```
<< Local`QFTToolKit`
```

AdS/CFT User Guide by Natsuume

2. General relativity and black holes

Energy-momentum tensor and Einstein equation of concern here

```
e227 = T[T, "dd", {μ, ν}] → (-Λ/(8 π G)) T[g, "dd", {μ, ν}]
e228 = T[R, "dd", {μ, ν}] - T[g, "dd", {μ, ν}] R / 2 + T[g, "dd", {μ, ν}] Λ → 0
```

$$T_{\mu\nu} \to -\frac{\Lambda\, g_{\mu\nu}}{8\, G\, \pi}$$

$$-\frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} + R_{\mu\nu} \to 0$$

Schwarzschild black hole

```
$ = (1 - 2 G M / r);
e230 = d[s]^2 → -($) d[t]^2 + (1 / $) d[r]^2 + (r^2) d[Ω2]^2
```

$$d[s]^2 \to \frac{d[r]^2}{1-\frac{2\,G\,M}{r}} + \left(-1+\frac{2\,G\,M}{r}\right) d[t]^2 + r^2\, d[\Omega_2]^2$$

p.24

3. Black holes and thermodynamics

4.2 Large - N_c gauge theory

```
PR["Example U[Nc] gauge theory, the gauge field: ",
 T[A, "d", {μ}] -> T[A, "dud", {μ, i, j}], back, Nc × Nc, " matrix",
 NL, "Define 't Hooft coupling: ", λ → gYM^2 Nc,
 " in theory {λ,Nc} are independent.  The large Nc limit: ",
 {Nc → ∞, λ → "fixed"}, imply, {λ ≫ 1 ⇒ "strong coupling"},
 NL, "For Lagrangian: ",
 ℒ → (1/gYM^2 (tuPartialD[A, _].tuPartialD[A, _] + A^2.tuPartialD[A, _] + A^4) → (Nc / λ).(…)),
 NL, "Amplitude factors: ",
 ($s = {"Propagator" → λ / Nc, "vertex" → Nc / λ, "loop" → Nc}) // Column,
 Yield, $ = "Amplitude" → "vertex"^Nv "Propagator"^Np "loop"^Nl,
 Yield, $ = $ /. $s,
 yield, $0 = $ = $ // PowerExpand,
 NL, "Resulting amplitude for planar diagrams: ", xSum[ai λ^i, {i, 0, ∞}] Nc^2,
 NL, "Non-planar and planar amplitude:", xSum[fi[λ] Nc^(-i + 2), {i, 0, ∞}],
 NL, "Relationship to topology via Euler characteristic: ",
 χ → V - "E" + F, " on a polygon to ", $0,
 NL, "the correspondence: ",
 {F, -"E", V} → $0[[2]] /. a_ Nc^i_ :→ Sort[Apply[List, i], Greater],
 NL, "Can get partition function: ",
 {$Zg = Log[Zgauge] → xSum[Nc^χ fh[λ], {h, 0, ∞}], h → "# of holes"} // FramedColumn
]
```

Example U[N_c] gauge theory, the gauge field: $A_\mu \to A_\mu{}^i{}_j \longleftarrow N_c \times N_c$ matrix

Define 't Hooft coupling: $\lambda \to g_{YM}^2 N_c$

in theory {λ,N_c} are independent. The large N_c limit:

$\{N_c \to \infty,\ \lambda \to \text{fixed}\} \Rightarrow \{\lambda \gg 1 \Rightarrow \text{strong coupling}\}$

For Lagrangian: $\mathcal{L} \to \dfrac{A^4 + A^2.\underline{\partial}\,[A] + \underline{\partial}\,[A].\underline{\partial}\,[A]}{g_{YM}^2} \to \dfrac{N_c}{\lambda}.\ldots$

Amplitude factors: $\begin{array}{l} \text{Propagator} \to \frac{\lambda}{N_c} \\ \text{vertex} \to \frac{N_c}{\lambda} \\ \text{loop} \to N_c \end{array}$

$\to$ $\text{Amplitude} \to \text{loop}^{N_l}\ \text{Propagator}^{N_p}\ \text{vertex}^{N_v}$

$\to$ $\text{Amplitude} \to \left(\frac{\lambda}{N_c}\right)^{N_p} N_c^{N_l} \left(\frac{N_c}{\lambda}\right)^{N_v} \longrightarrow \text{Amplitude} \to \lambda^{N_p - N_v} N_c^{N_l - N_p + N_v}$

Resulting amplitude for planar diagrams: $N_c^2 \sum_{\{i,0,\infty\}} [\lambda^i a_i]$

Non-planar and planar amplitude: $\sum_{\{i,0,\infty\}} [N_c^{2-i} f_i[\lambda]]$

Relationship to topology via Euler characteristic:

$\chi \to -E + F + V$ on a polygon to $\text{Amplitude} \to \lambda^{N_p - N_v} N_c^{N_l - N_p + N_v}$

the correspondence: $\{F, -E, V\} \to \{N_l, -N_p, N_v\}$

Can get partition function: $\boxed{\begin{array}{l} \text{Log}[Z_{gauge}] \to \sum_{\{h,0,\infty\}} [N_c^{\chi} f_h[\lambda]] \\ h \to \#\ \text{of holes} \end{array}}$

Supergravity

```
PR["(5.6): ",
 e56 = {S_SGravity → 1/(16 π G_10) tuIntegral[{{x^10}}, √(-g) R], G_10 -> " Newton's constant: "},
 NL, "with approximation: ",
 T[g, "dd", {μ, ν}] -> T[η, "dd", {μ, ν}] + T[h, "dd", {μ, ν}],
 Imply, $g1 = "graviton emission rate " ∝ √G_10 ,
 NL, $g2 = "Closed (graviton) string emission rate" ∝ g_s,
 Imply, G_10 ∝ g_s,
 NL, $g3 = "Open string emission rate" ∝ g̃_s ∝ √g_s ,
 NL, "Gauge field action: ",
 e58 = S_gauge → 1/g_YM^2 tuIntegral[{{x^(p+1)}}, tuDPartial[A, _]^2 + A^2 tuDPartial[A, _] + A^4],
 " for p+1-dim gauge theory on D-branes",
 Yield, $g4 = "gauge field emission rate" ∝ g_YM,
 NL, $G10 = dim[G_10] → L^8,
 and, "string scale" → l_s,
 imply, $G10 = G_10 ∝ g_s^2 l_s^2,
 NL, dim[A] → 1 / L, imply, dim[g_YM^2] → L^(p-3), imply, $2 = g_YM^2 ∝ g_s l_s^(p-3),
 Yield, {$1, $2} // FramedColumn, CG[" (5.10)"]
]
```

(5.6): $\{S_{SGravity} \to \frac{\int_{\{x^{10}\}}[\sqrt{-g}\,R]}{16\,\pi\,G_{10}}$, $G_{10} \to$ Newton's constant: }

with approximation: $g_{\mu\nu} \to h_{\mu\nu} + \eta_{\mu\nu}$

$\Rightarrow$ graviton emission rate $\propto \sqrt{G_{10}}$

Closed (graviton) string emission rate $\propto g_s$

$\Rightarrow$ $G_{10} \propto g_s$

Open string emission rate $\propto \tilde{g}_s \propto \sqrt{g_s}$

Gauge field action: $S_{gauge} \to \frac{\int_{\{x^{1+p}\}}[A^4 + A^2\,\underline{\partial}\,[A] + \underline{\partial}\,[A]^2]}{g_{YM}^2}$

  for p+1-dim gauge theory on D-branes

$\rightarrow$ gauge field emission rate $\propto g_{YM}$

$\dim[G_{10}] \to L^8$ and string scale $\to l_s$ $\Rightarrow$ $G_{10} \propto g_s^2\, l_s^2$

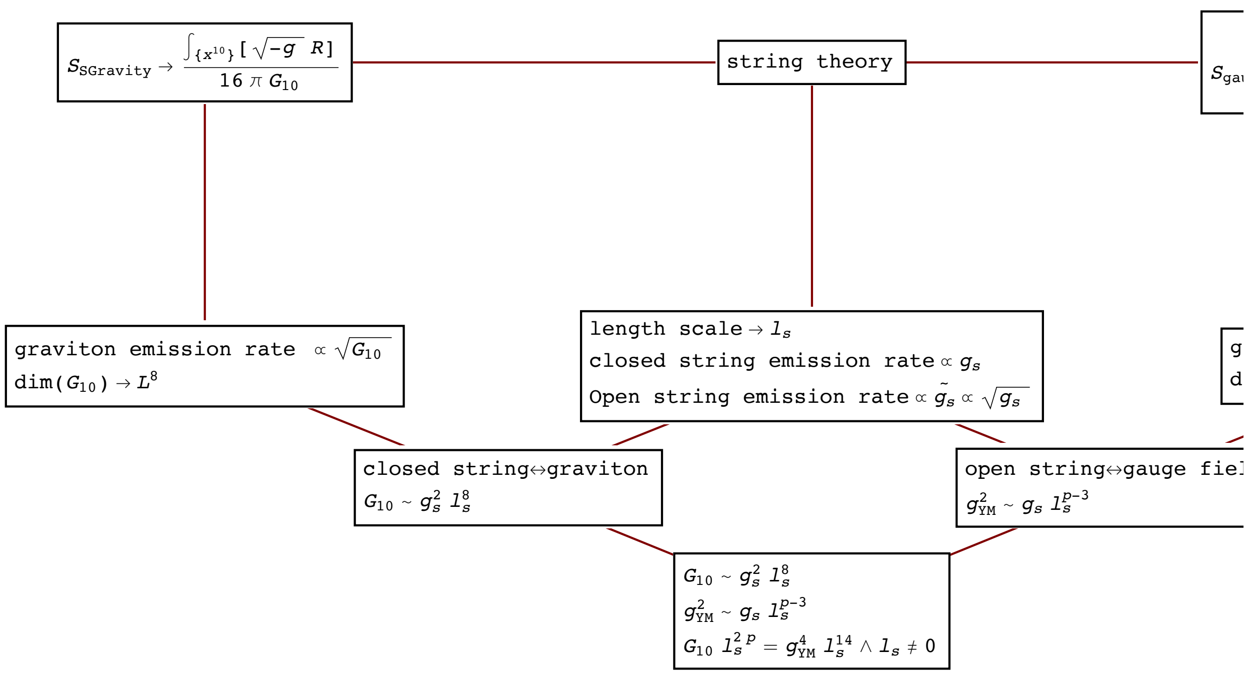
$\dim[A] \to \frac{1}{L}$ $\Rightarrow$ $\dim[g_{YM}^2] \to L^{-3+p}$ $\Rightarrow$ $g_{YM}^2 \propto g_s\, l_s^{-3+p}$

$\rightarrow$ $\boxed{\begin{array}{l} \$1 \\ g_{YM}^2 \propto g_s\, l_s^{-3+p} \end{array}}$ (5.10)

Graphic of relationship

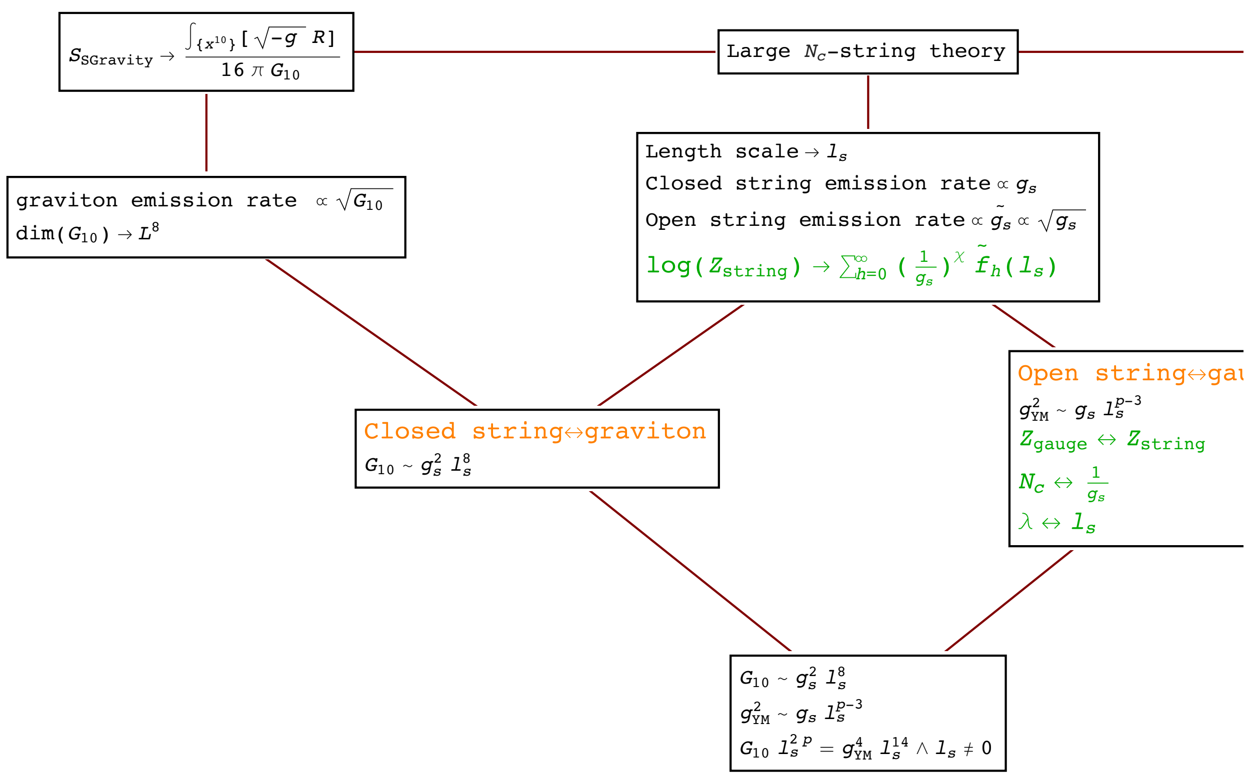
```
text[n_] := Switch[n,
   1, Framed[e56[[1]]],
   11, FramedColumn[{$g1, dim[G10] → L^8}],
   2, Framed["string theory"],
   21, FramedColumn[{"length scale" → ls, "closed string emission rate" ∝ gs, $g3}],
   3, Framed[e58],
   31, FramedColumn[{$g4, dim[gYM^2] → L^(p-3)}],
   1121, FramedColumn[{"closed string↔graviton", G10 ~ gs^2 ls^8}],
   2131, FramedColumn[{"open string↔gauge field", gYM^2 ~ gs ls^(p-3)}],
   11 212 131, FramedColumn[
    {G10 ~ gs^2 ls^8, gYM^2 ~ gs ls^(p-3), Eliminate[{G10 == gs^2 ls^8, gYM^2 == gs ls^(p-3)}, gs]}],
   _, n];

$wide = 2;
GraphPlot[{1 → 2, 2 → 3, 1 → 11, 2 → 21, 3 -> 31, 11 → 1121,
  1121 → 21, 21 → 2131, 31 → 2131, 2131 → 11 212 131, 1121 → 11 212 131},
 VertexCoordinateRules → {1 → {0, 0}, 2 → {$wide, 0}, 3 → {2 $wide, 0},
   11 → {0, -1}, 21 → {$wide, -1}, 31 → {2 $wide, -1},
   1121 → {.5 $wide, -1.4}, 2131 → {1.5 $wide, -1.4}},
 VertexRenderingFunction → (Text[text[#2], #1, Background → White] &)]
```



5.3.1 Comparison of partitian functions

```
$Zg;
$Zs = Log[Z_string] → Sum[(1 / g_s)^χ f̃_h[l_s], {h, 0, ∞}];
text[n_] := Switch[n,
  1, Framed[e56[[1]]],
  11, FramedColumn[{$g1, dim[G_10] → L^8}],
  2, Framed["Large N_c-string theory"],
  21, FramedColumn[
   {"Length scale" → l_s, "Closed string emission rate" ∝ g_s, $g3, CG[$Zs]}],
  3, Framed[e58],
  31, FramedColumn[{$g4, dim[g_YM^2] → L^(p-3), CG[$Zg]}],
  1121, FramedColumn[{CO["Closed string↔graviton"], G_10 ~ g_s^2 l_s^8}],
  2131, FramedColumn[{CO["Open string↔gauge field"],
    g_YM^2 ~ g_s l_s^(p-3), CG[Z_gauge ↔ Z_string, N_c ↔ 1 / g_s, λ ↔ l_s]}],
  11 212 131, FramedColumn[{G_10 ~ g_s^2 l_s^8, g_YM^2 ~ g_s l_s^(p-3),
    Eliminate[{G_10 == g_s^2 l_s^8, g_YM^2 == g_s l_s^(p-3)}, g_s]}],
  _, n]
GraphPlot[{1 → 2, 2 → 3, 1 → 11, 2 → 21, 3 -> 31, 11 → 1121,
  1121 → 21, 21 → 2131, 31 → 2131, 2131 → 11 212 131, 1121 → 11 212 131},
 VertexCoordinateRules → {1 → {0, 0}, 2 → {$wide, 0}, 3 → {2 $wide, 0},
   11 → {0, -.5}, 21 → {$wide, -.5}, 31 → {2 $wide, -.5},
   1121 → {.5 $wide, -1.2}, 2131 → {1.5 $wide, -1.2}, 11 212 131 → {$wide, -2}},
 VertexRenderingFunction → (Text[text[#2], #1, Background → White] &)]
```



```
PR["Quantum strings + Poincare invariance ISO[1,d-1]",
 imply, "spacetime", $dim = {dim[bosons] → d -> 26, dim[boson + fermion] → d -> 10};
 Column[$dim],
 NL, "String theory admits flat and curved spacetimes.  If only
   ISO[1,3]⇒flat spacetime, else⇒ISO[1,d-1], d>4, curved spacetime. ",
 NL, "•Here Large Nc gauge theories represented BY a 5-dim
   spacetime with ISO[1,3] symmetry(flat).",
 NL, "•Metric: ", $ds5 = d[s5]^2 → Ω[w]^2 (-d[t]^2 + d[x̄]^2) + d[w]^2,
 " with ISO[1,3] invariance on ", $ds4 = (-d[t]^2 + d[x̄]^2),
 NL, CG["●Demand scale invariance: "], $x = T[x, "u", {μ}] → a T[x, "u", {μ}],
 imply, $dA = T[A, "u", {μ}] -> T[A, "u", {μ}] / a, " as in Maxwell theory.",
 NL, CG["•Scale(μ) invariance for renomalized(1-loop) gauge theories."],
 NL, "β-function for SU[Nc] gauge theory: ",
 Yield, β[gYM] → μ tuDPartial[gYM, μ] → -11/(48 π^2) gYM^3 Nc (11 - 2 nfermion - nscalar / 2),
 yield, 0["for N=4 SYM", {nfermion → 4, nscalar → 6}],
 NL, CG["●The metric ", $ds5], " invariance to ", $x,
 imply, $ads = {$ds5[[1]] → (r / L)^2 $ds4 + L^2 d[r]^2 / r^2, r → L Exp[-w / L]};
 Framed[$ads],
 back, $ = {"AdS5spacetime", L → "AdS radius"},
 NL, $[[1]] ,
 " is solution to Einstein equation with negative cosmological constant.",
 Yield, {S5 → 1/(16 πG5) tuIntegral[{{x^5}}, √(-g5) (R5 - 2 Λ)], Λ → -12 / L^2},
 NL, CR["How are μ and a related?"]
]
```

Quantum strings + Poincare invariance ISO[1,d-1]

⇒ spacetime $\begin{array}{l}\text{dim[bosons]} \to d \to 26 \\ \text{dim[boson + fermion]} \to d \to 10\end{array}$

String theory admits flat and curved spacetimes. If only ISO[1,3]⇒flat spacetime, else⇒ISO[1,d-1], d>4, curved spacetime.

•Here Large N_c gauge theories represented BY a 5-dim spacetime with ISO[1,3] symmetry(flat).

•Metric: $d[s_5]^2 \to d[w]^2 + (-d[t]^2 + d[\bar{x}]^2)\,\Omega[w]^2$ with ISO[1,3] invariance on $-d[t]^2 + d[\bar{x}]^2$

●Demand scale invariance: $x^\mu \to a\, x^\mu$ ⇒ $A^\mu \to \frac{A^\mu}{a}$ as in Maxwell theory.

•Scale(μ) invariance for renomalized(1-loop) gauge theories.

β-function for SU[N_c] gauge theory:

$\to$ $\beta[g_{YM}] \to \mu\, \underset{\mu}{\partial}[g_{YM}] \to -\frac{11\, g_{YM}^3 \left(11 - 2\, n_{fermion} - \frac{n_{scalar}}{2}\right) N_c}{48\, \pi^2}$

$\longrightarrow$ 0[for $\mathcal{N}$=4 SYM, $\{n_{fermion} \to 4,\ n_{scalar} \to 6\}$]

●The metric $d[s_5]^2 \to d[w]^2 + (-d[t]^2 + d[\bar{x}]^2)\,\Omega[w]^2$ invariance to $x^\mu \to a\, x^\mu$ ⇒

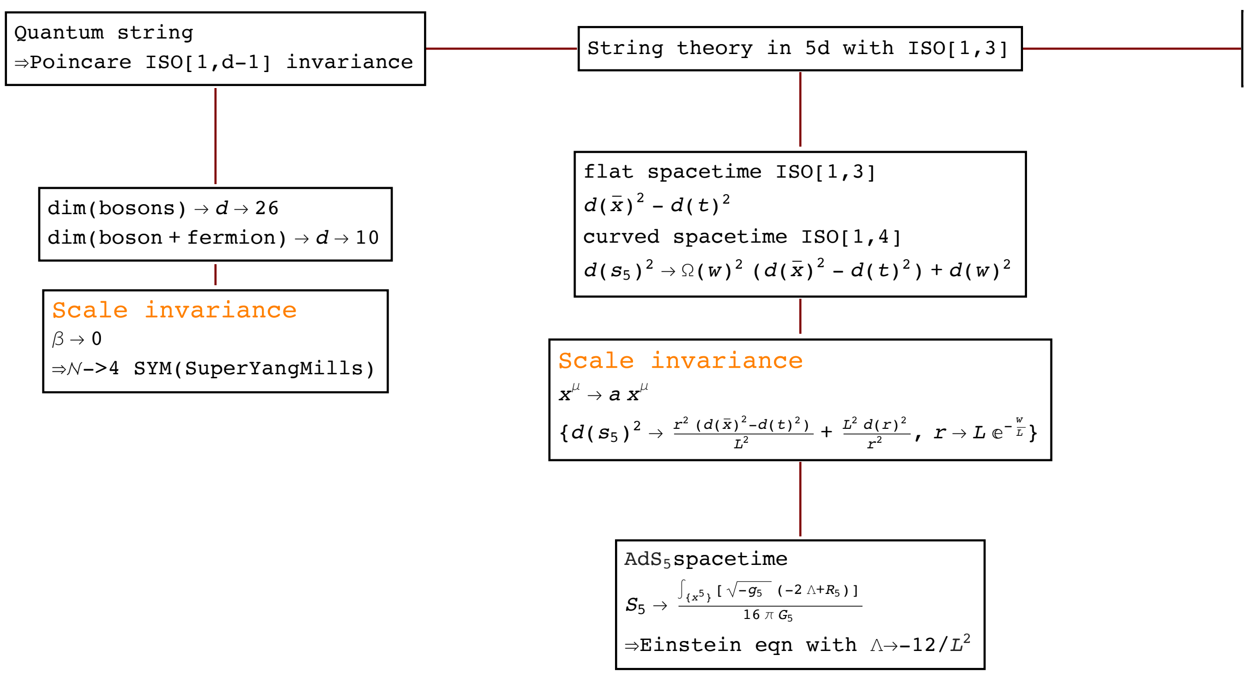
$$\boxed{\left\{d[s_5]^2 \to \frac{L^2\, d[r]^2}{r^2} + \frac{r^2\,(-d[t]^2 + d[\bar{x}]^2)}{L^2},\ r \to e^{-\frac{w}{L}}\, L\right\}} \longleftarrow \{\text{AdS}_5\text{spacetime},\ L \to \text{AdS radius}\}$$

AdS_5spacetime
is solution to Einstein equation with negative cosmological constant.

$\to$ $\left\{S_5 \to \frac{\int_{\{x^5\}}[\sqrt{-g_5}\ (-2\,\Lambda + R_5)]}{16\,\pi G_5},\ \Lambda \to -\frac{12}{L^2}\right\}$

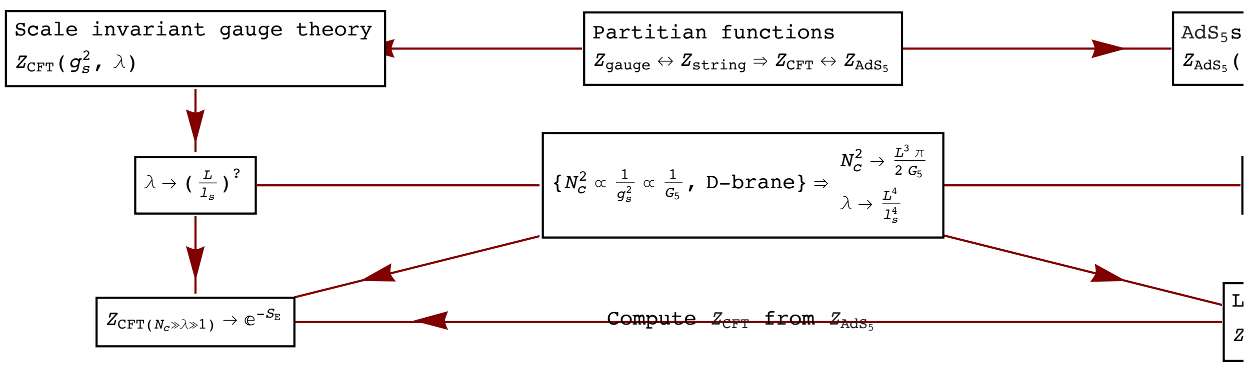
How are μ and a related?

```
text[n_] := Switch[n,
  1, FramedColumn[{"Quantum string", "⇒Poincare ISO[1,d-1] invariance"}],
  11, FramedColumn[$dim],
  12, FramedColumn[{CO["Scale invariance"], β → 0, "⇒N->4 SYM(SuperYangMills)"}],
  2, Framed["String theory in 5d with ISO[1,3]"],
  21,
  FramedColumn[{"flat spacetime ISO[1,3]", $ds4, "curved spacetime ISO[1,4]", $ds5}],
  3, FramedColumn[{"Large Nc Gauge theory", T[A, "u", {μ}]}],
  31,
  FramedColumn[{CO["Scale invariance"], $x,
    S → -1/(4 e^2) tuIntegral[{{x^4}}, T[F, "uu", {μ, ν}] T[F, "dd", {μ, ν}]], _ ⇒ $dA}],
  22, FramedColumn[{CO["Scale invariance"], $x, $ads}],
  23, FramedColumn[{"AdS5spacetime",
    S5 → 1/(16 π G5) tuIntegral[{{x^5}}, Sqrt[-g5] (R5 - 2 Λ)], "⇒Einstein eqn with Λ→-12/L^2"}],
  _, n]
GraphPlot[{1 → 2, 2 → 3, 1 → 11, 11 → 12, 2 → 21, 21 → 22, 22 → 23, 3 -> 31},
 VertexCoordinateRules →
  {1 → {0, 0}, 2 → {$wide, 0}, 3 → {2 $wide, 0}, 11 → {0, -.6}, 12 → {0, -1},
   21 → {$wide, -.6}, 22 → {$wide, -1.2}, 23 → {$wide, -1.9}, 31 → {2 $wide, -.6},
   22 → {$wide, -1.2}, 1121 → {.5 $wide, -1.6}, 2131 → {1.5 $wide, -1.6}},
 VertexRenderingFunction → (Text[text[#2], #1, Background → White] &)]
```



```
text[n_] := Switch[n,
  2, FramedColumn[{"Partitian functions", Zgauge ↔ Zstring ⇒ ZCFT ↔ ZAdS5}],
  1, FramedColumn[{"Scale invariant gauge theory", ZCFT[gs^2, λ]}],
  11, FramedColumn[{λ → (L / ls)^"?"}],
  3, FramedColumn[{"AdS5string theory", ZAdS5[G5, ls]}],
  31, FramedColumn[{Nc^2 → L^3 / G5}],

  32, FramedColumn[{"Large Nc", ZAdS5 -> Exp[-S"E"]}],
  21, FramedColumn[
   {{Nc^2 ∝ 1 / gs^2 ∝ 1 / G5, "D-brane"} ⇒ Column[e75 = {Nc^2 → π/2 L^3 / G5, λ → (L / ls)^4}]}],
  22, FramedColumn[{Nc^2 → π/2 L^3 / G5, λ → (L / ls)^4}],
  12, FramedColumn[{ZCFT[Nc≫λ≫1] → Exp[-S"E"]}],
  _, n]
GraphPlot[{2 → 1, 2 → 3, 1 → 11, 3 → 31, 11 → 21, 31 → 21,
  31 → 32, 11 → 12, {32 → 12, "Compute ZCFT from ZAdS5"}, 21 → 32, 21 → 12
 },
 VertexCoordinateRules →
  {1 → {0, 0}, 2 → {$wide, 0}, 3 → {2 $wide, 0}, 11 → {0, -.5}, 21 → {$wide, -.5},
   31 → {2 $wide, -.5}, 22 → {$wide, -1.6}, 32 → {2 $wide, -1}, 12 → {0, -1.}},
 VertexRenderingFunction → (Text[text[#2], #1, Background → White] &),
 DirectedEdges → True]
```



```
PR[CG["●5.5 Definitions"],
 NL, "▪Scale transform: ", $xa = tt : T[x, "u", {μ_}] → a tt,
 Imply, $0 = $ = d[s]^2 → T[η, "dd", {μ, ν}] d[T[x, "u", {μ}]] d[T[x, "u", {ν}]],
 yield, $ = $ /. $xa // tudExpand[d, {a}],
 NL, "▪Local Weyl transform: ",
 $ = T[g, "dd", {μ, ν}] d[T[x, "u", {μ}]] d[T[x, "u", {ν}]],
 yield, $ = $ /. $xa /. a → a[x] // tudExpand[d, {a[x]}],
 NL, "Invariance under this of: ", $ = S → -1/(4 e^2) tuIntegral[{{x^4}},
     √-g T[g, "uu", {μ, ν}] T[g, "uu", {ρ, σ}] T[F, "dd", {μ, ρ}] T[F, "dd", {ν, σ}]],
 Imply, T[A, "d", {ν}] -> T[A, "d", {ν}], CO[back, "factors:",
  √-g → a^2, imply, a^4.a^2.(A / a)^2 .a^-2.a^-2],
 NL, "▪{Local Weyl invariance", " + ", {T[g, "uu", {μ, ν}] -> T[η, "uu", {μ, ν}]}, "}",
 imply, "Conformal invariance ",
 NL, "▪▪Global Weyl invariance: ", $ = δ[T[g, "uu", {μ, ν}]] → -ε T[g, "uu", {μ, ν}],
 imply, δ[S] → ε/2 tuIntegral[{{x^4}}, √-g T[T, "ud", {μ, μ}]],
 imply, "If ", T[g, "uu", {μ, ν}] -> T[η, "uu", {μ, ν}],
 imply, T[T, "ud", {μ, μ}] → -tuDPartial[T[K, "u", {μ}], μ],
 NL, "▪Conformal invariance: ", T[η, "uu", {μ, ν}] → a[x]^2 T[η, "uu", {μ, ν}],
 imply, "∃ ", T[K, "u", {μ}] → -tuDPartial[T[L, "uu", {ν, μ}], ν],
 yield,
 T[T, "ud", {μ, μ}] → tuDPartial[tuDPartial[T[L, "uu", {ν, μ}], ν], μ], CG[" (5.47)"]
]
```

●5.5 Definitions

▪Scale transform: $tt : x^{\mu_{-}} \to a\, tt$

$\Rightarrow$ $d[s]^2 \to d[x^\mu]\, d[x^\nu]\, \eta_{\mu\nu} \longrightarrow d[s]^2 \to a^2\, d[x^\mu]\, d[x^\nu]\, \eta_{\mu\nu}$

▪Local Weyl transform: $d[x^\mu]\, d[x^\nu]\, g_{\mu\nu} \longrightarrow a[x]^2\, d[x^\mu]\, d[x^\nu]\, g_{\mu\nu}$

Invariance under this of: $S \to -\dfrac{\int_{\{x^4\}} [\sqrt{-g}\, F_{\mu\rho}\, F_{\nu\sigma}\, g^{\mu\nu}\, g^{\rho\sigma}]}{4\, e^2}$

$\Rightarrow$ $A_\nu \to A_\nu$ $\longleftarrow$factors: $\sqrt{-g} \to a^2 \Rightarrow a^4.a^2.\dfrac{A^2}{a^2}.\dfrac{1}{a^2}.\dfrac{1}{a^2}$

▪{Local Weyl invariance + $\{g^{\mu\nu} \to \eta^{\mu\nu}\}$} $\Rightarrow$ Conformal invariance

▪▪Global Weyl invariance: $\delta[g^{\mu\nu}] \to -\varepsilon\, g^{\mu\nu}$

$\Rightarrow$ $\delta[S] \to \dfrac{1}{2}\, \varepsilon \int_{\{x^4\}} [\sqrt{-g}\, T^\mu{}_\mu]$ $\Rightarrow$ If $g^{\mu\nu} \to \eta^{\mu\nu}$ $\Rightarrow$ $T^\mu{}_\mu \to -\underline{\partial}_\mu [K^\mu]$

▪Conformal invariance: $\eta^{\mu\nu} \to a[x]^2\, \eta^{\mu\nu}$ $\Rightarrow$ $\exists$ $K^\mu \to -\underline{\partial}_\nu [L^{\nu\mu}]$ $\longrightarrow$ $T^\mu{}_\mu \to \underline{\partial}_\mu [\underline{\partial}_\nu [L^{\nu\mu}]]$ (5.47)

```
PR["●Scalar field example: ",
 NL, "Weyl invariant: ",
 S → (-1/2) tuIntegral[{{x^4}}, √(-g) T[g, "uu", {μ, ν}] tuDPartial[ϕ, μ] tuDPartial[ϕ, ν]],
 NL, "Local Weyl invariant: ",
 {S → (-1/2) tuIntegral[{{x^4}}, √(-g) tuDCovariant[ϕ, μ] tuDCovariantu[ϕ, ν] + ξ R ϕ^2],
   ξ → 1 / 6, ϕ → ϕ / a[x]} // FramedColumn
]
```

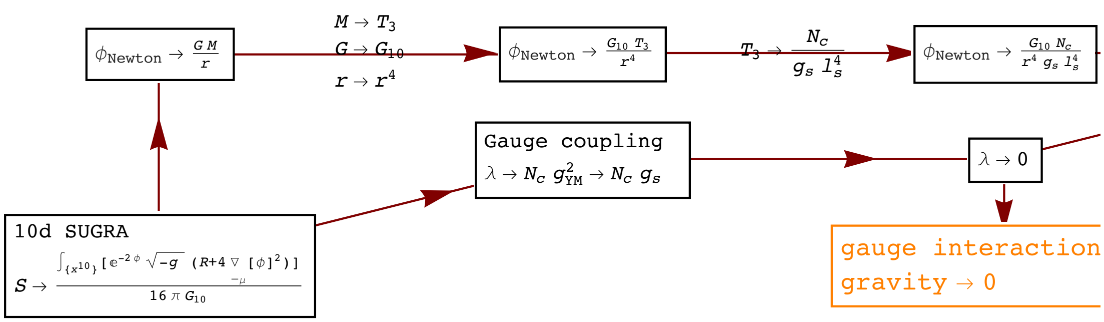
●Scalar field example:

Weyl invariant: $S \to -\frac{1}{2} \int_{\{x^4\}} [\sqrt{-g}\, g^{\mu\nu}\, \underline{\partial}_{\mu}[\phi]\, \underline{\partial}_{\nu}[\phi]]$

Local Weyl invariant:

$$\boxed{\begin{array}{l} S \to -\frac{1}{2} \int_{\{x^4\}} [R\,\xi\,\phi^2 + \sqrt{-g}\, \underline{\nabla}_{\mu}[\phi]\, \underline{\nabla}^{\nu}[\phi]] \\ \xi \to \frac{1}{6} \\ \phi \to \frac{\phi}{a[x]} \end{array}}$$

```
$wide = 2; $vert = -1;
$f1 = {"ϕNewton" → G M / r };
$s1 = {M → T3, G → G10, r → r^4}; $s1s = DisplayForm[Column[$s1]]; $s1s;
$f2 = $f1 /. $s1;
$s2 = T3 → Nc / (gs ls^4);
$f3 = $f2 /. $s2;
$s3 = G10 -> gs^2 ls^8;
$f4 = $f3 /. $s3;
text[n_] := Switch[n,
    1, FramedColumn[$f1],
    2, FramedColumn[$f2],
    3, FramedColumn[$f3],
    4, FramedColumn[CO[$f4]],
    50, FramedColumn[{"10d SUGRA",
      S → 1 / (16 π G10) tuIntegral[{{x^10}}, √(-g) Exp[-2 ϕ] (R + 4 tuDCovariant[ϕ, μ]^2)]}],
    5, FramedColumn[{"Gauge coupling ", λ → gYM^2 Nc → gs Nc}],
    6, FramedColumn[{λ → 0}],
    7, CO[FramedColumn[{"gauge interactions" → 0, "gravity" → 0}]],
    _, n];
GraphPlot[{{1 → 2, $s1s}, {2 → 3, $s2}, {3 → 4, $s3}, 5 → 6, 6 -> 7, 4 → 6, 50 → 5, 50 → 1},
 VertexCoordinateRules →
  {1 → {0, 0}, 2 → {$wide, 0}, 3 → {2 $wide, 0}, 4 → {3 $wide, 0}, 50 → {0 $wide, 1 $vert},
   5 → {1 $wide, .5 $vert}, 6 → {2 $wide, .5 $vert}, 7 → {2 $wide, $vert},
   11 → {0, -.5}, 21 → {$wide, -.5}, 31 → {2 $wide, -.5},
   22 → {$wide, -1.6}, 32 → {2 $wide, -1}, 12 → {0, -1.}},
 VertexRenderingFunction → (Text[text[#2], #1, Background → White] &),
 DirectedEdges → True, EdgeLabeling → All]
```



6.1 Spacetimes with constant curvature

```
{{, "S^3", },
  {"metric", $m = d[s]^2 → Sum[d[i]^2, {i, {X, Y, Z}}], },
  {"contraint", $c = Sum[i^2, {i, {X, Y, Z}}] → L^2, "SO3 invariant"},
  {"transform 1", $s = {X → L Sin[θ] Cos[φ], Y → L Sin[θ] Sin[φ], Z → L Cos[θ]}, },
  {"metric 1", $m /. $s // tudFnc[{θ, φ}, d, {L}] // Simplify},
  {, "H^2", },
  {"metric", $m = d[s]^2 → Sum[If[i === Z, -1, 1] d[i]^2, {i, {X, Y, Z}}], },
  {"contraint",
   $c = Sum[If[i === Z, -1, 1] i^2, {i, {X, Y, Z}}] → -L^2, "SO1,2 invariant"},
  {"transform 1", $s = {X → L Sinh[ρ] Cos[φ], Y → L Sinh[ρ] Sin[φ], Z → L Cosh[ρ]}, },
  {"metric 1", $m /. $s // tudFnc[{θ, φ}, d, {L}] // Simplify, "0 timelike coordinate"},
  {, CO["AdS2"], },
  {"metric", $m = d[s]^2 → Sum[If[MemberQ[{X, Z}, i], -1, 1] d[i]^2, {i, {X, Y, Z}}],
```

```
 "2 timelike coordinate"},
{"contraint", $c = Sum[If[MemberQ[{X, Z}, i], -1, 1] i^2, {i, {X, Y, Z}}] → -L^2,
 "SO2,1 invariant"},
{"transform 1", $s = {X → L Cosh[ρ] Sin[t̃], Y → L Sinh[ρ], Z → L Cosh[ρ] Cos[t̃]},
 "t̃ periodic"},
{"metric 1", $mAdS = $m /. $s // tudFnc[{ρ, t̃}, d, {L}] // Simplify,
 "1 timelike coordinate"},
{, CO["dS2"],},
{"metric", $m = d[s]^2 → Sum[If[MemberQ[{Z}, i], -1, 1] d[i]^2, {i, {X, Y, Z}}],
 "2 timelike coordinate"},
{"contraint", $c = Sum[If[MemberQ[{Z}, i], -1, 1] i^2, {i, {X, Y, Z}}] → L^2,
 "SO1,2 invariant"},
{"transform 1", $s = {X → L Cosh[t̃] Cos[θ], Y → L Cosh[t̃] Sin[θ], Z → L Sinh[t̃]},
 "t̃ periodic"},
{"metric 1(6.21)", $m /. $s // tudFnc[{θ, t̃}, d, {L}] // Simplify,
 "1 timelike coordinate"},

{, CO["AdS2 static"],},
{"AdS metric", $mAdS, "1 timelike coordinate"},
{"transform 1", $s = {ρ → ArcSinh[r̃]}, ""},
{"BH metric",
 $mAdS /. $s // tudFnc[{r̃, t̃}, d, {L}] // Simplify, "1 timelike coordinate"},

{, CO["AdS2 Conformal"],},
{"AdS metric", $mAdS, "1 timelike coordinate"},
{"transform", $s = {ρ → ArcSinh[Tan[θ]]}, ""},
{"Conformal metric",
 $mAdS /. $s // tudFnc[{θ, t̃}, d, {L}] // Simplify, "1 timelike coordinate"},

{, CO["AdS2 Poincare"],},
{"metric", $m = d[s]^2 → Sum[If[MemberQ[{X, Z}, i], -1, 1] d[i]^2, {i, {X, Y, Z}}],
 "2 timelike coordinate"},
{"contraint", $c = Sum[If[MemberQ[{X, Z}, i], -1, 1] i^2, {i, {X, Y, Z}}] → -L^2,
 "SO2,1 invariant"},
{"transform", $s = {X → L r t, Y → L r (-t^2 + 1 / r^2 - 1) / 2,
   Z → L r (-t^2 + 1 / r^2 + 1) / 2}},
{"Poincare metric", $m /. $s // tudFnc[{r, t}, d, {L}] // Simplify,
 "1 timelike coordinate"},
{, CO["------"],},
{, CO["AdSp+2"],},
{"metric",
 $m = d[s]^2 → Sum[If[MemberQ[{T[X, "d", {0}], T[X, "d", {p + 2}]}, i], -1, 1] d[i]^2,
     {i, {T[X, "d", {0}], T[X, "d", {p + 2}], T[X, "d", {1}],
       T[X, "d", {…}], T[X, "d", {p + 1}]}}], "2 timelike coordinate"},
{"contraint", $c = Sum[If[MemberQ[{T[X, "d", {0}], T[X, "d", {p + 2}]}, i], -1, 1] i^2,
     {i, {T[X, "d", {0}], T[X, "d", {p + 2}], T[X, "d", {1}],
       T[X, "d", {…}], T[X, "d", {p + 1}]}}] → -L^2, "SO2,p invariant"},
{"transform(6.47)", $s = {T[X, "d", {0}] → L Cosh[ρ] Cos[t̃], T[X, "d", {p + 2}] →
     L Cosh[ρ] Sin[t̃], T[X, "d", {i_}] → L Sinh[ρ] T[ω, "d", {i}]}, "t̃ periodic"},
{"constraint 1", $c1 = $c /. $s // FullSimplify;
 $ = {-# / L^2 & /@ $c1, ($t = Cosh[ρ]^2 - Sinh[ρ]^2) → TrigExpand[$t]};
 $ = tuEliminate[$, {Cosh[ρ]}] // Simplify;
```

```
  $c2 = # + 1 & /@ ($ /. Sinh[ρ] → 1) /. Equal → Rule;
  $c2d = d[#] & /@ $c2 // tudExpand[d, {}];
  $c2d = # / 2 & /@ $c2d // Expand;
  $c2 = {$c2, $c2d}
  ,},
 {"metric(6.48)",
  ($m = $m /. $s // tudFnc[{ρ, t̃, T[ω, "d", {1}], T[ω, "d", {1 + p}], T[ω, "d", {…}]},
        d, {L}] // FullSimplify;
    $mAdS = $m /. $c2 // Simplify) // ColumnSumExp
  , "1 timelike coordinate"},
 {"transform", $s = {Sinh[ρ]^2 -> r̃^2, Cosh[ρ]^2 -> r̃^2 + 1},},
 {"metric(6.49)", $mAdS /. $s // Simplify,}


} // Grid[#, Frame → All,
  Dividers → {{2 → Red, -2 → Blue}, {1 → Red, 6 → Red, 11 → Red, 16 → Red, 21 → Red}}] &
```

	S^3	
metric	$d[s]^2 \to d[X]^2 + d[Y]^2 + d[Z]^2$	
contraint	$X^2 + Y^2 + Z^2 \to L^2$	SO_3 invariant
transform 1	$\{X \to L\,Cos[\varphi]\,Sin[\theta],$ $Y \to L\,Sin[\theta]\,Sin[\varphi],\ Z \to L\,Cos[\theta]\}$	
metric 1	$d[s]^2 \to L^2\,(d[\theta]^2 + d[\varphi]^2\,Sin[\theta]^2)$	
	H^2	
metric	$d[s]^2 \to d[X]^2 + d[Y]^2 - d[Z]^2$	
contraint	$X^2 + Y^2 - Z^2 \to -L^2$	$SO_{1,2}$ invariant
transform 1	$\{X \to L\,Cos[\varphi]\,Sinh[\rho],$ $Y \to L\,Sin[\varphi]\,Sinh[\rho],\ Z \to L\,Cosh[\rho]\}$	
metric 1	$d[s]^2 \to L^2\,d[\varphi]^2\,Sinh[\rho]^2$	0 timelike coordinate
	AdS_2	
metric	$d[s]^2 \to -d[X]^2 + d[Y]^2 - d[Z]^2$	2 timelike coordinate
contraint	$-X^2 + Y^2 - Z^2 \to -L^2$	$SO_{2,1}$ invariant
transform 1	$\{X \to L\,Cosh[\rho]\,Sin[\tilde{t}],$ $Y \to L\,Sinh[\rho],\ Z \to L\,Cos[\tilde{t}]\,Cosh[\rho]\}$	$\tilde{t}$ periodic
metric 1	$d[s]^2 \to L^2\,(d[\rho]^2 - Cosh[\rho]^2\,d[\tilde{t}]^2)$	1 timelike coordinate
	dS_2	
metric	$d[s]^2 \to d[X]^2 + d[Y]^2 - d[Z]^2$	2 timelike coordinate
contraint	$X^2 + Y^2 - Z^2 \to L^2$	$SO_{1,2}$ invariant
transform 1	$\{X \to L\,Cos[\theta]\,Cosh[\tilde{t}],$ $Y \to L\,Cosh[\tilde{t}]\,Sin[\theta],\ Z \to L\,Sinh[\tilde{t}]\}$	$\tilde{t}$ periodic
metric 1(6.21)	$d[s]^2 \to L^2\,(Cosh[\tilde{t}]^2\,d[\theta]^2 - d[\tilde{t}]^2)$	1 timelike coordinate
	AdS_2 static	
AdS metric	$d[s]^2 \to L^2\,(d[\rho]^2 - Cosh[\rho]^2\,d[\tilde{t}]^2)$	1 timelike coordinate
transform 1	$\{\rho \to ArcSinh[\tilde{r}]\}$	

BH metric	$d[s]^2 \to \frac{L^2 (d[\tilde{r}]^2 - d[\tilde{t}]^2 (1+\tilde{r}^2)^2)}{1+\tilde{r}^2}$	1 timelike coordinate
	AdS_2 Conformal	
AdS metric	$d[s]^2 \to L^2 (d[\rho]^2 - \mathrm{Cosh}[\rho]^2 d[\tilde{t}]^2)$	1 timelike coordinate
transform	$\{\rho \to \mathrm{ArcSinh}[\mathrm{Tan}[\theta]]\}$	
Conformal metric	$d[s]^2 \to L^2 (d[\theta]^2 - d[\tilde{t}]^2) \mathrm{Sec}[\theta]^2$	1 timelike coordinate
	AdS_2 Poincare	
metric	$d[s]^2 \to -d[X]^2 + d[Y]^2 - d[Z]^2$	2 timelike coordinate
contraint	$-X^2 + Y^2 - Z^2 \to -L^2$	$SO_{2,1}$ invariant
transform	$\{X \to L\, r\, t,\ Y \to \frac{1}{2} L\, r (-1 + \frac{1}{r^2} - t^2),$ $Z \to \frac{1}{2} L\, r (1 + \frac{1}{r^2} - t^2)\}$	
Poincare metric	$d[s]^2 \to \frac{L^2 (d[r]^2 - r^4 d[t]^2)}{r^2}$	1 timelike coordinate
	------	
	AdS_{p+2}	
metric	$d[s]^2 \to$ $-d[X_0]^2 + d[X_1]^2 + d[X_{1+p}]^2 - d[X_{2+p}]^2 + d[X_{\dots}]^2$	2 timelike coordinate
contraint	$-(X_0)^2 + (X_1)^2 + (X_{1+p})^2 - (X_{2+p})^2 + (X_{\dots})^2 \to -L^2$	$SO_{2,p}$ invariant
transform(6.47)	$\{X_0 \to L\, \mathrm{Cos}[\tilde{t}]\, \mathrm{Cosh}[\rho],$ $X_{2+p} \to L\, \mathrm{Cosh}[\rho]\, \mathrm{Sin}[\tilde{t}],\ X_{i\_} \to L\, \mathrm{Sinh}[\rho]\, \omega_i\}$	$\tilde{t}$ periodic
constraint 1	$\{(\omega_1)^2 + (\omega_{1+p})^2 + (\omega_{\dots})^2 \to 1,$ $d[\omega_1]\, \omega_1 + d[\omega_{1+p}]\, \omega_{1+p} + d[\omega_{\dots}]\, \omega_{\dots} \to 0\}$	
metric(6.48)	$d[s]^2 \to$ $L^2 \sum[\ d[\rho]^2,\ -\mathrm{Cosh}[\rho]^2 d[\tilde{t}]^2,\ (d[\omega_1]^2 + d[\omega_{1+p}]^2 + d[\omega_{\dots}]^2) \mathrm{Sinh}[\rho]^2\ ]$	1 timelike coordinate
transform	$\{\mathrm{Sinh}[\rho]^2 \to \tilde{r}^2,\ \mathrm{Cosh}[\rho]^2 \to 1 + \tilde{r}^2\}$	
metric(6.49)	$d[s]^2 \to$ $L^2 (d[\rho]^2 + (d[\omega_1]^2 + d[\omega_{1+p}]^2 + d[\omega_{\dots}]^2)\, \tilde{r}^2 -$ $d[\tilde{t}]^2 (1 + \tilde{r}^2))$	

7.1 The AdS black hole

```
PR["SAdS₅ metric: ",
 $srL = (r / L)^2 h[r];
 Yield, $ds = {d[s]^2 → -$srL d[t]^2 + d[r]^2 / $srL + (r / L)^2 Sum[d[i]^2, {i, {x, y, z}}],
   h[r1_] → 1 - (r₀ / r1)^4},
 NL, "For ", $s = r₀ → 0,
 imply, $ds[[1]] /. $ds[[2]] /. $s // Simplify
]
```

$\mathrm{SAdS_5}$ metric:

$\to \{d[s]^2 \to \frac{r^2 (d[x]^2 + d[y]^2 + d[z]^2)}{L^2} + \frac{L^2 d[r]^2}{r^2 h[r]} - \frac{r^2 d[t]^2 h[r]}{L^2},\ h[r1\_] \to 1 - \frac{r_0^4}{r1^4}\}$

For $r_0 \to 0 \Rightarrow d[s]^2 \to \frac{L^4 d[r]^2 + r^4 (-d[t]^2 + d[x]^2 + d[y]^2 + d[z]^2)}{L^2 r^2}$

7.2 Thermodynamic quantities of AdS black holes

```
e75
PR["From: ",
 e76 = T[r] → tuDPartial[f[r], r] / (4 π),
 and,
 $ = f[r] → (r / L)^2 (1 - (r₀ / r)^4),
 Yield, $ = tuDPartial[#, r] & /@ $,
 yield, $ = $ // tuDerivativeExpand[{L, r₀}],
 Imply, $ = e76 /. $,
 Yield, e78 = $ = $ /. r₀ → r0 /. r → r0 /. r0 → r₀ // Simplify;
 Framed[$], CG["(7.8)"]
]
```

$\{N_c^2 \to \frac{L^3 \pi}{2 G_5},\ \lambda \to \frac{L^4}{l_s^4}\}$

From: $T[r] \to \frac{\partial_r[f[r]]}{4 \pi}$ and $f[r] \to \frac{r^2 (1 - \frac{r_0^4}{r^4})}{L^2}$

$\to \partial_r[f[r]] \to \partial_r[\frac{r^2 (1 - \frac{r_0^4}{r^4})}{L^2}] \longrightarrow \partial_r[f[r]] \to \frac{\frac{4 r_0^4}{r^3} + 2 r (1 - \frac{r_0^4}{r^4})}{L^2}$

$\Rightarrow T[r] \to \frac{\frac{4 r_0^4}{r^3} + 2 r (1 - \frac{r_0^4}{r^4})}{4 L^2 \pi}$

$\to \boxed{T[r_0] \to \frac{r_0}{L^2 \pi}}$ (7.8)

```
PR["From area law(3.21): ", $ = e321 = S → A k_B c^3/(4 G ℏ) → A k_B/(4 l_p^2),
 NL, "Area in 4d: ", $s = {A → (r_0 / L)^3 V_3, G → G_5},
 imply, $ = $ /. $s; $ = $[[1]] → $[[2, 1]],
 yield, "Density: ", $ = # / V_3 & /@ $,
 NL, "(7.8) (7.5): ", $ = {$, e78, e75} /. S → s V_3 // Flatten,
 NL, "with: ", $s = {ℏ → 1, k_B → 1, c → 1, T[r_0] → T},
 yield, $ = tuEliminate[$, {L, G_5, l_s}] /. $s /. N_1 → N_c;
 $ = tuRuleSolve[$, s] // First;
 Framed[e712 = $],
 NL, "From: ", $ = d[E] → T d[s],
 yield, $ = $ /. e712 // tudExpand[d, {N_c}]; $ = $ /. T^3 d[T] → d[T^4] / 4
]
```

From area law(3.21): $S \to \frac{A\, c^3\, k_B}{4\, G\, \hbar} \to \frac{A\, k_B}{4\, l_p^2}$

Area in 4d: $\{A \to \frac{r_0^3\, V_3}{L^3},\ G \to G_5\} \Rightarrow S \to \frac{c^3\, k_B\, r_0^3\, V_3}{4\, L^3\, \hbar\, G_5} \longrightarrow$ Density: $\frac{S}{V_3} \to \frac{c^3\, k_B\, r_0^3}{4\, L^3\, \hbar\, G_5}$

(7.8) (7.5): $\{s \to \frac{c^3\, k_B\, r_0^3}{4\, L^3\, \hbar\, G_5},\ T[r_0] \to \frac{r_0}{L^2\, \pi},\ N_c^2 \to \frac{L^3\, \pi}{2\, G_5},\ \lambda \to \frac{L^4}{l_s^4}\}$

with: $\{\hbar \to 1,\ k_B \to 1,\ c \to 1,\ T[r_0] \to T\} \longrightarrow \boxed{s \to \frac{1}{2}\, \pi^2\, T^3\, N_c^2}$

From: $d[E] \to T\, d[s] \longrightarrow d[E] \to \frac{3}{8}\, \pi^2\, d[T^4]\, N_c^2$

8.1 Wilson Loops

```
PR["Example U[1] gauge transformation: ",
 e81 =
  {ϕ[x] → Exp[I α[x]] ϕ[x], T[A, "d", {μ}][x] -> T[A, "d", {μ}][x] + tuDPartial[α[x], μ]};
 Column[$],
 NL, "Wilson loop: ", W_P[x, x] → Exp[I tuCIntegral[
     {{T[x, "u", {μ}]}}, T[x, "u", {μ}] T[A, "d", {μ}][x] / Abs[T[x, "u", {μ}]]]],
 NL, "Current of particle: ",
 $J = T[J, "u", {μ}][x] →
   tuCIntegral[{{λ}}, tuDPartial[T[y, "u", {μ}], λ] δ[T[x, "u", {μ}] - T[y, "u", {μ}][λ]]
   ],
 NL, "Partition function: ",
 $ = Z[J] → BraKet[f, Exp[-H 𝒯], i]

]
```

Example U[1] gauge transformation: Column[$Z[J] \to \langle f \mid e^S \mid i \rangle$]

Wilson loop: $W_P[x, x] \to e^{i \oint_{\{x^\mu\}} [\frac{x^\mu A_\mu[x]}{Abs[x^\mu]}]}$

Current of particle: $J^\mu[x] \to \oint_{\{\lambda\}} [\delta[x^\mu - y^\mu[\lambda]]\, \partial_\lambda[y^\mu]]$

Partition function: $Z[J] \to \langle f \mid e^{-H \mathcal{T}} \mid i \rangle$