```
<< Local`QFTToolKit`
Put[SaveFile = NBname["stub"] <> ".out"]

SetTensorValues[η@uu[i, j], DiagonalMatrix[{-1, 1, 1, 1}]]
SetTensorValues[η@dd[i, j], DiagonalMatrix[{-1, 1, 1, 1}]]

PR["Manipulations around (11): Given: ",
 Column[$s = {d[e] + ω.e -> 0, e -> Λ.ẽ, d[ẽ] + ω.ẽ}],
 Imply, $ = $s[[1]] /. $s[[2]],
 Yield, $ = $ // δExpand[d], (*this works because Λ is 0-form*)
 Yield, $ = Λ^-1.# & /@ $ // simpleDot3[{}], CK,
 Yield, $ = $ /. Λ^-1.Λ -> 1 // simpleDot3[{}],
 Yield, $ = $ /. a_ . e_ + b_ . e_ -> (a + b).e; Framed[$],
 Imply, $pass = ω̃ -> (ExtractPattern[a_ . ẽ][$] /. ẽ -> 1 // simpleDot3[{}] // First);
 Framed[$pass]
]
```

Manipulations around (11): Given: $\begin{array}{l} d[e]+\omega.e\to 0 \\ e\to\Lambda.\tilde{e} \\ d[\tilde{e}]+\omega.\tilde{e}\end{array}$

$\Rightarrow\ d[\Lambda.\tilde{e}]+\omega.\Lambda.\tilde{e}\to 0$

$\rightarrow\ \Lambda.d[\tilde{e}]+d[\Lambda].\tilde{e}+\omega.\Lambda.\tilde{e}\to 0$

$\rightarrow\ \frac{1}{\Lambda}.\Lambda.d[\tilde{e}]+\frac{1}{\Lambda}.d[\Lambda].\tilde{e}+\frac{1}{\Lambda}.\omega.\Lambda.\tilde{e}\to 0\longleftarrow$ CHECK

$\rightarrow\ d[\tilde{e}]+\frac{1}{\Lambda}.d[\Lambda].\tilde{e}+\frac{1}{\Lambda}.\omega.\Lambda.\tilde{e}\to 0$

$\rightarrow\ \boxed{d[\tilde{e}]+\left(\frac{1}{\Lambda}.d[\Lambda]+\frac{1}{\Lambda}.\omega.\Lambda\right).\tilde{e}\to 0}$

$\Rightarrow\ \boxed{\tilde{\omega}\to\frac{1}{\Lambda}.d[\Lambda]+\frac{1}{\Lambda}.\omega.\Lambda}$

```
PR["● Check: ", $ = R -> Λ.R̃.Λ^-1,
 NL, "With DifForm[]s: ", $s = R -> DifForm[ω] + ω.ω;
 $s = {$s /. tt : (R | ω) -> t̃t, $s},
 Yield, $ = $ /. $s /. ($pass /. d[a_] -> DifForm[a]),
 Yield, $ = $ // tuStdDifForm[{}, {Λ}, {{ω, 1}}, {NoSymmetric}];
 Framed[$], OK
]
```

● Check: $R\to\Lambda.\tilde{R}.\frac{1}{\Lambda}$

With DifForm[]s: $\{\tilde{R}\to\underline{d}[\tilde{\omega}]+\tilde{\omega}.\tilde{\omega},\ R\to\underline{d}[\omega]+\omega.\omega\}$

$\rightarrow\ \underline{d}[\omega]+\omega.\omega\to\Lambda.\left(\underline{d}\left[\frac{1}{\Lambda}.\underline{d}[\Lambda]+\frac{1}{\Lambda}.\omega.\Lambda\right]+\left(\frac{1}{\Lambda}.\underline{d}[\Lambda]+\frac{1}{\Lambda}.\omega.\Lambda\right).\left(\frac{1}{\Lambda}.\underline{d}[\Lambda]+\frac{1}{\Lambda}.\omega.\Lambda\right)\right).\frac{1}{\Lambda}$

$\rightarrow\ \boxed{\underline{d}[\omega]\to\underline{d}[\omega]}$ OK

```
PR["•For ", d[s]^2 -> (d[r]^2 + d[x]^2) / r^2,
 Imply, T[g, "dd"][r, r] -> 1 / r^2,
 imply, $s = T[e, "ud"][1, r] -> 1 / r,
 imply, $ = T[e, "u"][1] -> $s[[1]].DifForm[r],
 yield, $ = $ /. $s,
 Imply, $ = Map[DifForm[#] &, $],
 imply, $ = $ // tuStdDifForm[{}, {r, x}, {}]; Framed[$],
 NL, "And: ", T[g, "dd"][x, x] -> 1 / r^2,
 imply, $s = T[e, "ud"][2, x] -> 1 / r,
 imply, $ = T[e, "u"][2] -> $s[[1]].DifForm[x],
 yield, $ = $ /. $s,
 Imply, xtmp = $ = Map[DifForm[#] &, $],
 imply, $ = $ // tuStdDifForm[{}, {x, r}, {}]; Framed[$]
]
```

•For $d[s]^2 \to \frac{d[r]^2 + d[x]^2}{r^2}$

$\Rightarrow\ g_{r\,r} \to \frac{1}{r^2}\ \Rightarrow\ e^1{}_r \to \frac{1}{r}\ \Rightarrow\ e^1 \to e^1{}_r . \underline{d}[r]\ \longrightarrow\ e^1 \to \frac{1}{r} . \underline{d}[r]$

$\Rightarrow\ \underline{d}[e^1] \to \underline{d}[\frac{1}{r} . \underline{d}[r]]\ \Rightarrow\ \boxed{\underline{d}[e^1] \to 0}$

And: $g_{x\,x} \to \frac{1}{r^2}\ \Rightarrow\ e^2{}_x \to \frac{1}{r}\ \Rightarrow\ e^2 \to e^2{}_x . \underline{d}[x]\ \longrightarrow\ e^2 \to \frac{1}{r} . \underline{d}[x]$

$\Rightarrow\ \underline{d}[e^2] \to \underline{d}[\frac{1}{r} . \underline{d}[x]]\ \Rightarrow\ \boxed{\underline{d}[e^2] \to -\frac{\underline{d}[r] \wedge \underline{d}[x]}{r^2}}$

```
PR["p604.(22): ", $0 = $ = T[ω, "udd"][α, β, μ] → -T[e, "ud"][ν, β]
      (xPartialD[T[e, "ud"][α, ν], μ] - T[Γ, "udd"][λ, μ, ν] T[e, "ud"][α, λ]),
 Yield, $0 = $ = $ // Expand // (# DifForm[T[x, "u"][μ]]) & /@ # & // Expand, CK,
 Yield, $1 = $ /. Thread[{α, β, μ, ν, λ} -> {α1, β1, μ1, ν1, λ1}],
 Yield, ($ = TimesRules[{$, $1}] /. β → α1 /. β1 → β // Expand //
      tuStdDifForm[{}, {Tensor[_, _, _]}, {}]) // ColumnSumExp, CK,
 NL, "Compute: ", $2 = DifForm[#] & /@ $0,
 Yield, $2 = $2 // tuStdDifForm[{}, {Tensor[_, _, _]}, {}]; ColumnSumExp[$2]
]
```

p604.(22): $\omega^\alpha{}_{\beta\,\mu} \to -e^\nu{}_\beta\,(-e^\alpha{}_\lambda\,\Gamma^\lambda{}_{\mu\,\nu} + \underline{\partial}_\mu[e^\alpha{}_\nu])$

$\to\ \underline{d}[x^\mu]\,\omega^\alpha{}_{\beta\,\mu} \to \underline{d}[x^\mu]\,e^\alpha{}_\lambda\,e^\nu{}_\beta\,\Gamma^\lambda{}_{\mu\,\nu} - \underline{d}[x^\mu]\,e^\nu{}_\beta\,\underline{\partial}_\mu[e^\alpha{}_\nu]$ ⟵CHECK

$\to\ \underline{d}[x^{\mu 1}]\,\omega^{\alpha 1}{}_{\beta 1\,\mu 1} \to \underline{d}[x^{\mu 1}]\,e^{\alpha 1}{}_{\lambda 1}\,e^{\nu 1}{}_{\beta 1}\,\Gamma^{\lambda 1}{}_{\mu 1\,\nu 1} - \underline{d}[x^{\mu 1}]\,e^{\nu 1}{}_{\beta 1}\,\underline{\partial}_{\mu 1}[e^{\alpha 1}{}_{\nu 1}]$

$$\to\ \omega^\alpha{}_{\alpha 1\,\mu}\,\omega^{\alpha 1}{}_{\beta\,\mu 1}\,\underline{d}[x^\mu] \wedge \underline{d}[x^{\mu 1}] \to \sum\left[\begin{array}{l} \underline{\partial}_{-\mu}[e^\alpha{}_\nu].\underline{\partial}_{-\mu 1}[e^{\alpha 1}{}_{\nu 1}].(\underline{d}[x^\mu] \wedge \underline{d}[x^{\mu 1}])\,e^\nu{}_{\alpha 1}\,e^{\nu 1}{}_\beta \\ -\underline{\partial}_{-\mu 1}[e^{\alpha 1}{}_{\nu 1}].(\underline{d}[x^\mu] \wedge \underline{d}[x^{\mu 1}])\,e^\alpha{}_\lambda\,e^\nu{}_{\alpha 1}\,e^{\nu 1}{}_\beta\,\Gamma^\lambda{}_{\mu\,\nu} \\ -\underline{\partial}_{-\mu}[e^\alpha{}_\nu].(\underline{d}[x^\mu] \wedge \underline{d}[x^{\mu 1}])\,e^{\alpha 1}{}_{\lambda 1}\,e^\nu{}_{\alpha 1}\,e^{\nu 1}{}_\beta\,\Gamma^{\lambda 1}{}_{\mu 1\,\nu 1} \\ e^\alpha{}_\lambda\,e^{\alpha 1}{}_{\lambda 1}\,e^\nu{}_{\alpha 1}\,e^{\nu 1}{}_\beta\,\Gamma^\lambda{}_{\mu\,\nu}\,\Gamma^{\lambda 1}{}_{\mu 1\,\nu 1}\,\underline{d}[x^\mu] \wedge \underline{d}[x^{\mu 1}] \end{array}\right]$$ ⟵CHECK

Compute: $\underline{d}[\underline{d}[x^\mu]\,\omega^\alpha{}_{\beta\,\mu}] \to \underline{d}[\underline{d}[x^\mu]\,e^\alpha{}_\lambda\,e^\nu{}_\beta\,\Gamma^\lambda{}_{\mu\,\nu} - \underline{d}[x^\mu]\,e^\nu{}_\beta\,\underline{\partial}_\mu[e^\alpha{}_\nu]]$

$$\to\ -(\underline{d}[x^\mu] \wedge \underline{d}[\omega^\alpha{}_{\beta\,\mu}]) \to \sum\left[\begin{array}{l} -\underline{\partial}_{-\mu}[e^\alpha{}_\nu].(\underline{d}[e^\nu{}_\beta] \wedge \underline{d}[x^\mu]) \\ e^\nu{}_\beta\,\Gamma^\lambda{}_{\mu\,\nu}\,\underline{d}[e^\alpha{}_\lambda] \wedge \underline{d}[x^\mu] \\ e^\alpha{}_\lambda\,\Gamma^\lambda{}_{\mu\,\nu}\,\underline{d}[e^\nu{}_\beta] \wedge \underline{d}[x^\mu] \\ -e^\alpha{}_\lambda\,e^\nu{}_\beta\,\underline{d}[x^\mu] \wedge \underline{d}[\Gamma^\lambda{}_{\mu\,\nu}] \\ e^\nu{}_\beta\,\underline{d}[x^\mu] \wedge \underline{\partial}_{-\mu}[\underline{d}[e^\alpha{}_\nu]] \end{array}\right]$$

```
PR["p604.(22): ",
 $0 = $ = ω → -e.(DifForm[e] - Γ. e),
 Yield, xtmp = $ = DifForm[#] & /@ $,
 Yield, $1 = $ // tuStdDifForm[{}, {e}, {{e, 0}, {Γ, 1}, {ω, 1}}, {}], CK,
 NL, "For: ", $ = OpRules[{$0, $0}, Dot],
 Yield, $2 = $ // tuStdDifForm[{}, {}, {{e, 0}, {Γ, 1}, {ω, 1}}],
 Imply, (OpRules[{$2, $1}, Plus]) // ColumnSumExp
]
```

```
p604.(22): ω → -e.(d[e] - Γ.e)
→ d[ω] → d[-e.(d[e] - Γ.e)]
→ d[ω] → 0⟵CHECK
For: ω.ω → (-e.(d[e] - Γ.e)).(-e.(d[e] - Γ.e))
→ 0 → 0
⇒ d[ω] → 0
```

IX.7.1 Calculate curvature using differential forms.

```
PR["● From metric: ",
 Yield, $s = d[s]^2 → f[y]^2 d[x]^2 + g[x]^2 d[y]^2 /. d[a_] → DifForm[a],
 Imply, $e = {T[e, "u"][1] → f[y]. d[x], T[e, "u"][2] → g[x] .d[y]} /. d[a_] → DifForm[a],
 Yield, $ed = Map[DifForm[#] & /@ # &, $e] // tuStdDifForm[{}, {x, y}, {}],
 Yield, $ed = $ed /. DifForm[(ff : f | g)[x_]] → xPartialD[ff[x], x].DifForm[x],
 Yield, $ed = $ed // tuStdDifForm[{}, {}, {}],
 Framed[Column[$ed]],
 NL, "Definition: ", $ = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β],
 yield, $ = MapAt[Sum[#, {β, 1, 2}] &, $, 2],
 Yield, $ = {$ /. α → 1, $ /. α → 2},
 Yield, $ = Map[MapAt[(# /. $e) &, #, 2] &, $];
 NL, "Let: ", $s = {T[ω, "ud"][a_, a_] → 0, T[ω, "ud"][a_, b_] →
     T[ω, "udd"][a, b, x].DifForm[x] + T[ω, "udd"][a, b, y].DifForm[y]},
 Yield, $ = $ /. $s // tuStdDifForm[{}, {x, y}, {}]; Framed[Column[$]],
 NL, "Comparing these two and factoring out the common Wedge: ",
 Yield, $ = xEliminate[({$ed, $}), {DifForm[T[e, "u"][1]], DifForm[T[e, "u"][2]]}],
 Yield, $ = $ /. Wedge[_] → 1 // simpleDot3[{}],
 Yield, $ = Solve[$ /. Dot → Times, {T[ω, "udd"][1, 2, x], T[ω, "udd"][2, 1, y]}];
 Yield, $[[1, 2]] = IndexSwap[{1, 2}][$[[1, 2, 1]]] → -$[[1, 2, 2]];
 Framed[$],
 Yield, $o = T[ω, "ud"][1, 2] → (T[ω, "ud"][1, 2] /. $s),
 Yield, $o = $o /. $ // simpleDot3[{}] // First; Framed[$o],
 NL, "Then: ", $ = DifForm[#] & /@ $o // tuStdDifForm[{}, {x, y, f[y], g[x]}, {}],
 Yield, $ = $ /. DifForm[(ff : f | g)[x_]] → xPartialD[ff[x], x].DifForm[x] //
   tuStdDifForm[{}, {x, y, f[y], g[x], xPartialD[_, _]}, {}],
 Yield, $ = $ // DerivativeExpand[{DifForm[x | y]}] //
   tuStdDifForm[{}, {x, y, f[y], g[x], xPartialD[_, _]}, {}];
 Framed[$],
 NL, "The term: ", T[ω, "ud"][1, α] T[ω, "ud"][α, 2] → 0,
 NL, "Using (16): ", R → d[ω] + ω.ω,
 Imply, $ = T[R, "uddd"][1, 2, x, y] → $[[2]]; Framed[$]
]
```

- From metric:

→ $\underline{d}[s]^2 \to \underline{d}[x]^2 f[y]^2 + \underline{d}[y]^2 g[x]^2$

⇒ $\{e^1 \to f[y].\underline{d}[x],\ e^2 \to g[x].\underline{d}[y]\}$

→ $\{\underline{d}[e^1] \to -(\underline{d}[x] \wedge \underline{d}[f[y]]),\ \underline{d}[e^2] \to -(\underline{d}[y] \wedge \underline{d}[g[x]])\}$

→ $\{\underline{d}[e^1] \to -(\underline{d}[x] \wedge \underline{\partial}_y[f[y]].\underline{d}[y]),\ \underline{d}[e^2] \to -(\underline{d}[y] \wedge \underline{\partial}_x[g[x]].\underline{d}[x])\}$

→ $\{\underline{d}[e^1] \to -\underline{\partial}_y[f[y]].(\underline{d}[x] \wedge \underline{d}[y]),\ \underline{d}[e^2] \to \underline{\partial}_x[g[x]].(\underline{d}[x] \wedge \underline{d}[y])\}$

$$\boxed{\begin{array}{l} \underline{d}[e^1] \to -\underline{\partial}_y[f[y]].(\underline{d}[x] \wedge \underline{d}[y]) \\ \underline{d}[e^2] \to \underline{\partial}_x[g[x]].(\underline{d}[x] \wedge \underline{d}[y]) \end{array}}$$

Definition: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta.e^\beta \longrightarrow \underline{d}[e^\alpha] \to -\omega^\alpha{}_1.e^1 - \omega^\alpha{}_2.e^2$

→ $\{\underline{d}[e^1] \to -\omega^1{}_1.e^1 - \omega^1{}_2.e^2,\ \underline{d}[e^2] \to -\omega^2{}_1.e^1 - \omega^2{}_2.e^2\}$

→

Let: $\{\omega^{a\_}{}_{a\_} \to 0,\ \omega^{a\_}{}_{b\_} \to \omega^a{}_{b\,x}.\underline{d}[x] + \omega^a{}_{b\,y}.\underline{d}[y]\}$

→ $$\boxed{\begin{array}{l} \underline{d}[e^1] \to -g[x].\omega^1{}_{2\,x}.(\underline{d}[x] \wedge \underline{d}[y]) \\ \underline{d}[e^2] \to f[y].\omega^2{}_{1\,y}.(\underline{d}[x] \wedge \underline{d}[y]) \end{array}}$$

Comparing these two and factoring out the common Wedge:

→ $\underline{\partial}_y[f[y]].(\underline{d}[x] \wedge \underline{d}[y]) == g[x].\omega^1{}_{2\,x}.(\underline{d}[x] \wedge \underline{d}[y])\ \&\&\ \underline{\partial}_x[g[x]].(\underline{d}[x] \wedge \underline{d}[y]) == f[y].\omega^2{}_{1\,y}.(\underline{d}[x] \wedge \underline{d}[y])$

→ $\underline{\partial}_y[f[y]] == g[x].\omega^1{}_{2\,x}\ \&\&\ \underline{\partial}_x[g[x]] == f[y].\omega^2{}_{1\,y}$

→

→ $$\boxed{\left\{\left\{\omega^1{}_{2\,x} \to \frac{\underline{\partial}_y[f[y]]}{g[x]},\ \omega^1{}_{2\,y} \to -\frac{\underline{\partial}_x[g[x]]}{f[y]}\right\}\right\}}$$

→ $\omega^1{}_2 \to \omega^1{}_{2\,x}.\underline{d}[x] + \omega^1{}_{2\,y}.\underline{d}[y]$

→ $$\boxed{\omega^1{}_2 \to \frac{\underline{\partial}_y[f[y]]}{g[x]}.\underline{d}[x] - \frac{\underline{\partial}_x[g[x]]}{f[y]}.\underline{d}[y]}$$

Then: $\underline{d}[\omega^1{}_2] \to$

$$-\frac{\underline{\partial}_x[g[x]].(\underline{d}[y] \wedge \underline{d}[f[y]])}{f[y]^2} + \frac{\underline{\partial}_y[f[y]].(\underline{d}[x] \wedge \underline{d}[g[x]])}{g[x]^2} - \frac{\underline{d}[x] \wedge \underline{\partial}_y[\underline{d}[f[y]]]}{g[x]} + \frac{\underline{d}[y] \wedge \underline{\partial}_x[\underline{d}[g[x]]]}{f[y]}$$

→ $$\underline{d}[\omega^1{}_2] \to -\frac{\underline{d}[x] \wedge \underline{d}[y]\ \underline{\partial}_y[\underline{\partial}_y[f[y]]]}{g[x]} - \frac{\underline{d}[x] \wedge \underline{d}[y]\ \underline{\partial}_x[\underline{\partial}_x[g[x]]]}{f[y]}$$

→ $$\boxed{\underline{d}[\omega^1{}_2] \to -\frac{\underline{d}[x] \wedge \underline{d}[y]\ \underline{\partial}_y[\underline{\partial}_y[f[y]]]}{g[x]} - \frac{\underline{d}[x] \wedge \underline{d}[y]\ \underline{\partial}_x[\underline{\partial}_x[g[x]]]}{f[y]}}$$

The term: $\omega^1{}_\alpha\, \omega^\alpha{}_2 \to 0$

Using (16): $R \to d[\omega] + \omega.\omega$

⇒ $$\boxed{R^1{}_{2\,x\,y} \to -\frac{\underline{d}[x] \wedge \underline{d}[y]\ \underline{\partial}_y[\underline{\partial}_y[f[y]]]}{g[x]} - \frac{\underline{d}[x] \wedge \underline{d}[y]\ \underline{\partial}_x[\underline{\partial}_x[g[x]]]}{f[y]}}$$

IX.7.3

```
PR["●IX.7.3: Standard spherical coordinate metric: ",
 $ds = d[s]^2 → d[r]^2 + r^2 Sin[θ]^2 d[φ]^2 + r^2 d[θ]^2 /. d[a_] → DifForm[a],
 Imply, "•Vielbeins and their DifForm: ", $vb = {T[e, "u"][1] → DifForm[r],
   T[e, "u"][2] → r Sin[θ] DifForm[φ], T[e, "u"][3] → r DifForm[θ]},
 Yield, $de = Map[Map[DifForm[#] &, #] &, $vb];
 Yield, $de = $de // tuStdDifForm[{}, {θ, r, φ}, {}];
 Yield, $de = $de /. DifForm[Sin[θ]] → Cos[θ] DifForm[θ] //
   tuStdDifForm[{}, {θ, r, φ}, {{Tensor[ω, _, _], 1}}];
 Column[$de],
 NL, "•From the definition: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β],
 Yield, $ = Table[MapAt[Sum[#, {β, 1, 3}] &, $0, 2], {α, 1, 3}] /. T[ω, "ud"][a_, a_] → 0 //
   tuStdDifForm[{}, {θ, r, φ}, {{Tensor[_, _, _], 1}}];
 Column[$],
 Yield,
 $ = MapAt[# /. $vb &, #, 2] & /@ $ // tuStdDifForm[{}, {θ, r, φ}, {{Tensor[_, _, _], 1}}];
 Column[$],
 NL, "Since: ", $s =
  {T[ω, "ud"][i_, j_] -> T[ω, "udd"][i, j, k].DifForm[T[x, "u"][k]], T[x, "u"][a_] → a},
 Yield, $s[[1, 2]] = Sum[$s[[1, 2]], {k, {r, φ, θ}}] /. $s; $s,
 Imply, $pass = $ = $ /. $s // tuStdDifForm[{}, {θ, r, φ, Tensor[_, _, _]}, {}];
 Column[$]
]
```

●IX.7.3: Standard spherical coordinate metric: $\underline{d}[s]^2 \to \underline{d}[r]^2 + r^2\,\underline{d}[\theta]^2 + r^2\,\underline{d}[\varphi]^2\,\mathrm{Sin}[\theta]^2$

⇒ •Vielbeins and their DifForm: $\{e^1 \to \underline{d}[r],\ e^2 \to r\,\underline{d}[\varphi]\,\mathrm{Sin}[\theta],\ e^3 \to r\,\underline{d}[\theta]\}$

→

→

$\underline{d}[e^1] \to 0$

→ $\underline{d}[e^2] \to r\,\mathrm{Cos}[\theta].(\underline{d}[\theta]\wedge\underline{d}[\varphi]) + \mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])$

$\underline{d}[e^3] \to \underline{d}[r]\wedge\underline{d}[\theta]$

•From the definition: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta . e^\beta$

$\underline{d}[e^1] \to e^2\wedge\omega^1{}_2 + e^3\wedge\omega^1{}_3$

→ $\underline{d}[e^2] \to e^1\wedge\omega^2{}_1 + e^3\wedge\omega^2{}_3$

$\underline{d}[e^3] \to e^1\wedge\omega^3{}_1 + e^2\wedge\omega^3{}_2$

$\underline{d}[e^1] \to r\,\mathrm{Sin}[\theta].(\underline{d}[\varphi]\wedge\omega^1{}_2) + r\,\underline{d}[\theta]\wedge\omega^1{}_3$

→ $\underline{d}[e^2] \to \underline{d}[r]\wedge\omega^2{}_1 + r\,\underline{d}[\theta]\wedge\omega^2{}_3$

$\underline{d}[e^3] \to r\,\mathrm{Sin}[\theta].(\underline{d}[\varphi]\wedge\omega^3{}_2) + \underline{d}[r]\wedge\omega^3{}_1$

Since: $\{\omega^{i_-}{}_{j_-} \to \omega^i{}_{jk}.\underline{d}[x^k],\ x^{a_-} \to a\}$

→ $\{\omega^{i_-}{}_{j_-} \to \omega^i{}_{jr}.\underline{d}[r] + \omega^i{}_{j\theta}.\underline{d}[\theta] + \omega^i{}_{j\varphi}.\underline{d}[\varphi],\ x^{a_-} \to a\}$

⇒

$\underline{d}[e^1] \to -r\,\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^1{}_{2r} - r\,\mathrm{Sin}[\theta].(\underline{d}[\theta]\wedge\underline{d}[\varphi])\,\omega^1{}_{2\theta} - r\,\omega^1{}_{3r}\,\underline{d}[r]\wedge\underline{d}[\theta] + r\,\omega^1{}_{3\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]$

$\underline{d}[e^2] \to \omega^2{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\theta] - r\,\omega^2{}_{3r}\,\underline{d}[r]\wedge\underline{d}[\theta] + \omega^2{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi] + r\,\omega^2{}_{3\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]$

$\underline{d}[e^3] \to -r\,\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^3{}_{2r} - r\,\mathrm{Sin}[\theta].(\underline{d}[\theta]\wedge\underline{d}[\varphi])\,\omega^3{}_{2\theta} + \omega^3{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\theta] + \omega^3{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]$

```
PR["•Comparing: ", $ = {$pass, $de}; Column[$],
 NL, "Eliminating: ",
 $v = $ // ExtractPattern[DifForm[Tensor[e, _, _]]] // DeleteDuplicates,
 Imply, $ = xEliminate[$, $v],
 NL, "?If no Dot, can we Solve these equations for ω? ",
 Yield, $ = $ /. Dot → Times // Apply[List, #] &; Framed[Column[$]],
 NL, "Set coefficients of Wedge[]s ->0 : ",
 Yield, $ = $ /. a_ == b_ → a - b == 0 // Collect[#, Wedge[__], Zero[#] &] &;
 Yield, $ = $ // ExtractPattern[Zero[_]] // DeleteDuplicates;
 Yield, $ = $ /. Zero[a_] → (a → 0); Framed[$],
 NL, "ω[] are anti-symmetric: ",
 $s = tt : T[ω, "udd"][a_, b_, c_] :→ (tt -> -T[ω, "udd"][b, a, c]) /; b < a, "POFF",
 NL, "Extract ω[]s: ", $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates,
 Yield, $wr = $w /. $s, "PONdd",
 Yield, $wr = Cases[$wr, Rule[__]],
 Yield, $ = $ /. $wr,
 NL, "Solve for ω[]s: ", $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates,
 Yield, $sw = xRuleX[$, $w]; Framed[Column[$sw]]
]
PR["POFF", "•Check with: ", $ = $pass,
 Yield, $ = $ /. $wr /. $sw // simpleDot3[{}]; Column[$]
]
```

•Comparing:

```
{d[e^1] → -r Sin[θ].(d[r] ∧ d[φ]) ω^1_2r - r Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ - r ω^1_3r d[r] ∧ d[θ] + r ω^1_3φ d[θ] ∧ d[φ],
 d[e^2] → ω^2_1θ d[r] ∧ d[θ] - r ω^2_3r d[r] ∧ d[θ] + ω^2_1φ d[r] ∧ d[φ] + r ω^2_3φ d[θ] ∧ d[φ],
 d[e^3] → -r Sin[θ].(d[r] ∧ d[φ]) ω^3_2r - r Sin[θ].(d[θ] ∧ d[φ]) ω^3_2θ + ω^3_1θ d[r] ∧ d[θ] + ω^3_1φ d[r] ∧ d[φ]}
{d[e^1] → 0, d[e^2] → r Cos[θ].(d[θ] ∧ d[φ]) + Sin[θ].(d[r] ∧ d[φ]), d[e^3] → d[r] ∧ d[θ]}
```

Eliminating: {d[e^1], d[e^2], d[e^3]}

```
⇒ r Cos[θ].(d[θ] ∧ d[φ]) ==
   -Sin[θ].(d[r] ∧ d[φ]) + ω^2_1θ d[r] ∧ d[θ] - r ω^2_3r d[r] ∧ d[θ] + ω^2_1φ d[r] ∧ d[φ] + r ω^2_3φ d[θ] ∧ d[φ] &&
  r Sin[θ].(d[r] ∧ d[φ]) ω^1_2r == r (-Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ - ω^1_3r d[r] ∧ d[θ] + ω^1_3φ d[θ] ∧ d[φ]) &&
  ω^3_1θ d[r] ∧ d[θ] == r Sin[θ].(d[r] ∧ d[φ]) ω^3_2r +
    r Sin[θ].(d[θ] ∧ d[φ]) ω^3_2θ + d[r] ∧ d[θ] - ω^3_1φ d[r] ∧ d[φ] && ω^2_1θ d[r] ∧ d[θ]
    (Sin[θ].(d[r] ∧ d[φ]) ω^1_2r + Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ + ω^1_3r d[r] ∧ d[θ] - ω^1_3φ d[θ] ∧ d[φ]) ==
   (Sin[θ].(d[r] ∧ d[φ]))^2 ω^1_2r + Sin[θ].(d[r] ∧ d[φ]) Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ +
    Sin[θ].(d[r] ∧ d[φ]) ω^1_3r d[r] ∧ d[θ] - Sin[θ].(d[r] ∧ d[φ]) ω^1_2r ω^2_1φ d[r] ∧ d[φ] -
    Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ ω^2_1φ d[r] ∧ d[φ] - ω^1_3r ω^2_1φ d[r] ∧ d[θ] d[r] ∧ d[φ] -
    Sin[θ].(d[r] ∧ d[φ]) ω^1_3φ d[θ] ∧ d[φ] + ω^1_3φ ω^2_1φ d[r] ∧ d[φ] d[θ] ∧ d[φ] && ω^2_1θ ω^3_1φ d[r] ∧ d[φ]
    (Sin[θ].(d[r] ∧ d[φ]) ω^1_2r + Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ + ω^1_3r d[r] ∧ d[θ] - ω^1_3φ d[θ] ∧ d[φ]) ==
   (Sin[θ].(d[r] ∧ d[φ]))^2 ω^1_2r + Sin[θ].(d[r] ∧ d[φ]) Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ -
    (Sin[θ].(d[r] ∧ d[φ]))^2 ω^1_2r ω^3_1θ - Sin[θ].(d[r] ∧ d[φ]) Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ ω^3_1θ -
    r (Sin[θ].(d[r] ∧ d[φ]))^2 ω^1_3r ω^3_2r - r Sin[θ].(d[r] ∧ d[φ]) Sin[θ].(d[θ] ∧ d[φ]) ω^1_3r ω^3_2θ -
    Sin[θ].(d[r] ∧ d[φ]) ω^1_2r ω^2_1φ d[r] ∧ d[φ] - Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ ω^2_1φ d[r] ∧ d[φ] +
    Sin[θ].(d[r] ∧ d[φ]) ω^1_2r ω^2_1φ ω^3_1θ d[r] ∧ d[φ] + Sin[θ].(d[θ] ∧ d[φ]) ω^1_2θ ω^2_1φ ω^3_1θ d[r] ∧ d[φ] +
    Sin[θ].(d[r] ∧ d[φ]) ω^1_3r ω^3_1φ d[r] ∧ d[φ] + r Sin[θ].(d[r] ∧ d[φ]) ω^1_3r ω^2_1φ ω^3_2r d[r] ∧ d[φ] +
    r Sin[θ].(d[θ] ∧ d[φ]) ω^1_3r ω^2_1φ ω^3_2θ d[r] ∧ d[φ] - ω^1_3r ω^2_1φ ω^3_1φ (d[r] ∧ d[φ])^2 -
    Sin[θ].(d[r] ∧ d[φ]) ω^1_3φ d[θ] ∧ d[φ] + Sin[θ].(d[r] ∧ d[φ]) ω^1_3φ ω^3_1θ d[θ] ∧ d[φ] +
    ω^1_3φ ω^2_1φ d[r] ∧ d[φ] d[θ] ∧ d[φ] - ω^1_3φ ω^2_1φ ω^3_1θ d[r] ∧ d[φ] d[θ] ∧ d[φ]
```

?If no Dot, can we Solve these equations for ω?

→

$$r\,\mathrm{Cos}[\theta]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] ==$$
$$\omega^{2}{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\theta] - r\,\omega^{2}{}_{3r}\,\underline{d}[r]\wedge\underline{d}[\theta] - \mathrm{Sin}[\theta]\,\underline{d}[r]\wedge\underline{d}[\varphi] + \omega^{2}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi] + r\,\omega^{2}{}_{3\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]$$

$$r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\underline{d}[r]\wedge\underline{d}[\varphi] == r\,(-\omega^{1}{}_{3r}\,\underline{d}[r]\wedge\underline{d}[\theta] - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\underline{d}[\theta]\wedge\underline{d}[\varphi] + \omega^{1}{}_{3\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi])$$

$$\omega^{3}{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\theta] == \underline{d}[r]\wedge\underline{d}[\theta] - \omega^{3}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi] + r\,\mathrm{Sin}[\theta]\,\omega^{3}{}_{2r}\,\underline{d}[r]\wedge\underline{d}[\varphi] + r\,\mathrm{Sin}[\theta]\,\omega^{3}{}_{2\theta}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]$$

$$\omega^{2}{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\theta]\,(\omega^{1}{}_{3r}\,\underline{d}[r]\wedge\underline{d}[\theta] + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\underline{d}[r]\wedge\underline{d}[\varphi] + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\underline{d}[\theta]\wedge\underline{d}[\varphi] - \omega^{1}{}_{3\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]) ==$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\underline{d}[r]\wedge\underline{d}[\theta]\,\underline{d}[r]\wedge\underline{d}[\varphi] - \omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\theta]\,\underline{d}[r]\wedge\underline{d}[\varphi] +$$
$$\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\varphi}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} +$$
$$\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] - \mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] -$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] + \omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi]$$

$$\omega^{2}{}_{1\theta}\,\omega^{3}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]$$
$$(\omega^{1}{}_{3r}\,\underline{d}[r]\wedge\underline{d}[\theta] + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\underline{d}[r]\wedge\underline{d}[\varphi] + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\underline{d}[\theta]\wedge\underline{d}[\varphi] - \omega^{1}{}_{3\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]) ==$$
$$\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\varphi}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} - \mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r}\,\omega^{3}{}_{1\theta}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} +$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\theta}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\omega^{3}{}_{1\varphi}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} - \omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\varphi}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} -$$
$$r\,\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{3r}\,\omega^{3}{}_{2r}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} + r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{2r}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^{2} +$$
$$\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] - \mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] -$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] + \omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] -$$
$$\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta}\,\omega^{3}{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] + \mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi}\,\omega^{3}{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] +$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] - \omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] -$$
$$r\,\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{3r}\,\omega^{3}{}_{2\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi] + r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{2\theta}\,\underline{d}[r]\wedge\underline{d}[\varphi]\,\underline{d}[\theta]\wedge\underline{d}[\varphi]$$

Set coefficients of Wedge[]s ->0 :

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$$\{\mathrm{Sin}[\theta] - \omega^{2}{}_{1\varphi} \to 0,\ -\omega^{2}{}_{1\theta} + r\,\omega^{2}{}_{3r} \to 0,\ r\,\mathrm{Cos}[\theta] - r\,\omega^{2}{}_{3\varphi} \to 0,\ 0 \to 0,\ r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{2r} \to 0,$$
$$r\,\omega^{1}{}_{3r} \to 0,\ r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta} - r\,\omega^{1}{}_{3\varphi} \to 0,\ -1 + \omega^{3}{}_{1\theta} \to 0,\ \omega^{3}{}_{1\varphi} - r\,\mathrm{Sin}[\theta]\,\omega^{3}{}_{2r} \to 0,$$
$$-r\,\mathrm{Sin}[\theta]\,\omega^{3}{}_{2\theta} \to 0,\ \omega^{1}{}_{3r}\,\omega^{2}{}_{1\theta} \to 0,\ -\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\varphi} \to 0,$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\theta} - \omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\theta} \to 0,\ -\mathrm{Sin}[\theta]\,\omega^{1}{}_{3r} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\theta} + \omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi} \to 0,$$
$$-\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\varphi} - \omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\varphi} \to 0,\ \omega^{1}{}_{3r}\,\omega^{2}{}_{1\theta}\,\omega^{3}{}_{1\varphi} \to 0,$$
$$-\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\varphi} + \mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r}\,\omega^{3}{}_{1\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\omega^{3}{}_{1\varphi} +$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{2}{}_{1\theta}\,\omega^{3}{}_{1\varphi} + \omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\varphi} + r\,\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{3r}\,\omega^{3}{}_{2r} - r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{2r} \to 0,$$
$$-\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\varphi} - \omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\varphi} + \mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta}\,\omega^{3}{}_{1\theta} -$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi}\,\omega^{3}{}_{1\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\theta} + \omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{1\theta} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{2}{}_{1\theta}\,\omega^{3}{}_{1\varphi} -$$
$$\omega^{1}{}_{3\varphi}\,\omega^{2}{}_{1\theta}\,\omega^{3}{}_{1\varphi} + r\,\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{3r}\,\omega^{3}{}_{2\theta} - r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\omega^{2}{}_{1\varphi}\,\omega^{3}{}_{2\theta} \to 0\}$$

ω[] are anti-symmetric: $tt : \omega^{a\_}{}_{b\_\,c\_} \mapsto (tt \to -T[\omega, udd][b, a, c])\ /;\ b < a$

....

$$\to \{\omega^{2}{}_{1\varphi} \to -\omega^{1}{}_{2\varphi},\ \omega^{2}{}_{1\theta} \to -\omega^{1}{}_{2\theta},\ \omega^{3}{}_{1\theta} \to -\omega^{1}{}_{3\theta},\ \omega^{3}{}_{1\varphi} \to -\omega^{1}{}_{3\varphi},\ \omega^{3}{}_{2r} \to -\omega^{2}{}_{3r},\ \omega^{3}{}_{2\theta} \to -\omega^{2}{}_{3\theta}\}$$

$$\to \{\mathrm{Sin}[\theta] + \omega^{1}{}_{2\varphi} \to 0,\ \omega^{1}{}_{2\theta} + r\,\omega^{2}{}_{3r} \to 0,\ r\,\mathrm{Cos}[\theta] - r\,\omega^{2}{}_{3\varphi} \to 0,\ 0 \to 0,\ r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{2r} \to 0,$$
$$r\,\omega^{1}{}_{3r} \to 0,\ r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta} - r\,\omega^{1}{}_{3\varphi} \to 0,\ -1 - \omega^{1}{}_{3\theta} \to 0,\ -\omega^{1}{}_{3\varphi} + r\,\mathrm{Sin}[\theta]\,\omega^{2}{}_{3r} \to 0,$$
$$r\,\mathrm{Sin}[\theta]\,\omega^{2}{}_{3\theta} \to 0,\ -\omega^{1}{}_{2\theta}\,\omega^{1}{}_{3r} \to 0,\ -\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{1}{}_{2\varphi} \to 0,$$
$$-\mathrm{Sin}[\theta]\,(\omega^{1}{}_{2\theta})^{2} + \omega^{1}{}_{2\theta}\,\omega^{1}{}_{3\varphi} \to 0,\ -\mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{1}{}_{2\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{3r} - \omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3r} \to 0,$$
$$-\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{1}{}_{2\varphi} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi} + \omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3\varphi} \to 0,\ \omega^{1}{}_{2\theta}\,\omega^{1}{}_{3r}\,\omega^{1}{}_{3\varphi} \to 0,$$
$$-\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{1}{}_{2\varphi} - \mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2r}\,\omega^{1}{}_{3\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3\theta} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{2r}\,\omega^{1}{}_{2\theta}\,\omega^{1}{}_{3\varphi} +$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{3r}\,\omega^{1}{}_{3\varphi} + \omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3r}\,\omega^{1}{}_{3\varphi} - r\,\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{3r}\,\omega^{2}{}_{3r} - r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3r}\,\omega^{2}{}_{3r} \to 0,$$
$$-\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{1}{}_{2\varphi} - \mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{2\theta}\,\omega^{1}{}_{3\theta} - \mathrm{Sin}[\theta]\,\omega^{1}{}_{2\theta}\,\omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3\theta} +$$
$$\mathrm{Sin}[\theta]\,\omega^{1}{}_{3\varphi} + \mathrm{Sin}[\theta]\,(\omega^{1}{}_{2\theta})^{2}\,\omega^{1}{}_{3\varphi} + \omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3\varphi} + \mathrm{Sin}[\theta]\,\omega^{1}{}_{3\theta}\,\omega^{1}{}_{3\varphi} +$$
$$\omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3\theta}\,\omega^{1}{}_{3\varphi} - \omega^{1}{}_{2\theta}\,(\omega^{1}{}_{3\varphi})^{2} - r\,\mathrm{Sin}[\theta]^{2}\,\omega^{1}{}_{3r}\,\omega^{2}{}_{3\theta} - r\,\mathrm{Sin}[\theta]\,\omega^{1}{}_{2\varphi}\,\omega^{1}{}_{3r}\,\omega^{2}{}_{3\theta} \to 0\}$$

Solve for ω[]s: $\{\omega^{1}{}_{2\varphi},\ \omega^{1}{}_{2\theta},\ \omega^{2}{}_{3r},\ \omega^{2}{}_{3\varphi},\ \omega^{1}{}_{2r},\ \omega^{1}{}_{3r},\ \omega^{1}{}_{3\varphi},\ \omega^{1}{}_{3\theta},\ \omega^{2}{}_{3\theta}\}$

$$\rightarrow \boxed{\begin{array}{l} \omega^{1}{}_{2\,\varphi} \to -\mathrm{Sin}[\theta] \\ \omega^{1}{}_{2\,\theta} \to 0 \\ \omega^{2}{}_{3\,r} \to 0 \\ \omega^{2}{}_{3\,\varphi} \to \mathrm{Cos}[\theta] \\ \omega^{1}{}_{2\,r} \to 0 \\ \omega^{1}{}_{3\,r} \to 0 \\ \omega^{1}{}_{3\,\varphi} \to 0 \\ \omega^{1}{}_{3\,\theta} \to -1 \\ \omega^{2}{}_{3\,\theta} \to 0 \end{array}}$$

```
PR["•So in Cartesian coordinates: ", $ = F → g/(4 π) DifForm[Cos[θ]].DifForm[φ],
 Yield, $ = $ /. DifForm[Cos[θ]] → -Sin[θ] DifForm[θ],
 NL, "Inverting: ", $s = $vb,
 yield, $s = xRuleX[$s, {DifForm[r], DifForm[φ], DifForm[θ]}],
 Imply, $ = $ /. $s // tuStdDifForm[{}, {θ, r, φ}, {{Tensor[e, _, _], 1}}],
 NL, "Orthogonal coordinate: ",
 $s = Wedge[T[e, "u"][2], T[e, "u"][3]] -> T[e, "u"][3], "(radial)",
 Yield, $ = $ /. $s
]
```

•So in Cartesian coordinates: $F \to \frac{g\, \underline{d}[\mathrm{Cos}[\theta]].\underline{d}[\varphi]}{4\,\pi}$

$\rightarrow\ F \to \frac{g\,(-\underline{d}[\theta]\,\mathrm{Sin}[\theta]).\underline{d}[\varphi]}{4\,\pi}$

Inverting: $\{e^1 \to \underline{d}[r],\ e^2 \to r\,\underline{d}[\varphi]\,\mathrm{Sin}[\theta],\ e^3 \to r\,\underline{d}[\theta]\}$

$\longrightarrow\ \{\underline{d}[r] \to e^1,\ \underline{d}[\varphi] \to \frac{\mathrm{Csc}[\theta]\,e^2}{r},\ \underline{d}[\theta] \to \frac{e^3}{r}\}$

$\Rightarrow\ F \to \frac{g.(e^2 \wedge e^3)}{4\,\pi\,r^2}$

Orthogonal coordinate: $e^2 \wedge e^3 \to e^3$ (radial)

$\rightarrow\ F \to \frac{g.e^3}{4\,\pi\,r^2}$

IX.7.4

```
PR["●IX.7.4: Calculate curvature.  Using the previous algorithm on: ",
 $ds = d[s]^2 → Ω[x, y]^2 (d[x]^2 + d[y]^2) /. d[a_] → DifForm[a],
 Imply, "•Vielbeins and their DifForm: ",
 $vb = {T[e, "u"][1] → Ω[x, y] DifForm[x], T[e, "u"][2] → Ω[x, y] DifForm[y]},
 Yield, $de = Map[Map[DifForm[#] &, #] &, $vb];
 Yield, $de = $de // tuStdDifForm[{}, {x, y, Ω[_, _]}, {}]; Column[$de],
 NL, "•From the definition: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β],
 Yield, $ = Table[MapAt[Sum[#, {β, 1, 2}] &, $0, 2], {α, 1, 2}] /. T[ω, "ud"][a_, a_] → 0 //
   tuStdDifForm[{}, {x, y, Ω[_, _]}, {}];
 Column[$],
 yield, $ = MapAt[# /. $vb &, #, 2] & /@ $ // tuStdDifForm[{}, {x, y, Ω[_, _]}, {}];
 Column[$],
 NL, "Since: ",
 $sw0 = $s = {T[ω, "ud"][i_, j_] -> T[ω, "udd"][i, j, k].DifForm[T[x, "u"][k]],
    T[x, "u"][a_] → a},
 Yield, $s[[1, 2]] = Sum[$s[[1, 2]], {k, {x, y}}] /. $s; $s,
 Imply, $pass = $ = $ /. $s // tuStdDifForm[{}, {x, y, Ω[_, _]}, {}];
 Column[$]
]
```

●IX.7.4: Calculate curvature. Using the previous algorithm on:

$\underline{d}[s]^2 \to (\underline{d}[x]^2 + \underline{d}[y]^2)\,\Omega[x, y]^2$

$\Rightarrow$ •Vielbeins and their DifForm: $\{e^1 \to \underline{d}[x]\,\Omega[x, y],\ e^2 \to \underline{d}[y]\,\Omega[x, y]\}$

$\to$

$$\begin{array}{l} \underline{d}[e^1] \to -(\underline{d}[x] \wedge \underline{d}[\Omega[x, y]]) \\ \underline{d}[e^2] \to -(\underline{d}[y] \wedge \underline{d}[\Omega[x, y]]) \end{array}$$

•From the definition: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta . e^\beta$

$$\to \begin{array}{l} \underline{d}[e^1] \to -\omega^1{}_2 . e^2 \\ \underline{d}[e^2] \to -\omega^2{}_1 . e^1 \end{array} \longrightarrow \begin{array}{l} \underline{d}[e^1] \to -\omega^1{}_2 . \underline{d}[y]\,\Omega[x, y] \\ \underline{d}[e^2] \to -\omega^2{}_1 . \underline{d}[x]\,\Omega[x, y] \end{array}$$

Since: $\{\omega^{i_-}{}_{j_-} \to \omega^i{}_{jk} . \underline{d}[x^k],\ x^{a_-} \to a\}$

$\to \{\omega^{i_-}{}_{j_-} \to \omega^i{}_{jx} . \underline{d}[x] + \omega^i{}_{jy} . \underline{d}[y],\ x^{a_-} \to a\}$

$$\Rightarrow \begin{array}{l} \underline{d}[e^1] \to -\omega^1{}_{2x} . (\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y] \\ \underline{d}[e^2] \to \omega^2{}_{1y} . (\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y] \end{array}$$

```
PR["•Comparing: ", $ = {$pass, $de}; Column[$],
 NL, "Letting: ", $s =
  DifForm[Ω[x, y]] → xPartialD[Ω[x, y], x] DifForm[x] + xPartialD[Ω[x, y], y] DifForm[y],
 Yield, $ = $ /. $s // Flatten; Framed[$],
 NL, "Eliminating: ",
 $v = $ // ExtractPattern[DifForm[Tensor[e, _, _]]] // DeleteDuplicates,
 Imply, $ = xEliminate[$, $v] // tuStdDifForm[{}, {x, y, Ω[_, _]}, {}],
 NL, "?If no Dot, can we Solve these equations for ω? ",
 Yield, $ = $ /. Dot → Times // Apply[List, #] &; Framed[Column[$]],
 NL, "Set coefficients of Wedge[]s ->0 : ",
 Yield, $ = $ /. a_ == b_ → a - b == 0 // Collect[#, Wedge[__], Zero[#] &] &,
 Yield, $ = $ // ExtractPattern[Zero[_]] // DeleteDuplicates;
 Yield, $ = $ /. Zero[a_] → (a → 0); Framed[$],
 NL, "ω[] are anti-symmetric: ",
 $s = tt : T[ω, "udd"][a_, b_, c_] :→ (tt -> -T[ω, "udd"][b, a, c]) /; b < a, "POFF",
 NL, "Extract ω[]s: ", $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates,
 Yield, $wr = $w /. $s, "PONdd",
 Yield, $wr = Cases[$wr, Rule[__]],
 Yield, $ = $ /. $wr,
 NL, "Solve for ω[]s: ", $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates,
 Yield, $sw = xRuleX[$, $w]; Framed[Column[$sw]]
]
```

•Comparing: $\{\underline{d}[e^1] \to -\omega^1{}_{2x}.(\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y],\ \underline{d}[e^2] \to \omega^2{}_{1y}.(\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y]\}$
$\{\underline{d}[e^1] \to -(\underline{d}[x] \wedge \underline{d}[\Omega[x, y]]),\ \underline{d}[e^2] \to -(\underline{d}[y] \wedge \underline{d}[\Omega[x, y]])\}$

Letting: $\underline{d}[\Omega[x, y]] \to \underline{d}[x]\,\underline{\partial}_x[\Omega[x, y]] + \underline{d}[y]\,\underline{\partial}_y[\Omega[x, y]]$

$$\to \boxed{\begin{array}{l}\{\underline{d}[e^1] \to -\omega^1{}_{2x}.(\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y],\ \underline{d}[e^2] \to \omega^2{}_{1y}.(\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y],\\ \underline{d}[e^1] \to -(\underline{d}[x] \wedge (\underline{d}[x]\,\underset{-x}{\partial}[\Omega[x, y]])) - \underline{d}[x] \wedge (\underline{d}[y]\,\underset{-y}{\partial}[\Omega[x, y]]),\\ \underline{d}[e^2] \to -(\underline{d}[y] \wedge (\underline{d}[x]\,\underset{-x}{\partial}[\Omega[x, y]])) - \underline{d}[y] \wedge (\underline{d}[y]\,\underset{-y}{\partial}[\Omega[x, y]])\}\end{array}}$$

Eliminating: $\{\underline{d}[e^1], \underline{d}[e^2]\}$

$\Rightarrow$ $0 == -\underline{\partial}_y[\Omega[x, y]].(\underline{d}[x] \wedge \underline{d}[y]) + \omega^1{}_{2x}.(\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y]$ &&
$-\underline{\partial}_x[\Omega[x, y]].(\underline{d}[x] \wedge \underline{d}[y]) == -\omega^2{}_{1y}.(\underline{d}[x] \wedge \underline{d}[y])\,\Omega[x, y]$

?If no Dot, can we Solve these equations for ω?

$$\to \boxed{\begin{array}{l}0 == -(\underline{d}[x] \wedge \underline{d}[y])\,\underset{-y}{\partial}[\Omega[x, y]] + \omega^1{}_{2x}\,\underline{d}[x] \wedge \underline{d}[y]\,\Omega[x, y]\\ -(\underline{d}[x] \wedge \underline{d}[y])\,\underset{-x}{\partial}[\Omega[x, y]] == -\omega^2{}_{1y}\,\underline{d}[x] \wedge \underline{d}[y]\,\Omega[x, y]\end{array}}$$

Set coefficients of Wedge[]s ->0 :

$\to$ $\{\underline{d}[x] \wedge \underline{d}[y]\,\mathrm{Zero}[\underline{\partial}_y[\Omega[x, y]] - \omega^1{}_{2x}\,\Omega[x, y]] == \mathrm{Zero}[0],$
$\underline{d}[x] \wedge \underline{d}[y]\,\mathrm{Zero}[-\underline{\partial}_x[\Omega[x, y]] + \omega^2{}_{1y}\,\Omega[x, y]] == \mathrm{Zero}[0]\}$

$\to$

$$\to \boxed{\{\underset{-y}{\partial}[\Omega[x, y]] - \omega^1{}_{2x}\,\Omega[x, y] \to 0,\ 0 \to 0,\ -\underset{-x}{\partial}[\Omega[x, y]] + \omega^2{}_{1y}\,\Omega[x, y] \to 0\}}$$

ω[] are anti-symmetric: $tt : \omega^{a}{}_{b\_\,c\_} \mapsto (tt \to -T[\omega, udd][b, a, c])\ /;\ b < a$

....

$\to$ $\{\omega^2{}_{1y} \to -\omega^1{}_{2y}\}$

$\to$ $\{\underline{\partial}_y[\Omega[x, y]] - \omega^1{}_{2x}\,\Omega[x, y] \to 0,\ 0 \to 0,\ -\underline{\partial}_x[\Omega[x, y]] - \omega^1{}_{2y}\,\Omega[x, y] \to 0\}$

Solve for ω[]s: $\{\omega^1{}_{2x}, \omega^1{}_{2y}\}$

$$\to \boxed{\begin{array}{l}\omega^1{}_{2x} \to \dfrac{\underset{-y}{\partial}[\Omega[x,y]]}{\Omega[x,y]}\\ \omega^1{}_{2y} \to -\dfrac{\underset{-x}{\partial}[\Omega[x,y]]}{\Omega[x,y]}\end{array}}$$

```
PR["•Curvature form: ",
 $ = R → DifForm[ω] + ω.ω,
 Yield,
 $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] :→ Dot[(a /. β → β1), (b /. α → β1)],
 NL, "Since ", $sw0,
 Yield, $ = $ /. $sw0 /. Dot[a_, a1_, b_, b1_] :→ Dot[a, a1, (b /. k → k1), (b1 /. k → k1)],
 Yield, $ = $ // tuStdDifForm[{}, {Tensor[_, _, _]}, {}],
 Yield,
 $ = $ /. DifForm[tt : Tensor[ω, _, _]] → xPartialD[tt, k1].DifForm[T[x, "u"][k1]] //
   tuStdDifForm[{}, {Tensor[_, _, _]}, {}],
 Yield, $ = $ /. Dot → Times // Simplify,
 Imply, $0 =
  $ = $ /. T[R, "ud"][α, β] -> T[R, "uddd"][α, β, k, k1] /. Wedge[_] → 1 /. {k → x, k1 → y};
 Framed[$],
 NL, "ω is antisymmetric and α,β,β1 ∈ {1,2} : ",
 $s = tt : T[ω, "udd"][a_, β1, k_] T[ω, "udd"][β1, b_, k1_] :→ 0,
 Yield, $ = $ /. $s,
 NL, "Applying ", $sw,
 Yield, $rxy = $ /. {α → 1, β → 2} /. $sw,
 and, $ryx = Swap[{x, y}][$] /. {α → 1, β → 2} /. $sw,
 Yield, $ = T[R, "ud"][1, 2] →
   $rxy[[1]].DifForm[x].DifForm[y] + $ryx[[1]].DifForm[y].DifForm[x],
 Yield, $ = $ /. $rxy /. $ryx // tuStdDifForm[{}, {Tensor[_, _, _]}, {}],
 Yield, $ = $ /. xPartialD[-a_, b_] → -xPartialD[a, b] /. Dot → Times // Simplify,
 NL, "Inverting using vielbein: ", $vbi = xRuleX[$vb, {DifForm[x], DifForm[y]}],
 Yield, $ = $ /. $vbi // tuStdDifForm[{}, {Ω[x, y]}, {{Tensor[e, _, _] 1}}],
 Yield, $ = $ /. Dot → Times // Simplify,
 Yield,
 $ = $ /. T[R, "ud"][α_, β_] -> T[R, "uddd"][α, β, k, k1] /. Wedge[_] → 1 /. {k → 1, k1 → 2};
 Framed[$],
 NL, "This is the same as: ", $[[1]] → Swap[{1, 2}][$[[1]]],
 NL, "and the sum over up-down indices: ",
 yield, T[R, "dd"][1, 1] -> T[R, "dd"][2, 2] → R / 2
]
```

•Curvature form: $R \to \underline{d}[\omega] + \omega \cdot \omega$

$\to R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta}] + \omega^{\alpha}{}_{\beta 1} \cdot \omega^{\beta 1}{}_{\beta}$

Since $\{\omega^{i\_}{}_{j\_} \to \omega^{i}{}_{jk} \cdot \underline{d}[x^{k}],\ x^{a\_} \to a\}$

$\to R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta k} \cdot \underline{d}[x^{k}]] + \omega^{\alpha}{}_{\beta 1 k} \cdot \underline{d}[x^{k}] \cdot \omega^{\beta 1}{}_{\beta k1} \cdot \underline{d}[x^{k1}]$

$\to R^{\alpha}{}_{\beta} \to \omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}] - \underline{d}[x^{k}] \wedge \underline{d}[\omega^{\alpha}{}_{\beta k}]$

$\to R^{\alpha}{}_{\beta} \to -\underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta k}] \cdot (\underline{d}[x^{k}] \wedge \underline{d}[x^{k1}]) + \omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}]$

$\to R^{\alpha}{}_{\beta} \to \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}]\, (\omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1} - \underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta k}])$

$$\Rightarrow \boxed{R^{\alpha}{}_{\beta x y} \to \omega^{\alpha}{}_{\beta 1 x}\, \omega^{\beta 1}{}_{\beta y} - \underset{-y}{\partial}[\omega^{\alpha}{}_{\beta x}]}$$

ω is antisymmetric and $\alpha,\beta,\beta 1 \in \{1,2\}$: tt : $\omega^{\beta 1}{}_{b\_ k1\_}\, \omega^{a\_}{}_{\beta 1 k\_} :\to 0$

$\to R^{\alpha}{}_{\beta x y} \to -\underline{\partial}_{y}[\omega^{\alpha}{}_{\beta x}]$

Applying $\left\{\omega^{1}{}_{2x} \to \dfrac{\underline{\partial}_{y}[\Omega[x, y]]}{\Omega[x, y]},\ \omega^{1}{}_{2y} \to -\dfrac{\underline{\partial}_{x}[\Omega[x, y]]}{\Omega[x, y]}\right\}$

$\to R^{1}{}_{2xy} \to -\underline{\partial}_{y}\left[\dfrac{\underline{\partial}_{y}[\Omega[x, y]]}{\Omega[x, y]}\right]$ and $R^{1}{}_{2yx} \to -\underline{\partial}_{x}\left[-\dfrac{\underline{\partial}_{x}[\Omega[x, y]]}{\Omega[x, y]}\right]$

$\to R^{1}{}_{2} \to R^{1}{}_{2xy} \cdot \underline{d}[x] \cdot \underline{d}[y] + R^{1}{}_{2yx} \cdot \underline{d}[y] \cdot \underline{d}[x]$

$$\to R^{1}{}_{2} \to -\underline{\partial}_{x}[\underline{\partial}_{x}[\Omega[x, y]]] \cdot \frac{1}{\Omega[x, y]} \cdot (\underline{d}[x] \wedge \underline{d}[y]) - \underline{\partial}_{y}[\underline{\partial}_{y}[\Omega[x, y]]] \cdot \frac{1}{\Omega[x, y]} \cdot (\underline{d}[x] \wedge \underline{d}[y]) + \underline{\partial}_{x}[\Omega[x, y]]^{2} \cdot \frac{1}{\Omega[x, y]^{2}} \cdot (\underline{d}[x] \wedge \underline{d}[y]) + \underline{\partial}_{y}[\Omega[x, y]]^{2} \cdot \frac{1}{\Omega[x, y]^{2}} \cdot (\underline{d}[x] \wedge \underline{d}[y])$$

$$\to R^{1}{}_{2} \to \frac{\underline{d}[x] \wedge \underline{d}[y]\, (\underline{\partial}_{x}[\Omega[x, y]]^{2} + \underline{\partial}_{y}[\Omega[x, y]]^{2} - (\underline{\partial}_{x}[\underline{\partial}_{x}[\Omega[x, y]]] + \underline{\partial}_{y}[\underline{\partial}_{y}[\Omega[x, y]]])\, \Omega[x, y])}{\Omega[x, y]^{2}}$$

Inverting using vielbein: $\left\{\underline{d}[x] \to \dfrac{e^{1}}{\Omega[x, y]},\ \underline{d}[y] \to \dfrac{e^{2}}{\Omega[x, y]}\right\}$

$$\to R^{1}{}_{2} \to \frac{\underline{\partial}_{x}[\Omega[x, y]]^{2} \cdot (e^{1} \wedge e^{2})}{\Omega[x, y]^{4}} + \frac{\underline{\partial}_{y}[\Omega[x, y]]^{2} \cdot (e^{1} \wedge e^{2})}{\Omega[x, y]^{4}} - \frac{\underline{\partial}_{x}[\underline{\partial}_{x}[\Omega[x, y]]] \cdot (e^{1} \wedge e^{2})}{\Omega[x, y]^{3}} - \frac{\underline{\partial}_{y}[\underline{\partial}_{y}[\Omega[x, y]]] \cdot (e^{1} \wedge e^{2})}{\Omega[x, y]^{3}}$$

$$\to R^{1}{}_{2} \to \frac{e^{1} \wedge e^{2}\, (\underline{\partial}_{x}[\Omega[x, y]]^{2} + \underline{\partial}_{y}[\Omega[x, y]]^{2} - (\underline{\partial}_{x}[\underline{\partial}_{x}[\Omega[x, y]]] + \underline{\partial}_{y}[\underline{\partial}_{y}[\Omega[x, y]]])\, \Omega[x, y])}{\Omega[x, y]^{4}}$$

$$\to \boxed{R^{1}{}_{212} \to \frac{\underset{-x}{\partial}[\Omega[x, y]]^{2} + \underset{-y}{\partial}[\Omega[x, y]]^{2} - (\underset{-x}{\partial}[\underset{-x}{\partial}[\Omega[x, y]]] + \underset{-y}{\partial}[\underset{-y}{\partial}[\Omega[x, y]]])\, \Omega[x, y]}{\Omega[x, y]^{4}}}$$

This is the same as: $R^{1}{}_{212} \to R^{2}{}_{121}$

and the sum over up-down indices: $\longrightarrow R_{11} \to R_{22} \to \dfrac{R}{2}$

IX.8 examples

```
PR["●IX.8: Go through examples p.608: ",
 $ds = d[s]^2 → (d[r]^2 + d[x]^2) / r^2 /. d[a_] → DifForm[a],
 NL, "Vielbein and DifForm: ",
 $vb = {T[e, "u"][1] → d[r] / r, T[e, "u"][2] → d[x] / r} /. d[a_] → DifForm[a],
 Imply, xtmp = $de = Map[Map[DifForm[#] &, #] &, $vb];
 Yield, $de = $de // tuStdDifForm[{}, {x, r}, {}]; (*CHECK*)
 Column[$de],
 NL, "•From the definition: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β],
 yield, $ = Table[MapAt[Sum[#, {β, 1, 2}] &, $0, 2], {α, 1, 2}] /. T[ω, "ud"][a_, a_] → 0 //
   tuStdDifForm[{}, {x, r}, {}];
 Column[$],
 yield, $ = MapAt[# /. $vb &, #, 2] & /@ $ // tuStdDifForm[{}, {x, r}, {}];
 Column[$],
 NL, "Since: ",
 $sw0 = $s = {T[ω, "ud"][i_, j_] -> T[ω, "udd"][i, j, k].DifForm[T[x, "u"][k]],
    T[x, "u"][a_] → a},
 Yield, $s[[1, 2]] = Sum[$s[[1, 2]], {k, {x, r}}] /. $s; $s,
 Imply, $pass = $ = $ /. $s // tuStdDifForm[{}, {x, r}, {}]; Column[$]
]
```

●IX.8: Go through examples p.608: $\underline{d}[s]^2 \to \dfrac{\underline{d}[r]^2 + \underline{d}[x]^2}{r^2}$

Vielbein and DifForm: $\{e^1 \to \dfrac{\underline{d}[r]}{r},\ e^2 \to \dfrac{\underline{d}[x]}{r}\}$

⇒

$$\to \begin{array}{l} \underline{d}[e^1] \to 0 \\ \underline{d}[e^2] \to -\dfrac{\underline{d}[r]\wedge\underline{d}[x]}{r^2} \end{array}$$

•From the definition: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta . e^\beta \longrightarrow \begin{array}{l} \underline{d}[e^1] \to -\omega^1{}_2 . e^2 \\ \underline{d}[e^2] \to -\omega^2{}_1 . e^1 \end{array} \longrightarrow \begin{array}{l} \underline{d}[e^1] \to -\dfrac{\omega^1{}_2 . \underline{d}[x]}{r} \\ \underline{d}[e^2] \to -\dfrac{\omega^2{}_1 . \underline{d}[r]}{r} \end{array}$

Since: $\{\omega^{i\_}{}_{j\_} \to \omega^i{}_{jk} . \underline{d}[x^k],\ x^{a\_} \to a\}$

$\to \{\omega^{i\_}{}_{j\_} \to \omega^i{}_{jr} . \underline{d}[r] + \omega^i{}_{jx} . \underline{d}[x],\ x^{a\_} \to a\}$

$$\Rightarrow \begin{array}{l} \underline{d}[e^1] \to -\dfrac{\omega^1{}_{2r} . (\underline{d}[r]\wedge\underline{d}[x])}{r} \\ \underline{d}[e^2] \to \dfrac{\omega^2{}_{1x} . (\underline{d}[r]\wedge\underline{d}[x])}{r} \end{array}$$

```
PR["•Comparing: ", $ = {$pass, $de}; Column[$],
 NL, "Eliminating: ",
 $v = $ // ExtractPattern[DifForm[Tensor[e, _, _]]] // DeleteDuplicates,
 Imply, $ = xEliminate[$, $v] // tuStdDifForm[{}, {x, y, Ω[_, _]}, {}], CK,
 NL, "Solve these equations for ω ",
 Yield, $ = $[[1 ;; 2]] /. Dot → Times // Apply[List, #] &;
 Framed[Column[$]],
 NL, "Set coefficients of Wedge[]s ->0 : ",
 Yield, $ = $ /. a_ == b_ → a - b == 0 // Collect[#, Wedge[__], Zero[#] &] &;
 yield, $ = $ /. Zero[0] → 0 // ExtractPattern[Zero[_]] // DeleteDuplicates;
 yield, $ = $ /. Zero[a_] → (a → 0);
 (*Use antisymmetry of ω*)
 $ = $ /. T[ω, "udd"][a_, b_, c_] :→ -T[ω, "udd"][b, a, c] /; b < a;
 Framed[$],
 $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates;
 yield, $sw = xRuleX[$, $w];
 FramedColumn[$sw]
]
```

•Comparing: $\{d[\underline{e}^1] \to -\frac{\omega^1{}_{2\,r}\cdot(d[\underline{r}]\wedge d[\underline{x}])}{r},\ d[\underline{e}^2] \to \frac{\omega^2{}_{1\,x}\cdot(d[\underline{r}]\wedge d[\underline{x}])}{r}\}$

$\{d[\underline{e}^1] \to 0,\ d[\underline{e}^2] \to -\frac{d[\underline{r}]\wedge d[\underline{x}]}{r^2}\}$

Eliminating: $\{\underline{d}[e^1], \underline{d}[e^2]\}$

$\Rightarrow\ \omega^1{}_{2\,r}\cdot(\underline{d}[r]\wedge\underline{d}[x]) == 0\ \&\&\ \omega^2{}_{1\,x}\cdot(\underline{d}[r]\wedge\underline{d}[x]) == -\frac{1}{r}\cdot(\underline{d}[r]\wedge\underline{d}[x])\ \&\&\ r \neq 0 \longleftarrow$ CHECK

Solve these equations for ω

$\rightarrow \boxed{\begin{array}{l}\omega^1{}_{2\,r}\, d[\underline{r}]\wedge d[\underline{x}] == 0 \\ \omega^2{}_{1\,x}\, d[\underline{r}]\wedge d[\underline{x}] == -\frac{d[\underline{r}]\wedge d[\underline{x}]}{r}\end{array}}$

Set coefficients of Wedge[]s ->0 :

$\rightarrow\ \longrightarrow\ \longrightarrow \boxed{\{\omega^1{}_{2\,r} \to 0,\ \frac{1}{r} - \omega^1{}_{2\,x} \to 0\}} \longrightarrow \boxed{\begin{array}{l}\omega^1{}_{2\,r} \to 0 \\ \omega^1{}_{2\,x} \to \frac{1}{r}\end{array}}$

```
PR["•Curvature form: ",
 $ = R → DifForm[ω] + ω.ω,
 Yield,
 $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] :→ Dot[(a /. β → β1), (b /. α → β1)],
 NL, "Since ", $sw0,
 Yield, $ = $ /. $sw0 /. Dot[a_, a1_, b_, b1_] :→ Dot[a, a1, (b /. k → k1), (b1 /. k → k1)],
 Yield, $ = $ // tuStdDifForm[{}, {Tensor[_, _, _]}, {}],
 Yield,
 xtmp = $ = $ /. DifForm[tt : Tensor[ω, _, _]] → xPartialD[tt, k1].DifForm[T[x, "u"][k1]],
 Yield, $ = $ // tuStdDifForm[{}, {Tensor[ω, _, _], xPartialD[_, _]}, {}],
 Yield, $ = $ /. Dot → Times // Simplify,
 Imply, $0 =
  $ = $ /. T[R, "ud"][α, β] -> T[R, "uddd"][α, β, k, k1] /. Wedge[_] → 1 /. {k → x, k1 → r};
 Framed[$],
 NL, "ω is antisymmetric and α,β,β1 ∈ {1,2} : ",
 $s = tt : T[ω, "udd"][a_, β1, k_] T[ω, "udd"][β1, b_, k1_] :→ 0,
 Yield, $ = $ /. $s; Framed[$],
 NL, "Applying ", $sw,
 Yield, $rxy = {$, Swap[{x, r}][$]} /. {α → 1, β → 2} /. $sw // DerivativeExpand[{}],
 Yield, $ = T[R, "ud"][1, 2] →
   $rxy[[1, 1]].DifForm[x].DifForm[r] + $rxy[[2, 1]].DifForm[r].DifForm[x],
 Yield, $ = $ /. $rxy // tuStdDifForm[{}, {Tensor[_, _, _]}, {}],
 Yield, $ = $ /. Dot → Times // Simplify,
 NL, "Inverting using vielbein: ", $vbi = xRuleX[$vb, {DifForm[x], DifForm[r]}],

 Yield, $ = $ /. $vbi // tuStdDifForm[{}, {Ω[x, y]}, {{Tensor[e, _, _] 1}}], CK,
 Yield, $ = $ /. Dot → Times // Simplify,
 Yield,
 $ = $ /. T[R, "ud"][α_, β_] -> T[R, "uddd"][α, β, k, k1] /. Wedge[_] → 1 /. {k → 1, k1 → 2};
 Framed[$], CG["(9)"],
 NL, "This is the same as: ", $[[1]] → Swap[{1, 2}][$[[1]]],
 NL, "and the sum over up-down indices: ",
 yield, T[R, "dd"][1, 1] -> T[R, "dd"][2, 2] → R / 2,
 NL, "In world coordinates: ", $vb,
 imply, $vbw = $vb /. {(T[e, "u"][a_] → DifForm[x_] b_) → (T[e, "ud"][a, x] → b)},
 Yield, $ = T[e, "ud"][2, x].T[e, "ud"][2, x].# & /@ $,
 Yield, $[[1]] = T[R, "dd"][x, x];
 $ = $ /. $vbw /. Dot → Times;
 Framed[$], CG["(10)"]
]
```

•Curvature form: $R \to \underline{d}[\omega] + \omega . \omega$

$\to R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta}] + \omega^{\alpha}{}_{\beta 1} . \omega^{\beta 1}{}_{\beta}$

Since $\{\omega^{i\_}{}_{j\_} \to \omega^{i}{}_{j\,k} . \underline{d}[x^{k}],\ x^{a\_} \to a\}$

$\to R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta\,k} . \underline{d}[x^{k}]] + \omega^{\alpha}{}_{\beta 1\,k} . \underline{d}[x^{k}] . \omega^{\beta 1}{}_{\beta\,k1} . \underline{d}[x^{k1}]$

$\to R^{\alpha}{}_{\beta} \to \omega^{\alpha}{}_{\beta 1\,k}\, \omega^{\beta 1}{}_{\beta\,k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}] - \underline{d}[x^{k}] \wedge \underline{d}[\omega^{\alpha}{}_{\beta\,k}]$

$\to R^{\alpha}{}_{\beta} \to \omega^{\alpha}{}_{\beta 1\,k}\, \omega^{\beta 1}{}_{\beta\,k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}] - \underline{d}[x^{k}] \wedge \underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta\,k}] . \underline{d}[x^{k1}]$

$\to R^{\alpha}{}_{\beta} \to \omega^{\alpha}{}_{\beta 1\,k}\, \omega^{\beta 1}{}_{\beta\,k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}] - \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}]\, \underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta\,k}]$

$\to R^{\alpha}{}_{\beta} \to \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}]\, (\omega^{\alpha}{}_{\beta 1\,k}\, \omega^{\beta 1}{}_{\beta\,k1} - \underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta\,k}])$

$\Rightarrow \boxed{R^{\alpha}{}_{\beta\,x\,r} \to \omega^{\alpha}{}_{\beta 1\,x}\, \omega^{\beta 1}{}_{\beta\,r} - \underset{-r}{\partial}[\omega^{\alpha}{}_{\beta\,x}]}$

ω is antisymmetric and $\alpha,\beta,\beta 1 \in \{1,2\}$: $tt : \omega^{\beta 1}{}_{b\_\,k1\_}\, \omega^{a\_}{}_{\beta 1\,k\_} :\to 0$

$\to \boxed{R^{\alpha}{}_{\beta\,x\,r} \to -\underset{-r}{\partial}[\omega^{\alpha}{}_{\beta\,x}]}$

Applying $\{\omega^{1}{}_{2\,r} \to 0,\ \omega^{1}{}_{2\,x} \to \frac{1}{r}\}$

$\to \{R^{1}{}_{2\,x\,r} \to \frac{1}{r^{2}},\ R^{1}{}_{2\,r\,x} \to 0\}$

$\to R^{1}{}_{2} \to R^{1}{}_{2\,r\,x} . \underline{d}[r] . \underline{d}[x] + R^{1}{}_{2\,x\,r} . \underline{d}[x] . \underline{d}[r]$

$\to R^{1}{}_{2} \to -\frac{1}{r^{2}} . (\underline{d}[r] \wedge \underline{d}[x])$

$\to R^{1}{}_{2} \to -\frac{\underline{d}[r] \wedge \underline{d}[x]}{r^{2}}$

Inverting using vielbein: $\{\underline{d}[x] \to r\, e^{2},\ \underline{d}[r] \to r\, e^{1}\}$

$\to R^{1}{}_{2} \to -(e^{1} \wedge e^{2})$ $\longleftarrow$CHECK

$\to R^{1}{}_{2} \to -(e^{1} \wedge e^{2})$

$\to \boxed{R^{1}{}_{2\,1\,2} \to -1}$ (9)

This is the same as: $R^{1}{}_{2\,1\,2} \to R^{2}{}_{1\,2\,1}$

and the sum over up-down indices: $\longrightarrow R_{1\,1} \to R_{2\,2} \to \frac{R}{2}$

In world coordinates: $\{e^{1} \to \frac{\underline{d}[r]}{r},\ e^{2} \to \frac{\underline{d}[x]}{r}\} \Rightarrow \{e^{1}{}_{r} \to \frac{1}{r},\ e^{2}{}_{x} \to \frac{1}{r}\}$

$\to e^{2}{}_{x} . e^{2}{}_{x} . R^{1}{}_{2\,1\,2} \to e^{2}{}_{x} . e^{2}{}_{x} . (-1)$

$\to \boxed{R_{x\,x} \to -\frac{1}{r^{2}}}$ (10)

IX.8 example (11)-(24)

```
PR["●IX.8: Go through examples p.608 Expanding universe: ",
 $ds = d[s]^2 → -d[t]^2 + a[t]^2 ((d[x]^2 + d[y]^2 + d[z]^2)),
 Yield, $ds = $ds // Expand,
 NL, "Vielbein and DifForm: ",
 $vb = $ds // ExtractPattern[ (a_ d[b_])^2];
 $vb = √# & /@ $vb // PowerExpand // MapIndexed[T[e, "u"][#2[[1]]] → #1 &, #] &;
 $vb = Append[$vb, T[e, "u"][0] → d[t]] /. d[a_] → DifForm[a],
 Imply, $de = Map[Map[DifForm[#] &, #] &, $vb];
 $de = $de // tuStdDifForm[{}, {x, y, z, t}, {}];
 Column[$de],
 yield, $de = $de /. DifForm[a[t]] → xPartialD[a[t], t].DifForm[t] //
   tuStdDifForm[{}, {x, y, z, t}, {}];
 Column[$de],
 NL, "•From the definition: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β],
 yield, $ = Table[MapAt[Sum[#, {β, 0, 3}] &, $0, 2], {α, 0, 3}] /. T[ω, "ud"][a_, a_] → 0 //
   tuStdDifForm[{}, {x, y, z, t}, {}];
 Column[$],
 yield, $ = MapAt[# /. $vb &, #, 2] & /@ $ // tuStdDifForm[{}, {x, y, z, t}, {}];
 Column[$],
 NL, "Since: ",
 $sw0 = $s = {T[ω, "ud"][i_, j_] -> T[ω, "udd"][i, j, k].DifForm[T[x, "u"][k]],
    T[x, "u"][a_] → a},
 Yield, $s[[1, 2]] = Sum[$s[[1, 2]], {k, {t, x, y, z}}] /. $s; $s,
 Imply, $pass = $ = $ /. $s // tuStdDifForm[{}, {x, y, z, t}, {}]; Column[$]
]
```

●IX.8: Go through examples p.608 Expanding universe:

$d[s]^2 \to -d[t]^2 + a[t]^2 (d[x]^2 + d[y]^2 + d[z]^2)$

$\to d[s]^2 \to -d[t]^2 + a[t]^2 d[x]^2 + a[t]^2 d[y]^2 + a[t]^2 d[z]^2$

Vielbein and DifForm: $\{e^1 \to a[t]\, \underline{d}[x],\ e^2 \to a[t]\, \underline{d}[y],\ e^3 \to a[t]\, \underline{d}[z],\ e^0 \to \underline{d}[t]\}$

$$\Rightarrow \begin{array}{l} \underline{d}[e^1] \to -(\underline{d}[x] \wedge \underline{d}[a[t]]) \\ \underline{d}[e^2] \to -(\underline{d}[y] \wedge \underline{d}[a[t]]) \\ \underline{d}[e^3] \to -(\underline{d}[z] \wedge \underline{d}[a[t]]) \\ \underline{d}[e^0] \to 0 \end{array} \longrightarrow \begin{array}{l} \underline{d}[e^1] \to \partial_{\underline{t}}[a[t]].(\underline{d}[t] \wedge \underline{d}[x]) \\ \underline{d}[e^2] \to \partial_{\underline{t}}[a[t]].(\underline{d}[t] \wedge \underline{d}[y]) \\ \underline{d}[e^3] \to \partial_{\underline{t}}[a[t]].(\underline{d}[t] \wedge \underline{d}[z]) \\ \underline{d}[e^0] \to 0 \end{array}$$

•From the definition: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta . e^\beta \longrightarrow \begin{array}{l} \underline{d}[e^0] \to -\omega^0{}_1.e^1 - \omega^0{}_2.e^2 - \omega^0{}_3.e^3 \\ \underline{d}[e^1] \to -\omega^1{}_0.e^0 - \omega^1{}_2.e^2 - \omega^1{}_3.e^3 \\ \underline{d}[e^2] \to -\omega^2{}_0.e^0 - \omega^2{}_1.e^1 - \omega^2{}_3.e^3 \\ \underline{d}[e^3] \to -\omega^3{}_0.e^0 - \omega^3{}_1.e^1 - \omega^3{}_2.e^2 \end{array}$

$$\longrightarrow \begin{array}{l} \underline{d}[e^0] \to -a[t].\omega^0{}_1.\underline{d}[x] - a[t].\omega^0{}_2.\underline{d}[y] - a[t].\omega^0{}_3.\underline{d}[z] \\ \underline{d}[e^1] \to -\omega^1{}_0.\underline{d}[t] - a[t].\omega^1{}_2.\underline{d}[y] - a[t].\omega^1{}_3.\underline{d}[z] \\ \underline{d}[e^2] \to -\omega^2{}_0.\underline{d}[t] - a[t].\omega^2{}_1.\underline{d}[x] - a[t].\omega^2{}_3.\underline{d}[z] \\ \underline{d}[e^3] \to -\omega^3{}_0.\underline{d}[t] - a[t].\omega^3{}_1.\underline{d}[x] - a[t].\omega^3{}_2.\underline{d}[y] \end{array}$$

Since: $\{\omega^{i\_}{}_{j\_} \to \omega^i{}_{jk}.\underline{d}[x^k],\ x^{a\_} \to a\}$

$\to \{\omega^{i\_}{}_{j\_} \to \omega^i{}_{jt}.\underline{d}[t] + \omega^i{}_{jx}.\underline{d}[x] + \omega^i{}_{jy}.\underline{d}[y] + \omega^i{}_{jz}.\underline{d}[z],\ x^{a\_} \to a\}$

$$\Rightarrow \begin{array}{l}
\underline{d}[e^0] \to -a[t].\omega^0{}_{1t}.(\underline{d}[t] \wedge \underline{d}[x]) + a[t].\omega^0{}_{1y}.(\underline{d}[x] \wedge \underline{d}[y]) + a[t].\omega^0{}_{1z}.(\underline{d}[x] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^0{}_{2t}.(\underline{d}[t] \wedge \underline{d}[y]) - a[t].\omega^0{}_{2x}.(\underline{d}[x] \wedge \underline{d}[y]) + a[t].\omega^0{}_{2z}.(\underline{d}[y] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^0{}_{3t}.(\underline{d}[t] \wedge \underline{d}[z]) - a[t].\omega^0{}_{3x}.(\underline{d}[x] \wedge \underline{d}[z]) - a[t].\omega^0{}_{3y}.(\underline{d}[y] \wedge \underline{d}[z]) \\
\underline{d}[e^1] \to \omega^1{}_{0x}.(\underline{d}[t] \wedge \underline{d}[x]) + \omega^1{}_{0y}.(\underline{d}[t] \wedge \underline{d}[y]) + \omega^1{}_{0z}.(\underline{d}[t] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^1{}_{2t}.(\underline{d}[t] \wedge \underline{d}[y]) - a[t].\omega^1{}_{2x}.(\underline{d}[x] \wedge \underline{d}[y]) + a[t].\omega^1{}_{2z}.(\underline{d}[y] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^1{}_{3t}.(\underline{d}[t] \wedge \underline{d}[z]) - a[t].\omega^1{}_{3x}.(\underline{d}[x] \wedge \underline{d}[z]) - a[t].\omega^1{}_{3y}.(\underline{d}[y] \wedge \underline{d}[z]) \\
\underline{d}[e^2] \to \omega^2{}_{0x}.(\underline{d}[t] \wedge \underline{d}[x]) + \omega^2{}_{0y}.(\underline{d}[t] \wedge \underline{d}[y]) + \omega^2{}_{0z}.(\underline{d}[t] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^2{}_{1t}.(\underline{d}[t] \wedge \underline{d}[x]) + a[t].\omega^2{}_{1y}.(\underline{d}[x] \wedge \underline{d}[y]) + a[t].\omega^2{}_{1z}.(\underline{d}[x] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^2{}_{3t}.(\underline{d}[t] \wedge \underline{d}[z]) - a[t].\omega^2{}_{3x}.(\underline{d}[x] \wedge \underline{d}[z]) - a[t].\omega^2{}_{3y}.(\underline{d}[y] \wedge \underline{d}[z]) \\
\underline{d}[e^3] \to \omega^3{}_{0x}.(\underline{d}[t] \wedge \underline{d}[x]) + \omega^3{}_{0y}.(\underline{d}[t] \wedge \underline{d}[y]) + \omega^3{}_{0z}.(\underline{d}[t] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^3{}_{1t}.(\underline{d}[t] \wedge \underline{d}[x]) + a[t].\omega^3{}_{1y}.(\underline{d}[x] \wedge \underline{d}[y]) + a[t].\omega^3{}_{1z}.(\underline{d}[x] \wedge \underline{d}[z]) - \\
\quad a[t].\omega^3{}_{2t}.(\underline{d}[t] \wedge \underline{d}[y]) - a[t].\omega^3{}_{2x}.(\underline{d}[x] \wedge \underline{d}[y]) + a[t].\omega^3{}_{2z}.(\underline{d}[y] \wedge \underline{d}[z])
\end{array}$$

```
PR["•Comparing: ", $ = {$pass, $de}; Column[$],
 NL, "Eliminating: ",
 $v = $ // ExtractPattern[DifForm[Tensor[e, _, _]]] // DeleteDuplicates,
 Imply, $ = xEliminate[$, $v] // tuStdDifForm[{}, {x, y, z, t}, {}],
 NL, "Solve these equations for ω ",
 Yield, $ = $ /. Dot → Times // Apply[List, #] &; Framed[Column[$]],
 NL, "Set coefficients of Wedge[]s ->0 : ",
 Yield, $ = $ /. a_ == b_ → a - b == 0 // Collect[#, Wedge[__], Zero[#] &] &;
 yield, $ = $ /. Zero[0] → 0 // ExtractPattern[Zero[_]] // DeleteDuplicates;
 yield, $ = $ /. Zero[a_] → (a → 0);
 (*Use antisymmetry of ω*)
 $ = $ /. T[ω, "udd"][a_, b_, c_] :→ -T[ω, "udd"][b, a, c] /; b < a;
 Framed[$],
 $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates;
 yield, $sw = xRuleX[$, $w];
 Framed[Column[$sw]]
]
```

```
•Comparing: {d[e^0] → -a[t].ω^0_1t.(d[t]∧d[x]) + a[t].ω^0_1y.(d[x]∧d[y]) + a[t].ω^0_1z.(d[x]∧d[z]) -
               a[t].ω^0_2t.(d[t]∧d[y]) - a[t].ω^0_2x.(d[x]∧d[y]) + a[t].ω^0_2z.(d[y]∧d[z]) -
               a[t].ω^0_3t.(d[t]∧d[z]) - a[t].ω^0_3x.(d[x]∧d[z]) - a[t].ω^0_3y.(d[y]∧d[z]),
             d[e^1] → ω^1_0x.(d[t]∧d[x]) + ω^1_0y.(d[t]∧d[y]) + ω^1_0z.(d[t]∧d[z]) -
               a[t].ω^1_2t.(d[t]∧d[y]) - a[t].ω^1_2x.(d[x]∧d[y]) + a[t].ω^1_2z.(d[y]∧d[z]) -
               a[t].ω^1_3t.(d[t]∧d[z]) - a[t].ω^1_3x.(d[x]∧d[z]) - a[t].ω^1_3y.(d[y]∧d[z]),
             d[e^2] → ω^2_0x.(d[t]∧d[x]) + ω^2_0y.(d[t]∧d[y]) + ω^2_0z.(d[t]∧d[z]) -
               a[t].ω^2_1t.(d[t]∧d[x]) + a[t].ω^2_1y.(d[x]∧d[y]) + a[t].ω^2_1z.(d[x]∧d[z]) -
               a[t].ω^2_3t.(d[t]∧d[z]) - a[t].ω^2_3x.(d[x]∧d[z]) - a[t].ω^2_3y.(d[y]∧d[z]),
             d[e^3] → ω^3_0x.(d[t]∧d[x]) + ω^3_0y.(d[t]∧d[y]) + ω^3_0z.(d[t]∧d[z]) -
               a[t].ω^3_1t.(d[t]∧d[x]) + a[t].ω^3_1y.(d[x]∧d[y]) + a[t].ω^3_1z.(d[x]∧d[z]) -
               a[t].ω^3_2t.(d[t]∧d[y]) - a[t].ω^3_2x.(d[x]∧d[y]) + a[t].ω^3_2z.(d[y]∧d[z])}
            {d[e^1] → ∂_t[a[t]].(d[t]∧d[x]),
             d[e^2] → ∂_t[a[t]].(d[t]∧d[y]), d[e^3] → ∂_t[a[t]].(d[t]∧d[z]), d[e^0] → 0}
Eliminating: {d[e^0], d[e^1], d[e^2], d[e^3]}
⇒ ∂_t[a[t]].(d[t]∧d[x]) == ω^1_0x.(d[t]∧d[x]) + ω^1_0y.(d[t]∧d[y]) + ω^1_0z.(d[t]∧d[z]) -
     a[t].ω^1_2t.(d[t]∧d[y]) - a[t].ω^1_2x.(d[x]∧d[y]) + a[t].ω^1_2z.(d[y]∧d[z]) -
     a[t].ω^1_3t.(d[t]∧d[z]) - a[t].ω^1_3x.(d[x]∧d[z]) - a[t].ω^1_3y.(d[y]∧d[z]) &&
  ∂_t[a[t]].(d[t]∧d[y]) == ω^2_0x.(d[t]∧d[x]) + ω^2_0y.(d[t]∧d[y]) + ω^2_0z.(d[t]∧d[z]) -
     a[t].ω^2_1t.(d[t]∧d[x]) + a[t].ω^2_1y.(d[x]∧d[y]) + a[t].ω^2_1z.(d[x]∧d[z]) -
     a[t].ω^2_3t.(d[t]∧d[z]) - a[t].ω^2_3x.(d[x]∧d[z]) - a[t].ω^2_3y.(d[y]∧d[z]) &&
  ∂_t[a[t]].(d[t]∧d[z]) == ω^3_0x.(d[t]∧d[x]) + ω^3_0y.(d[t]∧d[y]) + ω^3_0z.(d[t]∧d[z]) -
     a[t].ω^3_1t.(d[t]∧d[x]) + a[t].ω^3_1y.(d[x]∧d[y]) + a[t].ω^3_1z.(d[x]∧d[z]) -
     a[t].ω^3_2t.(d[t]∧d[y]) - a[t].ω^3_2x.(d[x]∧d[y]) + a[t].ω^3_2z.(d[y]∧d[z]) &&
  a[t].ω^0_1t.(d[t]∧d[x]) == a[t].ω^0_1y.(d[x]∧d[y]) + a[t].ω^0_1z.(d[x]∧d[z]) -
     a[t].ω^0_2t.(d[t]∧d[y]) - a[t].ω^0_2x.(d[x]∧d[y]) + a[t].ω^0_2z.(d[y]∧d[z]) -
     a[t].ω^0_3t.(d[t]∧d[z]) - a[t].ω^0_3x.(d[x]∧d[z]) - a[t].ω^0_3y.(d[y]∧d[z])
Solve these equations for ω
```

$$\to \begin{aligned}
&\underline{d}[t]\wedge\underline{d}[x]\,\partial_{t}[a[t]] == \\
&\quad \omega^{1}{}_{0x}\,\underline{d}[t]\wedge\underline{d}[x]+\omega^{1}{}_{0y}\,\underline{d}[t]\wedge\underline{d}[y]-a[t]\,\omega^{1}{}_{2t}\,\underline{d}[t]\wedge\underline{d}[y]+\omega^{1}{}_{0z}\,\underline{d}[t]\wedge\underline{d}[z]-a[t]\,\omega^{1}{}_{3t}\,\underline{d}[t]\wedge\underline{d}[z]- \\
&\quad a[t]\,\omega^{1}{}_{2x}\,\underline{d}[x]\wedge\underline{d}[y]-a[t]\,\omega^{1}{}_{3x}\,\underline{d}[x]\wedge\underline{d}[z]+a[t]\,\omega^{1}{}_{2z}\,\underline{d}[y]\wedge\underline{d}[z]-a[t]\,\omega^{1}{}_{3y}\,\underline{d}[y]\wedge\underline{d}[z] \\
&\underline{d}[t]\wedge\underline{d}[y]\,\partial_{t}[a[t]] == \\
&\quad \omega^{2}{}_{0x}\,\underline{d}[t]\wedge\underline{d}[x]-a[t]\,\omega^{2}{}_{1t}\,\underline{d}[t]\wedge\underline{d}[x]+\omega^{2}{}_{0y}\,\underline{d}[t]\wedge\underline{d}[y]+\omega^{2}{}_{0z}\,\underline{d}[t]\wedge\underline{d}[z]-a[t]\,\omega^{2}{}_{3t}\,\underline{d}[t]\wedge\underline{d}[z]+ \\
&\quad a[t]\,\omega^{2}{}_{1y}\,\underline{d}[x]\wedge\underline{d}[y]+a[t]\,\omega^{2}{}_{1z}\,\underline{d}[x]\wedge\underline{d}[z]-a[t]\,\omega^{2}{}_{3x}\,\underline{d}[x]\wedge\underline{d}[z]-a[t]\,\omega^{2}{}_{3y}\,\underline{d}[y]\wedge\underline{d}[z] \\
&\underline{d}[t]\wedge\underline{d}[z]\,\partial_{t}[a[t]] == \\
&\quad \omega^{3}{}_{0x}\,\underline{d}[t]\wedge\underline{d}[x]-a[t]\,\omega^{3}{}_{1t}\,\underline{d}[t]\wedge\underline{d}[x]+\omega^{3}{}_{0y}\,\underline{d}[t]\wedge\underline{d}[y]-a[t]\,\omega^{3}{}_{2t}\,\underline{d}[t]\wedge\underline{d}[y]+\omega^{3}{}_{0z}\,\underline{d}[t]\wedge\underline{d}[z]+ \\
&\quad a[t]\,\omega^{3}{}_{1y}\,\underline{d}[x]\wedge\underline{d}[y]-a[t]\,\omega^{3}{}_{2x}\,\underline{d}[x]\wedge\underline{d}[y]+a[t]\,\omega^{3}{}_{1z}\,\underline{d}[x]\wedge\underline{d}[z]+a[t]\,\omega^{3}{}_{2z}\,\underline{d}[y]\wedge\underline{d}[z] \\
&a[t]\,\omega^{0}{}_{1t}\,\underline{d}[t]\wedge\underline{d}[x] == \\
&\quad -a[t]\,\omega^{0}{}_{2t}\,\underline{d}[t]\wedge\underline{d}[y]-a[t]\,\omega^{0}{}_{3t}\,\underline{d}[t]\wedge\underline{d}[z]+a[t]\,\omega^{0}{}_{1y}\,\underline{d}[x]\wedge\underline{d}[y]-a[t]\,\omega^{0}{}_{2x}\,\underline{d}[x]\wedge\underline{d}[y]+ \\
&\quad a[t]\,\omega^{0}{}_{1z}\,\underline{d}[x]\wedge\underline{d}[z]-a[t]\,\omega^{0}{}_{3x}\,\underline{d}[x]\wedge\underline{d}[z]+a[t]\,\omega^{0}{}_{2z}\,\underline{d}[y]\wedge\underline{d}[z]-a[t]\,\omega^{0}{}_{3y}\,\underline{d}[y]\wedge\underline{d}[z]
\end{aligned}$$

Set coefficients of Wedge[]s ->0 :

→ ⟶ ⟶

$$\begin{aligned}
\{&\omega^{0}{}_{1y}+a[t]\,\omega^{1}{}_{2t}\to 0,\ a[t]\,\omega^{1}{}_{2x}\to 0,\ \omega^{0}{}_{1z}+a[t]\,\omega^{1}{}_{3t}\to 0,\ a[t]\,\omega^{1}{}_{3x}\to 0,\ -a[t]\,\omega^{1}{}_{2z}+a[t]\,\omega^{1}{}_{3y}\to 0, \\
&\omega^{0}{}_{1x}+\partial_{t}[a[t]]\to 0,\ \omega^{0}{}_{2x}-a[t]\,\omega^{1}{}_{2t}\to 0,\ a[t]\,\omega^{1}{}_{2y}\to 0,\ \omega^{0}{}_{2z}+a[t]\,\omega^{2}{}_{3t}\to 0,\ a[t]\,\omega^{1}{}_{2z}+a[t]\,\omega^{2}{}_{3x}\to 0, \\
&a[t]\,\omega^{2}{}_{3y}\to 0,\ \omega^{0}{}_{2y}+\partial_{t}[a[t]]\to 0,\ \omega^{0}{}_{3x}-a[t]\,\omega^{1}{}_{3t}\to 0,\ a[t]\,\omega^{1}{}_{3z}\to 0,\ \omega^{0}{}_{3y}-a[t]\,\omega^{2}{}_{3t}\to 0, \\
&a[t]\,\omega^{1}{}_{3y}-a[t]\,\omega^{2}{}_{3x}\to 0,\ a[t]\,\omega^{2}{}_{3z}\to 0,\ \omega^{0}{}_{3z}+\partial_{t}[a[t]]\to 0,\ a[t]\,\omega^{0}{}_{1t}\to 0,\ a[t]\,\omega^{0}{}_{2t}\to 0, \\
&-a[t]\,\omega^{0}{}_{1y}+a[t]\,\omega^{0}{}_{2x}\to 0,\ a[t]\,\omega^{0}{}_{3t}\to 0,\ -a[t]\,\omega^{0}{}_{1z}+a[t]\,\omega^{0}{}_{3x}\to 0,\ -a[t]\,\omega^{0}{}_{2z}+a[t]\,\omega^{0}{}_{3y}\to 0\}
\end{aligned}$$

$$\longrightarrow \begin{array}{l}
\omega^{0}{}_{1y}\to 0 \\
\omega^{1}{}_{2t}\to 0 \\
\omega^{1}{}_{2x}\to 0 \\
\omega^{0}{}_{1z}\to 0 \\
\omega^{1}{}_{3t}\to 0 \\
\omega^{1}{}_{3x}\to 0 \\
\omega^{1}{}_{2z}\to 0 \\
\omega^{1}{}_{3y}\to 0 \\
\omega^{0}{}_{1x}\to -\partial_{t}[a[t]] \\
\omega^{0}{}_{2x}\to 0 \\
\omega^{1}{}_{2y}\to 0 \\
\omega^{0}{}_{2z}\to 0 \\
\omega^{2}{}_{3t}\to 0 \\
\omega^{2}{}_{3x}\to 0 \\
\omega^{2}{}_{3y}\to 0 \\
\omega^{0}{}_{2y}\to -\partial_{t}[a[t]] \\
\omega^{0}{}_{3x}\to 0 \\
\omega^{1}{}_{3z}\to 0 \\
\omega^{0}{}_{3y}\to 0 \\
\omega^{2}{}_{3z}\to 0 \\
\omega^{0}{}_{3z}\to -\partial_{t}[a[t]] \\
\omega^{0}{}_{1t}\to 0 \\
\omega^{0}{}_{2t}\to 0 \\
\omega^{0}{}_{3t}\to 0
\end{array}$$

```
PR[CG["•Curvature form: ",
  $ = R → DifForm[ω] + ω.ω],
 Yield,
 $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] ⧴ Dot[(a /. β → β1), (b /. α → β1)],
 NL, "Since ", $sw0,
 Yield, $ = $ /. $sw0 /. Dot[a_, a1_, b_, b1_] ⧴ Dot[a, a1, (b /. k → k1), (b1 /. k → k1)],
 Yield, $ = $ // tuStdDifForm[{}, {Tensor[_, _, _]}, {}],
 Yield,
 $ = $ /. DifForm[tt : Tensor[ω, _, _]] → xPartialD[tt, k1].DifForm[T[x, "u"][k1]],
 Yield, $ = $ // tuStdDifForm[{}, {Tensor[ω, _, _], xPartialD[_, _]}, {}],
 Yield, $ = $ /. Dot → Times // Simplify,
 Imply, $0 = $ = $ /. T[R, "ud"][α, β] -> T[R, "uddd"][α, β, k, k1] /. Wedge[_] → 1;
 Framed[xtmp = $], CK,
 NL, "Note that ", $sr = ($[[1]] /. k1 → k) → 0 // RuleX2PatternVar[#, {k, α, β}] &,
 NL, "Expanding the indices: ",
 Yield,
 $ = $ /. tt : T[ω, "udd"][a1_, β1, c1_] T[ω, "udd"][β1, a2_, b2_] ⧴ Sum[tt, {β1, 0, 3}];
 Yield, $ = Table[$ /. {k → i, k1 → j, α → a, β → b}, {i, {t, x, y, z}}, {j, {t, x, y, z}},
    {a, 0, 3}, {b, 0, 3}] /. T[ω, "udd"][a_, b_, c_] ⧴ -T[ω, "udd"][b, a, c] /; b < a;
 NL, "Applying ", $sw,
 $ = $ /. $sw /. T[ω, "udd"][a_, a_, c_] -> 0 // DerivativeExpand[{}] // Flatten;
 $ = $ /. xPartialD[_, x | y | z] → 0;
 $ = $ /. $sr;
 $ = Map[If[(#[[2]] =!= 0) && (#[[1]] =!= 0), #] &, $] // DeleteCases[Null],
 Yield, $ = $ /. Rule → xRule /. xRule[T[R, "uddd"][α_, β_, k_, k1_], cc_] ⧴
    xRule[T[R, "uddd"][β, α, k, k1], -cc] /; α > β,
 Yield, $ = $ /. xRule[T[R, "uddd"][α_, β_, k_, k1_], cc_] ⧴
     xRule[T[R, "uddd"][α, β, k1, k], -cc] /; OrderedQ[{k1, k}] // DeleteDuplicates;
 Yield, $ = $ /. xRule → Rule // Sort;
 Framed[Column[$]]
]
```

•Curvature form: $R \to \underline{d}[\omega] + \omega.\omega$

→ $R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta}] + \omega^{\alpha}{}_{\beta 1} . \omega^{\beta 1}{}_{\beta}$

Since $\{\omega^{i}{}_{-j\_} \to \omega^{i}{}_{j k} . \underline{d}[x^{k}],\ x^{a}{}_{-} \to a\}$

→ $R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta k} . \underline{d}[x^{k}]] + \omega^{\alpha}{}_{\beta 1 k} . \underline{d}[x^{k}] . \omega^{\beta 1}{}_{\beta k1} . \underline{d}[x^{k1}]$

→ $R^{\alpha}{}_{\beta} \to \omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}] - \underline{d}[x^{k}] \wedge \underline{d}[\omega^{\alpha}{}_{\beta k}]$

→ $R^{\alpha}{}_{\beta} \to \omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}] - \underline{d}[x^{k}] \wedge \underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta k}] . \underline{d}[x^{k1}]$

→ $R^{\alpha}{}_{\beta} \to \omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1}\, \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}] - \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}]\, \underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta k}]$

→ $R^{\alpha}{}_{\beta} \to \underline{d}[x^{k}] \wedge \underline{d}[x^{k1}]\, (\omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1} - \underline{\partial}_{k1}[\omega^{\alpha}{}_{\beta k}])$

⇒ $\boxed{R^{\alpha}{}_{\beta k k1} \to \omega^{\alpha}{}_{\beta 1 k}\, \omega^{\beta 1}{}_{\beta k1} - \partial_{-k1}[\omega^{\alpha}{}_{\beta k}]}$ ⟵CHECK

Note that $R^{\alpha}{}_{-\beta\_ k\_ k\_} \to 0$

Expanding the indices:

→

→

Applying $\{\omega^{0}{}_{1 y} \to 0,\ \omega^{1}{}_{2 t} \to 0,\ \omega^{1}{}_{2 x} \to 0,\ \omega^{0}{}_{1 z} \to 0,\ \omega^{1}{}_{3 t} \to 0,\ \omega^{1}{}_{3 x} \to 0,\ \omega^{1}{}_{2 z} \to 0,\ \omega^{1}{}_{3 y} \to 0,$
$\omega^{0}{}_{1 x} \to -\underline{\partial}_{t}[a[t]],\ \omega^{0}{}_{2 x} \to 0,\ \omega^{1}{}_{2 y} \to 0,\ \omega^{0}{}_{2 z} \to 0,\ \omega^{2}{}_{3 t} \to 0,\ \omega^{2}{}_{3 x} \to 0,\ \omega^{2}{}_{3 y} \to 0,\ \omega^{0}{}_{2 y} \to -\underline{\partial}_{t}[a[t]],$
$\omega^{0}{}_{3 x} \to 0,\ \omega^{1}{}_{3 z} \to 0,\ \omega^{0}{}_{3 y} \to 0,\ \omega^{2}{}_{3 z} \to 0,\ \omega^{0}{}_{3 z} \to -\underline{\partial}_{t}[a[t]],\ \omega^{0}{}_{1 t} \to 0,\ \omega^{0}{}_{2 t} \to 0,\ \omega^{0}{}_{3 t} \to 0\}$

$\{R^{0}{}_{1 x t} \to \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]],\ R^{1}{}_{0 x t} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]],\ R^{1}{}_{2 x y} \to -\underline{\partial}_{t}[a[t]]^{2},\ R^{1}{}_{3 x z} \to -\underline{\partial}_{t}[a[t]]^{2},$
$R^{0}{}_{2 y t} \to \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]],\ R^{2}{}_{0 y t} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]],\ R^{2}{}_{1 y x} \to -\underline{\partial}_{t}[a[t]]^{2},\ R^{2}{}_{3 y z} \to -\underline{\partial}_{t}[a[t]]^{2},$
$R^{0}{}_{3 z t} \to \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]],\ R^{3}{}_{0 z t} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]],\ R^{3}{}_{1 z x} \to -\underline{\partial}_{t}[a[t]]^{2},\ R^{3}{}_{2 z y} \to -\underline{\partial}_{t}[a[t]]^{2}\}$

→ $\{\mathrm{xRule}[R^{0}{}_{1 x t}, \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]]],\ \mathrm{xRule}[R^{0}{}_{1 x t}, \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]]],\ \mathrm{xRule}[R^{1}{}_{2 x y}, -\underline{\partial}_{t}[a[t]]^{2}],$
$\mathrm{xRule}[R^{1}{}_{3 x z}, -\underline{\partial}_{t}[a[t]]^{2}],\ \mathrm{xRule}[R^{0}{}_{2 y t}, \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]]],\ \mathrm{xRule}[R^{0}{}_{2 y t}, \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]]],$
$\mathrm{xRule}[R^{1}{}_{2 y x}, \underline{\partial}_{t}[a[t]]^{2}],\ \mathrm{xRule}[R^{2}{}_{3 y z}, -\underline{\partial}_{t}[a[t]]^{2}],\ \mathrm{xRule}[R^{0}{}_{3 z t}, \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]]],$
$\mathrm{xRule}[R^{0}{}_{3 z t}, \underline{\partial}_{t}[\underline{\partial}_{t}[a[t]]]],\ \mathrm{xRule}[R^{1}{}_{3 z x}, \underline{\partial}_{t}[a[t]]^{2}],\ \mathrm{xRule}[R^{2}{}_{3 z y}, \underline{\partial}_{t}[a[t]]^{2}]\}$

→

→ $\boxed{\begin{array}{l} R^{0}{}_{1 t x} \to -\partial_{-t}[\partial_{-t}[a[t]]] \\ R^{0}{}_{2 t y} \to -\partial_{-t}[\partial_{-t}[a[t]]] \\ R^{0}{}_{3 t z} \to -\partial_{-t}[\partial_{-t}[a[t]]] \\ R^{1}{}_{2 x y} \to -\partial_{-t}[a[t]]^{2} \\ R^{1}{}_{3 x z} \to -\partial_{-t}[a[t]]^{2} \\ R^{2}{}_{3 y z} \to -\partial_{-t}[a[t]]^{2} \end{array}}$

```
PR["•Apply world DifForm[]s: ",
 $s = xr : xRule[T[R, "uddd"][a_, b_, c_, d_], e_] ⧴ Map[# .DifForm[c].DifForm[d] &, xr],
 Yield, $x = $ /. Rule → xRule /. $s /. xRule → Rule; Column[$x],
 Yield, $x = Map[MapAt[
     (# /. T[R, "uddd"][a_, b_, t_, x_].DifForm[t_].DifForm[x_] :> T[R, "ud"][a, b]) &,
     #, 1] &, $x],
 Yield, $x = $x /. xRuleX[$vb, Thread[DifForm[{t, x, y, z}]]];
 Yield, $x = $x /. Dot → Times; Column[$x], CG["(16-17)"],
 yield,
 $x = $x /. Rule → xRule /. xRule[T[R, "ud"][a_, b_], c_ T[e, "u"][n_] T[e, "u"][m_]] ->
     xRule[T[R, "uddd"][a, b, n, m], c] /. xRule → Rule;
 FramedColumn[$x]
]
```

•Apply world DifForm[]s: $xr : xRule[R^{a}{}_{b\,c\,d}, e\_] ⧴ (\#1.\underline{d}[c].\underline{d}[d]\,\&)\,/@\,xr$

$R^0{}_{1\,t\,x}.\underline{d}[t].\underline{d}[x] \to (-\underline{\partial}_t[\underline{\partial}_t[a[t]]]).\underline{d}[t].\underline{d}[x]$

$R^0{}_{2\,t\,y}.\underline{d}[t].\underline{d}[y] \to (-\underline{\partial}_t[\underline{\partial}_t[a[t]]]).\underline{d}[t].\underline{d}[y]$

$R^0{}_{3\,t\,z}.\underline{d}[t].\underline{d}[z] \to (-\underline{\partial}_t[\underline{\partial}_t[a[t]]]).\underline{d}[t].\underline{d}[z]$

→ $R^1{}_{2\,x\,y}.\underline{d}[x].\underline{d}[y] \to (-\underline{\partial}_t[a[t]]^2).\underline{d}[x].\underline{d}[y]$

$R^1{}_{3\,x\,z}.\underline{d}[x].\underline{d}[z] \to (-\underline{\partial}_t[a[t]]^2).\underline{d}[x].\underline{d}[z]$

$R^2{}_{3\,y\,z}.\underline{d}[y].\underline{d}[z] \to (-\underline{\partial}_t[a[t]]^2).\underline{d}[y].\underline{d}[z]$

→

$\{R^0{}_1 \to (-\underline{\partial}_t[\underline{\partial}_t[a[t]]]).\underline{d}[t].\underline{d}[x],\ R^0{}_2 \to (-\underline{\partial}_t[\underline{\partial}_t[a[t]]]).\underline{d}[t].\underline{d}[y],\ R^0{}_3 \to (-\underline{\partial}_t[\underline{\partial}_t[a[t]]]).\underline{d}[t].\underline{d}[z],$
$R^1{}_2 \to (-\underline{\partial}_t[a[t]]^2).\underline{d}[x].\underline{d}[y],\ R^1{}_3 \to (-\underline{\partial}_t[a[t]]^2).\underline{d}[x].\underline{d}[z],\ R^2{}_3 \to (-\underline{\partial}_t[a[t]]^2).\underline{d}[y].\underline{d}[z]\}$

→

$R^0{}_1 \to -\frac{e^0\,e^1\,\underline{\partial}_t[\underline{\partial}_t[a[t]]]}{a[t]}$

$R^0{}_2 \to -\frac{e^0\,e^2\,\underline{\partial}_t[\underline{\partial}_t[a[t]]]}{a[t]}$

$R^0{}_3 \to -\frac{e^0\,e^3\,\underline{\partial}_t[\underline{\partial}_t[a[t]]]}{a[t]}$

→ $R^1{}_2 \to -\frac{e^1\,e^2\,\underline{\partial}_t[a[t]]^2}{a[t]^2}$ (16-17) ⟶

$R^1{}_3 \to -\frac{e^1\,e^3\,\underline{\partial}_t[a[t]]^2}{a[t]^2}$

$R^2{}_3 \to -\frac{e^2\,e^3\,\underline{\partial}_t[a[t]]^2}{a[t]^2}$

$$\boxed{\begin{array}{l}
R^0{}_{1\,1\,0} \to -\frac{\underline{\partial}_t[\underline{\partial}_t[a[t]]]}{a[t]} \\
R^0{}_{2\,2\,0} \to -\frac{\underline{\partial}_t[\underline{\partial}_t[a[t]]]}{a[t]} \\
R^0{}_{3\,3\,0} \to -\frac{\underline{\partial}_t[\underline{\partial}_t[a[t]]]}{a[t]} \\
R^1{}_{2\,2\,1} \to -\frac{\underline{\partial}_t[a[t]]^2}{a[t]^2} \\
R^1{}_{3\,3\,1} \to -\frac{\underline{\partial}_t[a[t]]^2}{a[t]^2} \\
R^2{}_{3\,3\,2} \to -\frac{\underline{\partial}_t[a[t]]^2}{a[t]^2}
\end{array}}$$

Maximally symmetric 3-space p.610

```
fnDifForm[DifForm[fn_[a__]]] :=
 Block[{}, Apply[Plus, Map[xPartialD[fn[a], #].DifForm[#] &, {a}]]]
fnDifForm[DifForm[F[r, t]]]
replaceRHS[exp_Rule, replace_Hold] :=
 Block[{}, exp[[1]] → (exp[[2]] /. replace) // ReleaseHold]
```

$\underline{\partial}_r[F[r, t]].\underline{d}[r] + \underline{\partial}_t[F[r, t]].\underline{d}[t]$

```
PR[CG["● Define the vielbein."],
  NL, "•World coordinates: ", $wc = {r, θ, φ}, $nc = Length[$wc];
  NL, "Vielbein values: ", $vbc = {F[r], r, r Sin[θ]},
  NL, "Non-zero vielbein: ", $vb = Table[T[e, "ud"][i, $wc[[i]]] → $vbc[[i]], {i, $nc}],
  NL, "Metric: ", $ds = d[s]^2 → T[η, "dd"][i, j] T[e, "ud"][i, $wc[[i1]]].d[$wc[[i1]]]
      T[e, "ud"][j, $wc[[j1]]].d[$wc[[j1]]] /. η → δ, (*Euclidean metric*)
  NL, "Sum over all indices: ", $ds = $ds[[1]] -> Sum[$ds[[2]], {i, 1, $nc},
          {j, 1, $nc}, {i1, 1, $nc}, {j1, 1, $nc}] /. $vb /. Tensor[e, _, _] → 0 /.
     T[δ, "dd"][i_, j_] → KroneckerDelta[i, j] // simpleDot3[{}],
  NL, "Vielbein 1-form: ", $se = T[e, "u"][i_] →
    Sum[T[e, "ud"][i, $wc[[i1]]]. DifForm[$wc[[i1]]], {i1, 1, $nc}],
  Yield, $ = Table[T[e, "u"][i], {i, 3}];
  $e = $ = $ → ($ /. $se /. $vb /. Tensor[e, _, _] → 0 // simpleDot3[{}]) // Thread[#] &;
  Column[$],
  NL, "• Cartan's 1st form: ",
  $ = Map[Map[d[#] &, #] &, $] //. d[a_] → DifForm[a];
  $ = $ // tuStdDifForm[{}, $wc, {}];
  $de =
   $ = $ /. {df : DifForm[Sin[θ]] → fnDifForm[df], df : DifForm[F[r]] → fnDifForm[df]} //
     tuStdDifForm[{}, $wc, {}];
  Column[$],
  NL, "•From the definition: ",
  $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β]. T[e, "u"][β],
  yield, $0 = replaceRHS[$0, Hold[a_ :> Sum[a, {β, $nc}]]],
  Yield, $0 = Table[$0, {α, 3}] /. ($sym = {T[ω, "ud"][a_, a_] → 0,
        T[ω, "ud"][a_, b_] :> -T[ω, "ud"][b, a] /; OrderedQ[{b, a}]}) // simpleDot3[{}];
  Column[$0],
  yield, $0 = Map[replaceRHS[#, Hold[$e]] &, $0] // tuStdDifForm[{}, $wc, {}];
  Column[$0],
  NL, "Since: ", $sw0 = $s = T[ω, "ud"][i_, j_] -> T[ω, "udd"][i, j, k].DifForm[k],
  yield, $s = replaceRHS[$s, Hold[a_ ⧴ Sum[a, {k, $wc}]]], CK,
  $wc1 = Join[$wc, {Tensor[_, _, _]}];
  Imply, $pass = $0 = $0 /. $s // tuStdDifForm[{}, $wc1, {}]
 ];
PR["•Comparing: ", $ = {$pass, $de}; Column[$],
 NL, "Eliminating: ",
 $v = $ // ExtractPattern[DifForm[Tensor[e, _, _]]] // DeleteDuplicates,
 Imply, $ = xEliminate[$, $v] // tuStdDifForm[{}, $wc1, {}],
 NL, "Solve these equations for ω ",
 Yield, $ = $ /. Dot → Times // Apply[List, #] &; Framed[Column[$]],
 NL, "Set coefficients of Wedge[]s ->0 : ",
 Yield, $ = $ /. a_ == b_ → a - b == 0 // Collect[#, Wedge[__], Zero[#] &] &;
 yield, $ = $ /. Zero[0] → 0 // ExtractPattern[Zero[_]] // DeleteDuplicates;
 yield, $ = $ /. Zero[a_] → (a → 0);
 (*Use antisymmetry of ω*)
 $ = $ /. T[ω, "udd"][a_, b_, c_] ⧴ -T[ω, "udd"][b, a, c] /; b < a;
 Framed[$],
 $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates;
 yield, $sw = xRuleX[$, $w];
 Framed[Column[$sw]], CG["(27)"]
]
```

● Define the vielbein.

•World coordinates: $\{r, \theta, \varphi\}$

Vielbein values: $\{F[r], r, r\,\mathrm{Sin}[\theta]\}$

Non-zero vielbein: $\{e^1{}_r \to F[r],\ e^2{}_\theta \to r,\ e^3{}_\varphi \to r\,\mathrm{Sin}[\theta]\}$

Metric: $d[s]^2 \to e^i{}_{\{r,\theta,\varphi\}[\![i1]\!]}.d[\{r, \theta, \varphi\}[\![i1]\!]]\ e^j{}_{\{r,\theta,\varphi\}[\![j1]\!]}.d[\{r, \theta, \varphi\}[\![j1]\!]]\ \delta_{i\,j}$

Sum over all indices: $d[s]^2 \to (r.d[\theta])^2 + (F[r].d[r])^2 + ((r\,\mathrm{Sin}[\theta]).d[\varphi])^2$

Vielbein 1-form: $\underline{e}^i \to e^i{}_r.\underline{d}[r] + e^i{}_\theta.\underline{d}[\theta] + e^i{}_\varphi.\underline{d}[\varphi]$

$$\to \begin{array}{l} e^1 \to F[r].\underline{d}[r] \\ e^2 \to r.\underline{d}[\theta] \\ e^3 \to (r\,\mathrm{Sin}[\theta]).\underline{d}[\varphi] \end{array}$$

• Cartan's 1st form:
$$\begin{array}{l} \underline{d}[e^1] \to 0 \\ \underline{d}[e^2] \to \underline{d}[r] \wedge \underline{d}[\theta] \\ \underline{d}[e^3] \to \mathrm{Sin}[\theta].(\underline{d}[r] \wedge \underline{d}[\varphi]) + r\,\partial_{\underline{\theta}}[\mathrm{Sin}[\theta]].(\underline{d}[\theta] \wedge \underline{d}[\varphi]) \end{array}$$

•From the definition: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta.e^\beta \longrightarrow \underline{d}[e^\alpha] \to -\omega^\alpha{}_1.e^1 - \omega^\alpha{}_2.e^2 - \omega^\alpha{}_3.e^3$

$$\to \begin{array}{l} \underline{d}[e^1] \to -\omega^1{}_2.e^2 - \omega^1{}_3.e^3 \\ \underline{d}[e^2] \to \omega^1{}_2.e^1 - \omega^2{}_3.e^3 \\ \underline{d}[e^3] \to \omega^1{}_3.e^1 + \omega^2{}_3.e^2 \end{array} \longrightarrow \begin{array}{l} \underline{d}[e^1] \to -r\,\omega^1{}_2.\underline{d}[\theta] - r\,\mathrm{Sin}[\theta].\omega^1{}_3.\underline{d}[\varphi] \\ \underline{d}[e^2] \to F[r].\omega^1{}_2.\underline{d}[r] - r\,\mathrm{Sin}[\theta].\omega^2{}_3.\underline{d}[\varphi] \\ \underline{d}[e^3] \to r\,\omega^2{}_3.\underline{d}[\theta] + F[r].\omega^1{}_3.\underline{d}[r] \end{array}$$

Since: $\omega^i{}_{\underline{j}} \to \omega^i{}_{j\,k}.\underline{d}[k] \longrightarrow \omega^i{}_{\underline{j}} \to \omega^i{}_{j\,r}.\underline{d}[r] + \omega^i{}_{j\,\theta}.\underline{d}[\theta] + \omega^i{}_{j\,\varphi}.\underline{d}[\varphi]$ ⟵CHECK

$\Rightarrow$

$\{\underline{d}[e^1] \to -r\,\mathrm{Sin}[\theta].(\underline{d}[r] \wedge \underline{d}[\varphi])\,\omega^1{}_{3\,r} - r\,\mathrm{Sin}[\theta].(\underline{d}[\theta] \wedge \underline{d}[\varphi])\,\omega^1{}_{3\,\theta} - r\,\omega^1{}_{2\,r}\,\underline{d}[r] \wedge \underline{d}[\theta] + r\,\omega^1{}_{2\,\varphi}\,\underline{d}[\theta] \wedge \underline{d}[\varphi],$

$\underline{d}[e^2] \to -F[r].(\underline{d}[r] \wedge \underline{d}[\theta])\,\omega^1{}_{2\,\theta} - F[r].(\underline{d}[r] \wedge \underline{d}[\varphi])\,\omega^1{}_{2\,\varphi} - r\,\mathrm{Sin}[\theta].(\underline{d}[r] \wedge \underline{d}[\varphi])\,\omega^2{}_{3\,r} - r\,\mathrm{Sin}[\theta].(\underline{d}[\theta] \wedge \underline{d}[\varphi])\,\omega^2{}_{3\,\theta},$

$\underline{d}[e^3] \to -F[r].(\underline{d}[r] \wedge \underline{d}[\theta])\,\omega^1{}_{3\,\theta} - F[r].(\underline{d}[r] \wedge \underline{d}[\varphi])\,\omega^1{}_{3\,\varphi} + r\,\omega^2{}_{3\,r}\,\underline{d}[r] \wedge \underline{d}[\theta] - r\,\omega^2{}_{3\,\varphi}\,\underline{d}[\theta] \wedge \underline{d}[\varphi]\}$

•Comparing:

$\{\underline{d}[e^1] \to -r\,\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^1{}_{3\,r} - r\,\mathrm{Sin}[\theta].(\underline{d}[\theta]\wedge\underline{d}[\varphi])\,\omega^1{}_{3\,\theta} - r\,\omega^1{}_{2\,r}\,\underline{d}[r]\wedge\underline{d}[\theta] + r\,\omega^1{}_{2\,\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi],$

$\underline{d}[e^2] \to -F[r].(\underline{d}[r]\wedge\underline{d}[\theta])\,\omega^1{}_{2\,\theta} - F[r].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^1{}_{2\,\varphi} -$

$r\,\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^2{}_{3\,r} - r\,\mathrm{Sin}[\theta].(\underline{d}[\theta]\wedge\underline{d}[\varphi])\,\omega^2{}_{3\,\theta},$

$\underline{d}[e^3] \to -F[r].(\underline{d}[r]\wedge\underline{d}[\theta])\,\omega^1{}_{3\,\theta} - F[r].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^1{}_{3\,\varphi} + r\,\omega^2{}_{3\,r}\,\underline{d}[r]\wedge\underline{d}[\theta] - r\,\omega^2{}_{3\,\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]\}$

$\{\underline{d}[e^1] \to 0,\ \underline{d}[e^2] \to \underline{d}[r]\wedge\underline{d}[\theta],\ \underline{d}[e^3] \to \mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi]) + r\,\partial_{\theta}[\mathrm{Sin}[\theta]].(\underline{d}[\theta]\wedge\underline{d}[\varphi])\}$

Eliminating: $\{\underline{d}[e^1], \underline{d}[e^2], \underline{d}[e^3]\}$

$\Rightarrow r\,\partial_{\theta}[\mathrm{Sin}[\theta]].(\underline{d}[\theta]\wedge\underline{d}[\varphi]) == -\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi]) - F[r].(\underline{d}[r]\wedge\underline{d}[\theta])\,\omega^1{}_{3\,\theta} -$
$F[r].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^1{}_{3\,\varphi} + r\,\omega^2{}_{3\,r}\,\underline{d}[r]\wedge\underline{d}[\theta] - r\,\omega^2{}_{3\,\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]\ \&\&\ F[r].(\underline{d}[r]\wedge\underline{d}[\theta])\,\omega^1{}_{2\,\theta} ==$
$-F[r].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^1{}_{2\,\varphi} - r\,\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^2{}_{3\,r} - r\,\mathrm{Sin}[\theta].(\underline{d}[\theta]\wedge\underline{d}[\varphi])\,\omega^2{}_{3\,\theta} - \underline{d}[r]\wedge\underline{d}[\theta]\ \&\&$
$r\,\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])\,\omega^1{}_{3\,r} == -r\,\mathrm{Sin}[\theta].(\underline{d}[\theta]\wedge\underline{d}[\varphi])\,\omega^1{}_{3\,\theta} - r\,\omega^1{}_{2\,r}\,\underline{d}[r]\wedge\underline{d}[\theta] + r\,\omega^1{}_{2\,\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi]\ \&\&$
$0 == -(\mathrm{Sin}[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi]))^2\,\omega^1{}_{3\,r}$

Solve these equations for ω

$\to$

$$\boxed{\begin{array}{l}
r\,\underline{d}[\theta]\wedge\underline{d}[\varphi]\,\partial_{\theta}[\mathrm{Sin}[\theta]] == \\
\quad -F[r]\,\omega^1{}_{3\,\theta}\,\underline{d}[r]\wedge\underline{d}[\theta] + r\,\omega^2{}_{3\,r}\,\underline{d}[r]\wedge\underline{d}[\theta] - \mathrm{Sin}[\theta]\,\underline{d}[r]\wedge\underline{d}[\varphi] - F[r]\,\omega^1{}_{3\,\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi] - r\,\omega^2{}_{3\,\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi] \\
F[r]\,\omega^1{}_{2\,\theta}\,\underline{d}[r]\wedge\underline{d}[\theta] == \\
\quad -(\underline{d}[r]\wedge\underline{d}[\theta]) - F[r]\,\omega^1{}_{2\,\varphi}\,\underline{d}[r]\wedge\underline{d}[\varphi] - r\,\mathrm{Sin}[\theta]\,\omega^2{}_{3\,r}\,\underline{d}[r]\wedge\underline{d}[\varphi] - r\,\mathrm{Sin}[\theta]\,\omega^2{}_{3\,\theta}\,\underline{d}[\theta]\wedge\underline{d}[\varphi] \\
r\,\mathrm{Sin}[\theta]\,\omega^1{}_{3\,r}\,\underline{d}[r]\wedge\underline{d}[\varphi] == -r\,\omega^1{}_{2\,r}\,\underline{d}[r]\wedge\underline{d}[\theta] + r\,\omega^1{}_{2\,\varphi}\,\underline{d}[\theta]\wedge\underline{d}[\varphi] - r\,\mathrm{Sin}[\theta]\,\omega^1{}_{3\,\theta}\,\underline{d}[\theta]\wedge\underline{d}[\varphi] \\
0 == -\mathrm{Sin}[\theta]^2\,\omega^1{}_{3\,r}\,(\underline{d}[r]\wedge\underline{d}[\varphi])^2
\end{array}}$$

Set coefficients of Wedge[ ]s ->0 :

$\to\ \longrightarrow\ \longrightarrow$
$$\boxed{\begin{array}{l}
\{\mathrm{Sin}[\theta] + F[r]\,\omega^1{}_{3\,\varphi} \to 0,\ F[r]\,\omega^1{}_{3\,\theta} - r\,\omega^2{}_{3\,r} \to 0,\ r\,\omega^2{}_{3\,\varphi} + r\,\partial_{\theta}[\mathrm{Sin}[\theta]] \to 0, \\
\ 1 + F[r]\,\omega^1{}_{2\,\theta} \to 0,\ F[r]\,\omega^1{}_{2\,\varphi} + r\,\mathrm{Sin}[\theta]\,\omega^2{}_{3\,r} \to 0,\ r\,\mathrm{Sin}[\theta]\,\omega^2{}_{3\,\theta} \to 0, \\
\ r\,\omega^1{}_{2\,r} \to 0,\ r\,\mathrm{Sin}[\theta]\,\omega^1{}_{3\,r} \to 0,\ -r\,\omega^1{}_{2\,\varphi} + r\,\mathrm{Sin}[\theta]\,\omega^1{}_{3\,\theta} \to 0,\ \mathrm{Sin}[\theta]^2\,\omega^1{}_{3\,r} \to 0\}
\end{array}}$$

$$\longrightarrow \boxed{\begin{array}{l}
\omega^1{}_{3\,\varphi} \to -\frac{\mathrm{Sin}[\theta]}{F[r]} \\
\omega^1{}_{3\,\theta} \to 0 \\
\omega^2{}_{3\,r} \to 0 \\
\omega^2{}_{3\,\varphi} \to -\partial_{\theta}[\mathrm{Sin}[\theta]] \\
\omega^1{}_{2\,\theta} \to -\frac{1}{F[r]} \\
\omega^1{}_{2\,\varphi} \to 0 \\
\omega^2{}_{3\,\theta} \to 0 \\
\omega^1{}_{2\,r} \to 0 \\
\omega^1{}_{3\,r} \to 0
\end{array}} \quad (27)$$

```
PR[CG["•Curvature form, Cartan's second form: ",
  $ = R → DifForm[ω] + ω.ω],
 NL, CO["May be more convenient in world coordinates."],
 NL, "With: ",
 $sw2 = {Map[(#[[1]] /. T[ω, "udd"][a_, b_, c_] -> T[ω, "udd"][b, a, c]) → -#[[2]] &, $sw],
     $sw} /. xPartialD[Sin[θ], θ] → Cos[θ] // Flatten,
 Yield, $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /.
   Dot[a_, b_] :→ Dot[(a /. β → β1), (b /. α → β1)],
 NL, "Sum over β1: ",
 Yield, $0 = $ = $ /. dd : Dot[_, _] :→ Sum[dd, {β1, $nc}] /; ! FreeQ[dd, β1] //
    tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}],
 NL, "In explicit 1-forms: ",
 $w1 = T[ω, "ud"][a_, b_] :→ Sum[T[ω, "udd"][a, b, k].DifForm[k], {k, $wc}],

 NL, "For ", $s = {α → 1, β → 2},
 Yield, $ = $0 /. $s,
 Yield, $ = $ /. $w1 /. $sw2 // tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}];
 Yield, $12 = $ /. DifForm[F[r]] → xPartialD[F[r], r].DifForm[r] /.
    ($sym = T[ω, "udd"][a_, a_, b_] → 0) // tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}],

 NL, "For ", $s = {α → 2, β → 3},
 Yield, $ = $0 /. $s,
 Yield, $ = $ /. $w1 /. $sw2 // tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}];
 Yield, $23 = $ /. DifForm[F[r]] → xPartialD[F[r], r].DifForm[r] /.
     DifForm[Cos[θ]] → -Sin[θ].DifForm[θ] /. ($sym = T[ω, "udd"][a_, a_, b_] → 0) //
   tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}],

 NL, "For ", $s = {α → 1, β → 3},
 Yield, $ = $0 /. $s,
 Yield, $ = $ /. $w1 /. $sw2 // tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}];
 Yield, $13 = $ /. DifForm[F[r]] → xPartialD[F[r], r].DifForm[r] /.
     DifForm[Sin[θ]] → Cos[θ].DifForm[θ] /. ($sym = T[ω, "udd"][a_, a_, b_] → 0) //
   tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}],
 NL, "Using: ", $s = xRuleX[$e /. Dot → Times, Table[DifForm[i], {i, $wc}]];
 Column[$s],
 NL, "We have: ", $ = {$12, $23, $13},
 Yield, $ = $ /. $s // tuStdDifForm[{}, $wc, {{T[e, "u"][_], 1}}];
 Yield, $R = $ /. Dot → Times // FullSimplify;
 FramedColumn[$R], CG["(28)"],
 NL, "Curvature tensor(non-zero): ", $ = $R /. Rule → xRule;
 $passr = Map[# /. xRule[T[R, "ud"][a_, b_], c_ Wedge[T[e, "u"][i_], T[e, "u"][j_]]] ->
       xRule[T[R, "uddd"][a, b, i, j], c ] &, $] /. xRule → Rule
]
```

•Curvature form, Cartan's second form: $R \to \underline{d}[\omega] + \omega . \omega$

May be more convenient in world coordinates.

With: $\{\omega^3{}_{1\varphi} \to \frac{\mathrm{Sin}[\theta]}{F[r]}, \omega^3{}_{1\theta} \to 0, \omega^3{}_{2r} \to 0, \omega^3{}_{2\varphi} \to \mathrm{Cos}[\theta],$

$\omega^2{}_{1\theta} \to \frac{1}{F[r]}, \omega^2{}_{1\varphi} \to 0, \omega^3{}_{2\theta} \to 0, \omega^2{}_{1r} \to 0, \omega^3{}_{1r} \to 0, \omega^1{}_{3\varphi} \to -\frac{\mathrm{Sin}[\theta]}{F[r]}, \omega^1{}_{3\theta} \to 0,$

$\omega^2{}_{3r} \to 0, \omega^2{}_{3\varphi} \to -\mathrm{Cos}[\theta], \omega^1{}_{2\theta} \to -\frac{1}{F[r]}, \omega^1{}_{2\varphi} \to 0, \omega^2{}_{3\theta} \to 0, \omega^1{}_{2r} \to 0, \omega^1{}_{3r} \to 0\}$

$\to R^\alpha{}_\beta \to \underline{d}[\omega^\alpha{}_\beta] + \omega^\alpha{}_{\beta 1} . \omega^{\beta 1}{}_\beta$

Sum over $\beta 1$:

$\to R^\alpha{}_\beta \to \underline{d}[\omega^\alpha{}_\beta] + \omega^\alpha{}_1 . \omega^1{}_\beta + \omega^\alpha{}_2 . \omega^2{}_\beta + \omega^\alpha{}_3 . \omega^3{}_\beta$

In explicit 1-forms: $\omega^{a\_}{}_{b\_} :\to \sum_k^{\$wc} T[\omega, udd][a, b, k] . \underline{d}[k]$

For $\{\alpha \to 1, \beta \to 2\}$

$\to R^1{}_2 \to \underline{d}[\omega^1{}_2] + \omega^1{}_1 . \omega^1{}_2 + \omega^1{}_2 . \omega^2{}_2 + \omega^1{}_3 . \omega^3{}_2$

$\to$

$\to R^1{}_2 \to \frac{1}{F[r]^2} . \underline{\partial}_r[F[r]] . (\underline{d}[r] \wedge \underline{d}[\theta])$

For $\{\alpha \to 2, \beta \to 3\}$

$\to R^2{}_3 \to \underline{d}[\omega^2{}_3] + \omega^2{}_1 . \omega^1{}_3 + \omega^2{}_2 . \omega^2{}_3 + \omega^2{}_3 . \omega^3{}_3$

$\to$

$\to R^2{}_3 \to \mathrm{Sin}[\theta] . (\underline{d}[\theta] \wedge \underline{d}[\varphi]) - \frac{1}{F[r]^2} . \mathrm{Sin}[\theta] . (\underline{d}[\theta] \wedge \underline{d}[\varphi])$

For $\{\alpha \to 1, \beta \to 3\}$

$\to R^1{}_3 \to \underline{d}[\omega^1{}_3] + \omega^1{}_1 . \omega^1{}_3 + \omega^1{}_2 . \omega^2{}_3 + \omega^1{}_3 . \omega^3{}_3$

$\to$

$\to R^1{}_3 \to \frac{1}{F[r]^2} . \mathrm{Sin}[\theta] . \underline{\partial}_r[F[r]] . (\underline{d}[r] \wedge \underline{d}[\varphi])$

Using: $\underline{d}[r] \to \frac{e^1}{F[r]}$, $\underline{d}[\theta] \to \frac{e^2}{r}$, $\underline{d}[\varphi] \to \frac{\mathrm{Csc}[\theta]\, e^3}{r}$

We have: $\{R^1{}_2 \to \frac{1}{F[r]^2} . \underline{\partial}_r[F[r]] . (\underline{d}[r] \wedge \underline{d}[\theta]),$

$R^2{}_3 \to \mathrm{Sin}[\theta] . (\underline{d}[\theta] \wedge \underline{d}[\varphi]) - \frac{1}{F[r]^2} . \mathrm{Sin}[\theta] . (\underline{d}[\theta] \wedge \underline{d}[\varphi]), R^1{}_3 \to \frac{1}{F[r]^2} . \mathrm{Sin}[\theta] . \underline{\partial}_r[F[r]] . (\underline{d}[r] \wedge \underline{d}[\varphi])\}$

$\to$

$$\to \boxed{\begin{array}{l} R^1{}_2 \to \frac{e^1 \wedge e^2\, \underline{\partial}_r[F[r]]}{r\, F[r]^3} \\ R^2{}_3 \to \frac{(-1+F[r]^2)\, e^2 \wedge e^3}{r^2\, F[r]^2} \\ R^1{}_3 \to \frac{e^1 \wedge e^3\, \underline{\partial}_r[F[r]]}{r\, F[r]^3} \end{array}} \quad (28)$$

Curvature tensor(non-zero): $\{R^1{}_{212} \to \frac{\underline{\partial}_r[F[r]]}{r\, F[r]^3}, R^2{}_{323} \to \frac{-1 + F[r]^2}{r^2\, F[r]^2}, R^1{}_{313} \to \frac{\underline{\partial}_r[F[r]]}{r\, F[r]^3}\}$

```
PR[CG["•The Ricci tensor, ",
   T[R, "dd"][i, j] → xSum[T[e, "ud"][a, i] T[e, "ud"][a, j] T[R, "dd"][a, a], {a, 3}],
   " where ", $R4 = T[R, "dd"][a, a] → xSum[T[R, "uddd"][c, a, c, a], {c, 3}]], CK,
  NL, "Evaluate: ", $R4 = $R4 /. xSum → Sum,
  NL, "Since: ", $passr,
  NL, "Compute: ", $ = $R4, " for a->{1,2,3} ",
  Yield, $ = Map[$ /. a → # &, {1, 2, 3}]; Column[$],
  NL, "Using: ", $s = {T[R, "uddd"][a_, b_, c_, c_] → 0,
    T[R, "uddd"][a_, b_, a_, b_] :> T[R, "uddd"][b, a, b, a] /; OrderedQ[{b, a}]},
  Yield, $ = $ /. $s,
  $ /. $passr,
  NL, CR["This is not (29)?"]
]
```

•The Ricci tensor, $R_{ij} \to \sum_{\{a,3\}} [e^a{}_i\, e^a{}_j\, R_{aa}]$ where $R_{aa} \to \sum_{\{c,3\}} [R^c{}_{aca}]$⟵CHECK

Evaluate: $R_{aa} \to R^1{}_{a1a} + R^2{}_{a2a} + R^3{}_{a3a}$

Since: $\{R^1{}_{212} \to \frac{\partial_r[F[r]]}{r\,F[r]^3},\ R^2{}_{323} \to \frac{-1+F[r]^2}{r^2\,F[r]^2},\ R^1{}_{313} \to \frac{\partial_r[F[r]]}{r\,F[r]^3}\}$

Compute: $R_{aa} \to R^1{}_{a1a} + R^2{}_{a2a} + R^3{}_{a3a}$ for a->{1,2,3}

$R_{11} \to R^1{}_{111} + R^2{}_{121} + R^3{}_{131}$

→ $R_{22} \to R^1{}_{212} + R^2{}_{222} + R^3{}_{232}$

$R_{33} \to R^1{}_{313} + R^2{}_{323} + R^3{}_{333}$

Using: $\{R^{a\_}{}_{b\_c\_c\_} \to 0,\ R^{a\_}{}_{b\_a\_b\_} :\to$ T[R, uddd][b, a, b, a] /; OrderedQ[{b, a}]$\}$

→ $\{R_{11} \to R^1{}_{212} + R^1{}_{313},\ R_{22} \to R^1{}_{212} + R^2{}_{323},\ R_{33} \to R^1{}_{313} + R^2{}_{323}\}$

$$\{R_{11} \to \frac{2\,\partial_r[F[r]]}{r\,F[r]^3},\ R_{22} \to \frac{-1+F[r]^2}{r^2\,F[r]^2} + \frac{\partial_r[F[r]]}{r\,F[r]^3},\ R_{33} \to \frac{-1+F[r]^2}{r^2\,F[r]^2} + \frac{\partial_r[F[r]]}{r\,F[r]^3}\}$$

This is not (29)?

Spherically symmetric static spacetimes p .610

```
fnDifForm[fn_[a__]] :=
 Block[{}, Apply[Plus, Map[xPartialD[fn[a], #].DifForm[#] &, {a}]]]

PR["●IX.8: Spherically symmetric static spacetimes p.610: ",
 NL, "World coordinates: ", $xw = {t, r, θ, φ}, $xwd = Thread[DifForm[$xw]];
 NL, "Indices: ", $i = {0, 1, 2, 3},
 NL, "Correspondence: ", $xwi = Association[Thread[$i → $xw]],
 NL, "Vielbein: ",
 $vb = Thread[$e = Table[T[e, "u"][i], {i, $i}] -> {Et[r], F[r], r, r Sin[θ]} $xwd],
 Imply, $ = Thread[Map[Thread[DifForm[#]] &, $e]] // tuStdDifForm[{}, $xw, {}];
 Column[$];
 Yield, $ed = $ = $ /. {DifForm[ff : fn_[aa_]] → xPartialD[ff, aa].DifForm[aa]} //
    tuStdDifForm[{}, $xw, {}];
 Column[$],
 NL, DifForm[T[e, "u"][n]], " in terms of itself: ",
 Yield, $vbi = xRuleX[$vb, $xwd],
 $ed1 = $ed /. $vbi // tuStdDifForm[{}, Flatten[{$xw}], {{T[e, "u"][_], 1}}],

 NL, "•Determine ω's from the definition: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β]. T[e, "u"][β],
 NL, "Expand ω's: ",
 Yield,
 $0 = MapAt[Sum[# . T[xη, "uu"][β, β], {β, $i}] &, $0, 2] /. xη → η // simpleDot3[{}],
 Yield, $0 = $0 /. T[ω, "ud"][a_, b_] :→ Sum[T[ω, "udd"][a, b, k].T[e, "u"][k], {k, $i}] //
   tuStdDifForm[{}, $xw, {{T[e, "u"][_], 1}}],
 Yield, $0 = $0 /. $vb // tuStdDifForm[{}, $xw, {{T[e, "u"][_], 1}}];
 $ = Map[$0 /. α → # &, $i];
 $ = xEliminate[Flatten[{$, $ed}], Keys[Association[$ed]]];
 $ = $ /. ($sym = T[ω, "udd"][a_, a_, b_] → 0) // tuStdDifForm[{}, $xw, {}] // Simplify,
 NL, "Set coefficients of Wedge[]s ->0, Solve for ω's: ",
 Yield, $ = $ /. Dot → Times /. a_ == b_ → a - b == 0 // Collect[#, Wedge[__], Zero[#] &] &;
 $ = $ /. Zero[0] → 0 // ExtractPattern[Zero[_]] // DeleteDuplicates;
 yield, $ = $ /. Zero[a_] → (a → 0);
 (*Use antisymmetry of ω*)
 $ = $ /. T[ω, "udd"][a_, b_, c_] :→ -T[ω, "udd"][b, a, c] /; b < a;
 Framed[$];
 $w = $ // ExtractPattern[Tensor[ω, _, _]] // DeleteDuplicates;
 yield, $sw = xRuleX[$, $w];
 Yield,
 $sw1 = Join[
   Map[(#[[1]] /. T[ω, "udd"][a_, b_, c_] → T[ω, "udd"][b, a, c]) → -#[[2]] &, $sw], $sw];
 Framed[Column[$sw1]]
]
```

●IX.8: Spherically symmetric static spacetimes p.610:
World coordinates: {t, r, θ, φ}
Indices: {0, 1, 2, 3}
Correspondence: ⟨|0 → t, 1 → r, 2 → θ, 3 → φ|⟩
Vielbein: $\{e^0 \to \underline{d}[t]\,Et[r],\ e^1 \to \underline{d}[r]\,F[r],\ e^2 \to r\,\underline{d}[\theta],\ e^3 \to r\,\underline{d}[\varphi]\,Sin[\theta]\}$
⇒

$$\rightarrow \begin{array}{l} \underline{d}[e^0] \to \partial_r[Et[r]].(\underline{d}[r] \wedge \underline{d}[t]) \\ \underline{d}[e^1] \to 0 \\ \underline{d}[e^2] \to \underline{d}[r] \wedge \underline{d}[\theta] \\ \underline{d}[e^3] \to Sin[\theta].(\underline{d}[r] \wedge \underline{d}[\varphi]) + r\,\partial_\theta[Sin[\theta]].(\underline{d}[\theta] \wedge \underline{d}[\varphi]) \end{array}$$

$\underline{d}[e^n]$ in terms of itself:

$$\rightarrow \{\underline{d}[t] \to \frac{e^0}{Et[r]},\ \underline{d}[r] \to \frac{e^1}{F[r]},\ \underline{d}[\theta] \to \frac{e^2}{r},\ \underline{d}[\varphi] \to \frac{Csc[\theta]\,e^3}{r}\}\{\underline{d}[e^0] \to -\frac{1}{Et[r]}.\frac{1}{F[r]}.\partial_r[Et[r]].(e^0 \wedge e^1),$$

$$\underline{d}[e^1]\to 0,\ \underline{d}[e^2]\to \frac{\frac{1}{F[r]}.(e^1\wedge e^2)}{r},\ \underline{d}[e^3]\to \frac{\frac{1}{F[r]}.(e^1\wedge e^3)}{r}+\frac{Csc[\theta].\underline{\partial}_\theta[Sin[\theta]].(e^2\wedge e^3)}{r}\}$$

•Determine ω's from the definition: $\underline{d}[e^\alpha]\to -\omega^\alpha{}_\beta.e^\beta$

Expand ω's:

$\to\ \underline{d}[e^\alpha]\to \omega^\alpha{}_0.e^0-\omega^\alpha{}_1.e^1-\omega^\alpha{}_2.e^2-\omega^\alpha{}_3.e^3$

$\to$

$$\underline{d}[e^\alpha]\to -\omega^\alpha{}_{01}.(e^0\wedge e^1)-\omega^\alpha{}_{02}.(e^0\wedge e^2)-\omega^\alpha{}_{03}.(e^0\wedge e^3)-\omega^\alpha{}_{10}.(e^0\wedge e^1)+\omega^\alpha{}_{12}.(e^1\wedge e^2)+\omega^\alpha{}_{13}.(e^1\wedge e^3)-\omega^\alpha{}_{20}.(e^0\wedge e^2)-\omega^\alpha{}_{21}.(e^1\wedge e^2)+\omega^\alpha{}_{23}.(e^2\wedge e^3)-\omega^\alpha{}_{30}.(e^0\wedge e^3)-\omega^\alpha{}_{31}.(e^1\wedge e^3)-\omega^\alpha{}_{32}.(e^2\wedge e^3)$$

$$\to\ Sin[\theta].(\underline{d}[r]\wedge\underline{d}[\varphi])+r\,(\underline{\partial}_\theta[Sin[\theta]].(\underline{d}[\theta]\wedge\underline{d}[\varphi])+Et[r].\omega^3{}_{02}.(\underline{d}[t]\wedge\underline{d}[\theta])+Et[r].\omega^3{}_{20}.(\underline{d}[t]\wedge\underline{d}[\theta])+F[r].\omega^3{}_{21}.(\underline{d}[r]\wedge\underline{d}[\theta])+Et[r].Sin[\theta].\omega^3{}_{03}.(\underline{d}[t]\wedge\underline{d}[\varphi]))==$$
$$r\,F[r].\omega^3{}_{12}.(\underline{d}[r]\wedge\underline{d}[\theta])+r^2\,Sin[\theta].\omega^3{}_{23}.(\underline{d}[\theta]\wedge\underline{d}[\varphi])+Et[r].F[r].\omega^3{}_{01}.(\underline{d}[r]\wedge\underline{d}[t])+Et[r].F[r].\omega^3{}_{10}.(\underline{d}[r]\wedge\underline{d}[t])+r\,F[r].Sin[\theta].\omega^3{}_{13}.(\underline{d}[r]\wedge\underline{d}[\varphi])\ \&\&\ \underline{\partial}_r[Et[r]].(\underline{d}[r]\wedge\underline{d}[t])+$$
$$r\,(Et[r].\omega^0{}_{20}.(\underline{d}[t]\wedge\underline{d}[\theta])+F[r].\omega^0{}_{21}.(\underline{d}[r]\wedge\underline{d}[\theta])+r\,Sin[\theta].\omega^0{}_{32}.(\underline{d}[\theta]\wedge\underline{d}[\varphi])+Et[r].Sin[\theta].\omega^0{}_{30}.(\underline{d}[t]\wedge\underline{d}[\varphi])+F[r].Sin[\theta].\omega^0{}_{31}.(\underline{d}[r]\wedge\underline{d}[\varphi]))==$$
$$r\,F[r].\omega^0{}_{12}.(\underline{d}[r]\wedge\underline{d}[\theta])+r^2\,Sin[\theta].\omega^0{}_{23}.(\underline{d}[\theta]\wedge\underline{d}[\varphi])+Et[r].F[r].\omega^0{}_{10}.(\underline{d}[r]\wedge\underline{d}[t])+r\,F[r].Sin[\theta].\omega^0{}_{13}.(\underline{d}[r]\wedge\underline{d}[\varphi])\ \&\&\ Et[r].F[r].\omega^1{}_{01}.(\underline{d}[r]\wedge\underline{d}[t])==$$
$$r\,(Et[r].\omega^1{}_{02}.(\underline{d}[t]\wedge\underline{d}[\theta])+Et[r].\omega^1{}_{20}.(\underline{d}[t]\wedge\underline{d}[\theta])+F[r].\omega^1{}_{21}.(\underline{d}[r]\wedge\underline{d}[\theta])-r\,Sin[\theta].\omega^1{}_{23}.(\underline{d}[\theta]\wedge\underline{d}[\varphi])+r\,Sin[\theta].\omega^1{}_{32}.(\underline{d}[\theta]\wedge\underline{d}[\varphi])+Et[r].Sin[\theta].\omega^1{}_{03}.(\underline{d}[t]\wedge\underline{d}[\varphi])+Et[r].Sin[\theta].\omega^1{}_{30}.(\underline{d}[t]\wedge\underline{d}[\varphi])+F[r].Sin[\theta].\omega^1{}_{31}.(\underline{d}[r]\wedge\underline{d}[\varphi]))\ \&\&$$
$$r\,Et[r].\omega^2{}_{02}.(\underline{d}[t]\wedge\underline{d}[\theta])+r^2\,Sin[\theta].\omega^2{}_{32}.(\underline{d}[\theta]\wedge\underline{d}[\varphi])+r\,Et[r].Sin[\theta].\omega^2{}_{03}.(\underline{d}[t]\wedge\underline{d}[\varphi])+r\,Et[r].Sin[\theta].\omega^2{}_{30}.(\underline{d}[t]\wedge\underline{d}[\varphi])+r\,F[r].Sin[\theta].\omega^2{}_{31}.(\underline{d}[r]\wedge\underline{d}[\varphi])+\underline{d}[r]\wedge\underline{d}[\theta]==$$
$$r\,F[r].\omega^2{}_{12}.(\underline{d}[r]\wedge\underline{d}[\theta])+Et[r].F[r].\omega^2{}_{01}.(\underline{d}[r]\wedge\underline{d}[t])+Et[r].F[r].\omega^2{}_{10}.(\underline{d}[r]\wedge\underline{d}[t])+r\,F[r].Sin[\theta].\omega^2{}_{13}.(\underline{d}[r]\wedge\underline{d}[\varphi])$$

Set coefficients of Wedge[]s ->0, Solve for ω's:

$\to\ \longrightarrow\ \longrightarrow$

$\omega^3{}_{03} \to 0$
$\omega^3{}_{01} \to 0$
$\omega^3{}_{10} \to 0$
$\omega^3{}_{13} \to \frac{1}{\text{r F[r]}}$
$\omega^3{}_{02} \to 0$
$\omega^3{}_{20} \to 0$
$\omega^3{}_{12} \to 0$
$\omega^3{}_{21} \to 0$
$\omega^3{}_{23} \to \frac{\text{Csc}[\theta]\,\partial_\theta[\text{Sin}[\theta]]}{\text{r}}$
$\omega^2{}_{00} \to 0$
$\omega^1{}_{02} \to 0$
$\omega^2{}_{01} \to 0$
$\omega^3{}_{00} \to 0$
$\omega^1{}_{03} \to 0$
$\omega^2{}_{03} \to 0$
$\omega^1{}_{00} \to -\frac{\partial_r[\text{Et[r]}]}{\text{Et[r] F[r]}}$
$\omega^1{}_{01} \to 0$
$\omega^2{}_{10} \to 0$
$\omega^2{}_{11} \to 0$
$\omega^3{}_{11} \to 0$
$\omega^2{}_{13} \to 0$
$\omega^2{}_{02} \to 0$
$\omega^2{}_{12} \to \frac{1}{\text{r F[r]}}$
→ $\omega^3{}_{22} \to 0$
$\omega^0{}_{33} \to 0$
$\omega^0{}_{31} \to 0$
$\omega^1{}_{30} \to 0$
$\omega^1{}_{33} \to -\frac{1}{\text{r F[r]}}$
$\omega^0{}_{32} \to 0$
$\omega^2{}_{30} \to 0$
$\omega^1{}_{32} \to 0$
$\omega^2{}_{31} \to 0$
$\omega^2{}_{33} \to -\frac{\text{Csc}[\theta]\,\partial_\theta[\text{Sin}[\theta]]}{\text{r}}$
$\omega^0{}_{20} \to 0$
$\omega^0{}_{12} \to 0$
$\omega^0{}_{21} \to 0$
$\omega^0{}_{30} \to 0$
$\omega^0{}_{13} \to 0$
$\omega^0{}_{23} \to 0$
$\omega^0{}_{10} \to \frac{\partial_r[\text{Et[r]}]}{\text{Et[r] F[r]}}$
$\omega^0{}_{11} \to 0$
$\omega^1{}_{20} \to 0$
$\omega^1{}_{21} \to 0$
$\omega^1{}_{31} \to 0$
$\omega^1{}_{23} \to 0$
$\omega^0{}_{22} \to 0$
$\omega^1{}_{22} \to -\frac{1}{\text{r F[r]}}$
$\omega^2{}_{32} \to 0$

```
PR["•Curvature form may be computed from the definition of ω: ",
 $st = T[ω, "ud"][a_, b_] →
    Sum[T[xη, "uu"][c, c].T[ω, "udd"][a, b, c].T[e, "u"][c], {c, $i}] /. xη → η,
 Yield, $t = Table[Inactivate[T[ω, "ud"]][i, j] -> T[ω, "ud"][i, j], {i, $i}, {j, $i}];
 $t = $t /. $st /. $sw1 /. T[ω, "udd"][a_, a_, b_] → 0 // simpleDot3[{}];
 $t = $t // Activate; MatrixForm[$t]
]
PR["•Cartan curvature form: ",
 $ = R → DifForm[ω] + ω.ω,
 Yield,
 $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] :→ Dot[(a /. β → β1), (b /. α → β1)];
 $ = $ // RuleX2PatternVar[{α, β}],
 Yield, $ = $ /. dd : Dot[_, _] :→ Sum[T[xη, "uu"][β1, β1] dd, {β1, $i}] /. xη → η,
 "POFF",
 Yield, $tR = Table[Inactivate[T[R, "ud"]][i, j] -> T[R, "ud"][i, j], {i, $i}, {j, $i}],
 Yield, $tR = $tR /. $,
 Yield, $tR = $tR // Activate // Flatten,
 Yield, $tR = $tR /. Flatten[$t] // simpleDot3[{}],
 Yield, $tR = $tR /. DifForm[0] → 0; MatrixForm[$tR],
 Yield, $tR = $tR // tuStdDifForm[{}, $xw, {{T[e, "u"][_], 1}}],
 Yield, $tR = $tR /. DifForm[f_[x_]] :→ xPartialD[f[x], x].DifForm[x] //
   tuStdDifForm[{}, $xw, {{T[e, "u"][_], 1}}],
 "PONdd",
 Yield, $tR = $tR /. $ed1 /. xPartialD[DifForm[x_], x_] → 0 /. $vbi //
   tuStdDifForm[{}, $xw, {{T[e, "u"][_], 1}}];
 Yield, $tR = $tR /. Dot → Times /. xPartialD → D; Column[$tR],
 NL, "Extract curvature tensor: ",
 $ = $tR /. Rule → xRule // Simplify;
 $tRpass = $ = $ /. Wedge → xWedge /.
    xRule[T[R, "ud"][a_, b_], cc : (c_ xWedge[T[e, "u"][d_], T[e, "u"][e_]]) | 0] :→
     If[cc === 0, T[R, "uddd"][a, b, i_, j_] → 0, T[R, "uddd"][a, b, d, e] → c];
 Framed[Column[$]]
]
```

•Curvature form may be computed from the definition of ω:

$\omega^{a_-}{}_{b_-} \to (-1).\omega^{a}{}_{b\,0}.e^0 + 1.\omega^{a}{}_{b\,1}.e^1 + 1.\omega^{a}{}_{b\,2}.e^2 + 1.\omega^{a}{}_{b\,3}.e^3$

$$\to \begin{pmatrix} \omega^0{}_0 \to 0 & \omega^0{}_1 \to -\frac{\partial_r[\mathrm{Et}[r]]}{\mathrm{Et}[r]\,F[r]}.e^0 & \omega^0{}_2 \to 0 & \omega^0{}_3 \to 0 \\ \omega^1{}_0 \to \frac{\partial_r[\mathrm{Et}[r]]}{\mathrm{Et}[r]\,F[r]}.e^0 & \omega^1{}_1 \to 0 & \omega^1{}_2 \to -\frac{1}{r\,F[r]}.e^2 & \omega^1{}_3 \to -\frac{1}{r\,F[r]}.e^3 \\ \omega^2{}_0 \to 0 & \omega^2{}_1 \to \frac{1}{r\,F[r]}.e^2 & \omega^2{}_2 \to 0 & \omega^2{}_3 \to -\frac{\mathrm{Csc}[\theta]\,\partial_\theta[\mathrm{Sin}[\theta]]}{r}.e^3 \\ \omega^3{}_0 \to 0 & \omega^3{}_1 \to \frac{1}{r\,F[r]}.e^3 & \omega^3{}_2 \to \frac{\mathrm{Csc}[\theta]\,\partial_\theta[\mathrm{Sin}[\theta]]}{r}.e^3 & \omega^3{}_3 \to 0 \end{pmatrix}$$

•Cartan curvature form: $R \to \underline{d}[\omega] + \omega.\omega$

$\to R^{\alpha}{}_{\beta\_} \to \underline{d}[\omega^{\alpha}{}_{\beta}] + \omega^{\alpha}{}_{\beta 1}.\omega^{\beta 1}{}_{\beta}$

$\to R^{\alpha}{}_{\beta\_} \to \underline{d}[\omega^{\alpha}{}_{\beta}] - \omega^{\alpha}{}_{0}.\omega^{0}{}_{\beta} + \omega^{\alpha}{}_{1}.\omega^{1}{}_{\beta} + \omega^{\alpha}{}_{2}.\omega^{2}{}_{\beta} + \omega^{\alpha}{}_{3}.\omega^{3}{}_{\beta}$

....

$\to$

$R^0{}_0 \to 0$

$R^0{}_1 \to -\frac{e^0 \wedge e^1\, Et'[r]\, F'[r]}{Et[r]\, F[r]^3} + \frac{e^0 \wedge e^1\, Et''[r]}{Et[r]\, F[r]^2}$

$R^0{}_2 \to \frac{e^0 \wedge e^2\, Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^0{}_3 \to \frac{e^0 \wedge e^3\, Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^1{}_0 \to \frac{e^0 \wedge e^1\, Et'[r]\, F'[r]}{Et[r]\, F[r]^3} - \frac{e^0 \wedge e^1\, Et''[r]}{Et[r]\, F[r]^2}$

$R^1{}_1 \to 0$

$R^1{}_2 \to \frac{e^1 \wedge e^2\, F'[r]}{r\, F[r]^3}$

$R^1{}_3 \to \frac{e^1 \wedge e^3\, F'[r]}{r\, F[r]^3}$

$\to$

$R^2{}_0 \to -\frac{e^0 \wedge e^2\, Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^2{}_1 \to -\frac{e^1 \wedge e^2\, F'[r]}{r\, F[r]^3}$

$R^2{}_2 \to 0$

$R^2{}_3 \to \frac{e^2 \wedge e^3}{r^2} - \frac{e^2 \wedge e^3}{r^2\, F[r]^2}$

$R^3{}_0 \to -\frac{e^0 \wedge e^3\, Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^3{}_1 \to -\frac{e^1 \wedge e^3\, F'[r]}{r\, F[r]^3}$

$R^3{}_2 \to -\frac{e^2 \wedge e^3}{r^2} + \frac{e^2 \wedge e^3}{r^2\, F[r]^2}$

$R^3{}_3 \to 0$

Extract curvature tensor:

$R^0{}_{0\, i\_\, j\_} \to 0$

$R^0{}_{1\,0\,1} \to \frac{-Et'[r]\, F'[r] + F[r]\, Et''[r]}{Et[r]\, F[r]^3}$

$R^0{}_{2\,0\,2} \to \frac{Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^0{}_{3\,0\,3} \to \frac{Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^1{}_{0\,0\,1} \to \frac{Et'[r]\, F'[r] - F[r]\, Et''[r]}{Et[r]\, F[r]^3}$

$R^1{}_{1\, i\_\, j\_} \to 0$

$R^1{}_{2\,1\,2} \to \frac{F'[r]}{r\, F[r]^3}$

$R^1{}_{3\,1\,3} \to \frac{F'[r]}{r\, F[r]^3}$

$R^2{}_{0\,0\,2} \to -\frac{Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^2{}_{1\,1\,2} \to -\frac{F'[r]}{r\, F[r]^3}$

$R^2{}_{2\, i\_\, j\_} \to 0$

$R^2{}_{3\,2\,3} \to \frac{-1 + F[r]^2}{r^2\, F[r]^2}$

$R^3{}_{0\,0\,3} \to -\frac{Et'[r]}{r\, Et[r]\, F[r]^2}$

$R^3{}_{1\,1\,3} \to -\frac{F'[r]}{r\, F[r]^3}$

$R^3{}_{2\,2\,3} \to -\frac{-1 + F[r]^2}{r^2\, F[r]^2}$

$R^3{}_{3\, i\_\, j\_} \to 0$

```
orderRicci[ricci_] := Block[{tmp}, tmp = ricci /. T[R, "uddd"][a_, b_, c_, d_] :>
       If[b === 0, 1, -1] T[R, "uddd"][b, a, c, d] /; OrderedQ[{b, a}];
   tmp = tmp /. T[R, "uddd"][a_, b_, c_, d_] :>
       -T[R, "uddd"][a, b, d, c] /; OrderedQ[{d, c}]
  ];
PR[$ = $tRpass,
 NL, "The Ricci tensor: ",
 $ = T[R, "dd"][a_, b_] → Sum[T[R, "uddd"][i, a, i, b], {i, $i}], CK,
 NL, "For: ",
 $Ri = Table[Inactive[T[R, "dd"]][i, i] -> T[R, "dd"][i, i], {i, $i}],
 Yield, $Ri = $Ri /. $,
 NL, "by ordering indices: ",
 $s = $tRpass /. tt : T[R, "uddd"][a_, b_, c_, d_] :> orderRicci[tt] /.
   Rule[-a_, b_] → Rule[a, -b];
 Yield,
 $Ri1 =
  $Ri /. tt : T[R, "uddd"][a_, b_, c_, d_] :> orderRicci[tt] /. $s // Simplify // Activate;
 Column[$Ri1],
 NL, "Verify statement: ",
 $ = T[R, "dd"][0, 0] + T[R, "dd"][1, 1],
 yield, $ = $ /. $Ri1 // Simplify
]
```

$\{R^0{}_{0\,i\_\,j\_} \to 0,\ R^0{}_{1\,0\,1} \to \frac{-Et'[r]\,F'[r] + F[r]\,Et''[r]}{Et[r]\,F[r]^3},\ R^0{}_{2\,0\,2} \to \frac{Et'[r]}{r\,Et[r]\,F[r]^2},$
$R^0{}_{3\,0\,3} \to \frac{Et'[r]}{r\,Et[r]\,F[r]^2},\ R^1{}_{0\,0\,1} \to \frac{Et'[r]\,F'[r] - F[r]\,Et''[r]}{Et[r]\,F[r]^3},\ R^1{}_{1\,i\_\,j\_} \to 0,\ R^1{}_{2\,1\,2} \to \frac{F'[r]}{r\,F[r]^3},$
$R^1{}_{3\,1\,3} \to \frac{F'[r]}{r\,F[r]^3},\ R^2{}_{0\,0\,2} \to -\frac{Et'[r]}{r\,Et[r]\,F[r]^2},\ R^2{}_{1\,1\,2} \to -\frac{F'[r]}{r\,F[r]^3},\ R^2{}_{2\,i\_\,j\_} \to 0,\ R^2{}_{3\,2\,3} \to \frac{-1 + F[r]^2}{r^2\,F[r]^2},$
$R^3{}_{0\,0\,3} \to -\frac{Et'[r]}{r\,Et[r]\,F[r]^2},\ R^3{}_{1\,1\,3} \to -\frac{F'[r]}{r\,F[r]^3},\ R^3{}_{2\,2\,3} \to -\frac{-1 + F[r]^2}{r^2\,F[r]^2},\ R^3{}_{3\,i\_\,j\_} \to 0\}$

The Ricci tensor: $R_{a\_\,b\_} \to R^0{}_{a\,0\,b} + R^1{}_{a\,1\,b} + R^2{}_{a\,2\,b} + R^3{}_{a\,3\,b}$ ⟵CHECK

For: $\{T[R, dd][0, 0] \to R_{0\,0},\ T[R, dd][1, 1] \to R_{1\,1},\ T[R, dd][2, 2] \to R_{2\,2},\ T[R, dd][3, 3] \to R_{3\,3}\}$

→ $\{T[R, dd][0, 0] \to R^0{}_{0\,0\,0} + R^1{}_{0\,1\,0} + R^2{}_{0\,2\,0} + R^3{}_{0\,3\,0},\ T[R, dd][1, 1] \to R^0{}_{1\,0\,1} + R^1{}_{1\,1\,1} + R^2{}_{1\,2\,1} + R^3{}_{1\,3\,1},$
$T[R, dd][2, 2] \to R^0{}_{2\,0\,2} + R^1{}_{2\,1\,2} + R^2{}_{2\,2\,2} + R^3{}_{2\,3\,2},\ T[R, dd][3, 3] \to R^0{}_{3\,0\,3} + R^1{}_{3\,1\,3} + R^2{}_{3\,2\,3} + R^3{}_{3\,3\,3}\}$

by ordering indices:

$R_{0\,0} \to \frac{r\,Et'[r]\,F'[r] - F[r]\,(2\,Et'[r] + r\,Et''[r])}{r\,Et[r]\,F[r]^3}$

$R_{1\,1} \to \frac{2\,Et[r]\,F'[r] - r\,Et'[r]\,F'[r] + r\,F[r]\,Et''[r]}{r\,Et[r]\,F[r]^3}$

→

$R_{2\,2} \to \frac{r\,F[r]\,Et'[r] + Et[r]\,(-F[r] + F[r]^3 + r\,F'[r])}{r^2\,Et[r]\,F[r]^3}$

$R_{3\,3} \to \frac{r\,F[r]\,Et'[r] + Et[r]\,(-F[r] + F[r]^3 + r\,F'[r])}{r^2\,Et[r]\,F[r]^3}$

Verify statement: $R_{0\,0} + R_{1\,1} \longrightarrow \frac{2\,(-F[r]\,Et'[r] + Et[r]\,F'[r])}{r\,Et[r]\,F[r]^3}$

IX.8.1 Warped polar coordinates

```
PR["● Calculate curvature for: ",
 $ds = d[s]^2 → d[r]^2 + f[r]^2 d[θ]^2, " where ", θ → θ + 2 π,
 NL, "World coordinates: ", $xw = {r, θ},
 NL, "Index correspondence: ", $i = {1, 2};
 $xwi = Association[Thread[$i → $xw]],
 NL, "• Vielbein: ",
 $vb = Table[$e = T[e, "u"][i], {i, 2}] -> {DifForm[r], f[r] DifForm[θ]} // Thread,
 NL, "Determine ", DifForm[T[e, "u"][i]], " in terms of itself: ",
 Yield, $ = $vb;
 Yield, $ =
  Map[Thread[DifForm[#], Rule] &, $] // δExpand[DifForm] // tuStdDifForm[{}, $xw, {}];
 Yield, $ = $ /. {DifForm[ff : fn_[aa_]] → xPartialD[ff, aa].DifForm[aa]} //
   tuStdDifForm[{}, $xw, {}];
 Column[$],
 Yield, $ = $ //. xRuleX[$vb, {DifForm[r], DifForm[θ]}] // tuStdDifForm[{},
     Append[$xw, xPartialD[_, r]], {}] // simpleDot3[{xPartialD[_, _]}],
 NL, ""
]
```

● Calculate curvature for: $d[s]^2 \to d[r]^2 + d[\theta]^2\, f[r]^2$ where $\theta \to 2\pi + \theta$
World coordinates: $\{r, \theta\}$
Index correspondence: $\langle| 1 \to r,\ 2 \to \theta |\rangle$
• Vielbein: $\{e^1 \to \underline{d}[r],\ e^2 \to \underline{d}[\theta]\, f[r]\}$
Determine $\underline{d}[e^i]$ in terms of itself:
→
→
→ $\underline{d}[e^1] \to 0$
$\underline{d}[e^2] \to -\underline{\partial}_r[f[r]].(\underline{d}[r] \wedge \underline{d}[\theta])$
→ $\{\underline{d}[e^1] \to 0,\ \underline{d}[e^2] \to -\frac{1}{f[r]}.e^1.e^2\, \underline{\partial}_r[f[r]]\}$

```
PR["● Calculate curvature for: ",
 $ds = d[s]^2 → d[r]^2 + f[r]^2 d[θ]^2, " where ", θ → θ + 2 π,
 NL, "World coordinates: ", $xw = {r, θ},
 NL, "Index correspondence: ", $i = {1, 2};
 $xwi = Association[Thread[$i → $xw]],
 NL, "• Vielbein: ",
 $vb = Table[$e = T[e, "u"][i], {i, 2}] -> {DifForm[r], f[r] DifForm[θ]} // Thread,
 NL, "Determine ", DifForm[T[e, "u"][i]], " in terms of itself: ",
 Yield, $ = $vb;
 Yield, $ = Map[Thread[DifForm[#], Rule] &, $] // δExpand[DifForm] //
   tuStdDifForm[{}, $xw, {T[e, "u"][1], T[e, "u"][2]}, {f[]}];
 Yield, $ = $ // tuStdDifForm[{}, $xw, {T[e, "u"][1], T[e, "u"][2]}, {f[]}];
 Column[$],
 Yield, $ = $ //. xRuleX[$vb, {DifForm[r], DifForm[θ]}],
 Yield, $de =
  $ /. ff : 1 / f[r] → -ff // tuStdDifForm[{}, $xw, {T[e, "u"][1], T[e, "u"][2]}, {f[]}],
 (*The order of Wedge produces matter to the curvature tensor.*)

 NL, "Determine ω from the first Cartan form: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β], "xPOFF",
 Yield, $01 = Table[$0, {α, 2}];
 $01 = Map[MapAt[Sum[#, {β, 2}] &, #, 2] &, $01],

 NL, "Antisymmetry of ω: ", $s = {T[ω, "ud"][b_, b_] → 0},
 Yield, $01 = $01 /. $s // simpleDot3[{}],
 Yield, $01 = $01 /. T[ω, "ud"][α_, β_] :→ Sum[T[ω, "udd"][α, β, i] T[e, "u"][i], {i, 2}] /.
    {T[ω, "udd"][a_, b_, b_] → 0} //
   tuStdDifForm[{}, $xw, {T[e, "u"][1], T[e, "u"][2]}, {f[]}],
 Yield, $1 = tuEliminate[Flatten[{$de, $01}], Table[DifForm[T[e, "u"][i]], {i, 2}]] //
   simpleDot3[{}],
 Yield, $1 = $1 /. Dot → Times /. Wedge[__] → 1; FramedColumn[$1],
 Yield, $1 = Apply[List, $1] /. Equal → Rule,
 Yield,
 $1 = $1 /. ( (tt : T[ω, "udd"][a_, b_, c_]) -> d_) → (tt T[e, "u"][c] -> d T[e, "u"][c]),
 Yield, $w1 = $1 /. (T[ω, "udd"][a_, b_, c_] T[e, "u"][c_]) -> T[ω, "ud"][a, b];
 FramedColumn[$w1]
]
```

● Calculate curvature for: $d[s]^2 \to d[r]^2 + d[\theta]^2\, f[r]^2$ where $\theta \to 2\pi + \theta$

World coordinates: $\{r, \theta\}$

Index correspondence: $\langle | 1 \to r,\ 2 \to \theta | \rangle$

• Vielbein: $\{e^1 \to \underline{d}[r],\ e^2 \to \underline{d}[\theta]\, f[r]\}$

Determine $\underline{d}[e^i]$ in terms of itself:

→

→

→ $\underline{d}[e^1] \to 0$
$\underline{d}[e^2] \to -\underline{\partial}_r[f[r]].(\underline{d}[r] \wedge \underline{d}[\theta])$

→ $\{\underline{d}[e^1] \to 0,\ \underline{d}[e^2] \to -\underline{\partial}_r[f[r]].(e^1 \wedge \frac{e^2}{f[r]})\}$

→ $\{\underline{d}[e^1] \to 0,\ \underline{d}[e^2] \to \frac{1}{f[r]}.\underline{\partial}_r[f[r]].(e^1 \wedge e^2)\}$

Determine ω from the first Cartan form: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta.e^\beta$xPOFF

→ $\{\underline{d}[e^1] \to -\omega^1{}_1.e^1 - \omega^1{}_2.e^2,\ \underline{d}[e^2] \to -\omega^2{}_1.e^1 - \omega^2{}_2.e^2\}$

Antisymmetry of ω: $\{\omega^{b\_}{}_{b\_} \to 0\}$

→ $\{\underline{d}[e^1] \to -\omega^1{}_2.e^2,\ \underline{d}[e^2] \to -\omega^2{}_1.e^1\}$

→ $\{\underline{d}[e^1] \to -\omega^1{}_{21}.(e^1 \wedge e^2),\ \underline{d}[e^2] \to \omega^2{}_{12}.(e^1 \wedge e^2)\}$

→ $\omega^1{}_{21}.(e^1 \wedge e^2) == 0\ \&\&\ \omega^2{}_{12}.(e^1 \wedge e^2) == \frac{1}{f[r]}.\underline{\partial}_r[f[r]].(e^1 \wedge e^2)$

→ $\boxed{\omega^1{}_{21} == 0\ \&\&\ \omega^2{}_{12} == \frac{\underline{\partial}_r[f[r]]}{f[r]}}$

→ $\{\omega^1{}_{21} \to 0,\ \omega^2{}_{12} \to \frac{\underline{\partial}_r[f[r]]}{f[r]}\}$

→ $\{e^1\,\omega^1{}_{21} \to 0,\ e^2\,\omega^2{}_{12} \to \frac{e^2\,\underline{\partial}_r[f[r]]}{f[r]}\}$

→ $\boxed{\begin{array}{l} \omega^1{}_2 \to 0 \\ \omega^2{}_1 \to \frac{e^2\,\underline{\partial}_r[f[r]]}{f[r]} \end{array}}$

```
PR["•Cartan curvature form: ",
 $ = R → DifForm[ω] + ω.ω,
 NL, "Add arguements α,β: ",
 $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] :→ Dot[(a /. β → β1), (b /. α → β1)];
 $ = $ /. dd : Dot[_, _] :→ Sum[T[xη, "uu"][β1, β1] dd, {β1, $i}] /. xη → η,
 NL, "Evaluate for ", $p = {α, β} → Permutations[{1, 1, 2, 2}, {2}],
 $ = Map[$ /. Thread[$p[[1]] → #] &, $p[[2]]],
 NL, "Apply symmetry and ω's: ", $s = {T[ω, "ud"][b_, b_] → 0},
 Yield, $ = $ /. $s /. $w1 // simpleDot3[{}] // tuStdDifForm[{},
    Flatten[{$xw, T[ω, "udd"][a_, b_, c_]}], {T[e, "u"][1], T[e, "u"][2]}, {f[]}];
 Column[$],
 Yield, $ = $ /. xRuleX[$vb, {DifForm[r], DifForm[θ]}] /. $de /. Dot → Times;
 FramedColumn[$], OK,
 NL, "Curvature tensor: ", $[[3]],
 imply, $ = T[R, "uddd"][2, 1, 2, 1] → ($[[3, 2]] /. Wedge[__] → -1),
 Yield, T[R, "uddd"][1, 2, 1, 2] -> T[R, "uddd"][2, 1, 2, 1],
 Imply, "Scalar curvature: ", yield, $ = R → 2 $[[2]]; Framed[$pass = $]
]
```

•Cartan curvature form: $R \to \underline{d}[\omega] + \omega.\omega$

Add arguements α,β: $R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta}] + \omega^{\alpha}{}_{1}.\omega^{1}{}_{\beta} + \omega^{\alpha}{}_{2}.\omega^{2}{}_{\beta}$

Evaluate for $\{\alpha, \beta\} \to \{\{1, 1\}, \{1, 2\}, \{2, 1\}, \{2, 2\}\}\{R^{1}{}_{1} \to \underline{d}[\omega^{1}{}_{1}] + \omega^{1}{}_{1}.\omega^{1}{}_{1} + \omega^{1}{}_{2}.\omega^{2}{}_{1},$
$R^{1}{}_{2} \to \underline{d}[\omega^{1}{}_{2}] + \omega^{1}{}_{1}.\omega^{1}{}_{2} + \omega^{1}{}_{2}.\omega^{2}{}_{2},\ R^{2}{}_{1} \to \underline{d}[\omega^{2}{}_{1}] + \omega^{2}{}_{1}.\omega^{1}{}_{1} + \omega^{2}{}_{2}.\omega^{2}{}_{1},\ R^{2}{}_{2} \to \underline{d}[\omega^{2}{}_{2}] + \omega^{2}{}_{1}.\omega^{1}{}_{2} + \omega^{2}{}_{2}.\omega^{2}{}_{2}\}$

Apply symmetry and ω's: $\{\omega^{b_{-}}{}_{b_{-}} \to 0\}$

$$\to \begin{array}{l} R^{1}{}_{1} \to 0 \\ R^{1}{}_{2} \to 0 \\ R^{2}{}_{1} \to -\frac{1}{f[r]^2}.\underline{\partial}_r[f[r]]^2.(\underline{d}[r] \wedge e^2) + \frac{1}{f[r]}.\underline{\partial}_r[f[r]].\underline{d}[e^2] + \frac{1}{f[r]}.\underline{\partial}_r[\underline{\partial}_r[f[r]]].(\underline{d}[r] \wedge e^2) \\ R^{2}{}_{2} \to 0 \end{array}$$

$$\to \boxed{\begin{array}{l} R^{1}{}_{1} \to 0 \\ R^{1}{}_{2} \to 0 \\ R^{2}{}_{1} \to \frac{e^1 \wedge e^2\ \underline{\partial}_r[\underline{\partial}_r[f[r]]]}{f[r]} \\ R^{2}{}_{2} \to 0 \end{array}} \quad \text{OK}$$

Curvature tensor: $R^{2}{}_{1} \to \dfrac{e^1 \wedge e^2\ \underline{\partial}_r[\underline{\partial}_r[f[r]]]}{f[r]} \Rightarrow R^{2}{}_{121} \to -\dfrac{\underline{\partial}_r[\underline{\partial}_r[f[r]]]}{f[r]}$

$\to R^{1}{}_{212} \to R^{2}{}_{121}$

⇒ Scalar curvature: $\longrightarrow \boxed{R \to -\dfrac{2\ \underline{\partial}_r[\underline{\partial}_r[f[r]]]}{f[r]}}$

```
PR["• If R → C[constant]",
 Imply, $ = $pass[[2]] → C[1],
 Yield, $ = f[r] # / 2 & /@ $ /. C[1] → 2 C[2],
 NL, "Which has a general solution: ", f[r] → C[4] Exp[√C[2] r],
 NL, "To preserve Flat metric at r->0 ",
 imply, Limit[f[r], r → 0] → r
]
```

• If R → C[constant]

$\Rightarrow -\dfrac{2\ \underline{\partial}_r[\underline{\partial}_r[f[r]]]}{f[r]} \to C[1]$

$\to -\underline{\partial}_r[\underline{\partial}_r[f[r]]] \to C[2]\ f[r]$

Which has a general solution: $f[r] \to e^{r\sqrt{C[2]}}\ C[4]$

To preserve Flat metric at r->0 $\Rightarrow$ $\mathrm{Limit}[f[r], r \to 0] \to r$

IX.8.2

```
PR["● Calculate curvature for: ",
 $ds = $ds = d[s]^2 → y^(2 p) d[x]^2 + x^(2 p) d[y]^2,
 NL, "World coordinates: ", $xw = {x, y},
 NL, "Index correspondence: ", $i = {1, 2};
 $xwi = Association[Thread[$i → $xw]],
 NL, "• Vielbein: ",
 $vb = Table[$e = T[e, "u"][i], {i, 2}] -> {y^p . DifForm[x], x^p . DifForm[y]} // Thread;
 Column[$vb],
 NL, "Determine ", DifForm[T[e, "u"][i]], " in terms of e's: ", $ = $vb;
 Yield, $ = Map[Thread[DifForm[#], Rule] &, $] // δExpand[DifForm],
 Yield, $ = $ // tuStdDifForm[{p}, Flatten[{$xw, p}], {T[e, "u"][1], T[e, "u"][2]}, {}];
 FramedColumn[$],
 NL, "• In terms of e's: ",
 Yield, $ = $ //. xRuleX[($vb /. Dot → Times), Map[DifForm[#] &, $xw]],
 Yield, $de = $ // tuStdDifForm[{}, Flatten[{$xw, p}], {T[e, "u"][1], T[e, "u"][2]}, {}];
 FramedColumn[$de], back,
 (*The order of Wedge produces matter to the curvature tensor.*)
 NL, "Definition of ω from the first Cartan form: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β],
 Yield, $01 = Table[$0, {α, 2}];
 $01 =
  Map[MapAt[Sum[#, {β, 2}] &, #, 2] &, $01] /. {T[ω, "ud"][b_, b_] → 0} // simpleDot3[{}];
 Yield, Column[$01],
 NL, "Add explicit e's and compare to determine ω's : ",
 Yield, $01 = $01 /. T[ω, "ud"][α_, β_] :→ Sum[T[ω, "udd"][α, β, i] T[e, "u"][i], {i, 2}] /.
    T[ω, "udd"][i_, j_, k_] :→ -T[ω, "udd"][j, i, k] /; OrderedQ[{j, i}] //
   simpleDot3[{T[ω, "udd"][i_, j_, k_]}];
 xtmp = $01 = $01 /. Dot → Wedge /. Wedge[a_, b_] :→ -Wedge[b, a] /; OrderedQ[{b, a}];
 FramedColumn[$01], back,
 $ = Flatten[{$de, $01}];
 $ = {SubtractRules[Select[$, FreeQ[DifForm[T[e, "u"][1]]]]],
   SubtractRules[Select[$, FreeQ[DifForm[T[e, "u"][2]]]]]};
 $ = xRuleX[$, {T[ω, "udd"][1, 2, 2], T[ω, "udd"][1, 2, 1]}],
 Yield, $ = $ /. Wedge[T[e, "u"][i_], T[e, "u"][i_]] → 0; FramedColumn[$],
 NL, "From the definition: ",
 $w = T[ω, "ud"][1, 2];
 $w = $w → ($w /. T[ω, "ud"][α_, β_] :→ Sum[T[ω, "udd"][α, β, i] T[e, "u"][i], {i, 2}]),
 Yield, $w = $w /. $,
 yield, $w = $w /. $vb /. Dot → Times; Framed[$w]
]
```

- Calculate curvature for: $d[s]^2 \to y^{2p}\, d[x]^2 + x^{2p}\, d[y]^2$

World coordinates: {x, y}

Index correspondence: $\langle | 1 \to x,\ 2 \to y | \rangle$

- Vielbein: $e^1 \to y^p . \underline{d}[x]$
  $e^2 \to x^p . \underline{d}[y]$

Determine $\underline{d}[e^i]$ in terms of e's:

$\to \{\underline{d}[e^1] \to y^p . \underline{d}[\underline{d}[x]] + (p\, y^{-1+p}\, \underline{d}[y]) . \underline{d}[x],\ \underline{d}[e^2] \to x^p . \underline{d}[\underline{d}[y]] + (p\, x^{-1+p}\, \underline{d}[x]) . \underline{d}[y]\}$

$$\to \boxed{\begin{array}{l} \underline{d}[e^1] \to -p\, y^{-1+p}\, \underline{d}[x] \wedge \underline{d}[y] \\ \underline{d}[e^2] \to p\, x^{-1+p}\, \underline{d}[x] \wedge \underline{d}[y] \end{array}}$$

- In terms of e's:

$\to \{\underline{d}[e^1] \to -p\, y^{-1+p}\, (y^{-p}\, e^1) \wedge (x^{-p}\, e^2),\ \underline{d}[e^2] \to p\, x^{-1+p}\, (y^{-p}\, e^1) \wedge (x^{-p}\, e^2)\}$

$$\to \boxed{\begin{array}{l} \underline{d}[e^1] \to -\frac{p\, x^{-p}\, e^1 \wedge e^2}{y} \\ \underline{d}[e^2] \to \frac{p\, y^{-p}\, e^1 \wedge e^2}{x} \end{array}} \longleftarrow$$

Definition of ω from the first Cartan form: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta . e^\beta$

$\to$

$$\to \begin{array}{l} \underline{d}[e^1] \to -\omega^1{}_2 . e^2 \\ \underline{d}[e^2] \to -\omega^2{}_1 . e^1 \end{array}$$

Add explicit e's and compare to determine ω's :

$$\to \boxed{\begin{array}{l} \underline{d}[e^1] \to -\omega^1{}_{21}\, e^1 \wedge e^2 + \omega^1{}_{22}\, e^2 \wedge e^2 \\ \underline{d}[e^2] \to -\omega^1{}_{21}\, e^1 \wedge e^1 - \omega^1{}_{22}\, e^1 \wedge e^2 \end{array}} \longleftarrow$$

$$\left\{\omega^1{}_{22} \to -\frac{p\, x^{-1-p}\, y^{-1-p}\, e^1 \wedge e^2\, (x\, y^p\, e^1 \wedge e^1 + x^p\, y\, e^1 \wedge e^2)}{(e^1 \wedge e^2)^2 + e^1 \wedge e^1\, e^2 \wedge e^2},\ \omega^1{}_{21} \to \frac{p\, x^{-1-p}\, y^{-1-p}\, e^1 \wedge e^2\, (x\, y^p\, e^1 \wedge e^2 - x^p\, y\, e^2 \wedge e^2)}{(e^1 \wedge e^2)^2 + e^1 \wedge e^1\, e^2 \wedge e^2}\right\}$$

$$\to \boxed{\begin{array}{l} \omega^1{}_{22} \to -\frac{p\, y^{-p}}{x} \\ \omega^1{}_{21} \to \frac{p\, x^{-p}}{y} \end{array}}$$

From the definition: $\omega^1{}_2 \to e^1\, \omega^1{}_{21} + e^2\, \omega^1{}_{22}$

$$\to \omega^1{}_2 \to \frac{p\, x^{-p}\, e^1}{y} - \frac{p\, y^{-p}\, e^2}{x} \longrightarrow \boxed{\omega^1{}_2 \to p\, x^{-p}\, y^{-1+p}\, \underline{d}[x] - p\, x^{-1+p}\, y^{-p}\, \underline{d}[y]}$$

```
PR["•Cartan curvature form: ",
 $ = R → DifForm[ω] + ω.ω,
 NL, "Add arguements α,β: ",
 $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] :→ Dot[(a /. β → β1), (b /. α → β1)];
 $ = $ /. dd : Dot[_, _] :→ Sum[T[xη, "uu"][β1, β1] dd, {β1, $i}] /. xη → η,
 NL, "Evaluate for ", $p = {α, β} → Permutations[{1, 1, 2, 2}, {2}],
 $ = Map[$ /. Thread[$p[[1]] → #] &, $p[[2]]],
 NL, "Apply symmetry and ω's: ", $s =
  {T[ω, "ud"][b_, b_] → 0, T[ω, "ud"][a_, b_] :→ -T[ω, "ud"][b, a] /; OrderedQ[{b, a}]},
 Yield, $ = $ //. $s // simpleDot3[{p, x, y}] // tuStdDifForm[{p},
    Flatten[{$xw, T[ω, "udd"][a_, b_, c_]}], {T[e, "u"][1], T[e, "u"][2]}, {}],
 Yield, $ = $ //. $s /. $w // simpleDot3[{p, x, y}]; Column[$], CK,
 Yield, $ = $ // tuStdDifForm[{p},
    Flatten[{$xw, T[ω, "udd"][a_, b_, c_]}], {T[e, "u"][1], T[e, "u"][2]}, {}];
 FramedColumn[$],
 Yield, $ = $ /. xRuleX[$vbt, Map[DifForm[#] &, {x, y}]] /. $de // tuStdDifForm[{p},
    Flatten[{$xw, T[ω, "udd"][a_, b_, c_]}], {T[e, "u"][1], T[e, "u"][2]}, {}];
 $ = $ /. Dot → Times;
 FramedColumn[$],

 NL, "Curvature tensors: ", $,
 imply, $ = T[R, "uddd"][2, 1, 2, 1] → ($[[3, 2]] /. Wedge[__] → -1),
 Yield, T[R, "uddd"][1, 2, 1, 2] -> T[R, "uddd"][2, 1, 2, 1],

 Imply, "Scalar curvature: ", yield, $ = R → 2 $[[2]] // Simplify;
 Framed[$pass = $],
 NL, "This space is flat for p->{0,1}."
]
```

•Cartan curvature form: $R \to \underline{d}[\omega] + \omega.\omega$

Add arguements α,β: $R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta}] + \omega^{\alpha}{}_{1}.\omega^{1}{}_{\beta} + \omega^{\alpha}{}_{2}.\omega^{2}{}_{\beta}$

Evaluate for $\{\alpha, \beta\} \to \{\{1, 1\}, \{1, 2\}, \{2, 1\}, \{2, 2\}\}$ $\{R^1{}_1 \to \underline{d}[\omega^1{}_1] + \omega^1{}_1.\omega^1{}_1 + \omega^1{}_2.\omega^2{}_1,$
$R^1{}_2 \to \underline{d}[\omega^1{}_2] + \omega^1{}_1.\omega^1{}_2 + \omega^1{}_2.\omega^2{}_2,\ R^2{}_1 \to \underline{d}[\omega^2{}_1] + \omega^2{}_1.\omega^1{}_1 + \omega^2{}_2.\omega^2{}_1,\ R^2{}_2 \to \underline{d}[\omega^2{}_2] + \omega^2{}_1.\omega^1{}_2 + \omega^2{}_2.\omega^2{}_2\}$

Apply symmetry and ω's: $\{\omega^{b\_}{}_{b\_} \to 0,\ \omega^{a\_}{}_{b\_} :\to -T[\omega, ud][b, a] /; OrderedQ[\{b, a\}]\}$

$\to \{R^1{}_1 \to -\omega^1{}_2.\omega^1{}_2,\ R^1{}_2 \to \underline{d}[\omega^1{}_2],\ R^2{}_1 \to -\underline{d}[\omega^1{}_2],\ R^2{}_2 \to -\omega^1{}_2.\omega^1{}_2\}$

$$R^1{}_1 \to -p^2 x^{-2p} y^{-2+2p}\,\underline{d}[x].\underline{d}[x] + \frac{p^2\,\underline{d}[x].\underline{d}[y]}{x\,y} + \frac{p^2\,\underline{d}[y].\underline{d}[x]}{x\,y} - p^2 x^{-2+2p} y^{-2p}\,\underline{d}[y].\underline{d}[y]$$

$\to$
$$R^1{}_2 \to \underline{d}[p\,x^{-p} y^{-1+p}\,\underline{d}[x] - p\,x^{-1+p} y^{-p}\,\underline{d}[y]]$$
$$R^2{}_1 \to -\underline{d}[p\,x^{-p} y^{-1+p}\,\underline{d}[x] - p\,x^{-1+p} y^{-p}\,\underline{d}[y]]$$
$$R^2{}_2 \to -p^2 x^{-2p} y^{-2+2p}\,\underline{d}[x].\underline{d}[x] + \frac{p^2\,\underline{d}[x].\underline{d}[y]}{x\,y} + \frac{p^2\,\underline{d}[y].\underline{d}[x]}{x\,y} - p^2 x^{-2+2p} y^{-2p}\,\underline{d}[y].\underline{d}[y]$$

⟵CHECK

$\to$

$$\boxed{\begin{array}{l} R^1{}_1 \to 0 \\ R^1{}_2 \to \\ \quad x^{-p} y^{-2+p}\,p.(\underline{d}[x] \wedge \underline{d}[y]) + x^{-2+p} y^{-p}\,p.(\underline{d}[x] \wedge \underline{d}[y]) - x^{-p} y^{-2+p}\,p^2.(\underline{d}[x] \wedge \underline{d}[y]) - x^{-2+p} y^{-p}\,p^2.(\underline{d}[x] \wedge \underline{d}[y]) \\ R^2{}_1 \to \\ \quad -x^{-p} y^{-2+p}\,p.(\underline{d}[x] \wedge \underline{d}[y]) - x^{-2+p} y^{-p}\,p.(\underline{d}[x] \wedge \underline{d}[y]) + x^{-p} y^{-2+p}\,p^2.(\underline{d}[x] \wedge \underline{d}[y]) + x^{-2+p} y^{-p}\,p^2.(\underline{d}[x] \wedge \underline{d}[y]) \\ R^2{}_2 \to 0 \end{array}}$$

$$\to \boxed{\begin{array}{l} R^1{}_1 \to 0 \\ R^1{}_2 \to \frac{p\,x^{-2p}\,e^1 \wedge e^2}{y^2} - \frac{p^2\,x^{-2p}\,e^1 \wedge e^2}{y^2} + \frac{p\,y^{-2p}\,e^1 \wedge e^2}{x^2} - \frac{p^2\,y^{-2p}\,e^1 \wedge e^2}{x^2} \\ R^2{}_1 \to -\frac{p\,x^{-2p}\,e^1 \wedge e^2}{y^2} + \frac{p^2\,x^{-2p}\,e^1 \wedge e^2}{y^2} - \frac{p\,y^{-2p}\,e^1 \wedge e^2}{x^2} + \frac{p^2\,y^{-2p}\,e^1 \wedge e^2}{x^2} \\ R^2{}_2 \to 0 \end{array}}$$

Curvature tensors: $\{R^1{}_1 \to 0,\ R^1{}_2 \to \frac{p\,x^{-2p}\,e^1 \wedge e^2}{y^2} - \frac{p^2\,x^{-2p}\,e^1 \wedge e^2}{y^2} + \frac{p\,y^{-2p}\,e^1 \wedge e^2}{x^2} - \frac{p^2\,y^{-2p}\,e^1 \wedge e^2}{x^2},$
$R^2{}_1 \to -\frac{p\,x^{-2p}\,e^1 \wedge e^2}{y^2} + \frac{p^2\,x^{-2p}\,e^1 \wedge e^2}{y^2} - \frac{p\,y^{-2p}\,e^1 \wedge e^2}{x^2} + \frac{p^2\,y^{-2p}\,e^1 \wedge e^2}{x^2},\ R^2{}_2 \to 0\}$

$\Rightarrow R^2{}_{121} \to \frac{p\,x^{-2p}}{y^2} - \frac{p^2\,x^{-2p}}{y^2} + \frac{p\,y^{-2p}}{x^2} - \frac{p^2\,y^{-2p}}{x^2}$

$\to R^1{}_{212} \to R^2{}_{121}$

$\Rightarrow$ Scalar curvature: $\longrightarrow$ $\boxed{R \to -2\,(-1 + p)\,p\,x^{-2\,(1+p)}\,y^{-2\,(1+p)}\,(x^{2p}\,y^2 + x^2\,y^{2p})}$

This space is flat for p->{0,1}.

IX.8.3 Curvature of torus

```
PR["● Calculate curvature for a torus: ",
  $ds = d[s]^2 → a^2 d[θ]^2 + (L + a Sin[θ])^2 d[φ]^2 /. d → DifForm,
  NL, "World coordinates: ", $xw = {θ, φ},
  NL, "Index correspondence: ", $i = {1, 2};
  $xwi = Association[Thread[$i → $xw]],
  NL, "• Vielbein: ", $ = Map[PowerExpand[# √Coefficient[$ds[[2]], #^2] ] &,
     {DifForm[θ], DifForm[φ]}] /. Times → Dot;
  yield, $vb = Table[$e = T[e, "u"][i], {i, 2}] -> $ // Thread; Column[$vb],
  $vbt = $vb /. Dot → Times;

  NL, "Determine ", DifForm[T[e, "u"][i]], " in terms of e's: ", $ = $vb;
  Yield, $ = Map[Thread[DifForm[#], Rule] &, $]; Column[$],
  Yield,
  $ = $ // tuStdDifForm[{a, L}, Flatten[{$xw}], {T[e, "u"][1], T[e, "u"][2]}, {Sin[]}];
  Column[$],
  NL, "• In terms of e's: ",
  Yield, $ = $ //. xRuleX[($vb /. Dot → Times), Map[DifForm[#] &, $xw]],
  Yield,
  $de = $ // tuStdDifForm[{}, Flatten[{$xw, p}], {T[e, "u"][1], T[e, "u"][2]}, {}];
  FramedColumn[$de], back,
  (*The order of Wedge produces matter to the curvature tensor.*)
  NL, "Definition of ω from the first Cartan form: ",
  $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β].T[e, "u"][β],
  Yield, $01 = Table[$0, {α, 2}];
  $01 = Map[MapAt[Sum[#, {β, 2}] &, #, 2] &, $01] /.
     {T[ω, "ud"][b_, b_] → 0} // simpleDot3[{}];
  Yield, Column[$01],
  NL, "Add explicit e's and compare to determine ω's : ",
  Yield, $01 = $01 /. T[ω, "ud"][α_, β_] :→ Sum[T[ω, "udd"][α, β, i] T[e, "u"][i], {i, 2}] /.
     T[ω, "udd"][i_, j_, k_] :→ -T[ω, "udd"][j, i, k] /; OrderedQ[{j, i}] //
    simpleDot3[{T[ω, "udd"][i_, j_, k_]}];
  $01 = $01 /. Dot → Wedge /. Wedge[a_, b_] :→ -Wedge[b, a] /; OrderedQ[{b, a}];
  FramedColumn[$01], back,
  $ = Flatten[{$de, $01}];
  $ = {SubtractRules[Select[$, FreeQ[DifForm[T[e, "u"][1]]]]],
    SubtractRules[Select[$, FreeQ[DifForm[T[e, "u"][2]]]]]};
  $ = xRuleX[$, {T[ω, "udd"][1, 2, 2], T[ω, "udd"][1, 2, 1]}],
  Yield, $ = $ /. Wedge[T[e, "u"][i_], T[e, "u"][i_]] → 0; FramedColumn[$],
  NL, "From the definition: ",
  $w = T[ω, "ud"][1, 2];
  $w = $w → ($w /. T[ω, "ud"][α_, β_] :→ Sum[T[ω, "udd"][α, β, i] T[e, "u"][i], {i, 2}]),
  Yield, $w = $w /. $,
  yield, $w = $w /. $vb /. Dot → Times; Framed[$w]
 ];
```

● Calculate curvature for a torus: $\underline{d}[s]^2 \to a^2\, \underline{d}[\theta]^2 + \underline{d}[\varphi]^2\, (L + a\, \mathrm{Sin}[\theta])^2$

World coordinates: $\{\theta, \varphi\}$

Index correspondence: $\langle\, |\, 1 \to \theta,\ 2 \to \varphi\, |\, \rangle$

• Vielbein: $\longrightarrow$ $\begin{matrix} e^1 \to a.\underline{d}[\theta] \\ e^2 \to \underline{d}[\varphi].(L + a.\mathrm{Sin}[\theta]) \end{matrix}$

Determine $\underline{d}[e^i]$ in terms of e's:

$\to$ $\begin{matrix} \underline{d}[e^1] \to \underline{d}[a.\underline{d}[\theta]] \\ \underline{d}[e^2] \to \underline{d}[\underline{d}[\varphi].(L + a.\mathrm{Sin}[\theta])] \end{matrix}$

$\to$ $\begin{matrix} \underline{d}[e^1] \to 0 \\ \underline{d}[e^2] \to a.\underline{\partial}_\theta[\mathrm{Sin}[\theta]].(\underline{d}[\theta] \wedge \underline{d}[\varphi]) \end{matrix}$

• In terms of e's:

$\to$ $\{\underline{d}[e^1] \to 0,\ \underline{d}[e^2] \to a.\underline{\partial}_\theta[\mathrm{Sin}[\theta]].(\frac{e^1}{a} \wedge \frac{e^2}{L + a\,\mathrm{Sin}[\theta]})\}$

$\to$ $\boxed{\begin{matrix} \underline{d}[e^1] \to 0 \\ \underline{d}[e^2] \to \frac{1}{L+a.\mathrm{Sin}[\theta]}.\underline{\partial}_\theta[\mathrm{Sin}[\theta]].(e^1 \wedge e^2) \end{matrix}}$ $\longleftarrow$

Definition of ω from the first Cartan form: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta.e^\beta$

$\to$

$\to$ $\begin{matrix} \underline{d}[e^1] \to -\omega^1{}_2.e^2 \\ \underline{d}[e^2] \to -\omega^2{}_1.e^1 \end{matrix}$

Add explicit e's and compare to determine ω's :

$\to$ $\boxed{\begin{matrix} \underline{d}[e^1] \to -\omega^1{}_{21}\, e^1 \wedge e^2 + \omega^1{}_{22}\, e^2 \wedge e^2 \\ \underline{d}[e^2] \to -\omega^1{}_{21}\, e^1 \wedge e^1 - \omega^1{}_{22}\, e^1 \wedge e^2 \end{matrix}}$ $\longleftarrow$

$$\{\omega^1{}_{22} \to -\frac{\frac{1}{L+a.\mathrm{Sin}[\theta]}.\underline{\partial}_\theta[\mathrm{Sin}[\theta]].(e^1 \wedge e^2)\, e^1 \wedge e^2}{(e^1 \wedge e^2)^2 + e^1 \wedge e^1\, e^2 \wedge e^2},\ \omega^1{}_{21} \to -\frac{\frac{1}{L+a.\mathrm{Sin}[\theta]}.\underline{\partial}_\theta[\mathrm{Sin}[\theta]].(e^1 \wedge e^2)\, e^2 \wedge e^2}{(e^1 \wedge e^2)^2 + e^1 \wedge e^1\, e^2 \wedge e^2}\}$$

$\to$ $\boxed{\begin{matrix} \omega^1{}_{22} \to -\frac{\frac{1}{L+a.\mathrm{Sin}[\theta]}.\underline{\partial}_\theta[\mathrm{Sin}[\theta]].(e^1 \wedge e^2)}{e^1 \wedge e^2} \\ \omega^1{}_{21} \to 0 \end{matrix}}$

From the definition: $\omega^1{}_2 \to e^1\, \omega^1{}_{21} + e^2\, \omega^1{}_{22}$

$\to$ $\omega^1{}_2 \to -\frac{\frac{1}{L+a.\mathrm{Sin}[\theta]}.\underline{\partial}_\theta[\mathrm{Sin}[\theta]].(e^1 \wedge e^2)\, e^2}{e^1 \wedge e^2}$ $\longrightarrow$ $\boxed{\omega^1{}_2 \to -\underline{d}[\varphi]\, \underline{\partial}_\theta[\mathrm{Sin}[\theta]]}$

```
PR["•Cartan curvature form: ",
  $ = R → DifForm[ω] + ω.ω,
  NL, "Add arguements α,β: ",
  $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] ⧴ Dot[(a /. β → β1), (b /. α → β1)];
  $ = $ /. dd : Dot[_, _] ⧴ Sum[T[xη, "uu"][β1, β1] dd, {β1, $i}] /. xη → η,
  NL, "Evaluate for ", $p = {α, β} → Permutations[{1, 1, 2, 2}, {2}],
  $ = Map[$ /. Thread[$p[[1]] → #] &, $p[[2]]],
  NL, "Apply symmetry and ω's: ", $s =
   {T[ω, "ud"][b_, b_] → 0, T[ω, "ud"][a_, b_] ⧴ -T[ω, "ud"][b, a] /; OrderedQ[{b, a}]},
  Yield, $ = $ //. $s // simpleDot3[{p, x, y}] // tuStdDifForm[{p},
     Flatten[{$xw, T[ω, "udd"][a_, b_, c_]}], {T[e, "u"][1], T[e, "u"][2]}, {}],
  Yield, $ = $ //. $s /. $w /. xPartialD → D // simpleDot3[{p, x, y}];
  Column[$],
  Yield, $ = $ // tuStdDifForm[{p},
     Flatten[{$xw, T[ω, "udd"][a_, b_, c_]}], {T[e, "u"][1], T[e, "u"][2]}, {Cos[]}];
  FramedColumn[$],
  Yield, $ = $ //. xRuleX[$vbt, Map[DifForm[#] &, {θ, φ}]],
  Yield, $ = $ /. $de // tuStdDifForm[{p},
     Flatten[{$xw, T[ω, "udd"][a_, b_, c_]}], {T[e, "u"][1], T[e, "u"][2]}, {Cos[]}];
  $ = $ /. Dot → Times /. xPartialD → D;
  FramedColumn[$],

  NL, "Curvature tensors: ",
  imply, $ = T[R, "uddd"][2, 1, 2, 1] → ($[[3, 2]] /. Wedge[__] → -1),
  Yield, T[R, "uddd"][1, 2, 1, 2] -> T[R, "uddd"][2, 1, 2, 1],

  Imply, "Scalar curvature: ", yield, $ = R → 2 $[[2]] // Simplify;
  Framed[$pass = $], OK
 ];
```

•Cartan curvature form: $R \to \underline{d}[\omega] + \omega.\omega$

Add arguements α,β: $R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta}] + \omega^{\alpha}{}_{1}.\omega^{1}{}_{\beta} + \omega^{\alpha}{}_{2}.\omega^{2}{}_{\beta}$

Evaluate for $\{\alpha, \beta\} \to \{\{1, 1\}, \{1, 2\}, \{2, 1\}, \{2, 2\}\}\{R^1{}_1 \to \underline{d}[\omega^1{}_1] + \omega^1{}_1.\omega^1{}_1 + \omega^1{}_2.\omega^2{}_1,$
$R^1{}_2 \to \underline{d}[\omega^1{}_2] + \omega^1{}_1.\omega^1{}_2 + \omega^1{}_2.\omega^2{}_2,\ R^2{}_1 \to \underline{d}[\omega^2{}_1] + \omega^2{}_1.\omega^1{}_1 + \omega^2{}_2.\omega^2{}_1,\ R^2{}_2 \to \underline{d}[\omega^2{}_2] + \omega^2{}_1.\omega^1{}_2 + \omega^2{}_2.\omega^2{}_2\}$

Apply symmetry and ω's: $\{\omega^{b\_}{}_{b\_} \to 0,\ \omega^{a\_}{}_{b\_} :\to -T[\omega, ud][b, a]$ /; OrderedQ[{b, a}]}

$\to \{R^1{}_1 \to -\omega^1{}_2.\omega^1{}_2,\ R^1{}_2 \to \underline{d}[\omega^1{}_2],\ R^2{}_1 \to -\underline{d}[\omega^1{}_2],\ R^2{}_2 \to -\omega^1{}_2.\omega^1{}_2\}$

$\to$
$R^1{}_1 \to -(\mathrm{Cos}[\theta]\,\underline{d}[\varphi]).(\mathrm{Cos}[\theta]\,\underline{d}[\varphi])$
$R^1{}_2 \to \underline{d}[-\mathrm{Cos}[\theta]\,\underline{d}[\varphi]]$
$R^2{}_1 \to -\underline{d}[-\mathrm{Cos}[\theta]\,\underline{d}[\varphi]]$
$R^2{}_2 \to -(\mathrm{Cos}[\theta]\,\underline{d}[\varphi]).(\mathrm{Cos}[\theta]\,\underline{d}[\varphi])$

$$\to \boxed{\begin{array}{l} R^1{}_1 \to 0 \\ R^1{}_2 \to -\underline{\partial}_{\theta}[\mathrm{Cos}[\theta]].(\underline{d}[\theta] \wedge \underline{d}[\varphi]) \\ R^2{}_1 \to \underline{\partial}_{\theta}[\mathrm{Cos}[\theta]].(\underline{d}[\theta] \wedge \underline{d}[\varphi]) \\ R^2{}_2 \to 0 \end{array}}$$

$$\to \{R^1{}_1 \to 0,\ R^1{}_2 \to -\underline{\partial}_{\theta}[\mathrm{Cos}[\theta]].(\frac{e^1}{a} \wedge \frac{e^2}{L + a\,\mathrm{Sin}[\theta]}),\ R^2{}_1 \to \underline{\partial}_{\theta}[\mathrm{Cos}[\theta]].(\frac{e^1}{a} \wedge \frac{e^2}{L + a\,\mathrm{Sin}[\theta]}),\ R^2{}_2 \to 0\}$$

$$\to \boxed{\begin{array}{l} R^1{}_1 \to 0 \\ R^1{}_2 \to \frac{\mathrm{Sin}[\theta]\, e^1 \wedge e^2}{a\,(L + a\,\mathrm{Sin}[\theta])} \\ R^2{}_1 \to -\frac{\mathrm{Sin}[\theta]\, e^1 \wedge e^2}{a\,(L + a\,\mathrm{Sin}[\theta])} \\ R^2{}_2 \to 0 \end{array}}$$

Curvature tensors: $\Rightarrow R^2{}_{1\,2\,1} \to \dfrac{\mathrm{Sin}[\theta]}{a\,(L + a\,\mathrm{Sin}[\theta])}$

$\to R^1{}_{2\,1\,2} \to R^2{}_{1\,2\,1}$

$\Rightarrow$ Scalar curvature: $\longrightarrow \boxed{R \to \dfrac{2\,\mathrm{Sin}[\theta]}{a\,(L + a\,\mathrm{Sin}[\theta])}}$ OK

IX.8.4 Kasner universe

```
PR["● Calculate curvature for the Kasner universe: ",

 NL, "World coordinates: ", $xw = {t, x, y, z},
 NL, "Metric: ", $ds =
  d[s]^2 → Apply[Plus, First[{DifForm[#]^2 & /@ $xw {-1, A[t]^2, B[t]^2, C[t]^2}}]],
 NL, "Index correspondence: ", $i = {0, 1, 2, 3};
 $xwi = Association[Thread[$i → $xw]],
 NL, "• Vielbein: ",
 $ = Map[PowerExpand[# √Coefficient[$ds[[2]], #^2] ] &, DifForm[#] & /@ $xw] /.
    Times → Dot;
 yield, $vb = Table[$e = T[e, "u"][i], {i, 0, 3}] -> $ // Thread;
 $vb = $vb /. I → 1;
 Column[$vb],
 $vbt = $vb /. Dot → Times;
 NL, "Determine ", DifForm[T[e, "u"][i]], " in terms of e's: ", $ = $vb;
 Yield, $ = Map[Thread[DifForm[#], Rule] &, $]; Column[$],
 Yield, $ = $ // tuStdDifForm[{a, L},
     Flatten[{$xw}], Table[T[e, "u"][i], {i, 0, 3}] , {A[], B[], C[]}];
 Column[$],
 NL, "• In terms of e's: ",
 Yield, $ = $ //. xRuleX[($vb /. Dot → Times), Map[DifForm[#] &, $xw]],
 Yield,
```

```
 $de = $ // tuStdDifForm[{}, Flatten[{$xw, p}], Table[T[e, "u"][i], {i, 0, 3}], {}];
 $de = $de /. Dot → Times;
 FramedColumn[$de], back,
 (*The order of Wedge produces matter to the curvature tensor.*)
 NL, "Definition of ω from the first Cartan form: ",
 $0 = DifForm[T[e, "u"][α]] → -T[ω, "ud"][α, β]. T[e, "u"][β],
 Yield, $01 = Table[$0, {α, 0, 3}];
 $01 = Map[MapAt[Sum[#, {β, 0, 3}] &, #, 2] &, $01] /.
    {T[ω, "ud"][b_, b_] → 0} // simpleDot3[{}];
 Yield, FramedColumn[$010 = $01]
]
PR["From: ", $de;
 NL, $ = $0 = $010 /. xDot → Wedge; Column[$];
 $ = {$de[[1]], $[[1]]},
 imply, $s = T[ω, "ud"][i_, 0] -> T[ω, "udd"][0, i, i] T[e, "u"][i];
 Framed[$s],
 Yield, $ = $0 /. $s,
 Yield, $ = $ //
   tuStdDifForm[{}, Flatten[{T[ω, "udd"][_, _, _]}], Table[T[e, "u"][i], {i, 0, 3}], {}];
 Column[$],
 NL, "Comparing: ", {$[[2]], $[[4]]} // Column,
 and, $de,
 Imply, $s = {T[ω, "ud"][1, 3] -> T[ω, "udd"][1, 3, 3] T[e, "u"][3], T[ω, "ud"][3, 1] ->
    T[ω, "udd"][3, 1, 1] T[e, "u"][1], T[ω, "ud"][1, 3] → -T[ω, "ud"][3, 1]},
 NL, "which is impossible unless ", T[ω, "ud"][1, 3] → 0,
 Imply, "All: ",
 $s = Apply[Alternatives, Flatten[Table[T[ω, "ud"][i, j], {i, 3}, {j, 3}]]] → 0,
 Yield, $ = $ /. $s // simpleDot3[{}],
 Yield, $ = {$[[2]], $de[[2]]},
 Yield, $ = SubtractRules[$] /. Wedge[__] → 1 // Simplify,
 Yield, $ = xRuleX[$, T[ω, "udd"][3, 1, 1]] // First,
 Imply, $ = T[ω, "ud"][0, 1] → $[[2]] T[e, "u"][1]; Framed[$],
 " Similarly for B,C.",
 $w = {$, $ /. {1 → 2, A → B}, $ /. {1 → 3, A → C}}; Framed[$w]
]
```

● Calculate curvature for the Kasner universe:
World coordinates: {t, x, y, z}
Metric: $d[s]^2 \to -\underline{d}[t]^2 + A[t]^2\,\underline{d}[x]^2 + B[t]^2\,\underline{d}[y]^2 + C[t]^2\,\underline{d}[z]^2$
Index correspondence: $\langle|\, 0 \to t,\ 1 \to x,\ 2 \to y,\ 3 \to z \,|\rangle$

• Vielbein: $\longrightarrow$
$e^0 \to 1.\underline{d}[t]$
$e^1 \to A[t].\underline{d}[x]$
$e^2 \to B[t].\underline{d}[y]$
$e^3 \to C[t].\underline{d}[z]$

Determine $\underline{d}[e^i]$ in terms of e's:

$\to$
$\underline{d}[e^0] \to \underline{d}[1.\underline{d}[t]]$
$\underline{d}[e^1] \to \underline{d}[A[t].\underline{d}[x]]$
$\underline{d}[e^2] \to \underline{d}[B[t].\underline{d}[y]]$
$\underline{d}[e^3] \to \underline{d}[C[t].\underline{d}[z]]$

$\to$
$\underline{d}[e^0] \to 0$
$\underline{d}[e^1] \to \underline{\partial}_t[A[t]].(\underline{d}[t] \wedge \underline{d}[x])$
$\underline{d}[e^2] \to \underline{\partial}_t[B[t]].(\underline{d}[t] \wedge \underline{d}[y])$
$\underline{d}[e^3] \to \underline{\partial}_t[C[t]].(\underline{d}[t] \wedge \underline{d}[z])$

• In terms of e's:
$\to$

$$\{\underline{d}[e^0] \to 0,\ \underline{d}[e^1] \to \underline{\partial}_t[A[t]].(e^0 \wedge \frac{e^1}{A[t]}),\ \underline{d}[e^2] \to \underline{\partial}_t[B[t]].(e^0 \wedge \frac{e^2}{B[t]}),\ \underline{d}[e^3] \to \underline{\partial}_t[C[t]].(e^0 \wedge \frac{e^3}{C[t]})\}$$

$\to$
$$\boxed{\begin{array}{l} \underline{d}[e^0] \to 0 \\ \underline{d}[e^1] \to \frac{e^0 \wedge e^1\, \underline{\partial}_t[A[t]]}{A[t]} \\ \underline{d}[e^2] \to \frac{e^0 \wedge e^2\, \underline{\partial}_t[B[t]]}{B[t]} \\ \underline{d}[e^3] \to \frac{e^0 \wedge e^3\, \underline{\partial}_t[C[t]]}{C[t]} \end{array}}$$
$\longleftarrow$

Definition of ω from the first Cartan form: $\underline{d}[e^\alpha] \to -\omega^\alpha{}_\beta . e^\beta$
$\to$

$\to$
$$\boxed{\begin{array}{l} \underline{d}[e^0] \to -\omega^0{}_1.e^1 - \omega^0{}_2.e^2 - \omega^0{}_3.e^3 \\ \underline{d}[e^1] \to -\omega^1{}_0.e^0 - \omega^1{}_2.e^2 - \omega^1{}_3.e^3 \\ \underline{d}[e^2] \to -\omega^2{}_0.e^0 - \omega^2{}_1.e^1 - \omega^2{}_3.e^3 \\ \underline{d}[e^3] \to -\omega^3{}_0.e^0 - \omega^3{}_1.e^1 - \omega^3{}_2.e^2 \end{array}}$$

From:

$\{\underline{d}[e^0]\to 0,\ \underline{d}[e^0]\to -\omega^0{}_1.e^1-\omega^0{}_2.e^2-\omega^0{}_3.e^3\}\ \Rightarrow\ \boxed{\omega^i{}_0\to e^i\,\omega^0{}_{i\,i}}$

$\to\ \{\underline{d}[e^0]\to -\omega^0{}_1.e^1-\omega^0{}_2.e^2-\omega^0{}_3.e^3,\ \underline{d}[e^1]\to -\omega^1{}_2.e^2-\omega^1{}_3.e^3-(e^1\,\omega^0{}_{1\,1}).e^0,$
$\underline{d}[e^2]\to -\omega^2{}_1.e^1-\omega^2{}_3.e^3-(e^2\,\omega^0{}_{2\,2}).e^0,\ \underline{d}[e^3]\to -\omega^3{}_1.e^1-\omega^3{}_2.e^2-(e^3\,\omega^0{}_{3\,3}).e^0\}$

$$\to\ \begin{array}{l}\underline{d}[e^0]\to -\omega^0{}_1.e^1-\omega^0{}_2.e^2-\omega^0{}_3.e^3\\ \underline{d}[e^1]\to -\omega^1{}_2.e^2-\omega^1{}_3.e^3-\omega^0{}_{1\,1}\,e^0\wedge e^1\\ \underline{d}[e^2]\to -\omega^2{}_1.e^1-\omega^2{}_3.e^3-\omega^0{}_{2\,2}\,e^0\wedge e^2\\ \underline{d}[e^3]\to -\omega^3{}_1.e^1-\omega^3{}_2.e^2-\omega^0{}_{3\,3}\,e^0\wedge e^3\end{array}$$

Comparing: $\begin{array}{l}\underline{d}[e^1]\to -\omega^1{}_2.e^2-\omega^1{}_3.e^3-\omega^0{}_{1\,1}\,e^0\wedge e^1\\ \underline{d}[e^3]\to -\omega^3{}_1.e^1-\omega^3{}_2.e^2-\omega^0{}_{3\,3}\,e^0\wedge e^3\end{array}$ and

$$\left\{\underline{d}[e^0]\to 0,\ \underline{d}[e^1]\to \frac{e^0\wedge e^1\,\underline{\partial}_t[A[t]]}{A[t]},\ \underline{d}[e^2]\to \frac{e^0\wedge e^2\,\underline{\partial}_t[B[t]]}{B[t]},\ \underline{d}[e^3]\to \frac{e^0\wedge e^3\,\underline{\partial}_t[C[t]]}{C[t]}\right\}$$

$\Rightarrow\ \{\omega^1{}_3\to e^3\,\omega^1{}_{3\,3},\ \omega^3{}_1\to e^1\,\omega^3{}_{1\,1},\ \omega^1{}_3\to -\omega^3{}_1\}$

which is impossible unless $\omega^1{}_3\to 0$

$\Rightarrow$ All: $\omega^1{}_1\ |\ \omega^1{}_2\ |\ \omega^1{}_3\ |\ \omega^2{}_1\ |\ \omega^2{}_2\ |\ \omega^2{}_3\ |\ \omega^3{}_1\ |\ \omega^3{}_2\ |\ \omega^3{}_3\to 0$

$\to\ \{\underline{d}[e^0]\to -\omega^0{}_1.e^1-\omega^0{}_2.e^2-\omega^0{}_3.e^3,\ \underline{d}[e^1]\to -\omega^0{}_{1\,1}\,e^0\wedge e^1,\ \underline{d}[e^2]\to -\omega^0{}_{2\,2}\,e^0\wedge e^2,\ \underline{d}[e^3]\to -\omega^0{}_{3\,3}\,e^0\wedge e^3\}$

$$\to\ \left\{\underline{d}[e^1]\to -\omega^0{}_{1\,1}\,e^0\wedge e^1,\ \underline{d}[e^1]\to \frac{e^0\wedge e^1\,\underline{\partial}_t[A[t]]}{A[t]}\right\}$$

$$\to\ 0\to -\frac{A[t]\,\omega^0{}_{1\,1}+\underline{\partial}_t[A[t]]}{A[t]}$$

$$\to\ \omega^0{}_{1\,1}\to -\frac{\underline{\partial}_t[A[t]]}{A[t]}$$

$\Rightarrow\ \boxed{\omega^0{}_1\to -\frac{e^1\,\underline{\partial}_t[A[t]]}{A[t]}}$ Similarly for B,C.

$$\boxed{\left\{\omega^0{}_1\to -\frac{e^1\,\underline{\partial}_t[A[t]]}{A[t]},\ \omega^0{}_2\to -\frac{e^2\,\underline{\partial}_t[B[t]]}{B[t]},\ \omega^0{}_3\to -\frac{e^3\,\underline{\partial}_t[C[t]]}{C[t]}\right\}}$$

```
CR["The ordering of DifForm is not correct."]
PR["•Cartan curvature form: ",
 $ = R → DifForm[ω] + ω.ω,
 NL, "Add arguements α,β: ",
 $ = $ /. rr : (R | ω) → T[rr, "ud"][α, β] /. Dot[a_, b_] :→ Dot[(a /. β → β1), (b /. α → β1)];
 $R = $ /. dd : Dot[_, _] :→ Sum[T[xη, "uu"][β1, β1] dd, {β1, $i}] /. xη → η,
 NL, "From above the only non-zero ω's are: ",
 $w1 = Map[#[[1]] | (#[[1]] /. T[a_, "ud"][b_, c_] → T[a, "ud"][c, b]) → #[[2]] &, $w],
 Yield, $w1vb = $w1 /. $vbt, $w0 = T[ω, "ud"][_, _] → 0;
 NL, "Then the different components of the curvature tensor: ",
 NL,
 $t = Table[{α → i, β → j}, {i, 0, 3}, {j, 0, 3}],
 Yield, $Rs =
  Map[tuStdDifForm[{}, Flatten[{A[_], B[_], C[_], $xw}], Table[T[e, "u"][i], {i, 0, 3}],
       {A[], B[], C[]}][($R /. # /. $w1vb /. $w0)] &, $t] // Flatten,
 Yield, $Res = $Rs /. xRuleX[($vb /. Dot → Times), Map[DifForm[#] &, $xw]] //
   tuStdDifForm[{}, Flatten[{A[_], B[_], C[_], $xw}],
    Table[T[e, "u"][i], {i, 0, 3}], {A[], B[], C[]}];
 $Res = $Res /. Dot → Times,
 NL, "The components of R: ",
 xR[exp_Rule] := Block[{$ = exp, $i}, $i = tuParseTermIndices[
      tuExtractPattern[Wedge[T[e, "u"][i_], T[e, "u"][j_]]][$]][[2, 1]];
   If[$i == {}, Return[{}]];
   $[[1]] = $[[1]] /. T[R, "ud"][a_, b_] :> T[R, "uddd"][a, b, Apply[Sequence, $i]];
   $ = $ /. Wedge[T[e, "u"][i_], T[e, "u"][j_]] → 1
  ];
 $Res = xR /@ $Res // DeleteCases[#, {}] &
]
```

Sequence[The ordering of DifForm is not correct.]

•Cartan curvature form: $R \to \underline{d}[\omega] + \omega.\omega$

Add arguements α,β: $R^{\alpha}{}_{\beta} \to \underline{d}[\omega^{\alpha}{}_{\beta}] - \omega^{\alpha}{}_{0}.\omega^{0}{}_{\beta} + \omega^{\alpha}{}_{1}.\omega^{1}{}_{\beta} + \omega^{\alpha}{}_{2}.\omega^{2}{}_{\beta} + \omega^{\alpha}{}_{3}.\omega^{3}{}_{\beta}$

From above the only non-zero ω's are:

$$\left\{\omega^{0}{}_{1} \mid \omega^{1}{}_{0} \to -\frac{e^{1}\,\underline{\partial}_{t}[A[t]]}{A[t]},\ \omega^{0}{}_{2} \mid \omega^{2}{}_{0} \to -\frac{e^{2}\,\underline{\partial}_{t}[B[t]]}{B[t]},\ \omega^{0}{}_{3} \mid \omega^{3}{}_{0} \to -\frac{e^{3}\,\underline{\partial}_{t}[C[t]]}{C[t]}\right\}$$

$\to$ $\{\omega^{0}{}_{1} \mid \omega^{1}{}_{0} \to -\underline{d}[x]\,\underline{\partial}_{t}[A[t]],\ \omega^{0}{}_{2} \mid \omega^{2}{}_{0} \to -\underline{d}[y]\,\underline{\partial}_{t}[B[t]],\ \omega^{0}{}_{3} \mid \omega^{3}{}_{0} \to -\underline{d}[z]\,\underline{\partial}_{t}[C[t]]\}$

Then the different components of the curvature tensor:

$\{\{\{\alpha \to 0, \beta \to 0\}, \{\alpha \to 0, \beta \to 1\}, \{\alpha \to 0, \beta \to 2\}, \{\alpha \to 0, \beta \to 3\}\},$
$\{\{\alpha \to 1, \beta \to 0\}, \{\alpha \to 1, \beta \to 1\}, \{\alpha \to 1, \beta \to 2\}, \{\alpha \to 1, \beta \to 3\}\},$
$\{\{\alpha \to 2, \beta \to 0\}, \{\alpha \to 2, \beta \to 1\}, \{\alpha \to 2, \beta \to 2\}, \{\alpha \to 2, \beta \to 3\}\},$
$\{\{\alpha \to 3, \beta \to 0\}, \{\alpha \to 3, \beta \to 1\}, \{\alpha \to 3, \beta \to 2\}, \{\alpha \to 3, \beta \to 3\}\}\}$

$\to$ $\{R^{0}{}_{0} \to 0,\ R^{0}{}_{1} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[A[t]]].(\underline{d}[t] \wedge \underline{d}[x]),\ R^{0}{}_{2} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[B[t]]].(\underline{d}[t] \wedge \underline{d}[y]),$
$R^{0}{}_{3} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[C[t]]].(\underline{d}[t] \wedge \underline{d}[z]),\ R^{1}{}_{0} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[A[t]]].(\underline{d}[t] \wedge \underline{d}[x]),\ R^{1}{}_{1} \to 0,$
$R^{1}{}_{2} \to -\underline{\partial}_{t}[A[t]].\underline{\partial}_{t}[B[t]].(\underline{d}[x] \wedge \underline{d}[y]),\ R^{1}{}_{3} \to -\underline{\partial}_{t}[A[t]].\underline{\partial}_{t}[C[t]].(\underline{d}[x] \wedge \underline{d}[z]),$
$R^{2}{}_{0} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[B[t]]].(\underline{d}[t] \wedge \underline{d}[y]),\ R^{2}{}_{1} \to \underline{\partial}_{t}[A[t]].\underline{\partial}_{t}[B[t]].(\underline{d}[x] \wedge \underline{d}[y]),\ R^{2}{}_{2} \to 0,$
$R^{2}{}_{3} \to -\underline{\partial}_{t}[B[t]].\underline{\partial}_{t}[C[t]].(\underline{d}[y] \wedge \underline{d}[z]),\ R^{3}{}_{0} \to -\underline{\partial}_{t}[\underline{\partial}_{t}[C[t]]].(\underline{d}[t] \wedge \underline{d}[z]),$
$R^{3}{}_{1} \to \underline{\partial}_{t}[A[t]].\underline{\partial}_{t}[C[t]].(\underline{d}[x] \wedge \underline{d}[z]),\ R^{3}{}_{2} \to \underline{\partial}_{t}[B[t]].\underline{\partial}_{t}[C[t]].(\underline{d}[y] \wedge \underline{d}[z]),\ R^{3}{}_{3} \to 0\}$

$$\to \left\{R^{0}{}_{0} \to 0,\ R^{0}{}_{1} \to -\frac{e^{0} \wedge e^{1}\,\underline{\partial}_{t}[\underline{\partial}_{t}[A[t]]]}{A[t]},\ R^{0}{}_{2} \to -\frac{e^{0} \wedge e^{2}\,\underline{\partial}_{t}[\underline{\partial}_{t}[B[t]]]}{B[t]},\ R^{0}{}_{3} \to -\frac{e^{0} \wedge e^{3}\,\underline{\partial}_{t}[\underline{\partial}_{t}[C[t]]]}{C[t]},\right.$$
$$R^{1}{}_{0} \to -\frac{e^{0} \wedge e^{1}\,\underline{\partial}_{t}[\underline{\partial}_{t}[A[t]]]}{A[t]},\ R^{1}{}_{1} \to 0,\ R^{1}{}_{2} \to -\frac{e^{1} \wedge e^{2}\,\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[B[t]]}{A[t]\,B[t]},\ R^{1}{}_{3} \to -\frac{e^{1} \wedge e^{3}\,\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[C[t]]}{A[t]\,C[t]},$$
$$R^{2}{}_{0} \to -\frac{e^{0} \wedge e^{2}\,\underline{\partial}_{t}[\underline{\partial}_{t}[B[t]]]}{B[t]},\ R^{2}{}_{1} \to \frac{e^{1} \wedge e^{2}\,\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[B[t]]}{A[t]\,B[t]},\ R^{2}{}_{2} \to 0,\ R^{2}{}_{3} \to -\frac{e^{2} \wedge e^{3}\,\underline{\partial}_{t}[B[t]]\,\underline{\partial}_{t}[C[t]]}{B[t]\,C[t]},$$
$$\left.R^{3}{}_{0} \to -\frac{e^{0} \wedge e^{3}\,\underline{\partial}_{t}[\underline{\partial}_{t}[C[t]]]}{C[t]},\ R^{3}{}_{1} \to \frac{e^{1} \wedge e^{3}\,\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[C[t]]}{A[t]\,C[t]},\ R^{3}{}_{2} \to \frac{e^{2} \wedge e^{3}\,\underline{\partial}_{t}[B[t]]\,\underline{\partial}_{t}[C[t]]}{B[t]\,C[t]},\ R^{3}{}_{3} \to 0\right\}$$

The components of R: $\left\{R^{0}{}_{101} \to -\frac{\underline{\partial}_{t}[\underline{\partial}_{t}[A[t]]]}{A[t]},\ R^{0}{}_{202} \to -\frac{\underline{\partial}_{t}[\underline{\partial}_{t}[B[t]]]}{B[t]},\ R^{0}{}_{303} \to -\frac{\underline{\partial}_{t}[\underline{\partial}_{t}[C[t]]]}{C[t]},\right.$

$$R^{1}{}_{001} \to -\frac{\underline{\partial}_{t}[\underline{\partial}_{t}[A[t]]]}{A[t]},\ R^{1}{}_{212} \to -\frac{\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[B[t]]}{A[t]\,B[t]},\ R^{1}{}_{313} \to -\frac{\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[C[t]]}{A[t]\,C[t]},$$
$$R^{2}{}_{002} \to -\frac{\underline{\partial}_{t}[\underline{\partial}_{t}[B[t]]]}{B[t]},\ R^{2}{}_{112} \to \frac{\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[B[t]]}{A[t]\,B[t]},\ R^{2}{}_{323} \to -\frac{\underline{\partial}_{t}[B[t]]\,\underline{\partial}_{t}[C[t]]}{B[t]\,C[t]},$$
$$\left.R^{3}{}_{003} \to -\frac{\underline{\partial}_{t}[\underline{\partial}_{t}[C[t]]]}{C[t]},\ R^{3}{}_{113} \to \frac{\underline{\partial}_{t}[A[t]]\,\underline{\partial}_{t}[C[t]]}{A[t]\,C[t]},\ R^{3}{}_{223} \to \frac{\underline{\partial}_{t}[B[t]]\,\underline{\partial}_{t}[C[t]]}{B[t]\,C[t]}\right\}$$

```
PR["Using the fact that: ",
 $Rx = {T[R, "uddd"][u_, a_, b_, c_] :→ -T[R, "uddd"][u, a, c, b]};
 $Rx0 = T[R, "uddd"][u_, a_, b_, b_] → 0, $R0 = T[R, "uddd"][u_, a_, c_, b_] → 0;
 NL, "The Sums: ",
 $sum = xSum[T[R, "uddd"][b, a, b, c], {b, 0, 3}] -> T[R, "dd"][a, c],
 NL, "The non-zero cases For: ",
 $s = Table[{a → i, c → j}, {i, 0, 3}, {j, 0, 3}],
 Yield,
 $ = Map[$sum /. # /. xSum → Sum /. $Res /. $Rx /. $Res /. $Rx0 /. $R0 &, $s] // Flatten //
   DeleteCases[#, 0 → _] &;
 Framed[$]
]
```

Using the fact that: $R^{u}{}_{a\_ b\_ b\_} \to 0$

The Sums: $\sum_{\{b,0,3\}} [R^{b}{}_{a b c}] \to R_{a c}$

The non-zero cases For: {{{a → 0, c → 0}, {a → 0, c → 1}, {a → 0, c → 2}, {a → 0, c → 3}}, {{a → 1, c → 0}, {a → 1, c → 1}, {a → 1, c → 2}, {a → 1, c → 3}}, {{a → 2, c → 0}, {a → 2, c → 1}, {a → 2, c → 2}, {a → 2, c → 3}}, {{a → 3, c → 0}, {a → 3, c → 1}, {a → 3, c → 2}, {a → 3, c → 3}}}

$$\to \boxed{\begin{array}{l} \{\frac{\partial_t[\partial_t[A[t]]]}{A[t]} + \frac{\partial_t[\partial_t[B[t]]]}{B[t]} + \frac{\partial_t[\partial_t[C[t]]]}{C[t]} \to R_{0\,0}, \\ -\frac{\partial_t[A[t]]\,\partial_t[B[t]]}{A[t]\,B[t]} - \frac{\partial_t[A[t]]\,\partial_t[C[t]]}{A[t]\,C[t]} - \frac{\partial_t[\partial_t[A[t]]]}{A[t]} \to R_{1\,1}, \\ -\frac{\partial_t[A[t]]\,\partial_t[B[t]]}{A[t]\,B[t]} - \frac{\partial_t[B[t]]\,\partial_t[C[t]]}{B[t]\,C[t]} - \frac{\partial_t[\partial_t[B[t]]]}{B[t]} \to R_{2\,2}, \\ -\frac{\partial_t[A[t]]\,\partial_t[C[t]]}{A[t]\,C[t]} - \frac{\partial_t[B[t]]\,\partial_t[C[t]]}{B[t]\,C[t]} - \frac{\partial_t[\partial_t[C[t]]]}{C[t]} \to R_{3\,3}\} \end{array}}$$