverse // tuPatternRemove //
    tuAddPatternVariable[{aa, bb}],
  Yield, $ = $ /. $s,
  Yield, $ = $ /. a_[Φ] → a; $ // Framed,
  CR["Puzzling role of operator product."]
 ];
```

•Calculate $[\mathcal{D}_\mu, \mathcal{D}_\nu]_-.\Phi$

→ $\mathcal{D}_\mu.\mathcal{D}_\nu.\Phi - \mathcal{D}_\nu.\mathcal{D}_\mu.\Phi \longrightarrow \mathcal{D}_\mu[\mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[\mathcal{D}_\mu[\Phi]]$

From the definition: $\{\mathcal{D}_\mu[a\_] \to ad[\nabla^E{}_\mu][a],\ ad[aa\_][bb\_] \to aa.bb - bb.aa\}$

→

→ $-\Phi.\nabla^E{}_\mu.\nabla^E{}_\nu + \Phi.\nabla^E{}_\nu.\nabla^E{}_\mu + \nabla^E{}_\mu.\nabla^E{}_\nu.\Phi - \nabla^E{}_\nu.\nabla^E{}_\mu.\Phi$

Use
$(a\_).\Phi - (b\_).\Phi \to (a - b).\Phi$
$\Phi.(a\_) - \Phi.(b\_) \to \Phi.(a - b)$
$(a\_).(b\_) - (b\_).(a\_) \to [a, b]_-$
$[a\_, b\_]_- :\to -[b, a]_- /; \text{OrderedQ}[\{b, a\}]$

→ $\boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu]_-.\Phi \to -[\Phi, [\nabla^E{}_\mu, \nabla^E{}_\nu]_-]_-}$

From $\Omega^E[\mu, \nu] \to \nabla^E{}_\mu.\nabla^E{}_\nu - \nabla^E{}_\nu.\nabla^E{}_\mu \longrightarrow [\nabla^E{}_{\mu\_}, \nabla^E{}_{\nu\_}]_- \to \Omega^E[\mu, \nu]$

⇒ $\boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu]_-.\Phi \to -[\Phi, \Omega^E[\mu, \nu]]_-} \longrightarrow \boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu]_-.\Phi \to ad[\Omega^E[\mu, \nu]][\Phi]}$ (3.4)

Since $\{\Omega^E[\mu, \nu] \to 1_N \otimes (i\, F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F},\ [\Omega^E[\mu, \nu], \Phi]_- \to 0\}$

→ $0 \to [1_N \otimes (i\, F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}, \Phi]_-$

→ $0 \to [1_N \otimes (i\, F_{\mu\nu}), \Phi]_- + [\Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}, \Phi]_-$

Using $[aa\_, bb\_]_- \to ad[aa][bb]$

→ $0 \to ad[1_N \otimes (i\, F_{\mu\nu})][\Phi] + ad[\Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}][\Phi]$

→ $\boxed{0 \to ad[1_N \otimes (i\, F_{\mu\nu})] + ad[\Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}]}$ Puzzling role of operator product.

```
PR["Calculate (3.5) from local coordinate Laplacian: ",
 $0 = $ = △"E" → -T[g, "uu", {μ, ν}].(T["∇""E", "d", {μ}].T["∇""E", "d", {ν}] -
        T[Γ, "udd", {ρ, μ, ν}].T["∇""E", "d", {ρ}]),
 NL, "Use definition ", $s = $sDA[[2]],
 Yield, $ = $ /. $s // tuDotSimplify[]; $ // ColumnSumExp,

 NL, "Combining tensor-product products: ",
 $combineProduct = {
   (*Combine tensor-product dot-products with possible ∇ operator on LHS.*)
   (a_⊗b_).(c_⊗d_) :→ a.c⊗b.d + c⊗tuCircleTimesInnerTerm[T["∇"S, "d", {_}]][a[d]],
   a_ . 1n_ → a, 1n_ . a_ → a,
   T["∇"S, "d", {n_}][a_] → tuPartialD[a, n],
   tuPartialD[1n_, a_] → 0,
   a_⊗(tt : Tensor[γ, _, _]).b_ → a.tt⊗b,
   a_⊗0 → 0,
   0⊗a_ → 0
  },
 CK,
 Yield, $ = $ // expandDC[$combineProduct] // Expand;
 $ // ColumnSumExp // Framed,
 (**)
 NL, "Define ", $s = $0 /. "E" → S,
 yield, $s = Map[#⊗1ℋF &, $s];
 $s =
  $s // tuRepeat[{tuOpDistribute[Dot], tuOpDistribute[CircleTimes], tuOpSimplify[Dot],
      tuOpSimplify[CircleTimes], (gg : Tensor[g, _, _]). a_ . b_ ⊗c_ → gg.(a.b⊗c),
      (gg : Tensor[Γ, _, _]). a_ ⊗c_ → gg.(a⊗c)}] // Reverse,
 Imply,
 $ =
  $ /.
   $s;
 $[[2, -2, 2, 2]] = $[[2, -2, 2, 2]] // tuIndexSwap[{μ, ν}];
 $e35 = $;
 $ // ColumnSumExp // Framed, CG[" (3.5)"]
]
```

Calculate (3.5) from local coordinate Laplacian: $\triangle^E \to -g^{\mu\nu}.(\nabla^E{}_\mu.\nabla^E{}_\nu - \Gamma^\rho{}_{\mu\nu}.\nabla^E{}_\rho)$

Use definition $\nabla^E{}_{\mu\_} \to i\, 1_N \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}$

$$\to \triangle^E \to \sum\left[\begin{array}{l} g^{\mu\nu}.(1_N\otimes B_\mu).(1_N\otimes B_\nu) \\ -i\, g^{\mu\nu}.(1_N\otimes B_\mu).(\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) \\ -i\, g^{\mu\nu}.(\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(1_N\otimes B_\nu) \\ -g^{\mu\nu}.(\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) \\ i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N\otimes B_\rho) \\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(\nabla^S{}_\rho\otimes 1_{\mathcal{H}_F}) \end{array}\right]$$

Combining tensor-product products:

$\{(a\_\otimes b\_).(c\_\otimes d\_) :\to a.c\otimes b.d + c\otimes \text{tuCircleTimesInnerTerm}[T[\nabla^S, d, \{\_\}]][a[d]],$
$(a\_).1_{n\_} \to a,\ 1_{n\_}.(a\_) \to a,\ \nabla^S{}_{n\_}[a\_] \to \underline{\partial}_n[a],\ \underline{\partial}_{a\_}[1_{n\_}] \to 0,$
$a\_\otimes(tt:\text{Tensor}[\gamma, \_, \_]).(b\_) \to a.tt\otimes b,\ a\_\otimes 0 \to 0,\ 0\otimes a\_ \to 0\}$ ⟵CHECK

$$\to \boxed{\triangle^E \to \sum\left[\begin{array}{l} -g^{\mu\nu}.(\nabla^S{}_\mu.\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) \\ g^{\mu\nu}.(1_N\otimes B_\mu.B_\nu) \\ -i\, g^{\mu\nu}.(1_N\otimes \underline{\partial}_{\_\mu}[B_\nu]) \\ -i\, g^{\mu\nu}.(\nabla^S{}_\mu\otimes B_\nu) \\ -i\, g^{\mu\nu}.(\nabla^S{}_\nu\otimes B_\mu) \\ i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N\otimes B_\rho) \\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(\nabla^S{}_\rho\otimes 1_{\mathcal{H}_F}) \end{array}\right]}$$

Define $\triangle^S \to -g^{\mu\nu}.(\nabla^S{}_\mu.\nabla^S{}_\nu - \Gamma^\rho{}_{\mu\nu}.\nabla^S{}_\rho) \longrightarrow -g^{\mu\nu}.(\nabla^S{}_\mu.\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) + g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(\nabla^S{}_\rho\otimes 1_{\mathcal{H}_F}) \to \triangle^S\otimes 1_{\mathcal{H}_F}$

$$\Rightarrow \boxed{\triangle^E \to \sum\left[\begin{array}{l} \triangle^S\otimes 1_{\mathcal{H}_F} \\ g^{\mu\nu}.(1_N\otimes B_\mu.B_\nu) \\ -i\, g^{\mu\nu}.(1_N\otimes \underline{\partial}_{\_\mu}[B_\nu]) \\ -2\, i\, g^{\mu\nu}.(\nabla^S{}_\mu\otimes B_\nu) \\ i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N\otimes B_\rho) \end{array}\right]} \quad (3.5)$$

```
PR["•Given the Lichnerowicz formula: ",
  $ = $L = {slash[𝒟].slash[𝒟] → △^S + s / 4,
     △^S[CG["Laplacian of spin connection ∇^S"], s[CG["scalar curvature of M"]]]};
  $ // ColumnBar,
  accumDef[$L];
  NL, "Prove(prop.3.1):",
  NL, $31 = $0 = $ =
     {𝒟_𝒜.𝒟_𝒜 → △^"E" - Q, Q → -( s ⊗ 1_ℋF) / 4 - 1_N ⊗ (Φ . Φ) + I / 2 (T[γ, "u", {μ}].T[γ, "u", {ν}]) ⊗
          T[F, "dd", {μ, ν}] - I T[γ, "u", {μ}]. T[γ, "d", {5}] ⊗ T[𝒟, "d", {μ}].Φ};
  $ // ColumnBar,
  NL, "●Compute: ", $ = $0[[1, 1]],
  Yield,
  $ = $ //. tuRuleSelect[$defall][{𝒟_𝒜, T["∇""E", "d", {μ_}]}] /. a_ . b_ :→ a. (b /. μ → ν),
  (*relabel dummy index of 2nd term*)
  Yield, $ = $ // expandDC[] // Expand; $ // ColumnSumExp,
  (****)
  NL, "Include γ n tensor product",
  $s = {(tt : Tensor[γ, _, _]) .(a_ ⊗ b_) :→ tt.a ⊗ b, (*γ act on ℳ space*)
     T[γ, "d", {5}].T[γ, "d", {5}] → 1_N
    } // tuRule;  $s // ColumnBar,

  Yield, $ = $ //. $s; $ // ColumnSumExp;

  NL, "Combine tensor-product products. ",
  Yield, $ = $ //. $combineProduct // expandDC[{$s}] // Expand;
  ColumnSumExp[$], CK,

  NL, "Apply ",
  $s = $ss = {
     T[γ, "d", {5}].T[γ, "d", {5}] → 1_N,
     (tt : T[γ, "u", {μ_}]). T["∇"^S, "d", {μ_}] :→ I slash[𝒟],
     1_n_ .a_ → a, a_ . 1_n_ → a,
     T[g, "uu", {μ, ν}] →
      1 / 2 (T[γ, "u", {μ}].T[γ, "u", {ν}] + T[γ, "u", {ν}].T[γ, "u", {μ}])
    },
  accumDef[$ss];
  Yield, $pass = $ = $ // tuRepeat[{$s, tuOpSimplify[CircleTimes]}, tuDotSimplify[]];
  ColumnSumExp[$] // Framed,
  CG[" p.29 with combined operator product. (Extra ∂ terms?)"]
 ];
```

•Given the Lichnerowicz formula:

$(\not{D}).(\not{D}) \to \frac{s}{4} + \triangle^S$

$\triangle^S$[Laplacian of spin connection $\nabla^S$, s[scalar curvature of M]]

Prove(prop.3.1):

$\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to -Q + \triangle^E$

$Q \to -\frac{1}{4} s \otimes 1_{\mathcal{H}_F} - i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi$

●Compute: $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}}$

→ $(\gamma_5 \otimes \Phi - i\,\gamma^\mu.(i\,1_N \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})).(\gamma_5 \otimes \Phi - i\,\gamma^\nu.(i\,1_N \otimes B_\nu + \nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}))$

→ $\sum[$
$(\gamma_5 \otimes \Phi).(\gamma_5 \otimes \Phi)$
$(\gamma_5 \otimes \Phi).\gamma^\nu.(1_N \otimes B_\nu)$
$-i\,(\gamma_5 \otimes \Phi).\gamma^\nu.(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F})$
$\gamma^\mu.(1_N \otimes B_\mu).(\gamma_5 \otimes \Phi)$
$-i\,\gamma^\mu.(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(\gamma_5 \otimes \Phi)$
$\gamma^\mu.(1_N \otimes B_\mu).\gamma^\nu.(1_N \otimes B_\nu)$
$-i\,\gamma^\mu.(1_N \otimes B_\mu).\gamma^\nu.(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F})$
$-i\,\gamma^\mu.(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).\gamma^\nu.(1_N \otimes B_\nu)$
$-\gamma^\mu.(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).\gamma^\nu.(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F})$
$]$

Include $\gamma$ n tensor product
$(tt : \mathrm{Tensor}[\gamma, \_, \_]).(a\_ \otimes b\_) \mapsto tt.a \otimes b$
$\gamma_5.\gamma_5 \to 1_N$

→

Combine tensor-product products.

→ $\sum[$
$-i\,\gamma_5.\gamma^\mu \otimes \partial_{-\mu}[\Phi]$
$\gamma_5.\gamma^\nu \otimes \Phi.B_\nu$
$\gamma^\mu.\gamma_5 \otimes B_\mu.\Phi$
$\gamma^\mu.\gamma^\nu \otimes B_\mu.B_\nu$
$-i\,\gamma^\nu.\gamma^\mu \otimes \partial_{-\mu}[B_\nu]$
$-i\,\gamma_5.\gamma^\nu.\nabla^S{}_\nu \otimes \Phi$
$-i\,\gamma^\mu.\nabla^S{}_\mu.\gamma_5 \otimes \Phi$
$-i\,\gamma^\mu.\nabla^S{}_\mu.\gamma^\nu \otimes B_\nu$
$-i\,\gamma^\mu.\gamma^\nu.\nabla^S{}_\nu \otimes B_\mu$
$-(\gamma^\mu.\nabla^S{}_\mu.\gamma^\nu.\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F})$
$1_N \otimes \Phi.\Phi$
$]$ ⟵CHECK

Apply $\{\gamma_5.\gamma_5 \to 1_N,\ (tt : \gamma^{\mu\_}).\nabla^S{}_{\mu\_} \mapsto i\,(\not{D}),\ 1_{n\_}.(a\_) \to a,\ (a\_).1_{n\_} \to a,\ g^{\mu\nu} \to \frac{1}{2}(\gamma^\mu.\gamma^\nu + \gamma^\nu.\gamma^\mu)\}$

→ $\boxed{\sum[$
$(\not{D}).(\not{D}) \otimes 1_{\mathcal{H}_F}$
$(\not{D}).\gamma_5 \otimes \Phi$
$(\not{D}).\gamma^\nu \otimes B_\nu$
$\gamma_5.(\not{D}) \otimes \Phi$
$-i\,\gamma_5.\gamma^\mu \otimes \partial_{-\mu}[\Phi]$
$\gamma_5.\gamma^\nu \otimes \Phi.B_\nu$
$\gamma^\mu.(\not{D}) \otimes B_\mu$
$\gamma^\mu.\gamma_5 \otimes B_\mu.\Phi$
$\gamma^\mu.\gamma^\nu \otimes B_\mu.B_\nu$
$-i\,\gamma^\nu.\gamma^\mu \otimes \partial_{-\mu}[B_\nu]$
$1_N \otimes \Phi.\Phi$
$]}$ p.29 with combined operator product. (Extra ∂ terms?)

```
Clear[$p];
PR["•Examine different terms of: ", $0 = $pass; ColumnSumExp[$0],
  $ns = Table[i, {i, Length[$0]}];
  $n = {1}; $pn = 0;
  NL, CO["•", $n, ": "], $ns = Complement[$ns, $n];
  $ = $0[[$n]] → "Lichnerowicz formula" → Framed[$p[++$pn] = $L[[1, 2]] ⊗ 1_ℋF], OK,
  $n = {2, 4};
  NL, CO["•", $n, ": "], $ns = Complement[$ns, $n];
  $ = $0[[$n]],
```

```
NL, "Use ", CommutatorM[T[γ, "d", {5}], slash[𝒟]] → 0,
imply, $ → Framed[0], OK,

$n = {3, 7, 10};
NL, CO["•", $n, ": "], $ns = Complement[$ns, $n];
$ = $0[[$n]] // MapAt[# /. ν → μ &, #, {1}] &;
$ = $ /. (tuRuleSelect[$defall][slash[𝒟]] /. μ → μ1) /. tuDs[dd : "∇"^S][_, a_] → dd_a //
   tuRepeat[{tuOpSimplify[Dot], tuOpDistribute[CircleTimes],
     tuOpSimplify[CircleTimes]}, {}],
NL, "Use: ", $s = ( a_ ⊗ b_) ⧴ (DeleteCases[a, ("∇"^S)_]) ⊗ ("∇"^S)_μ1[b] /; ! FreeQ[a, "∇"],
CK, CK,
Yield, $ = $(*/.$s*) //. tuOpCollect[CircleTimes] // Simplify,
NL, "Use: ", $s =
 (tuRuleSelect[$defall][T[g, "uu", {μ, ν}]] // First // tuRuleSolve[#, #[[2, 2]]] & //
     Reverse // tuAddPatternVariable[{ν, μ}]) // First,
Yield, $ = $ /. $s // expandDC[] // (# /. (gg : Tensor[g, _, _]) ⊗ b_ → gg.(1_N ⊗ b) &),
$p[++$pn] = $;

Framed[$], style[Cyan, "NOT able to compute. Missing Γ term."],

$n = {6, 8};
NL, CO["•", $n, ": "], $ns = Complement[$ns, $n];
$ = $0[[$n]],
NL, "Use ", $s = CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
yield, $s = $s /. CommutatorP → ACommutator,
yield, $s = -$s[[1, 2]] + # & /@ $s // tuAddPatternVariable[{μ}],
Imply, $ = $ /. $s /. tuOpSimplify[CircleTimes] /. ν → μ,
yield, $ = $ /. (a_ ⊗ b_ ) - (a_ ⊗ c_ ) → a ⊗ (b - c);
Framed[$p[++$pn] = $], OK,

$n = {9};
NL, CO["•", $n, ": "], $ns = Complement[$ns, $n];
$ = $0[[$n]],
NL, "Use symmetric and antisymmetric form: ",
$s = $[[2]] → 1 / 2 (Apply[CommutatorP, $[[2]]] + Apply[CommutatorM, $[[2]]]),
Yield, $ = $ /. $s // expandDC[],
NL, "Symmetrize term: ",
$s = $ // tuExtractPositionPattern[a_ ⊗ CommutatorP[_, _]] // Flatten,
Yield, $s = MapAt[tuIndexSymmetrize[{μ, ν}][#] &, $s, {1, 2, 1}],
NL, "Use: ",
$sg = tuRuleSelect[$defall][T[g, "uu", {μ, ν}]] // First, CK,
Yield, $s = $s /. Reverse[$sg],
Imply, $ = tuReplacePart[$, $s]; Framed[$p[++$pn] = $], OK,

$n = {5, 11};
NL, CO["•", $n, ": "], $ns = Complement[$ns, $n];
$ = $0[[$n]];
Framed[$p[++$pn] = $], OK,

NL, "●All terms: ", $pass1 = $31[[1, 1]] -> Sum[$p[i], {i, $pn}];
ColumnSumExp[$pass1] // Framed,
NL, CR["Missing Γ term. Not the same as in the text."],
NL, CB["Completion check ", $ns]
];
```

•Examine different terms of: $\sum\left[\begin{array}{l} (\not{D}).(\not{D})\otimes 1_{\mathcal{H}_F} \\ (\not{D}).\gamma_5\otimes\Phi \\ (\not{D}).\gamma^{\nu}\otimes B_{\nu} \\ \gamma_5.(\not{D})\otimes\Phi \\ -\mathrm{i}\,\gamma_5.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[\Phi] \\ \gamma_5.\gamma^{\nu}\otimes\Phi.B_{\nu} \\ \gamma^{\mu}.(\not{D})\otimes B_{\mu} \\ \gamma^{\mu}.\gamma_5\otimes B_{\mu}.\Phi \\ \gamma^{\mu}.\gamma^{\nu}\otimes B_{\mu}.B_{\nu} \\ -\mathrm{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[B_{\nu}] \\ 1_N\otimes\Phi.\Phi \end{array}\right]$

•{1}: $(\not{D}).(\not{D})\otimes 1_{\mathcal{H}_F}\to$ Lichnerowicz formula $\to \boxed{\left(\frac{s}{4}+\Delta^S\right)\otimes 1_{\mathcal{H}_F}}$ OK

•{2, 4}: $(\not{D}).\gamma_5\otimes\Phi+\gamma_5.(\not{D})\otimes\Phi$

Use $[\gamma_5,\ \not{D}]_-\to 0 \Rightarrow (\not{D}).\gamma_5\otimes\Phi+\gamma_5.(\not{D})\otimes\Phi\to\boxed{0}$ OK

•{3, 7, 10}: $-\mathrm{i}\,\gamma^{\mu}.(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1})\otimes B_{\mu}-\mathrm{i}\,(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1}).\gamma^{\mu}\otimes B_{\mu}-\mathrm{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[B_{\nu}]$

Use: a_⊗b_ :→ DeleteCases[a, $\nabla^S$_]⊗$\nabla^S{}_{\mu 1}$[b] /; ! FreeQ[a, $\nabla$] ⟵CHECK⟵CHECK

$\to$ $(-\mathrm{i}\,(\gamma^{\mu}.(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1})+(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1}).\gamma^{\mu}))\otimes B_{\mu}-\mathrm{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[B_{\nu}]$

Use: $\gamma^{\mu}\text{\_}.\gamma^{\nu}\text{\_}+\gamma^{\nu}\text{\_}.\gamma^{\mu}\text{\_}\to 2\,g^{\mu\nu}$

$\to$ $-\mathrm{i}\,\gamma^{\mu}.(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1})\otimes B_{\mu}-\mathrm{i}\,(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1}).\gamma^{\mu}\otimes B_{\mu}-\mathrm{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[B_{\nu}]$

$\boxed{-\mathrm{i}\,\gamma^{\mu}.(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1})\otimes B_{\mu}-\mathrm{i}\,(\nabla^S{}_{\mu 1}\,\gamma^{\mu 1}).\gamma^{\mu}\otimes B_{\mu}-\mathrm{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[B_{\nu}]}$ NOT able to compute. Missing Γ term.

•{6, 8}: $\gamma_5.\gamma^{\nu}\otimes\Phi.B_{\nu}+\gamma^{\mu}.\gamma_5\otimes B_{\mu}.\Phi$

Use $\{\gamma_5,\ \gamma^{\mu}\}_+\to 0 \longrightarrow \gamma_5.\gamma^{\mu}+\gamma^{\mu}.\gamma_5\to 0 \longrightarrow \gamma_5.\gamma^{\mu}\text{\_}\to -\gamma^{\mu}.\gamma_5$

$\Rightarrow$ $-(\gamma^{\mu}.\gamma_5\otimes\Phi.B_{\mu})+\gamma^{\mu}.\gamma_5\otimes B_{\mu}.\Phi \longrightarrow \boxed{\gamma^{\mu}.\gamma_5\otimes(-\Phi.B_{\mu}+B_{\mu}.\Phi)}$ OK

•{9}: $\gamma^{\mu}.\gamma^{\nu}\otimes B_{\mu}.B_{\nu}$

Use symmetric and antisymmetric form: $B_{\mu}.B_{\nu}\to\frac{1}{2}([B_{\mu},\ B_{\nu}]_-+\{B_{\mu},\ B_{\nu}\}_+)$

$\to$ $\frac{1}{2}\gamma^{\mu}.\gamma^{\nu}\otimes[B_{\mu},\ B_{\nu}]_-+\frac{1}{2}\gamma^{\mu}.\gamma^{\nu}\otimes\{B_{\mu},\ B_{\nu}\}_+$

Symmetrize term: $\{\{2,\ 2\}\to\gamma^{\mu}.\gamma^{\nu}\otimes\{B_{\mu},\ B_{\nu}\}_+\}$

$\to$ $\{\{2,\ 2\}\to(\frac{1}{2}(\gamma^{\mu}.\gamma^{\nu}+\gamma^{\nu}.\gamma^{\mu}))\otimes\{B_{\mu},\ B_{\nu}\}_+\}$

Use: $g^{\mu\nu}\to\frac{1}{2}(\gamma^{\mu}.\gamma^{\nu}+\gamma^{\nu}.\gamma^{\mu})$ ⟵CHECK

$\to$ $\{\{2,\ 2\}\to g^{\mu\nu}\otimes\{B_{\mu},\ B_{\nu}\}_+\}$

$\Rightarrow$ $\boxed{\frac{1}{2}\gamma^{\mu}.\gamma^{\nu}\otimes[B_{\mu},\ B_{\nu}]_-+\frac{1}{2}g^{\mu\nu}\otimes\{B_{\mu},\ B_{\nu}\}_+}$ OK

•{5, 11}: $\boxed{-\mathrm{i}\,\gamma_5.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[\Phi]+1_N\otimes\Phi.\Phi}$ OK

●All terms: $\mathcal{D}_{\mathcal{A}} \cdot \mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \left(\frac{s}{4} + \triangle^S\right) \otimes 1_{\mathcal{H}_F} \\ -i\, \gamma_5 \cdot \gamma^\mu \otimes \underset{-\mu}{\partial}\, [\Phi] \\ \gamma^\mu \cdot \gamma_5 \otimes (-\Phi \cdot B_\mu + B_\mu \cdot \Phi) \\ -i\, \gamma^\mu \cdot (\nabla^S{}_{\mu 1}\, \gamma^{\mu 1}) \otimes B_\mu \\ \frac{1}{2} \gamma^\mu \cdot \gamma^\nu \otimes [B_\mu, B_\nu]_- \\ -i\, (\nabla^S{}_{\mu 1}\, \gamma^{\mu 1}) \cdot \gamma^\mu \otimes B_\mu \\ -i\, \gamma^\nu \cdot \gamma^\mu \otimes \underset{-\mu}{\partial}\, [B_\nu] \\ 1_N \otimes \Phi \cdot \Phi \\ \frac{1}{2} g^{\mu\nu} \otimes \{B_\mu, B_\nu\}_+ \end{array}\right]$

Missing Γ term. Not the same as in the text.
Completion check {}

```
PR["In terms of △ᴱ: ", $e35,
 Yield,
 $ = $pass1 //. tuOpDistribute[CircleTimes]; $ // ColumnSumExp;
 Yield, $ = tuRuleEliminate[{ $[[2, 2]] }][{$e35, $}][[2]];
 $ // ColumnSumExp,
 Yield, $sF1 = T[γ, "u", {μ}].T[γ, "u", {ν}] ⊗ # & /@
    (tuRuleSelect[$defall][T[F, "dd", {μ, ν}]][[1]]) // expandDC[],
 NL, "Solve for: ", $s = -I $sF1[[2, 1]],
 Yield, $sF1 = tuRuleSolve[$sF1, $s],
 accumDef[$sF1];
 Yield, $ = $ /. $sF1 /. μ1 → ν // Expand; $ // ColumnSumExp,

 NL, "•Continuing: ",
 NL, "Expanding g's: ",
 $ = $ /. tuRuleSelect[$defall][T[g, "uu", {μ, ν}]] // expandDC[] // Simplify;
 $ // ColumnSumExp;
 NL, "Include coefficients in first term of CircleTimes: ",
 $s = {gg__ .( a_ ⊗ b_) → ((gg . a) ⊗ b)},
 Yield, $ = $ //. $s // Expand; $ // ColumnSumExp;
 Yield, $ = $ //. tuOpCollect[CircleTimes] //. $sgeneral[[-2 ;; -1]] // Simplify;
 $ // ColumnSumExp;
 NL, "Contract γ.γ terms: ",
 Yield, $ = $ //. tuOpCollect[Dot] // Simplify; $ // ColumnSumExp;
 NL, "Re-introduce g's ",
 $s = {(tuRuleSelect[$defall][T[g, "uu", {μ, ν}]] // First // tuRuleSolve[#,
        #[[2, 2, 1]]] & // Reverse), CommutatorP → ACommutator} // Flatten,
 Yield, $previous = $ = $ /. $s // Simplify // expandDC[];
 $ // ColumnSumExp, CK,
 NL, "Use symmetry of g's: ",
 $s = {ab : a_ ⊗ b_ :→ tuIndexSwap[{μ, ν}][ab] /; ! FreeQ[b, T[B, "d", {ν}]],
   T[g, "uu", {μ_, ν_}] :> T[g, "uu", {ν, μ}] /; OrderedQ[{ν, μ}]},
 Yield, $pass4 = $ = $ /. $s /. $s[[2]]; $ // ColumnSumExp
]
```

In terms of $\triangle^E$:

$\triangle^E \to \triangle^S \otimes 1_{\mathcal{H}_F} + g^{\mu\nu} \cdot (1_N \otimes B_\mu \cdot B_\nu) - i\, g^{\mu\nu} \cdot (1_N \otimes \underline{\partial}_\mu [B_\nu]) - 2\, i\, g^{\mu\nu} \cdot (\nabla^S{}_\mu \otimes B_\nu) + i\, g^{\mu\nu} \cdot \Gamma^\rho{}_{\mu\nu} \cdot (1_N \otimes B_\rho)$

→

$$\to\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^{E} \\ \frac{s}{4}\otimes 1_{\mathcal{H}_F} \\ -i\,\gamma_5.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[\Phi] \\ \gamma^{\mu}.\gamma_5\otimes -\Phi.B_{\mu} \\ \gamma^{\mu}.\gamma_5\otimes B_{\mu}.\Phi \\ -i\,\gamma^{\mu}.(\nabla^{S}{}_{\mu 1}\,\gamma^{\mu 1})\otimes B_{\mu} \\ \frac{1}{2}\gamma^{\mu}.\gamma^{\nu}\otimes[B_{\mu}, B_{\nu}]_{-} \\ -i\,(\nabla^{S}{}_{\mu 1}\,\gamma^{\mu 1}).\gamma^{\mu}\otimes B_{\mu} \\ -i\,\gamma^{\nu}.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[B_{\nu}] \\ 1_N\otimes\Phi.\Phi \\ \frac{1}{2}g^{\mu\nu}\otimes\{B_{\mu}, B_{\nu}\}_{+} \\ -g^{\mu\nu}.(1_N\otimes B_{\mu}.B_{\nu}) \\ i\,(g^{\mu\nu}.(1_N\otimes\underline{\partial}_{\mu}[B_{\nu}]) + 2\,g^{\mu\nu}.(\nabla^{S}{}_{\mu}\otimes B_{\nu}) - g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}.(1_N\otimes B_{\rho})) \end{array}\right]$$

$\to\ \gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \to i\,\gamma^{\mu}.\gamma^{\nu}\otimes[B_{\mu}, B_{\nu}]_{-} - \gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\nu}[B_{\mu}] + \gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\mu}[B_{\nu}]$

Solve for: $\gamma^{\mu}.\gamma^{\nu}\otimes[B_{\mu}, B_{\nu}]_{-}$

$\to\ \{\gamma^{\mu}.\gamma^{\nu}\otimes[B_{\mu}, B_{\nu}]_{-} \to i\,(-(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}) - \gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\nu}[B_{\mu}] + \gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\mu}[B_{\nu}])\}$

$$\to\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^{E} \\ \frac{s}{4}\otimes 1_{\mathcal{H}_F} \\ -i\,\gamma_5.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[\Phi] \\ \gamma^{\mu}.\gamma_5\otimes -\Phi.B_{\mu} \\ \gamma^{\mu}.\gamma_5\otimes B_{\mu}.\Phi \\ -\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\ -\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\nu}[B_{\mu}] \\ \frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\mu}[B_{\nu}] \\ -i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\ -i\,\gamma^{\nu}.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[B_{\nu}] \\ -i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\ 1_N\otimes\Phi.\Phi \\ \frac{1}{2}g^{\mu\nu}\otimes\{B_{\mu}, B_{\nu}\}_{+} \\ -g^{\mu\nu}.(1_N\otimes B_{\mu}.B_{\nu}) \\ i\,g^{\mu\nu}.(1_N\otimes\underline{\partial}_{\mu}[B_{\nu}]) \\ 2\,i\,g^{\mu\nu}.(\nabla^{S}{}_{\mu}\otimes B_{\nu}) \\ -i\,g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}.(1_N\otimes B_{\rho}) \end{array}\right]$$

•Continuing:

Expanding g's:

Include coefficients in first term of CircleTimes: `{(gg__).(a_⊗b_) → gg.a⊗b}`

→

→

Contract $\gamma.\gamma$ terms:

→

Re-introduce g's $\{\gamma^{\mu}.\gamma^{\nu} \to -\gamma^{\nu}.\gamma^{\mu} + 2\,g^{\mu\nu}$, CommutatorP → ACommutator$\}$

$$
\rightarrow\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \textstyle\sum\left[\;\begin{array}{|l}
\triangle^{E} \\
\frac{s\otimes 1_{\mathcal{H}_F}}{4} \\
2\,i\, g^{\mu\nu}.\nabla^{S}{}_{\mu}\otimes B_{\nu} \\
-i\, g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho} \\
-i\,\gamma_5.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[\Phi] \\
-(\gamma^{\mu}.\gamma_5\otimes\Phi.B_{\mu}) \\
\gamma^{\mu}.\gamma_5\otimes B_{\mu}.\Phi \\
-i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\
\frac{1}{2}\,i\,\gamma^{\nu}.\gamma^{\mu}\otimes F_{\mu\nu} \\
\frac{1}{2}\,i\,\gamma^{\nu}.\gamma^{\mu}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\
-\frac{3}{2}\,i\,\gamma^{\nu}.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[B_{\nu}] \\
-i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\
1_N\otimes\Phi.\Phi \\
-\frac{1}{2}\,g^{\mu\nu}\otimes B_{\mu}.B_{\nu} \\
\frac{1}{2}\,g^{\mu\nu}\otimes B_{\nu}.B_{\mu} \\
-i\, g^{\mu\nu}\otimes F_{\mu\nu} \\
-i\, g^{\mu\nu}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\
2\,i\, g^{\mu\nu}\otimes\underset{-\mu}{\partial}[B_{\nu}]
\end{array}\;\right] \longleftarrow \text{CHECK}
$$

Use symmetry of g's: `{ab : a_⊗b_ :→ tuIndexSwap[{μ, ν}][ab] /; ! FreeQ[b, T[B, d, {ν}]], g^μ_ ^ν_ :→ T[g, uu, {ν, μ}] /; OrderedQ[{ν, μ}]}`

$$
\rightarrow\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \textstyle\sum\left[\;\begin{array}{|l}
\triangle^{E} \\
\frac{s\otimes 1_{\mathcal{H}_F}}{4} \\
2\,i\, g^{\mu\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\
-i\, g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho} \\
-i\,\gamma_5.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[\Phi] \\
-(\gamma^{\mu}.\gamma_5\otimes\Phi.B_{\mu}) \\
\gamma^{\mu}.\gamma_5\otimes B_{\mu}.\Phi \\
-\frac{3}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\
-i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\
\frac{1}{2}\,i\,\gamma^{\nu}.\gamma^{\mu}\otimes F_{\mu\nu} \\
\frac{1}{2}\,i\,\gamma^{\nu}.\gamma^{\mu}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\
-i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\
1_N\otimes\Phi.\Phi \\
\frac{1}{2}\,g^{\mu\nu}\otimes B_{\mu}.B_{\nu} \\
-\frac{1}{2}\,g^{\mu\nu}\otimes B_{\nu}.B_{\mu} \\
-i\, g^{\mu\nu}\otimes F_{\mu\nu} \\
i\, g^{\mu\nu}\otimes\underset{-\nu}{\partial}[B_{\mu}]
\end{array}\;\right]
$$

```
PR["Use symmetry: ",
 $s1 = {a_ ⊗ (tt : T[F, "dd", {μ_, ν_}]) :→ a ⊗ tuIndexAntiSymmetrize[{μ, ν}][tt],
   tt : T[g, "uu", {μ_, ν_}] ⊗ Tensor[F, μ_, ν_] :→
    tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
   T[F, "dd", {μ_, ν_}] :→ -T[F, "dd", {ν, μ}] /; OrderedQ[{ν, μ}]
  },
 Yield, $ = $pass4 /. $s1 /. $s1[[-2 ;; -1]] // expandDC[] //
    (# //. tuOpCollect[CircleTimes] &) // Simplify;
 $ // ColumnSumExp;
 NL, "Substitute: ", $sg = tuRuleSelect[$defall][T[g, "uu", {μ, ν}]][[1]],
 Yield, $ = $ /. $sg // Simplify; $ // ColumnSumExp;
 Yield, $sg = tuRuleSolve[$sg, $sg[[2, 2]]][[1]],
 Yield, $ = $ /. $sg // expandDC[] // (# //. tuOpCollect[CircleTimes] &) // Simplify;
 $ // ColumnSumExp,
 NL, "Use: ", $s = a_ . b_ - b_ . a_ → CommutatorM[a, b],
 Yield, $p30 = $ = $ /. $s // expandDC[]; $ // ColumnSumExp,
 NL, CB["Compare with equation on p.30: "]
]
```

Use symmetry: $\{a\_\otimes(tt:F_{\mu\_\nu\_}) :\to a\otimes \text{tuIndexAntiSymmetrize}[\{\mu,\nu\}][tt],$
$tt : g^{\mu\_\nu\_}\otimes \text{Tensor}[F,\mu\_,\nu\_] :\to \text{tuIndexSwap}[\{\mu,\nu\}][tt] /; \text{OrderedQ}[\{\nu,\mu\}],$
$F_{\mu\_\nu\_} :\to -T[F, dd, \{\nu,\mu\}] /; \text{OrderedQ}[\{\nu,\mu\}]\}$

→

Substitute: $g^{\mu\nu} \to \frac{1}{2}(\gamma^\mu.\gamma^\nu + \gamma^\nu.\gamma^\mu)$

→

→ $\gamma^\mu.\gamma^\nu + \gamma^\nu.\gamma^\mu \to 2\, g^{\mu\nu}$

$$\to \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^E \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}\otimes B_\rho \\ -i\,\gamma_5.\gamma^\mu\otimes\underset{-\mu}{\partial}[\Phi] \\ \gamma^\mu.\gamma_5\otimes(-\Phi.B_\mu + B_\mu.\Phi) \\ -\frac{1}{2} i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ (-i\,(\gamma^\mu.\gamma^\nu - \gamma^\nu.\gamma^\mu))\otimes\underset{-\nu}{\partial}[B_\mu] \\ (i\,(2\,g^{\mu\nu}.\nabla^S{}_\nu - \gamma^\mu.(\nabla^S{}_\nu\,\gamma^\nu) - (\nabla^S{}_\nu\,\gamma^\nu).\gamma^\mu))\otimes B_\mu \\ 1_N\otimes\Phi.\Phi \\ g^{\mu\nu}\otimes(\frac{1}{2}(B_\mu.B_\nu - B_\nu.B_\mu)) \end{array}\right]$$

Use: $(a\_).(b\_) - (b\_).(a\_) \to [a, b]_-$

$$\to \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^E \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -i\,[\gamma^\mu,\gamma^\nu]_-\otimes\underset{-\nu}{\partial}[B_\mu] \\ 2\,i\,g^{\mu\nu}.\nabla^S{}_\nu\otimes B_\mu \\ -i\,g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}\otimes B_\rho \\ -i\,\gamma_5.\gamma^\mu\otimes\underset{-\mu}{\partial}[\Phi] \\ \gamma^\mu.\gamma_5\otimes[B_\mu,\Phi]_- \\ -\frac{1}{2} i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ -i\,\gamma^\mu.(\nabla^S{}_\nu\,\gamma^\nu)\otimes B_\mu \\ -i\,(\nabla^S{}_\nu\,\gamma^\nu).\gamma^\mu\otimes B_\mu \\ 1_N\otimes\Phi.\Phi \\ \frac{1}{2} g^{\mu\nu}\otimes[B_\mu,B_\nu]_- \end{array}\right]$$

Compare with equation on p.30:

```
PR["• Examine terms in: ", $ = $p30; $ // ColumnSumExp,
  NL, "● For the term ",
  $s = $ // tuTermSelect[T[g, "uu", {_, _}] ⊗ CommutatorM[_, _]] // First;
  $s = $s → 0, " by symmetry.",
  Imply, $ = $ /. $s; $ // ColumnSumExp,
  NL, "● ∇ commute with γ ", $s = tt : T["∇"-, "d", {_}] . T[γ, "u", {_}] → Reverse[tt],
  Imply, $ = $ //. $s; $ // ColumnSumExp,
  NL, "● For the terms: ", $s0 = $s = $ // tuTermSelect["∇"] // Apply[Plus, #] &,
  Yield, $s = $s //. tuOpCollect[CircleTimes] //. tuOpCollect[Dot] /.
      tuRuleSelect[$defall][{T[g, "uu", {_, _}]}] // Simplify // expandDC[];
  $s = $s0 → $s,
  Imply, $ = $ /. $s; $ // ColumnSumExp,

  NL, "● For the terms ", $s = $ // tuTermSelect[{T[γ, "d", {5}], T[γ, "u", {μ}]}],
  NL, "• Apply: ", $s1 = tuRuleSelect[$defall][T[𝒟, "d", {μ}][Φ]] /.
     T["∇"^S, "d", {μ}][Φ] → tuDPartial[Φ, μ] // First,
  Yield,
  $s1 = I T[γ, "u", {μ}].T[γ, "d", {5}].# & /@ $s1 // expandDC[] //
    (# //. gg__ .(a_ ⊗ b_) → ((gg . a) ⊗ b) &),
  Yield,
  $s1 = $s1 /. tuRuleSelect[$defall][{1_._, _ . 1_}] // Simplify,
  Yield,
  $s1 = # - $s1[[1]] & /@ $s1,
  Imply,
  $ = tuRuleAdd[{$s1, $}] /. $symmetries // expandDC[]; $ // ColumnSumExp,
  NL, CR["1 sign different and 2 Extra terms ", tuTermSelect[B][$] ]
 ];
tuSaveAllVariables[]
```

• Examine terms in: $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum[$

$$\begin{array}{|l}
\dfrac{\Delta^{E}_{s\otimes 1_{\mathcal{H}_F}}}{4} \\
-i\,[\gamma^{\mu},\gamma^{\nu}]_{-}\otimes\partial_{-\nu}[B_{\mu}] \\
2\,i\,g^{\mu\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\
-i\,g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho} \\
-i\,\gamma_5.\gamma^{\mu}\otimes\partial_{-\mu}[\Phi] \\
\gamma^{\mu}.\gamma_5\otimes[B_{\mu},\Phi]_{-} \\
-\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\
-i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\
-i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\
1_N\otimes\Phi.\Phi \\
\frac{1}{2}g^{\mu\nu}\otimes[B_{\mu},B_{\nu}]_{-}
\end{array}$$

$]$

● For the term $\frac{1}{2}g^{\mu\nu}\otimes[B_{\mu},B_{\nu}]_{-} \to 0$ by symmetry.

⇒ $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum[$

$$\begin{array}{|l}
\dfrac{\Delta^{E}_{s\otimes 1_{\mathcal{H}_F}}}{4} \\
-i\,[\gamma^{\mu},\gamma^{\nu}]_{-}\otimes\partial_{-\nu}[B_{\mu}] \\
2\,i\,g^{\mu\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\
-i\,g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho} \\
-i\,\gamma_5.\gamma^{\mu}\otimes\partial_{-\mu}[\Phi] \\
\gamma^{\mu}.\gamma_5\otimes[B_{\mu},\Phi]_{-} \\
-\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\
-i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\
-i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\
1_N\otimes\Phi.\Phi
\end{array}$$

$]$

- $\nabla$ commute with $\gamma$ tt : $\nabla\_\_.\gamma\_ \to$ Reverse[tt]

$$\Rightarrow \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^{E} \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -i\,[\gamma^{\mu},\gamma^{\nu}]_{-}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\ 2\,i\,g^{\mu\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\ -i\,g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho} \\ -i\,\gamma_5.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[\Phi] \\ \gamma^{\mu}.\gamma_5\otimes[B_{\mu},\Phi]_{-} \\ -\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\ -i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\ -i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\ 1_N\otimes\Phi.\Phi \end{array}\right]$$

- For the terms: $2\,i\,g^{\mu\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} - i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} - i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu}$

$\to$ $2\,i\,g^{\mu\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} - i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} - i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \to$
$-i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} - i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} + i\,\gamma^{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} + i\,\gamma^{\nu}.\gamma^{\mu}.\nabla^{S}{}_{\nu}\otimes B_{\mu}$

$$\Rightarrow \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^{E} \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -i\,[\gamma^{\mu},\gamma^{\nu}]_{-}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\ -i\,g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho} \\ -i\,\gamma_5.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[\Phi] \\ \gamma^{\mu}.\gamma_5\otimes[B_{\mu},\Phi]_{-} \\ -\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\ -i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\ -i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\ i\,\gamma^{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\ i\,\gamma^{\nu}.\gamma^{\mu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\ 1_N\otimes\Phi.\Phi \end{array}\right]$$

- For the terms $\{-i\,\gamma_5.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[\Phi],\ \gamma^{\mu}.\gamma_5\otimes[B_{\mu},\Phi]_{-}\}$
- Apply: $\mathcal{D}_{\mu}[\Phi] \to -i\,1_N\otimes[\Phi, B_{\mu}]_{-} + 1_N\otimes\underline{\partial}_{\mu}[\Phi]$

$\to$ $i\,\gamma^{\mu}.\gamma_5.\mathcal{D}_{\mu}[\Phi] \to i\,(-i\,\gamma^{\mu}.\gamma_5.1_N\otimes[\Phi, B_{\mu}]_{-} + \gamma^{\mu}.\gamma_5.1_N\otimes\underline{\partial}_{\mu}[\Phi])$

$\to$ $i\,\gamma^{\mu}.\gamma_5.\mathcal{D}_{\mu}[\Phi] \to \gamma^{\mu}.\gamma_5\otimes[\Phi, B_{\mu}]_{-} + i\,\gamma^{\mu}.\gamma_5\otimes\underline{\partial}_{\mu}[\Phi]$

$\to$ $0 \to \gamma^{\mu}.\gamma_5\otimes[\Phi, B_{\mu}]_{-} + i\,\gamma^{\mu}.\gamma_5\otimes\underline{\partial}_{\mu}[\Phi] - i\,\gamma^{\mu}.\gamma_5.\mathcal{D}_{\mu}[\Phi]$

$$\Rightarrow \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^{E} \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -i\,[\gamma^{\mu},\gamma^{\nu}]_{-}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\ -i\,g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho} \\ -\frac{1}{2}\,i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\ -i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu} \\ -i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu} \\ i\,\gamma^{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\ i\,\gamma^{\nu}.\gamma^{\mu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\ 1_N\otimes\Phi.\Phi \\ -i\,\gamma_5.\gamma^{\mu}.\mathcal{D}_{\mu}[\Phi] \end{array}\right]$$

1 sign different and 2 Extra terms $\{-i\,[\gamma^{\mu},\gamma^{\nu}]_{-}\otimes\underline{\partial}_{\nu}[B_{\mu}],\ -i\,g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}\otimes B_{\rho},$
$-i\,\gamma^{\mu}.(\nabla^{S}{}_{\nu}\,\gamma^{\nu})\otimes B_{\mu},\ -i\,(\nabla^{S}{}_{\nu}\,\gamma^{\nu}).\gamma^{\mu}\otimes B_{\mu},\ i\,\gamma^{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu},\ i\,\gamma^{\nu}.\gamma^{\mu}.\nabla^{S}{}_{\nu}\otimes B_{\mu}\}$

## 3.1.4 The heat expansion

```
PR["●Theorem 3.2. ",
  $t32 = {Tr[Exp[-t H]] -> xSum[t^((k - n) / 2) a_k[H], {k ≥ 0}],
    H → "Laplacian"["E"],
    n → dim[M],
    a_k[H] → xIntegral[a_k[x, H] √Det[g] , x ∈ M]
   }; Column[$t32]
 ];
```

●Theorem 3.2.

$\mathrm{Tr}[e^{-H\,t}] \to \sum_{\{k \geq 0\}} [t^{\frac{k-n}{2}} a_k[H]]$

$H \to \mathrm{Laplacian}[E]$

$n \to \mathrm{dim}[M]$

$a_k[H] \to \int_{x \in M} \sqrt{\mathrm{Det}[g]}\; a_k[x, H]$

```
PR["●Theorem 3.3. ",
  $t33 = {a_0[x, H] → (4 π)^(-n / 2) Tr_"E"_x[1_N],
    a_2[x, H] -> (4 π)^(-n / 2) Tr_"E"_x[s / 6 1_N + F],
    a_4[x, H] -> (4 π)^(-n / 2) (1 / 360)
      Tr_"E"_x[(-12 △[s] + 5 s.s - 2 T[R, "dd", {μ, ν}].T[R, "uu", {μ, ν}] +
          2 T[R, "dddd", {μ, ν, ρ, σ}]. T[R, "uuuu", {μ, ν, ρ, σ}] + 60 s. F +
          180 F . F - 60 △[F] + 30 T[Ω^"E", "dd", {μ, ν}]. T[Ω^"E", "uu", {μ, ν}])],
    H → "∇"^"E" - F,
    s → "scalar curvature of ∇",
    △ → "scalar Laplacian",
    T[Ω^"E", "dd", {μ, ν}] → "curvature of connnection ∇^E"
   }; Column[$t33]
 ];
```

●Theorem 3.3.

$a_0[x, H] \to 2^{-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N]$

$a_2[x, H] \to 2^{-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[F + \frac{s\,1_N}{6}]$

$a_4[x, H] \to$

$\frac{1}{45}\,2^{-3-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[180\,F.F + 60\,s.F + 5\,s.s - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[F] - 12\,\triangle[s]]$

$H \to \nabla^E - F$

$s \to$ scalar curvature of $\nabla$

$\triangle \to$ scalar Laplacian

$\Omega^{E}{}_{\mu\nu} \to$ curvature of connection $\nabla^E$

```
Clear[$s]
PR["●Proposition 3.4. ",
  $t34 =
   {Tr[f[𝒟𝒜 / Λ]] -> a4[𝒟𝒜^2] f[0] + 2 xSum[f4-k Λ^(4-k) ak[𝒟𝒜^2] / Γ[(4 - k) / 2], {k, 0, 4, even}],
    fj_ → xIntegral[v^(j-1) f[v], v]},
  Yield, $t34 = $t34 /. {k, 0, 4, even} → {k, {0, 2}} /. xSum → Sum,
  line,
  NL, "¶ Proof: Let: ", $g = $ = g[v] → xIntegral[Exp[-s v] h[s], s],
  Yield, $ = $ /. v → t iDA^2,
  Yield, $ = Tr /@ $ /. Tr[xIntegral[a_ h[s], b_]] → xIntegral[Tr[a] h[s], b],
  Yield,
  $ = $ /. (tuRule[$t32] // First // tuAddPatternVariable[t] // (# /. H → iDA^2 &)),
  Yield, $0 = $ = $ /. a_ xSum[b_, c_] → xSum[a b, c] /. tuOpSwitch[xIntegral, xSum] //
      PowerExpand // tuIntegralSimplify,
  NL, "Assume ", $s = t ≪ 1, imply, "keep only terms k≤4 ",
  NL, "● For: ", $s = {k → 4, n → 4, xSum[a_, _] → a, xIntegral[h[s], s] → g[0]},
  yield, $ = $0[[2]] //. $s,
  NL, "● For: ", $s = {k → 2, n → 4, xSum[a_, _] → a},
  yield, $ = $0[[2]] //. $s,
  line
 ];
```

●Proposition 3.4. $\{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]\to 2\sum_{\{k,0,4,\mathrm{even}\}}[\frac{\Lambda^{4-k}\, f_{4-k}\, a_k[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[\frac{4-k}{2}]}]+f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\ f_{j\_}\to\int v^{-1+j} f[v]\, dv\}$

$\rightarrow$ $\{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]\to 2\,(\frac{\Lambda^4 f_4\, a_0[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[2]}+\frac{\Lambda^2 f_2\, a_2[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[1]})+f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\ f_{j\_}\to\int v^{-1+j} f[v]\, dv\}$

---

¶ Proof: Let: $g[v]\to\int e^{-s v} h[s]\, ds$

$\rightarrow$ $g[t\, D_A^2]\to\int e^{-s t D_A^2} h[s]\, ds$

$\rightarrow$ $\mathrm{Tr}[g[t\, D_A^2]]\to\int h[s]\,\mathrm{Tr}[e^{-s t D_A^2}]\, ds$

$\rightarrow$ $\mathrm{Tr}[g[t\, D_A^2]]\to\int h[s]\sum_{\{k\geq 0\}}[(s\, t)^{\frac{k-n}{2}} a_k[D_A^2]]\, ds$

$\rightarrow$ $\mathrm{Tr}[g[t\, D_A^2]]\to\sum_{\{k\geq 0\}}[t^{\frac{k-n}{2}}\int s^{\frac{k-n}{2}} h[s]\, ds\; a_k[D_A^2]]$

Assume $t\ll 1$ ⇒ keep only terms k≤4

● For: $\{k\to 4,\ n\to 4,\ \sum[a\_]\to a,\ \int h[s]\, ds\to g[0]\}$ ⟶ $g[0]\, a_4[D_A^2]$

● For: $\{k\to 2,\ n\to 4,\ \sum[a\_]\to a\}$ ⟶ $\frac{\int\frac{h[s]}{s}\, ds\; a_2[D_A^2]}{t}$

---

```
PR["Calculation following from (3.11). Start with: ",
  Yield, $0 /. {n → 4},
  NL, "From (3.11): ", $ = Γ[z] -> xIntegral[r^(z-1) Exp[-r], {r, 0, ∞}], "POFF",
  Yield, $ = $ //. {r → s v, z → (4 - k) / 2},
  Yield, $ = $ /. xIntegral[a_, {v s, 0, ∞}] → xIntegral[a s, v] // PowerExpand,
  Yield, $ = $ // tuIntegralSimplify, "PONdd",
  yield, $ss = tuRuleSolve[$, $[[2, 1]]] // First // Map[1 / # &, #] &,
  NL, "● Apply to: ", $ = $0 /. {n → 4},
  Yield, $p = $ // tuExtractIntegrand,
  Yield, $p = $p /. $ss /. h[s] xIntegral[a_, b_] → xIntegral[h[s] a, b],
  Yield, $ = tuReplacePart[$, {$p}] // tuIntegralSimplify,
  Yield, $ = $ /. tuOpMerge[xIntegral],
  NL, "● Apply: ", $s = (Reverse[$g] // xIntegral[#, v] & /@ # & // tuIntegralSimplify),
  Yield, $s = $s /. tuOpMerge[] /. xIntegral[a_, b__] → xIntegral[A a, b] /.
      xIntegral[a_, b_, c_] → xIntegral[a, c, b] // tuAddPatternVariable[A],
  Yield, $pass5 = $ = $ /. $s; $ // Framed,
  NL, "Substitute: ", $s = {g[u_] → f[√u], v → u^2},
  Yield, $ = $ //. $s /. ii : xIntegral[_, _] :→ tuIntegralSwitchVar[d[u^2] → 2 u d[u]][ii] //
      PowerExpand // Simplify;
  $ // Framed,
  NL, "Substitute: ", $s = t → Λ^-2,
  Yield, $ = $ /. $s // PowerExpand // Simplify; $ // Framed
 ];
```

Calculation following from (3.11). Start with:

$\to$ $\mathrm{Tr}[g[t\, \mathcal{D}_A^2]] \to \sum_{\{k\geq 0\}} [t^{\frac{1}{2}(-4+k)} \int s^{\frac{1}{2}(-4+k)}\, h[s]\, ds\; a_k[\mathcal{D}_A^2]]$

From (3.11): $\Gamma[z] \to \int_0^\infty e^{-r}\, r^{-1+z}\, dr$

....... $\longrightarrow$ $s^{\frac{1}{2}(-4+k)} \to \dfrac{\int e^{-s\,v}\, v^{-1+\frac{4-k}{2}}\, dv}{\Gamma[\frac{4-k}{2}]}$

● Apply to: $\mathrm{Tr}[g[t\, \mathcal{D}_A^2]] \to \sum_{\{k\geq 0\}} [t^{\frac{1}{2}(-4+k)} \int s^{\frac{1}{2}(-4+k)}\, h[s]\, ds\; a_k[\mathcal{D}_A^2]]$

$\to$ $\{2, 1, 2, 1\} \to s^{\frac{1}{2}(-4+k)}\, h[s]$

$\to$ $\{2, 1, 2, 1\} \to \dfrac{\int e^{-s\,v}\, v^{-1+\frac{4-k}{2}}\, h[s]\, dv}{\Gamma[\frac{4-k}{2}]}$

$\to$ $\mathrm{Tr}[g[t\, \mathcal{D}_A^2]] \to \sum_{\{k\geq 0\}} [\dfrac{t^{\frac{1}{2}(-4+k)} \int\int e^{-s\,v}\, v^{-1+\frac{4-k}{2}}\, h[s]\, dv\, ds\; a_k[\mathcal{D}_A^2]}{\Gamma[\frac{4-k}{2}]}]$

$\to$ $\mathrm{Tr}[g[t\, \mathcal{D}_A^2]] \to \sum_{\{k\geq 0\}} [\dfrac{t^{\frac{1}{2}(-4+k)} \int\int e^{-s\,v}\, v^{-1+\frac{4-k}{2}}\, h[s]\, dv\, ds\; a_k[\mathcal{D}_A^2]}{\Gamma[\frac{4-k}{2}]}]$

● Apply: $\int\int e^{-s\,v}\, h[s]\, ds\, dv \to \int g[v]\, dv$

$\to$ $\int\int e^{-s\,v}\, h[s]\, A\_\, dv\, ds \to \int A\, g[v]\, dv$

$\to$ $\boxed{\mathrm{Tr}[g[t\, \mathcal{D}_A^2]] \to \sum_{\{k\geq 0\}} [\dfrac{t^{\frac{1}{2}(-4+k)} \int v^{-1+\frac{4-k}{2}}\, g[v]\, dv\; a_k[\mathcal{D}_A^2]}{\Gamma[\frac{4-k}{2}]}]}$

Substitute: $\{g[u\_] \to f[\sqrt{u}\,],\ v \to u^2\}$

$\to$ $\boxed{\mathrm{Tr}[f[\sqrt{t}\ \mathcal{D}_A]] \to \sum_{\{k\geq 0\}} [\dfrac{t^{-2+\frac{k}{2}} \int 2\, u^{3-k}\, f[u]\, du\; a_k[\mathcal{D}_A^2]}{\Gamma[2-\frac{k}{2}]}]}$

Substitute: $t \to \dfrac{1}{\Lambda^2}$

$\to$ $\boxed{\mathrm{Tr}[f[\dfrac{\mathcal{D}_A}{\Lambda}]] \to \sum_{\{k\geq 0\}} [\dfrac{\Lambda^{4-k} \int 2\, u^{3-k}\, f[u]\, du\; a_k[\mathcal{D}_A^2]}{\Gamma[2-\frac{k}{2}]}]}$

```
PR["●Proposition 3.5. For canonical triple ", {C"∞"[M], L²[M, S], slash[𝒟]},
  Yield,
  $p35 = $ = {Tr[f[slash[𝒟] / Λ]] → xIntegral[ℒM[T[g, "dd", {μ, ν}]] √Det[g], x⁴],
     ℒM[T[g, "dd", {μ, ν}]] → f₄ Λ⁴ / (2 π²) - f₂ Λ²
         / (24 π²) + f[0] / (16 π²) (△[s] / 30 -
          T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] / 20 + 11 / 360 R*.R*)};
  ColumnBar[$],

  line,
  NL, CO["Sketch proof: with ",
   $sdim = $s0 = {m → dim[M], dim[M] → 4, Tr"E"x[1N] → dim[S], dim[S] → 2^(m/2)}],
  NL, "■Evaluate terms in Theorem.3.4. ", $t34s = $t34 /. 𝒟𝒜 → slash[𝒟],

  next, "For ", $0 = $ = tuExtractPattern[a₀[_]][$t34s[[1, 2]]] // First,
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {M} → {x, x ∈ M} /. g → g[x] /.
    x ∈ M → x,
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[1]]],
  Yield, $a0 = $0 -> $ /. $t32[[3 ;; -1]] //. $sdim // tuIntegralSimplify;
  Framed[$a0],

  next, "For ", $0 = $ = tuExtractPattern[a₂[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = F → -s / 4 1N,
  Yield, $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /.
     {{M} → {x, x ∈ M}, g → g[x]} /. x ∈ M → x,
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[2]]] //. $sF,
  Yield, $ = ($ // tuArgSimplify[Tr"E"x, {s}]) /. s → s[x],
  Yield, $a2 = $0 -> $ /. $t32[[3 ;; -1]] //. $sdim // tuIntegralSimplify;
  Framed[$a2],

  next, "For ", $0 = $ = tuExtractPattern[a₄[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = {s → s . 1N, F → -s / 4 1N, Ω"E" → Ω^S},
  Yield, $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /.
     {{M} → {x, x ∈ M}, g → g[x]} /. x ∈ M → x,
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[3]]] /. $sF, "xPOFF",
  Yield, $ = $ // tuDotSimplify[{s}] // (# //. $sgeneral[[-2 ;; -1]] &),
  Yield, xtmp = $ = $ // tuArgSimplify[△, {1N}] // tuArgSimplify[Tr"E"x, {s, △[s]}],
  Yield, $ = $ /. s → s[x] // tuArgSimplify[Tr"E"x, {s, △[s]}] //
     tuIntegralSimplify // (# //. $sdim &),
  "PONdd", Framed[$a4b = $0 -> $ //. $sdim]
 ];
```

●Proposition 3.5. For canonical triple $\{C^{\infty}[M], L^2[M, S], \not{D}\}$

$$\to \left|\begin{array}{l} \mathrm{Tr}[f[\frac{\not{D}}{\Lambda}]] \to \int \sqrt{\mathrm{Det}[g]}\ \mathcal{L}_M[g_{\mu\nu}]\, dx^4 \\ \mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} + \frac{f[0]\,(\frac{11 R^*.R^*}{360} - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{\Delta[s]}{30})}{16\pi^2} \end{array}\right.$$

---

Sketch proof: with $\{m \to \dim[M], \dim[M] \to 4, \mathrm{Tr}_{E_x}[1_N] \to \dim[S], \dim[S] \to 2^{m/2}\}$

■Evaluate terms in Theorem.3.4.

$$\{\mathrm{Tr}[f[\frac{\not{D}}{\Lambda}]] \to 2\,(\frac{\Lambda^4 f_4\, a_0[(\not{D})^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2\, a_2[(\not{D})^2]}{\Gamma[1]}) + f[0]\, a_4[(\not{D})^2],\ f_{j\_} \to \int v^{-1+j} f[v]\, dv\}$$

◆For $a_0[(\not{D})^2]$

$$\to \int \sqrt{\mathrm{Det}[g[x]]}\ a_0[x, (\not{D})^2]\, dx$$

$$\to \int 2^{-n} \pi^{-n/2} \sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[1_N]\, dx$$

→ $\boxed{a_0[(D\!\!\!/)^2] \to \dfrac{\int \sqrt{Det[g[x]]}\, dx}{4\pi^2}}$

◆For $a_2[(D\!\!\!/)^2]$ using $F \to -\dfrac{s\, 1_N}{4}$

→ $\int \sqrt{Det[g[x]]}\; a_2[x, (D\!\!\!/)^2]\, dx$

→ $\int 2^{-n} \pi^{-n/2} \sqrt{Det[g[x]]}\; Tr_{E_x}[-\dfrac{s\, 1_N}{12}]\, dx$

→ $\int -\dfrac{1}{3} 2^{-2-n} \pi^{-n/2} \sqrt{Det[g[x]]}\; s[x]\, Tr_{E_x}[1_N]\, dx$

→ $\boxed{a_2[(D\!\!\!/)^2] \to -\dfrac{\int \sqrt{Det[g[x]]}\; s[x]\, dx}{48\pi^2}}$

◆For $a_4[(D\!\!\!/)^2]$ using $\{s \to s.1_N,\ F \to -\dfrac{s\, 1_N}{4},\ \Omega^E \to \Omega^S\}$

→ $\int \sqrt{Det[g[x]]}\; a_4[x, (D\!\!\!/)^2]\, dx$

→

$$\int \frac{1}{45} 2^{-3-n} \pi^{-n/2} \sqrt{Det[g[x]]}\; Tr_{E_x}[180\, (-\frac{s\, 1_N}{4}).(-\frac{s\, 1_N}{4}) - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu} + 60\; s.1_N.(-\frac{s\, 1_N}{4}) + 5\, s.1_N.s.1_N - 12\, \triangle[s.1_N] - 60\, \triangle[-\frac{s\, 1_N}{4}]]\, dx$$ xPOFF

→ $$\int \frac{1}{45} 2^{-3-n} \pi^{-n/2} \sqrt{Det[g[x]]}\; Tr_{E_x}[-2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu} + \frac{5\, s^2\, 1_N}{4} - 60\, \triangle[-\frac{s\, 1_N}{4}] - 12\, \triangle[s\, 1_N]]\, dx$$

→ $$\int \frac{1}{45} 2^{-3-n} \pi^{-n/2} \sqrt{Det[g[x]]}\; (-2\, Tr_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\, Tr_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\, Tr_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] + \frac{5}{4} s^2\, Tr_{E_x}[1_N] + 3\, \triangle[s]\, Tr_{E_x}[1_N])\, dx$$

→ $$\frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{Det[g[x]]}\; (5\, s[x]^2 + 12\, \triangle[s[x]] - 2\, Tr_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\, Tr_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\, Tr_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}])\, dx$$ PONdd

$$\boxed{a_4[(D\!\!\!/)^2] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{Det[g[x]]}\; (5\, s[x]^2 + 12\, \triangle[s[x]] - 2\, Tr_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\, Tr_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\, Tr_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}])\, dx}$$

```
PR["Using (3.14): ", $s = e314 =
    T[Ω^S, "dd", {μ, ν}] → 1 / 4 T[R, "dddd", {μ, ν, ρ, σ}] T[γ, "u", {ρ}].T[γ, "u", {σ}],
  yield, $s314 = {e314, e314 /. ρ → ρ1 /. σ → σ1 // tuIndicesRaise[{μ, ν}]} //
    tuAddPatternVariable[{μ, ν}], accumDef[$s314];
  NL, "Evaluate: ", $ = $a4b // tuExtractPattern[
      T[Ω^S, "dd", {μ, ν}].T[Ω^S, "uu", {μ, ν}]] // First;
  $t0 = $ = Tr[$],
  Yield, $ = $ /. $s314 // tuDotSimplify[{Tensor[R, __]}],
  NL, "Tr[] scalars: ", $s = {Tensor[R, _, _]},
  Yield, $ = $ // tuTrSimplify[$s],
  Yield, $ = $ /. subTraceGamma0,
  Yield, $ = $ // Expand // ContractUpDn[g],
  NL, "Use: ", $s = {T[R, "ddud", {μ_, ν_, ρ_, ρ_}] → 0, T[R, "dduu", {μ, ν, ρ1_, σ1_}] :>
     -T[R, "dduu", {μ, ν, σ1, ρ1}] /; OrderedQ[{σ1, ρ1}]},
  Yield, $t0 = $t0 -> $ /. $s /. Tr -> Tr"E"x; Framed[$t0], accumDef[$t0];
  Imply, $ = $a4b /. $t0; Framed[$],
  (**)
  NL, "Remaining Dot[] are scalars: ",
  Yield, $ = $ /. dd : HoldPattern[Dot[_]] → 1N dd /.
     tuOpSimplify[Tr"E"x, {HoldPattern[Dot[_]]}] //. $sdim,
  Yield, $ = UpDownIndexSwap[{ρ1, σ1}][$] /. ρ1 → ρ /. σ1 → σ /.
      tt : T[R, "dddd", {_, _, _, _}] ⧴ tuTensorAntiSymmetricOrdered[tt, {3, 4}] /. Dot →
      Times // Simplify;
  Framed[$a4c = $], CG[" (3.16)"],
  (**)
  NL, "▪Convert expression in terms of: ",
  NL, "•Weyl tensor: ", T[C, "dddd", {μ, ν, ρ, σ}],
  Yield,
  $ = T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] -> T[R, "dddd", {μ, ν, ρ, σ}]
      T[R, "uuuu", {μ, ν, ρ, σ}] - 2 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] + s[x]^2 / 3,
  NL, "•Pontryagin class ",
  $1 = R*.R* → s[x]^2 - 4 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] +
     T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
  NL, "The ",
  $2 = $a4c // tuExtractIntegrand;
  $2a0 = $2 // tuExtractPositionPattern[Plus[_, __]];
  $2a = integrandTerm → $2a0[[1, 2]],
  $ = {$, $1, $2a}; $ // ColumnBar,
  Imply,
  $ = tuEliminate[$, {T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
     T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}]}], CK,
  Yield, $ = tuRuleSolve[$, integrandTerm],
  Yield, $2a0[[1, 2]] = $[[1, 2]]; $2a0,
  Yield, $2 = tuReplacePart[$2, $2a0],
  Yield, $a4d = $ = tuReplacePart[$a4c, {$2}]; Framed[$], CG[" QED"]
 ];
```

Using (3.14): $\Omega^{S}{}_{\mu\nu} \to \frac{1}{4}\gamma^{\rho}.\gamma^{\sigma} R_{\mu\nu\rho\sigma} \longrightarrow \{\Omega^{S}{}_{\mu\_\nu\_} \to \frac{1}{4}\gamma^{\rho}.\gamma^{\sigma} R_{\mu\nu\rho\sigma},\ \Omega^{S\mu\_\nu\_} \to \frac{1}{4}\gamma^{\rho1}.\gamma^{\sigma1} R^{\mu\nu}{}_{\rho1\sigma1}\}$

Evaluate: $\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}]$

$\to \mathrm{Tr}[\frac{1}{16}\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\sigma1}]$

Tr[] scalars: {Tensor[R, _, _]}

$\to \frac{1}{16} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\sigma1} \mathrm{Tr}[\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1}]$

$\to \frac{1}{4}(g^{\rho\sigma} g^{\rho1\sigma1} + g^{\rho\sigma1} g^{\sigma\rho1} - g^{\rho\rho1} g^{\sigma\sigma1}) R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\sigma1}$

$\to -\frac{1}{4} R_{\mu\nu}{}^{\rho1\sigma1} R^{\mu\nu}{}_{\rho1\sigma1} + \frac{1}{4} R_{\mu\nu}{}^{\sigma1\rho1} R^{\mu\nu}{}_{\rho1\sigma1} + \frac{1}{4} R_{\mu\nu}{}^{\sigma}{}_{\sigma} R^{\mu\nu\sigma1}{}_{\sigma1}$

Use: $\{R_{\mu\_\nu\_}{}^{\rho\_}{}_{\rho\_} \to 0,\ R_{\mu\nu}{}^{\rho1\_\sigma1\_} :\to -T[R, dduu, \{\mu, \nu, \sigma1, \rho1\}] /; \mathrm{OrderedQ}[\{\sigma1, \rho1\}]\}$

$\to \boxed{\mathrm{Tr}_{E_x}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}] \to -\frac{1}{2} R_{\mu\nu}{}^{\rho1\sigma1} R^{\mu\nu}{}_{\rho1\sigma1}}$

$\Rightarrow$

$$\boxed{a_4[(\not{D})^2] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (5\, s[x]^2 - 15 R_{\mu\nu}{}^{\rho1\sigma1} R^{\mu\nu}{}_{\rho1\sigma1} + 12 \triangle[s[x]] - 2\, \mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\, \mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}])\, dx}$$

Remaining Dot[] are scalars:

$\to a_4[(\not{D})^2] \to$

$\frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (-8 R_{\mu\nu}.R^{\mu\nu} + 8 R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 5\, s[x]^2 - 15 R_{\mu\nu}{}^{\rho1\sigma1} R^{\mu\nu}{}_{\rho1\sigma1} + 12 \triangle[s[x]])\, dx$

$$\to \boxed{a_4[(\not{D})^2] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (5\, s[x]^2 - 8 R_{\mu\nu} R^{\mu\nu} - 7 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 12 \triangle[s[x]])\, dx}$$

(3.16)

▪Convert expression in terms of:

•Weyl tensor: $C_{\mu\nu\rho\sigma}$

$\to C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \to \frac{s[x]^2}{3} - 2 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

•Pontryagin class $R^{*}.R^{*} \to s[x]^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

The $\mathrm{integrandTerm} \to 5\, s[x]^2 - 8 R_{\mu\nu} R^{\mu\nu} - 7 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]$

$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \to \frac{s[x]^2}{3} - 2 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$
$R^{*}.R^{*} \to s[x]^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$
$\mathrm{integrandTerm} \to 5\, s[x]^2 - 8 R_{\mu\nu} R^{\mu\nu} - 7 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]$

$\Rightarrow \mathrm{integrandTerm} + 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - 12 \triangle[s[x]] == 11 R^{*}.R^{*} \longleftarrow$ CHECK

$\to \{\mathrm{integrandTerm} \to 11 R^{*}.R^{*} - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]\}$

$\to \{\{2, 2\} \to 11 R^{*}.R^{*} - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]]\}$

$\to \{2, 4, 1\} \to \sqrt{\mathrm{Det}[g[x]]}\ (11 R^{*}.R^{*} - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]])$

$$\to \boxed{a_4[(\not{D})^2] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \int \sqrt{\mathrm{Det}[g[x]]}\ (11 R^{*}.R^{*} - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + 12 \triangle[s[x]])\, dx}$$ QED

```
PR["•NOTE: In 4-dim compact orientable manifold M without boundary ",
  Yield,
  {IntegralOp[{{M}}, R*.R* νg] → 8 π² χ[M], χ[M] → "Euler Characteristic"} // Column,
  imply, "Topological term",
  yield, "Constant",
  yield, "Ignore",
  NL, "With no boundaries the ", △[s[x]], " term does not contribute."
 ];
```

•NOTE: In 4-dim compact orientable manifold M without boundary

→ $\int_{\{M\}}[R^*.R^*\ \nu_g] \to 8\pi^2\chi[M]$ ⇒ Topological term ⟶ Constant ⟶ Ignore
  $\chi[M] \to$ Euler Characteristic

With no boundaries the △[s[x]] term does not contribute.

```
PR["To derive Proposition 3.5.
• Insert a's into ", $ = $t34s; $ // ColumnSumExp,
  NL, "Using: ",
  $s = {R*.R* → 0, △[s[x]] → 0, tt : Tensor[C, _, _] → tt[x], n → 4, Γ → Gamma},
  Yield, $t34s1 = $ = $[[1]] /. {$a0, $a2, $a4d} /. $s //. tuIntegralGather // Simplify;
  $ // ColumnSumExp,
  NL, "•Comparing with (3.19). The relevant term in integrand: ",
  $ = $t34s1 // tuExtractIntegrand // Last // (# /. √_ → 1 &);
  $ // ColumnSumExp,
  Yield, $LM = ℒM[T[g, "dd", {μ, ν}]] -> $ // Expand, CG[" Agrees with (3.19)."]
 ];
```

To derive Proposition 3.5.

• Insert a's into $\{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \to \sum\left[\begin{array}{l} 2\left(\frac{\Lambda^4 f_4 a_0[(D\!\!\!/)^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2 a_2[(D\!\!\!/)^2]}{\Gamma[1]}\right) \\ f[0]\, a_4[(D\!\!\!/)^2] \end{array}\right],\ f_{j\_} \to \int v^{\sum[\,^{-1}_{j}\,]} f[v]\, dv\}$

Using: $\{R^*.R^* \to 0,\ \triangle[s[x]] \to 0,\ tt : \mathrm{Tensor}[C, \_, \_] \to tt[x],\ n \to 4,\ \Gamma \to \mathrm{Gamma}\}$

$$\to \mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \to \int \frac{\sum\left[\begin{array}{l} -40\,\Lambda^2\, s[x]\, f_2 \\ 480\,\Lambda^4\, f_4 \\ -3\, f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x] \end{array}\right]\sqrt{\mathrm{Det}[g[x]]}}{960\,\pi^2}\, dx$$

•Comparing with (3.19). The relevant term in integrand:

$$\frac{\sum\left[\begin{array}{l} -40\,\Lambda^2\, s[x]\, f_2 \\ 480\,\Lambda^4\, f_4 \\ -3\, f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x] \end{array}\right]}{960\,\pi^2}$$

$$\to \mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 s[x] f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} - \frac{f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x]}{320\pi^2}$$ Agrees with (3.19).

p.35

```
PR["●Proposition 3.7. The spectral action of the fluctuated Dirac operator is ",
  $p37 = $ = {Tr[f[𝒟A / Λ]] → xIntegral[√Det[g[x]] ℒ[T[g, "dd", {μ, ν}], Bμ, Φ], x ∈ M],
     ℒ[T[g, "dd", {μ, ν}], Bμ, Φ] →
      N ℒM[T[g, "dd", {μ, ν}]] + ℒB[Bμ] + ℒH[T[g, "dd", {μ, ν}], Bμ, Φ],
     $LM,
     N → dim[ℋF],
     ℒB[Bμ] → f[0] / (24 π^2) Tr[T[F, "dd", {μ, ν}] T[F, "uu", {μ, ν}]],
     ℒB[Bμ] → "Kinetic term gauge fields",
     ℒH[T[g, "dd", {μ, ν}], Bμ, Φ] →
      -2 f2 Λ^2 / (4 π^2) Tr[Φ.Φ] + f[0] / (8 π^2) Tr[Φ.Φ.Φ.Φ] + f[0] / (24 π^2) Δ[Tr[Φ.Φ]] +
       f[0] / (48 π^2) s[x] Tr[Φ.Φ] + f[0] / (8 π^2) Tr[T[𝒟, "d", {μ}][Φ].T[𝒟, "u", {μ}][Φ]],
     ℒH[T[g, "dd", {μ, ν}], Bμ, Φ] → "Higgs lagrangian",
     N → Tr[1ℋF]
    }; FramedColumn[$]
 ];
```

●Proposition 3.7. The spectral action of the fluctuated Dirac operator is

$$\mathrm{Tr}[f[\tfrac{\mathcal{D}_A}{\Lambda}]] \to \int_{x\in M} \sqrt{\mathrm{Det}[g[x]]}\ \mathcal{L}[g_{\mu\nu}, B_\mu, \Phi]$$

$$\mathcal{L}[g_{\mu\nu}, B_\mu, \Phi] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + N\,\mathcal{L}_M[g_{\mu\nu}]$$

$$\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2\, s[x]\, f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} - \frac{f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x]}{320\pi^2}$$

$$N \to \dim[\mathcal{H}_F]$$

$$\mathcal{L}_B[B_\mu] \to \frac{f[0]\, \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$$

$\mathcal{L}_B[B_\mu] \to$ Kinetic term gauge fields

$$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2\, \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_\mu[\Phi].\mathcal{D}^\mu[\Phi]]}{8\pi^2} + \frac{f[0]\, \mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\, \Delta[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$$

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$ Higgs lagrangian

$$N \to \mathrm{Tr}[1_{\mathcal{H}_F}]$$

```
PR["●Proof: Starting with the formulas from Theorem 3.3 ", $ = $t33[[1 ;; 3]];
  $ // ColumnBar,
  NL, "let ",
  $s = {F → Q, H → 𝒟A}, ". Using explicit tensor notation. ", H → S × ℋF,
  Yield,
  $t33a = {{$ /. $s, $31[[-1]]} /. (tt : Tr_)[1N] :→ tt[1N ⊗ 1ℋF] /. s 1N → s /. s ⊗ 1ℋF → s /.
       s → (s 1N ⊗ 1ℋF) /. 1Nx → 1N ⊗ 1ℋ
         1N → "Identity for spinor field"} // Flatten; FramedColumn[$t33a]
 ];
```

●Proof: Starting with the formulas from Theorem 3.3

$$a_0[x, H] \to 2^{-n} \pi^{-n/2}\, \mathrm{Tr}_{E_x}[1_N]$$

$$a_2[x, H] \to 2^{-n} \pi^{-n/2}\, \mathrm{Tr}_{E_x}[F + \tfrac{s\, 1_N}{6}]$$

$$a_4[x, H] \to \tfrac{1}{45} 2^{-3-n} \pi^{-n/2}\, \mathrm{Tr}_{E_x}[180\, F.F + 60\, s.F + 5$$

let $\{F \to Q, H \to \mathcal{D}_A\}$. Using explicit tensor notation. $H \to S \times \mathcal{H}_F$

$$a_0[x, \mathcal{D}_A] \to 2^{-n} \pi^{-n/2}\, \mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_2[x, \mathcal{D}_A] \to 2^{-n} \pi^{-n/2}\, \mathrm{Tr}_{E_x}[Q + \tfrac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]$$

$$\to\ a_4[x, \mathcal{D}_A] \to \tfrac{1}{45} 2^{-3-n} \pi^{-n/2}\, \mathrm{Tr}_{E_x}[180\, Q.Q + 60\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).Q + 5\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \Delta[Q] - 12\, \Delta[s\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

$$Q \to -i\, \gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \tfrac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \tfrac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$$

```
PR["●Compute the aₙ terms of ", $t34[[1, 1]], (*
  " relative to ",$p35[[1,1]],*)
  NL, "for ", $s04 = Join[$sdim, {Tr[1_N] → dim[S], n → dim[M]}],
  Yield, $t33a // FramedColumn
 ];
```

●Compute the $a_n$ terms of $\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

for $\{m \to \dim[M], \dim[M] \to 4, \mathrm{Tr}_{E_x}[1_N] \to \dim[S], \dim[S] \to 2^{m/2}, \mathrm{Tr}[1_N] \to \dim[S], n \to \dim[M]\}$

→

$$a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_2[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[Q + \frac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[180\, Q.Q + 60\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).Q + 5\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[Q] - 12\, \triangle[s\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

$$Q \to -\mathbb{i}\, \gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} \mathbb{i}\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$$

```
PR[next, "For ", $ = $t33a[[1]],

  next, "For ", $ = $t33a[[1]] /. Tr_ → Tr /. $t32[[3]] /. $sdim,
  Yield,
  $ = $ /. tuOpDistribute[Tr, CircleTimes] //. tuOpSimplify[CircleTimes, {Tr[_]}],
  " ", "Recall ", $s = $t33[[1]] //. Join[{H → slash[𝒟], Tr_ → Tr}, $s04[[{2, -1}]]],
  Imply, $a0a = tuRuleEliminate[{Tr[1_N]}][{$s, $}] // First; Framed[$a0a],

  next, "For ", $ = $t33a[[2]] /. Tr_ → Tr /. $t32[[3]] /. $sdim,
  Yield,
  $ = $ /. tuRuleSelect[$t33a][Q] //. tuOpDistribute[Tr] // tuArgSimplify[Tr, {s}] //
     tuOpDistributeF[Tr, CircleTimes] // tuOpSimplifyF[CircleTimes, {Tr[_]}],

  NL, "• ", T[F, "dd", {μ, ν}], " is anti-symmetric and ", $symmetries[[-1]], yield,
  $s = {Tr[T[γ, "u", {μ}].T[γ, "u", {ν}]] Tr[T[F, "dd", {μ, ν}]] → 0,
    Tr[T[γ, "u", {μ}].T[γ, "d", {5}]] → 0};
  $s // ColumnBar,
  Imply, $ = $ /. $s,
  NL, "Recall ",
  $s = $t33[[2]] //. Join[{H → slash[𝒟], Tr_ → Tr, $sF[[2]]}, $s04[[{2, -1}]]] //
    tuArgSimplify[Tr, {s}],
  Imply, $a2a = $ /. tuRuleSolve[$s, {s Tr[1_N]}] // Expand; Framed[$a2a]
 ];
```

◆For $a_0[x, \mathcal{D}_A] \to 2^{-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$

◆For $a_0[x, \mathcal{D}_A] \to \dfrac{\mathrm{Tr}[1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

$\to\ a_0[x, \mathcal{D}_A] \to \dfrac{\mathrm{Tr}[1_N]\,\mathrm{Tr}[1_{\mathcal{H}_F}]}{16\,\pi^2}$ Recall $a_0[x, \not{\mathcal{D}}] \to \dfrac{\mathrm{Tr}[1_N]}{16\,\pi^2}$

$\Rightarrow\ \boxed{a_0[x, \mathcal{D}_A] \to \mathrm{Tr}[1_{\mathcal{H}_F}]\, a_0[x, \not{\mathcal{D}}]}$

◆For $a_2[x, \mathcal{D}_A] \to \dfrac{\mathrm{Tr}[Q + \frac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

$\to\ a_2[x, \mathcal{D}_A] \to \dfrac{-i\,\mathrm{Tr}[\mathcal{D}_\mu . \Phi]\,\mathrm{Tr}[\gamma^\mu . \gamma_5] - \mathrm{Tr}[\Phi . \Phi]\,\mathrm{Tr}[1_N] - \frac{1}{12} s\,\mathrm{Tr}[1_N]\,\mathrm{Tr}[1_{\mathcal{H}_F}] + \frac{1}{2} i\,\mathrm{Tr}[\gamma^\mu . \gamma^\nu]\,\mathrm{Tr}[F_{\mu\nu}]}{16\,\pi^2}$

• $F_{\mu\nu}$ is anti-symmetric and $tt : \gamma^\mu . \gamma_5 :\to \mathrm{Reverse}[tt]$ $\longrightarrow$ $\left|\begin{array}{l} \mathrm{Tr}[\gamma^\mu . \gamma^\nu]\,\mathrm{Tr}[F_{\mu\nu}] \to 0 \\ \mathrm{Tr}[\gamma^\mu . \gamma_5] \to 0 \end{array}\right.$

$\Rightarrow\ a_2[x, \mathcal{D}_A] \to \dfrac{-\mathrm{Tr}[\Phi . \Phi]\,\mathrm{Tr}[1_N] - \frac{1}{12} s\,\mathrm{Tr}[1_N]\,\mathrm{Tr}[1_{\mathcal{H}_F}]}{16\,\pi^2}$

Recall $a_2[x, \not{\mathcal{D}}] \to -\dfrac{s\,\mathrm{Tr}[1_N]}{192\,\pi^2}$

$\Rightarrow\ \boxed{a_2[x, \mathcal{D}_A] \to -\dfrac{\mathrm{Tr}[\Phi . \Phi]\,\mathrm{Tr}[1_N]}{16\,\pi^2} + \mathrm{Tr}[1_{\mathcal{H}_F}]\, a_2[x, \not{\mathcal{D}}]}$

```
PR["■For: ", $ = $t33a[[3]] /. Tr_ → Tr /. $t32[[3]] /. $sdim; Framed[$],
 NL, "Let: ", $s = {tt : Tensor[R, _, _].Tensor[R, _, _] → tt 1N ⊗ 1ℋF},
 Yield, $ = $t33a[[3]] /. Tr_ → Tr /. $t32[[3]] /. $sdim /. $s,
 NL, "Scalars: ", $scal = {s, △[s], Tensor[R, _, _]},
 $sQ = {Map[#.(# /. {μ → μ1, ν → ν1}) &, $t33a[[4]]], $t33a[[4]]};
 NL, "Use: ", $s = Join[($sQ), {$s34}, $s314]; FramedColumn[$s],

 Yield, $ = $ //. $s; ColumnSumExp[$],
 Yield, $ = $ // tuDotSimplify[]; ColumnSumExp[$];
 Yield, $ = $ // tuArgSimplify[△] // tudExpand[△, {1_, Tensor[γ, _, _]}] // expandDC[];
 NL, "Combine product of operator product: ", $s = {};
 $ = $ //. tuOpSimplify[Dot, {s}] //. tuOpSimplify[CircleTimes] //.
    {(a_ ⊗ b_) .( c_ ⊗ d_) → a.c ⊗ b.d,
     1n_ . a_ → a, a_ . 1n_ → a} // Expand; $ // ColumnSumExp;
 NL, "Apply Tr[] over each space: ", $s = {Tr[a_ ⊗ b_] → Tr[a] Tr[b]},
 Yield,
 $ = $ //. tuOpDistribute[Tr] //
   tuArgSimplify[Tr, {s, △[s], Tensor[R, _, _].Tensor[R, _, _]}];
 $ = $ /. $s,
 NL, "Reduce Tr[γ's]: ",
 xtmp = $ = $ //. tuTrGamma;
 line,
 NL, "Evaluate g,F terms with symmetry: ",
 $ss = tt : T[g, "uu", {a_, b_}] A_ ⧴ 0 /; ! FreeQ[tt, T[F, "dd", {a, b}]],
 Yield, $s0 = $s = $ // tuTermSelect[{Tensor[g, _, _] , Tensor[F, _, _]}] // Flatten;
 Yield, $s = $s /. $ss,
 Yield, $s[[2]] = $s[[2]] // tuIndexSwap[{ν1, μ1}]; $s,
 NL, "Leaving terms: ",
 $s1 = $s = Apply[Plus, $s] /. T[F, "dd", {ν1, μ1}] → -T[F, "dd", {μ1, ν1}] // expandDC[] //
     tuArgSimplify[Tr] // Simplify,
 Yield, $s = Apply[Plus, $s0] → $s1,
 Yield, $s = tuRuleSolve[$s, $s[[1, -1]]],
 line,
 NL, "Do they cancel? ",
 $ = $ // tuRuleApply[$s] // Simplify; $ // ColumnSumExp
]
PR["Use ",
 $s = {dim[N] → 4, tt : T[γ, "u", {μ_}].T[γ, "d", {5}] . T[γ, "u", {μ1_}].T[γ, "d", {5}] →
    -T[γ, "u", {μ}].T[γ, "u", {μ1}]},
 Yield, $ = $ /. $s // tuArgSimplify[Tr] // (# /. tuTrGamma &) // Simplify,
 NL, "• Contracting indices: ",
 Yield, $s2 = $ // tuTermSelect[{T[g, "uu", {μ, μ1}]}] // Flatten;
 $s2 // ColumnBar,
 yield, $ss = tuIndexContractUpDn[g, {ν1, μ1}][#] & /@ $s2;
 yield, $s = Thread[$s2 → $ss] // tuRuleSimplify; $s // ColumnBar,
 Imply, $pass6 = $ = $ /. $s // Simplify; $ // ColumnSumExp,

 NL, "• Comparing to text(p.37)", CR[" there are 2 differences, but evaluate ",
  $oEE = tuTermSelect[Tr[Tensor[Ω"E", _, _].Tensor[Ω"E", _, _]]][$pass6]],
 Yield, $ = 360 (4 π)^2 # & /@ $;
 $p37a4 = $ = Collect[$, dim[_], Simplify];
 $ // ColumnFormOn[Plus] // Framed
]
```

■For: $\boxed{a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\mathrm{Tr}[180\,Q.Q + 60\,(s\,1_N\otimes 1_{\mathcal{H}_F}).Q + 5\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s\,1_N\otimes 1_{\mathcal{H}_F}]]}$

Let: {tt : Tensor[R, _, _].Tensor[R, _, _] → tt $1_N \otimes 1_{\mathcal{H}_F}$}

→ $a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}$ Tr[180 Q.Q + 60 (s $1_N \otimes 1_{\mathcal{H}_F}$).Q + 5 (s $1_N \otimes 1_{\mathcal{H}_F}$).(s $1_N \otimes 1_{\mathcal{H}_F}$) − 2 × $1_N \otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu}.R^{\mu\nu}$ + 2 × $1_N \otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}$ + 30 $\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu}$ − 60 △[Q] − 12 △[s $1_N \otimes 1_{\mathcal{H}_F}$]]

Scalars: {s, △[s], Tensor[R, _, _]}

Use:

$Q.Q \to (-i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}).(-i\,\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2} i\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F})$

$Q \to -i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$

$\Omega^E[\mu, \nu] \to 1_N \otimes (i\, F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}$

$\Omega^S{}_{\mu\_\,\nu\_} \to \frac{1}{4} \gamma^\rho.\gamma^\sigma R_{\mu\nu\rho\sigma}$

$\Omega^{S\mu\_\,\nu\_} \to \frac{1}{4} \gamma^{\rho 1}.\gamma^{\sigma 1} R^{\mu\nu}{}_{\rho 1\,\sigma 1}$

→ $a_4[x, \mathcal{D}_{\mathcal{A}}] \to$ Tr[∑[ 
5 (s $1_N \otimes 1_{\mathcal{H}_F}$).(s $1_N \otimes 1_{\mathcal{H}_F}$)
60 (s $1_N \otimes 1_{\mathcal{H}_F}$).($-i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$)
180 ($-i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$).($-i\,\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2} i\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$)
−2 $1_N \otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu}.R^{\mu\nu}$
2 × $1_N \otimes 1_{\mathcal{H}_F}$ $R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}$
30 $\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu}$
−12 △[s $1_N \otimes 1_{\mathcal{H}_F}$]
−60 △[$-i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$]
]] / $5760\,\pi^2$

→

→

Combine product of operator product:

Apply Tr[] over each space: {Tr[a_⊗b_] → Tr[a] Tr[b]}

→ $a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}$ (−15 i s Tr[$\mathcal{D}_\mu.\Phi$] Tr[$\gamma^\mu.\gamma_5$] + 60 i Tr[$\mathcal{D}_\mu$.△[Φ]] Tr[$\gamma^\mu.\gamma_5$] + 45 i s Tr[$\mathcal{D}_{\mu 1}.\Phi$] Tr[$\gamma^{\mu 1}.\gamma_5$] + 30 Tr[$\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu}$] + 60 i Tr[$\gamma^\mu.\gamma_5$] Tr[△[$\mathcal{D}_\mu$].Φ] − 90 i Tr[$\gamma^{\mu 1}.\gamma^{\nu 1}$] Tr[Φ.Φ.$F_{\mu 1\,\nu 1}$] − 90 i Tr[$\gamma^\mu.\gamma^\nu$] Tr[$F_{\mu\nu}$.Φ.Φ] + 180 i Tr[$\gamma^{\mu 1}.\gamma_5$] Tr[Φ.Φ.$\mathcal{D}_{\mu 1}$.Φ] + 180 i Tr[$\gamma^\mu.\gamma_5$] Tr[$\mathcal{D}_\mu$.Φ.Φ.Φ] − 180 Tr[$\mathcal{D}_\mu.\Phi.\mathcal{D}_{\mu 1}.\Phi$] Tr[$\gamma^\mu.\gamma_5.\gamma^{\mu 1}.\gamma_5$] + 90 Tr[$\mathcal{D}_\mu.\Phi.F_{\mu 1\,\nu 1}$] Tr[$\gamma^\mu.\gamma_5.\gamma^{\mu 1}.\gamma^{\nu 1}$] + 90 Tr[$F_{\mu\nu}.\mathcal{D}_{\mu 1}.\Phi$] Tr[$\gamma^\mu.\gamma^\nu.\gamma^{\mu 1}.\gamma_5$] − 45 Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$] Tr[$\gamma^\mu.\gamma^\nu.\gamma^{\mu 1}.\gamma^{\nu 1}$] + 30 s Tr[Φ.Φ] Tr[$1_N$] + 60 Tr[Φ.△[Φ]] Tr[$1_N$] + 60 Tr[△[Φ].Φ] Tr[$1_N$] + 180 Tr[Φ.Φ.Φ.Φ] Tr[$1_N$] + $\frac{5}{4}$ $s^2$ Tr[$1_N$] Tr[$1_{\mathcal{H}_F}$] − 2 $R_{\mu\nu}.R^{\mu\nu}$ Tr[$1_N$] Tr[$1_{\mathcal{H}_F}$] + 2 $R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}$ Tr[$1_N$] Tr[$1_{\mathcal{H}_F}$] + $\frac{15}{2}$ i s Tr[$\gamma^\mu.\gamma^\nu$] Tr[$F_{\mu\nu}$] − $\frac{45}{2}$ i s Tr[$\gamma^{\mu 1}.\gamma^{\nu 1}$] Tr[$F_{\mu 1\,\nu 1}$] − 30 i Tr[$\gamma^\mu.\gamma^\nu$] Tr[△[$F_{\mu\nu}$]] + 3 Tr[$1_N$] Tr[$1_{\mathcal{H}_F}$] △[s])

Reduce Tr[γ's]:

---

Evaluate g,F terms with symmetry: tt : A_ $g^{a\_\,b\_}$ :→ 0 /; ! FreeQ[tt, T[F, dd, {a, b}]]

→

→ {0, −180 $g^{\mu\,\nu 1}$ $g^{\nu\,\mu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$], 180 $g^{\mu\,\mu 1}$ $g^{\nu\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$], 0, 0, 0, 0, 0}

→ {0, −180 $g^{\mu\,\mu 1}$ $g^{\nu\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\nu 1\,\mu 1}$], 180 $g^{\mu\,\mu 1}$ $g^{\nu\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$], 0, 0, 0, 0, 0}

Leaving terms: 360 $g^{\mu\,\mu 1}$ $g^{\nu\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$]

→ −180 $g^{\mu\nu}$ $g^{\mu 1\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$] − 180 $g^{\mu\,\nu 1}$ $g^{\nu\,\mu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$] + 180 $g^{\mu\,\mu 1}$ $g^{\nu\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$] − 360 i $g^{\mu 1\,\nu 1}$ Tr[Φ.Φ.$F_{\mu 1\,\nu 1}$] − 360 i $g^{\mu\nu}$ Tr[$F_{\mu\nu}$.Φ.Φ] + 30 i s $g^{\mu\nu}$ Tr[$F_{\mu\nu}$] − 90 i s $g^{\mu 1\,\nu 1}$ Tr[$F_{\mu 1\,\nu 1}$] − 120 i $g^{\mu\nu}$ Tr[△[$F_{\mu\nu}$]] → 360 $g^{\mu\,\mu 1}$ $g^{\nu\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$]

→ {−120 i $g^{\mu\nu}$ Tr[△[$F_{\mu\nu}$]] → 180 ($g^{\mu\nu}$ $g^{\mu 1\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$] + $g^{\mu\,\nu 1}$ $g^{\nu\,\mu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$] + $g^{\mu\,\mu 1}$ $g^{\nu\,\nu 1}$ Tr[$F_{\mu\nu}.F_{\mu 1\,\nu 1}$]) + 30 i (12 $g^{\mu 1\,\nu 1}$ Tr[Φ.Φ.$F_{\mu 1\,\nu 1}$] + 12 $g^{\mu\nu}$ Tr[$F_{\mu\nu}$.Φ.Φ] − s $g^{\mu\nu}$ Tr[$F_{\mu\nu}$] + 3 s $g^{\mu 1\,\nu 1}$ Tr[$F_{\mu 1\,\nu 1}$])}

---

Do they cancel?

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{\sum\left[\begin{array}{l}120\,(12\,g^{\mu\,\mu 1}\,g^{\nu\,\nu 1}\,\mathrm{Tr}[F_{\mu\nu}.F_{\mu 1\,\nu 1}] + \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] - 6\,\mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}_{\mu 1}.\Phi]\,\mathrm{Tr}[\gamma^{\mu}.\gamma_5.\gamma^{\mu 1}.\gamma_5]) \\ \dim[N]\,(120\,(s\,\mathrm{Tr}[\Phi.\Phi] + 2\,(\mathrm{Tr}[\Phi.\triangle[\Phi]] + \mathrm{Tr}[\triangle[\Phi].\Phi] + 3\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]))) + \\ \quad \dim[\mathcal{H}_F]\,(5\,s^2 - 8\,R_{\mu\nu}.R^{\mu\nu} + 8\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 12\,\triangle[s]))\end{array}\right]}{23\,040\,\pi^2}$$

Use $\{\dim[N] \to 4,\ tt : \gamma^{\mu}-.\gamma_5.\gamma^{\mu 1}-.\gamma_5 \to -\gamma^{\mu}.\gamma^{\mu 1}\}$

→

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}(30\,(4\,s\,\mathrm{Tr}[\Phi.\Phi] + 8\,\mathrm{Tr}[\Phi.\triangle[\Phi]] + 12\,g^{\mu\,\mu 1}\,g^{\nu\,\nu 1}\,\mathrm{Tr}[F_{\mu\nu}.F_{\mu 1\,\nu 1}] + \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] + 8\,\mathrm{Tr}[\triangle[\Phi].\Phi] + 24\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] + 24\,g^{\mu\,\mu 1}\,\mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}_{\mu 1}.\Phi]) + \dim[\mathcal{H}_F]\,(5\,s^2 - 8\,R_{\mu\nu}.R^{\mu\nu} + 8\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 12\,\triangle[s]))$$

- Contracting indices:

$$\to \left|\begin{array}{l}360\,g^{\mu\,\mu 1}\,g^{\nu\,\nu 1}\,\mathrm{Tr}[F_{\mu\nu}.F_{\mu 1\,\nu 1}] \\ 720\,g^{\mu\,\mu 1}\,\mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}_{\mu 1}.\Phi]\end{array}\right. \longrightarrow \longrightarrow \left|\begin{array}{l}g^{\mu\,\mu 1}\,g^{\nu\,\nu 1}\,\mathrm{Tr}[F_{\mu\nu}.F_{\mu 1\,\nu 1}] \to \mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] \\ g^{\mu\,\mu 1}\,\mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}_{\mu 1}.\Phi] \to \mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}^{\mu}.\Phi]\end{array}\right.$$

$$\Rightarrow a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{\sum\left[\begin{array}{l}30\,(4\,s\,\mathrm{Tr}[\Phi.\Phi] + 8\,\mathrm{Tr}[\Phi.\triangle[\Phi]] + 12\,\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] + \\ \quad \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] + 8\,\mathrm{Tr}[\triangle[\Phi].\Phi] + 24\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] + 24\,\mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}^{\mu}.\Phi]) \\ \dim[\mathcal{H}_F]\,(5\,s^2 - 8\,R_{\mu\nu}.R^{\mu\nu} + 8\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 12\,\triangle[s])\end{array}\right]}{5760\,\pi^2}$$

- Comparing to text(p.37) there are 2 differences, but evaluate $\{30\,\mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}]\}$

$$\to \boxed{5760\,\pi^2\,a_4[x, \mathcal{D}_{\mathcal{A}}] \to \left|\begin{array}{l}30\left|\begin{array}{l}4\,s\,\mathrm{Tr}[\Phi.\Phi] \\ 8\,\mathrm{Tr}[\Phi.\triangle[\Phi]] \\ 12\,\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] \\ \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] \\ 8\,\mathrm{Tr}[\triangle[\Phi].\Phi] \\ 24\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] \\ 24\,\mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}^{\mu}.\Phi]\end{array}\right. \\ \dim[\mathcal{H}_F]\left|\begin{array}{l}5\,s^2 \\ -8\,R_{\mu\nu}.R^{\mu\nu} \\ 8\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} \\ 12\,\triangle[s]\end{array}\right.\end{array}\right.}$$

```
PR["Compute: ", $0 = $ = $oEE[[1]],
 Yield, $ = $ /. tuRuleSelect[$defall][{Tensor[Ω"E", _, _]}] // expandDC[],
 Yield, $ = $ //. $combineProduct // expandDC[],
 Yield,
 $ = $ //. tuOpDistribute[Tr] // tuArgSimplify[Tr] // tuOpDistributeF[Tr, CircleTimes] //
    tuIndexDummyOrdered // Simplify,
 Yield, $ = $ /. (tuRuleSelect[$defall][Tr"E"x[_]]
         /. Trx_ → Tr /. tt : Tensor[R, a_, b_] Tensor[R, a1_, b1_] :→ Apply[Dot, tt] //
       tuIndexSwapUpDown[{ρ1, σ1}] // tuIndexDummyOrdered),
 Yield, $s = $ = $0 → ($ /. CircleTimes → Times) // (# / 30 & /@ # &),
 NL, "Using ",
 $s1 = {Tr[1N] → 4, Tr[1n_] → dim[n], ρ1 → ρ, σ1 → σ, Tr[Tensor[F, _, _]] → 0},
 Yield, $s = $s //. $s1,
 NL, "• The above: ", $ = $p37a4; $[[1]],
 Yield, $ = $ /. $s;
 Yield, $ = $ // Simplify // ColumnFormOn[Plus] // Framed,
 NL, "Which is the expression on p.37."
]
```

Compute: $30\,\mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}]$

→ $30\,\mathrm{Tr}[-(1_N\otimes F_{\mu\nu}).(1_N\otimes F^{\mu\nu}) + i\,(1_N\otimes F_{\mu\nu}).(\Omega^{S\mu\nu}\otimes 1_{\mathcal{H}_F}) + i\,(\Omega^{S}{}_{\mu\nu}\otimes 1_{\mathcal{H}_F}).(1_N\otimes F^{\mu\nu}) + (\Omega^{S}{}_{\mu\nu}\otimes 1_{\mathcal{H}_F}).(\Omega^{S\mu\nu}\otimes 1_{\mathcal{H}_F})]$

→ $30\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}\otimes 1_{\mathcal{H}_F} - 1_N\otimes F_{\mu\nu}.F^{\mu\nu} + i\,\Omega^{S}{}_{\mu\nu}\otimes F^{\mu\nu} + i\,\Omega^{S\mu\nu}\otimes F_{\mu\nu}]$

→ $30\,(\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}]\otimes\mathrm{Tr}[1_{\mathcal{H}_F}] - \mathrm{Tr}[1_N]\otimes\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] + 2\,i\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}]\otimes\mathrm{Tr}[F^{\mu\nu}])$

→ $30\,((-\frac{1}{2}R_{\mu\nu\rho1\sigma1}.R^{\mu\nu\rho1\sigma1})\otimes\mathrm{Tr}[1_{\mathcal{H}_F}] - \mathrm{Tr}[1_N]\otimes\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] + 2\,i\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}]\otimes\mathrm{Tr}[F^{\mu\nu}])$

→ $\mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] \to -\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}]\,\mathrm{Tr}[1_N] - \frac{1}{2}R_{\mu\nu\rho1\sigma1}.R^{\mu\nu\rho1\sigma1}\,\mathrm{Tr}[1_{\mathcal{H}_F}] + 2\,i\,\mathrm{Tr}[F^{\mu\nu}]\,\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}]$

Using $\{\mathrm{Tr}[1_N]\to 4,\ \mathrm{Tr}[1_{n\_}]\to \mathrm{dim}[n],\ \rho1\to\rho,\ \sigma1\to\sigma,\ \mathrm{Tr}[\mathrm{Tensor}[F, \_, \_]]\to 0\}$

→ $\mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] \to -\frac{1}{2}\mathrm{dim}[\mathcal{H}_F]\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} - 4\,\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}]$

• The above: $5760\,\pi^2\,a_4[x, \mathcal{D}_{\mathcal{A}}]$

→

→ $5760\,\pi^2\,a_4[x, \mathcal{D}_{\mathcal{A}}] \to$
120 ( s Tr[Φ.Φ] + 2 ( Tr[Φ.Δ[Φ]] + Tr[$F_{\mu\nu}.F^{\mu\nu}$] + Tr[Δ[Φ].Φ] + 3 Tr[Φ.Φ.Φ.Φ] + 3 Tr[$\mathcal{D}_\mu$.Φ.$\mathcal{D}^\mu$.Φ] ) )
+ dim[$\mathcal{H}_F$] ( 5 $s^2$ − 8 $R_{\mu\nu}.R^{\mu\nu}$ − 7 $R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}$ + 12 Δ[s] )

Which is the expression on p.37.

Aside: Compute Q.Q

```
PR["•Evaluate: ", $ = $sQ[[1]],
  Yield, $ = $ // tuDotSimplify[],
  NL, CO["Is there a Logical order to the operations? "],
  $sX = { (a_ ⊗ b_) .( c_ ⊗ d_) → a.c ⊗ b.d,
    1n_ . a_ → a, a_ . 1n_ → a};
  Yield, $ = $ // tuRepeat[{$sX, tuOpSimplify[Dot, {s}]}] , $ // ColumnSumExp;
  Yield, $ = Tr[#] & /@ $ // tuTrSimplify[{s}]; $ // ColumnSumExp;
  Yield, $ = $ //. tuOpDistribute[Tr, CircleTimes] /. Tr[a_] ⊗ Tr[b_] → Tr[a] Tr[b];
  $ // ColumnSumExp;
  NL, "Use: ",
  $s = {dim[N] → 4, tt : T[γ, "u", {μ_}].T[γ, "d", {5}] . T[γ, "u", {μ1_}].T[γ, "d", {5}] →
     -T[γ, "u", {μ}].T[γ, "u", {μ1}]},
  Yield, $ = $ /. $s //. tuTrGamma // tuTrSimplify[]; $ // ColumnSumExp;
  NL, "Apply symmetries ",
  $ss = tt : T[g, "uu", {a_, b_}] A_ :→ 0 /; ! FreeQ[tt, T[F, "dd", {a, b}]],
  Yield, $ = $ /. $ss //. tuTrGamma // Expand;
  Yield, $ = $ // tuIndexContractUpDn[g, {ν1, μ1}]; $ // ColumnSumExp;
  NL, "Apply: ", $s = {aa : Tensor[g, _, _] A_ :> tuIndexContractUpDn[g, {ν1, μ1, ν}][aa],
    μ1 | ν1 → ν, tt : T[F, "du", {a_, b_}].Tensor[F, _, _] :→ tuIndexSwapUpDown[μ][tt],
    T[F, "ud", {a_, a_}] → 0},
  Yield, $sQQ = $ = $ //. $s /. $symmetries //. tuOpSimplify[Dot] // tuTrSimplify[];
  $ // ColumnSumExp // Framed
 ];
```

•Evaluate: $Q.Q \to (-i\,\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi + \frac{1}{2} i\,\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}).$
$(-i\,\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2} i\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F})$

$\to$ $Q.Q \to -(\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) + \frac{1}{2}(\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) +$
$i\,(\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(1_N \otimes \Phi.\Phi) + \frac{1}{4} i\,(\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + \frac{1}{2}(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) -$
$\frac{1}{4}(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) - \frac{1}{2} i\,(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(1_N \otimes \Phi.\Phi) -$
$\frac{1}{8} i\,(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + i\,(1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) - \frac{1}{2} i\,(1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) +$
$(1_N \otimes \Phi.\Phi).(1_N \otimes \Phi.\Phi) + \frac{1}{4}(1_N \otimes \Phi.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + \frac{1}{4} i\,(s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) -$
$\frac{1}{8} i\,(s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + \frac{1}{4}(s\, 1_N \otimes 1_{\mathcal{H}_F}).(1_N \otimes \Phi.\Phi) + \frac{1}{16}(s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F})$

Is there a Logical order to the operations?

$\to$ $Q.Q \to \frac{1}{4} i\, s\, \gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi + i\,\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi.\Phi.\Phi - \frac{1}{2} i\,\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}.\Phi.\Phi - \frac{1}{8} i\, s\, \gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} +$
$\frac{1}{4} i\, s\, \gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + i\,\gamma^{\mu 1}.\gamma_5 \otimes \Phi.\Phi.\mathcal{D}_{\mu 1}.\Phi - \frac{1}{2} i\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes \Phi.\Phi.F_{\mu 1\,\nu 1} - \frac{1}{8} i\, s\, \gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} -$
$\gamma^{\mu}.\gamma_5.\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi.\mathcal{D}_{\mu 1}.\Phi + \frac{1}{2}\gamma^{\mu}.\gamma_5.\gamma^{\mu 1}.\gamma^{\nu 1} \otimes \mathcal{D}_{\mu}.\Phi.F_{\mu 1\,\nu 1} + \frac{1}{2}\gamma^{\mu}.\gamma^{\nu}.\gamma^{\mu 1}.\gamma_5 \otimes F_{\mu\nu}.\mathcal{D}_{\mu 1}.\Phi -$
$\frac{1}{4}\gamma^{\mu}.\gamma^{\nu}.\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu\nu}.F_{\mu 1\,\nu 1} + \frac{1}{2} s\, 1_N \otimes \Phi.\Phi + 1_N \otimes \Phi.\Phi.\Phi.\Phi + \frac{1}{16} s^2\, 1_N \otimes 1_{\mathcal{H}_F}$

$\to$

$\to$

Use: {dim[N] → 4, tt : $\gamma^{\mu\_}.\gamma_5.\gamma^{\mu 1\_}.\gamma_5 \to -\gamma^{\mu}.\gamma^{\mu 1}$}

$\to$

Apply symmetries tt : A_ $g^{a\_\,b\_}$ :→ 0 /; ! FreeQ[tt, T[F, dd, {a, b}]]

$\to$

$\to$

Apply: {aa : A_Tensor[g, _, _] :→ tuIndexContractUpDn[g, {ν1, μ1, ν}][aa],
μ1 | ν1 → ν, tt : $F_{a\_}{}^{b\_}$.Tensor[F, _, _] :→ tuIndexSwapUpDown[μ][tt], $F^{a\_}{}_{a\_}$ → 0}

$\to$ $\boxed{\mathrm{Tr}[Q.Q] \to \sum\left[\begin{array}{l} \frac{1}{16} s^2\, \dim[N]\, \dim[\mathcal{H}_F] \\ \frac{1}{2} s\, \dim[N]\, \mathrm{Tr}[\Phi.\Phi] \\ 2\, \mathrm{Tr}[F^{\mu\nu}.F_{\mu\nu}] \\ \dim[N]\, \mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] \\ 4\, \mathrm{Tr}[\mathcal{D}_{\mu}.\Phi.\mathcal{D}^{\mu}.\Phi] \end{array}\right]}$

## 4. Electrodynamics (p.38)

### 4.1 A two-point space

```
PR["● Take the Two point space. ",
  {X -> {x, y}, C[X] → ℂ², C[CG["complex functions"]]},
  NL, "•Construct an even finite space ",
  {F_X → {C[X], ℋ_F, T[γ, "u", {v_}]_F, γ_F}, dim[ℋ_F] ≥ 2, γ_F[CG["ℤ²-grading"]] },
  Yield, γ_F ⇒ {ℋ_F → ℋ_F⁺ ⊕ ℋ_F⁻ → ℂ ⊕ ℂ, ℋ_F^"±" → {ψ ∈ ℋ_F | γ_F.ψ → ±ψ}},
  imply, $ = γ_F → {{1, 0}, {0, -1}}; MatrixForms[$],
```

```
NL, "• Since ", $sD0 = {CommutatorM[γF, a] → 0,
   CommutatorP[𝒟F, γF] → 0, 𝒟F[CG["offDiagonal"]], 𝒟F → {{0, du}, {dl, 0}}},
Imply, {a.ψ → Inactive[Dot][{{a+, 0}, {0, a-}}, {{ψ+}, {ψ-}}], a ∈ 𝒜F, ψ ∈ ℋF} //
 MatrixForms,

Imply, $sFX = FX → {{𝒜F, ℋF, 𝒟F, γF} → {ℂ^2, ℂ^2, {{0, t}, {t̄, 0}}, {{1, 0}, {0, -1}}}, t ∈ ℂ};
$sFX // MatrixForms,
line,
NL, "■ Prop.4.1. Only a real structure ", $ = JF ⇒ {𝒟F → 0}, " exists on FX.",
line,
NL, "Proof: Determine 𝒟F for even KO dimensions by requiring: ",
$def =
 tuRuleSelect[$defall][{CommutatorM[_, rghtA[b]], rghtA[b]}] // DeleteDuplicates;
$c = $ = Join[$J[[2]], $def]; ColumnBar[$],

NL, "■ KOdim→0: ", $sj = {JF → {{j+, 0}, {0, j-}}.cc, j"±" ∈ U[1]};
$sj // MatrixForms,
NL, "for ", $sa = ab : a | b -> {{ab+, 0}, {0, ab-}}; MatrixForms[$sa],
NL, "•Compute ", $0 = $ = tuRuleSelect[$c][{rghtA[b]}] // First,
yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];
yield, $[[2]] = $[[2]] // tuRepeat[$cc, tuConjugateTransposeSimplify[]];
MatrixForms[$],
yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
MatrixForms[$sb = $] // Framed, yield, b,
Yield, "This is diagonal hence satisfies 0-order condition: ",
tuRuleSelect[$c][CommutatorM[_, _]][[1]] // Framed,

NL, "• The 1-order condition ",
$ = tuRuleSelect[$c][{CommutatorM[CommutatorM[_, _], rghtA[b]]}] // First, "POFF",
$sa = ab : a | xb -> {{ab+, 0}, {0, ab-}};
Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
yield, $ = $ //. tuCommutatorExpand // expandDC[];
yield, $ = $ /. $sD0[[-1]] // Simplify;
MatrixForms[$],
$x = tuExtractPattern[du _][$][[1]] / du; "PONdd",
yield, $ = $x.(# / $x) & /@ $ /. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
imply, Framed[𝒟F → 0]
];

PR[
 "■ KOdim→2: ", $sj = {JF → {{0, j}, {-j, 0}}.cc, j ∈ U[1]};
 $sj // MatrixForms,
 NL, "for ", $sa = ab : a | b -> {{ab+, 0}, {0, ab-}}; MatrixForms[$sa],
 NL, "Compute ", $0 = $ = tuRuleSelect[$c][rghtA[b]] // DeleteDuplicates // First,
 yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];
 yield, $[[2]] = $[[2]] // tuRepeat[$cc, tuConjugateTransposeSimplify[]];
 MatrixForms[$sb = $],
 yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
 MatrixForms[$sb = $],
 Yield, "This is diagonal hence satisfies 0-order condition: ",
 tuRuleSelect[$c][CommutatorM[_, _]][[1]] // Framed, (**)

 NL, "• The 1-order condition ",
 $ = tuRuleSelect[$c][{CommutatorM[CommutatorM[_, _], rghtA[b]]}] // First, "POFF",
 $sa = ab : a | xb -> {{ab+, 0}, {0, ab-}};
 Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
 yield, $ = $ //. CommutatorM → MCommutator //
```

```
    tuRepeat[Join[tuOpSimplify[Dot], {tuOpDistribute[Dot]}]];
  yield, $ = $ /. $sD0[[-1]] // Simplify;
  MatrixForms[$],
  $x = tuExtractPattern[du _][$][[1]] / du; "PONdd",
  yield, $ = $x.(# / $x) & /@ $ //. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
  imply, Framed[𝒟F → 0]
 ];
PR["■ KOdim→4:",
  NL, "■ KOdim→6:"
 ];
```

● Take the Two point space. $\{X \to \{x, y\}, C[X] \to \mathbb{C}^2, C[\text{complex functions}]\}$

•Construct an even finite space $\{F_X \to \{C[X], \mathcal{H}_F, \gamma^{\vee}{}_{-F}, \gamma_F\}, \dim[\mathcal{H}_F] \geq 2, \gamma_F[\mathbb{Z}^2\text{-grading}]\}$

→ $\gamma_F \Rightarrow \{\mathcal{H}_F \to (\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^- \to \mathbb{C}\oplus\mathbb{C}, \mathcal{H}_F^{\pm} \to \{\psi \in \mathcal{H}_F \mid \gamma_F.\psi \to \pm\psi\}\}$ ⇒ $\gamma_F \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

• Since $\{[\gamma_F, a]_- \to 0, \{\mathcal{D}_F, \gamma_F\}_+ \to 0, \mathcal{D}_F[\text{offDiagonal}], \mathcal{D}_F \to \{\{0, du\}, \{dl, 0\}\}\}$

⇒ $\{a.\psi \to \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \circ \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, a \in \mathcal{A}_F, \psi \in \mathcal{H}_F\}$

⇒ $F_X \to \{\{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F\} \to \{\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}, t \in \mathbb{C}\}$

---

■ Prop.4.1. Only a real structure $J_F \Rightarrow \{\mathcal{D}_F \to 0\}$ exists on $F_X$.

---

Proof: Determine $\mathcal{D}_F$ for even KO dimensions by requiring:

$$\begin{array}{|l} J_F.J_F \to \varepsilon \\ J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F \\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F \\ [a, b^o]_- \to 0 \\ [[\mathcal{D}_F, a]_-, b^o]_- \to 0 \\ b^o \to J_F.b^\dagger.(J_F)^\dagger \end{array}$$

■ KOdim→0: $\{J_F \to \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix}.cc, j_\pm \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

•Compute $b^o \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow \longrightarrow b^o \to \begin{pmatrix} (j_+)^* b_+ j_+ & 0 \\ 0 & (j_-)^* b_- j_- \end{pmatrix} \longrightarrow \boxed{b^o \to \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}} \longrightarrow b$

→ This is diagonal hence satisfies 0-order condition: $\boxed{[a, b^o]_- \to 0}$

• The 1-order condition $[[\mathcal{D}_F, a]_-, b^o]_- \to 0$

....... $\longrightarrow ((a_- - a_+)(b_- - b_+)).\mathcal{D}_F \to 0$ ⇒ $\boxed{\mathcal{D}_F \to 0}$

■ KOdim→2: $\{J_F \to \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}.cc, j \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

Compute $b^o \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow \longrightarrow b^o \to \begin{pmatrix} j\, j^* b_- & 0 \\ 0 & j\, j^* b_+ \end{pmatrix} \longrightarrow b^o \to \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}$

→ This is diagonal hence satisfies 0-order condition: $\boxed{[a, b^o]_- \to 0}$

• The 1-order condition $[[\mathcal{D}_F, a]_-, b^o]_- \to 0$

....... $\longrightarrow -((a_- - a_+)(b_- - b_+)).\mathcal{D}_F \to 0$ ⇒ $\boxed{\mathcal{D}_F \to 0}$

■ KOdim→4:
■ KOdim→6:

### 4.1.2 The product space

```
PR["The product space  ",
 $ = {M × Fx → {𝒜 -> C"∞"[M, ℂ²], ℋ -> L²[M, S] ⊗ ℂ², 𝒟 -> slash[𝒟] ⊗ 1F, γ -> γ5 ⊗ γF, J -> JM ⊗ JF},
   M[CG["4-dim Riemann spin manifold"]],
   Fx[CG["two-point space"]],
   C∞[M, ℂ²] -> C∞[M] ⊕ C∞[M],
   ℋ -> L²[M, S] ⊕ L²[M, S],
   {(a ⊕ b).(ψ ⊕ ϕ) -> (a.ψ ⊕ b.ϕ), a ⊕ b ∈ C∞[M] ⊕ C∞[M], ψ ⊕ ϕ ∈ ℋ}
  }; $ // ColumnForms,
 accumDef[$]; ""
]
```

```
The product space   M × Fx →  𝒜 → C∞[M, ℂ²]
                              ℋ → L²[M, S]⊗ℂ²
                              𝒟 → (𝒟̸)⊗1F
                              γ → γ5⊗γF
                              J → JM⊗JF
                    M[4-dim Riemann spin manifold]
                    Fx[two-point space]
                    C∞[M, ℂ²] → C∞[M]⊕C∞[M]
                    ℋ → L²[M, S]⊕L²[M, S]
                     (a⊕b).(ψ⊕ϕ) → a.ψ⊕b.ϕ
                     a⊕b ∈ C∞[M]⊕C∞[M]
                     ψ⊕ϕ ∈ ℋ
```

Distance

```
PR["1•  Restrict distance formula to Fx: ",
 Yield, $0 = {d𝒟F[x, y] → sup[‖a[x] - a[y]‖], a ∈ 𝒜F, Abs[Det[CommutatorM[𝒟F, a]]] ≤ 1},
 NL, "Using: ", $s = $sFX;
 $s = Thread[$s[[2, 1]]]; $s // MatrixForms,
 NL, "Define algebra for the two points {x,y}: ", $s1 = a → {{a[x], 0}, {0, a[y]}};
 $s1 // MatrixForms,

 NL, "• Determine influence of: ", $ = Abs[Det[CommutatorM[𝒟F, a]]] ≤ 1,
 Imply, $ = ($ /. $s1 /. $s /. CommutatorM → MCommutator // Simplify),
 Yield, $ = $ /. t̄ → Conjugate[t] /. Abs[t^2 a_] → Abs[t^2] Abs[a],
 Yield, $ = # / Abs[t^2] & /@ $,
 NL, "A real structure JF (Prop.4.1)",
 imply, 𝒟F → 0, imply, t → 0, imply, $0[[1, 1]] → ∞,

 line,
 NL, "2•  For the case with points: ", {{p, x}, {p, y}, p ∈ M},
 NL, "Let ", $sa2 = {{a[n] → ax[p], ax_[p_] → a[p, x], ax[CG[C∞[M]]]},
    {dslash[iD]⊗1F[n_, m_] → sup[‖a[n] - a[m]‖ ],
     a ∈ 𝒜, Abs[Det[CommutatorM[slash[iD], a]]] ≤ 1, n_ | m_ ∈ N}
  }; $sa2 // ColumnForms,
 Yield, $ = tuRuleSelect[$sa2][d_[_, _]] // First,
 NL, "Define ", $s =
  tuRuleSelect[$sa2][a[n]] /. {x → x[n], p → p[n]} // tuAddPatternVariable[n] // First,
 Yield, $1 = $ = $ /. $s,
 NL, "• For ", $s = {x[m] | x[n] → g, CG["i.e. the same F-space points "]},
 Yield, $ = $ /. tuRule[$s],
 NL, "This can be identified with normal distance in M.",

 line,
 NL, "• For different F-space points the requirement: ",
 $1 = $ = Select[Flatten[$sa2], MatchQ[#, Abs[_] ≤ 1] &][[1]],
 NL, "implies different requirements depending on
    the definition of the algebra and Dirac operator.",
 next, "For: ", $s = {slash[iD] → iDM ⊗ iDF, a → aM ⊗ aF},
 Yield, $ = $ /. $s,
 NL, "• If the space is a disjoint product ",
 Yield, $s = Abs[Det[CommutatorM[a_ ⊗ b_, c_ ⊗ d_]]] →
    Abs[Det[CommutatorM[a, c]]] Abs[Det[CommutatorM[b, d]]],
 Yield, $ = $ /. $s, CR[
  " which is the same as the previous example so the distance is ∞. "],

 NL, "• If there is cross talk between the spaces ",
 $s = {slash[iD] → iDM ⊗ iDF, CG["only"]},
 Yield, $ = $1 /. tuRule[$s],
 NL, "Let ", $s = Abs[Det[CommutatorM[a_ ⊗ b_, c_]]] →
    Abs[Det[CommutatorM[a, c]]] Abs[Det[CommutatorM[b, c]]],
 Yield, $ /. $s, CO[" possible finite distance."]
]
```

1• Restrict distance formula to $F_X$:

→ $\{d_{\mathcal{D}_F}[x, y] \to \sup[\|a[x] - a[y]\|],\ a \in \mathcal{A}_F,\ \text{Abs}[\text{Det}[[\mathcal{D}_F, a]_-]] \le 1\}$

Using: $\{\mathcal{A}_F \to \mathbb{C}^2,\ \mathcal{H}_F \to \mathbb{C}^2,\ \mathcal{D}_F \to \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix},\ \gamma_F \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$

Define algebra for the two points {x,y}: $a \to \begin{pmatrix} a[x] & 0 \\ 0 & a[y] \end{pmatrix}$

• Determine influence of: $\text{Abs}[\text{Det}[[\mathcal{D}_F, a]_-]] \le 1$

⇒ $\text{Abs}[t\,(a[x] - a[y])^2\,\bar{t}] \le 1$

→ $\text{Abs}[t]^2\,\text{Abs}[a[x] - a[y]]^2 \le 1$

→ $\text{Abs}[a[x] - a[y]]^2 \le \dfrac{1}{\text{Abs}[t]^2}$

A real structure $J_F$ (Prop.4.1) ⇒ $\mathcal{D}_F \to 0$ ⇒ $t \to 0$ ⇒ $d_{\mathcal{D}_F}[x, y] \to \infty$

---

2• For the case with points: $\{\{p, x\}, \{p, y\}, p \in M\}$

Let
$a[n] \to a_x[p]$
$a_{x\_}[p\_] \to a[p, x]$
$a_x[C^\infty[M]]$
$d_{(\not{D})\otimes 1_F}[n\_, m\_] \to \sup[\|-a[m] + a[n]\|]$
$a \in \mathcal{A}$
$\text{Abs}[\text{Det}[[\not{D}, a]_-]] \le 1$
$n\_ \mid m\_ \in N$

→ $d_{(\not{D})\otimes 1_F}[n\_, m\_] \to \sup[\|-a[m] + a[n]\|]$

Define $a[n\_] \to a_{x[n]}[p[n]]$

→ $d_{(\not{D})\otimes 1_F}[n\_, m\_] \to \sup[\|-a_{x[m]}[p[m]] + a_{x[n]}[p[n]]\|]$

• For $\{x[m] \mid x[n] \to g$, i.e. the same F-space points $\}$

→ $d_{(\not{D})\otimes 1_F}[n\_, m\_] \to \sup[\|-a_g[p[m]] + a_g[p[n]]\|]$

This can be identified with normal distance in M.

---

• For different F-space points the requirement: $\text{Abs}[\text{Det}[[\not{D}, a]_-]] \le 1$ implies different requirements depending
on the definition of the algebra and Dirac operator.

◆For: $\{\not{D} \to D_M \otimes D_F,\ a \to a_M \otimes a_F\}$

→ $\text{Abs}[\text{Det}[[D_M \otimes D_F,\ a_M \otimes a_F]_-]] \le 1$

• If the space is a disjoint product

→ $\text{Abs}[\text{Det}[[a\_ \otimes b\_,\ c\_ \otimes d\_]_-]] \to \text{Abs}[\text{Det}[[a, c]_-]]\ \text{Abs}[\text{Det}[[b, d]_-]]$

→ $\text{Abs}[\text{Det}[[D_F, a_F]_-]]\ \text{Abs}[\text{Det}[[D_M, a_M]_-]] \le 1$
which is the same as the previous example so the distance is ∞.

• If there is cross talk between the spaces $\{\not{D} \to D_M \otimes D_F$, only$\}$

→ $\text{Abs}[\text{Det}[[D_M \otimes D_F, a]_-]] \le 1$

Let $\text{Abs}[\text{Det}[[a\_ \otimes b\_,\ c\_]_-]] \to \text{Abs}[\text{Det}[[a, c]_-]]\ \text{Abs}[\text{Det}[[b, c]_-]]$

→ $\text{Abs}[\text{Det}[[D_F, a]_-]]\ \text{Abs}[\text{Det}[[D_M, a]_-]] \le 1$ possible finite distance.

### 4.1.3 U[1] gauge theory

```
PR["U[1] gauge theory for: ", tuRuleSelect[$defall][M×Fx] // First,
 NL, "gauge group: ", 𝒢[𝒜] → Mod[U[𝒜], U[$sAt[[1]]]], U[𝒜] ≠ U[$sAt[[1]]],
 NL, "where ",
 {$t219[[1, -2]], U[𝒜] ≠ U[$sAt[[1]]][CG["non-trivial"]], $sAt} // ColumnBar,
 NL, "non-triviality", imply, "KOdim[JF]" → {2, 6},
 ", i.e., off diagonal.

Only KOdim→6 for Standard Model so used in this case. ",
 Imply, "Can use Def.2.17 for action functional ",
 $d217 = $ = {S → Sb + Sf, Sb → Tr[f[𝒟𝒜 / Λ]], Sf → 1 / 2 BraKet[J.ξ̃, 𝒟𝒜.ξ̃],
     ξ̃ ∈ ℋcl⁺, ℋcl⁺ → {ξ̃ | ξ ∈ ℋ⁺}, ξ̃[CG["GrassmannVariable"]]};
 $ // ColumnBar,
 NL, "•Consider ", $Fx = Fx → {ℂ², ℂ², 0, γF → {{1, 0}, {0, -1}}, JF → {{0, C}, {C, 0}}};
 MatrixForms[$Fx]
]
```

U[1] gauge theory for:

$M\times F_X \to \{\mathcal{A}\to C^\infty[M, \mathbb{C}^2], \mathcal{H}\to L^2[M, S]\otimes\mathbb{C}^2, \mathcal{D}\to(\not{D})\otimes 1_F, \gamma\to\gamma_5\otimes\gamma_F, J\to J_M\otimes J_F\}$

gauge group: $\mathcal{G}[\mathcal{A}]\to \mathrm{Mod}[U[\mathcal{A}], U[\tilde{\mathcal{A}}_J]]U[\mathcal{A}]\neq U[\tilde{\mathcal{A}}_J]$

where
$$\left|\begin{array}{l} (2.11)\Rightarrow \mathcal{G}[M\times F]\to\{U\to u.J.u.J^\dagger,\ u\in U[\mathcal{A}]\} \\ U[\mathcal{A}]\neq U[\tilde{\mathcal{A}}_J][\text{non-trivial}] \\ \tilde{\mathcal{A}}_J\to\{a\in\mathcal{A},\ a.J\to J.a^\dagger,\ a^o\to a\}\end{array}\right.$$

non-triviality ⇒ $\mathrm{KOdim}[J_F]\to\{2, 6\}$, i.e., off diagonal.

Only KOdim→6 for Standard Model so used in this case.

⇒ Can use Def.2.17 for action functional
$$\left|\begin{array}{l} S\to S_b+S_f \\ S_b\to \mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \\ S_f\to\frac{1}{2}\langle J.\tilde{\xi}\mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle \\ \tilde{\xi}\in(\mathcal{H}_{cl})^+ \\ (\mathcal{H}_{cl})^+\to\{\tilde{\xi}\mid \xi\in\mathcal{H}^+\} \\ \tilde{\xi}[\text{GrassmannVariable}]\end{array}\right.$$

•Consider $F_X\to\{\mathbb{C}^2, \mathbb{C}^2, 0, \gamma_F\to\begin{pmatrix}1&0\\0&-1\end{pmatrix}, J_F\to\begin{pmatrix}0&C\\C&0\end{pmatrix}\}$

```
PR["Prop.4.2. The gauge group of ", {𝒢[𝒜F] → U[1], 𝒜F[CG["2-point space"]]},
  line,
  NL, "Proof: Note: ", U[𝒜F] → U[1] × U[1],
  NL, "The subspace: ", $sAt // ColumnForms,
  yield, $ = ForAll[a,
    a ∈ ℂ^2 && a ∈ ($sAtj = ($sAt[[1]] /. J → F)_JF), (JF.ConjugateTranspose[a].JF → a)],
  NL, "Compute ", $0 = $ = tuExtractPattern[Rule[__]][$][[1]],
  yield, $ = $ /. $Fx[[2, -2 ;; -1]]; MatrixForms[$],
  NL, "The 2-point algebra ", $sCC = $s = {a -> DiagonalMatrix[{a1, a2}],
     C.a_ :> Conjugate[a].C /; FreeQ[a, C], Conjugate[C] → C, C.C → 1};
  $s // MatrixForms,
  Yield, $ = $ /. Dot → xDot /. $s // OrderedxDotMultiplyAll[];
  MatrixForms[$],
  yield, $ = $ // tuRepeat[$s, ConjugateCTSimplify1[{}]];
  MatrixForms[$] // Framed,
  imply, a1 → a2, imply, a ∝ "identity",
  imply, $pass4 = $ = $sAtj ≃ ℂ,
  imply, (U[$[[1]]] → U[1]) ⊂ U[𝒜F], CG[" QED"]
 ];
```

Prop.4.2. The gauge group of $\{\mathcal{G}[\mathcal{A}_F] \to U[1], \mathcal{A}_F[\text{2-point space}]\}$

---

Proof: Note: $U[\mathcal{A}_F] \to U[1] \times U[1]$

The subspace: $\tilde{\mathcal{A}}_J \to \left| \begin{array}{l} a \in \mathcal{A} \\ a.J \to J.a^\dagger \\ a^o \to a \end{array} \right. \longrightarrow \forall_{a, a \in \mathbb{C}^2 \&\& a \in \tilde{\mathcal{A}}_{F_{J_F}}} (J_F.a^\dagger.J_F \to a)$

Compute $J_F.a^\dagger.J_F \to a \longrightarrow \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}.a^\dagger.\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \to a$

The 2-point algebra $\{a \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix}, C.(a\_) :\to a^*.C \;/;\; \text{FreeQ}[a, C], C^* \to C, C.C \to 1\}$

$\to \begin{pmatrix} C.a2^*.C & 0 \\ 0 & C.a1^*.C \end{pmatrix} \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix} a2 & 0 \\ 0 & a1 \end{pmatrix} \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix}}$

$\Rightarrow a1 \to a2 \Rightarrow a \propto \text{identity} \Rightarrow \tilde{\mathcal{A}}_{F_{J_F}} \simeq \mathbb{C} \Rightarrow (U[\tilde{\mathcal{A}}_{F_{J_F}}] \to U[1]) \subset U[\mathcal{A}_F]$ QED

```
PR["■Determine Bμ of Prop.3.7: Since ", $pass4,
  yield, (hF → u[$sAtj]) ≃ I ℝ,
  NL, "Gauge field: ",
  Aμ[x] ∈ (I gF -> I Mod[u[($a = $sAt[[1]] /. J → F)], I ℝ]) → (Isu[$a] ≃ ℝ),
  NL, "Arbitrary hermitian field ",
  $sA = {Aμ → -I a tuDPartial[b, μ], Aμ → {{T[X"1", "d", {μ}], 0}, {0, T[X^2, "d", {μ}]}},
    {T[X"1", "d", {μ}], T[X^2, "d", {μ}]} ∈ C"∞"[M, ℝ], C.tt : T[X"1"|2, "d", {μ}] → tt.C};
  $sA // MatrixForms,
  NL, "Since ", Aμ , " is always in form ", $ = Bμ -> Aμ - JF.Aμ.inv[JF],
  Yield, $ = $ /. $Fx[[2, -1]] /. inv[cc : 0 | C] → cc /. Dot → xDot /.
       dd : xDot[__] :→ (dd /. $sA[[2]] //. $sA[[-1]]) /.
      Plus → xPlus /. $sA[[2]] // OrderedxDotMultiplyAll[];
  Yield, $ = $ /. xPlus → Plus /. $sA[[-1]] /. $sCC /. tuOpSimplify[Dot];
  MatrixForms[$B = $],
  " define ", $ = $ -> {{T[Y, "d", {μ}], 0}, {0, -T[Y, "d", {μ}]}};
  $ = Flatten /@ ($[[1, 2]] -> $[[-1]]);
  $sb = Thread[$] // DeleteCases[#, 0 → 0] & // First,
  imply, $B = $B /. {$sb, -1 # & /@ $sb};
  MatrixForms[$B -> T[Y, "d", {μ}] ⊗ γF] // Framed, CG[" (4.3)"]
 ];
```

■Determine $B_\mu$ of Prop.3.7: Since $\tilde{\mathcal{A}}_{F J_F} \simeq \mathbb{C}$ ⟶ $(h_F \to \mathfrak{u}[\tilde{\mathcal{A}}_{F J_F}]) \simeq i\,\mathbb{R}$

Gauge field: $A_\mu[x] \in (i\, g_F \to i\,\mathrm{Mod}[\mathfrak{u}[\tilde{\mathcal{A}}_F], i\,\mathbb{R}]) \to I\mathfrak{su}[\tilde{\mathcal{A}}_F] \simeq \mathbb{R}$

Arbitrary hermitian field

$\{A_\mu \to -i\, a\, \underline{\partial}_\mu[b],\ A_\mu \to \begin{pmatrix} X^1{}_\mu & 0 \\ 0 & X^2{}_\mu \end{pmatrix},\ \{X^1{}_\mu, X^2{}_\mu\} \in C^\infty[M, \mathbb{R}],\ C.(tt : X^{1|2}{}_\mu) \to tt.C\}$

Since $A_\mu$ is always in form $B_\mu \to -J_F.A_\mu.J_F^{-1} + A_\mu$

→

→ $B_\mu \to \begin{pmatrix} -X^2{}_\mu + X^1{}_\mu & 0 \\ 0 & X^2{}_\mu - X^1{}_\mu \end{pmatrix}$ define $-X^2{}_\mu + X^1{}_\mu \to Y_\mu$ ⇒ $\boxed{\left(B_\mu \to \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix}\right) \to Y_\mu \otimes \gamma_F}$ (4.3)

```
PR["●Prop.4.3. The inner fluctuations
    for ACM M×FX are parameterized by a U[1]-gauge field Yμ ",
  Yield, 𝒟 ↦ (𝒟' → 𝒟 + T[γ, "u", {μ}].T[Y, "d", {μ}] ⊗ γF),
  NL, "The action of gauge group ", 𝒢[𝒜] ≃ C"∞"[M, U[1]][𝒟'],
  Yield,
  {T[Y, "d", {μ}] -> T[Y, "d", {μ}] - I u.tuDPartial[ConjugateTranspose[u], μ], u ∈ 𝒢[𝒜]}
 ];
```

●Prop.4.3. The inner fluctuations
for ACM $M\times F_X$ are parameterized by a U[1]-gauge field $Y_\mu$

→ $\mathcal{D} \mapsto (\mathcal{D}' \to \mathcal{D} + \gamma^\mu.Y_\mu \otimes \gamma_F)$

The action of gauge group $\mathcal{G}[\mathcal{A}] \simeq C^\infty[M, U[1]][\mathcal{D}']$

→ $\{Y_\mu \to -i\, u.\underline{\partial}_\mu[u^\dagger] + Y_\mu,\ u \in \mathcal{G}[\mathcal{A}]\}$

## 4.2 Electrodynamics

```
$defEM = {};
accumDefEM[item_] := Block[{}, $defEM = tuAppendUniq[item][$defEM]];
```

```
PR["■Two modifications of ACM M×F_X needed for E-M: ",
 $ = {𝒟_F[CG["non-zero"]], S_fermion[CG["action"]] ⇒ "2 independent spinors",
   S[CG["action"]] → xIntegral[-I ψ̄.(T[γ, "u", {μ}].tuDPartial[_, μ] - m).ψ, x^4]};
 $ // ColumnBar,
 NL, "•Let ", $ = {{e, ē}[CG["basis of ℋ_F"]],
   e[CG["basis of ℋ_F^+"]],
   ē[CG["basis of ℋ_F^-"]],
   J_F.e -> ē,
   J_F.ē → e,
   γ_F.e → e,
   γ_F.ē → -ē
  }; $ // ColumnBar, accumDefEM[$];
 imply,
 $H = {ℋ → L^2[M, S]⊗ℋ_F, L^2[M, S] -> L^2[M, S]^+ ⊕L^2[M, S]^-,
   ℋ^+ → "positiveEigenSpace of γ→γ5⊗γ_F",
   ℋ^+ -> L^2[M, S]^+ ⊗ℋ_F^+ ⊕L^2[M, S]^- ⊗ℋ_F^-,
   {ξ[CG["arbitrary"]] ∈ ℋ^+,
    ξ → ψ_L⊗e + ψ_R⊗ē,
    ψ_L ∈ L^2[M, S]^+,
    ψ_R ∈ L^2[M, S]^-,
    ψ -> ψ_L + ψ_R,
    CG["⇒one Dirac spinor⇒too restrictive"]}
  }; $H // ColumnForms,
 NL, CO["Here OverBar → Conjugate"],
 line,
 NL, "• Solution is to Double space ", C^"∞"[M, ℂ^2] ↔ (N → M×X ≃ M⨆M),
 NL, "Let ", $se = {{e_R, e_L, ē_R, ē_L} → basis [ℋ_F → ℂ^4],
   γ_F.e_L → e_L, γ_F.e_R → -e_R, J_F.e_R → -ē_L, J_F.e_L → -ē_R, J_F.ē_L → -e_R, J_F.ē_R → -e_L,
   KOdim → 6, J_F.J_F → 1_F, J_F.γ_F → -γ_F.J_F}; $se // ColumnBar, accumDefEM[$se]
  NL, "Chirality ", $ =
  {J_F.γ_F.e_L → -γ_F.J_F.e_L, J_F.γ_F.e_R → -γ_F.J_F.e_R} // tuRepeat[Join[$se, tuOpSimplify[Dot]]];
 $ // ColumnBar, accumDefEM[$];
 Imply, $sgj = {γ_F → DiagonalMatrix[{-1, 1, 1, -1}],
    J_F → SparseArray[{Band[{1, 3}] → C, Band[{3, 1}] → C}, {4, 4}]} // Normal;
 $sgj // MatrixForms,
 NL, "•The elements ",
 $sa = {a ∈ (𝒜_F → ℂ^2), a[{e_R, e_L, ē_R, ē_L}] → DiagonalMatrix[{a_1, a_1, a_2, a_2}]};
 accumDefEM[{$sa, $sgj}]; MatrixForms[$sa]
]
PR["■Prop.4.5. ", F_ED → {ℂ^2, ℂ^4, 0, γ_F, J_F}, " is a real even finite space of KOdim→6."
]
```

■Two modifications of ACM $M\times F_X$ needed for E-M:

$\mathcal{D}_F$[non-zero]
$S_{fermion}$[action] ⇒ 2 independent spinors
S[action] → $\int -i\,\bar{\psi}.(-m+\gamma^{\mu}.\partial_{-\mu}[\_]).\psi\, dx^4$

•Let
{e, $\bar{e}$}[basis of $\mathcal{H}_F$]
e[basis of $\mathcal{H}_F{}^+$]
$\bar{e}$[basis of $\mathcal{H}_F{}^-$]
$J_F.e\to\bar{e}$
$J_F.\bar{e}\to e$
$\gamma_F.e\to e$
$\gamma_F.\bar{e}\to -\bar{e}$
⇒
$\mathcal{H}\to L^2[M, S]\otimes\mathcal{H}_F$
$L^2[M, S]\to L^2[M, S]^+\oplus L^2[M, S]^-$
$\mathcal{H}^+\to$ positiveEigenSpace of $\gamma\to\gamma_5\otimes\gamma_F$
$\mathcal{H}^+\to L^2[M, S]^+\otimes(\mathcal{H}_F)^+\oplus L^2[M, S]^-\otimes(\mathcal{H}_F)^-$
$\xi$[arbitrary] $\in\mathcal{H}^+$
$\xi\to\psi_L\otimes e+\psi_R\otimes\bar{e}$
$\psi_L\in L^2[M, S]^+$
$\psi_R\in L^2[M, S]^-$
$\psi\to\psi_L+\psi_R$
⇒one Dirac spinor⇒too restrictive

Here OverBar → Conjugate

---

• Solution is to Double space $C^\infty[M, \mathbb{C}^2]\leftrightarrow(N\to M\times X\simeq M\sqcup M)$

Let
$\{e_R, e_L, \bar{e}_R, \bar{e}_L\}\to$ basis$[\mathcal{H}_F\to\mathbb{C}^4]$
$\gamma_F.e_L\to e_L$
$\gamma_F.e_R\to -e_R$
$J_F.e_R\to -\bar{e}_L$
$J_F.e_L\to -\bar{e}_R$
$J_F.\bar{e}_L\to -e_R$
$J_F.\bar{e}_R\to -e_L$
KOdim → 6
$J_F.J_F\to 1_F$
$J_F.\gamma_F\to -\gamma_F.J_F$
{

{e, $\bar{e}$}[basis of $\mathcal{H}_F$],
e[basis of $\mathcal{H}_F{}^+$],
$\bar{e}$[basis of $\mathcal{H}_F{}^-$],
$(J_F.e\to\bar{e})$,
$(J_F.\bar{e}\to e)$,
$(\gamma_F.e\to e)$,
$(\gamma_F.\bar{e}\to -\bar{e})$,
$(\{e_R, e_L, \bar{e}_R, \bar{e}_L\}\to$ basis$[\mathcal{H}_F\to\mathbb{C}^4])$,
$(\gamma_F.e_L\to e_L)$,
$(\gamma_F.e_R\to -e_R)$,
$(J_F.e_R\to -\bar{e}_L)$,
$(J_F.e_L\to -\bar{e}_R)$,
$(J_F.\bar{e}_L\to -e_R)$,
$(J_F.\bar{e}_R\to -e_L)$,
(KOdim → 6),
$(J_F.J_F\to 1_F)$,
$(J_F.\gamma_F\to -\gamma_F.J_F)$}Chirality
$-\bar{e}_R\to\gamma_F.\bar{e}_R$
$\bar{e}_L\to\gamma_F.\bar{e}_L$

$$\Rightarrow \left\{\gamma_F\to\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\ J_F\to\begin{pmatrix}0&0&C&0\\0&0&0&C\\C&0&0&0\\0&C&0&0\end{pmatrix}\right\}$$

•The elements $\{a\in(\mathcal{A}_F\to\mathbb{C}^2),\ a[\{e_R, e_L, \bar{e}_R, \bar{e}_L\}]\to\begin{pmatrix}a_1&0&0&0\\0&a_1&0&0\\0&0&a_2&0\\0&0&0&a_2\end{pmatrix}\}$

■Prop.4.5. $F_{ED}\to\{\mathbb{C}^2, \mathbb{C}^4, 0, \gamma_F, J_F\}$ is a real even finite space of KOdim→6.

### 4.2.2 A non-trivial finite Dirac operator

```
$accum = {};
PR["■Determine non-trivial Dirac operator 𝒟F from constraints.",
```

```
next, "Hermitian condition: ", 𝒟F → ct[𝒟F],
"POFF",
NL, "General 𝒟F: ", $d = Table[di,j, {i, 4}, {j, 4}]; MatrixForms[$d],
NL, "ConjugateTranspose: ", $ct = ct[$d]; MatrixForms[$ct],
Yield, $ct = $d → $ct //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates;
$ct, CK, AppendTo[$accum, $ct]; "PONdd",
Yield, $ct = Select[$ct, ! OrderedQ[Apply[List, #[[1, 2 ;; 3]]]] &],

next,
$ = 𝒟F.γF → -γF.𝒟F, "POFF",
Yield, $d = 𝒟F → $d,
Yield, $ = $ /. $d /. $sgj; MatrixForms[$],
Yield, $ = $ //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates,
AppendTo[$accum, $];
Yield, $s = tuRuleSolve[$, Flatten[$d[[2]]]],
"PONdd",
Imply, $d0 = $d = $d /. $s /. $ct; MatrixForms[$d] // Framed,

next,
$ = 𝒟F.JF → JF.𝒟F, "POFF",
Yield,
$ = $ /. Dot → xDot /. $d /. $sgj // tuMatrixOrderedMultiply // tuOpSimplifyF[dotOps] //
   (# /. xDot → Dot &);
MatrixForms[$],
Yield, $ = $ /. C . d_ → Conjugate[d].C; MatrixForms[$],
Yield, $ = $ //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates;
Yield, $ = $ /. a_ .C → a // DeleteCases[#, a_ → a_] &, AppendTo[$accum, $];
Yield, $ =
 $ /. Rule → xRule /. aa : xRule[a_, b_] :→ Reverse[aa] /; FreeQ[a, 3 | 4] /. xRule → Rule //
   DeleteDuplicates,
"PONdd",
Imply, $d = $d /. $; MatrixForms[$d] // Framed,

next, "Order one condition: ",
$ord1 = tuRuleSelect[$c][CommutatorM[CommutatorM[_, _], _]][[1]],
NL, "• First compute: ",
$Da = $ = CommutatorM[𝒟F, a],
Yield, $ =
 $ /. $d /. (tuRuleSelect[$defEM][a[_]] /. a[_] → a) /. tuCommutatorExpand // Simplify;
$1 = $Da -> $; $1 // MatrixForms,
NL, "• Let: ", $s = {tuRuleSelect[$c][rghtA[b]] // DeleteDuplicates // First,
   b -> DiagonalMatrix[{b1, b1, b2, b2}]}, "POFF",
Yield, $ = $ord1 /. $1 /. $s /. $s /. Dot → xDot /. $sgj, CK,
Yield, $ = $ /. tuCommutatorExpand /. Dot → xDot,
$ = $ // tuMatrixOrderedMultiply // tuOpSimplifyF[dotOps] // (# /. xDot → Dot &);
"PONdd",
NL, "Let ", $s = { Dot[C , Shortest[e_]] :→ Dot[Conjugate[e], C] /; e =!= C,
   Conjugate[C] → C, C.C → 1};
$s // ColumnBar,
"POFF",
Yield, $ = $ // tuRepeat[$s, tuConjugateSimplify[]];
$ // MatrixForms,
"PONdd",
NL, "Determine dn,m for arbitrary a,b: ",
$ = $[[1]] // Flatten // DeleteCases[#, 0] &;
$ = # → 0 & /@ $,
NL, "Let ", $s = a2 → a12 + a1,
```

```
Yield, $ = $ /. $s //. tuOpSimplify[dotOps]; $ // Column,
Yield, $ = $ /. Dot → Times // Simplify; $ // ColumnBar,
NL, "Since the a,b's are arbitrary ",
Yield, $ = $ /. {a12 → 1, b1 - b2 → 1},
Imply, $e46 = $d = $d /. $;
accumDefEM[$d];
MatrixForms[$d] // Framed, CG[" (4.6)"]
]
```

■Determine non-trivial Dirac operator $\mathcal{D}_F$ from constraints.

◆Hermitian condition: $\mathcal{D}_F \to (\mathcal{D}_F)^\dagger$

.......

→ $\{d_{2,1} \to (d_{1,2})^*,\ d_{3,1} \to (d_{1,3})^*,\ d_{3,2} \to (d_{2,3})^*,\ d_{4,1} \to (d_{1,4})^*,\ d_{4,2} \to (d_{2,4})^*,\ d_{4,3} \to (d_{3,4})^*\}$

◆$\mathcal{D}_F . \gamma_F \to -\gamma_F . \mathcal{D}_F$

.......

⇒ $\boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d_{1,2} & d_{1,3} & 0 \\ (d_{1,2})^* & 0 & 0 & d_{2,4} \\ (d_{1,3})^* & 0 & 0 & d_{3,4} \\ 0 & (d_{2,4})^* & (d_{3,4})^* & 0 \end{pmatrix}}$

◆$\mathcal{D}_F . J_F \to J_F . \mathcal{D}_F$

.......

⇒ $\boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d_{1,2} & d_{1,3} & 0 \\ (d_{1,2})^* & 0 & 0 & d_{2,4} \\ (d_{1,3})^* & 0 & 0 & (d_{1,2})^* \\ 0 & (d_{2,4})^* & d_{1,2} & 0 \end{pmatrix}}$

◆Order one condition: $[[\mathcal{D}_F, a]_-, b^o]_- \to 0$

• First compute: $[\mathcal{D}_F, a]_-$

→ $[\mathcal{D}_F, a]_- \to \begin{pmatrix} 0 & 0 & (-a_1 + a_2)\, d_{1,3} & 0 \\ 0 & 0 & 0 & (-a_1 + a_2)\, d_{2,4} \\ (d_{1,3})^*\, (a_1 - a_2) & 0 & 0 & 0 \\ 0 & (d_{2,4})^*\, (a_1 - a_2) & 0 & 0 \end{pmatrix}$

• Let: $\{b^o \to J_F . b^\dagger . (J_F)^\dagger,\ b \to \{\{b_1, 0, 0, 0\}, \{0, b_1, 0, 0\}, \{0, 0, b_2, 0\}, \{0, 0, 0, b_2\}\}\}$

.......

Let
$C.\text{Shortest}[e\_] :\to e^*.C\ /;\ e =!= C$
$C^* \to C$
$C.C \to 1$

.......

Determine $d_{n,m}$ for arbitrary a,b:

$\{(-a_1 + a_2).d_{1,3}.b_1 - b_2.(-a_1 + a_2).d_{1,3} \to 0,\ (-a_1 + a_2).d_{2,4}.b_1 - b_2.(-a_1 + a_2).d_{2,4} \to 0,$
$(d_{1,3})^*.(a_1 - a_2).b_2 - b_1.(d_{1,3})^*.(a_1 - a_2) \to 0,\ (d_{2,4})^*.(a_1 - a_2).b_2 - b_1.(d_{2,4})^*.(a_1 - a_2) \to 0\}$

Let $a_2 \to a12 + a_1$

→
$a12.d_{1,3}.b_1 - b_2.a12.d_{1,3} \to 0$
$a12.d_{2,4}.b_1 - b_2.a12.d_{2,4} \to 0$
$-(d_{1,3})^*.a12.b_2 + b_1.(d_{1,3})^*.a12 \to 0$
$-(d_{2,4})^*.a12.b_2 + b_1.(d_{2,4})^*.a12 \to 0$

→
$a12\ (b_1 - b_2)\ d_{1,3} \to 0$
$a12\ (b_1 - b_2)\ d_{2,4} \to 0$
$a12\ (d_{1,3})^*\ (b_1 - b_2) \to 0$
$a12\ (d_{2,4})^*\ (b_1 - b_2) \to 0$

Since the a,b's are arbitrary

→ $\{d_{1,3} \to 0,\ d_{2,4} \to 0,\ (d_{1,3})^* \to 0,\ (d_{2,4})^* \to 0\}$

⇒ $\boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d_{1,2} & 0 & 0 \\ (d_{1,2})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & (d_{1,2})^* \\ 0 & 0 & d_{1,2} & 0 \end{pmatrix}}$ (4.6)

## 4.2.3 The almost commutative manifold

```
PR["●Then ", $ = tuRuleSelect[$defall][M×F_X][[1]]; $ // ColumnForms,
 " becomes ",
 $ = M×F_ED → {𝒜 -> C"∞"[M, ℂ^2], ℋ -> L^2[M, S]⊗ℂ^4,
    𝒟 -> slash[𝒟]⊗1_F + T[γ, "d", {5}]⊗𝒟_F, γ -> γ5⊗γ_F, J -> J_M⊗J_F};
 $ // ColumnForms, accumDefEM[$];
 NL, "Decompose ", $ = {𝒜 -> C"∞"[M, ℂ^2] -> C"∞"[M, ℂ]⊕C"∞"[M, ℂ],
   (ℋ -> L^2[M, S]⊗ℂ^4) -> L^2[M, S]⊗ℋ_e⊕L^2[M, S]⊗ℋ_e,
   a ∈ 𝒜 → $sa[[2]]
  }; $ // MatrixForms // ColumnBar, accumDefEM[$];
 NL, "Gauge group for 2-point space 𝒜_F (Prop.4.2): ", 𝒢[𝒜_F] ≃ U[1],
 Yield, $B = {T[B, "d", {μ}] → T[A, "d", {μ}] - J_F.T[A, "d", {μ}].ct[J_F], T[B, "d", {μ}] →
    DiagonalMatrix[{T[Y, "d", {μ}], T[Y, "d", {μ}], -T[Y, "d", {μ}], -T[Y, "d", {μ}]}],
   T[Y, "d", {μ}][x] ∈ ℝ};
 MatrixForms[$B] // ColumnBar, accumDefEM[$B]; ""
]
```

●Then $M\times F_X\rightarrow$ $\left|\begin{array}{l}\mathcal{A}\rightarrow C^{\infty}[M,\mathbb{C}^2]\\ \mathcal{H}\rightarrow L^2[M,S]\otimes\mathbb{C}^2\\ \mathcal{D}\rightarrow(\not{D})\otimes 1_F\\ \gamma\rightarrow\gamma_5\otimes\gamma_F\\ J\rightarrow J_M\otimes J_F\end{array}\right.$ becomes $M\times F_{ED}\rightarrow$ $\left|\begin{array}{l}\mathcal{A}\rightarrow C^{\infty}[M,\mathbb{C}^2]\\ \mathcal{H}\rightarrow L^2[M,S]\otimes\mathbb{C}^4\\ \mathcal{D}\rightarrow(\not{D})\otimes 1_F+\mathrm{Tensor}[\gamma,\ |\mathrm{Void}\ ,\ |5]\otimes\mathcal{D}_F\\ \gamma\rightarrow\gamma_5\otimes\gamma_F\\ J\rightarrow J_M\otimes J_F\end{array}\right.$

Decompose $\left|\begin{array}{l}\mathcal{A}\rightarrow C^{\infty}[M,\mathbb{C}^2]\rightarrow C^{\infty}[M,\mathbb{C}]\oplus C^{\infty}[M,\mathbb{C}]\\ (\mathcal{H}\rightarrow L^2[M,S]\otimes\mathbb{C}^4)\rightarrow L^2[M,S]\otimes\mathcal{H}_e\oplus L^2[M,S]\otimes\mathcal{H}_{\bar{e}}\\ a\in\mathcal{A}\rightarrow a[\{e_R,e_L,\bar{e}_R,\bar{e}_L\}]\rightarrow\begin{pmatrix}a_1&0&0&0\\0&a_1&0&0\\0&0&a_2&0\\0&0&0&a_2\end{pmatrix}\end{array}\right.$

Gauge group for 2-point space $\mathcal{A}_F$ (Prop.4.2): $\mathcal{G}[\mathcal{A}_F]\simeq U[1]$

$\rightarrow\left|\begin{array}{l}B_\mu\rightarrow -J_F.A_\mu.(J_F)^\dagger+A_\mu\\ B_\mu\rightarrow\begin{pmatrix}Y_\mu&0&0&0\\0&Y_\mu&0&0\\0&0&-Y_\mu&0\\0&0&0&-Y_\mu\end{pmatrix}\\ Y_\mu[x]\in\mathbb{R}\end{array}\right.$

## 4.2.4 The Lagrangian

● The spectral action

```
PR["●Prop. 4.6: The spectral action of ", tuRuleSelect[$defEM][M×F_ED],
  Yield,
  $p46 = $ = {Tr[f[𝒟_A / Λ]] → xIntegral[ℒ[T[g, "dd", {μ, ν}], T[Y, "d", {μ}]] √Det[g], x^4],
     ℒ[T[g, "dd", {μ, ν}], T[Y, "d", {μ}]] →
      4 ℒ_M[T[g, "dd", {μ, ν}]] + ℒ_Y[T[Y, "d", {μ}]] + ℒ_H[T[g, "dd", {μ, ν}], d],
     $p35[[2]],
     ℒ_Y[T[Y, "d", {μ}]] → f[0] / (6 π^2) T[ℱ, "dd", {μ, ν}] T[ℱ, "uu", {μ, ν}],
     T[ℱ, "dd", {μ, ν}] → tuDPartial[T[Y, "d", {ν}], μ] - tuDPartial[T[Y, "d", {μ}], ν],
     ℒ_H[T[g, "dd", {μ, ν}], d] →
      2 f_2 Λ^2 / π^2 Abs[d]^2 + f[0] / (2 π^2) Abs[d]^4 + f[0] / (12 π^2) s Abs[d]^2
    }; $ // ColumnSumExp // ColumnBar, accumDefEM[$]; ""
 ];
PR["● Proof: From Prop.3.7: ", $ = $p37; $ // ColumnBar,
 line,
 "Evaluate each part letting: ",
 $sPhi = { Φ → 𝒟_F, N → dim[ℋ_F], dim[ℋ_F] → 4, Tr[1_ℋ_F] → N, $B[[1]], $e46};
 MatrixForms[$sPhi],
 Yield, $ = #[[1]] → (#[[2]] /. $sPhi) & /@ $p37[[{2, 3, 5, 7}]];
 ColumnBar[$0 = $];

 line,
 next, "The term ", tuRuleSelect[$p37][ℒ_M[_]][[1]] // Framed, " is (3.19).",

 next, "Evaluate the term ", $0 = tuRuleSelect[$p37][ℒ_B[B_μ]] // First,
 NL, "where ", $ = $F,
 NL, "Using ",
 $s = (tuRuleSelect[$B][T[B, "d", {μ}]][[2]] // tuAddPatternVariable[μ]), "POFF",
 Yield, $ = $ /. $s /. Plus → Inactive[Plus] //. tt : tuDPartial[a_, b_] ⧴ Thread[tt] //.
     tuDExpand[DerivOps] /. tuCommutatorExpand // Activate,
 $u = $ // tuIndicesRaise[{μ, ν}], "PONdd",
 $ = Thread[$ . $u, Rule] // Simplify;
 Yield, $ = Tr[#] & /@ $; $,
 NL, "Defining ",
 $s = {$$ = tuRuleSelect[$p46][T[ℱ, "dd", {μ, ν}]][[1]], tuIndicesRaise[{μ, ν}][$$]},
 Imply, $s = $ /. Reverse /@ (-# & /@ # & /@ $s) /. Dot → Times,
 Imply, $ = $0 /. $s; Framed[$],

 next,
 "Evaluate term ", $ = $0 = tuRuleSelect[$p37][ℒ_H[__]] // First,
 Yield, $[[2]] = $[[2]] /. $sPhi; MatrixForms[$],
 NL, "Evaluate Tr[]'s (switch )", $s = d_1,2 → d, accumDefEM[$s];
 $1 = $ // tuExtractPositionPattern[Tr[_]];
 $1 =
  $1 /. $e46 /. $s //. tt : T[𝒟, "d", {μ}][_] | T[𝒟, "u", {μ}][_] ⧴ Thread[tt] /. a_[0] → 0,
 Yield, $ = tuReplacePart[$, $1]; Framed[$]
]
```

●Prop. 4.6: The spectral action of
$\{M\times F_{ED}\to\{\mathcal{A}\to C^{\infty}[M,\mathbb{C}^2],\ \mathcal{H}\to L^2[M,S]\otimes\mathbb{C}^4,\ \mathcal{D}\to(\not{D})\otimes 1_F+\gamma_5\otimes\mathcal{D}_F,\ \gamma\to\gamma_5\otimes\gamma_F,\ J\to J_M\otimes J_F\}\}$

→

$$\mathrm{Tr}[f[\tfrac{\mathcal{D}_A}{\Lambda}]]\to\int\sqrt{\mathrm{Det}[g]}\ \mathcal{L}[g_{\mu\nu},Y_\mu]\,dx^4$$

$$\mathcal{L}[g_{\mu\nu},Y_\mu]\to\sum\left[\begin{array}{l}\mathcal{L}_H[g_{\mu\nu},d]\\ 4\,\mathcal{L}_M[g_{\mu\nu}]\\ \mathcal{L}_Y[Y_\mu]\end{array}\right]$$

$$\mathcal{L}_M[g_{\mu\nu}]\to\sum\left[\begin{array}{l}-\frac{\Lambda^2 f_2}{24\pi^2}\\ \frac{\Lambda^4 f_4}{2\pi^2}\\ \frac{f[0]\left(\frac{11 R^*.R^*}{360}-\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+\frac{\Delta[s]}{30}\right)}{16\pi^2}\end{array}\right]$$

$$\mathcal{L}_Y[Y_\mu]\to\frac{f[0]\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}}{6\pi^2}$$

$$\mathcal{F}_{\mu\nu}\to\sum\left[\begin{array}{l}-\partial_{-\nu}[Y_\mu]\\ \partial_{-\mu}[Y_\nu]\end{array}\right]$$

$$\mathcal{L}_H[g_{\mu\nu},d]\to\sum\left[\begin{array}{l}\frac{s\,\mathrm{Abs}[d]^2 f[0]}{12\pi^2}\\ \frac{\mathrm{Abs}[d]^4 f[0]}{2\pi^2}\\ \frac{2\Lambda^2\mathrm{Abs}[d]^2 f_2}{\pi^2}\end{array}\right]$$

● Proof: From Prop.3.7:

$\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \to \int_{x\in M} \sqrt{\mathrm{Det}[g[x]]}\ \mathcal{L}[g_{\mu\nu}, B_\mu, \Phi]$

$\mathcal{L}[g_{\mu\nu}, B_\mu, \Phi] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + N\,\mathcal{L}_M[g_{\mu\nu}]$

$\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 s[x] f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} - \frac{f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x]}{320\pi^2}$

$N \to \dim[\mathcal{H}_F]$

$\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$

$\mathcal{L}_B[B_\mu] \to$ Kinetic term gauge fields

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2\, \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\,\mathrm{Tr}[\mathcal{D}_\mu[\Phi].\mathcal{D}^\mu[\Phi]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$ Higgs lagrangian

$N \to \mathrm{Tr}[1_{\mathcal{H}_F}]$

---

Evaluate each part letting: $\{\Phi \to \mathcal{D}_F,\ N \to \dim[\mathcal{H}_F],\ \dim[\mathcal{H}_F] \to 4,$

$\mathrm{Tr}[1_{\mathcal{H}_F}] \to N,\ B_\mu \to -J_F.A_\mu.(J_F)^\dagger + A_\mu,\ \mathcal{D}_F \to \begin{pmatrix} 0 & d_{1,2} & 0 & 0 \\ (d_{1,2})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & (d_{1,2})^* \\ 0 & 0 & d_{1,2} & 0 \end{pmatrix}\}$

→

---

◆The term $\boxed{\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 s[x] f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} - \frac{f[0]\, C_{\mu\nu\rho\sigma}[x]\, C^{\mu\nu\rho\sigma}[x]}{320\pi^2}}$ is (3.19).

◆Evaluate the term $\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$

where $F_{\mu\nu} \to i\,[B_\mu, B_\nu]_- - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

Using $B_{\mu\_} \to \{\{Y_\mu, 0, 0, 0\}, \{0, Y_\mu, 0, 0\}, \{0, 0, -Y_\mu, 0\}, \{0, 0, 0, -Y_\mu\}\}$

.......

→ $\mathrm{Tr}[F_{\mu\nu}.F^{\mu\nu}] \to 4\,(\underline{\partial}_\nu[Y_\mu] - \underline{\partial}_\mu[Y_\nu])\,(\underline{\partial}^\nu[Y^\mu] - \underline{\partial}^\mu[Y^\nu])$

Defining $\{\mathcal{F}_{\mu\nu} \to -\underline{\partial}_\nu[Y_\mu] + \underline{\partial}_\mu[Y_\nu],\ \mathcal{F}^{\mu\nu} \to -\underline{\partial}^\nu[Y^\mu] + \underline{\partial}^\mu[Y^\nu]\}$

⇒ $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 4\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$

⇒ $\boxed{\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}}{6\pi^2}}$

◆Evaluate term $\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$

$\frac{f[0]\, s[x]\, \mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2\, \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\,\mathrm{Tr}[\mathcal{D}_\mu[\Phi].\mathcal{D}^\mu[\Phi]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$

→ $\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]}{48\pi^2} - \frac{\Lambda^2 f_2\, \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]}{2\pi^2} +$

$\frac{f[0]\,\mathrm{Tr}[\mathcal{D}_\mu[\mathcal{D}_F].\mathcal{D}^\mu[\mathcal{D}_F]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F.\mathcal{D}_F.\mathcal{D}_F]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]]}{24\pi^2}$

Evaluate Tr[]'s (switch )$d_{1,2} \to d\{\{2, 1, 5\} \to 4\,d\,d^*,\ \{2, 2, 5\} \to 4\,d\,d^*,$
$\{2, 3, 4\} \to 2\,\mathcal{D}_\mu[d]\,\mathcal{D}^\mu[d] + 2\,\mathcal{D}_\mu[d^*]\,\mathcal{D}^\mu[d^*],\ \{2, 4, 4\} \to 4\,d^2\,d^{*2},\ \{2, 5, 4, 1\} \to 4\,d\,d^*\}$

→ $\boxed{\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{d^2\, d^{*2}\, f[0]}{2\pi^2} + \frac{d\, d^*\, f[0]\, s[x]}{12\pi^2} - \frac{2\, d\, \Lambda^2\, d^*\, f_2}{\pi^2} + \frac{f[0]\,\triangle[4\, d\, d^*]}{24\pi^2} + \frac{f[0]\,(2\,\mathcal{D}_\mu[d]\,\mathcal{D}^\mu[d] + 2\,\mathcal{D}_\mu[d^*]\,\mathcal{D}^\mu[d^*])}{8\pi^2}}$

### 4.2.5 Fermionic action

```
PR["The basis vectors for ℋF: ",
 $ = Select[$defEM, MatchQ[#, _ -> basis[_]] &][[1]], $basis = $[[1]];
 Yield, $H[[4]],
```

```
NL, "Spanning basis ", {𝓗F+[{eL, ēR}], 𝓗F-[{eR, ēL}]},
NL, "Arbitrary vector ",
$sξ = {ξ -> χR ⊗ eR + χL ⊗ eL + ψL ⊗ ēR + ψR ⊗ ēL, {χL, ψL} ∈ L2[M, S]+, {χR, ψR} ∈ L2[M, S]-};

$sξ // ColumnBar,
NL, "● Prop.4.7: The fermionic action for ", tuRuleSelect[$defEM][M×_],
$Sf = $ = Sf → -I BraKet[JM.χ̃, T[γ, "u", {μ}].(T["∇"S, "d", {μ}] - I T[Y, "d", {μ}]).ψ̃] +
    BraKet[JM.χ̃L, ct[d].ψ̃L] - BraKet[JM.χ̃R, d.ψ̃R];
Framed[$], CO["Prop.4.7"],
NL, "where the ~ means ", $sAt,
(****)
line,
NL, "■Proof: Compute: ", $00 = $d217[[3]], CG[" Definition 2.17"],
next, "Determine: The fluctuated Dirac operator ",
Yield,
$sDA1 = $ = $sDA[[1]] /. $sDA[[2]] /. N → M /. tuRuleSelect[$sPhi][Φ] // expandDC[],
"POFF", (*M?*)
Yield, $ = $ /. tuOpDistribute[Dot] //. tuOpSimplify[Dot] // Expand,
Yield, $sDA1 = $ = $ //. {a_ . (b_ ⊗ c_) → (a.b) ⊗ c, a_ . 1_ → a}, CK, "PON",
NL, "Since ", $s = $slashD[[1]] /. a_ tuDDown[tt : _][_, i_] ⧴ a .
     T[tt, "d", {i}] //. tuOpSimplify[Dot],
$slashd = $s = tuRuleSolve[$s, Dot[_, _]];
yield, $ = $ /. Reverse[$s] // expandDC[];
Framed[ColumnSumExp[$sDA0 = $]], CO["p.48"],

line,
NL, "▪Using ",
$sem = tuRuleSelect[$defEM][{𝒟F, d1,2, T[B, "d", {μ}]}] // Select[#, FreeQ[#, A] &] &,
Yield, $s1 = 𝒟F . # & /@ $basis;
$s2 = 𝒟F.Transpose[{$basis}] /. $sem // Transpose // First;
$sd = Thread[$s1 → $s2];
Yield, $s1 = T[B, "d", {μ}].# & /@ $basis;
$s2 = T[B, "d", {μ}].Transpose[{$basis}] /. $sem // Transpose // First;
$sb = Thread[$s1 → $s2];
NL, "Get Combined Rule[]s: ",
$s0J = {tuRuleSelect[$defEM][JF.(eL | eR | ēL | ēR)], $sd, $sb} // Flatten;
$s0J // ColumnBar,
(**)
$accum = {};
NL, "Compute ",
NL, "•", $ = J.ξ;
yield, $ = $ → ($ /. $sξ[[1]] /. tuRuleSelect[$defEM][J] //. tuOpDistribute[Dot] //.
      $combineProduct /. $s0J // expandDC[]);
AppendTo[$accum, $];
Framed[$],
NL, "•", $0 = $ = $sDA0[[2, 1]].ξ;
yield, $ = $ → ($ /. $sξ[[1]] /. tuRuleSelect[$defEM][J] //. tuOpDistribute[Dot] //.
      $combineProduct /. $s0J // expandDC[]);
AppendTo[$accum, $]; Framed[$],
NL, "•", $ = $sDA0[[2, 2]].ξ;
yield,
$ = $ → ($ /. $sξ[[1]] /. tuRuleSelect[$defEM][J] //. tuOpDistribute[Dot] //. $sX /.
    $s0J // expandDC[]);
AppendTo[$accum, $]; Framed[$],
NL, "•", $ = $sDA0[[2, 3]].ξ;
yield,
```

```
 $ = $ → ($ /. $sξ[[1]] /. tuRuleSelect[$defEM][J] //. tuOpDistribute[Dot] //. $sX /.
      $s0J // expandDC[]);
 AppendTo[$accum, $]; Framed[$]
]
```

The basis vectors for $\mathcal{H}_F$: $\{e_R, e_L, \bar{e}_R, \bar{e}_L\} \to \mathrm{basis}[\mathcal{H}_F \to \mathbb{C}^4]$

→ $\mathcal{H}^+ \to L^2[M, S]^+ \otimes (\mathcal{H}_F)^+ \oplus L^2[M, S]^- \otimes (\mathcal{H}_F)^-$

Spanning basis $\{(\mathcal{H}_F)^+[\{e_L, \bar{e}_R\}], (\mathcal{H}_F)^-[\{e_R, \bar{e}_L\}]\}$

Arbitrary vector $\left|\begin{array}{l} \xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \bar{e}_R + \psi_R \otimes \bar{e}_L \\ \{\chi_L, \psi_L\} \in L^2[M, S]^+ \\ \{\chi_R, \psi_R\} \in L^2[M, S]^- \end{array}\right.$

● Prop.4.7: The fermionic action for

$\{M \times F_{ED} \to \{\mathcal{A} \to C^\infty[M, \mathbb{C}^2], \mathcal{H} \to L^2[M, S] \otimes \mathbb{C}^4, \mathcal{D} \to (\not{D}) \otimes 1_F + \gamma_5 \otimes \mathcal{D}_F, \gamma \to \gamma_5 \otimes \gamma_F, J \to J_M \otimes J_F\}\}$

$\boxed{S_f \to -i \langle J_M.\tilde{\chi} \mid \gamma^\mu.(\nabla^S{}_\mu - i Y_\mu).\tilde{\psi}\rangle + \langle J_M.\tilde{\chi}_L \mid d^\dagger.\tilde{\psi}_L\rangle - \langle J_M.\tilde{\chi}_R \mid d.\tilde{\psi}_R\rangle}$ Prop.4.7

where the ~ means $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A}, a.J \to J.a^\dagger, a^o \to a\}$

---

■Proof: Compute: $S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$ Definition 2.17

◆Determine: The fluctuated Dirac operator

→ $\mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \mathcal{D}_F - i\,(i\,\gamma^\mu.(1_M \otimes B_\mu) + \gamma^\mu.(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}))$

Since $\not{D} \to -i\,\gamma^\mu.\nabla^S{}_\mu$ ⟶ $\boxed{\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} (\not{D}) \otimes 1_{\mathcal{H}_F} \\ \gamma_5 \otimes \mathcal{D}_F \\ \gamma^\mu \otimes B_\mu \end{array}\right]}$ p.48

---

■Using $\{\mathcal{D}_F \to \{\{0, d_{1,2}, 0, 0\}, \{(d_{1,2})^*, 0, 0, 0\}, \{0, 0, 0, (d_{1,2})^*\}, \{0, 0, d_{1,2}, 0\}\},$
$d_{1,2} \to d, B_\mu \to \{\{Y_\mu, 0, 0, 0\}, \{0, Y_\mu, 0, 0\}, \{0, 0, -Y_\mu, 0\}, \{0, 0, 0, -Y_\mu\}\}\}$

→

→

Get Combined Rule[]s: $\left|\begin{array}{l} J_F.e_R \to -\bar{e}_L \\ J_F.e_L \to -\bar{e}_R \\ J_F.\bar{e}_L \to -e_R \\ J_F.\bar{e}_R \to -e_L \\ \mathcal{D}_F.e_R \to e_L\, d_{1,2} \\ \mathcal{D}_F.e_L \to (d_{1,2})^*\, e_R \\ \mathcal{D}_F.\bar{e}_R \to (d_{1,2})^*\, \bar{e}_L \\ \mathcal{D}_F.\bar{e}_L \to \bar{e}_R\, d_{1,2} \\ B_\mu.e_R \to e_R\, Y_\mu \\ B_\mu.e_L \to e_L\, Y_\mu \\ B_\mu.\bar{e}_R \to -\bar{e}_R\, Y_\mu \\ B_\mu.\bar{e}_L \to -\bar{e}_L\, Y_\mu \end{array}\right.$

Compute

- ⟶ $\boxed{J.\xi \to -(J_M.\chi_L \otimes \bar{e}_R) - J_M.\chi_R \otimes \bar{e}_L - J_M.\psi_L \otimes e_L - J_M.\psi_R \otimes e_R}$
- ⟶ $\boxed{((\not{D}) \otimes 1_{\mathcal{H}_F}).\xi \to (\not{D}).\chi_L \otimes e_L + (\not{D}).\chi_R \otimes e_R + (\not{D}).\psi_L \otimes \bar{e}_R + (\not{D}).\psi_R \otimes \bar{e}_L}$
- ⟶ $\boxed{(\gamma_5 \otimes \mathcal{D}_F).\xi \to \gamma_5.\chi_L \otimes ((d_{1,2})^*\, e_R) + \gamma_5.\chi_R \otimes (e_L\, d_{1,2}) + \gamma_5.\psi_L \otimes ((d_{1,2})^*\, \bar{e}_L) + \gamma_5.\psi_R \otimes (\bar{e}_R\, d_{1,2})}$
- ⟶ $\boxed{(\gamma^\mu \otimes B_\mu).\xi \to \gamma^\mu.\chi_L \otimes (e_L\, Y_\mu) + \gamma^\mu.\chi_R \otimes (e_R\, Y_\mu) - \gamma^\mu.\psi_L \otimes (\bar{e}_R\, Y_\mu) - \gamma^\mu.\psi_R \otimes (\bar{e}_L\, Y_\mu)}$

```
PR["Substitute these terms into: ", $ = $00,
 Yield, $s = #.ξ̃ & /@ $sDA0 // expandDC[]; $s // ColumnSumExp,
 Yield, $ = $ /. $s /. ξ̃ → ξ /. $accum; $ // ColumnSumExp,
 NL, "Expand BraKet: ",
 Yield, $ = $ //. tuBraKetSimplify[];
 Yield, $ = $ //. BraKet[a_ ⊗ b_, c_ ⊗ d_] -> BraKet[a, c] ⊗ BraKet[b, d];
 Yield, $ = $ //. tuBraKetSimplify[{d1,2, Conjugate[d1,2], T[Y, "d", {_}]}] // Expand;

 NL, "Impose e orthogonality Using ",
 $s = {BraKet[a_ , a_] b_ : 1 → b, bb__ ⊗ (BraKet[a_, b_] y_ : 1) ⧴ 0 /; ! a === b},
 Yield, $ = $ /. $s /. CircleTimes → Times; $ // ColumnSumExp,

 NL, "Order χ,ψ (Y terms are symmetric, 𝒟 terms are antisymmetric) Using ",
 $s = {HoldPattern[(Times[cc___, BraKet[aa_ . ψa_, bb_ . χb_]])] ⧴
    - BraKet[aa . χb , bb . ψa] cc /; (! FreeQ[cc , Y]) || (! FreeQ[bb, T[γ, "d", {5}]]),
   HoldPattern[(Times[cc___, BraKet[aa_ . ψa_, bb_ . χb_]])] ⧴
    BraKet[aa . χb , bb . ψa] cc /; FreeQ[cc , Y]},
 Yield, $ = $ /. $s; $ // ColumnSumExp,

 NL, "If γ5 changes chirality: ", $s = {T[γ, "d", {5}].a_r_ ⧴ a<|R→L,L→R|>[r]},
 Yield, $ = $ /. $s; $ // ColumnSumExp,
 NL, "Collect the d terms: ", $s = Apply[Plus, tuTermSelect[d][$]];
 Yield, $[[2]] = $[[2]] - $s + ($s // Simplify); $ // ColumnSumExp,
 NL, "Let ", $s = {d1,2 → - I m},
 Yield, $ = $ /. $s // tuConjugateSimplify[{m}] // Simplify;
 $ // ColumnSumExp // Framed,
 CR["The mass terms not the same as text."]
]
```

Substitute these terms into: $S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$

→ $\mathcal{D}_{\mathcal{A}}.\tilde{\xi} \to \sum[\;\begin{array}{|l} ((\not{D})\otimes 1_{\mathcal{H}_F}).\tilde{\xi} \\ (\gamma_5\otimes\mathcal{D}_F).\tilde{\xi} \\ (\gamma^\mu\otimes B_\mu).\tilde{\xi} \end{array}\;]$

→ $S_f \to \frac{1}{2}\left\langle \sum[\;\begin{array}{|l} -(J_M.\chi_L\otimes \bar{e}_R) \\ -(J_M.\chi_R\otimes \bar{e}_L) \\ -(J_M.\psi_L\otimes e_L) \\ -(J_M.\psi_R\otimes e_R) \end{array}\;] \;\middle|\; \sum[\;\begin{array}{|l} (\not{D}).\chi_L\otimes e_L \\ (\not{D}).\chi_R\otimes e_R \\ (\not{D}).\psi_L\otimes \bar{e}_R \\ (\not{D}).\psi_R\otimes \bar{e}_L \\ \gamma_5.\chi_L\otimes((d_{1,2})^* e_R) \\ \gamma_5.\chi_R\otimes(e_L\, d_{1,2}) \\ \gamma_5.\psi_L\otimes((d_{1,2})^* \bar{e}_L) \\ \gamma_5.\psi_R\otimes(\bar{e}_R\, d_{1,2}) \\ \gamma^\mu.\chi_L\otimes(e_L\, Y_\mu) \\ \gamma^\mu.\chi_R\otimes(e_R\, Y_\mu) \\ -(\gamma^\mu.\psi_L\otimes(\bar{e}_R\, Y_\mu)) \\ -(\gamma^\mu.\psi_R\otimes(\bar{e}_L\, Y_\mu)) \end{array}\;]\right\rangle$

Expand BraKet:
→
→
→
Impose e orthogonality Using
$\{\langle a\_ \mid a\_\rangle\,(b\_ : 1) \to b,\; bb\_\_ \otimes (\langle a\_ \mid b\_\rangle\,(y\_ : 1)) \mapsto 0\;/;\; !\, a === b\}$

$$\to S_f \to \sum\left[\begin{array}{l} -\frac{1}{2}\langle J_M.\chi_L \mid (\not{D}).\psi_L\rangle \\ -\frac{1}{2}\langle J_M.\chi_R \mid (\not{D}).\psi_R\rangle \\ -\frac{1}{2}\langle J_M.\psi_L \mid (\not{D}).\chi_L\rangle \\ -\frac{1}{2}\langle J_M.\psi_R \mid (\not{D}).\chi_R\rangle \\ -\frac{1}{2}\langle J_M.\chi_R \mid \gamma_5.\psi_L\rangle (d_{1,2})^* \\ -\frac{1}{2}\langle J_M.\psi_R \mid \gamma_5.\chi_L\rangle (d_{1,2})^* \\ -\frac{1}{2}\langle J_M.\chi_L \mid \gamma_5.\psi_R\rangle d_{1,2} \\ -\frac{1}{2}\langle J_M.\psi_L \mid \gamma_5.\chi_R\rangle d_{1,2} \\ \frac{1}{2}\langle J_M.\chi_L \mid \gamma^\mu.\psi_L\rangle Y_\mu \\ \frac{1}{2}\langle J_M.\chi_R \mid \gamma^\mu.\psi_R\rangle Y_\mu \\ -\frac{1}{2}\langle J_M.\psi_L \mid \gamma^\mu.\chi_L\rangle Y_\mu \\ -\frac{1}{2}\langle J_M.\psi_R \mid \gamma^\mu.\chi_R\rangle Y_\mu \end{array}\right]$$

Order $\chi,\psi$ (Y terms are symmetric, $\mathcal{D}$ terms are antisymmetric) Using

{HoldPattern[cc___ $\langle$(aa_).$\psi_{a\_}$ | (bb_).$\chi_{b\_}\rangle$] :→
 −$\langle$aa.$\chi_b$ | bb.$\psi_a\rangle$ cc /; ! FreeQ[cc, Y] || ! FreeQ[bb, T[$\gamma$, d, {5}]],
 HoldPattern[cc___ $\langle$(aa_).$\psi_{a\_}$ | (bb_).$\chi_{b\_}\rangle$] :→ $\langle$aa.$\chi_b$ | bb.$\psi_a\rangle$ cc /; FreeQ[cc, Y]}

$$\to S_f \to \sum\left[\begin{array}{l} -\langle J_M.\chi_L \mid (\not{D}).\psi_L\rangle \\ -\langle J_M.\chi_R \mid (\not{D}).\psi_R\rangle \\ \frac{1}{2}\langle J_M.\chi_L \mid \gamma_5.\psi_R\rangle (d_{1,2})^* \\ -\frac{1}{2}\langle J_M.\chi_R \mid \gamma_5.\psi_L\rangle (d_{1,2})^* \\ -\frac{1}{2}\langle J_M.\chi_L \mid \gamma_5.\psi_R\rangle d_{1,2} \\ \frac{1}{2}\langle J_M.\chi_R \mid \gamma_5.\psi_L\rangle d_{1,2} \\ \langle J_M.\chi_L \mid \gamma^\mu.\psi_L\rangle Y_\mu \\ \langle J_M.\chi_R \mid \gamma^\mu.\psi_R\rangle Y_\mu \end{array}\right]$$

If $\gamma_5$ changes chirality: $\{\gamma_5.a\_r\_ :\to a_{\text{Association}[R\to L, L\to R][r]}\}$

$$\to S_f \to \sum\left[\begin{array}{l} -\langle J_M.\chi_L \mid (\not{D}).\psi_L\rangle \\ -\langle J_M.\chi_R \mid (\not{D}).\psi_R\rangle \\ \frac{1}{2}\langle J_M.\chi_L \mid \psi_L\rangle (d_{1,2})^* \\ -\frac{1}{2}\langle J_M.\chi_R \mid \psi_R\rangle (d_{1,2})^* \\ -\frac{1}{2}\langle J_M.\chi_L \mid \psi_L\rangle d_{1,2} \\ \frac{1}{2}\langle J_M.\chi_R \mid \psi_R\rangle d_{1,2} \\ \langle J_M.\chi_L \mid \gamma^\mu.\psi_L\rangle Y_\mu \\ \langle J_M.\chi_R \mid \gamma^\mu.\psi_R\rangle Y_\mu \end{array}\right]$$

Collect the d terms:

$$\to S_f \to \sum\left[\begin{array}{l} -\langle J_M.\chi_L \mid (\not{D}).\psi_L\rangle \\ -\langle J_M.\chi_R \mid (\not{D}).\psi_R\rangle \\ \frac{1}{2}(\langle J_M.\chi_L \mid \psi_L\rangle - \langle J_M.\chi_R \mid \psi_R\rangle)((d_{1,2})^* - d_{1,2}) \\ \langle J_M.\chi_L \mid \gamma^\mu.\psi_L\rangle Y_\mu \\ \langle J_M.\chi_R \mid \gamma^\mu.\psi_R\rangle Y_\mu \end{array}\right]$$

Let $\{d_{1,2} \to -i\, m\}$

$$\to \boxed{S_f \to \sum\left[\begin{array}{l} -\langle J_M.\chi_L \mid (\not{D}).\psi_L\rangle \\ i\, m \langle J_M.\chi_L \mid \psi_L\rangle \\ -\langle J_M.\chi_R \mid (\not{D}).\psi_R\rangle \\ -i\, m \langle J_M.\chi_R \mid \psi_R\rangle \\ \langle J_M.\chi_L \mid \gamma^\mu.\psi_L\rangle\, Y_\mu \\ \langle J_M.\chi_R \mid \gamma^\mu.\psi_R\rangle\, Y_\mu \end{array}\right]}$$ The mass terms not the same as text.

```
PR["●Theorem 4.9.  For ", tuRuleSelect[$defEM][M×F_ED] // Last,
 NL, "the full Lagrangian is: ",
 ℒgrav[T[g, "dd", {μ, ν}]] → 4 ℒM[T[g, "dd", {μ, ν}]] + ℒH[T[g, "dd", {μ, ν}]],
 CG[" Prop.4.6"],
 NL, "plus the E-M Lagrangian ",
 ℒEM[T[g, "dd", {μ, ν}]] →
  - I BraKet[JM.χ̃, (T[γ, "u", {μ}].(("∇")_μ - I T[Y, "d", {μ}] ) - m).ψ̃]_ℒ +
   f[0]/(6 π^2) T[ℱ, "dd", {μ, ν}] T[ℱ, "uu", {μ, ν}], CG[" Prop.4.7"],
 NL, "define ", BraKet[ξ, ψ] → xIntegral[√Abs[det[g]] BraKet[ξ, ψ]_ℒ, x ∈ M],
 NL, "to get theorem."
]
```

●Theorem 4.9. For
$M\times F_{ED} \to \{\mathcal{A}\to C^\infty[M, \mathbb{C}^2], \mathcal{H}\to L^2[M, S]\otimes\mathbb{C}^4, \mathcal{D}\to(\not{D})\otimes 1_F+\gamma_5\otimes\mathcal{D}_F, \gamma\to\gamma_5\otimes\gamma_F, J\to J_M\otimes J_F\}$

the full Lagrangian is: $\mathcal{L}_{grav}[g_{\mu\nu}]\to\mathcal{L}_H[g_{\mu\nu}]+4\,\mathcal{L}_M[g_{\mu\nu}]$ Prop.4.6

plus the E-M Lagrangian

$$\mathcal{L}_{EM}[g_{\mu\nu}]\to -i\,\langle J_M.\tilde{\chi} \mid (-m+\gamma^\mu.(\nabla_\mu - i\,Y_\mu)).\tilde{\psi}\rangle_{\mathcal{L}} + \frac{f[0]\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}}{6\,\pi^2}$$ Prop.4.7

define $\langle\xi \mid \psi\rangle \to \int_{x\in M}\sqrt{\mathrm{Abs}[\det[g]]}\,\langle\xi \mid \psi\rangle_{\mathcal{L}}$

to get theorem.

## 4.2.6 Fermionic degrees of freedom

```
PR["Grassmann variable definition: ",
 $grassmann = $ = { T[θ, "d", {i}],
    T[θ, "d", {i}] .T[θ, "d", {j}] → -T[θ, "d", {j}].T[θ, "d", {i}],
    xIntegral[1, T[θ, "d", {i}]] → 0,
    xIntegral[T[θ, "d", {i}], T[θ, "d", {i}]] → 1,
    iD[θ] -> xProduct[d[T[θ, "d", {i}]], {i, dim[F]}],
    iD[η, θ] -> xProduct[d[T[η, "d", {i}]] · d[T[θ, "d", {i}]], {i, dim[F]}],
    xIntegral[Exp[Transpose[θ].A.η], iD[η, θ]] → Det[A],
    Det[A] → 1 / dim[F] ! xSum[(-1)^(Abs[σ]+Abs[τ]) T[A, "dd", {σ[1], τ[1]}].
          … .T[A, "dd", {σ[dim[F]], τ[dim[F]]}], {σ, τ} ∈ Π_dim[F]],
    Π_dim[F][CG["permutations of {1,dim[F]}"]],
    {dim[F] → 2 n, θ → η, xIntegral[Exp[Transpose[η].A.η / 2], iD[η]] → Pf[A],
     Pf[A] → (-1)^n / (2^n n!)
       xSum[(-1)^Abs[σ] T[A, "dd", {σ[1], σ[2]}].….T[A, "dd", {σ[2 n - 1], σ[2 n]}]],
     {A[CG["skewsymmetric"]],
      Det[A] → Pf[A]^2
     }
    }

   }; $ // ColumnBar
]
```

Grassmann variable definition:

$\theta_i$

$\theta_i . \theta_j \to -\theta_j . \theta_i$

$\int 1 \, d\theta_i \to 0$

$\int \theta_i \, d\theta_i \to 1$

$D[\theta] \to \prod_{\{i,\overline{\dim}[F]\}} [d[\theta_i]]$

$D[\eta, \theta] \to \prod_{\{i,\overline{\dim}[F]\}} [d[\eta_i] \cdot d[\theta_i]]$

$\int e^{\theta^T . A . \eta} \, d D[\eta, \theta] \to \mathrm{Det}[A]$

$$\mathrm{Det}[A] \to \frac{\sum_{\{\sigma,\tau\}\in\overline{\Pi}_{\dim[F]}} [(-1)^{\mathrm{Abs}[\sigma]+\mathrm{Abs}[\tau]} A_{\sigma[1]\,\tau[1]} . \ldots . A_{\sigma[\dim[F]]\,\tau[\dim[F]]}]}{\dim[F]!}$$

$\Pi_{\dim[F]}$[permutations of {1,dim[F]}]

$\{\dim[F] \to 2\,n,\ \theta \to \eta,\ \int e^{\frac{1}{2}\eta^T . A . \eta} \, d D[\eta] \to \mathrm{Pf}[A],$

$$\mathrm{Pf}[A] \to \frac{(-\frac{1}{2})^n \,\mathrm{xSum}[(-1)^{\mathrm{Abs}[\sigma]} A_{\sigma[1]\,\sigma[2]} . \ldots . A_{\sigma[-1+2\,n]\,\sigma[2\,n]}]}{n!}, \{A[\text{skewsymmetric}], \mathrm{Det}[A] \to \mathrm{Pf}[A]^2\}\}$$

```
PR[$ = {U[ξ, ζ] → BraKet[J.ξ, 𝒟𝒜.ζ], {ξ, ζ} ∈ ℋ+,
    B[χ, ψ] → -I BraKet[JM.χ, (T[γ, "u", {μ}].(("∇"S)μ - I T[Y, "d", {μ}] ) - m).ψ],
    {χ, ψ} ∈ L2[M, S],
    $sξ, χ → χL + χR, ψ → ψL + ψR,
    $sDA1
   }; $ // ColumnBar,
  line,
  NL, "Show ", ($ = U[ξ, ξ]) → 2 B[χ, ψ],
  Yield, $,
  line,
  NL, "They get ",
  Yield, $ = {Pf[U] → (IntegralOp[{{D[ξ̃]}},
        Exp[1 / 2 U[ξ̃, ξ̃]]] →
       (IntegralOp[{{D[ξ̃]}, {D[ψ̃]}}, Exp[B[ξ̃, ψ̃]]] →
         Det[B]))}; $ // ColumnBar,
  NL, $s = D[η_, θ_] :→ (Table[d[T[η, "d", {i}]].d[T[θ, "d", {i}]], {i, dim[]}]),
  Yield, D[ξ, ψ] /. $s
 ]
```

$U[\xi, \zeta] \to \langle J.\xi \mid \mathcal{D}_{\mathcal{A}}.\zeta \rangle$
$\{\xi, \zeta\} \in \mathcal{H}^+$
$B[\chi, \psi] \to -i \langle J_M.\chi \mid (-m + \gamma^\mu.(\nabla^S{}_\mu - i\, Y_\mu)).\psi \rangle$
$\{\chi, \psi\} \in L^2[M, S]$
$\{\xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \overline{e_R} + \psi_R \otimes \overline{e_L}, \{\chi_L, \psi_L\} \in L^2[M, S]^+, \{\chi_R, \psi_R\} \in L^2[M, S]^-\}$
$\chi \to \chi_L + \chi_R$
$\psi \to \psi_L + \psi_R$
$\mathcal{D}_{\mathcal{A}} \to -i\, \gamma^\mu.\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F} + \gamma_5 \otimes \mathcal{D}_F + \gamma^\mu \otimes B_\mu$

---

Show $U[\xi, \xi] \to 2\, B[\chi, \psi]$
→ $U[\xi, \xi]$

---

They get
→ $Pf[U] \to \int_{\{D[\tilde{\xi}]\}} [e^{\frac{1}{2} U[\tilde{\xi}, \tilde{\xi}]}] \to \int_{\{D[\tilde{\xi}]\} \atop \{D[\tilde{\psi}]\}} [e^{B[\tilde{\xi}, \tilde{\psi}]}] \to Det[B]$

$D[\eta\_, \theta\_] :\to Table[d[T[\eta, d, \{i\}]].d[T[\theta, d, \{i\}]], \{i, dim[]\}]$
→ $Table[d[T[\xi, d, \{i\}]].d[T[\psi, d, \{i\}]], \{i, dim[]\}]$

```
tuSaveAllVariables[]
```