Young Tableaux, Irreducible Representations and Particles

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This notebook studies the Young Tableaux and the reducible representations of **SU[2]** and **SU[3]** and applications to particle physics.

Note: For most of the graphics and for one table, the input cells that produced the output are closed up. These input cells appear only as a thin empty cell. Select and evaluate the cell to obtain the output. Or simply evaluate the entire sections.

Initialization

Local Routines

SU[2] representations for the Spin

■ Two particles

A particle with spin 1/2 has two states

```
In[27]:= Sd[hat@i]
         {Spin[1 / 2, 1 / 2], Spin[1 / 2, -1 / 2]} = % // EinsteinArray[]
```

Out[27]= $S_{\hat{i}}$

Out[28]= $\{S_{\uparrow}, S_{\downarrow}\}$

where we use the following structure where the first argument is the angular momentum spin quantum number and the second argument is the spin projection of the z component.

```
In[29]:= Spin[S, Sz]
```

Out[29]= $\text{Spin}[S, S_z]$

According to quantum mechnics a particle with angular momentum J has $2J+1$ possible projections $\{J, J-1, J-2\ \ldots\ -J\}$

The coupling of two spins gives the following terms

```
In[30]:= Sdd[hat@i, hat@j]
         % // EinsteinArray[] // MatrixForm
```

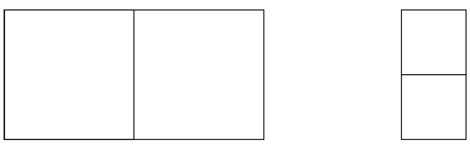
Out[30]= $S_{\hat{i}\hat{j}}$

Out[31]//MatrixForm=

$$\begin{pmatrix} S_{\uparrow\uparrow} & S_{\uparrow\downarrow} \\ S_{\downarrow\uparrow} & S_{\downarrow\downarrow} \end{pmatrix}$$

These four terms are said to be part of the space $2 \otimes 2$ but this must to be divided into irreducible multiplets as $2 \otimes 2 = 3 \oplus 1$. These mutiplets are found by calculating the possible permutations. In this case the triplet is symmetric and the singlet is antisymmetric. These results are summarized in the Young diagrams. The horizontal boxes represents the symmetric multiplet and the vertical boxes represent the antisymmetric singlet.

```
In[32]:= YoungTableauxDiagrams[2, 2]
```



Spin triplet S=1

The normalized triplet states are

```
In[33]:= Sdd[hat@i, hat@j] // SymmetrizePositions[S, 1, 2]
         TableForm[{Spin[1, -1], Spin[1, 0], Spin[1, 1]} = {#[[1]], #[[2]] / Sqrt[2], #[[3]]} &@
            Union@Flatten[% // EinsteinArray[] // ElliminateTensorNumericFactor]]
```

Out[33]= $S_{\hat{i}\hat{j}} + S_{\hat{j}\hat{i}}$

Out[34]//TableForm=

$S_{\downarrow\downarrow}$

$\frac{S_{\downarrow\uparrow}+S_{\uparrow\downarrow}}{\sqrt{2}}$

$S_{\uparrow\uparrow}$

where we see that the projections of the z component of the spin are $\{1, 0, -1\}$

Spin singlet S=0

The antisymmteric spin singlet that has combined spin 0 and therefore only one element

```
In[35]:= Sdd[hat@i, hat@j]

         TableForm[
          Spin[0, 0] =
           Union[1 / Sqrt[2] Flatten[% // AntiSymmetrizePositions[S, 1, 2] // EinsteinArray[] //
                   ElliminateTensorNumericFactor //
                  EliminateOppositeSignElement] /. 0 → Sequence[]] [[1]]]
```

Out[35]= $S_{\hat{i}\hat{j}}$

Out[36]//TableForm=

$$\frac{-S_{\downarrow\uparrow} + S_{\uparrow\downarrow}}{\sqrt{2}}$$

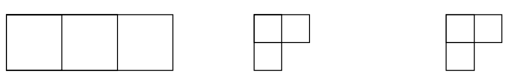
```
In[37]:= Spin[0, 0]
```

Out[37]= $$\frac{-S_{\downarrow\uparrow} + S_{\uparrow\downarrow}}{\sqrt{2}}$$

■ Three particles

The case with three particles is also divided into three irreducible representations. One completely symmetric and two with mixed symmetry

```
In[38]:= YoungTableauxDiagrams[3, 2]
```



● Completely Symmetric: Spin $\frac{3}{2}$

The completely normalized symmetric 4-plet that has combined spin $\frac{3}{2}$ is

```
In[39]:= Sddd[hat@i, hat@j, hat@k]
         TableForm[{Spin[3 / 2, -3 / 2], Spin[3 / 2, -1 / 2], Spin[3 / 2, 1 / 2], Spin[3 / 2, 3 / 2]} =
           {#[[1]], #[[2]] / Sqrt[3], #[[3]] / Sqrt[3], #[[4]]} &@
             Union@Flatten[% // SymmetrizePositions[S, 1, 2] // SymmetrizePositions[S, 2, 3] //
                   SymmetrizePositions[S, 3, 1] //
                  EinsteinArray[] // ElliminateTensorNumericFactor //
                EliminateOppositeSignElement] /. 0 → Sequence[]]
```

Out[39]= $S_{\hat{i}\hat{j}\hat{k}}$

Out[40]//TableForm=

$S_{\downarrow\downarrow\downarrow}$

$\frac{S_{\downarrow\downarrow\uparrow}+S_{\downarrow\uparrow\downarrow}+S_{\uparrow\downarrow\downarrow}}{\sqrt{3}}$

$\frac{S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow}}{\sqrt{3}}$

$S_{\uparrow\uparrow\uparrow}$

Two-particle Antisymmetric singlet + third particle: Spin $\frac{1}{2}$

The first multiplet with mixed symmetry is constructed by coupling the two particle singlet with the third particle. The combined spin of this multiplet is $\frac{1}{2}$. We could calculate this multiplet just by appending an index to the antisymmetrization of the first two indices because the two particle antisymetric combination is a singlet.

```
In[41]:= Sddd[hat@i, hat@j, hat@k]
         TableForm[{SpinMixed1[1 / 2, -1 / 2], SpinMixed1[1 / 2, 1 / 2]} =
           1 / Sqrt[2] Union@Flatten[% // AntiSymmetrizePositions[S, 1, 2] // EinsteinArray[] //
                 ElliminateTensorNumericFactor //
                EliminateOppositeSignElement] /. 0 → Sequence[]]
```

Out[41]= $S_{\hat{i}\hat{j}\hat{k}}$

Out[42]//TableForm=

$\frac{-S_{\downarrow\uparrow\downarrow}+S_{\uparrow\downarrow\downarrow}}{\sqrt{2}}$

$\frac{-S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}}{\sqrt{2}}$

Symmetric in the first two particles + third particle: Spin $\frac{1}{2}$

The second multiplet with mixed symmetry is more difficult to calculate. We have to use an explicit coupling of the symmetric two particle triplet with the third particle.

```
In[43]:= ClebschGordan[{1, m1 = 1}, {1/2, m2 = -1/2}, {1/2, 1/2}]
           Spin[1, 1] ** Spin[1/2, -1/2] +
          ClebschGordan[{1, m1 = 0}, {1/2, m2 = 1/2}, {1/2, 1/2}] Spin[1, 0] ** Spin[1/2, 1/2]

         SpinMixed2[1/2, 1/2] = Map[Distribute, %, 3] // CombineTensors[S]
```

Out[43]= $-\frac{\frac{S_{\downarrow\uparrow}+S_{\uparrow\downarrow}}{\sqrt{2}} ** S_{\uparrow}}{\sqrt{3}} + \sqrt{\frac{2}{3}}\, S_{\uparrow\uparrow} ** S_{\downarrow}$

Out[44]= $-\frac{S_{\downarrow\uparrow\uparrow}}{\sqrt{6}} - \frac{S_{\uparrow\downarrow\uparrow}}{\sqrt{6}} + \sqrt{\frac{2}{3}}\, S_{\uparrow\uparrow\downarrow}$

```
In[45]:= ClebschGordan[{1, m1 = -1}, {1/2, m2 = 1/2}, {1/2, -1/2}]
           Spin[1, -1] ** Spin[1/2, 1/2] +
          ClebschGordan[{1, m1 = 0}, {1/2, m2 = -1/2}, {1/2, -1/2}]
           Spin[1, 0] ** Spin[1/2, -1/2]

         SpinMixed2[1/2, -1/2] = Map[Distribute, %, 3] // CombineTensors[S]
```

Out[45]= $-\sqrt{\frac{2}{3}}\, S_{\downarrow\downarrow} ** S_{\uparrow} + \frac{\frac{S_{\downarrow\uparrow}+S_{\uparrow\downarrow}}{\sqrt{2}} ** S_{\downarrow}}{\sqrt{3}}$

Out[46]= $-\sqrt{\frac{2}{3}}\, S_{\downarrow\downarrow\uparrow} + \frac{S_{\downarrow\uparrow\downarrow}}{\sqrt{6}} + \frac{S_{\uparrow\downarrow\downarrow}}{\sqrt{6}}$

So, the mixed multiplet is finally

```
In[47]:= {SpinMixed2[1/2, 1/2], SpinMixed2[1/2, -1/2]} / Sqrt[6] // Expand
```

Out[47]= $\left\{-\frac{1}{6} S_{\downarrow\uparrow\uparrow} - \frac{1}{6} S_{\uparrow\downarrow\uparrow} + \frac{1}{3} S_{\uparrow\uparrow\downarrow},\ -\frac{1}{3} S_{\downarrow\downarrow\uparrow} + \frac{1}{6} S_{\downarrow\uparrow\downarrow} + \frac{1}{6} S_{\uparrow\downarrow\downarrow}\right\}$

```
In[48]:= TableForm[
          {SpinMixed2[1/2, 1/2], SpinMixed2[1/2, -1/2]}
         ]
```

Out[48]//TableForm=

$-\frac{S_{\downarrow\uparrow\uparrow}}{\sqrt{6}} - \frac{S_{\uparrow\downarrow\uparrow}}{\sqrt{6}} + \sqrt{\frac{2}{3}}\, S_{\uparrow\uparrow\downarrow}$

$-\sqrt{\frac{2}{3}}\, S_{\downarrow\downarrow\uparrow} + \frac{S_{\downarrow\uparrow\downarrow}}{\sqrt{6}} + \frac{S_{\uparrow\downarrow\downarrow}}{\sqrt{6}}$

Three Flavor Symmetry SU[3]: up-down-strange quarks

■ Introduction

The model of the nucleons, proton and neutron, cannot be extended to cover all the possible particles. Another approach is needed. This new approach is to define the quarks as the fundamental particles.

Quarks are classified according to two characteristics: Flavor and Color. There are six flavors divided in three generations and for each flavor there are three colors. In this section we study the approximate symmetry of thee flavors.

The following quantum numbers of **{Baryon number, Isospin, strangeness, Parity, Spin}** are associated to each quark

```
In[49]:= TableForm[
           QuantumNumbersRule = {u → {1 / 3, 1 / 2, 0, 1, 1 / 2}, d → {1 / 3, -1 / 2, 0, 1, 1 / 2},
             s → {1 / 3, 0, -2 / 3, 1, 1 / 2}, bar@u → -{1 / 3, 1 / 2, 0, 1, 1 / 2},
             bar@d → -{1 / 3, -1 / 2, 0, 1, 1 / 2}, bar@s → -{1 / 3, 0, -2 / 3, 1, 1 / 2}}
          ]
```

Out[49]//TableForm=

$$\begin{array}{l} u \to \{\frac{1}{3}, \frac{1}{2}, 0, 1, \frac{1}{2}\} \\ d \to \{\frac{1}{3}, -\frac{1}{2}, 0, 1, \frac{1}{2}\} \\ s \to \{\frac{1}{3}, 0, -\frac{2}{3}, 1, \frac{1}{2}\} \\ \bar{u} \to \{-\frac{1}{3}, -\frac{1}{2}, 0, -1, -\frac{1}{2}\} \\ \bar{d} \to \{-\frac{1}{3}, \frac{1}{2}, 0, -1, -\frac{1}{2}\} \\ \bar{s} \to \{-\frac{1}{3}, 0, \frac{2}{3}, -1, -\frac{1}{2}\} \end{array}$$

The Hypercharge is a function of the Baryon number ***B*** and strangeness ***S***. The most common graph is done plotting the particles in the Isospin-Hypercharge plane. This is done separately for Mesons and Baryons wich have defined baryon numbers so the axis in the Hypercharge varies according to the strageness of the particles.

```
In[50]:= Hypercharge[B_, S_] = B + S

Out[50]= B + S
```

The following facts are used to define the realizable multiplets in terms of particles

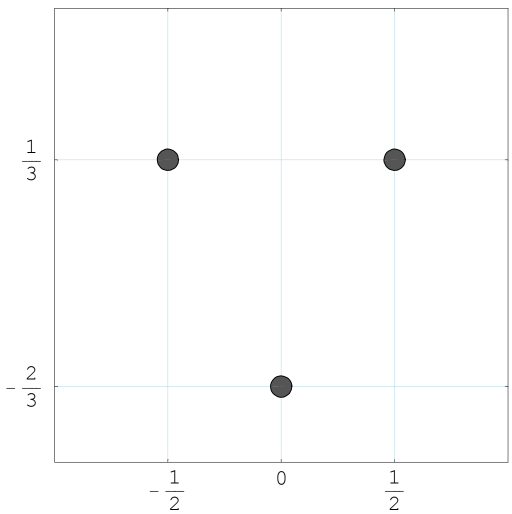
- The complete wave function must be antisymmetric according to the Pauli exclusion principle.
- Only particles with baryon number 0 or 1 are realizable in terms of particles according to quantum chromodynamics.
- The wavefunctions for the ground state must be symmetric. This is reasonable because the ground state will have zero orbital angular momentum, which is decribed by an isotropic gaussian function.
- Mesons defined to have $B = 0$ have to have antisymmetric spin-flavor wavefunction because the color part is symmetric.
- Baryons defined to have $B = 1$ have to have symmetric spin-flavor wavefunctions because the color part is antisymmetric.

■ First order tensor

The quark flavors in consideration are the up down and strange

Out[51]= $\mathbf{T_i}$

Out[52]= $\{\mathbf{T_u}, \mathbf{T_d}, \mathbf{T_s}\}$

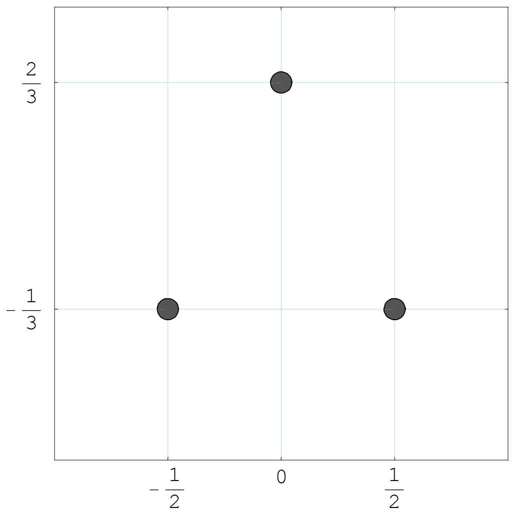


Quarks alone are not seen in nature because of the principle of quark confinement explained by quantum chromodynamics.

The antiquarks with opposite quantum numbers are then

Out[54]= $\mathbf{T}_{\bar{\mathbf{i}}}$

Out[55]= $\{\mathbf{T}_{\bar{\mathbf{u}}}, \mathbf{T}_{\bar{\mathbf{d}}}, \mathbf{T}_{\bar{\mathbf{s}}}\}$



■ Second order tensor

The second order tensor contains 9 elements

In[57]:= **Tdd[i, j]**
% // EinsteinArray[] // MatrixForm

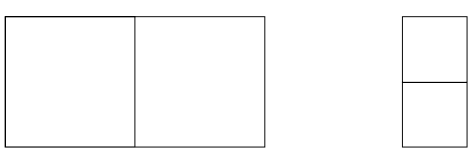
Out[57]= $\mathbf{T}_{\mathbf{i}\,\mathbf{j}}$

Out[58]//MatrixForm=

$$\begin{pmatrix} \mathbf{T}_{\mathbf{u}\,\mathbf{u}} & \mathbf{T}_{\mathbf{u}\,\mathbf{d}} & \mathbf{T}_{\mathbf{u}\,\mathbf{s}} \\ \mathbf{T}_{\mathbf{d}\,\mathbf{u}} & \mathbf{T}_{\mathbf{d}\,\mathbf{d}} & \mathbf{T}_{\mathbf{d}\,\mathbf{s}} \\ \mathbf{T}_{\mathbf{s}\,\mathbf{u}} & \mathbf{T}_{\mathbf{s}\,\mathbf{d}} & \mathbf{T}_{\mathbf{s}\,\mathbf{s}} \end{pmatrix}$$

The Young diagrams are

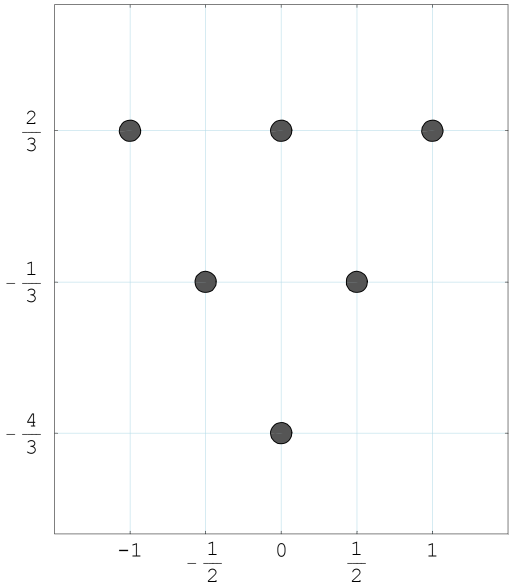
In[59]:= **YoungTableauxDiagrams[2, 3]**



and tell that there is a completely symmetric multiplet (two boxes in a row), and a completely antisymmetric muliplet (two boxes in a column)

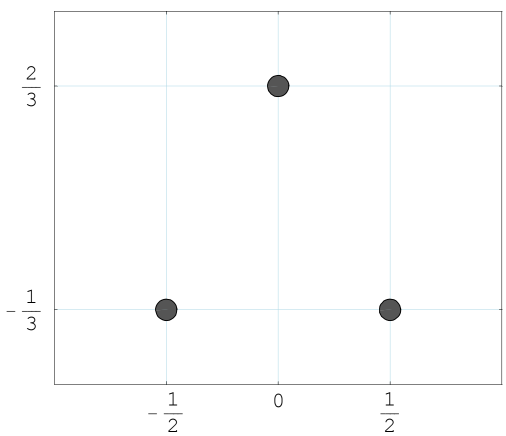
The completely symmetric multiplet in the isospin-hypercharge plane is actually an six-plet

Out[60]= $\{T_{dd}, T_{ds} + T_{sd}, T_{ss}, T_{du} + T_{ud}, T_{su} + T_{us}, T_{uu}\}$



The completely antisymmetric multiplet is a triplet.

Out[62]= $\{-T_{ds} + T_{sd}, -T_{du} + T_{ud}, -T_{su} + T_{us}\}$



The baryon number of these mutiplets is 2/3 so they are not realizable interms of particles.

■ Pseudo-scalar mesons: B=0

If we combine quarks and antiquarks we obtain mesons. There are 9 different elements

```
In[64]:= Tdd[i, bar@j]
         % // EinsteinArray[] // MatrixForm
```

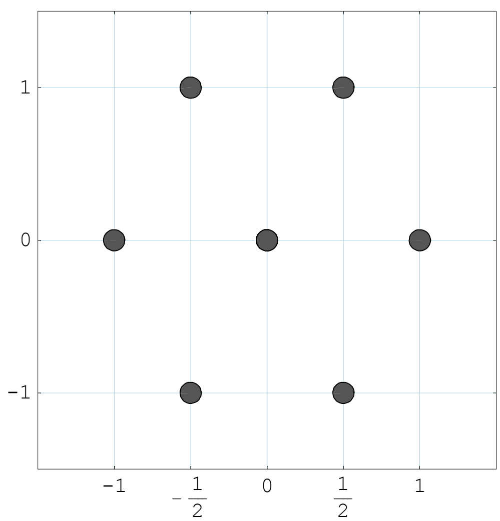
Out[64]= $T_{i\bar{j}}$

Out[65]//MatrixForm=

$$\begin{pmatrix} T_{u\bar{u}} & T_{u\bar{d}} & T_{u\bar{s}} \\ T_{d\bar{u}} & T_{d\bar{d}} & T_{d\bar{s}} \\ T_{s\bar{u}} & T_{s\bar{d}} & T_{s\bar{s}} \end{pmatrix}$$

The completely symmetric multiplet is associated with the Pseudo-scalar mesons with angular momentum $J = 0$ (antisymmetric spin singlet) and parity $P = -1$. The name Pseudoscalar comes from the fact that these mesons do not have angular momentum but change sign under a mirror reflection. Pure scalars are not suppose to change sign under a mirror reflection.

Out[66]= $\{T_{d\bar{d}} + T_{\bar{d}d}, T_{s\bar{d}} + T_{\bar{d}s}, T_{u\bar{d}} + T_{\bar{d}u}, T_{d\bar{s}} + T_{\bar{s}d}, T_{s\bar{s}} + T_{\bar{s}s}, T_{u\bar{s}} + T_{\bar{s}u}, T_{d\bar{u}} + T_{\bar{u}d}, T_{s\bar{u}} + T_{\bar{u}s}, T_{u\bar{u}} + T_{\bar{u}u}\}$



- ## π^+: $u\,\bar{d}$

```
In[68]:= SymmetrizePositions[T, 1, 2][Tdd[u, bar@d]] Spin[0, 0]
```

Out[68]= $\dfrac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,(T_{u\bar{d}} + T_{\bar{d}u})}{\sqrt{2}}$

- ## π^-: $\bar{u}\,d$

```
In[69]:= SymmetrizePositions[T, 1, 2][Tdd[d, bar@u]] Spin[0, 0]
```

Out[69]= $\dfrac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,(T_{d\bar{u}} + T_{\bar{u}d})}{\sqrt{2}}$

- ***K^+: $u\,\bar{s}$***

```
In[70]:= SymmetrizePositions[T, 1, 2][Tdd[u, bar@s]] Spin[0, 0]
```

Out[70]= $$\frac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,(T_{u\,\bar{s}} + T_{\bar{s}\,u})}{\sqrt{2}}$$

- ***K^0: $d\,\bar{s}$***

```
In[71]:= SymmetrizePositions[T, 1, 2][Tdd[d, bar@s]] Spin[0, 0]
```

Out[71]= $$\frac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,(T_{d\,\bar{s}} + T_{\bar{s}\,d})}{\sqrt{2}}$$

- ***$\overline{K}^0$: $s\,\overline{d}$***

```
In[72]:= SymmetrizePositions[T, 1, 2][Tdd[s, bar@d]] Spin[0, 0]
```

Out[72]= $$\frac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,\left(T_{s\,\bar{d}} + T_{\bar{d}\,s}\right)}{\sqrt{2}}$$

- ***K^-: $s\,\overline{u}$***

```
In[73]:= SymmetrizePositions[T, 1, 2][Tdd[s, bar@u]] Spin[0, 0]
```

Out[73]= $$\frac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,(T_{s\,\bar{u}} + T_{\bar{u}\,s})}{\sqrt{2}}$$

- ***π^0***

```
In[74]:= (Tdd[u, bar@u] - Tdd[d, bar@d]) Spin[0, 0]
```

Out[74]= $$\frac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,\left(-T_{d\,\bar{d}} + T_{u\,\bar{u}}\right)}{\sqrt{2}}$$

The following particles that are placed at the center of the graph belong to the same space and can be mixed as linear combinations. The coefficients are found experimentally

- ***η***

```
In[75]:= (Tdd[u, bar@u] - Tdd[d, bar@d] - 2 Tdd[s, bar@s]) Spin[0, 0]
```

Out[75]= $$\frac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\,\left(-T_{d\,\bar{d}} - 2\,T_{s\,\bar{s}} + T_{u\,\bar{u}}\right)}{\sqrt{2}}$$

- η'

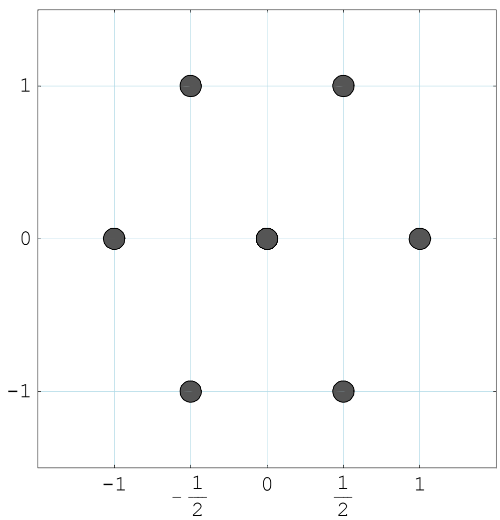
In[76]:= `(Tdd[u, bar@u] - Tdd[d, bar@d] + Tdd[s, bar@s]) Spin[0, 0]`

Out[76]= $$\frac{(-S_{\downarrow\uparrow} + S_{\uparrow\downarrow})\left(-T_{d\bar{d}} + T_{s\bar{s}} + T_{u\bar{u}}\right)}{\sqrt{2}}$$

Vector mesons: B=0

The antisymmetric multiplet is associated with the vector mesons with angular momentum $J = 1$ (symmetric spin triplet) and parity $P = -1$. The name vector comes from the fact that they have angular momentum.

Out[77]= $\{T_{d\bar{d}} - T_{\bar{d}d}, T_{s\bar{d}} - T_{\bar{d}s}, T_{u\bar{d}} - T_{\bar{d}u}, T_{d\bar{s}} - T_{\bar{s}d}, T_{s\bar{s}} - T_{\bar{s}s}, T_{u\bar{s}} - T_{\bar{s}u}, T_{d\bar{u}} - T_{\bar{u}d}, T_{s\bar{u}} - T_{\bar{u}s}, T_{u\bar{u}} - T_{\bar{u}u}\}$



Third order tensor: Hadrons B=1

These mutiplets fulfill the condition B=1, so each point in the diagram is associated to a certain particle.

The third order tensor contains 27 elements

```
In[79]:= Tddd[i, j, k]
         % // EinsteinArray[] // MatrixForm
```

Out[79]= $T_{i\,j\,k}$

Out[80]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} T_{uuu} \\ T_{uud} \\ T_{uus} \end{pmatrix} & \begin{pmatrix} T_{udu} \\ T_{udd} \\ T_{uds} \end{pmatrix} & \begin{pmatrix} T_{usu} \\ T_{usd} \\ T_{uss} \end{pmatrix} \\ \begin{pmatrix} T_{duu} \\ T_{dud} \\ T_{dus} \end{pmatrix} & \begin{pmatrix} T_{ddu} \\ T_{ddd} \\ T_{dds} \end{pmatrix} & \begin{pmatrix} T_{dsu} \\ T_{dsd} \\ T_{dss} \end{pmatrix} \\ \begin{pmatrix} T_{suu} \\ T_{sud} \\ T_{sus} \end{pmatrix} & \begin{pmatrix} T_{sdu} \\ T_{sdd} \\ T_{sds} \end{pmatrix} & \begin{pmatrix} T_{ssu} \\ T_{ssd} \\ T_{sss} \end{pmatrix} \end{pmatrix}$$

The Young diagrams are

```
In[81]:= YoungTableauxDiagrams[3, 3]
```

and tell that there is a completely symmetric multiplet (three boxes in a row), a completely asymmetric multiplet (three boxes in a column) and two multiplets with mixed symmetry.

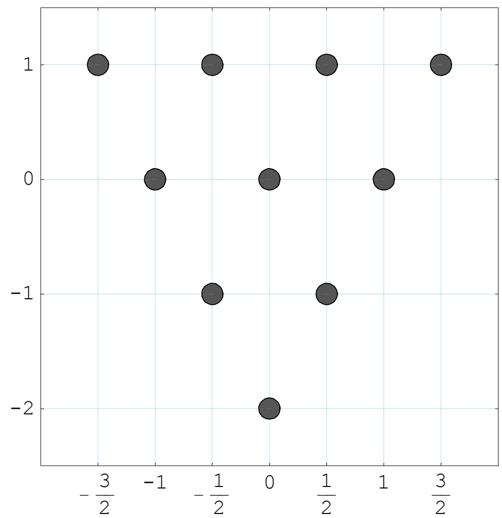
Completely symmetric 10-plet

```
In[82]:= PermutationRule[3] = MapThread[List[Rule[##]] &,
            {Array[{i_, j_, k_} &, 6],
             Part[{i, j, k}, #] & /@ Permutations[{1, 2, 3}]}
           ];
```

The completely symmetric 10-plet has to have angular momentum $J = 3/2$ made from three completely symmetric spins to have a symmetric flavor-spin wavefunction

Out[83]= $T_{ijk} + T_{ikj} + T_{jik} + T_{jki} + T_{kij} + T_{kji}$

Out[84]= $\{T_{ddd},\ T_{dds} + T_{dsd} + T_{sdd},\ T_{dss} + T_{sds} + T_{ssd},\ T_{sss},\ T_{ddu} + T_{dud} + T_{udd},\ T_{dsu} + T_{dus} + T_{sdu} + T_{sud} + T_{uds} + T_{usd},\ T_{ssu} + T_{sus} + T_{uss},\ T_{duu} + T_{udu} + T_{uud},\ T_{suu} + T_{usu} + T_{uus},\ T_{uuu}\}$



This means thay they could be mutiplied by any of the following spin wave functions

```
In[86]:= {Spin[3 / 2, -3 / 2], Spin[3 / 2, -1 / 2], Spin[3 / 2, 1 / 2], Spin[3 / 2, 3 / 2]}
```

Out[86]= $\left\{S_{\downarrow\downarrow\downarrow},\ \frac{S_{\downarrow\downarrow\uparrow} + S_{\downarrow\uparrow\downarrow} + S_{\uparrow\downarrow\downarrow}}{\sqrt{3}},\ \frac{S_{\downarrow\uparrow\uparrow} + S_{\uparrow\downarrow\uparrow} + S_{\uparrow\uparrow\downarrow}}{\sqrt{3}},\ S_{\uparrow\uparrow\uparrow}\right\}$

For example the wave function of the $\mathbf{\Delta^{++}}$ particle with spin projection 3/2 is

```
In[87]:= Tddd[u, u, u] Spin[3 / 2, -3 / 2]
```

Out[87]= $S_{\downarrow\downarrow\downarrow}\, T_{uuu}$

The list of spin 3/2 hadrons with flavor content is (evaluate the thin closed cell)

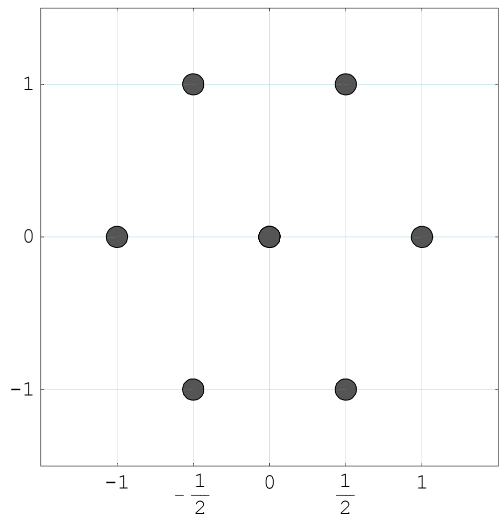
Out[88]//TableForm=

$\Delta^{++} \to T_{uuu}$
$\Delta^{+} \to T_{uud}$
$\Delta^{0} \to T_{udd}$
$\Delta^{-} \to T_{ddd}$
$\Sigma^{+} \to T_{uus}$
$\Sigma^{0} \to T_{uds}$
$\Sigma^{-} \to T_{dds}$
$\Xi^{*0} \to T_{uss}$
$\Xi^{*-} \to T_{dss}$
$\Omega^{-} \to T_{sss}$

First 8-plet with mixed symmetry

This 8-plet is antisymmetric under the the exchange of the first and second particles.

Out[89]= $\{-T_{dsd}+T_{sdd}, -T_{dss}+T_{sds}, -T_{dsu}+T_{sdu}, -T_{dud}+T_{udd}, -T_{dus}+T_{uds}, -T_{duu}+T_{udu}, -T_{sud}+T_{usd}, -T_{sus}+T_{uss}, -T_{suu}+T_{usu}\}$



Note.- It seems that there are nine elements but only eight are linearly independent.

There is second SU[3] 8-plet and an additional completely antysimmteric SU[3] singlet but they are not helpful to define the wave functons of the remaining baryons.

We already made use of the completely symmetric SU[3] 10-plet with spin 3/2. Now let us construct the wave functions with spin 1/2. It is not possible to construct spin 1/2 particles with three identical quarks because the quark wave function is definitely symmetric and there is no way to define a completely symmetric spin 1/2 wave function of three particles.

The practical way to construct the wave function of the hadrons is more manual and described as follows

Proton: up up down

The proton can be constructed as follows. Starting with a mixed spin wave function with the first and second indices symmetric and then applying a permutation over all the indices we obtain the proton with spin projection 1/2

In[91]:=
```
Tddd[u, u, d] SpinMixed2[1 / 2, 1 / 2] // Simplify
Plus @@ (% /. PermutationRule[3])
```

Out[91]= $-\frac{(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{uud}}{\sqrt{6}}$

Out[92]= $-\frac{2\,(-2\,S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{duu}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}-2\,S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{udu}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{uud}}{\sqrt{6}}$

HW: Find the proton with spin projection -1/2

Neutron: up down down

The neutron has spin 1/2 as well. The wave function with spin projection 1/2 is

```
In[93]:= Tddd[d, d, u] SpinMixed2[1 / 2, 1 / 2] // Simplify
         Plus @@ (% /. PermutationRule[3])
```

Out[93]= $-\frac{(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{ddu}}{\sqrt{6}}$

Out[94]= $-\frac{2\,(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{ddu}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}-2\,S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{dud}}{\sqrt{6}}-\frac{2\,(-2\,S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{udd}}{\sqrt{6}}$

Σ^+: up up strange

In simmilar way

```
In[95]:= Tddd[u, u, s] SpinMixed2[1 / 2, 1 / 2] // Simplify
         Plus @@ (% /. PermutationRule[3])
```

Out[95]= $-\frac{(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{uus}}{\sqrt{6}}$

Out[96]= $-\frac{2\,(-2\,S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{suu}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}-2\,S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{usu}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{uus}}{\sqrt{6}}$

Σ^-: down down strange

```
In[97]:= Tddd[d, d, s] SpinMixed2[1 / 2, 1 / 2] // Simplify
         Plus @@ (% /. PermutationRule[3])
```

Out[97]= $-\frac{(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{dds}}{\sqrt{6}}$

Out[98]= $-\frac{2\,(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{dds}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}-2\,S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{dsd}}{\sqrt{6}}-\frac{2\,(-2\,S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{sdd}}{\sqrt{6}}$

Ξ^0: up strange strange

In[99]:=
```
Tddd[s, s, u] SpinMixed2[1 / 2, 1 / 2] // Simplify
Plus @@ (% /. PermutationRule[3])
```

Out[99]=
$$-\frac{(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{ssu}}{\sqrt{6}}$$

Out[100]=
$$-\frac{2\,(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{ssu}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}-2\,S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{sus}}{\sqrt{6}}-\frac{2\,(-2\,S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{uss}}{\sqrt{6}}$$

Ξ^-: down strange strange

In[101]:=
```
Tddd[s, s, d] SpinMixed2[1 / 2, 1 / 2] // Simplify
Plus @@ (% /. PermutationRule[3])
```

Out[101]=
$$-\frac{(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{ssd}}{\sqrt{6}}$$

Out[102]=
$$-\frac{2\,(-2\,S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{dss}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}-2\,S_{\uparrow\downarrow\uparrow}+S_{\uparrow\uparrow\downarrow})\,T_{sds}}{\sqrt{6}}-\frac{2\,(S_{\downarrow\uparrow\uparrow}+S_{\uparrow\downarrow\uparrow}-2\,S_{\uparrow\uparrow\downarrow})\,T_{ssd}}{\sqrt{6}}$$

Σ^0: up down strange

Taking a spin wave function symmetric in the first and second indices we have to choose a symmetric flavor wave function in the first and second indicesas well.

In[103]:=
```
(Tddd[u, d, s] + Tddd[d, u, s]) SpinMixed2[1 / 2, 1 / 2]
Plus @@ (% /. PermutationRule[3])
```

Out[103]=
$$\left(-\frac{S_{\downarrow\uparrow\uparrow}}{\sqrt{6}}-\frac{S_{\uparrow\downarrow\uparrow}}{\sqrt{6}}+\sqrt{\frac{2}{3}}\,S_{\uparrow\uparrow\downarrow}\right)(T_{dus}+T_{uds})$$

Out[104]=
$$2\left(\sqrt{\frac{2}{3}}\,S_{\downarrow\uparrow\uparrow}-\frac{S_{\uparrow\downarrow\uparrow}}{\sqrt{6}}-\frac{S_{\uparrow\uparrow\downarrow}}{\sqrt{6}}\right)(T_{sdu}+T_{sud})+$$
$$2\left(-\frac{S_{\downarrow\uparrow\uparrow}}{\sqrt{6}}-\frac{S_{\uparrow\downarrow\uparrow}}{\sqrt{6}}+\sqrt{\frac{2}{3}}\,S_{\uparrow\uparrow\downarrow}\right)(T_{dus}+T_{uds})+2\left(-\frac{S_{\downarrow\uparrow\uparrow}}{\sqrt{6}}+\sqrt{\frac{2}{3}}\,S_{\uparrow\downarrow\uparrow}-\frac{S_{\uparrow\uparrow\downarrow}}{\sqrt{6}}\right)(T_{dsu}+T_{usd})$$

Λ^0: up down strange

Besides the previous Σ^0particle with the tree different flavors there is another possibility. Taking the spin wavefunction antisymmetric in the first and second indices and taking a flavor wave function antisymmetric in the first and second indices as well in order to cancel the anisymmetry. The last step to get the complete wavefunction is to add all the possible permutations to get a fully symmetric wave function.

In[105]:=

```
(Tddd[u, d, s] - Tddd[d, u, s]) SpinMixed1[1 / 2, 1 / 2]
Plus @@ (% /. PermutationRule[3] // Expand)
```

Out[105]=

$$\frac{(-S_{\downarrow\uparrow\uparrow} + S_{\uparrow\downarrow\uparrow})\,(-T_{dus} + T_{uds})}{\sqrt{2}}$$

Out[106]=

$$\frac{2\,S_{\downarrow\uparrow\uparrow}\,T_{dsu}}{\sqrt{2}} - \frac{2\,S_{\uparrow\uparrow\downarrow}\,T_{dsu}}{\sqrt{2}} + \frac{2\,S_{\downarrow\uparrow\uparrow}\,T_{dus}}{\sqrt{2}} - \frac{2\,S_{\uparrow\downarrow\uparrow}\,T_{dus}}{\sqrt{2}} + \frac{2\,S_{\uparrow\downarrow\uparrow}\,T_{sdu}}{\sqrt{2}} - \frac{2\,S_{\uparrow\uparrow\downarrow}\,T_{sdu}}{\sqrt{2}} - \frac{2\,S_{\uparrow\downarrow\uparrow}\,T_{sud}}{\sqrt{2}} + \frac{2\,S_{\uparrow\uparrow\downarrow}\,T_{sud}}{\sqrt{2}} - \frac{2\,S_{\downarrow\uparrow\uparrow}\,T_{uds}}{\sqrt{2}} + \frac{2\,S_{\uparrow\downarrow\uparrow}\,T_{uds}}{\sqrt{2}} - \frac{2\,S_{\downarrow\uparrow\uparrow}\,T_{usd}}{\sqrt{2}} + \frac{2\,S_{\uparrow\uparrow\downarrow}\,T_{usd}}{\sqrt{2}}$$

Acknowledgement

I would like to thank David Park for assistance on the graphics and some of the *Mathematica* constructions.

References

[1] Harry J. Lipkin, Lie Groups for Pedestrians, Dover Publications, 1966

[2] Greiner and Muller, *Quantum Mechanics Symmetries*, Springer-Verlag, 1994

[3] Kerson Huang, Quarks, Leptons & Gauge Fields, 2nd Ed., World Scientific, 1992