Fluctuations [Medium/Hard]

In this problem we derive the action for the curvature fluctuations used in the problem by Juan. We have to enter a deformed metric and use the diffgeo package to evaluate the action. After that we have to simplify it a bit to get the desired action. Using a parametrization from astro-ph/0210603 we write the metric as

```
ds2 = -Nt[t, x, y, z]^2 dt^2 +
   Sum[If[i == j, h[t, x, y, z], 0] (dx[i] + N[i][t, x, y, z] dt)
     (dx[j] + N[j][t, x, y, z] dt), {i, 3}, {j, 3}];
```

```
coord = {t, x, y, z};
dc[0] = dt;
dc[i_] = dx[i];
metric = Table[If[μ == ν, 1, 1 / 2] Coefficient[ds2, dc[μ] dc[ν]], {μ, 0, 3}, {ν, 0, 3}] //
   Simplify;
```

```
<< (NotebookDirectory[] <> "diffgeo.m");
```

- The first problem is to make the package work faster. As is it is too slow. The reason for that become obvious if you type ??RicciTensor Improve the functions RicciTensor, partial and tr by removing some parts of it. You can copy some parts of the code right from the output of the ??RicciTensor command
- After that you should be able to evaluate the action

$$S = \frac{1}{2}\int \sqrt{g}(R - (\nabla\phi)^2 - 2U(\phi))$$

and expand it to the second order in ϵ for

$$h = e^{2\rho(t)}(1 + 2\epsilon\,\zeta(t,\vec{x}))\quad,\quad N_t = 1 + \epsilon n_t(t,\vec{x})\quad,\quad N_i = \epsilon n_i(t,\vec{x})$$

use pure functions when making the replacement

```
S =
  1/2 SeriesCoefficient[
     Sqrt[-Det[metric]]
      (Expand[RicciScalar] - (covariant[ϕ[t]].inverse.covariant[ϕ[t]] + 2 U[ϕ[t]])) /.
    replacement, {ϵ, 0, 2}] // Simplify // PowerExpand;
```

- Make a function Variation[Action_,field_,variables__] which can compute first variation of an action with any number of derivatives. For example

```
Variation[1/2 D[x[t], t]^2 + 1/2 D[x[t], t, t, t]^2 + 1/2 x[t]^2, x, t]
```

$-x^{(6)}(t) - x''(t) + x(t)$

• Next find equations of motion comming from the variation in `n` and `nt` using `Variation`. Use the background equations

$$U(\phi) = 3\dot\rho^2 - \frac{1}{2}\phi^2 \quad , \quad \ddot\rho = -\frac{1}{2}\dot\phi^2$$

to simplify the result.

You should obtain the following equations for the auxiliary fields `n` and `nt`

```
2 Exp[-3 ρ[t]] Table[Variation[S, ns, t, x, y, z], {ns, {nt, n[1], n[2], n[3]}}] /.
    background // Expand // List // Transpose // Simplify
```

$$\begin{pmatrix} -2\, e^{-2\rho}\left(e^{2\rho}\, \mathfrak{n}_t\left(6\dot\rho^2-\dot\phi^2\right)+2\left(\partial_z^2\zeta+\partial_y^2\zeta+\partial_x^2\zeta+e^{2\rho}\dot\rho\left(\partial_z\mathfrak{n}_3+\partial_y\mathfrak{n}_2+\partial_x\mathfrak{n}_1-3\,\partial_t\zeta\right)\right)\right) \\ -\partial_z^2\mathfrak{n}_1-\partial_y^2\mathfrak{n}_1+4\dot\rho\,\partial_x\mathfrak{n}_t+\partial_x\partial_z\mathfrak{n}_3+\partial_x\partial_y\mathfrak{n}_2-4\,\partial_t\partial_x\zeta \\ -\partial_z^2\mathfrak{n}_2+4\dot\rho\,\partial_y\mathfrak{n}_t+\partial_y\partial_z\mathfrak{n}_3+\partial_x\partial_y\mathfrak{n}_1-\partial_x^2\mathfrak{n}_2-4\,\partial_t\partial_y\zeta \\ 4\dot\rho\,\partial_z\mathfrak{n}_t+\partial_y\partial_z\mathfrak{n}_2-\partial_y^2\mathfrak{n}_3+\partial_x\partial_z\mathfrak{n}_1-\partial_x^2\mathfrak{n}_3-4\,\partial_t\partial_z\zeta \end{pmatrix}$$

• Plug the following ansatz, which solves a part of the equations and find an equation for χ

$$n_i = \partial_i\psi \quad , \quad n_t = \frac{\dot\zeta}{\dot\rho} \quad , \quad \psi = -e^{-2\rho}\frac{\zeta}{\dot\rho} + \chi$$

(again use pure functions)

• Plug the same ansatz into the action and check that the variation of the action in χ is zero (on the background equations of motion)! This means that all terms containing χ assemble into a total derivative

```
dχAction = Variation[Sintegrated, χ, t, x, y, z] /. background // Simplify
```

0

• In the same way compute the first variation w.r.t ζ

$$-\frac{e^{\rho}\,\dot\phi\left(3\,e^{2\rho}\,\dot\rho^2\,\dot\phi\;\partial_t\zeta+e^{2\rho}\,\dot\phi^3\;\partial_t\zeta-\dot\rho\left(\dot\phi\left(-e^{2\rho}\;\partial_t^2\zeta+\partial_x^2\zeta+\partial_y^2\zeta+\partial_z^2\zeta\right)-2\,e^{2\rho}\,\ddot\phi\;\partial_t\zeta\right)\right)}{\dot\rho^3}$$

Check that it gives that same as the action

$$\frac{1}{2}\int\frac{\dot\phi^2}{\dot\rho^2}\left[e^{3\rho}\,\dot\zeta^2 - e^{\rho}\,\partial_i\zeta^2\right]$$

• Finally change the variables as

$$e^{\rho}\to a \quad , \quad dt \to a d\eta \quad , \quad \partial_t \to \frac{1}{a}\partial_\eta$$

To get the action from the problem of Juan

$$-\frac{\dot\phi^2\, a(t)^4\, d\eta\, dx\, dy\, dz\left(-\partial_\eta\zeta^2+\partial_x\zeta^2+\partial_y\zeta^2+\partial_z\zeta^2\right)}{2\, a'(t)^2}$$