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Physics 234A HW 1

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(1) Before beginning, let me note a few conventional things. The action given in the problem set (equation (1.1)) is

$$S(x,h) = \frac{1}{2}\int d\tau \sqrt{h_{\tau\tau}}\left(h^{\tau\tau}G_{\tau\tau} - m^2\right)$$

where G is the induced metric on the world line. With this convention, h has signature $(+)$, which confuses me a bit. Instead, I will do something closer to the conventions of Becker, Becker, and Schwarz, and put

$$S(x,h) = -\frac{1}{2}\int d\tau \sqrt{-h}\left(h^{\tau\tau}G_{\tau\tau} + m^2\right)$$

where h has signature $(-)$. Here, I put $h = \det h_{ab}$.

(1.1) Let us consider a variation in the functions x^μ: $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$. This puts $\delta g_{\mu\nu} = (\partial_\rho g_{\mu\nu})\,\delta x^\rho$. Then,

$$\begin{aligned}
\delta G_{\tau\tau} =& \delta\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) \\
&= \left(\left(\partial_\rho g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu\right)\delta x^\rho + 2g_{\mu\nu}\dot{x}^\mu\frac{d}{d\tau}\delta x^\nu.
\end{aligned}$$

The variation in the action is now

$$\begin{aligned}
\delta S =& -\frac{1}{2}\int d\tau\sqrt{-h}h^{\tau\tau}\delta G_{\tau\tau} \\
=& \frac{1}{2}\int d\tau\frac{1}{\sqrt{-h^{\tau\tau}}}\left(\left(\partial_\mu g_{\nu\rho}\right)\dot{x}^\nu\dot{x}^\rho\,\delta x^\mu + 2g_{\mu\nu}\dot{x}^\nu\frac{d}{d\tau}\delta x^\mu\right) \\
=& \frac{1}{2}\int d\tau\frac{1}{\sqrt{-h^{\tau\tau}}}\left(\left(\partial_\mu g_{\nu\rho}\right)\dot{x}^\nu\dot{x}^\rho - \frac{d}{d\tau}\left(2g_{\mu\nu}\dot{x}^\nu\right)\right)\delta x^\mu \\
=& \frac{1}{2}\int d\tau\frac{1}{\sqrt{-h^{\tau\tau}}}\left(\left(\partial_\mu g_{\nu\rho}\right)\dot{x}^\nu\dot{x}^\rho - 2\dot{x}^\rho(\partial_\rho g_{\mu\nu})\dot{x}^\nu - 2g_{\mu\nu}\ddot{x}^\nu\right)\delta x^\mu
\end{aligned}$$

By using the inverse spacetime metric, we now find that

$$\begin{aligned}
&\ddot{x}^\mu + g^{\mu\sigma}(\partial_\rho g_{\nu\sigma})\dot{x}^\nu \dot{x}^\rho - \frac{1}{2} g^{\mu\sigma}(\partial_\sigma g_{\nu\rho})\dot{x}^\nu \dot{x}^\rho \\
&= \ddot{x}^\mu + \frac{1}{2} g^{\mu\sigma}\left(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma}\right)\dot{x}^\nu \dot{x}^\rho - \frac{1}{2} g^{\mu\sigma}(\partial_\sigma g_{\nu\rho})\dot{x}^\nu \dot{x}^\rho \\
&= \ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu \dot{x}^\rho \\
&= 0.
\end{aligned}$$

Here, Γ is the Levi-Civita connection. This is exactly the geodesic equation.

(1.2) The momentum conjugate to x^μ is $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$. Noting that

$$\begin{aligned}
&\frac{\partial}{\partial \dot{x}^\mu} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \\
&= g_{\alpha\beta}\left(\delta^\alpha_\mu \dot{x}^\beta + \dot{x}^\alpha \delta^\beta_\mu\right) \\
&= 2 g_{\mu\nu}\dot{x}^\nu,
\end{aligned}$$

we find

$$p_\mu = \frac{1}{\sqrt{-h}} g_{\mu\nu}\dot{x}^\nu.$$

We can eventually change coordinate systems on the world line to make $h_{\tau\tau} = -1/m^2$. This would make $p^\mu = m\dot{x}^\mu$.

(1.3) We now consider adding to the action the term

$$\Delta S = \int dx^\mu e A_\mu.$$

Under a variation in x^μ, we find that ΔS changes by

$$\begin{aligned}
\delta\Delta S &= \int d\tau e\left(A_\mu \frac{d}{d\tau}\delta x^\mu + \dot{x}^\nu \partial_\mu A_\nu\, \delta x^\mu\right) \\
&= \int d\tau\, e\left(-\dot{x}^\nu \partial_\nu A_\mu\, \delta x^\mu + \dot{x}^\nu \partial_\mu A_\nu\, \delta x^\mu\right) \\
&= \int d\tau\, e F_{\mu\nu}\dot{x}^\nu\, \delta x^\mu
\end{aligned}$$

where F is field strength tensor. If we combine this result with the variation in the action computed in part (1.1), we find the condition

$$\frac{1}{\sqrt{-h}}\left(\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu \dot{x}^\rho\right) = eF^\mu{}_\nu \dot{x}^\nu$$

This looks like the usual Lorentz force law. It matches exactly if we parametrize τ so that $h_{\tau\tau} = -1/m^2$. In fact, given that the action puts

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2. $\boxed{\text{GAUGE FIXING}}$

2.1 DIFFEOMORPHISMS

$\tau \to \tau'(\tau)$. $h_{\tau\tau} \to h'_{\tau'\tau'}$

$dS_\Sigma^2 = -h_{\tau\tau}(d\tau)^2 = -h'_{\tau'\tau'}(d\tau')^2 \Rightarrow \boxed{h'_{\tau'\tau'} = \left(\frac{\partial\tau}{\partial\tau'}\right)^2 h_{\tau\tau}}$

while the length of the interval $\int_{\tau_1}^{\tau_2} dS_\Sigma$ does <u>NOT</u> change under diffeomorphisms, since the diffeo. are just reparametrizations.

$$S' = \frac{1}{2}\int d\tau' \sqrt{-h'}\left(h'^{\tau'\tau'}\partial_{\tau'}X^\mu \partial_{\tau'}X^\nu g_{\mu\nu} - m^2\right)$$

$$= \frac{1}{2}\int d\tau' \sqrt{h'_{\tau'\tau'}}\left[\frac{1}{h'_{\tau'\tau'}}\left(\frac{\partial\tau}{\partial\tau'}\right)^2 \partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu} - m^2\right]$$

$$= \frac{1}{2}\int d\tau' \frac{\partial\tau}{\partial\tau'}\sqrt{h_{\tau\tau}}\left[\frac{1}{h_{\tau\tau}}\left(\frac{\partial\tau'}{\partial\tau}\right)^2\left(\frac{\partial\tau}{\partial\tau'}\right)^2 \partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu} - m^2\right]$$

$$= \frac{1}{2}\int d\tau \sqrt{-h}\left(h^{\tau\tau}\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu} - m^2\right)$$

$$= S$$

Therefore, the action is $\boxed{\text{invariant}}$ under the diffeomorphisms.

2.2 GAUGE FIXING

By setting $h'_{\tau'\tau'} = 1$, it requires $1 = \left(\frac{\partial\tau}{\partial\tau'}\right)^2 h_{\tau\tau}$, i.e.

$$\tau' = \sqrt{h_{\tau\tau}}\,\tau = \sqrt{-h}\,\tau.$$

Therefore, the diffeomorphism that we are looking for is

$$\boxed{\tau \to \tau'(\tau) = \sqrt{-h}\,\tau}$$

2.3 EXTREMA OF S WRT $h_{\tau\tau}$

$\frac{\delta S}{\delta \eta} = \frac{1}{2}\left(-\frac{1}{\eta^2}\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} - m^2\right) = 0$ gives

$$-h^{\tau\tau}\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu} - m^2 = 0$$

i.e. $m^2 = -h^{\tau\tau}\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu} = -p^\mu p^\nu g_{\mu\nu}$

w/ p^μ the conjugate momentum we found in Promblem 1.

That is to say, the mass of such a particle is m.

3.

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(3.1)

$$\langle X|X'\rangle = \int_{X(0)=X}^{X(1)=X'} \mathcal{D}h\mathcal{D}X\, e^{-\frac{i}{2}\int_\Sigma d\tau e(e^{-2}\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} - m^2)} \tag{3.1}$$

To rewrite as an integral over L, we set $e(\tau) = L$ and note that up to an overall normalization coming from changing the measure of integration, the amplitude becomes

$$\langle X|X'\rangle \propto \int_0^\infty dL \int_{X(0)=X}^{X(1)=X'} \mathcal{D}X\, e^{-\frac{i}{2}\int_\Sigma d\tau (L^{-1}\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} - Lm^2)} \tag{3.2}$$

Finally, rotating $\tau \to i\tau$, $d\tau \to i\, d\tau$, $\partial_\tau x \to -i\partial_\tau x$, $X^0 \to iX^0$, $\eta_{\mu\nu} \to \delta_{\mu\nu}$, we get

$$\langle X|X'\rangle \propto \int_0^\infty dL \int_{X(0)=X}^{X(1)=X'} \mathcal{D}X\, e^{-\frac{1}{2}\int_\Sigma d\tau (L^{-1}\partial_\tau X^\mu \partial_\tau X^\nu \delta_{\mu\nu} + Lm^2)} \tag{3.3}$$

(3.2) We now need to discretize the path integral. We perturb around a classical path, so that $X^\mu = X^\mu(0) + \tau(X^\mu(1) - X^\mu(0)) + \delta X^\mu(\tau)$. The measure of the perturbed function is

$$\|\delta X^\mu\|^2 = \int_0^1 d\tau\, \sqrt{-h}(\delta X^\mu)^2 = L\int_0^1 d\tau\, (\delta X^\mu)^2 \tag{3.4}$$

so the path integral measure is

$$\mathcal{D}X \propto \prod_{\{\tau_n\}} \sqrt{L} d\delta X^\mu(\tau_n) \tag{3.5}$$

where τ_n is a discretization of $[0,1]$. Thus

$$\langle X|X'\rangle \propto \int_0^\infty dL \prod_{\{\tau_n\}} \int d\delta X^\mu(\tau_n)\, \sqrt{L} e^{-\frac{1}{2L}(X^\mu(1)-X^\mu(0))^2 - \frac{1}{2}Lm^2} e^{-\frac{1}{2L}\int_0^1 d\tau\, (\delta\partial_\tau X)^2} \tag{3.6}$$

Integrate by parts to find $\int_0^1 d\tau\, (\delta\partial_\tau X)^2 = -\int_0^1 d\tau\, \delta X \cdot \partial_\tau^2 \delta X$. Then we can do the Gaussian integral:

$$\int \prod_{\tau_n} d(\sqrt{L}\delta X^\mu(\tau_n)) e^{-\frac{1}{2}\int_0^1 d\tau\, (\sqrt{L}\delta X^\mu)\frac{-\partial_\tau^2}{L^2}(\sqrt{L}\delta X^\nu)\delta_{\mu\nu}} \propto \left[\det\left(-\frac{\partial_\tau^2}{L^2}\right)\right]^{-D/2} \tag{3.7}$$

The eigenfunctions of $-\partial_\tau^2/L^2$ are waves, and the boundary conditions $\delta X^\mu(0) = \delta X^\mu(1) = 0$ forces solutions of the form $a_n \sin(n\pi\tau)$ with eigenvalues $n^2\pi^2/L^2$. So

$$\det\left(-\frac{\partial_\tau^2}{L^2}\right) = \prod_{n=1}^\infty \left(\frac{n^2\pi^2}{L^2}\right) = \frac{\prod_{n=1}^\infty n^2}{\prod_{n=1}^\infty (L/\pi)^2} = \frac{(2\pi)^{-1}}{(L/\pi)^{-1}} \sim L \tag{3.8}$$

where I have used

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} \tag{3.9}$$

$$\zeta'(s) = \sum_{n=1}^\infty \partial_s e^{-s\ln n} = -\sum_{n=1}^\infty \ln n\, n^{-s} \tag{3.10}$$

$$\prod_{n=1}^\infty C = e^{\ln C\sum_{n=1}^\infty 1} = e^{\ln C\zeta(0)} = C^{\zeta(0)} = C^{-1/2} \tag{3.11}$$

$$\prod_{n=1}^\infty n^2 = e^{2\sum_{n=1}^\infty \ln n} = e^{2\zeta'(0)} = (2\pi)^{-1} \tag{3.12}$$