[Problem Set 2 for J. Maldacena's lectures for the Mathematica School]

1 Strong Sub-additivity [easy]

In this exercise we provide strong evidence for strong sub-additivity by checking it on a large set of random matrices.

Single Density Matrix Warm-up

- Generate a random complex rectangular matrix `X` of dimensions `dimPhy` $\times$ `dimAux`. You can take `dimPhy` and `dimAux` be some relatively small integers.
- We can now generate a random density matrix through

$$\rho = \frac{X \cdot X^\dagger}{\mathrm{tr}\,(X \cdot X^\dagger)} \tag{1}$$

  Check that this indeed looks like a reasonable density matrix. For instance, what is nice about its eigenvalues?

- Compute its entanglement entropy $S = -\mathrm{tr}\,(\rho \log \rho)$.

Strong Sub-additivity

Now we can generate a big density matrix in a product of three Hilbert traces $H_A \otimes H_B \otimes H_C$ and check sub-additivity by performing several partial traces.

- To do so, first take `dimPhy` to be a product of `dimA`, `dimB` and `dimC` for some small values for these dimensions (to start you can set them all equal to 2 and increase these values at the end when everything is working). The density matrix above is now denoted as ρ_{ABC}.
- Define a vector $e_i^{(dim)}$ $\equiv$`e[i_,dim_]` of dimension `dim` to be 1 at entry i and zero elsewhere.
- Define $\mathbb{I}^{(\mathrm{dim})}$ $\equiv$`id[dim_]` to be a density matrix of size `dim`.
- Then $\rho_{AC} = \mathrm{tr}_B\, \rho_{ABC}$ can be easily computed as

$$\rho_{AC} = \sum_{b=1}^{\mathrm{dim}_B} \left[\mathbb{I}^{(\mathrm{dim}_A)} \otimes e_b^{(\mathrm{dim}_B)} \otimes \mathbb{I}^{(\mathrm{dim}_C)}\right]^T \cdot \rho_{ABC} \cdot \left[\mathbb{I}^{(\mathrm{dim}_A)} \otimes e_b^{(\mathrm{dim}_B)} \otimes \mathbb{I}^{(\mathrm{dim}_C)}\right] \tag{2}$$

  where the tensor product $\otimes$ is implemented in Mathematica as `KroneckerProduct`. Understand why this expression is indeed correct and write down its analogue for ρ_{BC} and ρ_C.

- Implement the reduced density matrices ρ_{AC}, ρ_{BC} and ρ_C
- Compute its entanglement entropies
- Check strong sub-additivity

$$S_C + S_{ABC} \leq S_{AC} + S_{BC}\,. \tag{3}$$

  Repeat the check on many initial density matrices to convince yourself that this was not a coincidence.

- Now you can increase the sizes of the Hilbert Spaces and of the auxiliary dimension. Take for example $\dim_A = 6$, $\dim_B = 8$, $\dim_C = 10$. By changing $\dim_{\rm aux}$ you should observe that for very small values (as compared with the full dimension dim) and for very large values interesting things happen. For instance, in one limit the bound is almost saturated. Which one is which and why is this expected?

2 Page's Theorem [easy]

Here we use a similar set of tools to check Page's Theorem.

- Evaluate the proposed sum for $S_A = S_{\dim_A,\dim_B}$ as given in the abstract of Page's paper gr-qc/9305007.
- Check that the limit $1 \ll \dim_A \leq \dim_B$ agrees with the expression in the same abstract.
- Consider a several random *pure* states in $H_A \times H_B$ with $\dim_B > \dim_A$. Compute several hundreds of ρ_A and S_A following from these states (using the same sort of ideas employed in the previous exercise) and check the validity of Page's proposal for the average of S_A.
- Check also (7) in that same paper.