```
<< Local`QFTToolKit2`;
Get[$HomeDirectory <> "/Mathematica/NonCommutative/1204.0328
      ParticlePhysicsFromAlmostCommutativeSpacetime.2.redo.out"];

$defGWS = {};

"Notational definitions"
"Note that in the text the symbols may reference
  different Hilbert spaces. This has caused confusion in some of the
  calculations.  To address this problem we will try to label the
  variables by subscripts to designate the applicable Hilbert space.
  NOTE: Need to do notational change for .1,.2 notebooks."

rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := ∣a∣;
it[a_] := Style[a, Italic]
iD := it[D]
iI := it["I"]
C∞ := C^"∞"
B_x_ := T[B, "d", {x}]
("∇"^S)_n_ := T["∇"^S, "d", {n}]

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
    ""];
accumGWS[item_] := Block[{}, $defGWS = tuAppendUniq[item][$defGWS];
    ""];
selectGWS[heads_, with_: {}] := tuRuleSelect[$defGWS][Flatten[{heads}]] //
     Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // Last;

Clear[expandDC];
expandDC[sub_: {}] := tuRepeat[{sub, tuOpDistribute[Dot],
    tuOpSimplify[Dot], tuOpDistribute[CircleTimes], tuOpSimplify[CircleTimes]}]
Clear[expandCom]
expandCom[subs_: {}][exp_] := Block[{tmp = exp},
    tmp = tmp //. tuCommutatorExpand // expandDC[];
    tmp = tmp /. toxDot /. Flatten[{subs}];
    tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
    tmp
  ];
(**)
$sgeneral := {
  T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
  T[γ, "d", {5}].T[γ, "d", {5}] → 1,
  ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
  CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  T["∇", "d", {_}][1_n_] → 0, a_ . 1_n_ → a, 1_n_ . a_ → a}
$sgeneral // ColumnBar

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] ⧴ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
```

```
    tt : T[F, "dd", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] :→ CommutatorP[b, a] /; OrderedQ[{b, a}],
    tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
  };
$symmetries // ColumnBar

εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1,}}},
   {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
 ]
εRule[6]
```

Notational definitions

Note that in the text the symbols may reference different Hilbert spaces. This has caused confusion in some of the calculations. To address this problem we will try to label the variables by subscripts to designate the applicable Hilbert space. NOTE: Need to do notational change for .1,.2 notebooks.

$\gamma_5 \to \gamma^1 \gamma^2 \gamma^3 \gamma^4$
$\gamma_5 . \gamma_5 \to 1$
$(\gamma_5)^\dagger \to \gamma_5$
$\{\gamma_5, \gamma^\mu\}_+ \to 0$
$\nabla\_[1_{n\_}] \to 0$
$(a\_).1_{n\_} \to a$
$1_{n\_}.(a\_) \to a$

$tt : g^{\mu\_\nu\_} :\to tuIndexSwap[\{\mu, \nu\}][tt] /; OrderedQ[\{\nu, \mu\}]$
$tt : F^{\mu\_\nu\_} :\to -tuIndexSwap[\{\mu, \nu\}][tt] /; OrderedQ[\{\nu, \mu\}]$
$tt : F_{\mu\_\nu\_} :\to -tuIndexSwap[\{\mu, \nu\}][tt] /; OrderedQ[\{\nu, \mu\}]$
$[a\_, b\_]_- :\to -[b, a]_- /; OrderedQ[\{b, a\}]$
$\{a\_, b\_\}_+ :\to \{b, a\}_+ /; OrderedQ[\{b, a\}]$
$tt : \gamma^\mu . \gamma_5 :\to Reverse[tt]$

$\{\varepsilon \to 1, \varepsilon' \to 1, \varepsilon'' \to -1\}$

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5. Glashow-Weinberg-Salam Model

■ 5.1 Constructing the finite space F_{GWS}

```
PR[
  "Basis of finite space includes {e,ν}: ",
  $b = {($ = {eR, eL, ēR, ēL}), ($ /. e → ν)} // Flatten,
  NL, "Lepton basis ", $lep = ℋ1[CG[ℂ^4]] ->
    (Select[$b, Head[#] =!= OverBar &] // Sort[#] & // Permute[#, Cycles[{{1, 4}}]] &),
  NL, "AntiLepton basis ", $antilep = ℋ_1̄[CG[ℂ^4]] ->
    (Select[$b, Head[#] == OverBar &] // Sort[#] & // Permute[#, Cycles[{{1, 4}}]] &),
  NL, "Compose ", $h2 = ℋF2 -> ℋ1[CG[ℂ^4]] ⊕ ℋ_1̄[CG[ℂ^4]],
  NL, "with ",
  $basis = ℋF8 → $h2[[2]] /. {$lep, $antilep} /. CirclePlus[a__] :> Flatten[List[a]],

  NL, "● Algebra 𝒜F: Expand E-M algebra,
    ℂ[a1]⊕ℂ[a2], to accomodate weak interactions → ℂ⊕ℍ",
  $alg = $ = {𝒜F → ℂ ⊕ ℍ, ℍ[CG["quarterions"]], q ∈ ℍ, q → α + β j, {α, β} ∈ ℂ,
     q → {{α, β}, {-Conjugate[β], Conjugate[α]}},
     qλ → {{λ, 0}, {0, Conjugate[λ]}}, qλ[CG["embedding of ℂ in ℍ"]]};
  $ // MatrixForms // ColumnBar,
  NL, "1-Algebra definition: ", $alg1 = $ = {a1 ∈ 𝒜F1, a1 → {λ, q},
     a1 → ($ = {{qλ, 0}, {0, q}}),
     a1 → ($ /. tuRule[$alg][[-2 ;; -1]] // ArrayFlatten)
    }; $ // MatrixForms // ColumnBar,
  NL, CR["Not clear how one chooses the algebra and
     the connection between weak interactions and quaterions."],

  NL, "● For ", $h = ℋF4 → ℋ1R ⊕ ℋ1L ⊕ ℋ1̄R ⊕ ℋ1̄L,
  Yield, $alg2 = {($ = a1) → ($ /. tuRuleSelect[$alg1][a1][[-1]]),
    1 ∈ ℋ1, CG["By definition"], a1̄ → DiagonalMatrix[Table[λ, 4]], 1̄ ∈ ℋ1̄,
    a8 → ({{(a1 /. tuRuleSelect[$alg1][a1][[-1]]), 0},
         {0, DiagonalMatrix[Table[λ, 4]]}} // ArrayFlatten)
   }; MatrixForms[$alg2] // ColumnBar,

  NL, "● Choose ℤ2-grading and γF for KO-dimension 6.",
  NL, "So that: ", $sr = {JF.1 -> 1̄, JF.1̄ → 1,
    γF4 → DiagonalMatrix[{-1, 1, 1, -1}],
    γF8 → (DiagonalMatrix[{-1, 1, 1, -1}] /. 1 → {{1, 0}, {0, 1}} /.
         -1 → {{-1, 0}, {0, -1}} // ArrayFlatten),
    $s = JF4 → SparseArray[{Band[{1, 3}] → cc, Band[{3, 1}] → cc}, {4, 4}] // Normal,
    JF8 → ($s[[2]] /. cc → {{cc, 0}, {0, cc}} // ArrayFlatten)
   };
  MatrixForms[$sr] // Column // Framed,
  NL, CG[cc → "ComplexConjugate", ", F4 refers to ", $h, ", F8 refers to ", $basis],
  accumGWS[{$h2, $h, $basis, $alg, $alg1, $alg2, $sr}]; ""
 ];
```

Basis of finite space includes {e,ν}: $\{e_R, e_L, \bar{e}_R, \bar{e}_L, \nu_R, \nu_L, \bar{\nu}_R, \bar{\nu}_L\}$

Lepton basis $\mathcal{H}_1[\mathbb{C}^4] \to \{\nu_R, e_R, \nu_L, e_L\}$

AntiLepton basis $\mathcal{H}_{\bar{1}}[\mathbb{C}^4] \to \{\bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$

Compose $\mathcal{H}_{F_2} \to \mathcal{H}_1[\mathbb{C}^4] \oplus \mathcal{H}_{\bar{1}}[\mathbb{C}^4]$

with $\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\}$

- Algebra $\mathcal{A}_F$: Expand E-M algebra, $\mathbb{C}[a_1] \oplus \mathbb{C}[a_2]$, to accomodate weak interactions → $\mathbb{C} \oplus \mathbb{H}$

$\mathcal{A}_F \to \mathbb{C} \oplus \mathbb{H}$
$\mathbb{H}$[quarterions]
$q \in \mathbb{H}$
$q \to \alpha + j\,\beta$
$\{\alpha, \beta\} \in \mathbb{C}$
$$q \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$
$$q_\lambda \to \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$$
q_λ[embedding of $\mathbb{C}$ in $\mathbb{H}$]

1-Algebra definition:
$a_1 \in \mathcal{A}_{F_1}$
$a_1 \to \{\lambda, q\}$
$$a_1 \to \begin{pmatrix} q_\lambda & 0 \\ 0 & q \end{pmatrix}$$
$$a_1 \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$$

Not clear how one chooses the algebra
and the connection between weak interactions and quaterions.

- For $\mathcal{H}_{F_4} \to \mathcal{H}_{1_R} \oplus \mathcal{H}_{1_L} \oplus \mathcal{H}_{\overline{1}_R} \oplus \mathcal{H}_{\overline{1}_L}$

$$a_1 \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$$
$1 \in \mathcal{H}_1$
By definition
$$a_{\overline{1}} \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$
→
$\overline{1} \in \mathcal{H}_{\overline{1}}$
$$a_8 \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^* & \alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

- Choose $\mathbb{Z}_2$-grading and γ_F for KO-dimension 6.

So that:

$$J_F.1 \to \bar{1}$$
$$J_F.\bar{1} \to 1$$
$$\gamma_{F_4} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$\gamma_{F_8} \to \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$
$$J_{F_4} \to \begin{pmatrix} 0 & 0 & cc & 0 \\ 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 \\ 0 & cc & 0 & 0 \end{pmatrix}$$
$$J_{F_8} \to \begin{pmatrix} 0 & 0 & 0 & 0 & cc & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & cc & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & cc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & cc & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & cc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cc & 0 & 0 & 0 & 0 \end{pmatrix}$$

$cc \to$ ComplexConjugate, F_4 refers to $\mathcal{H}_{F_4} \to \mathcal{H}_{1_R} \oplus \mathcal{H}_{1_L} \oplus \mathcal{H}_{\bar{1}_R} \oplus \mathcal{H}_{\bar{1}_L}$, F_8 refers to $\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$

5.1.1 Finite Dirac Operator

```
PR["● Derive Hermitian Dirac operator in: ", tuRuleSelect[$defGWS][ℋF₂],
  NL, $df = $ = {𝒟F₂ → {{S , ct[T]}, {T, S '}}, {𝒟F₂, S, S '}[CG ["Hermitian"]] };
  MatrixForms[$],
  next, " 𝒟F₂ constrained by: ", $ = CommutatorM[𝒟F₂, JF₂] → 0,
  NL, "Since ", $s = {JF₂ → {{0, cc}, {cc, 0}}, a_ .cc :→ cc . Conjugate[a]},
  Yield, $ = $ /. tuCommutatorExpand /. Dot → xDot /. tuRule[$df] /. $s //
     tuMatrixOrderedMultiply // tuOpSimplifyF[xDot];
  Yield, $ = $ /. xDot → Dot /. $s /. tuOpCollect[]; $ // MatrixForms,

  Imply, $ = $ /. cc.a_ → a; $c1 = $ = Thread[Flatten[$[[1]]] → 0]; $,
  Imply, $df[[1]] = $df[[1]] /. tuRuleSolve[$c1, {ct[T], S '}];
  $df // MatrixForms // Framed, accumGWS[$df];

  next, " In ", tuRuleSelect[$defGWS][ℋF₄], " space, Let ",
  $s = {S -> Table[si,j, {i, 2}, {j, 2}], T -> Table[ti,j, {i, 2}, {j, 2}]};
  $s // MatrixForms, "POFF",
  Yield, $0 = $ = $df[[1, 2]] /. $s // ArrayFlatten; MatrixForm[$],
  Yield, $ht = ct[$]; MatrixForm[$ht],
  Yield, $ = $ → $ht //. rr : Rule[__] :→ Thread[rr]; MatrixForms[$], "PON",
  Yield, $s1 = tuRuleSolve[Flatten[$], {s2,1, t2,1}],
  Yield, $df44 = $ = 𝒟F₄ -> $0 /. $s1 /. Conjugate[si_,i_] → si,i;
  MatrixForms[$] // Framed,

  next, " The requirement: ", $ = CommutatorP[𝒟F, γF] → 0, "xPOFF",
  Yield, $ = $ /. F → F4,

  Yield, $ = $ /. $sr /. $df44; MatrixForms[$],
  Yield, $ = $ /. tuCommutatorExpand; MatrixForms[$], "PON",
  yield, $ = $ //. rr : Rule[__] :→ Thread[rr] // Flatten // DeleteDuplicates //
    tuRuleSolve[#, {s1,1, s2,2, t1,2}] &,
  Yield, $ = $df44 /. $; MatrixForms[$] // Framed, accumGWS[$], "PON",

  NL, "Using notation ", $s = {s1,2 → Conjugate[Y0], t1,1 → TR, t2,2 → TL},
  Imply, $df44 = $ = $ /. $s;
  MatrixForms[$] // Framed, accumGWS[{$s, $df44}],

  NL, "In the space ", $basis, yield,
  {Y0, TR, TL}, " are symmetric 2x2 matrices.",
  NL, "So in ", $ = {$df[[1]], S → $[[2, 1 ;; 2, 1 ;; 2]], T → $[[2, 3 ;; 4, 1 ;; 2]]};
  $ // MatrixForms, accumGWS[$]
 ];
```

● Derive Hermitian Dirac operator in: $\{\mathcal{H}_{F_2} \to \mathcal{H}_1[\mathbb{C}^4] \oplus \mathcal{H}_{\bar{I}}[\mathbb{C}^4]\}$

$\{\mathcal{D}_{F_2} \to \begin{pmatrix} S & T^\dagger \\ T & S' \end{pmatrix}\}$, $\{\mathcal{D}_{F_2}, S, S'\}$[Hermitian]}

◆ $\mathcal{D}_{F_2}$ constrained by: $[\mathcal{D}_{F_2}, J_{F_2}]_- \to 0$

Since $\{J_{F_2} \to \{\{0, cc\}, \{cc, 0\}\}, (a\_).cc :\to cc.a^*\}$

→

→ $\begin{pmatrix} cc.(-T + T^{\dagger *}) & cc.(S^* - S') \\ cc.(-S + (S')^*) & cc.(T^* - T^\dagger) \end{pmatrix} \to 0$

⇒ $\{-T + T^{\dagger *} \to 0,\ S^* - S' \to 0,\ -S + (S')^* \to 0,\ T^* - T^\dagger \to 0\}$

⇒ $\boxed{\{\mathcal{D}_{F_2} \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix}\}, \{\mathcal{D}_{F_2}, S, S'\}[\text{Hermitian}]\}}$

◆ In $\{\mathcal{H}_{F_4} \to \mathcal{H}_{1R} \oplus \mathcal{H}_{1L} \oplus \mathcal{H}_{\bar{I}R} \oplus \mathcal{H}_{\bar{I}L}\}$ space, Let $\{S \to \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{pmatrix}, T \to \begin{pmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{pmatrix}\}$

→ $\{s_{2,1} \to (s_{1,2})^*, t_{2,1} \to t_{1,2}\}$

→ $\boxed{\mathcal{D}_{F_4} \to \begin{pmatrix} s_{1,1} & s_{1,2} & (t_{1,1})^* & (t_{1,2})^* \\ (s_{1,2})^* & s_{2,2} & (t_{1,2})^* & (t_{2,2})^* \\ t_{1,1} & t_{1,2} & s_{1,1} & (s_{1,2})^* \\ t_{1,2} & t_{2,2} & s_{1,2} & s_{2,2} \end{pmatrix}}$

◆ The requirement: $\{\mathcal{D}_F, \gamma_F\}_+ \to 0$xPOFF

→ $\{\mathcal{D}_{F_4}, \gamma_{F_4}\}_+ \to 0$

→ $\{\begin{pmatrix} s_{1,1} & s_{1,2} & (t_{1,1})^* & (t_{1,2})^* \\ (s_{1,2})^* & s_{2,2} & (t_{1,2})^* & (t_{2,2})^* \\ t_{1,1} & t_{1,2} & s_{1,1} & (s_{1,2})^* \\ t_{1,2} & t_{2,2} & s_{1,2} & s_{2,2} \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\}_+ \to 0$

→ $\begin{pmatrix} -2\,s_{1,1} & 0 & 0 & -2\,(t_{1,2})^* \\ 0 & 2\,s_{2,2} & 2\,(t_{1,2})^* & 0 \\ 0 & 2\,t_{1,2} & 2\,s_{1,1} & 0 \\ -2\,t_{1,2} & 0 & 0 & -2\,s_{2,2} \end{pmatrix} \to 0 \longrightarrow \{s_{1,1} \to 0, s_{2,2} \to 0, t_{1,2} \to 0\}$

→ $\boxed{\mathcal{D}_{F_4} \to \begin{pmatrix} 0 & s_{1,2} & (t_{1,1})^* & 0 \\ (s_{1,2})^* & 0 & 0 & (t_{2,2})^* \\ t_{1,1} & 0 & 0 & (s_{1,2})^* \\ 0 & t_{2,2} & s_{1,2} & 0 \end{pmatrix}}$

Using notation $\{s_{1,2} \to (Y_0)^*, t_{1,1} \to T_R, t_{2,2} \to T_L\}$

⇒ $\boxed{\mathcal{D}_{F_4} \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix}}$

In the space $\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$
⟶ $\{Y_0, T_R, T_L\}$ are symmetric 2x2 matrices.

So in $\{\mathcal{D}_{F_2} \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix}, S \to \begin{pmatrix} 0 & (Y_0)^* \\ Y_0 & 0 \end{pmatrix}, T \to \begin{pmatrix} T_R & 0 \\ 0 & T_L \end{pmatrix}\}$

```
$basis8 = $basis[[2]]
PR["■ How does the restriction: ",
 $req = {T.$basis8[[1]] → Y_R.$basis8[[5]], T.l ⧴ 0 /; FreeQ[l, $basis8[[1]]]},
 " constrain T? ",
 NL, "where ", $t = T → DiagonalMatrix[{T_R, T_L}],
 NL, CO["Allows order-1 condition to be satisfied."],
 Yield, $t = $t /. tt : T_R_ -> Table[t[R]_{i,j}, {i, 2}, {j, 2}];
 $t[[2]] = $t[[2]] // ArrayFlatten;
 Yield, MatrixForms[$t], accumGWS[{$req, $t}];

 NL, "•Hermiticity of 𝒟_F", imply, $st = {t[L]_{2,1} -> t[L]_{1,2}, t[R]_{2,1} -> t[R]_{1,2}},
 Yield, $t44 = $t = $t /. $st; $t // MatrixForms, accumGWS[{$st, $t}];

 NL, "In the ", tuRuleSelect[$defGWS][ℋ_{F_8}], " space: ", "POFF",
 Yield, $ = {{0, Conjugate[T]}, {T, 0}},
 Yield, $t = T → ($ /. $t // ArrayFlatten); $t // MatrixForms,
 Yield, $ = T . Transpose[{$basis8}]; $ // MatrixForms,
 Yield, $ = $ → $; $ // MatrixForms,
 Yield, $[[2]] = $[[2]] /. $t; "PON",
 MatrixForms[$],
 NL, "The requirement ", $req, imply,
 "The only non-zero element of T: ", Conjugate[t[R]_{1,1}] // Framed,
 Yield, $t = $t /. tt : t[_]_{i_,j_} ⧴ 0 /; tt =!= t[R]_{1,1}; $t // MatrixForms,
 NL, "also ", $ = y_{2,1} → y_{1,2},
 NL, "Require ℋ_F to be mass eigenstates ",
 $Y = Y_0 → DiagonalMatrix[{Y_ν, Y_e}], accumGWS[{$t, $, $Y}];
 line,
 NL, "Rules for ", tuRuleSelect[$defGWS][ℋ_{F_8}], " space.",
 Yield, $df44[[1]],
 Yield, $sDAgws = $ = {$df44, tt : T_R_ → Table[t[R]_{i,j}, {i, 2}, {j, 2}],
    t[RL_]_{i_,j_} ⧴ 0 /; (i ≠ 1 || j ≠ 1 || RL =!= R), $Y};
 $ // MatrixForms,
 accumGWS[$];
 NL, $ = 𝒟_{F_8} -> 𝒟_{F_4} /. tuRuleSelect[$defGWS][𝒟_{F_4}][[-1]] //. $sDAgws;
 Yield, $[[2]] = $[[2]] /. t[R] → t_R // ArrayFlatten;
 $ // MatrixForms, accumGWS[$]
]
```

$\{\nu_R, e_R, \nu_L, e_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$

■ How does the restriction: $\{T.\nu_R \to Y_R.\overline{\nu_R},\ T.l :\to 0\ /;\ \text{FreeQ}[l, \$basis8[[1]]]\}$ constrain T? where $T \to \{\{T_R, 0\}, \{0, T_L\}\}$

Allows order-1 condition to be satisfied.

→

$$\to T \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} & 0 & 0 \\ t[R]_{2,1} & t[R]_{2,2} & 0 & 0 \\ 0 & 0 & t[L]_{1,1} & t[L]_{1,2} \\ 0 & 0 & t[L]_{2,1} & t[L]_{2,2} \end{pmatrix}$$

•Hermiticity of $\mathcal{D}_F$ ⇒ $\{t[L]_{2,1} \to t[L]_{1,2},\ t[R]_{2,1} \to t[R]_{1,2}\}$

$$\to T \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} & 0 & 0 \\ t[R]_{1,2} & t[R]_{2,2} & 0 & 0 \\ 0 & 0 & t[L]_{1,1} & t[L]_{1,2} \\ 0 & 0 & t[L]_{1,2} & t[L]_{2,2} \end{pmatrix}$$

In the $\{\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\}\}$ space: $T.\begin{pmatrix} \nu_R \\ e_R \\ \nu_L \\ e_L \\ \overline{\nu_R} \\ \overline{e_R} \\ \overline{\nu_L} \\ \overline{e_L} \end{pmatrix} \to \begin{pmatrix} (t[R]_{1,2})^* \overline{e_R} + (t[R]_{1,1})^* \overline{\nu_R} \\ (t[R]_{2,2})^* \overline{e_R} + (t[R]_{1,2})^* \overline{\nu_R} \\ (t[L]_{1,2})^* \overline{e_L} + (t[L]_{1,1})^* \overline{\nu_L} \\ (t[L]_{2,2})^* \overline{e_L} + (t[L]_{1,2})^* \overline{\nu_L} \\ \nu_R\, t[R]_{1,1} + e_R\, t[R]_{1,2} \\ \nu_R\, t[R]_{1,2} + e_R\, t[R]_{2,2} \\ \nu_L\, t[L]_{1,1} + e_L\, t[L]_{1,2} \\ \nu_L\, t[L]_{1,2} + e_L\, t[L]_{2,2} \end{pmatrix}$

The requirement $\{T.\nu_R \to Y_R.\overline{\nu_R},\ T.l :\to 0\ /;\ \text{FreeQ}[l, \$basis8[[1]]]\}$

⇒ The only non-zero element of T: $\boxed{(t[R]_{1,1})^*}$

$$\to T \to \begin{pmatrix} 0 & 0 & 0 & 0 & (t[R]_{1,1})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t[R]_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

also $y_{2,1} \to y_{1,2}$

Require $\mathcal{H}_F$ to be mass eigenstates $Y_0 \to \{\{Y_\nu, 0\}, \{0, Y_e\}\}$

Rules for $\{\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\}\}$ space.

→ $\mathcal{D}_{F_4}$

$$\to \left\{\mathcal{D}_{F_4} \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix},\right.$$

$$\left. tt : T_{R\_} \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} \\ t[R]_{2,1} & t[R]_{2,2} \end{pmatrix},\ t[RL\_]_{i\_,j\_} :\to 0\ /;\ i \neq 1\ ||\ j \neq 1\ ||\ RL =!= R,\ Y_0 \to \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}\right\}$$

$$\to \mathcal{D}_{F_8} \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 & (t_{R1,1})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & (Y_e)^* & 0 & 0 & 0 & 0 \\ Y_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{R1,1} & 0 & 0 & 0 & 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_e \\ 0 & 0 & 0 & 0 & (Y_\nu)^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (Y_e)^* & 0 & 0 \end{pmatrix}$$

```
(* the representation of l and l̄ must
 be distinquished in the following calculation.*)

$conditions = tuRuleSelect[Select[$defall, tuHasAnyQ[#, {γF, JF}] &]][
   {CommutatorM[__], CommutatorP[__], rghtA[_], Dot[_, _]}] // DeleteDuplicates;
$conditions // Column;
```

```
PR["Prop.5.1. ", $ = F_GWS → Map[# /. a_ → a_F &, {𝒜, ℋ, 𝒟, γ, J}],
 " define a real even KOdim→6 space.",
 imply, KOdim → 6,
 Imply, $se6 = εRule[6],

 line,
 NL, "Recall general conditions: ", $conditions // ColumnBar,
 next, "Check: ", $ = tuRuleSelect[$conditions][CommutatorM[γ_F, _]][[1]],
 imply, "OK γ_F diagonal. ",
 next, "Check: ", $ = tuRuleSelect[$conditions][J_F.J_F][[1]] /. $se6,
 imply, "OK",
 next, "Check: ",
 $ = tuRuleSelect[$conditions][J_F.𝒟_F] /. $se6 /. tuOpSimplify[Dot] // First,
 " by construction.",
 next, "Check: ",
 $ = tuRuleSelect[$conditions][J_F.γ_F] /. $se6 /. tuOpSimplify[Dot] // First,
 yield, $ = $ /. tuRuleSelect[$defGWS][{γ_F, J_F}] /. Rule → Equal,
 next, "Check: ", $ = tuRuleSelect[$conditions][CommutatorP[_, _]] // First,
 yield, $ = $ /. tuRuleSelect[$defGWS][{γ_F, 𝒟_F}] /. tuCommutatorExpand /. Rule → Equal,

 next, "Check order-0 condition: ",
 $ = tuRuleSelect[$conditions][CommutatorM[a, _]] // First,
 NL, CR["Need 8×8 space for correct computation."],
 Yield, $ = $ /. tuRuleSelect[$conditions][rghtA[_]] /. {aa : a | b → aa_8, F → F_8},
 NL, "for algebra's ",
 $s = tuRuleSelect[$defGWS][a_8] // Select[#, tuHasAnyQ[#, α] &] & // First,

 $s = {$s, ($s /. aa : λ | α | β → aa_b /. a → b)}; $s // MatrixForms, "POFF",
 Yield, $ = $ /. tuCommutatorExpand /. Dot → xDot;
 Yield, $ = $ /. $s /. tuRuleSelect[$defGWS][{J_F_8}]; $ // MatrixForms, CK,
 Yield, $ = $ // tuMatrixOrderedMultiply // (# /. xDot → Dot &),
 NL, "Using: ", $s = {cc . a_ → Conjugate[a].cc, Conjugate[cc] → cc, cc.cc → 1},
 Yield, $ = $ //. $s; $ // MatrixForms, CK, "PONdd",
 Yield, $ = $ // tuRepeat[
      {$s, tuOpSimplify[Dot, {λ, Conjugate[λ], α, β, Conjugate[α], Conjugate[β]}]},
      tuConjugateSimplify[{cc}]] // Simplify;
 $ // MatrixForms,
 next, "Check order-1 condition: ",
 $ = tuRuleSelect[$conditions][CommutatorM[CommutatorM[_, _], _]] // First,
 Yield, xtmp =
  $ = $ /. (tuRuleSelect[$conditions][rghtA[_]] // tuAddPatternVariable[b]) /.
    {aa : a | b → aa_8, F → F_8},
 NL, "for algebra's ",
 $s = tuRuleSelect[$defGWS][a_8] // Select[#, tuHasAnyQ[#, α] &] & // First;
 $s = {$s, ($s /. aa : λ | α | β → aa_b /. a → b)}; $s // MatrixForms, "POFF",
 Yield, $ = $ // expandCom[{$s, tuRuleSelect[$defGWS][{𝒟_F_8, J_F_8}]}];
 $ // MatrixForms, "PONdd",
 NL, "Using: ",
 $s = {cc . Shortest[a_] → Conjugate[a].cc, Conjugate[cc] → cc, cc.cc → 1},
 Yield, $ = $ // tuRepeat[
      {$s, tuOpSimplify[Dot, {λ, Conjugate[λ], α, β, Conjugate[α], Conjugate[β]}]},
      tuConjugateSimplify[{cc}]] // Simplify;
 $ = $ /. Dot → Times;
 $ // MatrixForms, yield, "OK"
]
```

Prop.5.1. $F_{GWS} \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F, J_F\}$ define a real even KOdim→6 space. ⇒ KOdim → 6
⇒ $\{\varepsilon \to 1, \varepsilon' \to 1, \varepsilon'' \to -1\}$

Recall general conditions:
$$\begin{array}{|l} [\gamma_F, a \in A_F]_- \to 0 \\ [a, b^o]_- \to 0 \\ [[\mathcal{D}_F, a]_-, b^o]_- \to 0 \\ \{\gamma_F, \mathcal{D}_F\}_+ \to 0 \\ b^o \to J_F.b^\dagger.(J_F)^\dagger \\ \gamma_F.\gamma_F \to 1_F \\ J_F.J_F \to \varepsilon \\ J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F \\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F \end{array}$$

◆Check: $[\gamma_F, a \in A_F]_- \to 0$ ⇒ OK γ_F diagonal.
◆Check: $J_F.J_F \to 1$ ⇒ OK
◆Check: $J_F.\mathcal{D}_F \to \mathcal{D}_F.J_F$ by construction.
◆Check: $J_F.\gamma_F \to -\gamma_F.J_F$ ⟶ $J_F.\gamma_F == -\gamma_F.J_F$
◆Check: $\{\gamma_F, \mathcal{D}_F\}_+ \to 0$ ⟶ $\mathcal{D}_F.\gamma_F + \gamma_F.\mathcal{D}_F == 0$
◆Check order-0 condition: $[a, b^o]_- \to 0$
Need 8×8 space for correct computation.
→ $[a_8, J_{F_8}.(b_8)^\dagger.(J_{F_8})^\dagger]_- \to 0$
for algebra's $a_8 \to \{\{\lambda, 0, 0, 0, 0, 0, 0, 0\}, \{0, \lambda^*, 0, 0, 0, 0, 0, 0\}, \{0, 0, \alpha, \beta, 0, 0, 0, 0\}, \{0, 0, -\beta^*, \alpha^*, 0, 0, 0, 0\}, \{0, 0, 0, 0, \lambda, 0, 0, 0\}, \{0, 0, 0, 0, 0, \lambda, 0, 0\}, \{0, 0, 0, 0, 0, 0, \lambda, 0\}, \{0, 0, 0, 0, 0, 0, 0, \lambda\}\}$

$$\left\{a_8 \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^* & \alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, b_8 \to \begin{pmatrix} \lambda_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\lambda_b)^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_b & \beta_b & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\beta_b)^* & (\alpha_b)^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b \end{pmatrix}\right\}$$

....

$$\to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to 0$$

◆Check order-1 condition: $[[\mathcal{D}_F, a]_-, b^o]_- \to 0$
→ $[[\mathcal{D}_{F_8}, a_8]_-, J_{F_8}.(b_8)^\dagger.(J_{F_8})^\dagger]_- \to 0$
for algebra's

$$\left\{a_8 \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^* & \alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, b_8 \to \begin{pmatrix} \lambda_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\lambda_b)^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_b & \beta_b & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\beta_b)^* & (\alpha_b)^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b \end{pmatrix}\right\}$$

....

Using: $\{cc.\text{Shortest}[a\_] \to a^*.cc, cc^* \to cc, cc.cc \to 1\}$

$$\to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to 0 \longrightarrow \text{OK}$$

5.2 The gauge theory

5.2.1 The gauge group

```
PR["• The Local gauge group from ", FGWS,
  NL, "Examine subalgebra ", $0 = $ = {𝒜̃FJF, {𝒜F → ℂ⊕ℍ, a ∈ 𝒜̃FJF, a.JF → JF.ct[a]}};
  $ // ColumnForms,
  NL, "For the above: ", $s = {a -> a8, JF → JF8},
  Yield, $ = tuRuleSelect[$][a.JF],
  Yield, $ = $ /. $s,
  Yield, $ = $ /. Dot → xDot /. tuRuleSelect[$defGWS][{a8, JF8}];
  Yield,
  $ = $ // tuMatrixOrderedMultiply // tuOpSimplifyF[xDot] // (# /. xDot → Dot /. cc.a_ →
         Conjugate[a].cc /. cc → 1 /. tuOpSimplify[Dot] &) // First;
  $ // MatrixForms,
  Yield, $ = Thread[$] /. rr : Rule[__] :→ Thread[rr] // Flatten // DeleteDuplicates //
     DeleteCases[#, Rule[a_, a_]] & // tuRule,
  Yield, $ = tuRuleSolve[$, {β, α, λ}, Complexes];
  NL, "Since ", $s = λ ∈ Reals,
  yield, $s1 = Refine[$, Assumptions → $s],
  Yield, $ = tuRuleSelect[$defGWS][{a8}] /. $s1 // Refine[#, Assumptions → $s] & // First;
  $ // MatrixForms,
  imply, $e54 = {$0[[1]] → λ 1ℋF, $0[[1]] ≃ ℝ}, CG[" (5.4)"]
 ];
```

• The Local gauge group from F_{GWS}

Examine subalgebra $\left|\begin{array}{l}\tilde{\mathcal{A}}_{F J_F}\\ \left|\begin{array}{l}\mathcal{A}_F \to \mathbb{C}\oplus\mathbb{H}\\ a \in \tilde{\mathcal{A}}_{F J_F}\\ a.J_F \to J_F.a^\dagger\end{array}\right.\end{array}\right.$

For the above: $\{a \to a_8,\ J_F \to J_{F_8}\}$

→ $\{a.J_F \to J_F.a^\dagger\}$

→ $\{a_8.J_{F_8} \to J_{F_8}.(a_8)^\dagger\}$

→

$$\to \begin{pmatrix} 0&0&0&0&\lambda&0&0&0\\ 0&0&0&0&0&\lambda^*&0&0\\ 0&0&0&0&0&0&\alpha&\beta\\ 0&0&0&0&0&0&-\beta^*&\alpha^*\\ \lambda&0&0&0&0&0&0&0\\ 0&\lambda&0&0&0&0&0&0\\ 0&0&\lambda&0&0&0&0&0\\ 0&0&0&\lambda&0&0&0&0 \end{pmatrix} \to \begin{pmatrix} 0&0&0&0&\lambda&0&0&0\\ 0&0&0&0&0&\lambda&0&0\\ 0&0&0&0&0&0&\lambda&0\\ 0&0&0&0&0&0&0&\lambda\\ \lambda&0&0&0&0&0&0&0\\ 0&\lambda^*&0&0&0&0&0&0\\ 0&0&\alpha&-\beta^*&0&0&0&0\\ 0&0&\beta&\alpha^*&0&0&0&0 \end{pmatrix}$$

→ $\{\lambda^* \to \lambda,\ \alpha \to \lambda,\ \beta \to 0,\ \beta^* \to 0,\ \alpha^* \to \lambda,\ \lambda \to \lambda^*,\ \lambda \to \alpha,\ 0 \to -\beta^*,\ 0 \to \beta,\ \lambda \to \alpha^*\}$

→

Since $\lambda \in$ Reals $\longrightarrow$ $\{\beta \to 0,\ \alpha \to \lambda\}$

$$\to a_8 \to \begin{pmatrix} \lambda&0&0&0&0&0&0&0\\ 0&\lambda&0&0&0&0&0&0\\ 0&0&\lambda&0&0&0&0&0\\ 0&0&0&\lambda&0&0&0&0\\ 0&0&0&0&\lambda&0&0&0\\ 0&0&0&0&0&\lambda&0&0\\ 0&0&0&0&0&0&\lambda&0\\ 0&0&0&0&0&0&0&\lambda \end{pmatrix} \Rightarrow \{\tilde{\mathcal{A}}_{F J_F} \to \lambda\, 1_{\mathcal{H}_F},\ \tilde{\mathcal{A}}_{F J_F} \simeq \mathbb{R}\}\ \ (5.4)$$

```
PR["", Ω𝒟^"1" → {xSum[aⱼ . CommutatorM[𝒟, bⱼ], {j}], aⱼ | bⱼ ∈ 𝒜}
]
```

$\Omega^1_{\mathcal{D}} \to \{\sum_{\{j\}} [a_j.[\mathcal{D}, b_j]_-], a_j \mid b_j \in \mathcal{A}\}$

```
PR["• Consider Lie algebra (2.11b) ", hF → u[$e54[[1, 1]]],
 Yield, {u[CG["anti-hermitian"]] ∈ u[𝒜F], u → {λ, q},
  λ ∈ I ℝ, q → -I xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 3}]},
 imply, Conjugate[λ] → -λ,
 Imply, {hF → u[$e54[[1, 1]]], {λ, Conjugate[λ], α, Conjugate[α]} → 0},
 imply, $lh = hF → {0},

 line,
 NL, "●Prop.5.2:  The local gauge group of FGWS is ",
 $G = 𝒢[FGWS] ≃ xMod[U[1] × SU[2], {1, -1}],
 NL, "■ Proof:
The unitary elements: ", U[𝒜F] ≃ U[1] × U[ℍ],
 NL, "• For ", {q ∈ ℍ, q → I xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 0, 3}]},
 and, {q[CG["Unitary"]], Abs[q]^2 → 1, imply, Det[q] → 1},
 imply, U[ℍ] ≃ SU[2],
 NL, "• Since ", $e54,
 imply, $s = {ℋF → U[$e54[[1, 1]]], ℋF → {1, -1}},
 Imply, $G,
 yield, $G /. Reverse[$s[[-1]]], CG[" QED"],
 line,
 next, " Since ", $lh,
 " the gauge field ", T[it[A], "d", {μ}],
 CR[" takes values"], " in the Lie subalgebra ",
 $ = {𝔤F → Mod[u[𝒜F], hF], Mod[u[𝒜F], hF] -> u[𝒜F], u[𝒜F] -> u[1] ⊕ su[2]};
 $ // ColumnBar
]
```

• Consider Lie algebra (2.11b) $h_F \to u[\tilde{\mathcal{A}}_{F J_F}]$

→ $\{u[\text{anti-hermitian}] \in u[\mathcal{A}_F], u \to \{\lambda, q\}, \lambda \in i\,\mathbb{R}, q \to -i \sum_{\{i,3\}} [q_i\, \sigma^i]\}$ ⇒ $\lambda^* \to -\lambda$

⇒ $\{h_F \to u[\tilde{\mathcal{A}}_{F J_F}], \{\lambda, \lambda^*, \alpha, \alpha^*\} \to 0\}$ ⇒ $h_F \to \{0\}$

●Prop.5.2: The local gauge group of F_{GWS} is $\mathcal{G}[F_{GWS}] \simeq \text{xMod}[U[1] \times SU[2], \{1, -1\}]$

■ Proof:

The unitary elements: $U[\mathcal{A}_F] \simeq U[1] \times U[\mathbb{H}]$

• For $\{q \in \mathbb{H}, q \to i \sum_{\{i,0,3\}} [q_i\, \sigma^i]\}$ and

$\{q[\text{Unitary}], \text{Abs}[q]^2 \to 1,$ ⇒ $, \text{Det}[q] \to 1\}$ ⇒ $U[\mathbb{H}] \simeq SU[2]$

• Since $\{\tilde{\mathcal{A}}_{F J_F} \to \lambda\, 1_{\mathcal{H}_F}, \tilde{\mathcal{A}}_{F J_F} \simeq \mathbb{R}\}$ ⇒ $\{\mathcal{H}_F \to U[\tilde{\mathcal{A}}_{F J_F}], \mathcal{H}_F \to \{1, -1\}\}$

⇒ $\mathcal{G}[F_{GWS}] \simeq \text{xMod}[U[1] \times SU[2], \{1, -1\}]$ ⟶ $\mathcal{G}[F_{GWS}] \simeq \text{xMod}[U[1] \times SU[2], \mathcal{H}_F]$ QED

◆ Since $h_F \to \{0\}$ the gauge field A_μ

takes values in the Lie subalgebra
$\mathfrak{g}_F \to \text{Mod}[u[\mathcal{A}_F], h_F]$
$\text{Mod}[u[\mathcal{A}_F], h_F] \to u[\mathcal{A}_F]$
$u[\mathcal{A}_F] \to u[1] \oplus su[2]$

5.2.2 The gauge fields and the Higgs field

```
PR["● For gauge and Higgs fields (2.13,2.14) ", {T[it[A], "d", {μ}], ϕ},
 NL, "Let ", {a → {λ, q}, b → {λ', q'}, {a, b} ∈ (𝒜 → C"∞"[M, ℂ⊕ℍ])},
 NL, "▪Calculate inner fluctuation (5.2) ",
 $tA = T[it[A], "d", {μ}] → -I a.tuDPartial[b, μ] /. {a → a₁, b → b₁},
 NL, "•Let ", $s = tuRuleSelect[$defGWS][a₁] // Select[#, tuHasAnyQ[#, α] &] &;
 $sb = $s /. {a → b, λ → λ', β → β', α → α'};
 $sab = {$s, $sb} // Flatten,
 Imply,
 $tA = $tA /. $sab //. tt : tuDDown["∂"][__] :→ Thread[tt] /. tuDDown["∂"][0, _] → 0 //
    tuConjugateSimplify[{μ}] // tuDerivativeExpand[];
 MatrixForms[$tA],
 NL, "•Hermiticity of ", $tA[[1]],
 imply, {$tA[[2, 1, 1]], $tA[[2, 2, 2]]} ∈ ℝ,

 NL, "• For the lower-right blocks ",
 $ = {$a = qa -> $sab[[1, 2, 3 ;; -1, 3 ;; -1]],
   $b = qb -> $sab[[2, 2, 3 ;; -1, 3 ;; -1]]}; $ // MatrixForms,
 Yield, $ = Thread[Inactive[Dot][$a, $b], Rule] // tuMatrixOrderedMultiply //
   tuOpSimplifyF[dotOps];
 $ // MatrixForms,
 Imply, $ = $tA[[2, 3 ;; -1, 3 ;; -1]] → -I ($[[1]] /. qb → tuDPartial[qb, μ]);
 $ // MatrixForms,
 NL, "Defining ", $ = {T[Λ, "d", {μ}] -> $tA[[2, 1, 1]], T[Q, "d", {μ}] → $[[-1]]};
 $ // ColumnBar, accumGWS[$];
 NL, "we can represent ", $A3 = $ = {T[it[A], "d", {μ}] →
     DiagonalMatrix[{T[Λ, "d", {μ}], -T[Λ, "d", {μ}], T[Q, "d", {μ}]}],
    T[Q, "d", {μ}] → I xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 0, 3}], T[q, "d", {i}] ∈ ℝ};
 $ // MatrixForms, accumGWS[$]
]
```

● For gauge and Higgs fields (2.13,2.14) $\{A_\mu, \phi\}$

Let $\{a \to \{\lambda, q\}, b \to \{\lambda', q'\}, \{a, b\} \in (\mathcal{A} \to C^\infty[M, \mathbb{C}\oplus\mathbb{H}])\}$

▪Calculate inner fluctuation (5.2) $A_\mu \to -i\, a_1 . \underline{\partial}_\mu[b_1]$

•Let $\{a_1 \to \{\{\lambda, 0, 0, 0\}, \{0, \lambda^*, 0, 0\}, \{0, 0, \alpha, \beta\}, \{0, 0, -\beta^*, \alpha^*\}\},$
$b_1 \to \{\{\lambda', 0, 0, 0\}, \{0, (\lambda')^*, 0, 0\}, \{0, 0, \alpha', \beta'\}, \{0, 0, -(\beta')^*, (\alpha')^*\}\}\}$

$$\Rightarrow A_\mu \to \begin{pmatrix} -i\lambda\underset{-\mu}{\partial}[\lambda'] & 0 & 0 & 0 \\ 0 & -i\lambda^*\underset{-\mu}{\partial}[\lambda']^* & 0 & 0 \\ 0 & 0 & i\beta\underset{-\mu}{\partial}[\beta']^* - i\alpha\underset{-\mu}{\partial}[\alpha'] & -i\beta\underset{-\mu}{\partial}[\alpha']^* - i\alpha\underset{-\mu}{\partial}[\beta'] \\ 0 & 0 & i\alpha^*\underset{-\mu}{\partial}[\beta']^* + i\beta^*\underset{-\mu}{\partial}[\alpha'] & -i\alpha^*\underset{-\mu}{\partial}[\alpha']^* + i\beta^*\underset{-\mu}{\partial}[\beta'] \end{pmatrix}$$

•Hermiticity of $A_\mu \Rightarrow \{-i\lambda\underline{\partial}_\mu[\lambda'], -i\lambda^*\underline{\partial}_\mu[\lambda']^*\} \in \mathbb{R}$

• For the lower-right blocks $\{q_a \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, q_b \to \begin{pmatrix} \alpha' & \beta' \\ -(\beta')^* & (\alpha')^* \end{pmatrix}\}$

$$\to q_a . q_b \to \begin{pmatrix} \alpha.\alpha' - \beta.(\beta')^* & \alpha.\beta' + \beta.(\alpha')^* \\ -\alpha^*.(\beta')^* - \beta^*.\alpha' & \alpha^*.(\alpha')^* - \beta^*.\beta' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} i\beta\underset{-\mu}{\partial}[\beta']^* - i\alpha\underset{-\mu}{\partial}[\alpha'] & -i\beta\underset{-\mu}{\partial}[\alpha']^* - i\alpha\underset{-\mu}{\partial}[\beta'] \\ i\alpha^*\underset{-\mu}{\partial}[\beta']^* + i\beta^*\underset{-\mu}{\partial}[\alpha'] & -i\alpha^*\underset{-\mu}{\partial}[\alpha']^* + i\beta^*\underset{-\mu}{\partial}[\beta'] \end{pmatrix} \to -i\, q_a . \underline{\partial}_\mu[q_b]$$

Defining
$$\Lambda_\mu \to -i\lambda\underset{-\mu}{\partial}[\lambda']$$
$$Q_\mu \to -i\, q_a . \underset{-\mu}{\partial}[q_b]$$

we can represent $\{A_\mu \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix}, Q_\mu \to i \sum_{\{i,0,3\}} [q_i\, \sigma^i], q_i \in \mathbb{R}\}$

```
PR["■From the definition ", ϕ → a . CommutatorM[𝒟F, b],
  NL, "For this case ", $ = {$df44, $Y}; MatrixForms[$],
  NL, "Previous calculation show that only the
    upper left quadrant (S) does not commute with the algebra. ",
  Imply, $sD = S → ($df44[[2, 1 ;; 2, 1 ;; 2]] /. $Y // ArrayFlatten);
  MatrixForms[$sD], accumGWS[$sD];
  NL, "• ", $ = ϕ -> a1 . CommutatorM[S, b1]; $,
  yield, $ = $ /. $sD /. $sab; MatrixForms[$],
  Yield, $ph = $ = $ /. tuCommutatorExpand // FullSimplify; MatrixForms[$],
  (**)
  NL, "Let ", $i = 1;
  $ph0 = $ph /. (yy : Y_ | cc[Y_]) _ :→ yy ϕ$i++;
  $ph0 // MatrixForms,
  NL, "By inspection: ", $sp = {ϕ4 → cc[ϕ1], ϕ8 → cc[ϕ5], ϕ3 → -cc[ϕ2], ϕ7 → -cc[ϕ6]},
  $ph0 = $ph0 /. $sp; $ph0 // MatrixForms, CG[" (5.6)"],
  NL, "Hermitian requirement: ", $ = ϕ → ct[ϕ],
  Yield, $ = $ /. $ph0 //. tt : Rule[__] :→ Thread[tt];
  Yield,
  $ = $ // tuConjugateSimplify[] // Flatten // DeleteDuplicates // DeleteCases[#, 0 → 0] &;
  $ // ColumnBar;
  Imply, $ = #[[2]] & /@ tuSolve /@ $ // Flatten; $ // Column;
  $ = Reduce[$, Table[ϕi, {i, 8}], Complexes];
  $ = Apply[List, $] /. {Equal → Rule},
  Yield, $e56 = $ph0 = $ph0 /. $;
  $ph0 // MatrixForms, CG[" (5.6)"], accumGWS[$e56];
  NL, "There only 2 independent relationships with equivalent formulas: ",
  Yield, $ = Thread[$ph0[[2]] -> $ph[[2]]] /. rr : Rule[__] :→ Thread[rr];
  Yield, $ph12 =
   $ = #[[2]] & /@ tuSolve /@ $ /. Equal → Rule // Flatten // DeleteCases[#, cc[_] -> _] &;
  $ // FramedColumn
 ];
```

▪From the definition $\phi \to a.[\mathcal{D}_F, b]_-$

For this case $\{\mathcal{D}_{F_4} \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix}, Y_0 \to \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}\}$

Previous calculation show that only the
upper left quadrant (S) does not commute with the algebra.

$$\Rightarrow S \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}$$

• $\phi \to a_1.[S, b_1]_- \longrightarrow \phi \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}.[\begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & (\lambda')^* & 0 & 0 \\ 0 & 0 & \alpha' & \beta' \\ 0 & 0 & -(\beta')^* & (\alpha')^* \end{pmatrix}]_-$

$$\to \phi \to \begin{pmatrix} 0 & 0 & \lambda (Y_\nu)^* (\alpha' - \lambda') & \lambda (Y_\nu)^* \beta' \\ 0 & 0 & -\lambda^* (Y_e)^* (\beta')^* & \lambda^* (Y_e)^* ((\alpha')^* - (\lambda')^*) \\ Y_\nu (\beta (\beta')^* + \alpha (-\alpha' + \lambda')) & -Y_e (\beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta') & 0 & 0 \\ Y_\nu (\alpha^* (\beta')^* + \beta^* (\alpha' - \lambda')) & Y_e (\alpha^* (-(\alpha')^* + (\lambda')^*) + \beta^* \beta') & 0 & 0 \end{pmatrix}$$

Let $\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & (Y_e)^* \phi_3 & (Y_e)^* \phi_4 \\ Y_\nu \phi_5 & Y_e \phi_6 & 0 & 0 \\ Y_\nu \phi_7 & Y_e \phi_8 & 0 & 0 \end{pmatrix}$

By inspection: $\{\phi_4 \to (\phi_1)^*, \phi_8 \to (\phi_5)^*, \phi_3 \to -(\phi_2)^*, \phi_7 \to -(\phi_6)^*\}$

$$\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ Y_\nu \phi_5 & Y_e \phi_6 & 0 & 0 \\ -(\phi_6)^* Y_\nu & (\phi_5)^* Y_e & 0 & 0 \end{pmatrix} \quad (5.6)$$

Hermitian requirement: $\phi \to \phi^\dagger$

→

→

⇒ $\{\phi_5 \to (\phi_1)^*, \phi_6 \to -\phi_2\}$

$$\to \phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 \end{pmatrix} \quad (5.6)$$

There only 2 independent relationships with equivalent formulas:

→

$$\to \boxed{\begin{array}{l} \phi_1 \to \lambda \alpha' - \lambda \lambda' \\ \phi_2 \to \lambda \beta' \\ \phi_2 \to \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta' \\ \phi_1 \to -\alpha^* (\alpha')^* + \alpha^* (\lambda')^* + \beta^* \beta' \end{array}}$$

```
PR["●Note: ϕ's is generally a sum of like terms: ",
 $ = Map[# /. tt : λ' | α' | β' | λ | α | β :→ T[tt, "d", {j}] &, $ph12];
 Yield, $ph12p = $ =
   Map[#[[1]] -> xSum[#[[2]], {j}] &, $] /. xSum[a_ -> b_, c_] → xSum[a, c] -> xSum[b, c];
 Column[$],
 NL, CR["Recall that ", ϕ → a . CommutatorM[𝒟F, b],
  " is the Higg's like field defined by the algebra and the Dirac
    operator.  What is the effect of different algebras on ϕ?"]
]
```

●Note: ϕ's is generally a sum of like terms:

$$\phi_1 \to \sum_{\{j\}} [\lambda_j \alpha'_j - \lambda_j \lambda'_j]$$

$$\phi_2 \to \sum_{\{j\}} [\lambda_j \beta'_j]$$

→

$$\phi_2 \to \sum_{\{j\}} [(\alpha'_j)^* \beta_j - (\lambda'_j)^* \beta_j + \alpha_j \beta'_j]$$

$$\phi_1 \to \sum_{\{j\}} [-(\alpha_j)^* (\alpha'_j)^* + (\alpha_j)^* (\lambda'_j)^* + (\beta_j)^* \beta'_j]$$

Recall that $\phi \to a.[\mathcal{D}_F, b]_-$ is the Higg's like field defined by the algebra and the Dirac operator. What is the effect of different algebras on ϕ?

```
PR["Summary: ",
  NL, $e57 = $ = {CG["On ℋ1"],
      $A3, T[Λ, "d", {μ}] ∈ ℝ,
      ϕ → {{0, ct[Y]}, {Y, 0}},
      $ph0,
      $ph12,
      T[Bℋ1, "d", {μ}] →
       {{0, 0, 0}, {0, -2 T[Λ, "d", {μ}], 0}, {0, 0, T[Q, "d", {μ}] - T[Λ, "d", {μ}] 12}},
      CG["On ℋI"],
      T[BℋI, "d", {μ}] → {{0, 0, 0}, {0, 2 T[Λ, "d", {μ}], 0},
         {0, 0, -T[Λ, "d", {μ}] 12 - Conjugate[T[Q, "d", {μ}]]}}
     } // Flatten;
  $ // MatrixForms // ColumnBar,
  NL, "● Calculate ", $ = tuRuleSelect[$defEM][T[B, "d", {μ}]][[1]], accumGWS[$e57],

  NL, "In 8x8 representation ", "POFF",
  Yield, $q = T[Q, "d", {μ}] → Table[T[q, "d", {μ}]i,j, {i, 2}, {j, 2}];
  MatrixForms[$q],
  Yield, $b = tuRuleSelect[$defEM][T[B, "d", {μ}]][[1]] /. toxDot /. A → it[A] /. F → F8 /.
    Plus → Inactive[Plus],
  Yield, $a = tuRuleSelect[$e57][T[it[A], "d", {μ}]] /. it[A] → it[A]1 /. $q // First;
  Yield, $a = MapAt[ArrayFlatten[#] &, $a, -1]; $a // MatrixForms,
  Yield, $aa = ($a[[1]] /. l → I) → (DiagonalMatrix[Table[T[Λ, "d", {μ}], 4]]) // Normal,
  Yield, $aaa = {{$a[[1]], 0}, {0, $aa[[1]]}},
  Yield, $aaa = T[it[A], "d", {μ}] -> ($aaa /. $a /. $aa // ArrayFlatten);
  $aaa // MatrixForms, accumGWS[{$a, $aa, $aaa, $q}];
  $j8 = tuRuleSelect[$defGWS][JF8],
  Yield, $b = $b /. $aaa /. $j8 // tuMatrixOrderedMultiply // (# /. toDot &) //
    tuRepeat[{tuOpSimplify[Dot], Dot[cc, a_] :→ Dot[cc[a], cc]}];
  $b // MatrixForms,
  Yield,
  $b = $b // tuRepeat[{tuOpSimplify[dotOps], Dot[cc, a_] :→ Dot[cc[a], cc], cc[cc] → cc,
       cc. cc → 1}] // tuConjugateSimplify[{T[Λ, "d", {μ}]}],
  "PONdd",
  Yield, $e58 = $ = $b // Activate;
  $ // MatrixForms // Framed, CG[" (5.8)"],
  NL, "Coefficients of Λ associated with hyper-charge."; accumGWS[$e58]
 ];
```

Summary:

On $\mathcal{H}_1$

$$A_\mu \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix}$$

$$Q_\mu \to i \sum_{\{i,\bar{0},3\}} [q_i \, \sigma^i]$$

$q_i \in \mathbb{R}$

$\Lambda_\mu \in \mathbb{R}$

$$\phi \to \begin{pmatrix} 0 & Y^\dagger \\ Y & 0 \end{pmatrix}$$

$$\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 \end{pmatrix}$$

$\phi_1 \to \lambda \alpha' - \lambda \lambda'$

$\phi_2 \to \lambda \beta'$

$\phi_2 \to \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta'$

$\phi_1 \to -\alpha^* (\alpha')^* + \alpha^* (\lambda')^* + \beta^* \beta'$

$$B_{\mathcal{H}_1\,\mu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu - 1_2 \Lambda_\mu \end{pmatrix}$$

On $\mathcal{H}_{\bar{1}}$

$$B_{\mathcal{H}_{\bar{1}}\,\mu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\Lambda_\mu & 0 \\ 0 & 0 & -(Q_\mu)^* - 1_2 \Lambda_\mu \end{pmatrix}$$

- Calculate $B_\mu \to -J_F . A_\mu . (J_F)^\dagger + A_\mu$

In 8x8 representation

....

$$\to \boxed{B_\mu \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 1,1} - \Lambda_\mu & q_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 2,1} & q_{\mu 2,2} - \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 1,1})^* + \Lambda_\mu & -(q_{\mu 1,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 2,1})^* & -(q_{\mu 2,2})^* + \Lambda_\mu \end{pmatrix}} \quad (5.8)$$

```
PR["● Higgs field ",
 $ = Φ → Inactivate[ 𝒟F + {{ϕ, 0}, {0, 0}} + JF.{{ϕ, 0}, {0, 0}}.ct[JF], Plus];
 MatrixForms[$],
 NL, "In 8x8 representation: ",
 $sϕ = {{ϕ, 0}, {0, 0}} → ArrayFlatten[DiagonalMatrix[{ϕ, 0, 0, 0, 0}] /. $ph0];
 MatrixForms[$sϕ], accumGWS[$sϕ], "POFF",

 $ = $ /. JF → JF8;
 $ = $ /. Dot → xDot /. $j8 /. $sϕ; MatrixForms[$],
 Yield,
 $ = $ // tuMatrixOrderedMultiply // (# /. xDot → Dot &) //
   tuRepeat[{tuOpSimplify[Dot], Dot[cc, a_] :→ Dot[cc[a], cc], cc[cc] → cc, cc.cc → 1}];
 "PON",
 MatrixForms[$], accumGWS[$];
 NL, "From (5.6) ", $e56 // MatrixForms,
 NL, "Condense into space ", tuRuleSelect[$defGWS][ℋF2][[1]],
 Yield, $s = {$[[2, -1]] → {{0, 0}, {0, cc[ϕ]}}, $[[2, -2]] → {{ϕ, 0}, {0, 0}}};
 $s // MatrixForms,
 Yield, $ = $ /. $s /. F → F2, CK,
 NL, "From ", $s = tuRuleSelect[$defGWS][𝒟F2]; MatrixForms[$s],
 Yield, $ = $ /. $s // Activate;
 MatrixForms[$e59 = $] // Framed, CG[" (5.9)"]; accumGWS[$e59]
]
```

- **Higgs field** $\Phi \to \mathcal{D}_F + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F . \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} . (J_F)^\dagger$

In 8x8 representation: $\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$\Phi \to \mathcal{D}_F + \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_\nu . (\phi_1)^* & Y_\nu . (\phi_2)^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y_e . \phi_2 & Y_e . \phi_1 \\ 0 & 0 & 0 & 0 & \phi_1 . (Y_\nu)^* & -(Y_e)^* . (\phi_2)^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_2 . (Y_\nu)^* & (Y_e)^* . (\phi_1)^* & 0 & 0 \end{pmatrix}$$

From (5.6) $\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 \end{pmatrix}$

Condense into space $\mathcal{H}_{F_2} \to \mathcal{H}_1[\mathbb{C}^4] \oplus \mathcal{H}_{\bar{1}}[\mathbb{C}^4]$

$$\to \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_\nu . (\phi_1)^* & Y_\nu . (\phi_2)^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y_e . \phi_2 & Y_e . \phi_1 \\ 0 & 0 & 0 & 0 & \phi_1 . (Y_\nu)^* & -(Y_e)^* . (\phi_2)^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_2 . (Y_\nu)^* & (Y_e)^* . (\phi_1)^* & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & \phi^* \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$\to$ $\Phi \to \mathcal{D}_{F_2} + \{\{\phi, 0\}, \{0, 0\}\} + \{\{0, 0\}, \{0, \phi^*\}\} \longleftarrow$ CHECK

From $\left\{ \mathcal{D}_{F_2} \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix} \right\}$

$$\to \boxed{\Phi \to \begin{pmatrix} S + \phi & T^* \\ T & S^* + \phi^* \end{pmatrix}}$$

Prop.5.3.

```
PR["●Prop.5.3. The action of the gauge group ",
  𝒢[M × F_GWS][𝒟_𝒜 → slash[𝒟] ⊗ 𝕀 + T[γ, "u", {μ}] ⊗ T[B, "d", {μ}] + T[γ, "d", {5}] ⊗ Φ],
  NL, "is given by: ",
  $ = {T[Λ, "d", {μ}] -> T[Λ, "d", {μ}] - I λ.tuDPartial[Conjugate[λ], μ],
    T[Q, "d", {μ}] → q.T[Q, "d", {μ}].ct[q] - I q.tuDPartial[ct[q], μ],
    {{ϕ1}, {ϕ2}} → Conjugate[λ].q.{{ϕ1}, {ϕ2}} + (Conjugate[λ].q - 1).{{1}, {0}},
    λ ∈ C^"∞"[M, U[1]], q ∈ C^"∞"[M, SU[2]]
   }; MatrixForms[$e221a = $] // ColumnBar,
  line,
  NL, "For the fields (5.7) compute the transformations (2.21): ",
  $e221 = $ = {T[it[A], "d", {μ}] → u.T[it[A], "d", {μ}].ct[u] - I u.tuDPartial[ct[u], μ],
     ϕ -> u.ϕ.ct[u] + u.CommutatorM[𝒟_F, ct[u]],
     {u → {λ, q}} ∈ C^"∞"[M, U[1] × SU[2]][CG["gauge transformation"]]
    }; $ // ColumnBar, accumGWS[{$e221a, $e221}], accumGWS[{$e221, $e221a}],

  NL, "The 8x8 representation: ", $a88 = $ =
    tuRuleSelect[$defGWS][T[it[A], "d", {μ}]] // Select[#, tuHasAnyQ[#, q] &] & // First;
  $ // MatrixForms,
  NL, "and(from ", $a88[[1]], "): " ,
  $u = u -> $a88[[2]] /. {T[Λ, "d", {μ}] -> λ, q → uq}; MatrixForms[$u],

  NL, "It is easy to see that ", $0 = u. $a88[[1]] . ct[u],
  imply, T[Q, "d", {μ}] → u.T[Q, "d", {μ}].ct[u],
  " since the block diagonal elements are independant.", "POFF",
  Yield, $1 = $ = $0 -> $u [[2]].$a88[[2]].ct[$u[[2]]];
  MatrixForms[$], "PON",

  NL, "Similarly for ", $0 = $ = I u.tuDPartial[ct[u], μ], "POFF",
  Yield,
  $ = $0 → ($ /. $u //. tt : tuDDown["∂"][_, _] ⧴ Thread[tt] /. tuDDown["∂"][0, _] → 0);
  MatrixForms[$], CK, "PON",
  Imply,
  {$e221a[[2]][CG["over the q's"]], $e221a[[1]][CG["over the λ's"]]} // ColumnBar
 ];
```

●Prop.5.3. The action of the gauge group $\mathcal{G}[M\times F_{GWS}][\mathcal{D}_{\mathcal{A}}\to(\not{D})\otimes\mathbb{I}+\gamma_5\otimes\Phi+\gamma^\mu\otimes B_\mu]$

is given by:

$$\begin{cases}\Lambda_\mu\to -i\,\lambda.\underset{-\mu}{\partial}[\lambda^*]+\Lambda_\mu\\ Q_\mu\to -i\,q.\underset{-\mu}{\partial}[q^\dagger]+q.Q_\mu.q^\dagger\\ \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\to(-1+\lambda^*.q).\begin{pmatrix}1\\ 0\end{pmatrix}+\lambda^*.q.\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\\ \lambda\in C^\infty[M,U[1]]\\ q\in C^\infty[M,SU[2]]\end{cases}$$

For the fields (5.7) compute the transformations (2.21):

$$\begin{cases}\mathcal{A}_\mu\to -i\,u.\underset{-\mu}{\partial}[u^\dagger]+u.\mathcal{A}_\mu.u^\dagger\\ \phi\to u.[\mathcal{D}_F,u^\dagger]_- +u.\phi.u^\dagger\\ \{u\to\{\lambda,q\}\}\in C^\infty[M,U[1]\times SU[2]][\text{gauge transformation}]\end{cases}$$

The 8x8 representation: $\mathcal{A}_\mu\to\begin{pmatrix}\Lambda_\mu&0&0&0&0&0&0&0\\0&-\Lambda_\mu&0&0&0&0&0&0\\0&0&q_{\mu 1,1}&q_{\mu 1,2}&0&0&0&0\\0&0&q_{\mu 2,1}&q_{\mu 2,2}&0&0&0&0\\0&0&0&0&\Lambda_\mu&0&0&0\\0&0&0&0&0&\Lambda_\mu&0&0\\0&0&0&0&0&0&\Lambda_\mu&0\\0&0&0&0&0&0&0&\Lambda_\mu\end{pmatrix}$

and(from $\mathcal{A}_\mu$): $u\to\begin{pmatrix}\lambda&0&0&0&0&0&0&0\\0&-\lambda&0&0&0&0&0&0\\0&0&uq_{\mu 1,1}&uq_{\mu 1,2}&0&0&0&0\\0&0&uq_{\mu 2,1}&uq_{\mu 2,2}&0&0&0&0\\0&0&0&0&\lambda&0&0&0\\0&0&0&0&0&\lambda&0&0\\0&0&0&0&0&0&\lambda&0\\0&0&0&0&0&0&0&\lambda\end{pmatrix}$

It is easy to see that $u.\mathcal{A}_\mu.u^\dagger \Rightarrow Q_\mu\to u.Q_\mu.u^\dagger$
since the block diagonal elements are independant.
Similarly for $i\,u.\underline{\partial}_\mu[u^\dagger]$

$$\Rightarrow\begin{cases}(Q_\mu\to -i\,q.\underset{-\mu}{\partial}[q^\dagger]+q.Q_\mu.q^\dagger)[\text{over the q's}]\\ (\Lambda_\mu\to -i\,\lambda.\underset{-\mu}{\partial}[\lambda^*]+\Lambda_\mu)[\text{over the }\lambda\text{'s}]\end{cases}$$

```
PR["Check Higg's field gauge transformation ", $ = $e221[[2]],
 NL, "Collect relevant pieces ", "POFF",
 Yield,
 $s08 = {ϕ -> $sϕ[[2]], u -> $aa, $df44[[1]] → ($df44[[1]] //. $sDAgws // ArrayFlatten)};
 MatrixForms[$s08],
 line,
 NL, "Calculate RHS:",
 Yield, $[[2]] = $[[2]] /. Plus → Inactive[Plus] /. $s08;
 MatrixForms[$0 = $], "PON",
 line,
 NL, "The commutator term: ", $ = $0 // tuExtractPositionPattern[CommutatorM[__]];
 Yield, $ = $ /. CommutatorM → MCommutator;
 MatrixForms[$],
 NL, "Recombine ",
 Yield, $pht = $ = tuReplacePart[$0, $] // Activate // Simplify;
 MatrixForms[$]
]
```

Check Higg's field gauge transformation $\phi \to u.[\mathcal{D}_F, u^\dagger]_- + u.\phi.u^\dagger$

Collect relevant pieces

The commutator term:

$$\rightarrow \{\{2,1,2\} \to -((\mathcal{A}_{I_\mu})^\dagger \to \begin{pmatrix} (\Lambda_\mu)^* & 0 & 0 & 0 \\ 0 & (\Lambda_\mu)^* & 0 & 0 \\ 0 & 0 & (\Lambda_\mu)^* & 0 \\ 0 & 0 & 0 & (\Lambda_\mu)^* \end{pmatrix}).\mathcal{D}_F + \mathcal{D}_F.((\mathcal{A}_{I_\mu})^\dagger \to \begin{pmatrix} (\Lambda_\mu)^* & 0 & 0 & 0 \\ 0 & (\Lambda_\mu)^* & 0 & 0 \\ 0 & 0 & (\Lambda_\mu)^* & 0 \\ 0 & 0 & 0 & (\Lambda_\mu)^* \end{pmatrix})\}$$

Recombine

$$\rightarrow \phi \to (\mathcal{A}_{I_\mu} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 & 0 \\ 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & \Lambda_\mu \end{pmatrix}).$$

$$(-((\mathcal{A}_{I_\mu})^\dagger \to \begin{pmatrix} (\Lambda_\mu)^* & 0 & 0 & 0 \\ 0 & (\Lambda_\mu)^* & 0 & 0 \\ 0 & 0 & (\Lambda_\mu)^* & 0 \\ 0 & 0 & 0 & (\Lambda_\mu)^* \end{pmatrix}).\mathcal{D}_F + \mathcal{D}_F.((\mathcal{A}_{I_\mu})^\dagger \to \begin{pmatrix} (\Lambda_\mu)^* & 0 & 0 & 0 \\ 0 & (\Lambda_\mu)^* & 0 & 0 \\ 0 & 0 & (\Lambda_\mu)^* & 0 \\ 0 & 0 & 0 & (\Lambda_\mu)^* \end{pmatrix})) +$$

$$(\mathcal{A}_{I_\mu} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 & 0 \\ 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & \Lambda_\mu \end{pmatrix}).\begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$((\mathcal{A}_{I_\mu})^\dagger \to \begin{pmatrix} (\Lambda_\mu)^* & 0 & 0 & 0 \\ 0 & (\Lambda_\mu)^* & 0 & 0 \\ 0 & 0 & (\Lambda_\mu)^* & 0 \\ 0 & 0 & 0 & (\Lambda_\mu)^* \end{pmatrix})$$

```
PR["Compute Higg's field gauge transformation ", xtmp = $ = $e221[[2]],
 line,
 NL, "Higg's non-zero for ℋ1: ",
 $ph = tuRuleSelect[$defGWS][ϕ] // Select[#, tuHasAnyQ[#, ν] &] & // Last;
 $ph[[1]] = $ph[[1]] /. ϕ → ϕ1; $ph,

 $u = u -> $a88[[2]] /. {T[Λ, "d", {μ}] -> λ, q → uq}; MatrixForms[$u];
 NL, "Use ", $u[[2]] = $u[[2, 1 ;; 4, 1 ;; 4]];
 $u = $u /. u → u1;
 $u // MatrixForms,
 NL, "Use ", $d = tuRuleSelect[$defGWS][𝒟F8][[1]];
 $d[[2]] = $d[[2, 1 ;; 4, 1 ;; 4]]; $d = $d /. F8 → F1; $d // MatrixForms,
 Imply, "POFF",
 Imply, $ = $ /. {ϕ → ϕ1, u → u1, F → F1},
 $[[2]] = $[[2]] // expandCom[{$d, $u, $ph}];
 "PONdd",
 Yield, $ = $ /. Dot → Times // Simplify; $ // MatrixForms,
 NL, CR["Find transform of ϕ maintain separation of ϕ's. "]
]
PR["Is there a transformation in ",
 $e221[[2]],
 NL, "From: ",
 $ = $[[2]]; $ // MatrixForms,
 NL, "Assume transformed form ",
 $s = $ph; $s[[2]] = $s[[2]] /. ϕ → ϕt; $s // MatrixForms, CK,
 Imply,
 $ = Thread[$ → $s[[2]]] /. rr : Rule[__] :→ Thread[rr] // Flatten // DeleteDuplicates;
 $ // ColumnBar;
 $ = tuSolve[$[[2 ;; 5]]];
 Yield, $[[2]] // ColumnBar,
 CR["Apply symmetries."]
]
```

Compute Higg's field gauge transformation $\phi \to u.[\mathcal{D}_F, u^\dagger]_- + u.\phi.u^\dagger$

Higg's non-zero for $\mathcal{H}_1$: $\phi_1 \to \{\{0, 0, (Y_\nu)^* \phi_1, (Y_\nu)^* \phi_2\}, \{0, 0, -(Y_e)^* (\phi_2)^*, (Y_e)^* (\phi_1)^*\}, \{(\phi_1)^* Y_\nu, -Y_e \phi_2, 0, 0\}, \{(\phi_2)^* Y_\nu, Y_e \phi_1, 0, 0\}\}$

Use $u_1 \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & uq_{\mu 1,1} & uq_{\mu 1,2} \\ 0 & 0 & uq_{\mu 2,1} & uq_{\mu 2,2} \end{pmatrix}$

Use $\mathcal{D}_{F_1} \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}$

⇒

....

$\to \phi_1 \to \begin{pmatrix} 0 & \\ 0 & \\ Y_\nu (-(uq_{\mu 1,1})^* uq_{\mu 1,1} - (uq_{\mu 1,2})^* uq_{\mu 1,2} + \lambda^* ((1 + (\phi_1)^*) uq_{\mu 1,1} + (\phi_2)^* uq_{\mu 1,2})) & -Y_e ((uq_{\mu 2,1})^* uq_{\mu 1,1} + (\\ Y_\nu (-(uq_{\mu 1,1})^* uq_{\mu 2,1} - (uq_{\mu 1,2})^* uq_{\mu 2,2} + \lambda^* ((1 + (\phi_1)^*) uq_{\mu 2,1} + (\phi_2)^* uq_{\mu 2,2})) & -Y_e ((uq_{\mu 2,1})^* uq_{\mu 2,1} + (\end{pmatrix}$

Find transform of ϕ maintain separation of ϕ's.

Is there a transformation in $\phi \to u.[\mathcal{D}_F, u^\dagger]_- + u.\phi.u^\dagger$

From: $\begin{pmatrix} 0 \\ 0 \\ Y_\nu\,(-(uq_{\mu 1,1})^* uq_{\mu 1,1} - (uq_{\mu 1,2})^* uq_{\mu 1,2} + \lambda^*((1+(\phi_1)^*)\,uq_{\mu 1,1} + (\phi_2)^* uq_{\mu 1,2})) & -Y_e\,((uq_{\mu 2,1})^* uq_{\mu 1,1} + (\\ Y_\nu\,(-(uq_{\mu 1,1})^* uq_{\mu 2,1} - (uq_{\mu 1,2})^* uq_{\mu 2,2} + \lambda^*((1+(\phi_1)^*)\,uq_{\mu 2,1} + (\phi_2)^* uq_{\mu 2,2})) & -Y_e\,((uq_{\mu 2,1})^* uq_{\mu 2,1} + (\end{pmatrix}$

Assume transformed form $\phi_1 \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi t_1 & (Y_\nu)^* \phi t_2 \\ 0 & 0 & -(Y_e)^* (\phi t_2)^* & (Y_e)^* (\phi t_1)^* \\ (\phi t_1)^* Y_\nu & -Y_e\,\phi t_2 & 0 & 0 \\ (\phi t_2)^* Y_\nu & Y_e\,\phi t_1 & 0 & 0 \end{pmatrix}$ ⟵CHECK

⇒

$$\to \left|\begin{array}{l} \phi t_1 == \lambda\,(-\lambda^* + (uq_{\mu 1,1})^* + (uq_{\mu 1,1})^* \phi_1 + (uq_{\mu 1,2})^* \phi_2) \\ \phi t_2 == \lambda\,((uq_{\mu 2,1})^* + (uq_{\mu 2,1})^* \phi_1 + (uq_{\mu 2,2})^* \phi_2) \\ (\phi t_2)^* == \lambda\,(-(\phi_2)^* (uq_{\mu 1,1})^* + (uq_{\mu 1,2})^* + (\phi_1)^* (uq_{\mu 1,2})^*) \\ (\phi t_1)^* == -\lambda\,(\lambda^* - (\phi_2)^* (uq_{\mu 2,1})^* + (uq_{\mu 2,2})^* + (\phi_1)^* (uq_{\mu 2,2})^*) \end{array}\right.$$

Apply symmetries.

5.3 Spectral Action

```
$p37;
$e57;
$e58;
$F;
PR["●Lemma 5.4: ",
   $l54 = $ = {Tr[T[F, "uu", {μ, ν}] T[F, "dd", {μ, ν}]] → 12 T[Λ, "dd", {μ, ν}]
         T[Λ, "uu", {μ, ν}] + 2 Tr[T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]],
      T[Λ, "dd", {μ, ν}] → tuDPartial[T[Λ, "d", {ν}], μ] -
        tuDPartial[T[Λ, "d", {μ}], ν],
      T[Q, "dd", {μ, ν}] → tuDPartial[T[Q, "d", {ν}], μ] - tuDPartial[T[Q, "d", {μ}], ν] +
        I CommutatorM[T[Q, "d", {μ}], T[Q, "d", {ν}]]
     }; FramedColumn[$], accumGWS[$l54]
  ];
```

●Lemma 5.4: $\boxed{\begin{array}{l} \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 12\,\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 2\,\mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}] \\ \Lambda_{\mu\nu} \to -\underset{-\nu}{\partial}[\Lambda_\mu] + \underset{-\mu}{\partial}[\Lambda_\nu] \\ Q_{\mu\nu} \to i\,[Q_\mu, Q_\nu]_- - \underset{-\nu}{\partial}[Q_\mu] + \underset{-\mu}{\partial}[Q_\nu] \end{array}}$

```
PR["● From the definition: ", $ = tuRuleSelect[$defall][T[F, "dd", {μ, ν}]][[1]],
 NL, "Use above ",
 $sb = tuRuleSelect[$defGWS][T[B, "d", {μ}]][[-1]] // tuAddPatternVariable[μ],
 Yield, $ = $ /. tuCommutatorExpand /. Plus → Inactive[Plus] /. $sb //.
     tuOpDistribute[tuDDown["∂"], List] // tuDerivativeExpand[] // Activate;
 $ // MatrixForms,
 ,
 NL, "q's are hermitian and tracelist: ",
 $sq = {Conjugate[(qq : T[q, "d", {μ_}])i_,j_] :> qqj,i,
   Conjugate[(qq : T[q, "u", {μ_}])i_,j_] :> qqj,i, Conjugate[qq : q_i_,i_] → qi,i,
   T[q, "d", {μ_}]1,1 + T[q, "d", {μ_}]2,2 → 0,
   T[q, "u", {μ_}]1,1 + T[q, "u", {μ_}]2,2 → 0
  };

 $sq // ColumnBar,
 Yield, $ = $ //. $sq // tuConjugateSimplify[{μ, ν}]; $ // MatrixForms,
 NL, "Tr[] of Product: ", $u = $ // tuIndicesRaise[{μ, ν}];
 $ = Thread[Dot[$ , $u], Rule]; $ // MatrixForms;
```

```
Yield, $trff = $ = Tr /@ $ //. $sq // Expand;
NL, "Common index substitutions: ",
$sqsub = {aa : a_ b__ :→ tuIndexSwapUpDown[{μ}][aa] /; ! FreeQ[aa, T[q, "d", {μ}]],
  aa : a_ b__ :→ tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, T[q, "d", {ν}]],
  aa : a_ b__ :→ tuIndexSwap[{ν, μ}][aa] /; ! FreeQ[aa, T[q, "u", {ν}]], aa : a_ b__ :→
   tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, tuDDown["∂"][T[q, "u", {i_}]_,_, ν_]],
  aa : a_ tuDDown["∂"][T[q, "u", {i_}]_,_, ν_] :→ tuIndexSwapUpDown[{i}][aa],
  aa : a_ tuDUp["∂"][T[q, "u", {i_}]_,_, ν_] :→ tuIndexSwapUpDown[{i, ν}][aa],
  aa : a__ tuDUp["∂"][T[q, "d", {i_}]_,_, ν_] :→ tuIndexSwapUpDown[{ν}][aa],
  aa : a_ T[q, "d", {μ_}]i_,i_ :> tuIndexSwapUpDown[{μ}][aa],
  aa : a_ T[q, "u", {ν}]i_,i_ :> tuIndexSwap[{μ, ν}][aa]
 },

next, "The ΛΛ terms: ", $ll = (Apply[Plus, ($trff // tuTermSelect[Λ, q])] // Simplify);
$ll // Framed,

next, "The Λq terms: ", $lq = $ = Apply[Plus, ($trff // tuTermSelect[{Λ, q}])];
yield, $lq = $ = $ //. $sqsub[[1 ;; 4]] /. $sqsub // Simplify //
    (# //. tuOpCollect[tuDDown["∂"]] /. $sq &) // tuDerivativeExpand[];
$ // Framed,
(**)
next, "The qq terms: ", $qq = $ = Apply[Plus, ($trff // tuTermSelect[q, Λ])];
yield, $qq = $ = $ /. $sqsub // Simplify; $ // Framed,
NL, "Too many terms to find text
  relationship directly. Compare with direct computation of ",

$ = tuRuleSelect[$defGWS][T[Q, "dd", {_, _}]][[1]];
$s = $ // tuIndicesRaise[{μ, ν}];
Yield, $ = $ . $s // Thread[#, Rule] &, "POFF",
Yield, $ = $ /. tuCommutatorExpand // expandDC[],
NL, "Use ",
$s = tuRuleSelect[$defGWS][T[Q, "d", {_}]] // Select[#, tuHasAnyQ[#, {2}] &] & // First,
$s = {$s, $s // tuIndicesRaise[μ]} // tuAddPatternVariable[μ];
Yield,
$ = $ /. toxDot //. $s //. tt : (tuDDown["∂"][_, _] | tuDUp["∂"][_, _]) :→ Thread[tt] //
   tuMatrixOrderedMultiply // (# /. xDot → Times &);
"PONdd",
Yield, $qq0 = $ = Tr /@ $ // Simplify,
NL, "Comparing ", 2 $qq0[[1]], " with FF calculation ", imply,
2 $qq0[[2]] == $qq // tuIndicesLower[{μ, ν}] // Simplify // Framed
]
```

● From the definition: $F_{\mu\nu} \to i\,[B_\mu, B_\nu]_- - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

Use above

$B_{\mu\_} \to \{\{0, 0, 0, 0, 0, 0, 0, 0\}, \{0, -2\Lambda_\mu, 0, 0, 0, 0, 0, 0\}, \{0, 0, q_{\mu 1,1} - \Lambda_\mu, q_{\mu 1,2}, 0, 0, 0, 0\}, \{0, 0, q_{\mu 2,1}, q_{\mu 2,2} - \Lambda_\mu, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 2\Lambda_\mu, 0, 0\}, \{0, 0, 0, 0, 0, 0, -(q_{\mu 1,1})^* + \Lambda_\mu, -(q_{\mu 1,2})^*\}, \{0, 0, 0, 0, 0, 0, -(q_{\mu 2,1})^*, -(q_{\mu 2,2})^* + \Lambda_\mu\}\}$

$$\to F_{\mu\nu} \to \left(\begin{array}{llll}
0 & 0 & & 0 \\
0 & 2\,\partial_{-\nu}[\Lambda_\mu] - 2\,\partial_{-\mu}[\Lambda_\nu] & & 0 \\
0 & 0 & i\,(-q_{\mu 2,1}\, q_{\nu 1,2} + q_{\mu 1,2}\, q_{\nu 2,1}) - \partial_{-\nu}[q_{\mu 1,1}] + \partial_{-\mu}[q_{\nu 1,1}] + \partial_{-\nu}[\Lambda_\mu] - \partial_{-\mu} & \\
0 & 0 & i\,(-q_{\nu 2,1}\,(q_{\mu 1,1} - \Lambda_\mu) + q_{\nu 2,1}\,(q_{\mu 2,2} - \Lambda_\mu) + q_{\mu 2,1}\,(q_{\nu 1,1} - \Lambda_\nu) - q_{\mu 2,1}\,(q_{\nu 2,2} - \Lambda_\nu)) - & \\
0 & 0 & & 0 \\
0 & 0 & & 0 \\
0 & 0 & & 0 \\
0 & 0 & & 0
\end{array}\right.$$

q's are hermitian and tracelist: $\left|\begin{array}{l}(qq : q_{\mu\_i\_,j\_})^* \mapsto qq_{j,i} \\ (qq : q^{\mu}{}_{-i\_,j\_})^* \mapsto qq_{j,i} \\ (qq : q_{\_i\_,i\_})^* \to q_{i,i} \\ q_{\mu\_1,1} + q_{\mu\_2,2} \to 0 \\ q^{\mu}{}_{-1,1} + q^{\mu}{}_{-2,2} \to 0\end{array}\right.$

$$\rightarrow F_{\mu\nu} \to \left(\begin{array}{llll} 0 & 0 & 0 \\ 0 & 2\underset{-\nu}{\partial}[\Lambda_\mu] - 2\underset{-\mu}{\partial}[\Lambda_\nu] & 0 \\ 0 & 0 & -i\, q_{\mu 2,1} q_{\nu 1,2} + i\, q_{\mu 1,2} q_{\nu 2,1} - \underset{-\nu}{\partial}[q_{\mu 1,1}] + \underset{-\mu}{\partial}[q_{\nu 1,1}] + \underset{-\nu}{\partial}[\Lambda_\mu] - \underset{-\mu}{\partial}[\Lambda_\nu] & -i \\ 0 & 0 & i\, q_{\mu 2,1} q_{\nu 1,1} - i\, q_{\mu 1,1} q_{\nu 2,1} + i\, q_{\mu 2,2} q_{\nu 2,1} - i\, q_{\mu 2,1} q_{\nu 2,2} - \underset{-\nu}{\partial}[q_{\mu 2,1}] + \underset{-\mu}{\partial}[q_{\nu 2,1}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right.$$

Tr[] of Product:
→
Common index substitutions:

```
{aa : a_ b__ :→ tuIndexSwapUpDown[{μ}][aa] /; ! FreeQ[aa, T[q, d, {μ}]],
  aa : a_ b__ :→ tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, T[q, d, {ν}]],
  aa : a_ b__ :→ tuIndexSwap[{ν, μ}][aa] /; ! FreeQ[aa, T[q, u, {ν}]],
  aa : a_ b__ :→ tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, ∂_ν[q^i-_,_]],
  aa : a_ ∂_ν_[q^i-_,_] :→ tuIndexSwapUpDown[{i}][aa], aa : a_ ∂^ν_[q^i-_,_] :→ tuIndexSwapUpDown[{i, ν}][aa],
  aa : a__ ∂^ν_[q_i_,_] :→ tuIndexSwapUpDown[{ν}][aa],
  aa : a_ q_μ_i_,i_ :→ tuIndexSwapUpDown[{μ}][aa], aa : a_ q^ν_i_,i_ :→ tuIndexSwap[{μ, ν}][aa]}
```

◆The ΛΛ terms: $\boxed{12\,(\underset{-\nu}{\partial}[\Lambda_\mu] - \underset{-\mu}{\partial}[\Lambda_\nu])\,(\underset{-}{\partial}^\nu[\Lambda^\mu] - \underset{-}{\partial}^\mu[\Lambda^\nu])}$

◆The Λq terms: ⟶ $\boxed{0}$

◆The qq terms: ⟶

$$
\begin{aligned}
&-2\,(-q_{\mu 1,1}\,q_{\nu 2,1}\,q^{\mu}_{1,1}\,q^{\nu}_{1,2}+q_{\mu 2,2}\,q_{\nu 2,1}\,q^{\mu}_{1,1}\,q^{\nu}_{1,2}+q_{\mu 1,1}\,q_{\nu 1,1}\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}-\\
&\quad q_{\mu 2,2}\,q_{\nu 1,1}\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}-q_{\mu 1,1}\,q_{\nu 2,2}\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}+q_{\mu 2,2}\,q_{\nu 2,2}\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}+q_{\mu 1,1}\,q_{\nu 2,1}\,q^{\mu}_{2,2}\,q^{\nu}_{1,2}-\\
&\quad q_{\mu 2,2}\,q_{\nu 2,1}\,q^{\mu}_{2,2}\,q^{\nu}_{1,2}-q_{\mu 1,1}\,q_{\nu 1,2}\,q^{\mu}_{1,1}\,q^{\nu}_{2,1}+q_{\mu 2,2}\,q_{\nu 1,2}\,q^{\mu}_{1,1}\,q^{\nu}_{2,1}+q_{\mu 1,1}\,q_{\nu 1,1}\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}-\\
&\quad q_{\mu 2,2}\,q_{\nu 1,1}\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}-q_{\mu 1,1}\,q_{\nu 2,2}\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}+q_{\mu 2,2}\,q_{\nu 2,2}\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}+q_{\mu 1,1}\,q_{\nu 1,2}\,q^{\mu}_{2,2}\,q^{\nu}_{2,1}-\\
&\quad q_{\mu 2,2}\,q_{\nu 1,2}\,q^{\mu}_{2,2}\,q^{\nu}_{2,1}-i\,q_{\mu 1,1}\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q^{\mu}_{1,2}]+i\,q_{\mu 2,2}\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q^{\mu}_{1,2}]+i\,q_{\mu 1,1}\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q^{\mu}_{2,1}]-\\
&\quad i\,q_{\mu 2,2}\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q^{\mu}_{2,1}]+q_{\mu 2,1}\,(q_{\nu 2,2}\,q^{\mu}_{1,2}\,q^{\nu}_{1,1}-2\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,q^{\nu}_{1,2}-q_{\nu 2,2}\,q^{\mu}_{1,2}\,q^{\nu}_{2,2}+\\
&\qquad q_{\nu 1,1}\,q^{\mu}_{1,2}\,(-q^{\nu}_{1,1}+q^{\nu}_{2,2})+q_{\nu 1,2}\,(2\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}+q^{\mu}_{1,1}\,(q^{\nu}_{1,1}-q^{\nu}_{2,2})+q^{\mu}_{2,2}\,(-q^{\nu}_{1,1}+q^{\nu}_{2,2}))-\\
&\qquad i\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q^{\mu}_{1,1}]+i\,q^{\nu}_{1,1}\,\underline{\partial}_{\nu}[q^{\mu}_{1,2}]-i\,q^{\nu}_{2,2}\,\underline{\partial}_{\nu}[q^{\mu}_{1,2}]+i\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q^{\mu}_{2,2}])+\\
&\quad q_{\mu 1,2}\,(q_{\nu 2,2}\,q^{\mu}_{2,1}\,q^{\nu}_{1,1}-2\,q_{\nu 1,2}\,q^{\mu}_{2,1}\,q^{\nu}_{2,1}-q_{\nu 2,2}\,q^{\mu}_{2,1}\,q^{\nu}_{2,2}+q_{\nu 1,1}\,q^{\mu}_{2,1}\,(-q^{\nu}_{1,1}+q^{\nu}_{2,2})+\\
&\qquad q_{\nu 2,1}\,(2\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}+q^{\mu}_{1,1}\,(q^{\nu}_{1,1}-q^{\nu}_{2,2})+q^{\mu}_{2,2}\,(-q^{\nu}_{1,1}+q^{\nu}_{2,2}))+i\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q^{\mu}_{1,1}]-\\
&\qquad i\,q^{\nu}_{1,1}\,\underline{\partial}_{\nu}[q^{\mu}_{2,1}]+i\,q^{\nu}_{2,2}\,\underline{\partial}_{\nu}[q^{\mu}_{2,1}]-i\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q^{\mu}_{2,2}])-2\,i\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,\underline{\partial}_{\mu}[q^{\nu}_{1,1}]+\\
&\quad 2\,i\,q_{\nu 1,2}\,q^{\mu}_{2,1}\,\underline{\partial}_{\mu}[q^{\nu}_{1,1}]+2\,\underline{\partial}_{\nu}[q^{\mu}_{1,1}]\,\underline{\partial}_{\mu}[q^{\nu}_{1,1}]+2\,i\,q_{\nu 2,1}\,q^{\mu}_{1,1}\,\underline{\partial}_{\mu}[q^{\nu}_{1,2}]-2\,i\,q_{\nu 1,1}\,q^{\mu}_{2,1}\,\underline{\partial}_{\mu}[q^{\nu}_{1,2}]+\\
&\quad 2\,i\,q_{\nu 2,2}\,q^{\mu}_{2,1}\,\underline{\partial}_{\mu}[q^{\nu}_{1,2}]-2\,i\,q_{\nu 2,1}\,q^{\mu}_{2,2}\,\underline{\partial}_{\mu}[q^{\nu}_{1,2}]+2\,\underline{\partial}_{\nu}[q^{\mu}_{2,1}]\,\underline{\partial}_{\mu}[q^{\nu}_{1,2}]-\\
&\quad 2\,i\,q_{\nu 1,2}\,q^{\mu}_{1,1}\,\underline{\partial}_{\mu}[q^{\nu}_{2,1}]+2\,i\,q_{\nu 1,1}\,q^{\mu}_{1,2}\,\underline{\partial}_{\mu}[q^{\nu}_{2,1}]-2\,i\,q_{\nu 2,2}\,q^{\mu}_{1,2}\,\underline{\partial}_{\mu}[q^{\nu}_{2,1}]+\\
&\quad 2\,i\,q_{\nu 1,2}\,q^{\mu}_{2,2}\,\underline{\partial}_{\mu}[q^{\nu}_{2,1}]+2\,\underline{\partial}_{\nu}[q^{\mu}_{1,2}]\,\underline{\partial}_{\mu}[q^{\nu}_{2,1}]+2\,i\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,\underline{\partial}_{\mu}[q^{\nu}_{2,2}]-\\
&\quad 2\,i\,q_{\nu 1,2}\,q^{\mu}_{2,1}\,\underline{\partial}_{\mu}[q^{\nu}_{2,2}]+2\,\underline{\partial}_{\nu}[q^{\mu}_{2,2}]\,\underline{\partial}_{\mu}[q^{\nu}_{2,2}]+i\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,\underline{\partial}^{\nu}[q_{\mu 1,1}]-\\
&\quad i\,q_{\nu 1,2}\,q^{\mu}_{2,1}\,\underline{\partial}^{\nu}[q_{\mu 1,1}]-\underline{\partial}_{\nu}[q^{\mu}_{1,1}]\,\underline{\partial}^{\nu}[q_{\mu 1,1}]-i\,q_{\nu 2,1}\,q^{\mu}_{1,1}\,\underline{\partial}^{\nu}[q_{\mu 1,2}]+i\,q_{\nu 1,1}\,q^{\mu}_{2,1}\,\underline{\partial}^{\nu}[q_{\mu 1,2}]-\\
&\quad i\,q_{\nu 2,2}\,q^{\mu}_{2,1}\,\underline{\partial}^{\nu}[q_{\mu 1,2}]+i\,q_{\nu 2,1}\,q^{\mu}_{2,2}\,\underline{\partial}^{\nu}[q_{\mu 1,2}]-\underline{\partial}_{\nu}[q^{\mu}_{2,1}]\,\underline{\partial}^{\nu}[q_{\mu 1,2}]+i\,q_{\nu 1,2}\,q^{\mu}_{1,1}\,\underline{\partial}^{\nu}[q_{\mu 2,1}]-\\
&\quad i\,q_{\nu 1,1}\,q^{\mu}_{1,2}\,\underline{\partial}^{\nu}[q_{\mu 2,1}]+i\,q_{\nu 2,2}\,q^{\mu}_{1,2}\,\underline{\partial}^{\nu}[q_{\mu 2,1}]-i\,q_{\nu 1,2}\,q^{\mu}_{2,2}\,\underline{\partial}^{\nu}[q_{\mu 2,1}]-\\
&\quad \underline{\partial}_{\nu}[q^{\mu}_{1,2}]\,\underline{\partial}^{\nu}[q_{\mu 2,1}]-i\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,\underline{\partial}^{\nu}[q_{\mu 2,2}]+i\,q_{\nu 1,2}\,q^{\mu}_{2,1}\,\underline{\partial}^{\nu}[q_{\mu 2,2}]-\underline{\partial}_{\nu}[q^{\mu}_{2,2}]\,\underline{\partial}^{\nu}[q_{\mu 2,2}]-\\
&\quad \underline{\partial}_{\mu}[q^{\nu}_{1,1}]\,\underline{\partial}^{\mu}[q_{\nu 1,1}]-\underline{\partial}_{\mu}[q^{\nu}_{2,1}]\,\underline{\partial}^{\mu}[q_{\nu 1,2}]-\underline{\partial}_{\mu}[q^{\nu}_{1,2}]\,\underline{\partial}^{\mu}[q_{\nu 2,1}]-\underline{\partial}_{\mu}[q^{\nu}_{2,2}]\,\underline{\partial}^{\mu}[q_{\nu 2,2}])
\end{aligned}
$$

Too many terms to find text relationship directly. Compare with direct computation of

$$\to Q_{\mu\nu}.Q^{\mu\nu} \to (i\,[Q_{\mu},Q_{\nu}]_{-}-\underline{\partial}_{\nu}[Q_{\mu}]+\underline{\partial}_{\mu}[Q_{\nu}]).(i\,[Q^{\mu},Q^{\nu}]_{-}-\underline{\partial}^{\nu}[Q^{\mu}]+\underline{\partial}^{\mu}[Q^{\nu}])$$

....

$$
\begin{aligned}
&\to \mathrm{Tr}[Q_{\mu\nu}.Q^{\mu\nu}] \to q_{\mu 2,2}\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,q^{\nu}_{1,1}+q_{\mu 1,2}\,q_{\nu 1,1}\,q^{\mu}_{2,1}\,q^{\nu}_{1,1}+\\
&\quad q_{\mu 2,2}\,q_{\nu 1,2}\,q^{\mu}_{2,1}\,q^{\nu}_{1,1}-q_{\mu 1,2}\,q_{\nu 2,2}\,q^{\mu}_{2,1}\,q^{\nu}_{1,1}-q_{\mu 2,2}\,q_{\nu 2,1}\,q^{\mu}_{1,1}\,q^{\nu}_{1,2}+2\,q_{\mu 1,2}\,q_{\nu 2,1}\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}+\\
&\quad q_{\mu 2,2}\,q_{\nu 2,1}\,q^{\mu}_{2,2}\,q^{\nu}_{1,2}-q_{\mu 1,2}\,q_{\nu 1,1}\,q^{\mu}_{1,1}\,q^{\nu}_{2,1}-q_{\mu 2,2}\,q_{\nu 1,2}\,q^{\mu}_{1,1}\,q^{\nu}_{2,1}+q_{\mu 1,2}\,q_{\nu 2,2}\,q^{\mu}_{1,1}\,q^{\nu}_{2,1}-\\
&\quad 2\,q_{\mu 1,2}\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}+q_{\mu 1,2}\,q_{\nu 1,1}\,q^{\mu}_{2,2}\,q^{\nu}_{2,1}+q_{\mu 2,2}\,q_{\nu 1,2}\,q^{\mu}_{2,2}\,q^{\nu}_{2,1}-q_{\mu 1,2}\,q_{\nu 2,2}\,q^{\mu}_{2,2}\,q^{\nu}_{2,1}-\\
&\quad q_{\mu 2,2}\,q_{\nu 2,1}\,q^{\mu}_{1,2}\,q^{\nu}_{2,2}-q_{\mu 1,2}\,q_{\nu 1,1}\,q^{\mu}_{2,1}\,q^{\nu}_{2,2}-q_{\mu 2,2}\,q_{\nu 1,2}\,q^{\mu}_{2,1}\,q^{\nu}_{2,2}+q_{\mu 1,2}\,q_{\nu 2,2}\,q^{\mu}_{2,1}\,q^{\nu}_{2,2}+\\
&\quad i\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q_{\mu 1,1}]-i\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q_{\mu 1,1}]-i\,q^{\mu}_{2,1}\,q^{\nu}_{1,1}\,\underline{\partial}_{\nu}[q_{\mu 1,2}]+i\,q^{\mu}_{1,1}\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q_{\mu 1,2}]-\\
&\quad i\,q^{\mu}_{2,2}\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q_{\mu 1,2}]+i\,q^{\mu}_{2,1}\,q^{\nu}_{2,2}\,\underline{\partial}_{\nu}[q_{\mu 1,2}]+i\,q^{\mu}_{1,2}\,q^{\nu}_{1,1}\,\underline{\partial}_{\nu}[q_{\mu 2,1}]-i\,q^{\mu}_{1,1}\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q_{\mu 2,1}]+\\
&\quad i\,q^{\mu}_{2,2}\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q_{\mu 2,1}]-i\,q^{\mu}_{1,2}\,q^{\nu}_{2,2}\,\underline{\partial}_{\nu}[q_{\mu 2,1}]-i\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}\,\underline{\partial}_{\nu}[q_{\mu 2,2}]+i\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}\,\underline{\partial}_{\nu}[q_{\mu 2,2}]-\\
&\quad i\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}\,\underline{\partial}_{\mu}[q_{\nu 1,1}]+i\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}\,\underline{\partial}_{\mu}[q_{\nu 1,1}]+i\,q^{\mu}_{2,1}\,q^{\nu}_{1,1}\,\underline{\partial}_{\mu}[q_{\nu 1,2}]-i\,q^{\mu}_{1,1}\,q^{\nu}_{2,1}\,\underline{\partial}_{\mu}[q_{\nu 1,2}]+\\
&\quad i\,q^{\mu}_{2,2}\,q^{\nu}_{2,1}\,\underline{\partial}_{\mu}[q_{\nu 1,2}]-i\,q^{\mu}_{2,1}\,q^{\nu}_{2,2}\,\underline{\partial}_{\mu}[q_{\nu 1,2}]-i\,q^{\mu}_{1,2}\,q^{\nu}_{1,1}\,\underline{\partial}_{\mu}[q_{\nu 2,1}]+i\,q^{\mu}_{1,1}\,q^{\nu}_{1,2}\,\underline{\partial}_{\mu}[q_{\nu 2,1}]-\\
&\quad i\,q^{\mu}_{2,2}\,q^{\nu}_{1,2}\,\underline{\partial}_{\mu}[q_{\nu 2,1}]+i\,q^{\mu}_{1,2}\,q^{\nu}_{2,2}\,\underline{\partial}_{\mu}[q_{\nu 2,1}]+i\,q^{\mu}_{2,1}\,q^{\nu}_{1,2}\,\underline{\partial}_{\mu}[q_{\nu 2,2}]-i\,q^{\mu}_{1,2}\,q^{\nu}_{2,1}\,\underline{\partial}_{\mu}[q_{\nu 2,2}]-\\
&\quad i\,q_{\mu 1,2}\,q_{\nu 2,1}\,\underline{\partial}^{\nu}[q^{\mu}_{1,1}]+\underline{\partial}_{\nu}[q_{\mu 1,1}]\,\underline{\partial}^{\nu}[q^{\mu}_{1,1}]-\underline{\partial}_{\mu}[q_{\nu 1,1}]\,\underline{\partial}^{\nu}[q^{\mu}_{1,1}]-i\,q_{\mu 2,2}\,q_{\nu 2,1}\,\underline{\partial}^{\nu}[q^{\mu}_{1,2}]+\\
&\quad \underline{\partial}_{\nu}[q_{\mu 2,1}]\,\underline{\partial}^{\nu}[q^{\mu}_{1,2}]-\underline{\partial}_{\mu}[q_{\nu 2,1}]\,\underline{\partial}^{\nu}[q^{\mu}_{1,2}]+i\,q_{\mu 1,2}\,q_{\nu 1,1}\,\underline{\partial}^{\nu}[q^{\mu}_{2,1}]+i\,q_{\mu 2,2}\,q_{\nu 1,2}\,\underline{\partial}^{\nu}[q^{\mu}_{2,1}]-\\
&\quad i\,q_{\mu 1,2}\,q_{\nu 2,2}\,\underline{\partial}^{\nu}[q^{\mu}_{2,1}]+\underline{\partial}_{\nu}[q_{\mu 1,2}]\,\underline{\partial}^{\nu}[q^{\mu}_{2,1}]-\underline{\partial}_{\mu}[q_{\nu 1,2}]\,\underline{\partial}^{\nu}[q^{\mu}_{2,1}]+i\,q_{\mu 1,2}\,q_{\nu 2,1}\,\underline{\partial}^{\nu}[q^{\mu}_{2,2}]+\\
&\quad \underline{\partial}_{\nu}[q_{\mu 2,2}]\,\underline{\partial}^{\nu}[q^{\mu}_{2,2}]-\underline{\partial}_{\mu}[q_{\nu 2,2}]\,\underline{\partial}^{\nu}[q^{\mu}_{2,2}]+i\,q_{\mu 1,2}\,q_{\nu 2,1}\,\underline{\partial}^{\mu}[q^{\nu}_{1,1}]-\underline{\partial}_{\nu}[q_{\mu 1,1}]\,\underline{\partial}^{\mu}[q^{\nu}_{1,1}]+\\
&\quad \underline{\partial}_{\mu}[q_{\nu 1,1}]\,\underline{\partial}^{\mu}[q^{\nu}_{1,1}]+i\,q_{\mu 2,2}\,q_{\nu 2,1}\,\underline{\partial}^{\mu}[q^{\nu}_{1,2}]-\underline{\partial}_{\nu}[q_{\mu 2,1}]\,\underline{\partial}^{\mu}[q^{\nu}_{1,2}]+\underline{\partial}_{\mu}[q_{\nu 2,1}]\,\underline{\partial}^{\mu}[q^{\nu}_{1,2}]+\\
&\quad q_{\mu 1,1}\,(q_{\nu 2,1}\,(q^{\mu}_{1,1}\,q^{\nu}_{1,2}-q^{\mu}_{2,2}\,q^{\nu}_{1,2}+q^{\mu}_{1,2}\,(-q^{\nu}_{1,1}+q^{\nu}_{2,2})+i\,\underline{\partial}^{\nu}[q^{\mu}_{1,2}]-i\,\underline{\partial}^{\mu}[q^{\nu}_{1,2}])+\\
&\qquad q_{\nu 1,2}\,(q^{\mu}_{1,1}\,q^{\nu}_{2,1}-q^{\mu}_{2,2}\,q^{\nu}_{2,1}+q^{\mu}_{2,1}\,(-q^{\nu}_{1,1}+q^{\nu}_{2,2})-i\,\underline{\partial}^{\nu}[q^{\mu}_{2,1}]+i\,\underline{\partial}^{\mu}[q^{\nu}_{2,1}]))-\\
&\quad i\,q_{\mu 1,2}\,q_{\nu 1,1}\,\underline{\partial}^{\mu}[q^{\nu}_{2,1}]-i\,q_{\mu 2,2}\,q_{\nu 1,2}\,\underline{\partial}^{\mu}[q^{\nu}_{2,1}]+i\,q_{\mu 1,2}\,q_{\nu 2,2}\,\underline{\partial}^{\mu}[q^{\nu}_{2,1}]-\underline{\partial}_{\nu}[q_{\mu 1,2}]\,\underline{\partial}^{\mu}[q^{\nu}_{2,1}]+\\
&\quad \underline{\partial}_{\mu}[q_{\nu 1,2}]\,\underline{\partial}^{\mu}[q^{\nu}_{2,1}]-i\,q_{\mu 1,2}\,q_{\nu 2,1}\,\underline{\partial}^{\mu}[q^{\nu}_{2,2}]-\underline{\partial}_{\nu}[q_{\mu 2,2}]\,\underline{\partial}^{\mu}[q^{\nu}_{2,2}]+\underline{\partial}_{\mu}[q_{\nu 2,2}]\,\underline{\partial}^{\mu}[q^{\nu}_{2,2}]+\\
&\quad q_{\mu 2,1}\,(q_{\nu 2,2}\,(q^{\mu}_{1,1}\,q^{\nu}_{1,2}-q^{\mu}_{2,2}\,q^{\nu}_{1,2}+q^{\mu}_{1,2}\,(-q^{\nu}_{1,1}+q^{\nu}_{2,2})+i\,\underline{\partial}^{\nu}[q^{\mu}_{1,2}]-i\,\underline{\partial}^{\mu}[q^{\nu}_{1,2}])+
\end{aligned}
$$

$q_{\nu 1,1}\,(-q^{\mu}{}_{1,1}\,q^{\nu}{}_{1,2} + q^{\mu}{}_{2,2}\,q^{\nu}{}_{1,2} + q^{\mu}{}_{1,2}\,(q^{\nu}{}_{1,1} - q^{\nu}{}_{2,2}) - \mathbb{i}\,\underline{\partial}^{\nu}[q^{\mu}{}_{1,2}] + \mathbb{i}\,\underline{\partial}^{\mu}[q^{\nu}{}_{1,2}]) +$

$q_{\nu 1,2}\,(-2\,q^{\mu}{}_{2,1}\,q^{\nu}{}_{1,2} + 2\,q^{\mu}{}_{1,2}\,q^{\nu}{}_{2,1} + \mathbb{i}\,(\underline{\partial}^{\nu}[q^{\mu}{}_{1,1}] - \underline{\partial}^{\nu}[q^{\mu}{}_{2,2}] - \underline{\partial}^{\mu}[q^{\nu}{}_{1,1}] + \underline{\partial}^{\mu}[q^{\nu}{}_{2,2}])))$

Comparing $2\,\mathrm{Tr}[Q_{\mu\nu}.Q^{\mu\nu}]$ with FF calculation ⇒ $\boxed{\text{True}}$

Lemma 5.5

```
PR["●Lemma 5.5: ",
 $155 = $ = {Tr[Φ^2] → 4 a Abs[H']^2 + 2 c,
    Tr[Φ^4] → 4 b Abs[H']^4 + 8 e Abs[H']^2 + 2 d,
    H' → {ϕ1 + 1, ϕ2},
    a → Abs[Yν]^2 + Abs[Ye]^2,
    b → Abs[Yν]^4 + Abs[Ye]^4,
    c -> Abs[YR]^2, d -> Abs[YR]^4, e -> Abs[YR]^2 Abs[Yν]^2
   }; ColumnBar[$],
 $155a = Association[$155];
 Imply, $155x = $ = {
     Abs[H']^2 → (H'.Conjugate[H'] /. $155) // FullSimplify // Reverse,
     t[RL_]i_,j_ :→ 0 /; i ≠ 1 || j ≠ 1 || RL == L,
     t[_]i,j[CG["GWS basis"]]
    } /. Re[x_] → (x + Conjugate[x]) / 2; $ // ColumnBar,
 NL, "The requirement ", tuRuleSelect[$defGWS][T.νR],
 Yield,
 $ = ($ = ct[T].T) → ($ /. tuRuleSelect[$defGWS][T][[-1]] /. tuRule[$155x] // Simplify);
 MatrixForms[$],
 Yield, $ = $[[2, 1, 1]] -> Abs[YR]^2; $ // Framed,
 AppendTo[$155x, $];

 line,
 NL, "Proof: ",
 NL, "Use the 8x8 ℋF8 representation of: ",
 xtmp = $ = tuRuleSelect[$defGWS][Φ] // Select[#, tuHasAllQ[#, S] &] & // First;
 $ // MatrixForms,
 NL, "Using ", $s = {tuRuleSelect[$defGWS][S] // Select[#, tuHasAllQ[#, ν] &] &,
     tuRuleSelect[$defGWS][ϕ] // Select[#, tuHasAllQ[#, ν] &] & // First,
     tuRuleSelect[$defGWS][T] // Select[#, tuHasAllQ[#, 2] &] & // Last} /.
    tuRule[$155x] // Flatten;
 $s // MatrixForms,
 Yield, $[[2]] = $[[2]] /. $s // ArrayFlatten;
 ($sΦ1 = $) // MatrixForms,

 next, "Compute: ", $01 = $ = Inactive[Tr][Φ.Φ], "POFF",
 Yield, $ = $ /. $sΦ1; MatrixForms[$];
 Yield, $ = $01 -> $ // Activate // FullSimplify,
 Yield, $ = $ //. tuRule[($155x // FullSimplify)] // Simplify,
 Yield, $ = {$, $155[[{-5, -3}]]} // Flatten; $ // ColumnBar,
 Yield, $ = tuEliminate[$, {Abs[Ye]^2, Abs[Yν]^2}] /. And → List; "PONdd",
 Yield, $ = $ // tuRuleSolve[#, Tr[Φ.Φ]] & // First // (# /. $155[[{-5, -3}]] &);
 $ // Framed
]
```

●Lemma 5.5: $\left|\begin{array}{l} \text{Tr}[\Phi^2] \to 2\,c + 4\,a\,\text{Abs}[H']^2 \\ \text{Tr}[\Phi^4] \to 2\,d + 8\,e\,\text{Abs}[H']^2 + 4\,b\,\text{Abs}[H']^4 \\ H' \to \{1+\phi_1,\ \phi_2\} \\ a \to \text{Abs}[Y_e]^2 + \text{Abs}[Y_\nu]^2 \\ b \to \text{Abs}[Y_e]^4 + \text{Abs}[Y_\nu]^4 \\ c \to \text{Abs}[Y_R]^2 \\ d \to \text{Abs}[Y_R]^4 \\ e \to \text{Abs}[Y_R]^2\,\text{Abs}[Y_\nu]^2 \end{array}\right.$

⇒ $\left|\begin{array}{l} 1 + \text{Abs}[\phi_1]^2 + \text{Abs}[\phi_2]^2 + (\phi_1)^* + \phi_1 \to \text{Abs}[H']^2 \\ t[RL\_]_{i\_,j\_} :\to 0\ /;\ i \neq 1\ ||\ j \neq 1\ ||\ RL == L \\ t[\_]_{i,j}[\text{GWS basis}] \end{array}\right.$

The requirement $\{T.\nu_R \to Y_R.\overline{\nu_R}\}$

→ $T^\dagger.T \to \begin{pmatrix} (t[R]_{1,1})^*\,t[R]_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (t[R]_{1,1})^*\,t[R]_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

→ $\boxed{(t[R]_{1,1})^*\,t[R]_{1,1} \to \text{Abs}[Y_R]^2}$

Proof:

Use the 8x8 $\mathcal{H}_{F_8}$ representation of: $\Phi \to \begin{pmatrix} S+\phi & T^* \\ T & S^*+\phi^* \end{pmatrix}$

Using

$\left\{S \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix},\ \phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 \\ 0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^*\,(\phi_1)^* \\ (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 \\ (\phi_2)^*\,Y_\nu & Y_e\,\phi_1 & 0 & 0 \end{pmatrix},\ T \to \begin{pmatrix} t[R]_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$

→ $\Phi \to \left(\begin{array}{lllllll} 0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 & (t[R]_{1,1})^* & 0 & 0 \\ 0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^* + (Y_e)^*\,(\phi_1)^* & 0 & 0 & 0 \\ Y_\nu + (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^*\,Y_\nu & Y_e + Y_e\,\phi_1 & 0 & 0 & 0 & 0 & 0 \\ t[R]_{1,1} & 0 & 0 & 0 & 0 & 0 & Y_\nu + (\phi_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y_e\,\phi \\ 0 & 0 & 0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & -(Y_e\,\phi_2)^* & 0 \\ 0 & 0 & 0 & 0 & (Y_\nu)^*\,\phi_2 & (Y_e)^* + (Y_e\,\phi_1)^* & 0 \end{array}\right.$

◆Compute: $\text{Tr}[\Phi.\Phi]$

....

→ $\boxed{\text{Tr}[\Phi.\Phi] \to 2\,(\text{Abs}[Y_R]^2 + 2\,(\text{Abs}[Y_e]^2 + \text{Abs}[Y_\nu]^2)\,\text{Abs}[H']^2)}$

```
$sexp = {Conjugate[a_ b_] → Conjugate[a] Conjugate[b], Abs[a_ b_] → Abs[a] Abs[b],
   a_ Conjugate[a_] → Abs[a]^2, a_^2 Conjugate[a_]^2 → Abs[a]^4}
PR[next, "In the same way Compute: ", $01 = $ = Inactive[Tr][Φ.Φ.Φ.Φ],
  Yield, $ = $ /. $sΦ1 /. tX[R | L]_,_ → Y_R // Activate // Simplify;
  Yield, $ = Expand[$] //. tuRule[$155x] //. $sexp;
  Yield, $ = $01 -> $ //. tuRule[$155x] // tuTrSimplify[{Abs[_]}] // Simplify;
  Yield, $ = $ /.
       tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] /.
      tuRuleSolve[$155x[[1]], $155x[[1, 1, 2]]] // Simplify;
  Yield, $ = $ /. tuRuleSolve[$155x[[1]], $155x[[1, 1, 4]]] /.
      tuRuleSolve[$155x[[1]], $155x[[1, 1, 3]]] // Simplify;
  Yield, $ = $ /. (#^2 & /@ tuRuleSolve[$155x[[1]], $155x[[1, 1, 3]]][[1]] // Expand) /.
      $sexp // Simplify;
  Yield, $ = $ /. $sexp /. tuRuleSolve[$155x[[1]], $155x[[1, 1, 2]]];
  Yield, $ =
   $ /. tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] // Collect[#, Abs[H'], Simplify] &;
  ColumnSumExp[$];
  Yield,
  $trpppp = $ = $ /. tuRuleSolve[Expand[#^2 & /@ $155x[[-1]]], Abs[t[L]_1,1]^4] // Collect[#,
        {Abs[H'], Tr[Abs[Y_R]^2], Conjugate[Y_ν], Y_ν, t[L]_1,1, Abs[t[R]_1,1]}, Simplify] &;
  ColumnSumExp[$] // Framed
  (*t[L]_1,1 → in this case.*)
 ];
```

{(a_ b_)* → a* b*, Abs[a_ b_] → Abs[a] Abs[b], a_* a_ → Abs[a]^2, a_^{*2} a_^{2} → Abs[a]^4}

◆In the same way Compute: Tr[Φ.Φ.Φ.Φ]
→
→
→
→
→
→
→
→

→ $\boxed{\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] \to \sum\left[\begin{array}{l} 2\,\mathrm{Abs}[t[R]_{1,1}]^4 \\ 8\,\mathrm{Abs}[Y_R]^2\,\mathrm{Abs}[Y_\nu]^2\,\mathrm{Abs}[H']^2 \\ 4\,(\mathrm{Abs}[Y_e]^4 + \mathrm{Abs}[Y_\nu]^4)\,\mathrm{Abs}[H']^4 \end{array}\right]}$

Lemma 5.6

```
PR["●Lemma 5.6. ",
 $156 = $ = {Tr[tuDDown[iD][Φ, μ] tuDUp[iD][Φ, μ]] → 4 a Abs[tuDDown[iD̃][H', μ]]^2,
    tuDDown[iD̃][H', μ] → tuDDown["∂"][H', μ] +
      I xSum[T[Q, "du", {μ, j}] T[σ, "d", {j}], {j, 3}].H' - I T[Λ, "d", {μ}].H',
    $e31 = tuDDown[iD][Φ, μ] → tuDPartial[Φ, μ] + I CommutatorM[T[B, "d", {μ}], Φ],
    tuRuleSelect[$defGWS][Φ] // Select[#, tuHasAllQ[#, S] &] & // First,
    H' → {ϕ1 + 1, ϕ2},
    T[Q, "d", {μ}] -> xSum[T[Q, "du", {μ, j}] T[σ, "d", {j}], {j, 3}],
    T[Q, "du", {μ, j}][CG["ℝ"]]

   }; ColumnBar[$], accumGWS[$],
 NL, "Recall ", $156[[3, 1]], back, tuRuleSelect[$defall][T[𝒟, "d", {μ}][Φ]][[1]],

 next, " In 8x8 space Calculate ", $ = $156[[3, 2, 1]], "xPOFF",
 NL, "Use: ", $s = {$sΦ1, $e58}; $s // MatrixForms // ColumnBar,
 Yield, $part[1] = $ = $ // expandCom[$s] // Simplify;
 MatrixForms[$], CK,
 "PON",
 next, " Calculate ", $ = $156[[3, 2, 2]], "POFF",
 Yield, $ = $ /. $s //. tt : tuDDown["∂"][a_, b_] :→ Thread[tt];
 Yield, $ = $ // tuDerivativeExpand[{μ, ν}], "PONdd",
 Yield, $ // MatrixForms,

 NL, "Summing: ",
 Yield, $d = $ = $e31[[1]] -> $part[1] + $ // Simplify; MatrixForms[$], CK,

 Yield, $u = $d // tuIndicesRaise[{ν, μ}];
 NL, "Compute: ", $ = Thread[$u $d, Rule] // Simplify,
 NL, "The Tr[] is: ", $ = Tr /@ $; $ // Framed, CR[" BUG?"]

]
```

●Lemma 5.6.

$\mathrm{Tr}[\underset{-\mu}{D}[\Phi]\,\underset{-}{D}^{\mu}[\Phi]] \to 4\,a\,\mathrm{Abs}[\underset{-\mu}{\tilde{D}}[H']]^2$

$\underset{-\mu}{\tilde{D}}[H'] \to -i\,\Lambda_\mu.H' + i \sum_{\{\bar{j},3\}} [Q_\mu{}^j\,\sigma_j].H' + \underset{-\mu}{\partial}[H']$

$\underset{-\mu}{D}[\Phi] \to i\,[B_\mu, \Phi]_- + \underset{-\mu}{\partial}[\Phi]$

$\Phi \to \{\{S + \phi, T^*\}, \{T, S^* + \phi^*\}\}$

$H' \to \{1 + \phi_1, \phi_2\}$

$Q_\mu \to \sum_{\{\bar{j},3\}} [Q_\mu{}^j\,\sigma_j]$

$Q_\mu{}^j[\mathbb{R}]$

Recall $\underline{D}_\mu[\Phi]$ ⟵ $\mathcal{D}_\mu[\Phi] \to -i\,1_N \otimes [\Phi, B_\mu]_- + 1_N \otimes \nabla^S{}_\mu[\Phi]$

◆ In 8x8 space Calculate $i\,[B_\mu, \Phi]_-$xPOFF

Use:

$$\Phi \to \left(\begin{array}{ccccccc}
0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 & (t[R]_{1,1})^* & 0 & \\
0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^* + (Y_e)^*\,(\phi_1)^* & 0 & 0 & \\
Y_\nu + (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 & 0 & 0 & \\
(\phi_2)^*\,Y_\nu & Y_e + Y_e\,\phi_1 & 0 & 0 & 0 & 0 & \\
t[R]_{1,1} & 0 & 0 & 0 & 0 & 0 & Y_\nu + \\
0 & 0 & 0 & 0 & 0 & 0 & - \\
0 & 0 & 0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & -(Y_e\,\phi_2)^* & \\
0 & 0 & 0 & 0 & (Y_\nu)^*\,\phi_2 & (Y_e)^* + (Y_e\,\phi_1)^* &
\end{array}\right.$$

$$B_\mu \to \left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2\,\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_{\mu 1,1} - \Lambda_\mu & q_{\mu 1,2} & 0 & 0 & 0 & 0 \\
0 & 0 & q_{\mu 2,1} & q_{\mu 2,2} - \Lambda_\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\,\Lambda_\mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 1,1})^* + \Lambda_\mu & -(q_{\mu 1,2})^* \\
0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 2,1})^* & -(q_{\mu 2,2})^* + \Lambda_\mu
\end{array}\right)$$

$$\to \left(\begin{array}{cc}
0 & \\
0 & \\
i\,(q_{\mu 1,1}.Y_\nu - \Lambda_\mu.Y_\nu + q_{\mu 1,1}.(\phi_1)^*.Y_\nu + q_{\mu 1,2}.(\phi_2)^*.Y_\nu - \Lambda_\mu.(\phi_1)^*.Y_\nu) & i\,(q_{\mu 1,2}.Y_e - 2\,Y_e.\phi_2.\Lambda_\mu - q_{\mu 1,1} \\
i\,(q_{\mu 2,1}.Y_\nu + q_{\mu 2,1}.(\phi_1)^*.Y_\nu + q_{\mu 2,2}.(\phi_2)^*.Y_\nu - \Lambda_\mu.(\phi_2)^*.Y_\nu) & i\,(2\,Y_e.\Lambda_\mu + q_{\mu 2,2}.Y_e - \Lambda_\mu.Y_e + 2\,Y_e.\phi_1.\Lambda \\
0 & \\
0 & \\
0 & \\
0 &
\end{array}\right.$$

◆ **Calculate** $\underline{\partial}_\mu[\Phi]$

....

$$\to \left(\begin{array}{cccc}
0 & 0 & \partial_{\underline{\mu}}[Y_\nu]^* + \partial_{\underline{\mu}}[Y_\nu]^*\,\phi_1 + (Y_\nu)^*\,\partial_{\underline{\mu}}[\phi_1] & \\
0 & 0 & -(\phi_2)^*\,\partial_{\underline{\mu}}[Y_e]^* - (Y_e)^*\,\partial_{\underline{\mu}}[\phi_2]^* & \partial_{\underline{\mu}}[Y_e] \\
\partial_{\underline{\mu}}[\phi_1]^*\,Y_\nu + \partial_{\underline{\mu}}[Y_\nu] + (\phi_1)^*\,\partial_{\underline{\mu}}[Y_\nu] & -\phi_2\,\partial_{\underline{\mu}}[Y_e] - Y_e\,\partial_{\underline{\mu}}[\phi_2] & 0 & \\
\partial_{\underline{\mu}}[\phi_2]^*\,Y_\nu + (\phi_2)^*\,\partial_{\underline{\mu}}[Y_\nu] & \partial_{\underline{\mu}}[Y_e] + \phi_1\,\partial_{\underline{\mu}}[Y_e] + Y_e\,\partial_{\underline{\mu}}[\phi_1] & 0 & \\
\partial_{\underline{\mu}}[t[R]_{1,1}] & 0 & 0 & \\
0 & 0 & 0 & \\
0 & 0 & 0 & \\
0 & 0 & 0 &
\end{array}\right.$$

Summing:

$$\to \underline{D}_\mu[\Phi] \to \left(\begin{array}{c}
0 \\
0 \\
i\,(q_{\mu 1,1}.Y_\nu - \Lambda_\mu.Y_\nu + q_{\mu 1,1}.(\phi_1)^*.Y_\nu + q_{\mu 1,2}.(\phi_2)^*.Y_\nu - \Lambda_\mu.(\phi_1)^*.Y_\nu) + \partial_{\underline{\mu}}[\phi_1]^*\,Y_\nu + \partial_{\underline{\mu}}[Y_\nu] + (\phi_1)^*\,\partial_{\underline{\mu}}[\\
i\,(q_{\mu 2,1}.Y_\nu + q_{\mu 2,1}.(\phi_1)^*.Y_\nu + q_{\mu 2,2}.(\phi_2)^*.Y_\nu - \Lambda_\mu.(\phi_2)^*.Y_\nu) + \partial_{\underline{\mu}}[\phi_2]^*\,Y_\nu + (\phi_2)^*\,\partial_{\underline{\mu}}[Y_\nu] \\
\partial_{\underline{\mu}}[t[R]_{1,1}] \\
0 \\
0 \\
0
\end{array}\right.$$

→

Compute:

$\underline{D}_\mu[\Phi]\,\underline{D}^\mu[\Phi] \to \{\{0,\ 0,\ (\underline{\partial}_\mu[Y_\nu]^*\,(1+\phi_1) - i\,((Y_\nu)^*.q_{\mu 1,1} - (Y_\nu)^*.\Lambda_\mu + (Y_\nu)^*.\phi_1.q_{\mu 1,1} - (Y_\nu)^*.\phi_1.\Lambda_\mu +$

$(Y_\nu)^*.\phi_2.q_{\mu 2,1} + i\,(Y_\nu)^*\,\underline{\partial}_\mu[\phi_1]))\ (\underline{\partial}^\mu[Y_\nu]^*\,(1+\phi_1) -$

$i\,((Y_\nu)^*.q^\mu{}_{1,1} - (Y_\nu)^*.\Lambda^\mu + (Y_\nu)^*.\phi_1.q^\mu{}_{1,1} - (Y_\nu)^*.\phi_1.\Lambda^\mu + (Y_\nu)^*.\phi_2.q^\mu{}_{2,1} + i\,(Y_\nu)^*\,\underline{\partial}^\mu[\phi_1])),$

$(-i\,((Y_\nu)^*.q_{\mu 1,2} + (Y_\nu)^*.\phi_1.q_{\mu 1,2} + (Y_\nu)^*.\phi_2.q_{\mu 2,2} - (Y_\nu)^*.\phi_2.\Lambda_\mu) + \underline{\partial}_\mu[Y_\nu]^*\,\phi_2 + (Y_\nu)^*\,\underline{\partial}_\mu[\phi_2])$

$$(-\mathrm{i}\,((Y_\nu)^*.q^\mu{}_{1,2} + (Y_\nu)^*.\phi_1.q^\mu{}_{1,2} + (Y_\nu)^*.\phi_2.q^\mu{}_{2,2} - (Y_\nu)^*.\phi_2.\Lambda^\mu) + \underline{\partial}^\mu[Y_\nu]^*\,\phi_2 + (Y_\nu)^*\,\underline{\partial}^\mu[\phi_2]),$$

$$\underline{\partial}_\mu[t[R]_{1,1}]^*\,\underline{\partial}^\mu[t[R]_{1,1}]^*,\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ (-(\phi_2)^*\,\underline{\partial}_\mu[Y_e]^* - (Y_e)^*\,\underline{\partial}_\mu[\phi_2]^* - \mathrm{i}\,((Y_e)^*.q_{\mu2,1} + (Y_e)^*.(\phi_1)^*.q_{\mu2,1} - (Y_e)^*.(\phi_2)^*.q_{\mu1,1} +$$
$$(Y_e)^*.(\phi_2)^*.\Lambda_\mu - 2\,\Lambda_\mu.(Y_e)^*.(\phi_2)^*))\,(-(\phi_2)^*\,\underline{\partial}^\mu[Y_e]^* - (Y_e)^*\,\underline{\partial}^\mu[\phi_2]^* -$$
$$\mathrm{i}\,((Y_e)^*.q^\mu{}_{2,1} + (Y_e)^*.(\phi_1)^*.q^\mu{}_{2,1} - (Y_e)^*.(\phi_2)^*.q^\mu{}_{1,1} + (Y_e)^*.(\phi_2)^*.\Lambda^\mu - 2\,\Lambda^\mu.(Y_e)^*.(\phi_2)^*)),$$
$$((1 + (\phi_1)^*)\,\underline{\partial}_\mu[Y_e]^* + (Y_e)^*\,\underline{\partial}_\mu[\phi_1]^* - \mathrm{i}\,((Y_e)^*.q_{\mu2,2} - (Y_e)^*.\Lambda_\mu + 2\,\Lambda_\mu.(Y_e)^* +$$
$$(Y_e)^*.(\phi_1)^*.q_{\mu2,2} - (Y_e)^*.(\phi_1)^*.\Lambda_\mu - (Y_e)^*.(\phi_2)^*.q_{\mu1,2} + 2\,\Lambda_\mu.(Y_e)^*.(\phi_1)^*))$$
$$((1 + (\phi_1)^*)\,\underline{\partial}^\mu[Y_e]^* + (Y_e)^*\,\underline{\partial}^\mu[\phi_1]^* - \mathrm{i}\,((Y_e)^*.q^\mu{}_{2,2} - (Y_e)^*.\Lambda^\mu + 2\,\Lambda^\mu.(Y_e)^* + (Y_e)^*.(\phi_1)^*.q^\mu{}_{2,2} -$$
$$(Y_e)^*.(\phi_1)^*.\Lambda^\mu - (Y_e)^*.(\phi_2)^*.q^\mu{}_{1,2} + 2\,\Lambda^\mu.(Y_e)^*.(\phi_1)^*)),\ 0,\ 0,\ 0,\ 0\},$$

$$\{(\mathrm{i}\,(q_{\mu1,1}.Y_\nu - \Lambda_\mu.Y_\nu + q_{\mu1,1}.(\phi_1)^*.Y_\nu + q_{\mu1,2}.(\phi_2)^*.Y_\nu - \Lambda_\mu.(\phi_1)^*.Y_\nu) + \underline{\partial}_\mu[\phi_1]^*\,Y_\nu + \underline{\partial}_\mu[Y_\nu] + (\phi_1)^*\,\underline{\partial}_\mu[Y_\nu])$$
$$(\mathrm{i}\,(q^\mu{}_{1,1}.Y_\nu - \Lambda^\mu.Y_\nu + q^\mu{}_{1,1}.(\phi_1)^*.Y_\nu + q^\mu{}_{1,2}.(\phi_2)^*.Y_\nu - \Lambda^\mu.(\phi_1)^*.Y_\nu) +$$
$$\underline{\partial}^\mu[\phi_1]^*\,Y_\nu + \underline{\partial}^\mu[Y_\nu] + (\phi_1)^*\,\underline{\partial}^\mu[Y_\nu]),$$

$$(\mathrm{i}\,(q_{\mu1,2}.Y_e - 2\,Y_e.\phi_2.\Lambda_\mu - q_{\mu1,1}.Y_e.\phi_2 + q_{\mu1,2}.Y_e.\phi_1 + \Lambda_\mu.Y_e.\phi_2) - \phi_2\,\underline{\partial}_\mu[Y_e] - Y_e\,\underline{\partial}_\mu[\phi_2])$$
$$(\mathrm{i}\,(q^\mu{}_{1,2}.Y_e - 2\,Y_e.\phi_2.\Lambda^\mu - q^\mu{}_{1,1}.Y_e.\phi_2 + q^\mu{}_{1,2}.Y_e.\phi_1 + \Lambda^\mu.Y_e.\phi_2) - \phi_2\,\underline{\partial}^\mu[Y_e] - Y_e\,\underline{\partial}^\mu[\phi_2]),\ 0,\ 0,\ 0,\ 0,$$

$$0,\ 0\},\ \{(\mathrm{i}\,(q_{\mu2,1}.Y_\nu + q_{\mu2,1}.(\phi_1)^*.Y_\nu + q_{\mu2,2}.(\phi_2)^*.Y_\nu - \Lambda_\mu.(\phi_2)^*.Y_\nu) + \underline{\partial}_\mu[\phi_2]^*\,Y_\nu + (\phi_2)^*\,\underline{\partial}_\mu[Y_\nu])$$
$$(\mathrm{i}\,(q^\mu{}_{2,1}.Y_\nu + q^\mu{}_{2,1}.(\phi_1)^*.Y_\nu + q^\mu{}_{2,2}.(\phi_2)^*.Y_\nu - \Lambda^\mu.(\phi_2)^*.Y_\nu) + \underline{\partial}^\mu[\phi_2]^*\,Y_\nu + (\phi_2)^*\,\underline{\partial}^\mu[Y_\nu]),$$

$$(\mathrm{i}\,(2\,Y_e.\Lambda_\mu + q_{\mu2,2}.Y_e - \Lambda_\mu.Y_e + 2\,Y_e.\phi_1.\Lambda_\mu - q_{\mu2,1}.Y_e.\phi_2 + q_{\mu2,2}.Y_e.\phi_1 - \Lambda_\mu.Y_e.\phi_1) + \underline{\partial}_\mu[Y_e] + \phi_1\,\underline{\partial}_\mu[Y_e] +$$
$$Y_e\,\underline{\partial}_\mu[\phi_1])\,(\mathrm{i}\,(2\,Y_e.\Lambda^\mu + q^\mu{}_{2,2}.Y_e - \Lambda^\mu.Y_e + 2\,Y_e.\phi_1.\Lambda^\mu - q^\mu{}_{2,1}.Y_e.\phi_2 + q^\mu{}_{2,2}.Y_e.\phi_1 - \Lambda^\mu.Y_e.\phi_1) +$$
$$\underline{\partial}^\mu[Y_e] + \phi_1\,\underline{\partial}^\mu[Y_e] + Y_e\,\underline{\partial}^\mu[\phi_1]),\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\},\ \{\underline{\partial}_\mu[t[R]_{1,1}]\,\underline{\partial}^\mu[t[R]_{1,1}],\ 0,\ 0,\ 0,\ 0,\ 0,$$

$$(\mathrm{i}\,(Y_\nu.(q_{\mu1,1})^* - Y_\nu.\Lambda_\mu + (\phi_1)^*.Y_\nu.(q_{\mu1,1})^* - (\phi_1)^*.Y_\nu.\Lambda_\mu + (\phi_2)^*.Y_\nu.(q_{\mu2,1})^*) + \underline{\partial}_\mu[\phi_1]^*\,Y_\nu + \underline{\partial}_\mu[Y_\nu] +$$
$$(\phi_1)^*\,\underline{\partial}_\mu[Y_\nu])\,(\mathrm{i}\,(Y_\nu.(q^\mu{}_{1,1})^* - Y_\nu.\Lambda^\mu + (\phi_1)^*.Y_\nu.(q^\mu{}_{1,1})^* - (\phi_1)^*.Y_\nu.\Lambda^\mu + (\phi_2)^*.Y_\nu.(q^\mu{}_{2,1})^*) +$$
$$\underline{\partial}^\mu[\phi_1]^*\,Y_\nu + \underline{\partial}^\mu[Y_\nu] + (\phi_1)^*\,\underline{\partial}^\mu[Y_\nu]),$$

$$(\mathrm{i}\,(Y_\nu.(q_{\mu1,2})^* + (\phi_1)^*.Y_\nu.(q_{\mu1,2})^* + (\phi_2)^*.Y_\nu.(q_{\mu2,2})^* - (\phi_2)^*.Y_\nu.\Lambda_\mu) + \underline{\partial}_\mu[\phi_2]^*\,Y_\nu + (\phi_2)^*\,\underline{\partial}_\mu[Y_\nu])$$
$$(\mathrm{i}\,(Y_\nu.(q^\mu{}_{1,2})^* + (\phi_1)^*.Y_\nu.(q^\mu{}_{1,2})^* + (\phi_2)^*.Y_\nu.(q^\mu{}_{2,2})^* - (\phi_2)^*.Y_\nu.\Lambda^\mu) +$$
$$\underline{\partial}^\mu[\phi_2]^*\,Y_\nu + (\phi_2)^*\,\underline{\partial}^\mu[Y_\nu])\},\ \{0,\ 0,\ 0,\ 0,\ 0,\ 0,$$

$$(\mathrm{i}\,(Y_e.(q_{\mu2,1})^* + Y_e.\phi_1.(q_{\mu2,1})^* - Y_e.\phi_2.(q_{\mu1,1})^* + Y_e.\phi_2.\Lambda_\mu - 2\,\Lambda_\mu.Y_e.\phi_2) - \phi_2\,\underline{\partial}_\mu[Y_e] - Y_e\,\underline{\partial}_\mu[\phi_2])$$
$$(\mathrm{i}\,(Y_e.(q^\mu{}_{2,1})^* + Y_e.\phi_1.(q^\mu{}_{2,1})^* - Y_e.\phi_2.(q^\mu{}_{1,1})^* + Y_e.\phi_2.\Lambda^\mu - 2\,\Lambda^\mu.Y_e.\phi_2) - \phi_2\,\underline{\partial}^\mu[Y_e] - Y_e\,\underline{\partial}^\mu[\phi_2]),$$

$$(\mathrm{i}\,(Y_e.(q_{\mu2,2})^* - Y_e.\Lambda_\mu + 2\,\Lambda_\mu.Y_e + Y_e.\phi_1.(q_{\mu2,2})^* - Y_e.\phi_1.\Lambda_\mu - Y_e.\phi_2.(q_{\mu1,2})^* + 2\,\Lambda_\mu.Y_e.\phi_1) +$$
$$\underline{\partial}_\mu[Y_e] + \phi_1\,\underline{\partial}_\mu[Y_e] + Y_e\,\underline{\partial}_\mu[\phi_1])$$
$$(\mathrm{i}\,(Y_e.(q^\mu{}_{2,2})^* - Y_e.\Lambda^\mu + 2\,\Lambda^\mu.Y_e + Y_e.\phi_1.(q^\mu{}_{2,2})^* - Y_e.\phi_1.\Lambda^\mu - Y_e.\phi_2.(q^\mu{}_{1,2})^* + 2\,\Lambda^\mu.Y_e.\phi_1) +$$
$$\underline{\partial}^\mu[Y_e] + \phi_1\,\underline{\partial}^\mu[Y_e] + Y_e\,\underline{\partial}^\mu[\phi_1])\},$$

$$\{0,\ 0,\ 0,\ 0,\ (\underline{\partial}_\mu[Y_\nu]^*\,(1 + \phi_1) - \mathrm{i}\,((q_{\mu1,1})^*.(Y_\nu)^* - \Lambda_\mu.(Y_\nu)^* + (q_{\mu1,1})^*.(Y_\nu)^*.\phi_1 +$$
$$(q_{\mu1,2})^*.(Y_\nu)^*.\phi_2 - \Lambda_\mu.(Y_\nu)^*.\phi_1 + \mathrm{i}\,(Y_\nu)^*\,\underline{\partial}_\mu[\phi_1]))\,(\underline{\partial}^\mu[Y_\nu]^*\,(1 + \phi_1) -$$
$$\mathrm{i}\,((q^\mu{}_{1,1})^*.(Y_\nu)^* - \Lambda^\mu.(Y_\nu)^* + (q^\mu{}_{1,1})^*.(Y_\nu)^*.\phi_1 + (q^\mu{}_{1,2})^*.(Y_\nu)^*.\phi_2 - \Lambda^\mu.(Y_\nu)^*.\phi_1 + \mathrm{i}\,(Y_\nu)^*\,\underline{\partial}^\mu[\phi_1])),$$

$$(-(\phi_2\,\underline{\partial}_\mu[Y_e] + Y_e\,\underline{\partial}_\mu[\phi_2])^* - \mathrm{i}\,((q_{\mu1,2})^*.(Y_e)^* - 2\,(Y_e)^*.(\phi_2)^*.\Lambda_\mu -$$
$$(q_{\mu1,1})^*.(Y_e)^*.(\phi_2)^* + (q_{\mu1,2})^*.(Y_e)^*.(\phi_1)^* + \Lambda_\mu.(Y_e)^*.(\phi_2)^*))$$
$$(-(\phi_2\,\underline{\partial}^\mu[Y_e] + Y_e\,\underline{\partial}^\mu[\phi_2])^* - \mathrm{i}\,((q^\mu{}_{1,2})^*.(Y_e)^* - 2\,(Y_e)^*.(\phi_2)^*.\Lambda^\mu - (q^\mu{}_{1,1})^*.(Y_e)^*.(\phi_2)^* +$$
$$(q^\mu{}_{1,2})^*.(Y_e)^*.(\phi_1)^* + \Lambda^\mu.(Y_e)^*.(\phi_2)^*)),\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,$$
$$(-\mathrm{i}\,((q_{\mu2,1})^*.(Y_\nu)^* + (q_{\mu2,1})^*.(Y_\nu)^*.\phi_1 + (q_{\mu2,2})^*.(Y_\nu)^*.\phi_2 - \Lambda_\mu.(Y_\nu)^*.\phi_2) + \underline{\partial}_\mu[Y_\nu]^*\,\phi_2 + (Y_\nu)^*\,\underline{\partial}_\mu[\phi_2])$$
$$(-\mathrm{i}\,((q^\mu{}_{2,1})^*.(Y_\nu)^* + (q^\mu{}_{2,1})^*.(Y_\nu)^*.\phi_1 + (q^\mu{}_{2,2})^*.(Y_\nu)^*.\phi_2 - \Lambda^\mu.(Y_\nu)^*.\phi_2) + \underline{\partial}^\mu[Y_\nu]^*\,\phi_2 + (Y_\nu)^*\,\underline{\partial}^\mu[\phi_2]),$$

$$(\underline{\partial}_\mu[Y_e]^* + (\phi_1\,\underline{\partial}_\mu[Y_e] + Y_e\,\underline{\partial}_\mu[\phi_1])^* + \mathrm{i}\,(-2\,(Y_e)^*.\Lambda_\mu - (q_{\mu2,2})^*.(Y_e)^* + \Lambda_\mu.(Y_e)^* -$$
$$2\,(Y_e)^*.(\phi_1)^*.\Lambda_\mu + (q_{\mu2,1})^*.(Y_e)^*.(\phi_2)^* - (q_{\mu2,2})^*.(Y_e)^*.(\phi_1)^* + \Lambda_\mu.(Y_e)^*.(\phi_1)^*))$$
$$(\underline{\partial}^\mu[Y_e]^* + (\phi_1\,\underline{\partial}^\mu[Y_e] + Y_e\,\underline{\partial}^\mu[\phi_1])^* + \mathrm{i}\,(-2\,(Y_e)^*.\Lambda^\mu - (q^\mu{}_{2,2})^*.(Y_e)^* + \Lambda^\mu.(Y_e)^* -$$
$$2\,(Y_e)^*.(\phi_1)^*.\Lambda^\mu + (q^\mu{}_{2,1})^*.(Y_e)^*.(\phi_2)^* - (q^\mu{}_{2,2})^*.(Y_e)^*.(\phi_1)^* + \Lambda^\mu.(Y_e)^*.(\phi_1)^*)),\ 0,\ 0\}\}$$

The Tr[] is: $\boxed{\mathrm{Tr}[\underline{D}_\mu[\Phi]\,\underline{D}^\mu[\Phi]] \to 0}$

BUG?

```
$156;
PR["Follow text and Calculate in 1-space: ", $0 = $ = $156[[3, 2, 1]],
 NL, "Use ", $s = tuRuleSelect[$156][{Φ}],
```

```
NL, "Since ", $xB = $e58; $xB // MatrixForms,
NL, "In l,I space ", $xB = $xB[[1]] -> {{T[B1, "d", {μ}], 0}, {0, T[BI, "d", {μ}]}};
$xB // MatrixForms,
NL, "where ", $xBs =
 {T[B1, "d", {μ}] → $e58[[2, 1 ;; 4, 1 ;; 4]], T[BI, "d", {μ}] → $e58[[2, 5 ;; 8, 5 ;; 8]]},
$xBss = {$xBs, $xBs // tuIndicesRaise[μ]} // Flatten;
aside,
NL, "Check that ",
$ss = tuRuleSelect[$defGWS][{𝒟F2}][[1]] /. S → 0 /. dd : 𝒟F2 → dd[CG["OffDiagonal"]];
$00 = $ = CommutatorM[$ss[[1]], $xB[[1]]] → 0,
NL, "where ", {$xB, $ss} // MatrixForms // ColumnBar, CK,
Yield, $ = $ // expandCom[{$xB, $ss}];
$ // MatrixForms,
NL, "where we use ",
$t = tuRuleSelect[$defGWS][T] // Select[#, tuHasNoneQ[#, L] &] & // First;
$t[[2]] = $t[[2, 1 ;; 4, 5 ;; -1]]; $t // MatrixForms,
Yield, $ = $ /. Flatten[{$t, $xBs}]; $ // MatrixForms,
CG[" Verifies ", $00],
asideout,

next, "Compute ", $ = $0, CK,
Yield, $ /. $xB /. $s, CK,
Yield,
$ = $0 → ($ /. tuCommutatorExpand /. toxDot /. $xB /. $s // tuMatrixOrderedMultiply //
      (# /. toDot &) // expandDC[]);
NL, "Set off-diagonal →0 ", $1 = $ = $ /. T → 0 // expandDC[];
$ // MatrixForms, CK,
line,
next, "Check calculation in l,I space(4x4) ", $ = $1; $ // MatrixForms,
NL, "Using: ",
$sall = $s = {$xBss,
     tuRuleSelect[$defGWS][{S}] // Select[#, tuHasAnyQ[#, ν] &] &,
     tuRuleSelect[$defGWS][{ϕ}] // Select[#, tuHasAnyQ[#, ν] &] &,
     $t} // Flatten; $s // MatrixForms // ColumnBar;
Yield, $ = $ /. $s // MapAt[ArrayFlatten[#] &, #, 2] & // Collect[#, Y_, Simplify] &;
$ // MatrixForms,
NL, "• To see the relationship with text Extract
  the l-space block and relate the terms ", T[q, "d", {μ}]i,j,
" to ", T[Q, "d", {μ}] → Sum[T[Q, "du", {μ, i}] T[σ, "u", {i}], {i, 3}],
Yield, $s = {T[q, "d", {μ}]1,1 → T[Q, "du", {μ, 3}],
   T[q, "d", {μ}]2,2 → -T[Q, "du", {μ, 3}],
   T[q, "d", {μ}]1,2 →  (T[Q, "du", {μ, 1}] + I T[Q, "du", {μ, 2}]) / 2,
   T[q, "d", {μ}]2,1 → (T[Q, "du", {μ, 1}] - I T[Q, "du", {μ, 2}]) / 2},
Imply, $⟦2⟧ = $⟦2, 1 ;; 4, 1 ;; 4⟧ /. $s // Simplify;
$1a = $ = $ /. tt : CommutatorM[__] → tt1;
$ // MatrixForms,
NL, CR[" Differ by factor of I,1/2 and Conjugate[ϕ] "],
next, "Add ", $0 = $ = $156[[3, 2, 2]],
Yield, $2 = $ = $0 → ($ /. tuRuleSelect[$156][{Φ}] // tuDerivativeExpand[]);
$ // MatrixForms,
Yield, $ = $ /. $sall // tuDerivativeExpand[] // MapAt[ArrayFlatten[#] &, #, 2] &;
$⟦2⟧ = $⟦2, 1 ;; 4, 1 ;; 4⟧; $⟦1⟧ = $⟦1⟧1; ($2a = $) // MatrixForms,
Imply, $ = tuRuleAdd[{$1a, $2a}] // Collect[#, Y_, Simplify] &;
$ // MatrixForms,
NL, CR["Y's are constant."],
Yield, $12 = $ = $ // tuDerivativeExpand[{Y_}] // Collect[#, Y_, Simplify] &;
$ // MatrixForms,
```

```
line, "Determine the relationship between coefficients of Y's in ",
$1a // MatrixForms,
NL, "Extract the Coefficients: ",
$coef0 = $coef = $ = CoefficientList[$1a[[2]], {Yν, Ye, cc[Yν], cc[Ye]}] // Flatten //
      DeleteDuplicates // DeleteCases[#, 0] & // Simplify,

NL,
"Determine which coefficients are related(Factor,ConjugateFactor). Assume Reals: ",
$real = {{Tensor[Q, _, _], Tensor[Λ, _, _]}},
NL, "Coefficient indices groups ", $tlist = tuRelatedElements[$, , $real];
$tlist // ColumnBar;
$tlist = #[[1]] & /@ Select[$tlist, tuHasNoneQ[#, If | {_, _, 0}] &];
$related = tuConnectedPairs[$tlist]; $related // ColumnBar,

NL, "Only two independent coefficients ",
$coef = $coef0⟦{1, 2}⟧; $coef // ColumnBar,
NL, "Which correspond (up to factors of  {I,1/2} ) to the defined ", {χ1, χ2},
next, "Substitute χ's into ", $1a[[1]],
yield, $ = $1a[[2]];
NL, "With transformation: ",
$s = Thread[$coef -> {χ1, χ2}];
$sχ = $s = tuRuleSolve[$s, {ϕ1, ϕ2}]; $s // ColumnBar,
$ = $ /. $s;
Yield, $ = $1a[[1]] → ($ // tuConjugateSimplify[$real] // Simplify);
$ // MatrixForms
]
```

Equal test

Numeric factor test

Conjugate factor test

Follow text and Calculate in l-space: $\mathbb{i}\,[B_\mu, \Phi]_-$

Use $\{\Phi \to \{\{S+\phi, T^*\}, \{T, S^*+\phi^*\}\}\}$

Since $B_\mu \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu1,1}-\Lambda_\mu & q_{\mu1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu2,1} & q_{\mu2,2}-\Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu1,1})^*+\Lambda_\mu & -(q_{\mu1,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu2,1})^* & -(q_{\mu2,2})^*+\Lambda_\mu \end{pmatrix}$

In l,I space $B_\mu \to \begin{pmatrix} B_{1\mu} & 0 \\ 0 & B_{I\mu} \end{pmatrix}$

where $\{B_{1\mu} \to \{\{0, 0, 0, 0\}, \{0, -2\Lambda_\mu, 0, 0\}, \{0, 0, q_{\mu1,1}-\Lambda_\mu, q_{\mu1,2}\}, \{0, 0, q_{\mu2,1}, q_{\mu2,2}-\Lambda_\mu\}\}, B_{I\mu} \to$
$\{\{0, 0, 0, 0\}, \{0, 2\Lambda_\mu, 0, 0\}, \{0, 0, -(q_{\mu1,1})^*+\Lambda_\mu, -(q_{\mu1,2})^*\}, \{0, 0, -(q_{\mu2,1})^*, -(q_{\mu2,2})^*+\Lambda_\mu\}\}\}$

←←←←←←←←Side Note

Check that $[\mathcal{D}_{F_2}[\text{OffDiagonal}], B_\mu]_- \to 0$

where $B_\mu \to \begin{pmatrix} B_{1\mu} & 0 \\ 0 & B_{I\mu} \end{pmatrix}$, $\mathcal{D}_{F_2}[\text{OffDiagonal}] \to \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ ←CHECK

$\to \begin{pmatrix} 0 & T^*.B_{\mathrm{I}_\mu} - B_{1_\mu}.T^* \\ T.B_{1_\mu} - B_{\mathrm{I}_\mu}.T & 0 \end{pmatrix} \to 0$

where we use $T \to \begin{pmatrix} (t[R]_{1,1})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 0 & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & 0 \end{pmatrix} \to 0$ Verifies $[\mathcal{D}_{F_2}[\text{OffDiagonal}], B_\mu]_- \to 0$

→→→→→→→→

◆Compute $i\,[B_\mu, \Phi]_-$ ⟵CHECK

$\to i\,[\{\{B_{1_\mu}, 0\}, \{0, B_{\mathrm{I}_\mu}\}\}, \{\{S+\phi, T^*\}, \{T, S^*+\phi^*\}\}]_-$ ⟵CHECK

→

Set off-diagonal →0

$i\,[B_\mu, \Phi]_- \to \begin{pmatrix} i\,(-S.B_{1_\mu} - \phi.B_{1_\mu} + B_{1_\mu}.S + B_{1_\mu}.\phi) & 0 \\ 0 & i\,(-S^*.B_{\mathrm{I}_\mu} - \phi^*.B_{\mathrm{I}_\mu} + B_{\mathrm{I}_\mu}.S^* + B_{\mathrm{I}_\mu}.\phi^*) \end{pmatrix}$ ⟵CHECK

◆Check calculation in 1,I space(4x4)

$i\,[B_\mu, \Phi]_- \to \begin{pmatrix} i\,(-S.B_{1_\mu} - \phi.B_{1_\mu} + B_{1_\mu}.S + B_{1_\mu}.\phi) & 0 \\ 0 & i\,(-S^*.B_{\mathrm{I}_\mu} - \phi^*.B_{\mathrm{I}_\mu} + B_{\mathrm{I}_\mu}.S^* + B_{\mathrm{I}_\mu}.\phi^*) \end{pmatrix}$

Using:

$\to i\,[B_\mu, \Phi]_- \to \begin{pmatrix} 0 & 0 & -i\,(\\ 0 & 0 & i\,(Y \\ i\,Y_\nu\,((1+(\phi_1)^*)\,q_{\mu 1,1} + (\phi_2)^*\,q_{\mu 1,2} - (1+(\phi_1)^*)\,\Lambda_\mu) & i\,Y_e\,((1+\phi_1)\,q_{\mu 1,2} - \phi_2\,(q_{\mu 1,1}+\Lambda_\mu)) & \\ i\,Y_\nu\,((1+(\phi_1)^*)\,q_{\mu 2,1} + (\phi_2)^*\,(q_{\mu 2,2} - \Lambda_\mu)) & -i\,Y_e\,(\phi_2\,q_{\mu 2,1} - (1+\phi_1)\,(q_{\mu 2,2}+\Lambda_\mu)) & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \end{pmatrix}$

• To see the relationship with text Extract the l-space block and relate the terms $q_{\mu i,j}$ to $Q_\mu \to Q_\mu{}^1\,\sigma^1 + Q_\mu{}^2\,\sigma^2 + Q_\mu{}^3\,\sigma^3$

$\to \{q_{\mu 1,1} \to Q_\mu{}^3,\ q_{\mu 2,2} \to -Q_\mu{}^3,\ q_{\mu 1,2} \to \frac{1}{2}\,(Q_\mu{}^1 + i\,Q_\mu{}^2),\ q_{\mu 2,1} \to \frac{1}{2}\,(Q_\mu{}^1 - i\,Q_\mu{}^2)\}$

$\Rightarrow i\,[B_\mu, \Phi]_{-1} \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{2}\,i\,Y_\nu\,((\phi_2)^*\,(Q_\mu{}^1 + i\,Q_\mu{}^2) + 2\,(1+(\phi_1)^*)\,(Q_\mu{}^3 - \Lambda_\mu)) & i\,Y_e\,(\frac{1}{2}\,(1+\phi_1)\,(Q_\mu{}^1 + i\,Q_\mu{}^2)\ \cdot \\ \frac{1}{2}\,Y_\nu\,(i\,(1+(\phi_1)^*)\,Q_\mu{}^1 + (1+(\phi_1)^*)\,Q_\mu{}^2 - 2\,i\,(\phi_2)^*\,(Q_\mu{}^3 + \Lambda_\mu)) & -\frac{1}{2}\,i\,Y_e\,(\phi_2\,(Q_\mu{}^1 - i\,Q_\mu{}^2) + 2\,(1\ \cdot \end{pmatrix}$

Differ by factor of I,1/2 and Conjugate[ϕ]

◆Add $\underline{\partial}_\mu[\Phi]$

$\to \underline{\partial}_\mu[\Phi] \to \begin{pmatrix} \partial_{-\mu}[S] + \partial_{-\mu}[\phi] & \partial_{-\mu}[T]^* \\ \partial_{-\mu}[T] & \partial_{-\mu}[S]^* + \partial_{-\mu}[\phi]^* \end{pmatrix}$

$$\to \underline{\partial}_\mu[\Phi]_1 \to \left(\begin{array}{ccc} 0 & 0 & \underset{-\mu}{\partial}[Y_\nu]^* + \underset{-\mu}{\partial}[Y_\nu]^*\phi_1 + (Y_\nu)^*\underset{-\mu}{\partial}[\phi_1 \\ 0 & 0 & -(\phi_2)^*\underset{-\mu}{\partial}[Y_e]^* - (Y_e)^*\underset{-\mu}{\partial}[\phi_2]^* \\ \underset{-\mu}{\partial}[\phi_1]^* Y_\nu + \underset{-\mu}{\partial}[Y_\nu] + (\phi_1)^*\underset{-\mu}{\partial}[Y_\nu] & -\phi_2\underset{-\mu}{\partial}[Y_e] - Y_e\underset{-\mu}{\partial}[\phi_2] & 0 \\ \underset{-\mu}{\partial}[\phi_2]^* Y_\nu + (\phi_2)^*\underset{-\mu}{\partial}[Y_\nu] & \underset{-\mu}{\partial}[Y_e] + \phi_1\underset{-\mu}{\partial}[Y_e] + Y_e\underset{-\mu}{\partial}[\phi_1] & 0 \end{array}\right.$$

$$\Rightarrow \mathrm{i}[B_\mu, \Phi]_{-1} + \underline{\partial}_\mu[\Phi]_1 \to \left(\begin{array}{c} 0 \\ 0 \\ Y_\nu\left(\underset{-\mu}{\partial}[\phi_1]^* + \frac{1}{2}\mathrm{i}\left((\phi_2)^*(Q_\mu{}^1 + \mathrm{i}\,Q_\mu{}^2) + 2(1+(\phi_1)^*)(Q_\mu{}^3 - \Lambda_\mu)\right)\right) + (1+(\phi_1)^*)\underset{-\mu}{\partial}[Y_\nu \\ Y_\nu\left(\underset{-\mu}{\partial}[\phi_2]^* + \frac{1}{2}\left(\mathrm{i}(1+(\phi_1)^*)Q_\mu{}^1 + (1+(\phi_1)^*)Q_\mu{}^2 - 2\,\mathrm{i}(\phi_2)^*(Q_\mu{}^3 + \Lambda_\mu)\right)\right) + (\phi_2)^*\underset{-\mu}{\partial}[Y \end{array}\right.$$

Y's are constant.

$$\to \mathrm{i}[B_\mu, \Phi]_{-1} + \underline{\partial}_\mu[\Phi]_1 \to \left(\begin{array}{cc} 0 & \\ 0 & \\ Y_\nu\left(\underset{-\mu}{\partial}[\phi_1]^* + \frac{1}{2}\mathrm{i}\left((\phi_2)^*(Q_\mu{}^1 + \mathrm{i}\,Q_\mu{}^2) + 2(1+(\phi_1)^*)(Q_\mu{}^3 - \Lambda_\mu)\right)\right) & Y_e(\mathrm{i}(\frac{1}{2} \\ Y_\nu\left(\underset{-\mu}{\partial}[\phi_2]^* + \frac{1}{2}\left(\mathrm{i}(1+(\phi_1)^*)Q_\mu{}^1 + (1+(\phi_1)^*)Q_\mu{}^2 - 2\,\mathrm{i}(\phi_2)^*(Q_\mu{}^3 + \Lambda_\mu)\right)\right) & Y_e(-\frac{1}{2}\mathrm{i}((\end{array}\right.$$

Determine the relationship between coefficients of Y's in $\mathrm{i}[B_\mu, \Phi]_{-1} \to \left(\begin{array}{c} \frac{1}{2}\mathrm{i}\,Y_\nu((\phi_2)^*((\\ \frac{1}{2}Y_\nu(\mathrm{i}(1+(\phi_1)^*)(\end{array}\right.$

Extract the Coefficients:

$$\{-\frac{1}{2}\mathrm{i}\left(\phi_2(Q_\mu{}^1 - \mathrm{i}\,Q_\mu{}^2) + 2(1+\phi_1)(Q_\mu{}^3 - \Lambda_\mu)\right), \frac{1}{2}\left(-\mathrm{i}(1+\phi_1)Q_\mu{}^1 + (1+\phi_1)Q_\mu{}^2 + 2\,\mathrm{i}\,\phi_2(Q_\mu{}^3 + \Lambda_\mu)\right),$$
$$\mathrm{i}\left(-\frac{1}{2}(1+(\phi_1)^*)(Q_\mu{}^1 - \mathrm{i}\,Q_\mu{}^2) + (\phi_2)^*(Q_\mu{}^3 + \Lambda_\mu)\right), \frac{1}{2}\mathrm{i}\left((\phi_2)^*(Q_\mu{}^1 + \mathrm{i}\,Q_\mu{}^2) + 2(1+(\phi_1)^*)(Q_\mu{}^3 - \Lambda_\mu)\right),$$
$$\mathrm{i}\left(\frac{1}{2}(1+\phi_1)(Q_\mu{}^1 + \mathrm{i}\,Q_\mu{}^2) - \phi_2(Q_\mu{}^3 + \Lambda_\mu)\right), \frac{1}{2}\left(\mathrm{i}(1+(\phi_1)^*)Q_\mu{}^1 + (1+(\phi_1)^*)Q_\mu{}^2 - 2\,\mathrm{i}(\phi_2)^*(Q_\mu{}^3 + \Lambda_\mu)\right)\}$$

Determine which coefficients are related(Factor,ConjugateFactor). Assume Reals:

{{Tensor[Q, _, _], Tensor[Λ, _, _]}}

Coefficient indices groups $\left|\begin{array}{l} \{\{1, 4\}\} \\ \{\{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{5, 6\}\} \end{array}\right.$

Only two independent coefficients $\left|\begin{array}{l} -\frac{1}{2}\mathrm{i}\left(\phi_2(Q_\mu{}^1 - \mathrm{i}\,Q_\mu{}^2) + 2(1+\phi_1)(Q_\mu{}^3 - \Lambda_\mu)\right) \\ \frac{1}{2}\left(-\mathrm{i}(1+\phi_1)Q_\mu{}^1 + (1+\phi_1)Q_\mu{}^2 + 2\,\mathrm{i}\,\phi_2(Q_\mu{}^3 + \Lambda_\mu)\right) \end{array}\right.$

Which correspond (up to factors of {I,1/2}) to the defined $\{\chi_1, \chi_2\}$

◆Substitute χ's into $\mathrm{i}[B_\mu, \Phi]_{-1}$ ⟶

With transformation: $\left|\begin{array}{l} \phi_1 \to -\frac{-2\,\mathrm{i}\,\chi_2 Q_\mu{}^1 + (Q_\mu{}^1)^2 - 2\chi_2 Q_\mu{}^2 + (Q_\mu{}^2)^2 - 4\,\mathrm{i}\,\chi_1 Q_\mu{}^3 + 4(Q_\mu{}^3)^2 - 4\,\mathrm{i}\,\chi_1\Lambda_\mu - 4(\Lambda_\mu)^2}{(Q_\mu{}^1)^2 + (Q_\mu{}^2)^2 + 4(Q_\mu{}^3)^2 - 4(\Lambda_\mu)^2} \\ \phi_2 \to -\frac{2(-\mathrm{i}\,\chi_1 Q_\mu{}^1 + \chi_1 Q_\mu{}^2 + 2\,\mathrm{i}\,\chi_2 Q_\mu{}^3 - 2\,\mathrm{i}\,\chi_2\Lambda_\mu)}{(Q_\mu{}^1)^2 + (Q_\mu{}^2)^2 + 4(Q_\mu{}^3)^2 - 4(\Lambda_\mu)^2} \end{array}\right.$

$$\to \mathrm{i}[B_\mu, \Phi]_{-1} \to \begin{pmatrix} 0 & 0 & (Y_\nu)^*\chi_1 & (Y_\nu)^*\chi_2 \\ 0 & 0 & -(Y_e)^*(\chi_2)^* & (Y_e)^*(\chi_1)^* \\ (\chi_1)^* Y_\nu & -Y_e\chi_2 & 0 & 0 \\ (\chi_2)^* Y_\nu & Y_e\chi_1 & 0 & 0 \end{pmatrix}$$

```
PR[
 next, "Compute ", $0 = $ = $156[[3, 2, 2]],
 Yield, $2 = $ = $0 → ($ /. $s // tuDerivativeExpand[]); $ // MatrixForms,


 Imply, $ = $156[[3]] /. {$1, $2}; $ // MatrixForms,

 next, "Compute the product: ",
 $s = $ // tuIndicesRaise[{μ}];
 Yield, $ = Thread[xDot[$ , $s], Rule] /. toxDot; $ // MatrixForms;
 Yield, $ = $ // expandCom[]; $ // MatrixForms,
 next, "Expand l-space diagonal block for Tr[] ", $ = $01 = $[[2, 1, 1]]
]
```

◆Compute $\underline{\partial}_\mu[\Phi]$

→ $\underline{\partial}_\mu[\Phi] \to \underline{\partial}_\mu[\Phi]$

⇒ $\underline{D}_\mu[\Phi] \to \begin{pmatrix} i\,(-S.B_{1\mu} - \phi.B_{1\mu} + B_{1\mu}.S + B_{1\mu}.\phi) + \underset{-\mu}{\partial}[\Phi] & \underset{-\mu}{\partial}[\Phi] \\ \underset{-\mu}{\partial}[\Phi] & i\,(-S^*.B_{\mathrm{I}\mu} - \phi^*.B_{\mathrm{I}\mu} + B_{\mathrm{I}\mu}.S^* + B_{\mathrm{I}\mu}.\phi^*) + \underset{-\mu}{\partial}[\Phi] \end{pmatrix}$

◆Compute the product:

→

→ $\underline{D}_\mu[\Phi].\underline{D}^\mu[\Phi] \to (2\,\underset{-\mu}{\partial}[\Phi].\underline{\partial}^\mu[\Phi] - i\,S.B_{1\mu}.\underline{\partial}^\mu[\Phi] - i\,\phi.B_{1\mu}.\underline{\partial}^\mu[\Phi] + i\,B_{1\mu}.S.\underline{\partial}^\mu[\Phi] + i\,B_{1\mu}.\phi.\underline{\partial}^\mu[\Phi] - i\,\underset{-\mu}{\partial}[\Phi].S.B_1$

◆Expand l-space diagonal block for Tr[]

$2\,\underline{\partial}_\mu[\Phi].\underline{\partial}^\mu[\Phi] - i\,S.B_{1\mu}.\underline{\partial}^\mu[\Phi] - i\,\phi.B_{1\mu}.\underline{\partial}^\mu[\Phi] + i\,B_{1\mu}.S.\underline{\partial}^\mu[\Phi] + i\,B_{1\mu}.\phi.\underline{\partial}^\mu[\Phi] - i\,\underline{\partial}_\mu[\Phi].S.B_1{}^\mu -$
$i\,\underline{\partial}_\mu[\Phi].\phi.B_1{}^\mu + i\,\underline{\partial}_\mu[\Phi].B_1{}^\mu.S + i\,\underline{\partial}_\mu[\Phi].B_1{}^\mu.\phi - S.B_{1\mu}.S.B_1{}^\mu - S.B_{1\mu}.\phi.B_1{}^\mu + S.B_{1\mu}.B_1{}^\mu.S +$
$S.B_{1\mu}.B_1{}^\mu.\phi - \phi.B_{1\mu}.S.B_1{}^\mu - \phi.B_{1\mu}.\phi.B_1{}^\mu + \phi.B_{1\mu}.B_1{}^\mu.S + \phi.B_{1\mu}.B_1{}^\mu.\phi + B_{1\mu}.S.S.B_1{}^\mu +$
$B_{1\mu}.S.\phi.B_1{}^\mu - B_{1\mu}.S.B_1{}^\mu.S - B_{1\mu}.S.B_1{}^\mu.\phi + B_{1\mu}.\phi.S.B_1{}^\mu + B_{1\mu}.\phi.\phi.B_1{}^\mu - B_{1\mu}.\phi.B_1{}^\mu.S - B_{1\mu}.\phi.B_1{}^\mu.\phi$

Proposition 5.7. The spectral action of the AC-manifold

```
PR["Proposition 5.7. The spectral action of the AC-manifold ",
 $p57 = $ = M×F_GWS → {C∞[M, ℂ⊕ℍ], L²[M, S]⊗(ℂ⁴⊕ℂ⁴),
     slash[iD]⊗1_F + T[γ, "d", {5}]⊗iD_F, T[γ, "d", {5}]⊗T[γ, "d", {F}], J_M⊗J_F};
 $ // ColumnForms,
 NL, "is ", $p57 = $ = {Tr[f[iD_A / Λ]] → xIntegral[
      ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] Sqrt[Det[g]], x⁴],
    ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] →
     8 ℒ_M[T[g, "dd", {μ, ν}]] + ℒ_A[T[Λ, "d", {μ}], T[Q, "d", {μ}]] +
      ℒ_H[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'],
    ℒ_A[T[Λ, "d", {μ}], T[Q, "d", {μ}]] → f[0] / (12 π²) (6 T[Λ, "dd", {μ, ν}]
         T[Λ, "uu", {μ, ν}] + Tr[ T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]]),
    ℒ_H[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] →
     b f[0] / (2 π^2) Abs[H']^4 + (-2 a f₂ Λ² + e f[0]) / π² Abs[H']² -
      c f₂ Λ² / π² + d f[0] / (4 π^2) + a f[0] s Abs[H']^2 / (12 π^2) +
      c f[0] s / (24 π²) + a f[0] Abs[T[iD̄, "d", {μ}][H']]^2 / (2 π^2),
    $p35[[-1]]
   }; $ // ColumnBar,
 line,
 NL, "Prop 3.5", yield, $ = tuRuleSelect[$p35][ℒ_M[_]][[1]], AppendTo[$p57, $];
 NL, "Prop 3.7, Lemma 5.4", yield, $ = {tuRuleSelect[$p37][ℒ_B[_]][[1]], $154};
 $ // ColumnBar, AppendTo[$p57, $];
 NL, "Prop 3.5, Lemma 5.5", yield, $ = tuRule[{tuRuleSelect[$p37][ℒ_H[__]][[1]],
     $155, $156}] // Flatten; $ // ColumnBar, AppendTo[$p57, $];
 accumGWS[prop57 -> $p57]
]
```

Proposition 5.7. The spectral action of the AC-manifold

$$M\times F_{GWS}\to \left|\begin{array}{l} C^\infty[M,\ \mathbb{C}\oplus\mathbb{H}] \\ L^2[M,\ S]\otimes(\mathbb{C}^4\oplus\mathbb{C}^4) \\ (\not{D})\otimes 1_F+\text{Tensor}[\gamma,\ |\text{Void},\ |5]\otimes\mathcal{D}_F \\ \text{Tensor}[\gamma,\ |\text{Void},\ |5]\otimes\text{Tensor}[\gamma,\ |\text{Void},\ |F] \\ J_M\otimes J_F \end{array}\right.$$

is

$$\left|\begin{array}{l} \text{Tr}[f[\frac{D_A}{\Lambda}]]\to\int\sqrt{\text{Det}[g]}\ \mathcal{L}[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']\, dx^4 \\ \mathcal{L}[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']\to\mathcal{L}_A[\Lambda_\mu,\ Q_\mu]+\mathcal{L}_H[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']+8\,\mathcal{L}_M[g_{\mu\nu}] \\ \mathcal{L}_A[\Lambda_\mu,\ Q_\mu]\to\frac{f[0]\,(6\,\Lambda_{\mu\nu}\Lambda^{\mu\nu}+\text{Tr}[Q_{\mu\nu}Q^{\mu\nu}])}{12\,\pi^2} \\ \mathcal{L}_H[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']\to \\ \quad \frac{d\,f[0]}{4\,\pi^2}+\frac{c\,s\,f[0]}{24\,\pi^2}+\frac{a\,s\,\text{Abs}[H']^2 f[0]}{12\,\pi^2}+\frac{b\,\text{Abs}[H']^4 f[0]}{2\,\pi^2}+\frac{a\,\text{Abs}[\tilde{\mathcal{D}}_\mu[H']]^2 f[0]}{2\,\pi^2}-\frac{c\,\Lambda^2 f_2}{\pi^2}+\frac{\text{Abs}[H']^2\,(e\,f[0]-2\,a\,\Lambda^2 f_2)}{\pi^2} \\ \mathcal{L}_M[g_{\mu\nu}]\to-\frac{\Lambda^2 f_2}{24\,\pi^2}+\frac{\Lambda^4 f_4}{2\,\pi^2}+\frac{f[0]\,(\frac{11\,R^*.R^*}{360}-\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+\frac{\Delta[s]}{30})}{16\,\pi^2} \end{array}\right.$$

Prop 3.5 $\longrightarrow$ $\mathcal{L}_M[g_{\mu\nu}]\to-\frac{\Lambda^2 f_2}{24\,\pi^2}+\frac{\Lambda^4 f_4}{2\,\pi^2}+\frac{f[0]\,(\frac{11\,R^*.R^*}{360}-\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+\frac{\Delta[s]}{30})}{16\,\pi^2}$

Prop 3.7, Lemma 5.4 $\longrightarrow$

$$\left|\begin{array}{l} \mathcal{L}_B[B_\mu]\to\frac{f[0]\,\text{Tr}[F_{\mu\nu}F^{\mu\nu}]}{24\,\pi^2} \\ \{\text{Tr}[F_{\mu\nu}F^{\mu\nu}]\to 12\,\Lambda_{\mu\nu}\Lambda^{\mu\nu}+2\,\text{Tr}[Q_{\mu\nu}Q^{\mu\nu}],\ \Lambda_{\mu\nu}\to-\underset{-\nu}{\partial}[\Lambda_\mu]+\underset{-\mu}{\partial}[\Lambda_\nu],\ Q_{\mu\nu}\to i\,[Q_\mu,\ Q_\nu]_- -\underset{-\nu}{\partial}[Q_\mu]+\underset{-\mu}{\partial}[Q_\nu]\} \end{array}\right.$$

Prop 3.5, Lemma 5.5 $\longrightarrow$

$$\left|\begin{array}{l} \mathcal{L}_H[g_{\mu\nu},\ B_\mu,\ \Phi]\to\frac{f[0]\,s[x]\,\text{Tr}[\Phi.\Phi]}{48\,\pi^2}-\frac{\Lambda^2 f_2\,\text{Tr}[\Phi.\Phi]}{2\,\pi^2}+\frac{f[0]\,\text{Tr}[\mathcal{D}_\mu[\Phi].\mathcal{D}^\mu[\Phi]]}{8\,\pi^2}+\frac{f[0]\,\text{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\,\pi^2}+\frac{f[0]\,\Delta[\text{Tr}[\Phi.\Phi]]}{24\,\pi^2} \\ \text{Tr}[\Phi^2]\to 2\,c+4\,a\,\text{Abs}[H']^2 \\ \text{Tr}[\Phi^4]\to 2\,d+8\,e\,\text{Abs}[H']^2+4\,b\,\text{Abs}[H']^4 \\ H'\to\{1+\phi_1,\ \phi_2\} \\ a\to\text{Abs}[Y_e]^2+\text{Abs}[Y_\nu]^2 \\ b\to\text{Abs}[Y_e]^4+\text{Abs}[Y_\nu]^4 \\ c\to\text{Abs}[Y_R]^2 \\ d\to\text{Abs}[Y_R]^4 \\ e\to\text{Abs}[Y_R]^2\,\text{Abs}[Y_\nu]^2 \\ \text{Tr}[\underset{-\mu}{D}[\Phi]\,\underset{-}{D}^\mu[\Phi]]\to 4\,a\,\text{Abs}[\underset{-\mu}{\tilde{D}}[H']]^2 \\ \underset{-\mu}{\tilde{D}}[H']\to-i\,\Lambda_\mu.H'+i\sum_{\{j,\bar{3}\}}[Q_\mu{}^j\,\sigma_j].H'+\underset{-\mu}{\partial}[H'] \\ \underset{-\mu}{D}[\Phi]\to i\,[B_\mu,\ \Phi]_-+\underset{-\mu}{\partial}[\Phi] \\ \Phi\to\{\{S+\phi,\ T^*\},\ \{T,\ S^*+\phi^*\}\} \\ H'\to\{1+\phi_1,\ \phi_2\} \\ Q_\mu\to\sum_{\{j,\bar{3}\}}[Q_\mu{}^j\,\sigma_j] \end{array}\right.$$

5.4 Normalization of kinetic terms

5.4.1 Rescaling the Higgs field

```
PR["The canonical kinetic energy term is of form ",
 ℒ → xIntegral[1 / 2 tuDPartial[ψ, μ] tuDPartialu[ψ, μ] √Abs[Det[g]] , x^4],
 NL, "Identify within ",
 $ = tuRuleSelect[$defGWS][ℒH[__]] // Select[#, tuHasNoneQ[#, Φ] &] & // Last,
 Yield, $ = $ // tuTermSelect[iD] // Last,
 " as the kinetic energy Lagrangian density for
   Higg's and rescale to canonical form by ",
 $ = H → √ (a f[0] / π^2) H'; $ // Framed, accumGWS[$]
]
```

The canonical kinetic energy term is of form $\mathcal{L}\to\int \frac{1}{2}\sqrt{\mathrm{Abs}[\mathrm{Det}[g]]}\,\partial_{\mu}[\psi]\,\partial^{\mu}[\psi]\,\mathrm{d}x^4$

Identify within $\mathcal{L}_H[g_{\mu\nu}, \Lambda_\mu, Q_\mu, H'] \to \frac{d\,f[0]}{4\pi^2} + \frac{c\,s\,f[0]}{24\pi^2} + \frac{a\,s\,\mathrm{Abs}[H']^2 f[0]}{12\pi^2} + \frac{b\,\mathrm{Abs}[H']^4 f[0]}{2\pi^2} + \frac{a\,\mathrm{Abs}[\mathcal{D}_\mu[H']]^2 f[0]}{2\pi^2} - \frac{c\,\Lambda^2 f_2}{\pi^2} + \frac{\mathrm{Abs}[H']^2 (e\,f[0] - 2\,a\,\Lambda^2 f_2)}{\pi^2}$

$\to \frac{a\,\mathrm{Abs}[\mathcal{D}_\mu[H']]^2 f[0]}{2\pi^2}$ as the kinetic energy Lagrangian density

for Higg's and rescale to canonical form by $\boxed{H \to \frac{\sqrt{a\,f[0]}\;H'}{\pi}}$

5.4.2 The coupling constants

```
PR["Rescale Gauge fields: ", $gaugeRescaled = $ = {
    T[Λ, "d", {μ}] -> g1[CG["coupling"]] / 2 T[B, "d", {μ}],
    T[Q, "du", {μ, a}] -> T[W, "du", {μ, a}] g2[CG["coupling"]] / 2,
    T[Q, "d", {μ}] -> T[W, "d", {μ}] g2 / 2 ,
    g1[CG["coupling"]],
    g2[CG["coupling"]],
    T[B, "d", {μ}][CG["U[1] hypercharge field"]],
    T[Λ, "dd", {μ, ν}] -> g1 / 2 T[B, "dd", {μ, ν}],
    T[Q, "ddu", {μ, ν, a}] -> g2 / 2 T[W, "ddu", {μ, ν, a}]
  }; $ // ColumnBar, accumGWS[$gaugeRescaled], CR["Why?"],
 NL, "Previously ",
 $pass =
  $ = tuRuleSelect[$defGWS][{T[Q, "dd", {μ, ν}], T[Λ, "dd", {μ, ν}]}] // DeleteDuplicates;
 $ // ColumnBar
]
PR["From Lemma 5.6: ",
 $s = tuRuleSelect[$defGWS][T[Q, "d", {_}]] // Select[#, ! FreeQ[#, xSum] &] & // Last //
    tuAddPatternVariable[μ] // (# /. xSum[a_, _] → a &),
 Yield, $ = $pass /. $s /. CommutatorM[a_, b_] :→ CommutatorM[a, (b /. j → i)] //
    tuCommutatorSimplify[{Tensor[Q, _, _]}] // tuDerivativeExpand[{Tensor[σ, _, _]}],
 Yield, $ = $ /. tuSU2commutation[σ] // tuIndexSwapUpDown[c$];
 NL, "In σ components: ",
 $[[1, 2, 1]] = $[[1, 2, 1]] /. {j → k, c$ → j};
 $ = $ /. Tensor[σ, _, _] → 1 /. tt : T[Q, "dd", {μ, ν}] :→ tuIndexAdd[-1, j][tt];
 $ // ColumnBar
]
PR["The rescaled relationships ",
 $ = $ /. (tuRule[$gaugeRescaled] // tuAddPatternVariable[{a, μ, ν}]) //
   tuDerivativeExpand[{g_}];
 $ = tuRuleSolve[$, {T[W, "ddu", {_, _, _}], T[B, "dd", {_, _}]}] // Expand;
 $ // ColumnBar, accumGWS[$]
]
```

Rescale Gauge fields:

$\Lambda_\mu \to \frac{1}{2} B_\mu\, g_1[\text{coupling}]$

$Q_\mu{}^a \to \frac{1}{2} W_\mu{}^a\, g_2[\text{coupling}]$

$Q_\mu \to \frac{1}{2} g_2\, W_\mu$

$g_1[\text{coupling}]$

$g_2[\text{coupling}]$

$B_\mu[\text{U[1] hypercharge field}]$

$\Lambda_{\mu\nu} \to \frac{1}{2} g_1\, B_{\mu\nu}$

$Q_{\mu\nu}{}^a \to \frac{1}{2} g_2\, W_{\mu\nu}{}^a$

Why?

Previously

$Q_{\mu\nu} \to i\,[Q_\mu, Q_\nu]_- - \underset{-\nu}{\partial}[Q_\mu] + \underset{-\mu}{\partial}[Q_\nu]$

$\Lambda_{\mu\nu} \to -\underset{-\nu}{\partial}[\Lambda_\mu] + \underset{-\mu}{\partial}[\Lambda_\nu]$

$\Lambda_{\mu\nu} \to \frac{1}{2} g_1\, B_{\mu\nu}$

From Lemma 5.6: $Q_{\mu\_} \to Q_\mu{}^j \sigma_j$

$\to \{Q_{\mu\nu} \to i [\sigma_j, \sigma_i]_- Q_\nu{}^i Q_\mu{}^j - \sigma_j \underline{\partial}_\nu[Q_\mu{}^j] + \sigma_j \underline{\partial}_\mu[Q_\nu{}^j],\ \Lambda_{\mu\nu} \to -\underline{\partial}_\nu[\Lambda_\mu] + \underline{\partial}_\mu[\Lambda_\nu],\ \Lambda_{\mu\nu} \to \frac{1}{2} g_1 B_{\mu\nu}\}$

$\to$

In σ components:

$$Q_{\mu\nu}{}^j \to -2 Q_\nu{}^i Q_\mu{}^k \epsilon_{ki}{}^j - \underset{-\nu}{\partial}[Q_\mu{}^j] + \underset{-\mu}{\partial}[Q_\nu{}^j]$$
$$\Lambda_{\mu\nu} \to -\underset{-\nu}{\partial}[\Lambda_\mu] + \underset{-\mu}{\partial}[\Lambda_\nu]$$
$$\Lambda_{\mu\nu} \to \frac{1}{2} g_1 B_{\mu\nu}$$

The rescaled relationships

$$W_{\mu\nu}{}^j \to -g_2 W_\nu{}^i W_\mu{}^k \epsilon_{ki}{}^j - \underset{-\nu}{\partial}[W_\mu{}^j] + \underset{-\mu}{\partial}[W_\nu{}^j]$$
$$B_{\mu\nu} \to -\underset{-\nu}{\partial}[B_\mu] + \underset{-\mu}{\partial}[B_\nu]$$

```
PR["Evaluate ",
 $ = selectGWS[Tr[_], {F, T[Λ, "dd", {μ, ν}], Q, μ, ν}],
 NL, "Apply ",
 $s = selectGWS[{Tensor[Λ, __]}, {B, μ, ν}];
 $s = {$s, tuIndicesRaise[{μ, ν}][$s]},
 Yield, $ = $ /. $s,
 NL, "Apply ",
 $s = selectGWS[Tensor[Q, __], σ],
 $s = T[σ, "d", {a}] # & /@ $s,
 $s = {$s, tuIndicesRaise[{μ, ν}][$s]} // tuAddPatternVariable[a] // Flatten,
 Yield,
 $ = $ /. tt : Tensor[Q, _, _] :> tuIndexAdd[-1, a][tt] /.
     tt : T[Q, "uuu", {i_, j_, a}] :> ((tt /. a → b) T[σ, "d", {b}]) /.
   tt : T[Q, "ddu", {i_, j_, a}] → tt T[σ, "d", {a}],
 Yield, $ = $ // tuTrSimplify[{Tensor[Q, _, _]}],
 NL, "Apply ",
 $s = Tr[T[σ, "d", {a}] T[σ, "d", {b}]] → 2 T[δ, "dd", {a, b}],
 Yield, $ = $ /. $s;
 $[[2]] = tuIndexContractUpDn[δ, {b}] /@ $[[2]]; $,
 NL, "Apply ", $s = selectGWS[Tensor[Q, __], W];
 $s = {$s, $s // tuIndicesRaise[{μ, ν}] // tuIndicesLower[{a}]} //
   tuAddPatternVariable[{μ, ν, a}],
 Yield, $ = $ /. $s; $ // Framed, CG[" (5.14)"], accumGWS[$]
]
```

Evaluate $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 12 \Lambda_{\mu\nu} \Lambda^{\mu\nu} + 2\,\mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}]$

Apply $\{\Lambda_{\mu\nu} \to \frac{1}{2} g_1 B_{\mu\nu},\ \Lambda^{\mu\nu} \to \frac{1}{2} g_1 B^{\mu\nu}\}$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3 g_1^2 B_{\mu\nu} B^{\mu\nu} + 2\,\mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}]$

Apply $Q_\mu \to \sum_{\{j,3\}} [Q_\mu{}^j \sigma_j] Q_\mu \sigma_a \to \sigma_a \sum_{\{j,3\}} [Q_\mu{}^j \sigma_j] \{Q_\mu \sigma_{a\_} \to \sigma_a \sum_{\{j,3\}} [Q_\mu{}^j \sigma_j],\ Q^\mu \sigma_{a\_} \to \sigma_a \sum_{\{j,3\}} [Q^{\mu j} \sigma_j]\}$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3 g_1^2 B_{\mu\nu} B^{\mu\nu} + 2\,\mathrm{Tr}[Q_{\mu\nu}{}^a Q^{\mu\nu b} \sigma_a \sigma_b]$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3 g_1^2 B_{\mu\nu} B^{\mu\nu} + 2 Q_{\mu\nu}{}^a Q^{\mu\nu b}\,\mathrm{Tr}[\sigma_a \sigma_b]$

Apply $\mathrm{Tr}[\sigma_a \sigma_b] \to 2 \delta_{ab}$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3 g_1^2 B_{\mu\nu} B^{\mu\nu} + 4 Q_{\mu\nu}{}^a Q^{\mu\nu}{}_a$

Apply $\{Q_{\mu\_\nu\_}{}^{a\_} \to \frac{1}{2} g_2 W_{\mu\nu}{}^a,\ Q^{\mu\_\nu\_}{}_{a\_} \to \frac{1}{2} g_2 W^{\mu\nu}{}_a\}$

$\to \boxed{\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3 g_1^2 B_{\mu\nu} B^{\mu\nu} + g_2^2 W_{\mu\nu}{}^a W^{\mu\nu}{}_a}$ (5.14)

5.4.3 Electroweak unification

```
PR["The canonical form of gauge field Kinetic term: ",
 $0 = $ = selectGWS[Tr[ T[F, "dd", {μ, ν}] _ ]];
 $0 = ℒ → -1 / 2 $[[1]],
 NL, "Here ", $f = $,
 NL, "Since ",
 $ = {selectGWS[ℒA[__], {f[0]}], selectGWS[Tr[_], {F, T[Λ, "dd", {μ, ν}], Q, μ, ν}]};
 $ // ColumnBar,
 $s = tuTermSelect[Q][$][[1]] / 2,
 NL, $ = {$} // Flatten; $ // ColumnBar,

 Yield, $1 = $ = tuEliminate[$, {$s}] // tuRuleSolve[#, ℒA[__]] & // Last,
 Imply, $1 = $0[[2]] → $[[2]],
 Yield, $ = tuRuleSolve[$1, f[0]] // Last; $ // Framed, accumGWS[$],
 Imply, $ = $1 /. $,
 Yield, $ = $1 /. $f // Expand,
 NL, "Imposing ", $s = {f[0] g1^2 / (8 π^2) → 1 / 4, f[0] g2^2 / (24 π^2) → 1 / 4},
 Yield, $ = $ /. $s; $ // Framed, accumGWS[$],
 Yield, $ = tuEliminate[$s, f[0]] // Simplify;
 ($ = $ /. Equal → Rule) // Framed, accumGWS[$],
 CR["This relationship stems from the imposed condition and may be arbitrary. "]
]
```

The canonical form of gauge field Kinetic term: $\mathcal{L}\to -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$

Here $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to 3\,g_1^2 B_{\mu\nu}B^{\mu\nu}+g_2^2 W_{\mu\nu}{}^{a}W^{\mu\nu}{}_{a}$

Since $\left|\begin{array}{l}\mathcal{L}_A[\Lambda_\mu,Q_\mu]\to \frac{f[0]\,(6\Lambda_{\mu\nu}\Lambda^{\mu\nu}+\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}])}{12\pi^2} \\ \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to 12\Lambda_{\mu\nu}\Lambda^{\mu\nu}+2\,\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}]\end{array}\right.$ $\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}]$

$$\left|\begin{array}{l}\mathcal{L}_A[\Lambda_\mu,Q_\mu]\to \frac{f[0]\,(6\Lambda_{\mu\nu}\Lambda^{\mu\nu}+\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}])}{12\pi^2} \\ \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to 12\Lambda_{\mu\nu}\Lambda^{\mu\nu}+2\,\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}]\end{array}\right.$$

$\to$ $\mathcal{L}_A[\Lambda_\mu,Q_\mu]\to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]}{24\pi^2}$

$\Rightarrow$ $-\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]}{24\pi^2}$

$\to$ $\boxed{f[0]\to -12\pi^2}$

$\Rightarrow$ $\mathcal{L}_A[\Lambda_\mu,Q_\mu]\to -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$

$\to$ $\mathcal{L}_A[\Lambda_\mu,Q_\mu]\to \frac{f[0]\,g_1^2 B_{\mu\nu}B^{\mu\nu}}{8\pi^2}+\frac{f[0]\,g_2^2 W_{\mu\nu}{}^{a}W^{\mu\nu}{}_{a}}{24\pi^2}$

Imposing $\left\{\frac{f[0]\,g_1^2}{8\pi^2}\to\frac{1}{4},\ \frac{f[0]\,g_2^2}{24\pi^2}\to\frac{1}{4}\right\}$

$\to$ $\boxed{\mathcal{L}_A[\Lambda_\mu,Q_\mu]\to \frac{1}{4}B_{\mu\nu}B^{\mu\nu}+\frac{1}{4}W_{\mu\nu}{}^{a}W^{\mu\nu}{}_{a}}$

$\to$ $\boxed{3\,g_1^2\to g_2^2}$

This relationship stems from the imposed condition and may be arbitrary.

```
PR["• Evaluate: ", $ = selectGWS[{tuDDown[iD̃][_, μ]}, {}] /. xSum[a_, _] → a,
 NL, "The scaling for H' drops out and using ",
 $s = tuRule[selectGWS[#, {"coupling"}] & /@ {T[Λ, "d", {_}], T[Q, "du", {_, _}]}] //
   tuAddPatternVariable[{a, μ}],
 Yield, $e515 = $ = $ /. $s /. H' → H;
 $ // Framed, accumGWS[$];
 CG[" (5.15)"], accumGWS[$]
]
```

• Evaluate: $\underline{\tilde{D}}_{\mu}[H] \to -i\,(\frac{1}{2} g_1 B_{\mu}).H + i\,(\frac{1}{2} g_2 W_{\mu}{}^{j}\,\sigma_j).H + \underline{\partial}_{\mu}[H]$

The scaling for H' drops out and using $\{\Lambda_{\mu\_} \to \frac{1}{2} g_1 B_{\mu},\ Q_{\mu\_}{}^{a\_} \to \frac{1}{2} g_2 W_{\mu}{}^{a}\}$

→ $\boxed{\tilde{D}_{-\mu}[H] \to -i\,(\frac{1}{2} g_1 B_{\mu}).H + i\,(\frac{1}{2} g_2 W_{\mu}{}^{j}\,\sigma_j).H + \partial_{-\mu}[H]}$ (5.15)

5.5 The Higgs mechanism

```
PR["● The Higgs portion of the Lagrangian ",
 $ = tuRuleSelect[$defGWS][ℒH[__]] // Select[#, tuHasNoneQ[#, Φ] &] & // Last;
 $ = selectGWS[ℒH[__], H'];
 Yield,
 $higgsL = $ = $ /. tuRuleSolve[tuRuleSelect[$defGWS][H], H'] /. T[iD̄, "d", {m_}][a_] →
       tuDDown[iD][a, m](*convert earlier for consistency*) //
     tuDerivativeExpand[{f[0], a}] // tuOpSimplifyF[Abs, {1 / √(a f[0]) }];
 $ // ColumnSumExp,
 NL, "Assume scalar curvature ", $s = s → 0, ", minimize the Potential wrt H: ", $ =
  ℒHpot → (Apply[Plus, tuTermSelect[H][$]] /. tuDDown[iD][_, _] → 0) /. $s, accumGWS[$],
 Imply, "The non-zero minima is ",
 $ = 0 -> tuDPartial[$[[2]], Abs[H]] // tuDerivOps2D;
 $ = tuRuleSolve[$, Abs[H]];
 $ = #^2 & /@ $[[2]]; $ // Framed, CG[" (5.18)"], accumGWS[$],


 NL, " which is identified with the vacuum state of the Higgs field ",
 {v, 0} ⇒ ($ = v² → $[[1]]), accumGWS[$],
 line,
 next, "Simplify Higgs potential by unitary transform: ",
 $u = {H -> u.H, u[CG["U[1]×SU[2]"]], u → {{a, -cc[b]}, {b, cc[a]}}, a cc[a] + cc[b] b → 1};
 $u // MatrixForms // ColumnBar,
 NL, "For general Higgs doublet: ", $ = {{h1, h2} → u .{Abs[H], 0}, h1|2[CG[ℂ]]},
 yield, $ = $ /. tuRuleSelect[$u][u]; $ // ColumnForms,
 Imply, "Can express ", $e519 = $ = {H → u[x].{{v + h[x]}, {0}},
    u[x] -> {{a[x], -cc[b[x]]}, {b[x], cc[a[x]]}}, h[x] → Abs[H[x]] - v};
 $ // ColumnBar,
 CR["u[x] transform is the gauge freedom of H."],

 NL, "Re-express ", $0 = selectGWS[ℒHpot],
 NL, "in terms of ", $h2 = $ = Abs[H];
 $ = ($ /. $e519 /. Abs → xAbs /. tuRuleSelect[$e519][u[x]] /.
      xAbs[vv : {a_, b_}] → √(ct[{{a}, {b}}].{{a}, {b}}) //
     tuConjugateTransposeSimplify[{v, h[x]}, {a[x], b[x], h[x], v}]) // Simplify;
 $h2 = $h2 → ($ /. (tuRuleSelect[$u][a cc[a] + b cc[b]] /. {a → a[x], b → b[x]}) // Flatten //
     Last),
 Yield, $ = $0 /. $h2,
 NL, "Substituting ", v², yield, $s = selectGWS[Abs[H]^2],
 yield, $s[[1]] = v^2; $s,
 Yield, $ = tuEliminate[{$, $s}, f2],
 Yield, $ = Solve[$, ℒHpot][[1, 1]] // Collect[#, v] &;
 $ // Framed, CG[" (5.20)"], accumGWS[{$, $e519, $u, $h2}],
 NL, "Note {mass,interaction,cosmological} terms with ", {h[x]², h[x]^(n>2), h[x]^"0"}
]
```

● The Higgs portion of the Lagrangian

$$\to \mathcal{L}_H[g_{\mu\nu}, \Lambda_\mu, Q_\mu, \frac{H\pi}{\sqrt{a\,f[0]}}] \to \sum\left[\begin{array}{l} \frac{1}{12} s\,\text{Abs}[H]^2 \\ \frac{1}{2}\text{Abs}[D_{-\mu}[H]]^2 \\ \frac{b\pi^2\,\text{Abs}[H]^4}{2a^2 f[0]} \\ \frac{d\,f[0]}{4\pi^2} \\ \frac{c\,s\,f[0]}{24\pi^2} \\ -\frac{c\Lambda^2 f_2}{\pi^2} \\ \frac{\text{Abs}[H]^2 (e\,f[0]-2a\Lambda^2 f_2)}{a\,f[0]} \end{array}\right]$$

Assume scalar curvature $s \to 0$

, minimize the Potential wrt H: $\mathcal{L}_{Hpot} \to \frac{e\,\text{Abs}[H]^2}{a} + \frac{b\pi^2\,\text{Abs}[H]^4}{2a^2 f[0]} - \frac{2\Lambda^2\,\text{Abs}[H]^2 f_2}{f[0]}$

⇒ The non-zero minima is $\boxed{\text{Abs}[H]^2 \to \frac{a(-e\,f[0]+2a\Lambda^2 f_2)}{b\pi^2}}$ (5.18)

which is identified with the vacuum state of the Higgs field $\{v, 0\} \Rightarrow (v^2 \to \text{Abs}[H]^2)$

◆Simplify Higgs potential by unitary transform: $\left|\begin{array}{l} H \to u.H \\ u[U[1]\times SU[2]] \\ u \to \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \\ a\,a^* + b\,b^* \to 1 \end{array}\right.$

For general Higgs doublet: $\{\{h_1, h_2\} \to u.\{\text{Abs}[H], 0\}, h_{1|2}[\mathbb{C}]\} \longrightarrow \left|\begin{array}{l} \begin{array}{l} h_1 \\ h_2 \end{array} \to \begin{array}{l} a\,\text{Abs}[H] \\ b\,\text{Abs}[H] \end{array} \\ h_{1|2}[\mathbb{C}] \end{array}\right.$

⇒ Can express $\left|\begin{array}{l} H \to u[x].\{\{v+h[x]\}, \{0\}\} \\ u[x] \to \{\{a[x], -b[x]^*\}, \{b[x], a[x]^*\}\} \\ h[x] \to -v + \text{Abs}[H[x]] \end{array}\right.$

u[x] transform is the gauge freedom of H.

Re-express $\mathcal{L}_{Hpot} \to -\frac{b\pi^2 v^4}{2a^2 f[0]} + \frac{2b\pi^2 v^2 h[x]^2}{a^2 f[0]} + \frac{2b\pi^2 v\,h[x]^3}{a^2 f[0]} + \frac{b\pi^2 h[x]^4}{2a^2 f[0]}$

in terms of $\text{Abs}[H] \to \sqrt{(v+h[x])^2}$

$\to \mathcal{L}_{Hpot} \to -\frac{b\pi^2 v^4}{2a^2 f[0]} + \frac{2b\pi^2 v^2 h[x]^2}{a^2 f[0]} + \frac{2b\pi^2 v\,h[x]^3}{a^2 f[0]} + \frac{b\pi^2 h[x]^4}{2a^2 f[0]}$

Substituting $v^2 \longrightarrow \text{Abs}[H]^2 \to \frac{a(-e\,f[0]+2a\Lambda^2 f_2)}{b\pi^2} \longrightarrow v^2 \to \frac{a(-e\,f[0]+2a\Lambda^2 f_2)}{b\pi^2}$

$\to 2a^2\,\mathcal{L}_{Hpot} == \frac{b\pi^2(-v^4 + 4v^2 h[x]^2 + 4v\,h[x]^3 + h[x]^4)}{f[0]}$ && $b \neq 0$ && $f[0] \neq 0$

$\to \boxed{\mathcal{L}_{Hpot} \to -\frac{b\pi^2 v^4}{2a^2 f[0]} + \frac{2b\pi^2 v^2 h[x]^2}{a^2 f[0]} + \frac{2b\pi^2 v\,h[x]^3}{a^2 f[0]} + \frac{b\pi^2 h[x]^4}{2a^2 f[0]}}$ (5.20)

Note {mass,interaction,cosmological} terms with $\{h[x]^2, h[x]^{n>2}, h[x]^0\}$

5.5.1 Massive gauge bosons

```
PR["● The Higgs Lagrangian at mininum potential ",
 $ = $higgsL;
 $ = $ /. s → 0 /. (√# & /@ selectGWS[Abs[H]^2]) // Expand,
 NL, "The kinetic energy portion: ", $[[2]] = $[[2]] // tuTermSelect[H] // First;
 $ = $ /. ℒH → ℒkin,
 NL, "and the gauge fields must be invariant under gauge transform ",
 $e519 // ColumnBar,
 NL, "Examine ", $ = selectGWS[tuDDown[iD̃][H, μ]],
 NL, "• ", T[B, "d", {μ}],
 " is a scalar so is unaffected by u[x].  The W term transforms as ",
 $ = {selectGWS[T[Q, "d", {_}], q], selectGWS[T[Q, "d", {_}], W]};
 $ // ColumnBar,
 Imply, $ = $[[1]] /. $[[2]] /. q → u //. tuOpSimplify[Dot, {g_}];
 xtmp = $ = 2 / g2 # & /@ $ // Simplify, accumGWS[$]
]
PR[line, "Check invariance of: ",
 $ = xtmp,
 Yield, $ = #.u.H & /@ $ // expandDC[] // tuOpSimplifyF[Dot, {g_}],
 NL, "Since ",
 $s = $s0 = ct[u].u → 1,
 yield,
 $s = tuDPartial[#, μ] & /@ $s // tuDerivativeExpand[] // tuConjugateTransposeSimplify[
      {μ}] // tuRuleSolve[#, tuDPartial[_, μ].u] & // First,
 Yield, $ = $ /. $s /. $s0;
 Yield, $ = $ // expandDC[], CR[" ??"],
 line
]
```

● The Higgs Lagrangian at mininum potential

$$\mathcal{L}_H[g_{\mu\nu}, \Lambda_\mu, Q_\mu, \frac{H\pi}{\sqrt{a\,f[0]}}] \to \frac{1}{2}\mathrm{Abs}[\underline{D}_\mu[H]]^2 + \frac{d\,f[0]}{4\pi^2} - \frac{e^2 f[0]}{2 b \pi^2} - \frac{c\,\Lambda^2 f_2}{\pi^2} + \frac{2\,a\,e\,\Lambda^2 f_2}{b\,\pi^2} - \frac{2\,a^2 \Lambda^4 f_2^2}{b\,\pi^2 f[0]}$$

The kinetic energy portion: $\mathcal{L}_{kin}[g_{\mu\nu}, \Lambda_\mu, Q_\mu, \frac{H\pi}{\sqrt{a\,f[0]}}] \to \frac{1}{2}\mathrm{Abs}[\underline{D}_\mu[H]]^2$

and the gauge fields must be invariant under gauge transform

$H \to u[x].\{\{v + h[x]\}, \{0\}\}$
$u[x] \to \{\{a[x], -b[x]^*\}, \{b[x], a[x]^*\}\}$
$h[x] \to -v + \mathrm{Abs}[H[x]]$

Examine $\underline{\tilde{D}}_\mu[H] \to -i\,(\frac{1}{2} g_1 B_\mu).H + i\,(\frac{1}{2} g_2 W_\mu{}^j \sigma_j).H + \underline{\partial}_\mu[H]$

• B_μ is a scalar so is unaffected by u[x]. The W term transforms as

$Q_\mu \to -i\,q.\partial_{-\mu}[q^\dagger] + q.Q_\mu.q^\dagger$
$Q_\mu \to \frac{1}{2} g_2 W_\mu$

⇒ $W_\mu \to u.W_\mu.u^\dagger - \frac{2\,i\,u.\underline{\partial}_\mu[u^\dagger]}{g_2}$

Check invariance of: $W_\mu \to u.W_\mu.u^\dagger - \frac{2\,i\,u.\underline{\partial}_\mu[u^\dagger]}{g_2}$

→ $W_\mu.u.H \to u.W_\mu.u^\dagger.u.H - \frac{2\,i\,u.\underline{\partial}_\mu[u^\dagger].u.H}{g_2}$

Since $u^\dagger.u \to 1 \longrightarrow \underline{\partial}_\mu[u^\dagger].u \to -u^\dagger.\underline{\partial}_\mu[u]$

→

→ $W_\mu.u.H \to u.W_\mu.H + \frac{2\,i\,u.u^\dagger.\underline{\partial}_\mu[u].H}{g_2}$??

```
PR[next, "From ", $ = $e515, $real = {v, h[x], g_, Tensor[W | B, _, _], μ};
 Yield, $ = $ // tuIndexSum[{j}, {1, 2, 3}],
 Yield, $ = $ // expandDC[] // (# //. tuOpSimplify[Dot, {g_, Tensor[W | B, _, _]}] &),
 NL, "Take ", $s = selectGWS[H, u[x]] /. u[x] → 1 // expandDC[],
 Yield, $[[2]] =
  $[[2]] /. Plus → Inactive[Plus] /. $s /. tuPauliExpand // tuDerivativeExpand[{v}];
 $ // MatrixForms // ColumnSumExp,
 $ = {$, tuIndicesRaise[μ][$]};
 NL, "Compute ",
 $ = ct[$[[1, 1]]].($[[2, 1]]) → (ct[$[[1, 1]]].$[[2, 1]] /. $) // Activate,
 Yield, $ = $ // tuConjugateSimplify[$real] // tuIndexDummyOrdered // Simplify;
 $[[1]] = Abs[tuDDown[iD̃][H, μ]]^2;
 $[[2]] = Flatten[$[[2]]] // Last;
 ($d2 = $) // ColumnSumExp // Framed,
 NL, CR[ Plus @@ tuTermSelect[{B, W}][Expand[$]] // Simplify,
  " gives the electro-weak mixing angle between the gauge fields."],
 NL, "defined as ", $ = {c_w → Cos[θ_w], Cos[θ_w] → g₂ / √(g₁^2 + g₂^2),
   s_w → Sin[θ_w], Sin[θ_w] → g₁ / √(g₁^2 + g₂^2)};
 $ // ColumnBar, accumGWS[{$d2, $}],
 NL, "Given the relation ", $sg = tuRuleSolve[selectGWS[a_ g₁^2], g₂] // Last,
 Yield, $ = tuRuleSelect[$ /. $sg][{Cos[_], Sin[_]}];
 $ = Map[#^2 & /@ # &, $];
 $ // ColumnBar, accumGWS[$],
 CR["at the electroweak unification scale ", Λ_EW, ". Why at this scale?"]
]
```

◆From $\tilde{\underline{D}}_\mu[H] \to -i\left(\frac{1}{2} g_1 B_\mu\right).H + i\left(\frac{1}{2} g_2 W_\mu{}^j \sigma_j\right).H + \underline{\partial}_\mu[H]$

→ $\tilde{\underline{D}}_\mu[H] \to -i\left(\frac{1}{2} g_1 B_\mu\right).H + i\left(\frac{1}{2} g_2 W_\mu{}^1 \sigma_1 + \frac{1}{2} g_2 W_\mu{}^2 \sigma_2 + \frac{1}{2} g_2 W_\mu{}^3 \sigma_3\right).H + \underline{\partial}_\mu[H]$

→ $\tilde{\underline{D}}_\mu[H] \to -\frac{1}{2} i H g_1 B_\mu + i\left(\frac{1}{2}\sigma_1.H g_2 W_\mu{}^1 + \frac{1}{2}\sigma_2.H g_2 W_\mu{}^2 + \frac{1}{2}\sigma_3.H g_2 W_\mu{}^3\right) + \underline{\partial}_\mu[H]$

Take $H \to \{\{v + h[x]\}, \{0\}\}$

→ $\tilde{\underline{D}}_\mu[H] \to \sum\left[\begin{array}{l} \begin{pmatrix} -\frac{1}{2} i (v+h[x]) g_1 B_\mu \\ 0 \end{pmatrix} \\ i\left(\begin{pmatrix} 0 \\ \frac{1}{2}(v+h[x]) g_2 W_\mu{}^1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} i (v+h[x]) g_2 W_\mu{}^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(v+h[x]) g_2 W_\mu{}^3 \\ 0 \end{pmatrix}\right) \\ \begin{pmatrix} \underline{\partial}_\mu[h[x]] \\ 0 \end{pmatrix} \end{array}\right]$

Compute $\tilde{\underline{D}}_\mu[H]^\dagger.\tilde{\underline{D}}^\mu[H] \to$

$\{\{(\frac{1}{2}(v+h[x]) g_2 W_\mu{}^1 + \frac{1}{2} i (v+h[x]) g_2 W_\mu{}^2)^* (\frac{1}{2}(v+h[x]) g_2 W^{\mu 1} + \frac{1}{2} i (v+h[x]) g_2 W^{\mu 2}) +$

$(-\frac{1}{2} i (v+h[x]) g_1 B_\mu + \frac{1}{2} i (v+h[x]) g_2 W_\mu{}^3 + \underline{\partial}_\mu[h[x]])^*$

$(-\frac{1}{2} i (v+h[x]) g_1 B^\mu + \frac{1}{2} i (v+h[x]) g_2 W^{\mu 3} + \underline{\partial}^\mu[h[x]])\}\}$

→ $\boxed{\mathrm{Abs}[\underline{D}_\mu[H]]^2 \to \frac{1}{4}\sum\left[\begin{array}{l} (v+h[x])^2 g_1^2 B_\mu B^\mu \\ -2 (v+h[x])^2 g_1 g_2 B^\mu W_\mu{}^3 \\ (v+h[x])^2 g_2^2 (W_\mu{}^1 W^{\mu 1} + W_\mu{}^2 W^{\mu 2} + W_\mu{}^3 W^{\mu 3}) \\ 4\, \underline{\partial}_\mu[h[x]]\, \underline{\partial}^\mu[h[x]] \end{array}\right]}$

$-\frac{1}{2}(v+h[x])^2 g_1 g_2 B^\mu W_\mu{}^3$

gives the electro-weak mixing angle between the gauge fields.

defined as $\left|\begin{array}{l} c_w \to \mathrm{Cos}[\theta_w] \\ \mathrm{Cos}[\theta_w] \to \frac{g_2}{\sqrt{g_1^2+g_2^2}} \\ s_w \to \mathrm{Sin}[\theta_w] \\ \mathrm{Sin}[\theta_w] \to \frac{g_1}{\sqrt{g_1^2+g_2^2}} \end{array}\right.$

Given the relation $g_2 \to \sqrt{3}\, g_1$

→ $\left|\begin{array}{l} \mathrm{Cos}[\theta_w]^2 \to \frac{3}{4} \\ \mathrm{Sin}[\theta_w]^2 \to \frac{1}{4} \end{array}\right.$ at the electroweak unification scale Λ_{EW}. Why at this scale?

```
PR["Define ", $e521 = $ = {T[W, "d", {μ}] → (T[W, "du", {μ, 1}] + I T[W, "du", {μ, 2}]) / √2 ,
    cc[T[W, "d", {μ}]] → (T[W, "du", {μ, 1}] - I T[W, "du", {μ, 2}]) / √2 ,
    T[Z, "d", {μ}] → cw T[W, "du", {μ, 3}] - sw T[B, "d", {μ}],
    T[A, "d", {μ}] → sw T[W, "du", {μ, 3}] + cw T[B, "d", {μ}]
   }; $ // ColumnBar,
 NL, "From ", $d2 = selectGWS[{Abs[_]^2, iD̃}],
 NL, "we see that ", {T[W, "du", {μ, 1}], T[W, "du", {μ, 2}]}, " are mass eigenstates.",
 NL, "Inverting ",
 $s = $ = tuRuleSolve[$e521, {T[W, "du", {μ, 1}], T[W, "du", {μ, 2}], T[W, "du", {μ, 3}],
       T[B, "d", {μ}]}] /. Map[#^2 & /@ # &, selectGWS[#] & /@ {cw, sw}] // Simplify;

 $s = {$s, $s // tuIndicesRaise[{μ}]} // Flatten; $s // ColumnBar,
 Imply, $ = $d2 /. $s // Expand // Simplify; $ // ColumnSumExp,
 Yield, $ = $ // tuIndexDummyOrdered // Collect[#, {Tensor[Z, _, _],
      Tensor[A, _, _], Tensor[W, _, _], cc[Tensor[W, _, _]]}, Simplify] &;
 $ // ColumnSumExp,
 NL, "Given ", $s = (selectGWS[#] & /@ {sw, cw, Cos[θw], Sin[θw]}) /. $sg // PowerExpand,
 Imply, $ = $ /. (tuRuleSolve[selectGWS[a_ g1^2], g1] // Last) //. $s // Simplify;
 $ // ColumnSumExp, CK,
 NL, "So W's, and Z's acquire a mass term, but A's do not.",
 CR["Do A's interact with h's? Consider interaction terms."],
 NL, "Let the masses be ", {MW → v g2 / 2, MZ → v g2 / (2 cw)}

]
```

Define $\left|\begin{array}{l} W_\mu \to \frac{W_\mu{}^1 + i\, W_\mu{}^2}{\sqrt{2}} \\ (W_\mu)^* \to \frac{W_\mu{}^1 - i\, W_\mu{}^2}{\sqrt{2}} \\ Z_\mu \to -s_w B_\mu + c_w W_\mu{}^3 \\ A_\mu \to c_w B_\mu + s_w W_\mu{}^3 \end{array}\right.$

From $\text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to \frac{1}{4}((v+h[x])^2 g_1^2 B_\mu B^\mu - 2(v+h[x])^2 g_1 g_2 B^\mu W_\mu{}^3 + (v+h[x])^2 g_2^2 (W_\mu{}^1 W^{\mu 1} + W_\mu{}^2 W^{\mu 2} + W_\mu{}^3 W^{\mu 3}) + 4\, \underline{\partial}_\mu[h[x]]\, \underline{\partial}^\mu[h[x]])$

we see that $\{W_\mu{}^1, W_\mu{}^2\}$ are mass eigenstates.

Inverting $\left|\begin{array}{l} W_\mu{}^1 \to \frac{(W_\mu)^* + W_\mu}{\sqrt{2}} \\ W_\mu{}^2 \to \frac{i((W_\mu)^* - W_\mu)}{\sqrt{2}} \\ W_\mu{}^3 \to s_w A_\mu + c_w Z_\mu \\ B_\mu \to c_w A_\mu - s_w Z_\mu \\ W^{\mu 1} \to \frac{(W^\mu)^* + W^\mu}{\sqrt{2}} \\ W^{\mu 2} \to \frac{i((W^\mu)^* - W^\mu)}{\sqrt{2}} \\ W^{\mu 3} \to s_w A^\mu + c_w Z^\mu \\ B^\mu \to c_w A^\mu - s_w Z^\mu \end{array}\right.$

$$\Rightarrow \text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to \frac{1}{4}\sum\left[\left|\begin{array}{l} (v+h[x])^2 g_2^2 (s_w^2 A_\mu A^\mu + (W^\mu)^* W_\mu + (W_\mu)^* W^\mu) \\ 2(v+h[x])^2 g_1 g_2 s_w^2 A_\mu Z^\mu \\ v^2 g_1^2 s_w^2 Z_\mu Z^\mu \\ 2\, v\, h[x]\, g_1^2 s_w^2 Z_\mu Z^\mu \\ h[x]^2 g_1^2 s_w^2 Z_\mu Z^\mu \\ (v+h[x])^2 c_w^2 (g_1^2 A_\mu A^\mu - 2 g_1 g_2 A^\mu Z_\mu + g_2^2 Z_\mu Z^\mu) \\ -(v+h[x])^2 c_w s_w (g_1^2 (A^\mu Z_\mu + A_\mu Z^\mu) - g_2^2 (A^\mu Z_\mu + A_\mu Z^\mu) + 2 g_1 g_2 (A_\mu A^\mu - Z_\mu Z^\mu)) \\ 4\, \underset{-\mu}{\partial}[h[x]]\, \underline{\partial}^\mu[h[x]] \end{array}\right.\right]$$

$$\to \text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to$$

$$\sum\left[\left|\begin{array}{l} \frac{1}{4}(v+h[x])^2 (c_w g_1 - g_2 s_w)^2 A_\mu A^\mu \\ \frac{1}{2}(W^\mu)^* (v+h[x])^2 g_2^2 W_\mu \\ Z_\mu \left(-\frac{1}{2}(v+h[x])^2 (c_w^2 g_1 g_2 + c_w (g_1^2 - g_2^2) s_w - g_1 g_2 s_w^2) A^\mu + \frac{1}{4}(v+h[x])^2 (c_w g_2 + g_1 s_w)^2 Z^\mu\right) \\ \underset{-\mu}{\partial}[h[x]]\, \underline{\partial}^\mu[h[x]] \end{array}\right.\right]$$

Given $\{s_w \to \text{Sin}[\theta_w],\ c_w \to \text{Cos}[\theta_w],\ \text{Cos}[\theta_w] \to \frac{\sqrt{3}}{2},\ \text{Sin}[\theta_w] \to \frac{1}{2}\}$

$$\Rightarrow \text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to \frac{1}{18}\sum\left[\left|\begin{array}{l} 9\,(W^\mu)^* (v+h[x])^2 g_2^2 W_\mu \\ 6\,(v+h[x])^2 g_2^2 Z_\mu Z^\mu \\ 18\, \underset{-\mu}{\partial}[h[x]]\, \underline{\partial}^\mu[h[x]] \end{array}\right.\right] \longleftarrow \text{CHECK}$$

So W's, and Z's acquire a mass term, but A's do not.

Do A's interact with h's? Consider interaction terms.

Let the masses be $\{M_W \to \frac{v\, g_2}{2},\ M_Z \to \frac{v\, g_2}{2\, c_w}\}$