Easy

Critical $O(N)$ model in $d = 6 - \epsilon$

Consider the $O(N)$ symmetric cubic theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3 \,. \tag{1}$$

in dimensions $d = 6 - \epsilon$. It can be shown that this theory has perturbative IR stable fixed points for sufficiently large N (the perturbative parameter being ϵ, as in the familiar Wilson-Fisher fixed point). In $d = 5$ ($\epsilon = 1$), the fixed point is expected to describe an interacting $O(N)$ model dual to Vasiliev higher spin theory in AdS_6 with alternate boundary condition ($\Delta = 2$) on the bulk scalar field. In this problem we study the fixed points of this theory at the one loop order.

I. IR fixed points at large N

The one-loop β functions of this theory in $d = 6 - \epsilon$ are

$$\beta_1 = -\frac{\epsilon}{2}g_1 + \frac{(N-8)g_1^3 - 12g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3} \tag{2}$$

$$\beta_2 = -\frac{\epsilon}{2}g_2 + \frac{-4Ng_1^3 + Ng_1^2 g_2 - 3g_2^3}{4(4\pi)^3} \tag{3}$$

It is not difficult to derive these from a one-loop calculation. If you are interested and have time, try to do it!

It is easy to see that at large N, there is a solution of $\beta_1 = 0, \beta_2 = 0$ given by

$$g_1^* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}}(1 + \mathcal{O}(\frac{1}{N}))\,, \qquad g_2^* = 6g_1^*(1 + \mathcal{O}(\frac{1}{N})) \tag{4}$$

a) Write a Mathematica code to find analytically the corrections to this large N solution up to order $1/N^8$. It will be convenient to rescale the coupling constants and introduce the variables

$$g_1^* = \sqrt{\frac{\epsilon(4\pi)^3}{N}}x\,, \qquad g_2^* = \sqrt{\frac{\epsilon(4\pi)^3}{N}}y$$

so that the equations to be solved perturbatively in N are[1]

$$E_1 = N\left(x^2 - 6\right) - 8x^2 - 12xy + y^2 = 0 \tag{5}$$

$$E_2 = 4Nx^3 - N\left(x^2 - 2\right)y + 3y^3 = 0 \tag{6}$$

and you are looking for a solution in the form

$$\begin{aligned} x &= \sqrt{6} + \frac{x_1}{N} + \frac{x_2}{N^2} + \dots \\ y &= 6\sqrt{6} + \frac{y_1}{N} + \frac{y_2}{N^2} + \dots \end{aligned} \tag{7}$$

[1] We are not interested in the solution $x = 0$, so we can factor out x from β_1.

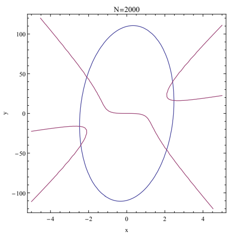

Figure 1: Zeroes of the β functions at $N = 2000$.

(Hint: 'LogicalExpand' is probably the most economical way of dealing with such expansions, but you can also design your own solver.)

b) Compute the matrix of derivatives of the β functions, $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$, and show that when evaluated on the large N solution above it has two positive eigenvalues. This implies that the fixed point is an IR stable fixed point.

II. Numerical analysis of fixed points at finite N

From the solution of part a) above, you should notice that the coefficients in the $1/N$ expansion of the fixed point couplings are quite large. This suggests that the $1/N$ expansion may break down at some value of N. In this part we want to see by some numerical analysis that the IR fixed point which exist at large N disappears at around $N = 1038$.

a) Make contour plots of the equations $E_1 = 0, E_2 = 0$ for $N = 500, 1000, 1040, 2000$ (you are welcome to explore other values). You should see that at around $N = 1040$ four new real solutions appear, while for lower values of N there are only two real solutions (these are not IR stable fixed points, see below). An example of the type of plot you should obtain is given in Fig. 1. Try to also use 'Manipulate' to follow these solutions as you vary N. (Hint: You might want to use 'ControlActive' when dealing with plots inside a 'Manipulate').

b) Solve numerically the system $E_1 = 0, E_2 = 0$ for $N = 1037, 1038, 1039, 1040$ (and explore other values if you are curious). There are six solutions (only three are physically distinct since the solutions related by $g_1, g_2 \to -g_1, -g_2$ are equivalent). Also compute the eigenvalues of the $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$ matrix at each of these solutions. Verify that for $N \leq 1038$ there are only two real solutions, and they are not IR stable (they have a negative eigenvalue), while for $N \geq 1039$ there are six real solutions, and two (physically equivalent) of them are IR stable. This is the solution that matches upon the large N solution found in part I.

c) Make plots of the IR stable numerical solutions for g_1 and g_2 as a function of N for $1039 \leq N \leq 1800$, and on the same plot compare to the analytic $1/N$ expansion found above. Try comparing both keeping up to $1/N^4$ and up to $1/N^8$. An example of the type of plot you should obtain is shown in Fig. 2.

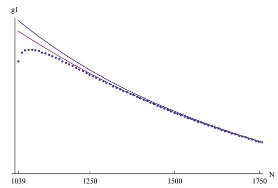

Figure 2: Comparing numerical solution and large N expansion.

III. Anomalous dimensions of ϕ^i and σ and comparison to the large N $O(N)$ vector model

The anomalous dimensions of ϕ^i and σ to the one-loop order are given by the equations (again, it is not difficult to derive these from the one-loop correction to the propagator)

$$\gamma_\phi = \frac{g_1^2}{6(4\pi)^3} \qquad \gamma_\sigma = \frac{Ng_1^2 + g_2^2}{12(4\pi)^3} \tag{8}$$

The conformal dimensions of ϕ and σ at the IR CFT are then given by

$$\Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi \tag{9}$$

$$\Delta_\sigma = \frac{d}{2} - 1 + \gamma_\sigma \tag{10}$$

where one should plug in the fixed point values of g_1 and g_2, and it is understood that $d = 6 - \epsilon$.

a) Using the analytic large N solution for g_1 and g_2 found in part I, compute the dimensions Δ_ϕ and Δ_σ to order $1/N^2$.

b) Various results for conformal dimensions of operators are known in the large N expansion of the critical $O(N)$ vector model in arbitrary dimension d. This model was historically studied assuming $2 < d < 4$, with $d = 3$ being the interesting physical dimension. However, one can formally extend those results to $4 < d < 6$ without obvious issues, and we are now going to check that close to $d = 6$ they match upon the RG analysis of the cubic theory above.

In the critical $O(N)$ model, the lowest lying scalar primaries are the fundamental $O(N)$ vector ϕ^i, and a $O(N)$ singlet scalar operator $\mathcal{O}$ of dimension $2 + O(1/N)$. Their conformal dimensions have been computed long ago to be

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{\eta_1}{N} + \frac{\eta_2}{N^2} + \dots \tag{11}$$

$$\eta_1 = \frac{2^{d-3}(d-4)\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}+1\right)} \tag{12}$$

$$\eta_2 = 2\eta_1^2\left(f_1 + f_2 + f_3\right)\ ;$$
$$f_1 = v'(\mu) + \frac{\mu^2+\mu-1}{2\mu(\mu-1)}\,, \quad f_2 = \frac{\mu}{2-\mu}v'(\mu) + \frac{\mu(3-\mu)}{2(2-\mu)^2}\,, \quad f_3 = \frac{\mu(2\mu-3)}{2-\mu}v'(\mu) + \frac{2\mu(\mu-1)}{2-\mu}\ ;$$
$$v'(\mu) = \psi(2-\mu) + \psi(2\mu-2) - \psi(\mu-2) - \psi(2)\,, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\,, \qquad \mu = \frac{d}{2} \tag{13}$$

and for the $O(N)$ scalar singlet[2]

$$\Delta_{\mathcal{O}} = 2 + \frac{1}{N}\frac{4(d-1)(d-2)}{d-4}\eta_1 \tag{14}$$

Expand Δ_ϕ and $\Delta_{\mathcal{O}}$ in $d = 6 - \epsilon$ to linear order in ϵ, and show that the result agrees with the conformal dimensions found in question a), provided $\mathcal{O}$ is identified with σ.

[2] The order $1/N^3$ for Δ_ϕ and order $1/N^2$ for $\Delta_{\mathcal{O}}$ are also known, but we omit them here.