```
<< Local`QFTToolKit`
ct := ConjugateTranspose
```

■ Krein spectral triples and the fermionic action

● 2 Krein spectral triples

```
PR["Definitions: ",
  $ = {ℋ → {"Krein space", "indefinite inner product" -> BraKet["·", "·"]},
    𝒥 → {"fundamental symmetry", 𝒥[ℋ] → ℋ, (1 + 𝒥)[ℋ] > 0, (1 - 𝒥)[ℋ] < 0},
    ℋ𝒥 →
     {"Hilbert space|Positive-definite", BraKet["·", "·"]𝒥 -> BraKet[𝒥["·"], "·"]},
    T⁺ → {"Krein adjoint wrt", BraKet["·", "·"]},
    T* → {"Hilbert space adjoint wrt", BraKet["·", "·"]𝒥},
    T⁺ → 𝒥 · T* · 𝒥}; $ // ColumnBar,
  NL, "Definition 2.1",
  $ = {"Krein space {ℋ,𝒥} is ℤ2-graded" ⟸ {ℋ𝒥 → "ℤ2-graded", 𝒥 → "homogeneous"},
    {ℋ𝒥 → "ℤ2-graded"} ⟹ {ℋ → ℋ^"0" ⊕ ℋ^"1", ForAll[ψ0 ∈ ℋ^0 && ψ1 ∈ ℋ^1, BraKet[ψ0, ψ1]𝒥 → 0]},
    {ℬ[ℋ] -> ℬ^"0"[ℋ] ⊕ ℬ^"1"[ℋ], ℬ[ℋ] → "bounded operators"},
    {𝒥 → "homogeneous"} ⟹ {𝒥 → "even" || "odd"},
    {𝒥 → "odd"} ⟹ {ℋ^"0" ≃ ℋ^"1", Γ → "Krein-anti-self-adjoint",
     Γ⁺ → 𝒥 · Γ · 𝒥 → -Γ · 𝒥 · 𝒥 → -Γ, Γ → "grading operator"} ⟹ {Γ[ℋ^j] -> (-1)^j, j ∈ ℤ2},
    {"combined graph inner product", BraKet[ψ, ϕ]S,T -> BraKet[ψ, ϕ]𝒥 +
       BraKet[S[ψ], S[ϕ]]𝒥 + BraKet[T[ψ], T[ϕ]]𝒥, {S, T} ∈ "closed operators",
     BraKet["·", "·"]𝒥 → BraKet[𝒥["·"], "·"] ⟵ "positive definite",
     {ψ, ϕ} ∈ Inactive[Intersection][ Dom[S], Dom[T]]},
    {"combined graph norm", Norm["·"]S,T},
    {𝒟 → "Krein self-adjoint operator"} ⟹
     {𝒥 · 𝒟* → 𝒟 · 𝒥, Dom[𝒟*] → Dom[𝒟 · 𝒥] → 𝒥 · Dom[𝒟],
      BraKet["·", "·"]𝒟,𝒟* -> BraKet["·", "·"]𝒟·𝒥,𝒥·𝒟[Inactive[Intersection][
          Dom[𝒟], Dom[𝒟*]] -> Inactive[Intersection][ Dom[𝒟], 𝒥 · Dom[𝒟]]]}
   };
  $ // ColumnBar,
  NL, "Definition 2.2:",
  NL, $ = {"Even Krein spectral triple: ", {𝒜, ℋ, 𝒟, 𝒥}, ℋ → "ℤ2-graded Krein space",
    {𝒜 -> "*-algebra", π → "*-algebra representation" → π[𝒜] → B^"0"[ℋ]},
    {𝒥 → "fundamental symmetry", 𝒥* → 𝒥, 𝒥 · 𝒥 → 1},
    {𝒟 → "closed,odd operator", 𝒟[Dom[𝒟]] → ℋ,
     {Exists[ℰ, ℰ ⊂ Inactive[Intersection][ Dom[𝒟], 𝒥 · Dom[𝒟]] && ℰ →
        {"dense wrt", Norm["·"]𝒟·𝒥,𝒥·𝒟}]},
     {𝒟 → "Krein-self-adjoint on ℰ" → 𝒥 · 𝒟[Dom[𝒟] → ℋ𝒥]},
     {π[𝒜] · ℰ ⊂ Inactive[Intersection][ Dom[𝒟], 𝒥 · Dom[𝒟]],
      CommutatorM[𝒟, π[a]] → "Bounded on ℰ for all a∈𝒜 "},
     {π[a] ∘ ı [Inactive[Intersection][ Dom[𝒟], 𝒥 · Dom[𝒟]]] → ℋ,
      ForAll[a ∈ 𝒜, ℋ → "compact"], ı → "natural inclusion map",
      Inactive[Intersection][ Dom[𝒟], 𝒥 · Dom[𝒟]] →
       {"Hilbert space with", BraKet["·", "·"]𝒟·𝒥,𝒥·𝒟 }
     }
    },
    {{𝒜, ℋ, 𝒟, 𝒥 → "odd"}["even"] → "Lorentz-type"}
   };
  $ // ColumnBar
 ];
```

Definitions:

$\mathcal{H} \to$ {Krein space, indefinite inner product $\to \langle \cdot \mid \cdot \rangle$}
$\mathcal{J} \to$ {fundamental symmetry, $\mathcal{J}[\mathcal{H}] \to \mathcal{H}$, $(1+\mathcal{J})[\mathcal{H}] > 0$, $(1-\mathcal{J})[\mathcal{H}] < 0$}
$\mathcal{H}_{\mathcal{J}} \to$ {Hilbert space|Positive-definite, $\langle \cdot \mid \cdot \rangle_{\mathcal{J}} \to \langle \mathcal{J}[\cdot] \mid \cdot \rangle$}
$T^{+} \to$ {Krein adjoint wrt, $\langle \cdot \mid \cdot \rangle$}
$T^{*} \to$ {Hilbert space adjoint wrt, $\langle \cdot \mid \cdot \rangle_{\mathcal{J}}$}
$T^{+} \to \mathcal{J} \cdot T^{*} \cdot \mathcal{J}$

Definition 2.1

Krein space $\{\mathcal{H},\mathcal{J}\}$ is $\mathbb{Z}_2$-graded $\Longleftarrow$ $\{\mathcal{H}_{\mathcal{J}} \to \mathbb{Z}_2$-graded, $\mathcal{J} \to$ homogeneous$\}$
$\{\mathcal{H}_{\mathcal{J}} \to \mathbb{Z}_2$-graded$\} \Longrightarrow \{\mathcal{H} \to \mathcal{H}^0 \oplus \mathcal{H}^1$, $\forall_{\psi_0 \in \mathcal{H}^0 \,\&\&\, \psi_1 \in \mathcal{H}^1} (\langle \psi_0 \mid \psi_1 \rangle_{\mathcal{J}} \to 0)\}$
$\{\mathcal{B}[\mathcal{H}] \to \mathcal{B}^0[\mathcal{H}] \oplus \mathcal{B}^1[\mathcal{H}]$, $\mathcal{B}[\mathcal{H}] \to$ bounded operators$\}$
$\{\mathcal{J} \to$ homogeneous$\} \Longrightarrow \{\mathcal{J} \to$ even || odd$\}$
$\{\mathcal{J} \to$ odd$\} \Longrightarrow \{\mathcal{H}^0 \simeq \mathcal{H}^1$, $\Gamma \to$ Krein-anti-self-adjoint, $\Gamma^{+} \to \mathcal{J} \cdot \Gamma \cdot \mathcal{J} \to -\Gamma \cdot \mathcal{J} \cdot \mathcal{J} \to -\Gamma$, $\Gamma \to$ grading operator$\} \Longrightarrow$
$\{\Gamma[\mathcal{H}^j] \to (-1)^j$, $j \in \mathbb{Z}_2\}$
{combined graph inner product, $\langle \psi \mid \phi \rangle_{S,T} \to \langle \psi \mid \phi \rangle_{\mathcal{J}} + \langle S[\psi] \mid S[\phi] \rangle_{\mathcal{J}} + \langle T[\psi] \mid T[\phi] \rangle_{\mathcal{J}}$,
$\{S, T\} \in$ closed operators, $\langle \cdot \mid \cdot \rangle_{\mathcal{J}} \to \langle \mathcal{J}[\cdot] \mid \cdot \rangle \longleftarrow$ positive definite, $\{\psi, \phi\} \in \mathrm{Dom}[S] \cap \mathrm{Dom}[T]$}
{combined graph norm, $\mathrm{Norm}[\cdot]_{S,T}$}
$\{\mathcal{D} \to$ Krein self-adjoint operator$\} \Longrightarrow \{\mathcal{J} \cdot \mathcal{D}^{*} \to \mathcal{D} \cdot \mathcal{J}$, $\mathrm{Dom}[\mathcal{D}^{*}] \to \mathrm{Dom}[\mathcal{D} \cdot \mathcal{J}] \to \mathcal{J} \cdot \mathrm{Dom}[\mathcal{D}]$,
$\langle \cdot \mid \cdot \rangle_{\mathcal{D},\mathcal{D}^{*}} \to \langle \cdot \mid \cdot \rangle_{\mathcal{D}\cdot\mathcal{J},\mathcal{J}\cdot\mathcal{D}}[\mathrm{Dom}[\mathcal{D}] \cap \mathrm{Dom}[\mathcal{D}^{*}] \to \mathrm{Dom}[\mathcal{D}] \cap \mathcal{J} \cdot \mathrm{Dom}[\mathcal{D}]]\}$

Definition 2.2:

Even Krein spectral triple:
$\{\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$
$\mathcal{H} \to \mathbb{Z}_2$-graded Krein space
$\{\mathcal{A} \to$ *-algebra, $\pi \to$ *-algebra representation $\to \pi[\mathcal{A}] \to B^0[\mathcal{H}]\}$
$\{\mathcal{J} \to$ fundamental symmetry, $\mathcal{J}^{*} \to \mathcal{J}$, $\mathcal{J} \cdot \mathcal{J} \to 1\}$
$\{\mathcal{D} \to$ closed,odd operator, $\mathcal{D}[\mathrm{Dom}[\mathcal{D}]] \to \mathcal{H}$,
$\{\exists_{\mathcal{E}} (\mathcal{E} \subset \mathrm{Dom}[\mathcal{D}] \cap \mathcal{J} \cdot \mathrm{Dom}[\mathcal{D}]$ && $\mathcal{E} \to$ {dense wrt, $\mathrm{Norm}[\cdot]_{\mathcal{D}\cdot\mathcal{J},\mathcal{J}\cdot\mathcal{D}}\})\}$,
$\{\mathcal{D} \to$ Krein-self-adjoint on $\mathcal{E} \to \mathcal{J} \cdot \mathcal{D}[\mathrm{Dom}[\mathcal{D}] \to \mathcal{H}_{\mathcal{J}}]\}$,
$\{\pi[\mathcal{A}] \cdot \mathcal{E} \subset \mathrm{Dom}[\mathcal{D}] \cap \mathcal{J} \cdot \mathrm{Dom}[\mathcal{D}]$, $[\mathcal{D}, \pi[a]] \to$ Bounded on $\mathcal{E}$ for all $a \in \mathcal{A}$ $\}$,
$\{\pi[a] \circ i[\mathrm{Dom}[\mathcal{D}] \cap \mathcal{J} \cdot \mathrm{Dom}[\mathcal{D}]] \to \mathcal{H}$, $\forall_{a \in \mathcal{A}} (\mathcal{H} \to$ compact$)$, $i \to$ natural inclusion map,
$\mathrm{Dom}[\mathcal{D}] \cap \mathcal{J} \cdot \mathrm{Dom}[\mathcal{D}] \to$ {Hilbert space with, $\langle \cdot \mid \cdot \rangle_{\mathcal{D}\cdot\mathcal{J},\mathcal{J}\cdot\mathcal{D}}\}\}\}$
$\{\{\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J} \to$ odd$\}$[even] $\to$ Lorentz-type$\}$

```
PR["■Definitions: ",
 $ = {"quadratic forms on ℋ sesquillinear map"⟶ q[Dom[q] × Dom[q]] → ℂ,
   ForAll[{ψ1, ψ2} ∈ ℋ, q[ψ1, ψ2] → Conjugate[q[ψ2, ψ1]]] ⇒ "ℋ symmetric"
  };
 $ // ColumnBar,
 NL, "♁Sesquillinear map φ: ", $ = {φ[x + y, w + z] -> φ[x, w] + φ[x, z] + φ[y, w] + φ[y, z],
   φ[a x, b y] → Conjugate[a] b φ[x, y]};
 $ // ColumnBar,
 NL, "●Proposition 2.3: ",
 NL, "For an even Krein spectral triple: ", {𝒜, ℋ, 𝒟, 𝒥},
 Yield, $ = {ℱ[ψ1_, ψ2_] → BraKet[ψ1, 𝒟[ψ2]], ℱ[ψ1_, ψ2_] → BraKet[𝒥 · ψ1, 𝒟[ψ2]]𝒥};
 $ // ColumnBar,
 "defines a symmetric quadratic form ℱ where Dom[ℱ]→Dom[𝒟].
   If Krein spectral triple is Lorentz-time ⇒ ℱ is ℤ2-graded.",
 NL, "¶Proof: ", 𝒟 → "Krein self-adjoint operator",
 imply, $ = {Conjugate[BraKet[ψ1_, 𝒟 · ψ2_]] → BraKet[𝒟 · ψ2, ψ1],
   Conjugate[BraKet[ψ1_, 𝒟 · ψ2_]] → BraKet[ψ2, 𝒟 · ψ1]};
 $ // ColumnBar,
 NL, "Lorentz-type", imply, {Γ → "Krein-anti-self-adjoint"},
 Imply, xtmp =
  $ = ForAll[ψ0 ∈ ℋ^"0" && ψ1 ∈ ℋ^"1", {BraKet[ψ0_, 𝒟 · ψ1_] → BraKet[Γ · ψ0, 𝒟 · ψ1], BraKet[ψ0_,
        𝒟 · ψ1_] → -BraKet[ψ0, Γ · 𝒟 · ψ1], BraKet[ψ0_, 𝒟 · ψ1_] → BraKet[ψ0, 𝒟 · Γ · ψ1],
      BraKet[ψ0_, 𝒟 · ψ1_] → -BraKet[ψ0, 𝒟 · ψ1]}]; $ // ColumnFormOn[List]
]
```

■Definitions:
$$\left|\begin{array}{l} \text{quadratic forms on } \mathcal{H} \text{ sesquillinear map} \longrightarrow q[\mathrm{Dom}[q]\times \mathrm{Dom}[q]] \to \mathbb{C} \\ \forall_{\{\psi_1,\psi_2\}\in\mathcal{H}} \left(q[\psi_1, \psi_2] \to q[\psi_2, \psi_1]^*\right) \Rightarrow \mathcal{H} \text{ symmetric} \end{array}\right.$$

♁Sesquillinear map φ:
$$\left|\begin{array}{l} \varphi[x+y,\, w+z] \to \varphi[x, w] + \varphi[x, z] + \varphi[y, w] + \varphi[y, z] \\ \varphi[a\,x,\, b\,y] \to b\, a^*\, \varphi[x, y] \end{array}\right.$$

●Proposition 2.3:

For an even Krein spectral triple: $\{\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$

$$\Rightarrow \left|\begin{array}{l} \mathcal{F}[\psi1\_, \psi2\_] \to \langle \psi1 \mid \mathcal{D}[\psi2] \rangle \\ \mathcal{F}[\psi1\_, \psi2\_] \to \langle \mathcal{J}\cdot\psi1 \mid \mathcal{D}[\psi2] \rangle_{\mathcal{J}} \end{array}\right.$$

defines a symmetric quadratic form $\mathcal{F}$ where Dom[$\mathcal{F}$]→Dom[$\mathcal{D}$].
If Krein spectral triple is Lorentz-time ⇒ $\mathcal{F}$ is $\mathbb{Z}_2$-graded.

¶Proof: $\mathcal{D}$ → Krein self-adjoint operator ⇒
$$\left|\begin{array}{l} \langle \psi1\_ \mid \mathcal{D}\cdot\psi2\_ \rangle^* \to \langle \mathcal{D}\cdot\psi2 \mid \psi1 \rangle \\ \langle \psi1\_ \mid \mathcal{D}\cdot\psi2\_ \rangle^* \to \langle \psi2 \mid \mathcal{D}\cdot\psi1 \rangle \end{array}\right.$$

Lorentz-type ⇒ {Γ → Krein-anti-self-adjoint}

$$\Rightarrow \forall_{\psi_0\in\mathcal{H}^0 \,\&\&\, \psi_1\in\mathcal{H}^1} \left|\begin{array}{l} \langle \psi0\_ \mid \mathcal{D}\cdot\psi1\_ \rangle \to \langle \Gamma\cdot\psi0 \mid \mathcal{D}\cdot\psi1 \rangle \\ \langle \psi0\_ \mid \mathcal{D}\cdot\psi1\_ \rangle \to -\langle \psi0 \mid \Gamma\cdot\mathcal{D}\cdot\psi1 \rangle \\ \langle \psi0\_ \mid \mathcal{D}\cdot\psi1\_ \rangle \to \langle \psi0 \mid \mathcal{D}\cdot\Gamma\cdot\psi1 \rangle \\ \langle \psi0\_ \mid \mathcal{D}\cdot\psi1\_ \rangle \to -\langle \psi0 \mid \mathcal{D}\cdot\psi1 \rangle \end{array}\right.$$

```
PR["■Definition 2.4: ",
 NL, "For a Lorentz-type spectral triple: ", $st = {𝒜, ℋ, 𝒟, 𝒥},
 " define Krein action ",
 $ = {S𝒦[ℋ^"0"] → ℂ, S𝒦[ψ] → ℱ[ψ, ψ], ℱ[ψ, ψ] → BraKet[ψ, 𝒟[ψ]]};
 $ // ColumnBar,
 NL, CR["Where is Lorentzian signature ℱ?"]
]
```

■Definition 2.4:

For a Lorentz-type spectral triple:

$\{\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$ define Krein action
$$\left|\begin{array}{l} S_{\mathcal{K}}[\mathcal{H}^0] \to \mathbb{C} \\ S_{\mathcal{K}}[\psi] \to \mathcal{F}[\psi, \psi] \\ \mathcal{F}[\psi, \psi] \to \langle \psi \mid \mathcal{D}[\psi] \rangle \end{array}\right.$$

Where is Lorentzian signature $\mathcal{F}$?

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3 Gauge Theory

```
PR["Let 𝒜 → trivially graded unital *-algebra. Define opposite algebra of 𝒜: ",
 𝒜^op → {a^op, a ∈ 𝒜, a^op · b^op → (b · a)^op},
 NL, "Let ", {ℋ, 𝒥 → "fundamental symmetry"}, " be a ℤ2-graded Krein space",
 NL, "Let two commuting even representations ", {π[𝒜] → ℬ^"0"[ℋ], π^op[𝒜^op] → ℬ^"0"[ℋ]},
 Imply, $ = {representation[𝒜⊙𝒜^op][ℋ] → π̄[a ⊗ b^op], π̄[a ⊗ b^op] → π[a] · π^op[b^op]};
 $ // ColumnBar,
 NL, "Then the Krein spectral triple ", {𝒜⊙𝒜^op, ℋ, 𝒟, 𝒥},
 " satisfies ", CG["order-one condition"], " if ",
 $e1 = ForAll[{a, b} ∈ 𝒜, CommutatorM[π[a], CommutatorM[𝒟, π^op[b^op]]] → 0]
]
```

Let $\mathcal{A}$ → trivially graded unital *-algebra. Define opposite algebra of $\mathcal{A}$:
$\mathcal{A}^{op} \to \{a^{op},\ a \in \mathcal{A},\ a^{op} \cdot b^{op} \to (b \cdot a)^{op}\}$
Let $\{\mathcal{H}, \mathcal{J} \to \text{fundamental symmetry}\}$ be a $\mathbb{Z}_2$-graded Krein space
Let two commuting even representations $\{\pi[\mathcal{A}] \to \mathcal{B}^0[\mathcal{H}],\ \pi^{op}[\mathcal{A}^{op}] \to \mathcal{B}^0[\mathcal{H}]\}$
⇒ | representation$[\mathcal{A}\odot\mathcal{A}^{op}][\mathcal{H}] \to \bar{\pi}[a\otimes b^{op}]$
| $\bar{\pi}[a\otimes b^{op}] \to \pi[a] \cdot \pi^{op}[b^{op}]$
Then the Krein spectral triple $\{\mathcal{A}\odot\mathcal{A}^{op}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$
satisfies order-one condition if $\forall_{\{a,b\}\in\mathcal{A}} ([\pi[a], [\mathcal{D}, \pi^{op}[b^{op}]]] \to 0)$

3.1 Inner perturbations

```
PR["●Fluctuations of 𝒟",
  NL, "Consider ", $ = {𝒜⊙𝒜^op, 𝒜 → "trivially graded unital *-algebra",
    A ∈ 𝒜⊙𝒜^op,
    A → Sum[T[a, "d"][j] ⊗ T[b, "d"][j]^op, j],
    "anti-linear involution A→Ā ",
    Sum[T[a, "d"][j] ⊗ T̄[b, "d"][j]^op, j] -> Sum[T[b, "d"][j]^* ⊗ T[a, "d"][j]^*^op, j],
    {(λ ·̄ A) → λ̄ · Ā,
     Ā̄ → A,
     (A ·̄ A') → Ā · Ā',
     λ ∈ ℂ, {A, A'} ∈ 𝒜⊙𝒜^op},
    "A real" ⟸ (A → Ā),
    "A normalized" ⟸ Sum[T[a, "d"][j] T[b, "d"][j], j] → (1 ∈ 𝒜)
   }; $ // ColumnBar
 ];
```

●Fluctuations of $\mathcal{D}$

Consider
| $\mathcal{A}\odot\mathcal{A}^{op}$
| $\mathcal{A} \to$ trivially graded unital *-algebra
| $A \in \mathcal{A}\odot\mathcal{A}^{op}$
| $A \to \sum_j a_j \otimes (b_j)^{op}$
| anti-linear involution A→Ā
| $\overline{\sum_j a_j \otimes (b_j)^{op}} \to \sum_j (b_j)^* \otimes ((a_j)^*)^{op}$
| $\{\overline{\lambda \cdot A} \to \bar{\lambda}\cdot\bar{A},\ \bar{\bar{A}} \to A,\ \overline{A \cdot A'} \to \bar{A}\cdot\bar{A'},\ \lambda \in \mathbb{C},\ \{A, A'\} \in \mathcal{A}\odot\mathcal{A}^{op}\}$
| A real $\Longleftarrow (A \to \bar{A})$
| A normalized $\Longleftarrow \sum_j a_j\, b_j \to 1 \in \mathcal{A}$

```
PR["■Definition 3.1: ",
 NL, "Perturbation semi-group Pert[𝒜] ", yield,
 {A ∈ 𝒜⊙𝒜^op, A["real, normalized"], "algebra of 𝒜⊙𝒜^op"},
 NL, "The Krein spectral triple ", {ℬ, ℋ, 𝒟, 𝒥},
 " the generalized one-forms ",
 Ω𝒟^"1"[ℬ] → {xSum[T[a, "d"][j] · CommutatorM[𝒟, T[b, "d"][j]], j],
   {T[a, "d"][j], T[b, "d"][j]} ∈ ℬ},
 NL, "Define ", $ = {η𝒟[ℬ -> 𝒜⊙𝒜^op] → {Ω𝒟^"1"[𝒜⊙𝒜^op] ⊂ ℬ[ℋ]},
    η𝒟[Sum[T[a, "d"][j] ⊗ T[b, "d"][j]^op, j]] →
     Sum[π̃[T[a, "d"][j] ⊗ T[a, "d"][k]^*^op] ·
       CommutatorM[𝒟, π̃[T[b, "d"][j] ⊗ T[b, "d"][k]^*^op]], j, k],
    η𝒟[Ā] -> η𝒟[A]^+,
    η𝒟[Pert[𝒜]] -> Ω𝒟^"1"[𝒜⊙𝒜^op][CG["self-adjoint"]],
    CO["For order-one condition"],
    η𝒟[Sum[T[a, "d"][j] ⊗ T[b, "d"][j]^op, j]] -> Sum[T[a, "d"][j] ·
        CommutatorM[𝒟, T[b, "d"][j]], j] +
      Sum[T[a, "d"][k]^*^op ·
        CommutatorM[𝒟, T[b, "d"][k]^*^op], k]
  }; $ // ColumnBar
]
```

■Definition 3.1:
Perturbation semi-group Pert[$\mathcal{A}$] $\longrightarrow$ {$A \in \mathcal{A}\odot\mathcal{A}^{op}$, A[real, normalized], algebra of $\mathcal{A}\odot\mathcal{A}^{op}$}
The Krein spectral triple {$\mathcal{B}$, $\mathcal{H}$, $\mathcal{D}$, $\mathcal{J}$}
the generalized one-forms $\Omega^1_{\mathcal{D}}[\mathcal{B}] \to \{\sum_j [a_j \cdot [\mathcal{D}, b_j]], \{a_j, b_j\} \in \mathcal{B}\}$

Define
$\eta_{\mathcal{D}}[\mathcal{B} \to \mathcal{A}\odot\mathcal{A}^{op}] \to \{\Omega^1_{\mathcal{D}}[\mathcal{A}\odot\mathcal{A}^{op}] \subset \mathcal{B}[\mathcal{H}]\}$
$\eta_{\mathcal{D}}[\sum_j a_j \otimes (b_j)^{op}] \to \sum_j \sum_k \tilde{\pi}[a_j \otimes ((a_k)^*)^{op}] \cdot [\mathcal{D}, \tilde{\pi}[b_j \otimes ((b_k)^*)^{op}]]$
$\eta_{\mathcal{D}}[\bar{A}] \to \eta_{\mathcal{D}}[A]^+$
$\eta_{\mathcal{D}}[\mathrm{Pert}[\mathcal{A}]] \to \Omega^1_{\mathcal{D}}[\mathcal{A}\odot\mathcal{A}^{op}]$[self-adjoint]
For order-one condition
$\eta_{\mathcal{D}}[\sum_j a_j \otimes (b_j)^{op}] \to \sum_k ((a_k)^*)^{op} \cdot [\mathcal{D}, ((b_k)^*)^{op}] + \sum_j a_j \cdot [\mathcal{D}, b_j]$

```
PR["■Definition 3.2:",
 NL, "Fluctuation of 𝒟 by A∈Pert[𝒜] or Fluctuated Dirac Operator: ",
 $da = 𝒟A_ → 𝒟 + η𝒟[A],
 NL, "•Proposition: ", $ = {(𝒟A)A' → (𝒟 + η𝒟[A])A', (𝒟A)A' -> 𝒟A' + η𝒟[A]A',
    (𝒟A)A' → 𝒟 + η𝒟[A'] + η𝒟[A]A', CR[(𝒟A)A' → 𝒟 + η𝒟[A' · A]],
    (𝒟A)A' → 𝒟A'·A
  }; $ // ColumnBar
]
```

■Definition 3.2:
Fluctuation of $\mathcal{D}$ by A∈Pert[$\mathcal{A}$] or Fluctuated Dirac Operator: $\mathcal{D}_{A\_} \to \mathcal{D} + \eta_{\mathcal{D}}[A]$

•Proposition:
$\mathcal{D}_{AA'} \to \mathcal{D} + \eta_{\mathcal{D}}[A]_{A'}$
$\mathcal{D}_{AA'} \to \mathcal{D}_{A'} + \eta_{\mathcal{D}}[A]_{A'}$
$\mathcal{D}_{AA'} \to \mathcal{D} + \eta_{\mathcal{D}}[A]_{A'} + \eta_{\mathcal{D}}[A']$
$\mathcal{D}_{AA'} \to \mathcal{D} + \eta_{\mathcal{D}}[A' \cdot A]$
$\mathcal{D}_{AA'} \to \mathcal{D}_{A' \cdot A}$

3.2 Gauge Action

```
PR["A Semi-group homomorphism ",
  $dsh = $ = {△[𝒰[𝒜]["unitary"]] → Pert[𝒜]["perturbation semi-group"],
     △[u] → u ⊗ u*op,
     △[u] · (A -> Sum[(T[a, "d"][j]) ⊗ (T[b, "d"][j]op), j]) →
      Sum[(u · T[a, "d"][j]) ⊗ (u*op · (T[b, "d"][j]op)), j],
     △[u] · (A -> Sum[(T[a, "d"][j]) ⊗ (T[b, "d"][j]op), j]) →
      Sum[(u · T[a, "d"][j]) ⊗ (((T[b, "d"][j] · u*)op)), j],
     "group representation ρ",
     ρ → π̃ ∘ △[𝒰[𝒜]] → ℬ[ℋ]
    }; $ // ColumnBar
 ];
```

A Semi-group homomorphism

$\triangle[\mathcal{U}[\mathcal{A}][\text{unitary}]] \to \text{Pert}[\mathcal{A}][\text{perturbation semi-group}]$

$\triangle[u] \to u \otimes (u^*)^{\text{op}}$

$\triangle[u] \cdot (A \to \sum_j a_j \otimes (b_j)^{\text{op}}) \to \sum_j (u \cdot a_j) \otimes ((u^*)^{\text{op}} \cdot (b_j)^{\text{op}})$

$\triangle[u] \cdot (A \to \sum_j a_j \otimes (b_j)^{\text{op}}) \to \sum_j (u \cdot a_j) \otimes (b_j \cdot u^*)^{\text{op}}$

group representation ρ

$\rho \to \tilde{\pi} \circ \triangle[\mathcal{U}[\mathcal{A}]] \to \mathcal{B}[\mathcal{H}]$

```
PR["■Definition 3.4:",
 NL, "Gauge group: ",
 $ga = $ = {𝒢[𝒜] → {ρ[u], u ∈ 𝒰[𝒜]} ≃ Mod[𝒰[𝒜], Ker[ρ]],
    CO["Dfn:action of γ of 𝒰[𝒜] on " Ω𝒟"1"[𝒜⊙𝒜op]],
    γu[T_] → ρ[u] · T · ρ[u*] + η𝒟 ∘ △[u],
    {γu[T_] → ρ[u] · T · ρ[u*] + ρ[u] · CommutatorM[𝒟, ρ[u*]],
     T ∈ Ω𝒟"1"[𝒜⊙𝒜op],
     u ∈ 𝒰[𝒜]},
    γu ∘ η𝒟[𝒜] -> η𝒟[△[u] · 𝒜],
    {ρ[u][𝒟𝒜] -> 𝒟△[u]·𝒜, ρ[u] ∈ 𝒢[𝒜],
     {u ∈ Ker[ρ] ⇒ {η𝒟[△[u] · 𝒜] → η𝒟[𝒜], 𝒟△[u]·𝒜 -> 𝒟𝒜, ρ[u][𝒟𝒜] → "independant of u"}}}
   }; $ // ColumnBar,
 NL, "■Proposition 3.5: ",
 NL, "The Krein action ",
 $sk = {𝒮𝒦[ψ, A] → BraKet[ψ, 𝒟A · ψ], 𝒟A → "fluctuated Dirac operator"},
 " is invariant under gauge group ", {ρ[u] · ψ, △[u] · A},
 NL, "¶Proof: Since η𝒟 is covariant under gauge group: ",
 NL, "Using Rule[]s: ",
 $s = {Reverse[tuRuleSelect[$ga][γu ∘ η𝒟[_]][[1]]],
   (tuRuleSelect[$ga][γu[T_]][[2]] /. γu[a_] -> γu ∘ a),
   ρ[a_] · ρ[a_*] → 1,
   rr : ρ[a_*] · 𝒟 · ρ[a_] :> Reverse[rr],
   Reverse[$da /. A → 𝒜] // tuPatternRemove
  }; $s // ColumnBar,
 Yield, $1 = {$ = ($ = 𝒟△[u]·𝒜) → ($ /. $da),
   $ = $ /. $s,
   $ = $ /. $s,
   $ = $ /. CommutatorM → MCommutator /. Dot → CenterDot //.
      tuOpDistribute[CenterDot] /. tuOpSimplify[CenterDot],
   $ = $ /. $s //. tuOpSimplify[CenterDot],
   $ = $ /. $s /. a_ · b1_ · c_ + a_ · b2_ · c_ -> a · (b1 + b2) · c /. tuOpSimplify[CenterDot],
   $2 = $ = $ /. $s
  };
 $1 // ColumnBar,

 NL, "Using ", $s = {BraKet[a_, b_] → BraKet[ρ[u] · a, ρ[u] · b],
   # · ρ[u] & /@ $2 /. ρ[a_*] · ρ[a_] → 1 /. tuOpSimplify[CenterDot] /. 𝒜 → A // Reverse,
   Reverse[$sk[[1]]] // tuAddPatternVariable[{ψ, A}]
  }; $s // ColumnBar,
 NL, "We compute: ",
 $1 = {$ = $sk[[1]],
   $ = $ /. $s,
   $ = $ /. $s[[2]],
   $ = $ /. $s[[3]]
  }; $1 // ColumnBar, CG[" QED"]
]
```

■Definition 3.4:

Gauge group:
$\mathcal{G}[\mathcal{A}] \to \{\rho[u],\ u \in \mathcal{U}[\mathcal{A}]\} \simeq \mathrm{Mod}[\mathcal{U}[\mathcal{A}],\ \mathrm{Ker}[\rho]]$
Dfn:action of γ of $\mathcal{U}[\mathcal{A}]$ on $\Omega^1_{\mathcal{D}}[\mathcal{A}\odot\mathcal{A}^{\mathrm{op}}]$
$\gamma_u[\mathrm{T\_}] \to \rho[u]\cdot \mathrm{T}\cdot \rho[u^*] + \eta_{\mathcal{D}}\circ \Delta[u]$
$\{\gamma_u[\mathrm{T\_}] \to \rho[u]\cdot[\mathcal{D},\ \rho[u^*]] + \rho[u]\cdot \mathrm{T}\cdot \rho[u^*],\ \mathrm{T}\in \Omega^1_{\mathcal{D}}[\mathcal{A}\odot\mathcal{A}^{\mathrm{op}}],\ u \in \mathcal{U}[\mathcal{A}]\}$
$\gamma_u \circ \eta_{\mathcal{D}}[\mathcal{A}] \to \eta_{\mathcal{D}}[\Delta[u]\cdot \mathcal{A}]$
$\{\rho[u][\mathcal{D}_{\mathcal{A}}] \to \mathcal{D}_{\Delta[u]\cdot\mathcal{A}},\ \rho[u] \in \mathcal{G}[\mathcal{A}],$
$\{u \in \mathrm{Ker}[\rho] \Rightarrow \{\eta_{\mathcal{D}}[\Delta[u]\cdot\mathcal{A}] \to \eta_{\mathcal{D}}[\mathcal{A}],\ \mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \mathcal{D}_{\mathcal{A}},\ \rho[u][\mathcal{D}_{\mathcal{A}}] \to \text{independant of } u\}\}$

■Proposition 3.5:
The Krein action $\{\mathcal{S}_{\mathcal{K}}[\psi,\ \mathrm{A}] \to \langle \psi \mid \mathcal{D}_{\mathrm{A}}\cdot\psi\rangle,\ \mathcal{D}_{\mathrm{A}} \to \text{fluctuated Dirac operator}\}$
is invariant under gauge group $\{\rho[u]\cdot\psi,\ \Delta[u]\cdot \mathrm{A}\}$

¶Proof: Since $\eta_{\mathcal{D}}$ is covariant under gauge group:

Using Rule[]s:
$\eta_{\mathcal{D}}[\Delta[u]\cdot\mathcal{A}] \to \gamma_u \circ \eta_{\mathcal{D}}[\mathcal{A}]$
$\gamma_u \circ \mathrm{T\_} \to \rho[u]\cdot[\mathcal{D},\ \rho[u^*]] + \rho[u]\cdot\mathrm{T}\cdot\rho[u^*]$
$\rho[\mathrm{a\_}]\cdot\rho[\mathrm{a\_}^*] \to 1$
$\mathrm{rr} : \rho[\mathrm{a\_}^*]\cdot\mathcal{D}\cdot\rho[\mathrm{a\_}] :\to \mathrm{Reverse}[\mathrm{rr}]$
$\mathcal{D} + \eta_{\mathcal{D}}[\mathcal{A}] \to \mathcal{D}_{\mathcal{A}}$

→
$\mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \mathcal{D} + \eta_{\mathcal{D}}[\Delta[u]\cdot\mathcal{A}]$
$\mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \mathcal{D} + \gamma_u\circ\eta_{\mathcal{D}}[\mathcal{A}]$
$\mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \mathcal{D} + \rho[u]\cdot[\mathcal{D},\ \rho[u^*]] + \rho[u]\cdot\eta_{\mathcal{D}}[\mathcal{A}]\cdot\rho[u^*]$
$\mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \mathcal{D} + \rho[u]\cdot\mathcal{D}\cdot\rho[u^*] - \rho[u]\cdot\rho[u^*]\cdot\mathcal{D} + \rho[u]\cdot\eta_{\mathcal{D}}[\mathcal{A}]\cdot\rho[u^*]$
$\mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \rho[u]\cdot\mathcal{D}\cdot\rho[u^*] + \rho[u]\cdot\eta_{\mathcal{D}}[\mathcal{A}]\cdot\rho[u^*]$
$\mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \rho[u]\cdot(\mathcal{D} + \eta_{\mathcal{D}}[\mathcal{A}])\cdot\rho[u^*]$
$\mathcal{D}_{\Delta[u]\cdot\mathcal{A}} \to \rho[u]\cdot\mathcal{D}_{\mathcal{A}}\cdot\rho[u^*]$

Using
$\langle \mathrm{a\_} \mid \mathrm{b\_}\rangle \to \langle \rho[u]\cdot\mathrm{a} \mid \rho[u]\cdot\mathrm{b}\rangle$
$\rho[u]\cdot\mathcal{D}_{\mathrm{A}} \to \mathcal{D}_{\Delta[u]\cdot\mathrm{A}}\cdot\rho[u]$
$\langle \psi\_ \mid \mathcal{D}_{\mathrm{A}}\_\cdot\psi\_\rangle \to \mathcal{S}_{\mathcal{K}}[\psi,\ \mathrm{A}]$

We compute:
$\mathcal{S}_{\mathcal{K}}[\psi,\ \mathrm{A}] \to \langle \psi \mid \mathcal{D}_{\mathrm{A}}\cdot\psi\rangle$
$\mathcal{S}_{\mathcal{K}}[\psi,\ \mathrm{A}] \to \langle \rho[u]\cdot\psi \mid \rho[u]\cdot\mathcal{D}_{\mathrm{A}}\cdot\psi\rangle$
$\mathcal{S}_{\mathcal{K}}[\psi,\ \mathrm{A}] \to \langle \rho[u]\cdot\psi \mid \mathcal{D}_{\Delta[u]\cdot\mathrm{A}}\cdot\rho[u]\cdot\psi\rangle$
$\mathcal{S}_{\mathcal{K}}[\psi,\ \mathrm{A}] \to \mathcal{S}_{\mathcal{K}}[\rho[u]\cdot\psi,\ \Delta[u]\cdot\mathrm{A}]$
QED

4 Almost-commutative manifolds

```
PR["●Finite space: ", $ = {F → (#F & /@ $st),
    ΓF["ℤ2-grading"], ℋF["finite dimensional"],
    "→even Krein spectral triple"}; $ // ColumnBar,
  NL, "Almost-commutative manifold" →
   M["pseudo-Riemannian spin manifold"] ⊗ F["Finite space"],
  NL, "Finite space Krein spectral triple",
  NL, "Let ",
  $ = {M, g}["n-dimensional, time,space-oriented pseudo-Riemannian spin manifold with
      signature ", {t["#time-dimensions", g < 0], s["#space-dimensions", g > 0]}];
  $ // ColumnForms,
  NL, "Orthogonal decomposition of tangent bundle ",
  $tm = $ = {TM → "E"t ⊕ "E"s, "E"t → "purely time-like", "E"s → "purely space-like",
     CO["Dfn time-like projection"],
     T["E"t ⊕ "E"s] -> "E"t,
     r → 1 - 2 T,
     r → "space-like reflection",
     r["E"t ⊕ "E"s] → (-1 ⊕ 1)["E"t ⊕ "E"s],
     CO["Wick rotation"],
     gr[v_, w_] → g[r[v], w],
     {M, gr}["Riemannian"]
    }; $ // ColumnBar
 ];
PR[
 NL, "•real Clifford algebra wrt g ",
 $ = {Cl[TM, g],
    γ -> "Clifford representation",
    γ[TM] → Cl[TM, g],
    γ[v_] · γ[w_] + γ[w_] · γ[v_] → -2 g[v, w],
    CO["Let"],
    h[TM*] → TM,
    h[α ∈ TM*] → dual[TM],
    ForAll[w ∈ TM, (h[α] → v) ⟺ α[w] → g[v, w]]
   }; $ // ColumnBar
]
```

●Finite space:
| $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F\}$
| Γ_F[$\mathbb{Z}_2$-grading]
| $\mathcal{H}_F$[finite dimensional]
| →even Krein spectral triple

Almost-commutative manifold → M[pseudo-Riemannian spin manifold] ⊗ F[Finite space]

Finite space Krein spectral triple

Let {M, g}[n-dimensional, time,space-oriented pseudo-Riemannian spin manifold with signature , {t[#time-dimensions, g < 0], s[#space-dimensions, g > 0]}]

Orthogonal decomposition of tangent bundle
| $TM \to E_t \oplus E_s$
| $E_t \to$ purely time-like
| $E_s \to$ purely space-like
| Dfn time-like projection
| $T[E_t \oplus E_s] \to E_t$
| $r \to 1 - 2\,T$
| $r \to$ space-like reflection
| $r[E_t \oplus E_s] \to (-1 \oplus 1)[E_t \oplus E_s]$
| Wick rotation
| $g_r[v\_, w\_] \to g[r[v], w]$
| $\{M, g_r\}$[Riemannian]

```
                                      Cl[TM, g]
                                      γ → Clifford representation
                                      γ[TM] → Cl[TM, g]
•real Clifford algebra wrt g          γ[v_] · γ[w_] + γ[w_] · γ[v_] → -2 g[v, w]
                                      Let
                                      h[TM*] → TM
                                      h[α ∈ TM*] → dual[TM]
                                      ∀w∈TM ((h[α] → v) ⟺ α[w] → g[v, w])

PR[$1 = {S[CG["spinor bundle"]] -> M[CG["w/spin structure"]],
    T[Γ, "du"][c, ∞][S] → "compact smooth sections",

    CO[c → "pseudo-Riemannian Clifford multiply"],
    c[α ⊗ ψ] → γ[h[α]] · ψ,
    c · T[Γ, "du"][c, ∞][TM* ⊗ S] -> T[Γ, "du"][c, ∞][S],
    CO["Dirac operator" ⟶ ($sD = slash[D] → c · "∇"^S)],
    ($ = slash[D] · T[Γ, "du"][c, ∞][S]) →
     ($ /. $sD) → c · T[Γ, "du"][c, ∞][TM* ⊗ S] -> T[Γ, "du"][c, ∞][S]
   };
  $1 // ColumnBar,
  NL, "●Locally, choose pseudo-orthonormal frame ",
  {T[e, "d"][j], {j, 1, n}} ∋ {$tm[[1]],
    {T[e, "d"][j] ∈ $tm[[1, 2, 1]], j ≤ t}, {T[e, "d"][j] ∈ $tm[[1, 2, 2]], j > t}},
  NL, "Metric: ",
  {g[T[e, "d"][i], T[e, "d"][j]] → T[δ, "dd"][i, j] κ[j], κ[j] :> If[j ≤ t, -1, 1]},
  NL,
  $ = {{T[θ, "u"][i], {i, 1, n}} → {"basis of TM* dual to" -> {T[e, "d"][j], {j, 1, n}}},
    T[θ, "u"][i][T[e, "d"][j]] -> T[δ, "ud"][i, j],
    h[T[θ, "u"][j]] → κ[j] T[e, "d"][j],
    CO["Dirac operator"],
    slash[D] → c ∘ "∇"^S,
    slash[D] → Sum[κ[j] · γ[T[e, "d"][j]] · T["∇", "du"][T[e, "d"][j], S], {j, n}]
   };
  $ // ColumnBar,
  NL, "•Given ", $tm[[1]],
  Yield, $ = {∃ {BraKet[_, _]_𝒥M} ∋ {BraKet[_, _]_𝒥M → "positive-definite,Hermitian",
      BraKet[_, _]_𝒥M[T[Γ, "du"][c, ∞][S] × T[Γ, "du"][c, ∞][S]] -> T[C, "du"][c, ∞][M]},
    BraKet[ψ1, ψ2]_𝒥M → tuIntegral[{vol[M, g]}, BraKet[ψ1, ψ2]_𝒥M],
    L^2[S] → "completion of Γc^∞[S] wrt inner product",
    CO["Dfn"],
    𝒥M[L^2[S]] → I^(t (t-1)/2) Product[γ[T[e, "d"][j]], {j, t}],
    𝒥M → "self-adjoint,unitary",
    𝒥M · γ[v] · 𝒥M → (-1)^t γ[r · v],
    L^2[S] ⟶ {"Krein space with indefinite product",
      BraKet[_, _] → BraKet[𝒥M · _, _]_𝒥M, 𝒥M → "fundamental symmetry",
      "Independent of decomposition" → $tm[[1]]}};
  $ // ColumnBar
 ];
```

```
S[spinor bundle] → M[w/spin structure]
Γc∞[S] → compact smooth sections
c → pseudo-Riemannian Clifford multiply
c[α⊗ψ] → γ[h[α]] · ψ
c · Γc∞[TM*⊗S] → Γc∞[S]
Dirac operator ⟶ (D̸ → c · ∇S)
(D̸) · Γc∞[S] → c · ∇S · Γc∞[S] → c · Γc∞[TM*⊗S] → Γc∞[S]
```

●Locally, choose pseudo-orthonormal frame

$\{e_j, \{j, 1, n\}\} \ni \{TM \to E_t \oplus E_s, \{e_j \in E_t, j \le t\}, \{e_j \in E_s, j > t\}\}$

Metric: $\{g[e_i, e_j] \to \delta_{ij}\, \kappa[j], \kappa[j] \mapsto \mathrm{If}[j \le t, -1, 1]\}$

$\{\theta^i, \{i, 1, n\}\} \to \{\text{basis of } TM^* \text{ dual to} \to \{e_j, \{j, 1, n\}\}\}$

$\theta^i[e_j] \to \delta^i{}_j$

$h[\theta^j] \to e_j\, \kappa[j]$

Dirac operator

$D\!\!\!/ \to c \circ \nabla^S$

$D\!\!\!/ \to \sum_j^n \kappa[j] \cdot \gamma[e_j] \cdot \nabla_{e_j}{}^S$

•Given $TM \to E_t \oplus E_s$

$\mathrm{Exists}[\{\langle \_ \mid \_ \rangle_{\mathcal{J}_M}\}] \ni \{\langle \_ \mid \_ \rangle_{\mathcal{J}_M} \to \text{positive-definite,Hermitian}, \langle \_ \mid \_ \rangle_{\mathcal{J}_M}[\Gamma_c{}^\infty[S] \times \Gamma_c{}^\infty[S]] \to C_c{}^\infty[M]\}$

$\langle \psi_1 \mid \psi_2 \rangle_{\mathcal{J}_M} \to \int_{\mathrm{vol}[M,g]} [\langle \psi_1 \mid \psi_2 \rangle_{\mathcal{J}_M}]$

$L^2[S] \to$ completion of $\Gamma_c{}^\infty[S]$ wrt inner product

Dfn

→ $\mathcal{J}_M[L^2[S]] \to (-1)^{\frac{1}{4}(-1+t)\,t} \prod_j^t \gamma[e_j]$

$\mathcal{J}_M \to$ self-adjoint,unitary

$\mathcal{J}_M \cdot \gamma[v] \cdot \mathcal{J}_M \to (-1)^t\, \gamma[r \cdot v]$

$L^2[S] \longrightarrow \{\text{Krein space with indefinite product}, \langle \_ \mid \_ \rangle \to \langle \mathcal{J}_M \cdot \_ \mid \_ \rangle_{\mathcal{J}_M},$

$\mathcal{J}_M \to \text{fundamental symmetry, Independent of decomposition} \to TM \to E_t \oplus E_s\}$

```
PR["■Proposition 4.1:", "  Let ", {M, g},
 "n-dimensional time-/space-oriented pseudo-Riemannian spin manifold of signature",
 {t, s}, ". Let r be a spacelike reflection such that the associated Riemannian
   metric gr is complete. We obtain an even Krein spectral triple ",
 {T[C, "du"][c, ∞][M], L²[S], I^t slash[D], 𝒥M},
 " with grading operator ", ΓM, ".", " t[odd]⇒ a Lorentz-type spectral triple.",
 NL, "¶ •Take ", {ℰ → Dom[slash[D]] ⋂ (𝒥M · Dom[slash[D]]),
  T[Γ, "du"][c, ∞][S] ⊂ ℰ, ℰ → Style["a core for slash[D]", Red]},
 NL, "•To show local compactness of ", ℰ ↦ L²[S], " define ",
 slash[D]"±" → (slash[D] + slash[D]*) / 2 - I ((slash[D] - slash[D]*)) / 2,
 NL, CR[slash[D]"±", " are elliptic hence have locally compact resolvents.",
  imply, Dom[slash[D]]+ ⋂ Dom[slash[D]]- → ℰ, imply,
  ℰ ↦ slash[D]"±" ↦ L²[S], " is locally compact"],
 NL, "• ", {M → "even dimensional", ΓM · 𝒥M → (-1)^t 𝒥M · ΓM},
 imply, "Lorentz-type spectral triple" ⇔ "t is odd"
]
```

■Proposition 4.1: Let $\{M, g\}$ n-dimensional time-/space-oriented pseudo-Riemannian spin manifold of signature $\{t, s\}$. Let r be a spacelike reflection such that the associated Riemannian metric g_r is complete. We obtain an even Krein spectral triple $\{C_c{}^\infty[M], L^2[S], i^t\,(D\!\!\!/), \mathcal{J}_M\}$ with grading operator Γ_M. t[odd]⇒ a Lorentz-type spectral triple.

¶ •Take $\{\mathcal{E} \to \mathrm{Dom}[D\!\!\!/] \bigcap \mathcal{J}_M \cdot \mathrm{Dom}[D\!\!\!/], \Gamma_c{}^\infty[S] \subset \mathcal{E}, \mathcal{E} \to \text{a core for slash[D]}\}$

•To show local compactness of $\mathcal{E} \mapsto L^2[S]$ define $D\!\!\!/_\pm \to -\frac{1}{2} i\,(D\!\!\!/ - (D\!\!\!/)^*) + \frac{1}{2}(D\!\!\!/ + (D\!\!\!/)^*)$

$D\!\!\!/_\pm$ are elliptic hence have locally compact resolvents.

⇒ $(\mathrm{Dom}[D\!\!\!/])_+ \bigcap (\mathrm{Dom}[D\!\!\!/])_- \to \mathcal{E}$ ⇒ $\mathcal{E} \mapsto D\!\!\!/_\pm \mapsto L^2[S]$ is locally compact

• $\{M \to \text{even dimensional}, \Gamma_M \cdot \mathcal{J}_M \to (-1)^t \mathcal{J}_M \cdot \Gamma_M\}$ ⇒ Lorentz-type spectral triple ⇔ t is odd

```
PR[
  "●Definition 4.2: Given {M,g}[even-dimensional pseudo-Riemannian spin manifold] an
    almost-commutative pseudo-Riemannian manifold F×M is the product
    of a finite space F with the manifold M: ",
  $acm = $ = {$st → { T[C, "du"][c, ∞][M, 𝒜F], ℋF ⊗ (L^2[S]),
        (I^deg[𝒥F]) (1) ⊗ (I^t slash[D]) + I^deg[𝒥M] 𝒟F ⊗ (1), I^deg[𝒥F] deg[𝒥M] 𝒥F ⊗ 𝒥M},
      Γ → ΓF ⊗ ΓM, CR["⊗ is the graded tensor product"]};
  $ // ColumnBar
 ];
```

●Definition 4.2: Given {M,g}[even-dimensional pseudo-Riemannian spin manifold] an almost-commutative pseudo-Riemannian manifold F×M is the product of a finite space F with the manifold M:

$\{\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}\} \to \{C_c{}^{\infty}[M, \mathcal{A}_F], \mathcal{H}_F \otimes L^2[S], i^{\deg[\mathcal{J}_F]}\, 1 \otimes (i^t\, (\not{D})) + i^{\deg[\mathcal{J}_M]}\, \mathcal{D}_F \otimes 1, i^{\deg[\mathcal{J}_F]\deg[\mathcal{J}_M]}\, \mathcal{J}_F \otimes \mathcal{J}_M\}$

$\Gamma \to \Gamma_F \otimes \Gamma_M$

⊗ is the graded tensor product

```
PR["●Proposition 4.3: An almost-commutative
    pseudo-Riemannian manifold is an even Krein spectral triple.",
  NL, "¶: ",
  NL, "• ", I^deg[𝒥F] deg[𝒥M] 𝒥F ⊗ 𝒥M ⇒ 𝒥 → "self-adjoint and unitary",
  NL, "• ", {CommutatorM[𝒥F, 𝒜] → 0, CommutatorM[𝒥M, 𝒜] → 0} ⇒ CommutatorM[𝒥, 𝒜] -> 0,
  NL, "• ", ({𝒥F, 𝒥M} → "homogeneous") ⇒ {𝒥 → "homogeneous", deg[𝒥] -> deg[𝒥F] + deg[𝒥M]},
  NL, "• ", {(I^deg[𝒥F]) (1) ⊗ (I^t slash[D]), I^deg[𝒥M] 𝒟F ⊗ (1)} ⇒ {𝒟 → "Krein symmetric"},
  NL, "• ",
  {I^t slash[D] → "Krein self-adjoint", 𝒟F → "bounded"} ⇒ {𝒟 → "Krein self-adjoint"},
  line,
  NL,
  "■For their examples, they use even-dimensional Lorentzian manifold with 𝒥M odd.
    The Krein action AC-manifold is Lorentzian ⇒ Finite
    space NOT Lorentz-type ⇒ 𝒥F even.",
  Yield, $e2 = F × M → {T[C, "du"][c, ∞][M, 𝒜F], ℋF ⊗ (L^2[S]),
     (1) ⊗ (I^t slash[D]) + I 𝒟F ⊗ (1), 𝒥F ⊗ 𝒥M},
  NL, "•Compare with ACM above ", $acm
 ];
```

●Proposition 4.3: An almost-commutative pseudo-Riemannian manifold is an even Krein spectral triple.

¶:

- $i^{\deg[\mathcal{J}_F]\deg[\mathcal{J}_M]}\, \mathcal{J}_F \otimes \mathcal{J}_M \Rightarrow \mathcal{J} \to$ self-adjoint and unitary
- $\{[\mathcal{J}_F, \mathcal{A}] \to 0, [\mathcal{J}_M, \mathcal{A}] \to 0\} \Rightarrow [\mathcal{J}, \mathcal{A}] \to 0$
- $(\{\mathcal{J}_F, \mathcal{J}_M\} \to \text{homogeneous}) \Rightarrow \{\mathcal{J} \to \text{homogeneous}, \deg[\mathcal{J}] \to \deg[\mathcal{J}_F] + \deg[\mathcal{J}_M]\}$
- $\{i^{\deg[\mathcal{J}_F]}\, 1 \otimes (i^t\, (\not{D})), i^{\deg[\mathcal{J}_M]}\, \mathcal{D}_F \otimes 1\} \Rightarrow \{\mathcal{D} \to \text{Krein symmetric}\}$
- $\{i^t\, (\not{D}) \to \text{Krein self-adjoint}, \mathcal{D}_F \to \text{bounded}\} \Rightarrow \{\mathcal{D} \to \text{Krein self-adjoint}\}$

■For their examples, they use even-dimensional Lorentzian manifold with $\mathcal{J}_M$ odd. The Krein action AC-manifold is Lorentzian ⇒ Finite space NOT Lorentz-type ⇒ $\mathcal{J}_F$ even.

→ $F \times M \to \{C_c{}^{\infty}[M, \mathcal{A}_F], \mathcal{H}_F \otimes L^2[S], 1 \otimes (i^t\, (\not{D})) + i\, \mathcal{D}_F \otimes 1, \mathcal{J}_F \otimes \mathcal{J}_M\}$

•Compare with ACM above

$\{\{\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}\} \to \{C_c{}^{\infty}[M, \mathcal{A}_F], \mathcal{H}_F \otimes L^2[S], i^{\deg[\mathcal{J}_F]}\, 1 \otimes (i^t\, (\not{D})) + i^{\deg[\mathcal{J}_M]}\, \mathcal{D}_F \otimes 1, i^{\deg[\mathcal{J}_F]\deg[\mathcal{J}_M]}\, \mathcal{J}_F \otimes \mathcal{J}_M\},$
$\Gamma \to \Gamma_F \otimes \Gamma_M,$ ⊗ is the graded tensor product}

5 Electrodynamics

```
PR["Consider even finite space and algebra: ",
 $ = $fed = {F_ED → {(𝒜_F → ℂ⊕ℂ) ⊙ 𝒜_F^op, ℋ_F → ℂ⊗ℂ, 𝒟_F → {{0, -I m}, {I m, 0}}, 𝒥_F → 1},
    ℋ_F → {e_R[odd], e_L[even]},
    𝒜_F → {ℂ⊕ℂ, Commutative},
    𝒜_F^op ≃ 𝒜_F,
    {π, π^op} → representation,
    {π, π^op}[ℂ⊗ℂ] → ℬ^"0"[ℋ_F],
    ℬ^"0" → "bounded even operators",
    π[λ, μ] → λ 1_2,
    π^op[λ, μ] → μ 1_2,
    imply,
    π̃[{λ, μ}⊗{λ', μ'}] → (λ μ') · 1_2 ,
    π̃[ℋ_F] → Style[ℋ_F, Red],
    {π, π^op} → {"Satisfy order-1 condition", $e1}
   }; $ // ColumnForms
]
```

Consider even finite space and algebra:

$F_{ED} \to \left| \begin{array}{l} (\mathcal{A}_F \to \mathbb{C}\oplus\mathbb{C}) \odot \mathcal{A}_F^{op} \\ \mathcal{H}_F \to \mathbb{C}\otimes\mathbb{C} \\ \mathcal{D}_F \to \left| \begin{array}{l} \left| \begin{array}{l} 0 \\ -i\, m \end{array} \right. \\ \left| \begin{array}{l} i\, m \\ 0 \end{array} \right. \end{array} \right. \\ \mathcal{J}_F \to 1 \end{array} \right.$

$\mathcal{H}_F \to \left| \begin{array}{l} e_R[odd] \\ e_L[even] \end{array} \right.$

$\mathcal{A}_F \to \left| \begin{array}{l} \mathbb{C}\oplus\mathbb{C} \\ \text{Commutative} \end{array} \right.$

$\mathcal{A}_F^{op} \simeq \mathcal{A}_F$

$\left| \begin{array}{l} \pi \\ \pi^{op} \end{array} \right. \to \text{representation}$

$\left| \begin{array}{l} \pi \\ \pi^{op} \end{array} \right. [\mathbb{C}\otimes\mathbb{C}] \to \mathcal{B}^0[\mathcal{H}_F]$

$\mathcal{B}^0 \to$ bounded even operators

$\pi[\lambda, \mu] \to \lambda\, 1_2$

$\pi^{op}[\lambda, \mu] \to \mu\, 1_2$

$\Rightarrow$

$\tilde{\pi}\left[\left| \begin{array}{l} \lambda \\ \mu \end{array} \right. \otimes \left| \begin{array}{l} \lambda' \\ \mu' \end{array} \right. \right] \to (\lambda\, \mu') \cdot 1_2$

$\tilde{\pi}[\mathcal{H}_F] \to \mathcal{H}_F$

$\left| \begin{array}{l} \pi \\ \pi^{op} \end{array} \right. \to \left| \begin{array}{l} \text{Satisfy order-1 condition} \\ \forall_{\text{ColumnBar}[\{a,b\}]\in\mathcal{A}} ([\pi[a], [\mathcal{D}, \pi^{op}[b^{op}]]] \to 0) \end{array} \right.$

```
PR["■Proposition 5.1: The gauge group of the finite space F_ED is ", 𝒢[F_ED] → U[1],
 NL, "¶ ",
 $ = {𝒰[𝒜_F] → U[1] × U[1],
    Ker[ρ] → {{λ, λ} ∈ 𝒰[𝒜_F], λ ∈ U[1]},
    Ker[ρ] ≃ U[1],
    ρ[𝒰[𝒜_F]] → ℬ[ℋ_F],
    imply,
    𝒢[F_ED] → Mod[𝒰[𝒜_F], Ker[ρ]],
    𝒢[F_ED] ≃ U[1]
   };
 $ // ColumnBar
]
```

■Proposition 5.1: The gauge group of the finite space F_{ED} is $\mathcal{G}[F_{ED}] \to U[1]$

¶
$\mathcal{U}[\mathcal{A}_F] \to U[1] \times U[1]$
$Ker[\rho] \to \{\{\lambda, \lambda\} \in \mathcal{U}[\mathcal{A}_F], \lambda \in U[1]\}$
$Ker[\rho] \simeq U[1]$
$\rho[\mathcal{U}[\mathcal{A}_F]] \to \mathcal{B}[\mathcal{H}_F]$
$\Rightarrow$
$\mathcal{G}[F_{ED}] \to Mod[\mathcal{U}[\mathcal{A}_F], Ker[\rho]]$
$\mathcal{G}[F_{ED}] \simeq U[1]$

```
PR["Let ", $ = {{M, g} → "even-dim-pseudo-Riemannian spin manifold", t[odd],
    F_ED × M → {T[C, "du"][c, ∞][M, 𝒜_F ⊙ 𝒜_F^op], ℋ_F ⊗ (L^2[S]),
      (1) ⊗ (I^t slash[D]) + I 𝒟_F ⊗ (1), 1 ⊗ 𝒥_M}, CR["⊗ is the graded tensor product"],
    A ∈ Pert[T[C, "du"][c, ∞][M, 𝒜_F]],
    {𝒜 -> T[C, "du"][c, ∞][M, 𝒜_F][CG[Commutative]], 𝒜^op ≃ 𝒜}, imply,
    {A → Sum[T[a, "d"][j] ⊗ T[b, "d"][j], j],
     T[a, "d"][j] → {λ_j, μ_j},
     T[b, "d"][j] → {λ'_j, μ'_j}},
    CO["order-1 condition ⇒"],
    η_D[A] → Sum[λ_j CommutatorM[I^t slash[D], (λ')_j], j] +
      Sum[μ_j CommutatorM[I^t slash[D], (μ')_j], j],
    A_μ ⊗ (I^t γ_μ) -> η_D[A],
    {A_μ → Sum[λ_j · tuDPartial[(λ')_j, μ] + μ_j · tuDPartial[(μ')_j, μ], j],
     A_μ ∈ T[C, "du"][c, ∞][M]},
    {A[CG[Real]], η_D[CG["involutive"]]} ⇒ {η_D[A][CG["Krein-self adjoint"]]},
    {I^t T[γ, "u"][μ][CG["Krein-anti-symmetric"]] ⇒ A_μ[CG["Krein-anti-symmetric"]]},
    imply,
    {A_μ ∈ T[C, "du"][c, ∞][M, I ℝ]},
    CO["fluctuated Dirac operator"],
    $e3 = 𝒟_A → 1 ⊗ (I^t slash[D]) + I 𝒟_F ⊗ 1 + A_μ ⊗ (I^t T[γ, "u"][μ]),
    F_ED × M[CG["Lorentz-type"]],
    $e4 = {ξ[CG["any vector"]] ∈ ℋ^"0" -> (ℋ^"0")_F ⊗ L^2[S]^"0" ⊕ (ℋ^"1")_F ⊗ L^2[S]^"1",
      ξ → e_R ⊗ ψ_R + e_L ⊗ ψ_L, ψ_R ∈ L^2[S]^"1", ψ_L ∈ L^2[S]^"0"}
  }; $ // ColumnBar
];
```

```
    {M, g} → even-dim-pseudo-Riemannian spin manifold
    t[odd]
    F_ED × M → {C_c^∞[M, 𝒜_F ⊙ 𝒜_F^op], ℋ_F ⊗ L^2[S], 1 ⊗ (ⅈ^t (D̸)) + ⅈ 𝒟_F ⊗ 1, 1 ⊗ 𝒥_M}
    ⊗ is the graded tensor product
    A ∈ Pert[C_c^∞[M, 𝒜_F]]
    {𝒜 → C_c^∞[M, 𝒜_F][Commutative], 𝒜^op ≃ 𝒜}
     ⇒
    {A → ∑_j a_j ⊗ b_j, a_j → {λ_j, μ_j}, b_j → {λ'_j, μ'_j}}
    order-1 condition ⇒
Let η_D[A] → ∑_j [ⅈ^t (D̸), λ'_j] λ_j + ∑_j [ⅈ^t (D̸), μ'_j] μ_j
    A_μ ⊗ (ⅈ^t γ_μ) → η_D[A]
    {A_μ → ∑_j (λ_j · ∂_{-μ} [λ'_j] + μ_j · ∂_{-μ} [μ'_j]), A_μ ∈ C_c^∞[M]}
    {A[Real], η_D[involutive]} ⇒ {η_D[A][Krein-self adjoint]}
    {ⅈ^t γ^μ[Krein-anti-symmetric] ⇒ A_μ[Krein-anti-symmetric]}
     ⇒
    {A_μ ∈ C_c^∞[M, ⅈ ℝ]}
    fluctuated Dirac operator
    𝒟_A → 1 ⊗ (ⅈ^t (D̸)) + A_μ ⊗ (ⅈ^t γ^μ) + ⅈ 𝒟_F ⊗ 1
    F_ED × M[Lorentz-type]
    {ξ[any vector] ∈ ℋ^0 → ℋ^0_F ⊗ L^2[S]^0 ⊕ ℋ^1_F ⊗ L^2[S]^1, ξ → e_L ⊗ ψ_L + e_R ⊗ ψ_R, ψ_R ∈ L^2[S]^1, ψ_L ∈ L^2[S]^0}
```

```
PR["●Proposition 5.2:  The Krein action for ", F_ED × M, " is given by ",
  S_ED[ψ, A] → BraKet[ψ, (I^t (slash[D] + T[γ, "u"][μ] · A_μ) - m) · ψ],
  NL, CO["For computing graded tensor products we use the definition
     found in Ref: https://en.wikipedia.org/wiki/Superalgebra"],

  NL, "¶ Calculate ", {BraKet[𝒥 · ξ, 𝒟_A · ξ]_𝒥, $e4},
  NL, "•For ", $j = 𝒥 → 1 ⊗ 𝒥_M,
  NL, "•Compute: ", $0 = $ = 𝒥 · ξ,
  Yield, $ = $ /. $j /. tuRuleSelect[$e4][ξ],
  Yield, $ = $ //. tuOpDistribute[CenterDot] //. tuOpSimplify[CenterDot],
  Yield, $ = $ /. (a_ ⊗ b_) · (a1_ ⊗ b1_) → (-1)^(deg[b] deg[a1]) (a · a1) ⊗ (b · b1) //.
    tuOpSimplify[CenterDot],
  NL, "with ", $deg = $s = {deg[e_R] → 1, deg[e_L] → 0, deg[𝒥_M] → 1},
  Yield, $jx = $ = $0 → ($ /. $s); $ // Framed,
  line,
  NL, "•For ", $ = $e3,
  NL, "Compute: ", $0 = $ = # · ξ & /@ $; MapAt[Framed[#] &, $, 1],
  Yield, $ = $[[2]] /. tuRuleSelect[$e4][ξ],
  Yield, $ = $ /. tuOpDistribute[CenterDot] /. tuOpSimplify[CenterDot];
  $ // ColumnSumExp,
  Yield, $ = $ /. (a_ ⊗ b_) · (a1_ ⊗ b1_) → (-1)^(deg[b] deg[a1]) (a · a1) ⊗ (b · b1) //.
    tuOpSimplify[CenterDot];
  $ // ColumnSumExp,
  NL, "Use: ",
  $deg = $s = {deg[ν_R] → 1, deg[ν_L] → 0, deg[I^t T[γ, "u"][μ] ] → 1, deg[I^t slash[D] ] → 1,
       deg[1] → 1, $deg} // Flatten // DeleteDuplicates,
  Yield, $ = $ /. $s; $ // ColumnSumExp,
  NL, CR["Perhaps the ", 𝒟_F · e_R|L, " terms could be expressed ",
   $1 = 𝒟_F.e, Imply, $1 = $1 /. tuRuleSelect[$fed][𝒟_F] /. e → {{e_R}, {e_L}},
   Yield, $1 = Thread[{𝒟_F · e_R, 𝒟_F · e_L} → Flatten[$1]]],
  Yield, $ = $0[[1]] → ($ /. $1 //. tuOpSimplify[CircleTimes, {m}]);
  Yield, $dx =
   $ = $ //. tuOpSimplify[CenterDot, {A_m_}] /. a_ ((a1_ : 1) ee : e_L|R) ⊗ b_ → ee ⊗ (a b) //.
     (ee : e_L|R) ⊗ b_ + (ee : e_L|R) ⊗ c_ → ee ⊗ (b + c);
  $ // Framed, CR["I am uncertain on the handling of the graded
     tensor product and the deg[] value of the different terms."]
 ];
PR["●Compute: ", $ = BraKet[$jx[[1]], $dx[[1]]],
  Yield, $ = $ /. {$jx, $dx},
  Yield, $ = $ //. tuOpDistribute[BraKet] //. tuOpSimplify[BraKet];
  Yield, $ = $ /. BraKet[(n_ : 1) a_ ⊗ b_, (m_ : 1) c_ ⊗ d_] :→
     (-1)^(deg[c] deg[b]) n m BraKet[a, c] ⊗ BraKet[b, d];
  $ = $ /. BraKet[e_r_, e_l_] :→ 0 /; r =!= l /. BraKet[e_r_, e_l_] :→ 1 /; r == l //. tuOpSimplify[
       CircleTimes] /. BraKet[aa : 𝒥_M · ψ_R, b1_] → - BraKet[aa, - b1] /. deg[_] → 0;
  $ // ColumnSumExp,
  NL, CR["In order to achieve their expression, m is a 2×2 matrix similar to 𝒟_F"]
 ];
```

●Proposition 5.2: The Krein action for
$F_{ED} \times M$ is given by $S_{ED}[\psi, A] \to \langle \psi \mid (-m + i^t (\gamma^\mu \cdot A_\mu + \not{D})) \cdot \psi \rangle$
For computing graded tensor products we use the definition
found in Ref: https://en.wikipedia.org/wiki/Superalgebra
¶ Calculate $\{\langle \mathcal{J} \cdot \xi \mid \mathcal{D}_A \cdot \xi \rangle_{\mathcal{J}}$, $\{\xi[$any vector$] \in \mathcal{H}^0 \to \mathcal{H}^0{}_F \otimes L^2[S]^0 \oplus \mathcal{H}^1{}_F \otimes L^2[S]^1$,
$\xi \to e_L \otimes \psi_L + e_R \otimes \psi_R$, $\psi_R \in L^2[S]^1$, $\psi_L \in L^2[S]^0\}\}$

•For $\mathcal{J} \to 1 \otimes \mathcal{J}_M$
•Compute: $\mathcal{J} \cdot \xi$
→ $1 \otimes \mathcal{J}_M \cdot (e_L \otimes \psi_L + e_R \otimes \psi_R)$
→ $1 \otimes \mathcal{J}_M \cdot e_L \otimes \psi_L + 1 \otimes \mathcal{J}_M \cdot e_R \otimes \psi_R$
→ $(-1)^{\deg[e_L] \deg[\mathcal{J}_M]} e_L \otimes (\mathcal{J}_M \cdot \psi_L) + (-1)^{\deg[e_R] \deg[\mathcal{J}_M]} e_R \otimes (\mathcal{J}_M \cdot \psi_R)$
with $\{\deg[e_R] \to 1, \deg[e_L] \to 0, \deg[\mathcal{J}_M] \to 1\}$
→ $\boxed{\mathcal{J} \cdot \xi \to e_L \otimes (\mathcal{J}_M \cdot \psi_L) - e_R \otimes (\mathcal{J}_M \cdot \psi_R)}$

•For $\mathcal{D}_A \to 1 \otimes (i^t (\not{D})) + A_\mu \otimes (i^t \gamma^\mu) + i\, \mathcal{D}_F \otimes 1$
Compute: $\boxed{\mathcal{D}_A \cdot \xi} \to (1 \otimes (i^t (\not{D})) + A_\mu \otimes (i^t \gamma^\mu) + i\, \mathcal{D}_F \otimes 1) \cdot \xi$
→ $(1 \otimes (i^t (\not{D})) + A_\mu \otimes (i^t \gamma^\mu) + i\, \mathcal{D}_F \otimes 1) \cdot (e_L \otimes \psi_L + e_R \otimes \psi_R)$

$$\to \sum\left[\begin{array}{|l} 1 \otimes (i^t (\not{D})) \cdot e_L \otimes \psi_L \\ 1 \otimes (i^t (\not{D})) \cdot e_R \otimes \psi_R \\ A_\mu \otimes (i^t \gamma^\mu) \cdot e_L \otimes \psi_L \\ A_\mu \otimes (i^t \gamma^\mu) \cdot e_R \otimes \psi_R \\ i\, \mathcal{D}_F \otimes 1 \cdot e_L \otimes \psi_L \\ i\, \mathcal{D}_F \otimes 1 \cdot e_R \otimes \psi_R \end{array}\right]$$

$$\to \sum\left[\begin{array}{|l} (-1)^{\deg[e_L] \deg[i^t \gamma^\mu]} (A_\mu \cdot e_L) \otimes (i^t \gamma^\mu \cdot \psi_L) \\ (-1)^{\deg[e_R] \deg[i^t \gamma^\mu]} (A_\mu \cdot e_R) \otimes (i^t \gamma^\mu \cdot \psi_R) \\ i\, (-1)^{\deg[1] \deg[e_L]} (\mathcal{D}_F \cdot e_L) \otimes \psi_L \\ i\, (-1)^{\deg[1] \deg[e_R]} (\mathcal{D}_F \cdot e_R) \otimes \psi_R \\ (-1)^{\deg[i^t (\not{D})] \deg[e_L]} e_L \otimes (i^t (\not{D}) \cdot \psi_L) \\ (-1)^{\deg[i^t (\not{D})] \deg[e_R]} e_R \otimes (i^t (\not{D}) \cdot \psi_R) \end{array}\right]$$

Use: $\{\deg[\nu_R] \to 1, \deg[\nu_L] \to 0, \deg[i^t \gamma^\mu] \to 1,$
$\deg[i^t (\not{D})] \to 1, \deg[1] \to 1, \deg[e_R] \to 1, \deg[e_L] \to 0, \deg[\mathcal{J}_M] \to 1\}$

$$\to \sum\left[\begin{array}{|l} (A_\mu \cdot e_L) \otimes (i^t \gamma^\mu \cdot \psi_L) \\ -((A_\mu \cdot e_R) \otimes (i^t \gamma^\mu \cdot \psi_R)) \\ i\, (\mathcal{D}_F \cdot e_L) \otimes \psi_L \\ -i\, (\mathcal{D}_F \cdot e_R) \otimes \psi_R \\ e_L \otimes (i^t (\not{D}) \cdot \psi_L) \\ -(e_R \otimes (i^t (\not{D}) \cdot \psi_R)) \end{array}\right]$$

Perhaps the $\mathcal{D}_F \cdot e_{R|L}$ terms could be expressed $\mathcal{D}_F$.e
⇒ $\{\{-i\, m\, e_L\}, \{i\, m\, e_R\}\}$
→ $\{\mathcal{D}_F \cdot e_R \to -i\, m\, e_L, \mathcal{D}_F \cdot e_L \to i\, m\, e_R\}$
→
→ $\boxed{\mathcal{D}_A \cdot \xi \to e_L \otimes (i^t (\not{D}) \cdot \psi_L + i^t \gamma^\mu \cdot \psi_L - m\, \psi_R) + e_R \otimes (-i^t (\not{D}) \cdot \psi_R - i^t \gamma^\mu \cdot \psi_R - m\, \psi_L)}$
I am uncertain on the handling of the graded
tensor product and the deg[] value of the different terms.

●Compute: $\langle \mathcal{J} \cdot \xi \mid \mathcal{D}_A \cdot \xi \rangle$
→ $\langle e_L \otimes (\mathcal{J}_M \cdot \psi_L) - e_R \otimes (\mathcal{J}_M \cdot \psi_R) \mid e_L \otimes (i^t (\not{D}) \cdot \psi_L + i^t \gamma^\mu \cdot \psi_L - m\, \psi_R) + e_R \otimes (-i^t (\not{D}) \cdot \psi_R - i^t \gamma^\mu \cdot \psi_R - m\, \psi_L) \rangle$
→

$$\to \sum\left[\begin{array}{|l} \langle \mathcal{J}_M \cdot \psi_L \mid i^t (\not{D}) \cdot \psi_L + i^t \gamma^\mu \cdot \psi_L - m\, \psi_R \rangle \\ \langle \mathcal{J}_M \cdot \psi_R \mid i^t (\not{D}) \cdot \psi_R + i^t \gamma^\mu \cdot \psi_R + m\, \psi_L \rangle \end{array}\right]$$

In order to achieve their expression, m is a 2×2 matrix similar to $\mathcal{D}_F$

```
PR["In 4-d Lorentzian signature {1,3}; ", 𝒥M → T[γ, "u"][0],
 NL, "The indefinite inner product ",
 {BraKet[ψ, ϕ] → tuIntegral[{{d[vol[M, g]]}}, ψ̄ · ϕ], ψ̄ → ct[ψ] · T[γ, "u"][0]},
 NL, "The Lagrangian can be written: ",
 Yield, ℒED[ψ, A] → ψ̄ · (I T[γ, "u"][μ] · (tuDs["∇"][_, μ] + Aμ) - m) · ψ
]
```

In 4-d Lorentzian signature {1,3}; $\mathcal{J}_M \to \gamma^0$
The indefinite inner product $\{\langle \psi \mid \phi \rangle \to \int_{\{d[vol[M,g]]\}} [\overline{\psi} \cdot \phi], \overline{\psi} \to \psi^\dagger \cdot \gamma^0\}$
The Lagrangian can be written:
$\to \mathcal{L}_{ED}[\psi, A] \to \overline{\psi} \cdot (-m + i\, \gamma^\mu \cdot (A_\mu + \underline{\nabla}_\mu[\_])) \cdot \psi$

Electro-weak theory

```
PR["finite-dimensional Z₂-graded Hilbert space: ",
  $dEW = $ = {ℋF → ℋR ⊕ ℋL, ℋR|L → ℂ ⊗ ℂ, ℋF^"0"[even] → ℋL, ℋF^"1"[odd] → ℋR,
     ℋF[basis] → {νR, eR, νL, eL},
     CO["algebra[Real]"],
     𝒜F → ℂ ⊕ ℍ,
     π["even-representations"][𝒜F] → ℬ[ℋR] ⊕ ℬ[ℋL],
     π[λ, q] → qλ ⊕ q,
     qλ → {{λ, 0}, {0, ct[λ]}},
     q → {{α, β}, {-ct[β], ct[α]}},
     π^op[{λ, q}^op] → λ ⊕ λ,
     λ ∈ ℂ,
     q → α + β j ∈ ℍ,
     π̃[𝒜F ⊙ 𝒜F^op] → π ⊗ π^op ,
     π̃[{λ, q} ⊗ {λ', q'}^op] → λ' · qλ ⊕ λ' · q,
     CR["Is the connection: 𝒜F->{λ,q} from 𝒜F→ℂ⊕ℍ"]
    };
  $ // ColumnBar
 ];
PR["Define mass matrix for basis: ", tuRuleSelect[$dEW][ℋF[basis]],
  Yield, $ = 𝒟F → {{0, 0, -I mν, 0}, {0, 0, 0, -I me}, {I mν, 0, 0, 0}, {0, I me, 0, 0}};
  $ // MatrixForms,
  NL, "even finite space: ", FEW → (#F & /@ $st) /. jj : 𝒥F → (jj → 0)
 ];
AppendTo[$dEW, $];
```

finite-dimensional Z_2-graded Hilbert space:

$\mathcal{H}_F \to \mathcal{H}_R \oplus \mathcal{H}_L$
$\mathcal{H}_{R|L} \to \mathbb{C} \otimes \mathbb{C}$
$\mathcal{H}_F^0[\text{even}] \to \mathcal{H}_L$
$\mathcal{H}_F^1[\text{odd}] \to \mathcal{H}_R$
$\mathcal{H}_F[\text{basis}] \to \{\nu_R, e_R, \nu_L, e_L\}$
algebra[Real]
$\mathcal{A}_F \to \mathbb{C} \oplus \mathbb{H}$
$\pi[\text{even-representations}][\mathcal{A}_F] \to \mathcal{B}[\mathcal{H}_R] \oplus \mathcal{B}[\mathcal{H}_L]$
$\pi[\lambda, q] \to q_\lambda \oplus q$
$q_\lambda \to \{\{\lambda, 0\}, \{0, \lambda^\dagger\}\}$
$q \to \{\{\alpha, \beta\}, \{-\beta^\dagger, \alpha^\dagger\}\}$
$\pi^{op}[\{\lambda^{op}, q^{op}\}] \to \lambda \oplus \lambda$
$\lambda \in \mathbb{C}$
$q \to \alpha + j\,\beta \in \mathbb{H}$
$\tilde{\pi}[\mathcal{A}_F \odot \mathcal{A}_F^{op}] \to \pi \otimes \pi^{op}$
$\tilde{\pi}[\{\lambda, q\} \otimes \{(\lambda')^{op}, (q')^{op}\}] \to \lambda' \cdot q_\lambda \oplus \lambda' \cdot q$
Is the connection: $\mathcal{A}_F$->{λ,q} from $\mathcal{A}_F \to \mathbb{C}\oplus\mathbb{H}$

Define mass matrix for basis: $\{\mathcal{H}_F[\text{basis}] \to \{\nu_R, e_R, \nu_L, e_L\}\}$

$$\to \mathcal{D}_F \to \begin{pmatrix} 0 & 0 & -i\, m_\nu & 0 \\ 0 & 0 & 0 & -i\, m_e \\ i\, m_\nu & 0 & 0 & 0 \\ 0 & i\, m_e & 0 & 0 \end{pmatrix}$$

even finite space: $F_{EW} \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F \to 0\}$

```
PR["●Proposition 6.1: The gauge group of F_EW is ", 𝒢[F_EW] → Mod[U[1] × SU[2], ℤ2],
  NL, "¶ ",
  $ = {𝒰[𝒜F] -> U[1] × SU[2][CG[ℍ]],
    ρ → (π̃ ∘ △)[𝒰[𝒜F]] → ℬ[ℋF], Ker[ρ] → {{±1, ±1} ∈ 𝒰[𝒜F]} ≃ ℤ2,
    imply,
    Framed[𝒢[F_EW] → Mod[𝒰[𝒜F], Ker[ρ]] -> Mod[U[1] × SU[2], ℤ2]]
   }; $ // ColumnBar,
  NL, "Recall ", $dsh // ColumnBar
 ];
```

●Proposition 6.1: The gauge group of F_{EW} is $\mathcal{G}[F_{EW}] \to \mathrm{Mod}[U[1] \times SU[2], \mathbb{Z}_2]$

¶
$\mathcal{U}[\mathcal{A}_F] \to U[1] \times SU[2][\mathbb{H}]$
$\rho \to (\tilde{\pi} \circ \triangle)[\mathcal{U}[\mathcal{A}_F]] \to \mathcal{B}[\mathcal{H}_F]$
$\mathrm{Ker}[\rho] \to \{\{\pm 1, \pm 1\} \in \mathcal{U}[\mathcal{A}_F]\} \simeq \mathbb{Z}_2$
$\Rightarrow$
$\boxed{\mathcal{G}[F_{EW}] \to \mathrm{Mod}[\mathcal{U}[\mathcal{A}_F], \mathrm{Ker}[\rho]] \to \mathrm{Mod}[U[1] \times SU[2], \mathbb{Z}_2]}$

Recall
$\triangle[\mathcal{U}[\mathcal{A}][\text{unitary}]] \to \mathrm{Pert}[\mathcal{A}][\text{perturbation semi-group}]$
$\triangle[u] \to u \otimes (u^*)^{op}$
$\triangle[u] \cdot (A \to \sum_j a_j \otimes (b_j)^{op}) \to \sum_j (u \cdot a_j) \otimes ((u^*)^{op} \cdot (b_j)^{op})$
$\triangle[u] \cdot (A \to \sum_j a_j \otimes (b_j)^{op}) \to \sum_j (u \cdot a_j) \otimes (b_j \cdot u^*)^{op}$
group representation ρ
$\rho \to \tilde{\pi} \circ \triangle[\mathcal{U}[\mathcal{A}]] \to \mathcal{B}[\mathcal{H}]$

```
PR["Let ",
  {M, g} → "even-dimensional pseudo-Riemannian spin manifold, t[odd]",
  NL, "The representations ",
  {π, π^op} ⟶ {rep[{T[C, "du"][c, ∞][M, 𝒜F], T[C, "du"][c, ∞][M, 𝒜F]^op}]},
  NL, CR["Show that it satisfies the order-one condition ", $e1],
  NL, "The ACM: ",
  F_EW × M →
   {T[C, "du"][c, ∞][M, 𝒜F ⊙ 𝒜F^op], ℋF ⊗ (L^2[S]), (1) ⊗ (I^t slash[D]) + (I 𝒟F) ⊗ (1), 1 ⊗ 𝒥M},
  NL, "●Proposition 6.2: ", $p62 = $ = {fluctuation[A ∈ Pert[T[C, "du"][c, ∞][M, 𝒜F]]][
       𝒟 → 1 ⊗ (I^t slash[D]) + (I 𝒟F) ⊗ 1] → 𝒟A,
     𝒟A → 𝒟 + η𝒟[A],
     𝒟A → 1 ⊗ (I^t slash[D]) + T[A, "d"][μ] ⊗ (I^t T[γ, "u"][μ]) + (I 𝒟F + ϕ) ⊗ 1,
     T[A, "d"][μ][CG["gauge field"]],
     ϕ[CG["Higgs field"]],
     T[A, "d"][μ] ->
      SparseArray[{{2, 2} → -2 T[Λ, "d"][μ] , {3, 3} -> T[Q, "d"][μ] - 1_2 T[Λ, "d"][μ]}],
     {T[Λ, "d"][μ], T[Q, "d"][μ]} → T[C, "du"][c, ∞][M, I R ⊕ su[2]],
     ϕ → {{0, 0, m_ν ct[ϕ1], m_e ct[ϕ2]}, {0, 0, -m_ν ϕ2, m_e ϕ1},
        {-m_ν ϕ1, m_e ct[ϕ2], 0, 0}, {-m_ν ϕ2, -m_e ct[ϕ1], 0, 0}}, CR["Changed defn of ϕ"],
     {T[Λ, "d"][μ], T[Q, "d"][μ]} ∈ T[C, "du"][c, ∞][M, I ℝ ⊕ su[2]],
     {ϕ1, ϕ2} ∈ T[C, "du"][c, ∞][M, ℂ ⊗ ℂ]
    }; $ // MatrixForms // ColumnBar
 ];
PR["¶",
 $p62p = $ = {A → Sum[a_j ⊗ b_j^op, j],
    A -> Sum[{λ_j, q_j} ⊗ {λ_j ', q_j '}^op, j],
    Sum[{λ_j, q_j} ⊗ {λ_j ', q_j '}^op, j] ∈ Pert[T[C, "du"][c, ∞][M, 𝒜F]],
    "fluctuation" → η𝒟[A],
    η𝒟[A] → a_j · CommutatorM[𝒟, b_j] + ct[a_j^op] · CommutatorM[𝒟, ct[b_j^op]],
    𝒟 → 1 ⊗ (I^t slash[D]) + (I 𝒟F) ⊗ 1,
    {ϕ → Sum[a_j · CommutatorM[I 𝒟F, b_j], j], CR["Check with above"]},
    ϕ → {{0, 0, m_ν ϕ1 ', m_ν ϕ2 '}, {0, 0, -m_e ct[ϕ2 '], m_e ct[ϕ1 ']},
      {-m_ν ϕ1, m_e ct[ϕ2], 0, 0}, {-m_ν ϕ2, -m_e ct[ϕ1], 0, 0}},
    ϕ1 → Sum[α_j · (λ_j ' - α_j ') + β_j · ct[β_j '], j],
    ϕ2 → Sum[-ct[β_j] · (λ_j ' - α_j ') + ct[α_j] · ct[β_j '], j],
    ϕ1 ' → Sum[λ_j · (α_j ' - λ_j '), j],
    ϕ2 ' → Sum[λ_j · β_j ', j],
    CommutatorM[𝒟F, π^op] → 0,
    ct[a_j^op] · CommutatorM[I 𝒟F, ct[b_j^op]] → 0,
    CO["Since"],
    A[CG["Real"]] ⇒ ϕ[CG["Krein self-adjoint"]],
    (I ϕ)[CG["self-adjoint"]],
    ϕ1 ' → ct[ϕ1],
    ϕ2 ' → ct[ϕ2], imply,
    Sum[a_j · CommutatorM[I slash[𝒟], b_j], j] -> SparseArray[{{1, 1} → T[Λ, "d"][μ],
        {2, 2} → -T[Λ, "d"][μ] , {3, 3} -> T[Q, "d"][μ]}] ⊗ (I^t T[γ, "u"][μ]),
    Sum[ct[a_j^op] · CommutatorM[I^t slash[𝒟], ct[b_j^op]], j] →
     -(T[Λ, "d"][μ] 1_4) ⊗ (I^t T[γ, "u"][μ]),
    (T[Λ, "d"][μ] → Sum[λ_j tuDPartial[λ_j ', μ], j]) ∈ T[C, "du"][c, ∞][M, I ℝ],
    (T[Q, "d"][μ] -> Sum[q_j tuDPartial[q_j ', μ], j]) ∈ T[C, "du"][c, ∞][M, su[2]],
    η𝒟[A] → T[A, "d"][μ] ⊗ (I^t T[γ, "u"][μ]) + ϕ ⊗ 1, imply, CR["Show this"],
    T[A, "d"][μ] ->
     SparseArray[{{2, 2} → -2 T[Λ, "d"][μ] , {3, 3} -> T[Q, "d"][μ] - 1_2 T[Λ, "d"][μ]}]
   };
 $ // MatrixForms // ColumnBar
]
```

Let $\{M, g\} \to$ even-dimensional pseudo-Riemannian spin manifold, t[odd]

The representations $\{\pi, \pi^{op}\} \longrightarrow \{\mathrm{rep}[\{C_c{}^{\infty}[M, \mathcal{A}_F], (C_c{}^{\infty}[M, \mathcal{A}_F])^{op}\}]\}$

Show that it satisfies the order-one condition $\forall_{\{a,b\}\in\mathcal{A}} ([\pi[a], [\mathcal{D}, \pi^{op}[b^{op}]]] \to 0)$

The ACM: $F_{EW} \times M \to \{C_c{}^{\infty}[M, \mathcal{A}_F \odot \mathcal{A}_F^{op}], \mathcal{H}_F \otimes L^2[S], 1 \otimes (\mathbb{i}^t (\not{D})) + (\mathbb{i}\, \mathcal{D}_F) \otimes 1, 1 \otimes \mathcal{J}_M\}$

●Proposition 6.2:

$\text{fluctuation}[A \in \text{Pert}[C_c{}^{\infty}[M, \mathcal{A}_F]]][\mathcal{D} \to 1 \otimes (\mathbb{i}^t (\not{D})) + (\mathbb{i}\, \mathcal{D}_F) \otimes 1] \to \mathcal{D}_A$

$\mathcal{D}_A \to \mathcal{D} + \eta_{\mathcal{D}}[A]$

$\mathcal{D}_A \to 1 \otimes (\mathbb{i}^t (\not{D})) + (\phi + \mathbb{i}\, \mathcal{D}_F) \otimes 1 + A_\mu \otimes (\mathbb{i}^t \gamma^\mu)$

A_μ[gauge field]

ϕ[Higgs field]

$A_\mu \to$ SparseArray[Specifiedelements 2 Dimensions {3, 3}]

$\{\Lambda_\mu, Q_\mu\} \to C_c{}^{\infty}[M, \mathbb{i}\, R \oplus su[2]]$

$$\phi \to \begin{pmatrix} 0 & 0 & (\phi_1)^\dagger m_\nu & (\phi_2)^\dagger m_e \\ 0 & 0 & -m_\nu\, \phi_2 & m_e\, \phi_1 \\ -m_\nu\, \phi_1 & (\phi_2)^\dagger m_e & 0 & 0 \\ -m_\nu\, \phi_2 & -(\phi_1)^\dagger m_e & 0 & 0 \end{pmatrix}$$

Changed defn of ϕ

$\{\Lambda_\mu, Q_\mu\} \in C_c{}^{\infty}[M, \mathbb{i}\, \mathbb{R} \oplus su[2]]$

$\{\phi_1, \phi_2\} \in C_c{}^{\infty}[M, \mathbb{C} \otimes \mathbb{C}]$

$A \to \sum_j a_j \otimes b_j^{op}$

$A \to \sum_j \{\lambda_j, q_j\} \otimes \{(\lambda_j{}')^{op}, (q_j{}')^{op}\}$

$\sum_j \{\lambda_j, q_j\} \otimes \{(\lambda_j{}')^{op}, (q_j{}')^{op}\} \in \mathrm{Pert}[C_c{}^{\infty}[M, \mathcal{A}_F]]$

$\mathrm{fluctuation} \to \eta_{\mathcal{D}}[A]$

$\eta_{\mathcal{D}}[A] \to (a_j^{op})^{\dagger} \cdot [\mathcal{D}, (b_j^{op})^{\dagger}] + a_j \cdot [\mathcal{D}, b_j]$

$\mathcal{D} \to 1 \otimes (i^t (\not{D})) + (i\, \mathcal{D}_F) \otimes 1$

$\{\phi \to \sum_j a_j \cdot [i\, \mathcal{D}_F, b_j],$ Check with above$\}$

$$\phi \to \begin{pmatrix} 0 & 0 & m_{\nu}\, \phi_1{}' & m_{\nu}\, \phi_2{}' \\ 0 & 0 & -(\phi_2{}')^{\dagger}\, m_e & (\phi_1{}')^{\dagger}\, m_e \\ -m_{\nu}\, \phi_1 & (\phi_2)^{\dagger}\, m_e & 0 & 0 \\ -m_{\nu}\, \phi_2 & -(\phi_1)^{\dagger}\, m_e & 0 & 0 \end{pmatrix}$$

$\phi_1 \to \sum_j (\alpha_j \cdot (-\alpha_j{}' + \lambda_j{}') + \beta_j \cdot (\beta_j{}')^{\dagger})$

$\phi_2 \to \sum_j ((\alpha_j)^{\dagger} \cdot (\beta_j{}')^{\dagger} + -(\beta_j)^{\dagger} \cdot (-\alpha_j{}' + \lambda_j{}'))$

$\phi_1{}' \to \sum_j \lambda_j \cdot (\alpha_j{}' - \lambda_j{}')$

$\phi_2{}' \to \sum_j \lambda_j \cdot \beta_j{}'$

$[\mathcal{D}_F, \pi^{op}] \to 0$

$(a_j^{op})^{\dagger} \cdot [i\, \mathcal{D}_F, (b_j^{op})^{\dagger}] \to 0$

Since

A[Real] ⇒ ϕ[Krein self-adjoint]

($i\, \phi$)[self-adjoint]

$\phi_1{}' \to (\phi_1)^{\dagger}$

$\phi_2{}' \to (\phi_2)^{\dagger}$

⇒

$\sum_j a_j \cdot [i\, (\not{D}), b_j] \to$ SparseArray[Specifiedelements 3 Dimensions {3, 3}] $\otimes (i^t \gamma^{\mu})$

$\sum_j (a_j^{op})^{\dagger} \cdot [i^t (\not{D}), (b_j^{op})^{\dagger}] \to (-1_4\, \Lambda_{\mu}) \otimes (i^t \gamma^{\mu})$

$(\Lambda_{\mu} \to \sum_j \lambda_j\, \underset{-\mu}{\partial}\, [\lambda_j{}']) \in C_c{}^{\infty}[M, i\,\mathbb{R}]$

$(Q_{\mu} \to \sum_j q_j\, \underset{-\mu}{\partial}\, [q_j{}']) \in C_c{}^{\infty}[M, su[2]]$

$\eta_{\mathcal{D}}[A] \to \phi \otimes 1 + A_{\mu} \otimes (i^t \gamma^{\mu})$

⇒

Show this

$A_{\mu} \to$ SparseArray[Specifiedelements 2 Dimensions {3, 3}]

```
PR[(F_EW × M)[CG["ACM Lorentz-type "]] ⇒ S[CG["Krein-action"]],
 imply, $e5 = $ = {ξ[CG["arbitrary vector"]] ∈ (ℋ^"0" → ℋ_L ⊗ L^2[S]^"0" ⊕ ℋ_R ⊗ L^2[S]^"1"),
     ξ → ν_R ⊗ (ψ^ν)_R + e_R ⊗ (ψ^e)_R + ν_L ⊗ (ψ^ν)_L + e_L ⊗ (ψ^e)_L,
     {(ψ^ν)_L, (ψ^e)_L}[CG["Weyl spinors"]] ∈ L^2[S]^"0",
     {(ψ^ν)_R, (ψ^e)_R}[CG["Weyl spinors"]] ∈ L^2[S]^"1",
     ξ[ψ^ν -> (ψ^ν)_L + (ψ^ν)_R, ψ^e -> (ψ^e)_L + (ψ^e)_R],
     CO["Combine"],
     (Ψ_L → {{(ψ^ν)_L}, {(ψ^e)_L}}) ∈ L^2[S]^"0" ⊗ (ℂ ⊗ ℂ),
     (Ψ_R → {{(ψ^ν)_R}, {(ψ^e)_R}}) ∈ L^2[S]^"1" ⊗ (ℂ ⊗ ℂ),
     (Ψ → Ψ_L + Ψ_R) ∈ L^2[S] ⊗ (ℂ ⊗ ℂ)
    }; $ // ColumnBar
]
```

$(F_{EW} \times M)$[ACM Lorentz-type] ⇒ S[Krein-action]

	ξ[arbitrary vector] $\in (\mathcal{H}^0 \to \mathcal{H}_L \otimes L^2[S]^0 \oplus \mathcal{H}_R \otimes L^2[S]^1)$
	$\xi \to e_L \otimes \psi^e{}_L + e_R \otimes \psi^e{}_R + \nu_L \otimes \psi^\nu{}_L + \nu_R \otimes \psi^\nu{}_R$
	$\{\psi^\nu{}_L, \psi^e{}_L\}$[Weyl spinors] $\in L^2[S]^0$
	$\{\psi^\nu{}_R, \psi^e{}_R\}$[Weyl spinors] $\in L^2[S]^1$
⇒	$\xi[\psi^\nu \to \psi^\nu{}_L + \psi^\nu{}_R,\ \psi^e \to \psi^e{}_L + \psi^e{}_R]$
	Combine
	$(\Psi_L \to \{\{\psi^\nu{}_L\}, \{\psi^e{}_L\}\}) \in L^2[S]^0 \otimes (\mathbb{C} \otimes \mathbb{C})$
	$(\Psi_R \to \{\{\psi^\nu{}_R\}, \{\psi^e{}_R\}\}) \in L^2[S]^1 \otimes (\mathbb{C} \otimes \mathbb{C})$
	$(\Psi \to \Psi_L + \Psi_R) \in L^2[S] \otimes (\mathbb{C} \otimes \mathbb{C})$

```
PR["●Proposition 6.3. The Krein action for ", F_EW×M, " is given by ",
  $p63 = $ = {S_EW[Ψ, A] →
      BraKet[Ψ, I^t slash[D]·Ψ] + BraKet[(ψ^e)_R, -2 I^t T[γ, "u"][μ]·T[Λ, "d"][μ]·(ψ^e)_R] +
       BraKet[Ψ_L, I^t T[γ, "u"][μ]·(T[Q, "d"][μ] - T[Λ, "d"][μ])·Ψ_L] +
       BraKet[Ψ_R, Φ·Ψ_L] + BraKet[Ψ_L, ct[Φ]·Ψ_R],
     {T[Λ, "d"][μ], T[Q, "d"][μ]}[CG["gauge fields"]],
     Φ → {{-m_ν ct[ϕ1 + 1], -m_ν ct[ϕ2]}, {m_e ϕ2, -m_e (ϕ1 + 1)}},
     {ϕ1, ϕ2}[CG["Higgs field"]]
    }; $ // MatrixForms // ColumnBar,

  line,
  NL, "¶roof: Compute ", $action = $ = BraKet[𝒥·ξ, 𝒟_A·ξ]_𝒥,
  NL, "■determine: ", $0 = $ = 𝒥·ξ, " with ", $s = 𝒥 → 1⊗𝒥_M,
  Imply, $ = $ /. $s /. tuRuleSelect[$e5][ξ],
  Yield, $ = $ //. tuOpDistribute[CenterDot] /.
     (a_⊗b_)·(a1_⊗b1_) → (-1)^(deg[b] deg[a1]) (a·a1)⊗(b·b1) /. $deg,
  Yield, $ = $ // tuOpSimplifyF[CenterDot];
  $sjx = $ = $0 → $; $ // Framed
 ];
```

●Proposition 6.3. The Krein action for $F_{EW} \times M$ is given by

$S_{EW}[\Psi, A] \to \langle \Psi \mid i^t (\not{D})\cdot\Psi \rangle + \langle \psi^e{}_R \mid -2\, i^t \gamma^\mu \cdot \Lambda_\mu \cdot \psi^e{}_R \rangle + \langle \Psi_L \mid \Phi^\dagger \cdot \Psi_R \rangle + \langle \Psi_L \mid i^t \gamma^\mu \cdot (Q_\mu - \Lambda_\mu)\cdot \Psi_L \rangle + \langle \Psi_R \mid \Phi \cdot \Psi_L \rangle$

$\{\Lambda_\mu, Q_\mu\}$[gauge fields]

$\Phi \to \begin{pmatrix} -(1+(\phi_1)^\dagger)\, m_\nu & -(\phi_2)^\dagger\, m_\nu \\ m_e\, \phi_2 & -m_e\,(1+\phi_1) \end{pmatrix}$

$\{\phi_1, \phi_2\}$[Higgs field]

¶roof: Compute $\langle \mathcal{J}\cdot\xi \mid \mathcal{D}_A \cdot \xi \rangle_{\mathcal{J}}$

■determine: $\mathcal{J}\cdot\xi$ with $\mathcal{J} \to 1 \otimes \mathcal{J}_M$

⇒ $1\otimes\mathcal{J}_M \cdot (e_L \otimes \psi^e{}_L + e_R \otimes \psi^e{}_R + \nu_L \otimes \psi^\nu{}_L + \nu_R \otimes \psi^\nu{}_R)$

→ $(1\cdot e_L)\otimes(\mathcal{J}_M \cdot \psi^e{}_L) - (1\cdot e_R)\otimes(\mathcal{J}_M \cdot \psi^e{}_R) + (1\cdot \nu_L)\otimes(\mathcal{J}_M \cdot \psi^\nu{}_L) - (1\cdot \nu_R)\otimes(\mathcal{J}_M \cdot \psi^\nu{}_R)$

→ $\boxed{\mathcal{J}\cdot\xi \to e_L \otimes (\mathcal{J}_M \cdot \psi^e{}_L) - e_R \otimes (\mathcal{J}_M \cdot \psi^e{}_R) + \nu_L \otimes (\mathcal{J}_M \cdot \psi^\nu{}_L) - \nu_R \otimes (\mathcal{J}_M \cdot \psi^\nu{}_R)}$

```
PR["•In vector form: ", $sjx,
  Yield, $ = $sjx[[2]]; $ // ColumnSumExp;
  $p = $ // tuExtractPattern[CenterDot[__], 1];
  Yield, $p0 = $p = $ - Apply[Plus, $p]; $p = Apply[List, $p],
  Yield, $p1 = Extract[$p,
     Position[$p, Apply[Alternatives, tuRuleSelect[$dEW][ℋ_F[basis]][[1, 2]]]]],
  Yield, $s = tuRuleSelect[$dEW][ℋ_F[basis]][[1, 2]],
  Yield, $s = FindPermutation[$s, $p1],
  Yield, $sjx1 = $sjx[[1]] -> ($p = {Permute[$p, $s]} // Transpose);
  $sjx1 // MatrixForms // Framed
 ];
```

•In vector form: $\mathcal{J}\cdot\xi \to e_L \otimes (\mathcal{J}_M \cdot \psi^e{}_L) - e_R \otimes (\mathcal{J}_M \cdot \psi^e{}_R) + \nu_L \otimes (\mathcal{J}_M \cdot \psi^\nu{}_L) - \nu_R \otimes (\mathcal{J}_M \cdot \psi^\nu{}_R)$

→

→ $\{e_L \otimes (\mathcal{J}_M \cdot \psi^e{}_L),\ -(e_R \otimes (\mathcal{J}_M \cdot \psi^e{}_R)),\ \nu_L \otimes (\mathcal{J}_M \cdot \psi^\nu{}_L),\ -(\nu_R \otimes (\mathcal{J}_M \cdot \psi^\nu{}_R))\}$

→ $\{e_L, e_R, \nu_L, \nu_R\}$

→ $\{\nu_R, e_R, \nu_L, e_L\}$

→ Cycles[{{1, 4}}]

→ $\boxed{\mathcal{J}\cdot\xi \to \begin{pmatrix} -(\nu_R \otimes (\mathcal{J}_M \cdot \psi^\nu{}_R)) \\ -(e_R \otimes (\mathcal{J}_M \cdot \psi^e{}_R)) \\ \nu_L \otimes (\mathcal{J}_M \cdot \psi^\nu{}_L) \\ e_L \otimes (\mathcal{J}_M \cdot \psi^e{}_L) \end{pmatrix}}$

```
PR["■determine: ", $0a = $0 = $ = 𝒟A · ξ,
  NL, "with ", $s = {tuRuleSelect[$p62][𝒟A][[2]], tuRuleSelect[$e5][ξ]} // Flatten;
  $s // ColumnBar,
  Yield, $ = $ /. $s,
  Yield, $ = $ //. tuOpDistribute[CenterDot],
  Yield, $ = $ /. (a_ ⊗ b_) · (a1_ ⊗ b1_) → (-1)^(deg[b] deg[a1]) (a · a1) ⊗ (b · b1),
  Yield, $pass = $ = $ //. $deg //. tuOpSimplify[CenterDot];
  $ // ColumnSumExp
 ];
```

■determine: $\mathcal{D}_A \cdot \xi$

with $\mathcal{D}_A \to 1\otimes(\mathrm{i}^t\,(\not{D})) + (\phi + \mathrm{i}\,\mathcal{D}_F)\otimes 1 + A_\mu\otimes(\mathrm{i}^t\,\gamma^\mu)$
$\xi \to e_L\otimes\psi^e{}_L + e_R\otimes\psi^e{}_R + \nu_L\otimes\psi^\nu{}_L + \nu_R\otimes\psi^\nu{}_R$

→ $(1\otimes(\mathrm{i}^t\,(\not{D})) + (\phi + \mathrm{i}\,\mathcal{D}_F)\otimes 1 + A_\mu\otimes(\mathrm{i}^t\,\gamma^\mu))\cdot(e_L\otimes\psi^e{}_L + e_R\otimes\psi^e{}_R + \nu_L\otimes\psi^\nu{}_L + \nu_R\otimes\psi^\nu{}_R)$

→ $1\otimes(\mathrm{i}^t\,(\not{D}))\cdot e_L\otimes\psi^e{}_L + 1\otimes(\mathrm{i}^t\,(\not{D}))\cdot e_R\otimes\psi^e{}_R + 1\otimes(\mathrm{i}^t\,(\not{D}))\cdot \nu_L\otimes\psi^\nu{}_L + 1\otimes(\mathrm{i}^t\,(\not{D}))\cdot \nu_R\otimes\psi^\nu{}_R +$
$(\phi + \mathrm{i}\,\mathcal{D}_F)\otimes 1\cdot e_L\otimes\psi^e{}_L + (\phi + \mathrm{i}\,\mathcal{D}_F)\otimes 1\cdot e_R\otimes\psi^e{}_R + (\phi + \mathrm{i}\,\mathcal{D}_F)\otimes 1\cdot \nu_L\otimes\psi^\nu{}_L + (\phi + \mathrm{i}\,\mathcal{D}_F)\otimes 1\cdot \nu_R\otimes\psi^\nu{}_R +$
$A_\mu\otimes(\mathrm{i}^t\,\gamma^\mu)\cdot e_L\otimes\psi^e{}_L + A_\mu\otimes(\mathrm{i}^t\,\gamma^\mu)\cdot e_R\otimes\psi^e{}_R + A_\mu\otimes(\mathrm{i}^t\,\gamma^\mu)\cdot \nu_L\otimes\psi^\nu{}_L + A_\mu\otimes(\mathrm{i}^t\,\gamma^\mu)\cdot \nu_R\otimes\psi^\nu{}_R$

→ $(-1)^{\deg[\mathrm{i}^t\,(\not{D})]\deg[e_L]}\,(1\cdot e_L)\otimes((\mathrm{i}^t\,(\not{D}))\cdot\psi^e{}_L) + (-1)^{\deg[\mathrm{i}^t\,(\not{D})]\deg[e_R]}\,(1\cdot e_R)\otimes((\mathrm{i}^t\,(\not{D}))\cdot\psi^e{}_R) +$
$(-1)^{\deg[\mathrm{i}^t\,(\not{D})]\deg[\nu_L]}\,(1\cdot \nu_L)\otimes((\mathrm{i}^t\,(\not{D}))\cdot\psi^\nu{}_L) + (-1)^{\deg[\mathrm{i}^t\,(\not{D})]\deg[\nu_R]}\,(1\cdot \nu_R)\otimes((\mathrm{i}^t\,(\not{D}))\cdot\psi^\nu{}_R) +$
$(-1)^{\deg[1]\deg[e_L]}\,((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot e_L)\otimes(1\cdot\psi^e{}_L) + (-1)^{\deg[1]\deg[e_R]}\,((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot e_R)\otimes(1\cdot\psi^e{}_R) +$
$(-1)^{\deg[1]\deg[\nu_L]}\,((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot \nu_L)\otimes(1\cdot\psi^\nu{}_L) + (-1)^{\deg[1]\deg[\nu_R]}\,((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot \nu_R)\otimes(1\cdot\psi^\nu{}_R) +$
$(-1)^{\deg[e_L]\deg[\mathrm{i}^t\,\gamma^\mu]}\,(A_\mu\cdot e_L)\otimes((\mathrm{i}^t\,\gamma^\mu)\cdot\psi^e{}_L) + (-1)^{\deg[e_R]\deg[\mathrm{i}^t\,\gamma^\mu]}\,(A_\mu\cdot e_R)\otimes((\mathrm{i}^t\,\gamma^\mu)\cdot\psi^e{}_R) +$
$(-1)^{\deg[\nu_L]\deg[\mathrm{i}^t\,\gamma^\mu]}\,(A_\mu\cdot \nu_L)\otimes((\mathrm{i}^t\,\gamma^\mu)\cdot\psi^\nu{}_L) + (-1)^{\deg[\nu_R]\deg[\mathrm{i}^t\,\gamma^\mu]}\,(A_\mu\cdot \nu_R)\otimes((\mathrm{i}^t\,\gamma^\mu)\cdot\psi^\nu{}_R)$

→ ∑[
$((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot e_L)\otimes\psi^e{}_L$
$-(((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot e_R)\otimes\psi^e{}_R)$
$((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot \nu_L)\otimes\psi^\nu{}_L$
$-(((\phi + \mathrm{i}\,\mathcal{D}_F)\cdot \nu_R)\otimes\psi^\nu{}_R)$
$(A_\mu\cdot e_L)\otimes(\mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_L)$
$-((A_\mu\cdot e_R)\otimes(\mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_R))$
$(A_\mu\cdot \nu_L)\otimes(\mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L)$
$-((A_\mu\cdot \nu_R)\otimes(\mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_R))$
$e_L\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_L)$
$-(e_R\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_R))$
$\nu_L\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_L)$
$-(\nu_R\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_R))$
]

```
PR["Put Aμ coefficients in: ",
  $s = tuRuleSelect[$dEW][ℋF[basis]][[1, 2]], " basis order ",
  Yield,
  $ = Select[$pass, ! FreeQ[#, Tensor[A, _, _]] &];
  $ = $ /. (c_: 1) ((aa : Tensor[A, _, _]) · e_) ⊗ a_ → c aa · (e ⊗ a) //.
    tuOpSimplify[CenterDot];
  $ // ColumnSumExp;
  $ = $ //. (c1_: 1) (aa : Tensor[A, _, _]) · a1_ + (c2_: 1) (aa : Tensor[A, _, _]) · a2_ ->
      aa · (c1 a1 + c2 a2) //. tuOpSimplify[CenterDot]; $ // ColumnSumExp;
  $p = Apply[List, $[[2]]];
  Yield,
  $p1 = Extract[$p,
    Position[$p, Apply[Alternatives, tuRuleSelect[$dEW][ℋF[basis]][[1, 2]]]]];
  $s = FindPermutation[$s, $p1];
  $p = {Permute[$p, $s]} // Transpose;
  $p = T[A, "d"][μ] · $p; $p // MatrixForms,
  NL, "Full expression: ",
  $pass1 = $ = Select[$pass, FreeQ[#, Tensor[A, _, _]] &] + $p;
  $ // MatrixForms // ColumnSumExp
 ];
PR["•Expand ", $ = tuRuleSelect[$p62p][T[A, "d"][μ]],
  NL, "with ", $s = {T[Q, "d"][μ] → Table[T[Q, "duu"][μ, i, j], {i, 2}, {j, 2}],
    12 → DiagonalMatrix[{1, 1}]},
  Yield, $ = tuRuleSelect[$p62p][T[A, "d"][μ]] // Normal,
  Yield, $ = $ /. $s // First,
  Yield, $[[2]] = $[[2]] // ArrayFlatten; $ // MatrixForms,
  $sA = $;
 ];
```

Put A_μ coefficients in: $\{\nu_R, e_R, \nu_L, e_L\}$ basis order

→

$$\rightarrow A_\mu \cdot \begin{pmatrix} -(\nu_R \otimes (i^t\, \gamma^\mu \cdot \psi^\nu{}_R)) \\ -(e_R \otimes (i^t\, \gamma^\mu \cdot \psi^e{}_R)) \\ \nu_L \otimes (i^t\, \gamma^\mu \cdot \psi^\nu{}_L) \\ e_L \otimes (i^t\, \gamma^\mu \cdot \psi^e{}_L) \end{pmatrix}$$

Full expression: $$\sum\left[\begin{array}{l} A_\mu \cdot \begin{pmatrix} -(\nu_R \otimes (i^t\, \gamma^\mu \cdot \psi^\nu{}_R)) \\ -(e_R \otimes (i^t\, \gamma^\mu \cdot \psi^e{}_R)) \\ \nu_L \otimes (i^t\, \gamma^\mu \cdot \psi^\nu{}_L) \\ e_L \otimes (i^t\, \gamma^\mu \cdot \psi^e{}_L) \end{pmatrix} \\ ((\phi + i\, \mathcal{D}_F) \cdot e_L) \otimes \psi^e{}_L \\ -(((\phi + i\, \mathcal{D}_F) \cdot e_R) \otimes \psi^e{}_R) \\ ((\phi + i\, \mathcal{D}_F) \cdot \nu_L) \otimes \psi^\nu{}_L \\ -(((\phi + i\, \mathcal{D}_F) \cdot \nu_R) \otimes \psi^\nu{}_R) \\ e_L \otimes (i^t\, (\not{D}) \cdot \psi^e{}_L) \\ -(e_R \otimes (i^t\, (\not{D}) \cdot \psi^e{}_R)) \\ \nu_L \otimes (i^t\, (\not{D}) \cdot \psi^\nu{}_L) \\ -(\nu_R \otimes (i^t\, (\not{D}) \cdot \psi^\nu{}_R)) \end{array}\right]$$

•Expand $\{A_\mu \to$ SparseArray[Specifiedelements 2 Dimensions {3, 3}]}

with $\{Q_\mu \to \{\{Q_\mu{}^{11}, Q_\mu{}^{12}\}, \{Q_\mu{}^{21}, Q_\mu{}^{22}\}\}, 1_2 \to \{\{1, 0\}, \{0, 1\}\}\}$

→ $\{A_\mu \to \{\{0, 0, 0\}, \{0, -2\,\Lambda_\mu, 0\}, \{0, 0, Q_\mu - 1_2\,\Lambda_\mu\}\}\}$

→ $A_\mu \to \{\{0, 0, 0\}, \{0, -2\,\Lambda_\mu, 0\}, \{0, 0, \{\{Q_\mu{}^{11} - \Lambda_\mu, Q_\mu{}^{12}\}, \{Q_\mu{}^{21}, Q_\mu{}^{22} - \Lambda_\mu\}\}\}\}$

→ $A_\mu \to \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 & 0 \\ 0 & 0 & Q_\mu{}^{11} - \Lambda_\mu & Q_\mu{}^{12} \\ 0 & 0 & Q_\mu{}^{21} & Q_\mu{}^{22} - \Lambda_\mu \end{array}\right)$Null

```
PR[$pass1;
 "Compute the ϕ+i𝒟F terms: ",
 $ = Select[$pass1, ! FreeQ[#, ϕ] &],
 Yield, $ = $ /. (c_ : 1) (aa_ · (ee : eL | eR | νL | νR)) ⊗ a_ → c aa · (ee ⊗ a),
 Yield, $00 = $0 = $ = $ //. (c1_ : 1) aa_ · a1_ + (c2_ : 1) aa_ · a2_ ->
       aa · (c1 a1 + c2 a2) //. tuOpSimplify[CenterDot],
 NL, $p = Apply[List, $[[2]]];
 Yield,
 $p1 = Extract[$p,
   Position[$p, Apply[Alternatives, $s = tuRuleSelect[$dEW][ℋF[basis]][[1, 2]]]]];
 $s = FindPermutation[$s, $p1];
 $p = {Permute[$p, $s]} // Transpose;
 $0 = $ = $0[[1]] · $p; $ // MatrixForms,

 NL, "Using: ", $s = {tuRuleSelect[$dEW][𝒟F], tuRuleSelect[$p62][ϕ]};
 $s // MatrixForms,
 NL, "Compute: ", $ = $0[[1]] /. Plus → Inactive[Plus],
 Yield, $ = $ /. tuRuleSelect[$dEW][𝒟F] /. tuRuleSelect[$p62][ϕ] // Activate;
 $ // MatrixForms,
 Imply, $0[[1]] = $; $00 -> $0 // MatrixForms, CK,
 line,
 NL, "•Complete expression for: ",
 $ = Select[$pass1, FreeQ[#, ϕ] &] + $0 /. $sA // Simplify;
 $sDx = $ = $0a -> $;
 $ // MatrixForms // Framed
]
```

Compute the ϕ+i$\mathcal{D}_F$ terms:

$$((\phi+i\,\mathcal{D}_F)\cdot e_L)\otimes\psi^e{}_L-((\phi+i\,\mathcal{D}_F)\cdot e_R)\otimes\psi^e{}_R+((\phi+i\,\mathcal{D}_F)\cdot \nu_L)\otimes\psi^\nu{}_L-((\phi+i\,\mathcal{D}_F)\cdot \nu_R)\otimes\psi^\nu{}_R$$

$$\to (\phi+i\,\mathcal{D}_F)\cdot e_L\otimes\psi^e{}_L-(\phi+i\,\mathcal{D}_F)\cdot e_R\otimes\psi^e{}_R+(\phi+i\,\mathcal{D}_F)\cdot \nu_L\otimes\psi^\nu{}_L-(\phi+i\,\mathcal{D}_F)\cdot \nu_R\otimes\psi^\nu{}_R$$

$$\to (\phi+i\,\mathcal{D}_F)\cdot(e_L\otimes\psi^e{}_L-e_R\otimes\psi^e{}_R+\nu_L\otimes\psi^\nu{}_L-\nu_R\otimes\psi^\nu{}_R)$$

$$\to (\phi+i\,\mathcal{D}_F)\cdot\begin{pmatrix}-(\nu_R\otimes\psi^\nu{}_R)\\-(e_R\otimes\psi^e{}_R)\\\nu_L\otimes\psi^\nu{}_L\\e_L\otimes\psi^e{}_L\end{pmatrix}$$

Using: $\left(\begin{array}{c}\mathcal{D}_F\to\begin{pmatrix}0&0&-i\,m_\nu&0\\0&0&0&-i\,m_e\\i\,m_\nu&0&0&0\\0&i\,m_e&0&0\end{pmatrix}\\ \phi\to\begin{pmatrix}0&0&(\phi_1)^\dagger m_\nu&(\phi_2)^\dagger m_e\\0&0&-m_\nu\,\phi_2&m_e\,\phi_1\\-m_\nu\,\phi_1&(\phi_2)^\dagger m_e&0&0\\-m_\nu\,\phi_2&-(\phi_1)^\dagger m_e&0&0\end{pmatrix}\end{array}\right)$

Compute: $\phi+i\,\mathcal{D}_F$

$$\to\begin{pmatrix}0&0&m_\nu+(\phi_1)^\dagger m_\nu&(\phi_2)^\dagger m_e\\0&0&-m_\nu\,\phi_2&m_e+m_e\,\phi_1\\-m_\nu-m_\nu\,\phi_1&(\phi_2)^\dagger m_e&0&0\\-m_\nu\,\phi_2&-m_e-(\phi_1)^\dagger m_e&0&0\end{pmatrix}$$

$$\Rightarrow (\phi+i\,\mathcal{D}_F)\cdot(e_L\otimes\psi^e{}_L-e_R\otimes\psi^e{}_R+\nu_L\otimes\psi^\nu{}_L-\nu_R\otimes\psi^\nu{}_R)\to$$

$$\begin{pmatrix}0&0&m_\nu+(\phi_1)^\dagger m_\nu&(\phi_2)^\dagger m_e\\0&0&-m_\nu\,\phi_2&m_e+m_e\,\phi_1\\-m_\nu-m_\nu\,\phi_1&(\phi_2)^\dagger m_e&0&0\\-m_\nu\,\phi_2&-m_e-(\phi_1)^\dagger m_e&0&0\end{pmatrix}\cdot\begin{pmatrix}-(\nu_R\otimes\psi^\nu{}_R)\\-(e_R\otimes\psi^e{}_R)\\\nu_L\otimes\psi^\nu{}_L\\e_L\otimes\psi^e{}_L\end{pmatrix}\longleftarrow\text{CHECK}$$

•Complete expression for:

$$\boxed{\begin{array}{l}\mathcal{D}_A\cdot\xi\to\begin{pmatrix}0&0&0&0\\0&-2\,\Lambda_\mu&0&0\\0&0&Q_\mu{}^{11}-\Lambda_\mu&Q_\mu{}^{12}\\0&0&Q_\mu{}^{21}&Q_\mu{}^{22}-\Lambda_\mu\end{pmatrix}\cdot\begin{pmatrix}-(\nu_R\otimes(i^t\,\gamma^\mu\cdot\psi^\nu{}_R))\\-(e_R\otimes(i^t\,\gamma^\mu\cdot\psi^e{}_R))\\\nu_L\otimes(i^t\,\gamma^\mu\cdot\psi^\nu{}_L)\\e_L\otimes(i^t\,\gamma^\mu\cdot\psi^e{}_L)\end{pmatrix}+\\ \begin{pmatrix}0&0&(1+(\phi_1)^\dagger)\,m_\nu&(\phi_2)^\dagger m_e\\0&0&-m_\nu\,\phi_2&m_e\,(1+\phi_1)\\-m_\nu\,(1+\phi_1)&(\phi_2)^\dagger m_e&0&0\\-m_\nu\,\phi_2&-(1+(\phi_1)^\dagger)\,m_e&0&0\end{pmatrix}\cdot\begin{pmatrix}-(\nu_R\otimes\psi^\nu{}_R)\\-(e_R\otimes\psi^e{}_R)\\\nu_L\otimes\psi^\nu{}_L\\e_L\otimes\psi^e{}_L\end{pmatrix}+\\ e_L\otimes(i^t\,(\not{D})\cdot\psi^e{}_L)-e_R\otimes(i^t\,(\not{D})\cdot\psi^e{}_R)+\nu_L\otimes(i^t\,(\not{D})\cdot\psi^\nu{}_L)-\nu_R\otimes(i^t\,(\not{D})\cdot\psi^\nu{}_R)\end{array}}$$

```
PR["•Put remaining terms in vector form: ",
  Yield, $ = $sDx[[2]]; $ // ColumnSumExp;
  $p = $ // tuExtractPattern[CenterDot[__], 1];
  Yield, $p0 = $p = $ - Apply[Plus, $p]; $p = Apply[List, $p],
  Yield, $p1 = Extract[$p,
    Position[$p, Apply[Alternatives, tuRuleSelect[$dEW][ℋF[basis]][[1, 2]]]]],
  Yield, $s = tuRuleSelect[$dEW][ℋF[basis]][[1, 2]],
  Yield, $s = FindPermutation[$s, $p1],
  Yield, $p = {Permute[$p, $s]} // Transpose; $p // MatrixForm // Framed,
  Yield, $ = $sDx[[2]] - $p0 ; $ // MatrixForms // ColumnSumExp;
  Yield, $ = Inactive[Plus][$, $p];
  $ // MatrixForms // ColumnSumExp // Framed,
  NL, "Put in basis form: ",
  Yield, $sDx1 = $ = $0a -> $ /. CenterDot → Dot /. Dot → CenterDot // Activate;
  $ // MatrixForms // ColumnSumExp
 ];
PR[$ = $sDx1[[2]];
  $ = Apply[Plus, $][[1]];
  NL, "Apply Rules[]: Normal form: ",
  NL, $s = c (# ⊗ b) → # ⊗ (c b) & /@ $p1 // tuAddPatternVariable[{b, c}],
  Yield, $ = $ //. $s //. a_ ⊗ b_ + a_ ⊗ c_ → a ⊗ (b + c);
  $ // MatrixForms // ColumnSumExp,


  NL, "Order according to basis: ",
  Yield, $ = Apply[List, $],
  $p1 = Extract[$,
    Position[$, Apply[Alternatives, tuRuleSelect[$dEW][ℋF[basis]][[1, 2]]]]];
  $s = tuRuleSelect[$dEW][ℋF[basis]][[1, 2]];
  $s = FindPermutation[$s, $p1];
  $ = Permute[$, $s],
  $sDx2 = $ = $0a → $; $ // MatrixForms // ColumnSumExp
 ];
```

•Put remaining terms in vector form:

→

→ $\{e_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_L),\ -(e_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_R)),\ \nu_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_L),\ -(\nu_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_R))\}$

→ $\{e_L,\ e_R,\ \nu_L,\ \nu_R\}$

→ $\{\nu_R,\ e_R,\ \nu_L,\ e_L\}$

→ Cycles[{{1, 4}}]

$$\rightarrow\ \boxed{\begin{pmatrix} -(\nu_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_R)) \\ -(e_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_R)) \\ \nu_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_L) \\ e_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_L) \end{pmatrix}}$$

→

$$\rightarrow\ \boxed{\sum\left[\begin{array}{l} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 & 0 \\ 0 & 0 & Q_\mu{}^{11}-\Lambda_\mu & Q_\mu{}^{12} \\ 0 & 0 & Q_\mu{}^{21} & Q_\mu{}^{22}-\Lambda_\mu \end{pmatrix}\cdot\begin{pmatrix} -(\nu_R\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^\nu{}_R)) \\ -(e_R\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^e{}_R)) \\ \nu_L\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L) \\ e_L\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^e{}_L) \end{pmatrix} + \\ \begin{pmatrix} 0 & 0 & (1+(\phi_1)^\dagger)\,m_\nu & (\phi_2)^\dagger\,m_e \\ 0 & 0 & -m_\nu\,\phi_2 & m_e\,(1+\phi_1) \\ -m_\nu\,(1+\phi_1) & (\phi_2)^\dagger\,m_e & 0 & 0 \\ -m_\nu\,\phi_2 & -(1+(\phi_1)^\dagger)\,m_e & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} -(\nu_R\otimes\psi^\nu{}_R) \\ -(e_R\otimes\psi^e{}_R) \\ \nu_L\otimes\psi^\nu{}_L \\ e_L\otimes\psi^e{}_L \end{pmatrix} \\ \begin{pmatrix} -(\nu_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_R)) \\ -(e_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_R)) \\ \nu_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_L) \\ e_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_L) \end{pmatrix} \end{array}\right]}$$

Put in basis form:

$$\rightarrow\ \mathcal{D}_A\cdot\xi\to\begin{pmatrix} \sum\left[\begin{array}{l} -(\nu_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_R)) \\ e_L\otimes\psi^e{}_L\,(\phi_2)^\dagger\,m_e \\ \nu_L\otimes\psi^\nu{}_L\,(1+(\phi_1)^\dagger)\,m_\nu \end{array}\right] \\ \sum\left[\begin{array}{l} -(e_R\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_R)) \\ e_L\otimes\psi^e{}_L\,m_e\,(1+\phi_1) \\ -(\nu_L\otimes\psi^\nu{}_L)\,m_\nu\,\phi_2 \\ 2\,e_R\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^e{}_R)\,\Lambda_\mu \end{array}\right] \\ \sum\left[\begin{array}{l} \nu_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^\nu{}_L) \\ -(e_R\otimes\psi^e{}_R)\,(\phi_2)^\dagger\,m_e \\ \nu_R\otimes\psi^\nu{}_R\,m_\nu\,(1+\phi_1) \\ e_L\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^e{}_L)\,Q_\mu{}^{12} \\ \nu_L\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L)\,(Q_\mu{}^{11}-\Lambda_\mu) \end{array}\right] \\ \sum\left[\begin{array}{l} e_L\otimes(\mathbb{i}^t\,(\not{D})\cdot\psi^e{}_L) \\ e_R\otimes\psi^e{}_R\,(1+(\phi_1)^\dagger)\,m_e \\ \nu_R\otimes\psi^\nu{}_R\,m_\nu\,\phi_2 \\ \nu_L\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L)\,Q_\mu{}^{21} \\ e_L\otimes(\mathbb{i}^t\,\gamma^\mu\cdot\psi^e{}_L)\,(Q_\mu{}^{22}-\Lambda_\mu) \end{array}\right] \end{pmatrix}$$

Apply Rules[]: Normal form:

$\{e_L \otimes b\_\ c\_ \to e_L \otimes (b\,c),\ e_R \otimes b\_\ c\_ \to e_R \otimes (b\,c),\ \nu_L \otimes b\_\ c\_ \to \nu_L \otimes (b\,c),\ \nu_R \otimes b\_\ c\_ \to \nu_R \otimes (b\,c)\}$

$$\to \sum\left[\begin{array}{|l} e_L \otimes (\mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_L + (\phi_2)^\dagger\, m_e\, \psi^e{}_L + m_e\,(1+\phi_1)\,\psi^e{}_L + \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\, Q_\mu{}^{1\,2} + \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\,(Q_\mu{}^{2\,2} - \Lambda_\mu)) \\ e_R \otimes (-\mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_R + (1 + (\phi_1)^\dagger)\, m_e\, \psi^e{}_R - (\phi_2)^\dagger\, m_e\, \psi^e{}_R + 2\,\mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_R\,\Lambda_\mu) \\ \nu_L \otimes (\mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_L + (1 + (\phi_1)^\dagger)\, m_\nu\, \psi^\nu{}_L - m_\nu\,\phi_2\,\psi^\nu{}_L + \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\, Q_\mu{}^{2\,1} + \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\,(Q_\mu{}^{1\,1} - \Lambda_\mu)) \\ \nu_R \otimes (-\mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_R + m_\nu\,(1+\phi_1)\,\psi^\nu{}_R + m_\nu\,\phi_2\,\psi^\nu{}_R) \end{array}\right]$$

Order according to basis:

$$\begin{aligned} \to \{ & e_L \otimes (\mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_L + (\phi_2)^\dagger\, m_e\, \psi^e{}_L + m_e\,(1+\phi_1)\,\psi^e{}_L + \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\, Q_\mu{}^{1\,2} + \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\,(Q_\mu{}^{2\,2} - \Lambda_\mu)), \\ & e_R \otimes (-\mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_R + (1 + (\phi_1)^\dagger)\, m_e\, \psi^e{}_R - (\phi_2)^\dagger\, m_e\, \psi^e{}_R + 2\,\mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_R\,\Lambda_\mu), \\ & \nu_L \otimes (\mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_L + (1 + (\phi_1)^\dagger)\, m_\nu\, \psi^\nu{}_L - m_\nu\,\phi_2\,\psi^\nu{}_L + \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\, Q_\mu{}^{2\,1} + \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\,(Q_\mu{}^{1\,1} - \Lambda_\mu)), \\ & \nu_R \otimes (-\mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_R + m_\nu\,(1+\phi_1)\,\psi^\nu{}_R + m_\nu\,\phi_2\,\psi^\nu{}_R)\}\{\nu_R \otimes (-\mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_R + m_\nu\,(1+\phi_1)\,\psi^\nu{}_R + m_\nu\,\phi_2\,\psi^\nu{}_R), \\ & e_R \otimes (-\mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_R + (1 + (\phi_1)^\dagger)\, m_e\, \psi^e{}_R - (\phi_2)^\dagger\, m_e\, \psi^e{}_R + 2\,\mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_R\,\Lambda_\mu), \\ & \nu_L \otimes (\mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_L + (1 + (\phi_1)^\dagger)\, m_\nu\, \psi^\nu{}_L - m_\nu\,\phi_2\,\psi^\nu{}_L + \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\, Q_\mu{}^{2\,1} + \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\,(Q_\mu{}^{1\,1} - \Lambda_\mu)), \\ & e_L \otimes (\mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_L + (\phi_2)^\dagger\, m_e\, \psi^e{}_L + m_e\,(1+\phi_1)\,\psi^e{}_L + \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\, Q_\mu{}^{1\,2} + \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\,(Q_\mu{}^{2\,2} - \Lambda_\mu))\} \end{aligned}$$

$$\mathcal{D}_A \cdot \xi \to \{\nu_R \otimes \sum\left[\begin{array}{|l} -\mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_R \\ m_\nu\,(1+\phi_1)\,\psi^\nu{}_R \\ m_\nu\,\phi_2\,\psi^\nu{}_R \end{array}\right],\ e_R \otimes \sum\left[\begin{array}{|l} -\mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_R \\ (1 + (\phi_1)^\dagger)\, m_e\, \psi^e{}_R \\ -(\phi_2)^\dagger\, m_e\, \psi^e{}_R \\ 2\,\mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_R\,\Lambda_\mu \end{array}\right],$$

$$\nu_L \otimes \sum\left[\begin{array}{|l} \mathbb{i}^t\,(\not{D}) \cdot \psi^\nu{}_L \\ (1 + (\phi_1)^\dagger)\, m_\nu\, \psi^\nu{}_L \\ -m_\nu\,\phi_2\,\psi^\nu{}_L \\ \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\, Q_\mu{}^{2\,1} \\ \mathbb{i}^t\,\gamma^\mu \cdot \psi^\nu{}_L\,(Q_\mu{}^{1\,1} - \Lambda_\mu) \end{array}\right],\ e_L \otimes \sum\left[\begin{array}{|l} \mathbb{i}^t\,(\not{D}) \cdot \psi^e{}_L \\ (\phi_2)^\dagger\, m_e\, \psi^e{}_L \\ m_e\,(1+\phi_1)\,\psi^e{}_L \\ \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\, Q_\mu{}^{1\,2} \\ \mathbb{i}^t\,\gamma^\mu \cdot \psi^e{}_L\,(Q_\mu{}^{2\,2} - \Lambda_\mu) \end{array}\right]\}$$

```
PR["●Now compute: ", $ = $action,
 Yield, $ = $ /. $sDx2 /. $sjx1; $ // MatrixForms,
 Yield, $ = $ /. BraKet[a_, b_] → a · b,
 Yield, $[[1, 1]] = $[[1, 1]] // Transpose, CK,
 Yield, $ = $ /. CenterDot → xDot;
 Yield, $ = $ // OrderedxDotMultiplyAll[{m_, Tensor[Q, _, _]}];
 $ = $ /. Dot → CenterDot;
 Yield, $ = $ /. a_ ⊗ b_ · c_ ⊗ d_ → (a · c) ⊗ (b · d); $ // ColumnSumExp;
 NL, "Apply orthornormality of ", $p1,
 Yield, $ = $ //. (a_ · b_) ⊗ (c_ · d_) ⧴ 0 /; a =!= b /.
     (a_ · b_) ⊗ (c_ · d_) ⧴ c · d /; a === b /. tuOpSimplify[CenterDot];
 $ // ColumnSumExp
]
```

●Now compute: $\langle \mathcal{J}\cdot\xi \mid \mathcal{D}_A\cdot\xi\rangle_{\mathcal{J}}$

$$\to \left\langle \begin{pmatrix} -(\nu_R\otimes(\mathcal{J}_M\cdot\psi^\nu{}_R)) \\ -(e_R\otimes(\mathcal{J}_M\cdot\psi^e{}_R)) \\ \nu_L\otimes(\mathcal{J}_M\cdot\psi^\nu{}_L) \\ e_L\otimes(\mathcal{J}_M\cdot\psi^e{}_L) \end{pmatrix} \middle| \{\nu_R\otimes(-\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_R + m_\nu\,(1+\phi_1)\,\psi^\nu{}_R + m_\nu\,\phi_2\,\psi^\nu{}_R),\right.$$

$$e_R\otimes(-\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_R + (1+(\phi_1)^\dagger)\,m_e\,\psi^e{}_R - (\phi_2)^\dagger\,m_e\,\psi^e{}_R + 2\,\mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_R\,\Lambda_\mu),$$
$$\nu_L\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_L + (1+(\phi_1)^\dagger)\,m_\nu\,\psi^\nu{}_L - m_\nu\,\phi_2\,\psi^\nu{}_L + \mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L\,Q_\mu{}^{21} + \mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L\,(Q_\mu{}^{11}-\Lambda_\mu)),$$
$$\left. e_L\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_L + (\phi_2)^\dagger\,m_e\,\psi^e{}_L + m_e\,(1+\phi_1)\,\psi^e{}_L + \mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_L\,Q_\mu{}^{12} + \mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_L\,(Q_\mu{}^{22}-\Lambda_\mu))\}\right\rangle_{\mathcal{J}}$$

$$\to \{\{-(\nu_R\otimes(\mathcal{J}_M\cdot\psi^\nu{}_R))\},\{-(e_R\otimes(\mathcal{J}_M\cdot\psi^e{}_R))\},\{\nu_L\otimes(\mathcal{J}_M\cdot\psi^\nu{}_L)\},\{e_L\otimes(\mathcal{J}_M\cdot\psi^e{}_L)\}\}\cdot$$
$$\{\nu_R\otimes(-\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_R + m_\nu\,(1+\phi_1)\,\psi^\nu{}_R + m_\nu\,\phi_2\,\psi^\nu{}_R),$$
$$e_R\otimes(-\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_R + (1+(\phi_1)^\dagger)\,m_e\,\psi^e{}_R - (\phi_2)^\dagger\,m_e\,\psi^e{}_R + 2\,\mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_R\,\Lambda_\mu),$$
$$\nu_L\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_L + (1+(\phi_1)^\dagger)\,m_\nu\,\psi^\nu{}_L - m_\nu\,\phi_2\,\psi^\nu{}_L + \mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L\,Q_\mu{}^{21} + \mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L\,(Q_\mu{}^{11}-\Lambda_\mu)),$$
$$e_L\otimes(\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_L + (\phi_2)^\dagger\,m_e\,\psi^e{}_L + m_e\,(1+\phi_1)\,\psi^e{}_L + \mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_L\,Q_\mu{}^{12} + \mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_L\,(Q_\mu{}^{22}-\Lambda_\mu))\}_{\mathcal{J}}$$

$\to \{\{-(\nu_R\otimes(\mathcal{J}_M\cdot\psi^\nu{}_R)), -(e_R\otimes(\mathcal{J}_M\cdot\psi^e{}_R)), \nu_L\otimes(\mathcal{J}_M\cdot\psi^\nu{}_L), e_L\otimes(\mathcal{J}_M\cdot\psi^e{}_L)\}\}$ ⟵CHECK

→

→

→

Apply orthornormality of $\{e_L, e_R, \nu_L, \nu_R\}$

$$\to \{\textstyle\sum[\left| \begin{array}{l} -((-\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_R + 2\,\mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_R\cdot\Lambda_\mu + (1+(\phi_1)^\dagger)\cdot\psi^e{}_R\,m_e - (\phi_2)^\dagger\cdot\psi^e{}_R\,m_e)\cdot\mathcal{J}_M\cdot\psi^e{}_R) \\ -((-\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_R + (1+\phi_1)\cdot\psi^\nu{}_R\,m_\nu + \phi_2\cdot\psi^\nu{}_R\,m_\nu)\cdot\mathcal{J}_M\cdot\psi^\nu{}_R) \\ (\mathrm{i}^t\,(\not{D})\cdot\psi^e{}_L + \mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_L\cdot(Q_\mu{}^{22}-\Lambda_\mu) + (\phi_2)^\dagger\cdot\psi^e{}_L\,m_e + (1+\phi_1)\cdot\psi^e{}_L\,m_e + \mathrm{i}^t\,\gamma^\mu\cdot\psi^e{}_L\,Q_\mu{}^{12})\cdot\mathcal{J}_M\cdot\psi^e{}_L \\ (\mathrm{i}^t\,(\not{D})\cdot\psi^\nu{}_L + \mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L\cdot(Q_\mu{}^{11}-\Lambda_\mu) + (1+(\phi_1)^\dagger)\cdot\psi^\nu{}_L\,m_\nu - \phi_2\cdot\psi^\nu{}_L\,m_\nu + \mathrm{i}^t\,\gamma^\mu\cdot\psi^\nu{}_L\,Q_\mu{}^{21})\cdot\mathcal{J}_M\cdot\psi^\nu{}_L \end{array}\right.]\}_{\mathcal{J}}$$

```
PR["•The Higgs terms that were missed: ",
 $0 = BraKet[𝒥M · ΨR, Φ · ΨL] + BraKet[𝒥M · ΨL, ct[Φ] · ΨR];
 Yield, $ = $0[[1]],
 NL, "Where: ",
 $s = {tuRuleSelect[$e5][ΨR|L], tuRuleSelect[$p63][Φ]} // Flatten[#, 1] & //
    tuConjugateTransposeSimplify[{}, {ϕ_}] // Simplify,
 Yield, $ = $ /. $s; $ // MatrixForms;
 Yield, $ = $ /. CenterDot → Dot // tuConjugateSimplify[{me|ν}] // Simplify;
 $ // MatrixForms,
 NL, "The term: ", $ = $0[[2]],
 Yield, $ = $ /. $s; $ // MatrixForms;
 Yield, $ = $ /. CenterDot → Dot // tuConjugateSimplify[{me|ν}] // Simplify;
 $ // MatrixForms
]
```

•The Higgs terms that were missed:

→ $\left\langle \mathcal{J}_M \cdot \Psi_L \mid \Phi^\dagger \cdot \Psi_R \right\rangle$

Where:

$\{\Psi_L \to \{\{\psi^\nu{}_L\}, \{\psi^e{}_L\}\}, \Psi_R \to \{\{\psi^\nu{}_R\}, \{\psi^e{}_R\}\}, \Phi \to \{\{-(1 + (\phi_1)^*)\, m_\nu, -(\phi_2)^*\, m_\nu\}, \{m_e\, \phi_2, -m_e\, (1 + \phi_1)\}\}\}$

→

→ $\left\langle \mathcal{J}_M \cdot \begin{pmatrix} \psi^\nu{}_L \\ \psi^e{}_L \end{pmatrix} \middle| \begin{pmatrix} (\phi_2)^*\, m_e\, \psi^e{}_R - m_\nu\, (1 + \phi_1)\, \psi^\nu{}_R \\ -(1 + (\phi_1)^*)\, m_e\, \psi^e{}_R - m_\nu\, \phi_2\, \psi^\nu{}_R \end{pmatrix} \right\rangle$

The term: $\left\langle \mathcal{J}_M \cdot \Psi_R \mid \Phi \cdot \Psi_L \right\rangle$

→

→ $\left\langle \mathcal{J}_M \cdot \begin{pmatrix} \psi^\nu{}_R \\ \psi^e{}_R \end{pmatrix} \middle| \begin{pmatrix} m_\nu\, (-(\phi_2)^*\, \psi^e{}_L - (1 + (\phi_1)^*)\, \psi^\nu{}_L) \\ m_e\, (-(1 + \phi_1)\, \psi^e{}_L + \phi_2\, \psi^\nu{}_L) \end{pmatrix} \right\rangle$

```
PR["●Now compute(Explore the meaning of the BraKet[]): ", $ = $action,
 Yield, $ = $ /. $sDx1 /. $sjx1; $ // MatrixForms;
 Yield, $ = $ /. BraKet[a_, b_] → Transpose[a] · b // First, $ // ColumnSumExp;
 Yield, $ = $ // tuMatrixOrderedMultiply // tuOpSimplifyF[CenterDot];
 Yield, $ = $ //. tuOpDistribute[CenterDot] //.
    tuOpSimplify[CenterDot, {Tensor[∧ | Q, _, _], m_}];
 $ // ColumnSumExp;
 Yield, $00 = $ = $ /. a_ ⊗ b_ · c_ ⊗ d_ → (a · c) ⊗ (b · d); $ // ColumnSumExp;

 NL, "Look at Higgs terms: ",
 $ = Select[$[[1, 1]], FreeQ[#, D | ∧] && FreeQ[#, Q] &]; $ // ColumnSumExp,
 NL, "Ignore finite space.",
 $ = $ /. (a_ ⊗ b_) → b //. tuOpSimplify[CenterDot, {ϕ_, ct[ϕ_]}];
 $ // ColumnSumExp;
 $ = $ /. (a_ · b_) c__ → a · (c b);
 NL, "Select and combine terms with ", $s = 𝒥M • (ψ−)L,
 $ = Select[$, ! FreeQ[#, $s] &];
 Yield, $ = $ //. CenterDot[a_, b_] + CenterDot[a_, c_] → a · (b + c) // Simplify;
 $ // ColumnSumExp,
 NL, CR["This is the same as above. Does this mean
     we can ignore finite space in this part of the calculation?"]
]
```

●Now compute(Explore the meaning of the BraKet[]): $\langle \mathcal{J}\cdot\xi \mid \mathcal{D}_A\cdot\xi\rangle_{\mathcal{J}}$

→

→ $\{\{-(\nu_R\otimes(\mathcal{J}_M\cdot\psi^\nu{}_R)),\ -(e_R\otimes(\mathcal{J}_M\cdot\psi^e{}_R)),\ \nu_L\otimes(\mathcal{J}_M\cdot\psi^\nu{}_L),\ e_L\otimes(\mathcal{J}_M\cdot\psi^e{}_L)\}\}\cdot$
$\{\{-(\nu_R\otimes(i^t\,(\not{D})\cdot\psi^\nu{}_R))+e_L\otimes\psi^e{}_L\,(\phi_2)^\dagger\,m_e+\nu_L\otimes\psi^\nu{}_L\,(1+(\phi_1)^\dagger)\,m_\nu\},$
$\{-(e_R\otimes(i^t\,(\not{D})\cdot\psi^e{}_R))+e_L\otimes\psi^e{}_L\,m_e\,(1+\phi_1)-\nu_L\otimes\psi^\nu{}_L\,m_\nu\,\phi_2+2\,e_R\otimes(i^t\,\gamma^\mu\cdot\psi^e{}_R)\,\Lambda_\mu\},$
$\{\nu_L\otimes(i^t\,(\not{D})\cdot\psi^\nu{}_L)-e_R\otimes\psi^e{}_R\,(\phi_2)^\dagger\,m_e+\nu_R\otimes\psi^\nu{}_R\,m_\nu\,(1+\phi_1)+e_L\otimes(i^t\,\gamma^\mu\cdot\psi^e{}_L)\,Q_\mu{}^{12}+$
$\nu_L\otimes(i^t\,\gamma^\mu\cdot\psi^\nu{}_L)\,(Q_\mu{}^{11}-\Lambda_\mu)\},\ \{e_L\otimes(i^t\,(\not{D})\cdot\psi^e{}_L)+e_R\otimes\psi^e{}_R\,(1+(\phi_1)^\dagger)\,m_e+$
$\nu_R\otimes\psi^\nu{}_R\,m_\nu\,\phi_2+\nu_L\otimes(i^t\,\gamma^\mu\cdot\psi^\nu{}_L)\,Q_\mu{}^{21}+e_L\otimes(i^t\,\gamma^\mu\cdot\psi^e{}_L)\,(Q_\mu{}^{22}-\Lambda_\mu)\}\}$

→

→

→

Look at Higgs terms: $\sum\left[\begin{array}{|l}
(e_L\cdot e_R)\otimes(\mathcal{J}_M\bullet\psi^e{}_L\cdot\psi^e{}_R)\cdot(\phi_1)^\dagger\,m_e\\
-((e_R\cdot e_L)\otimes(\mathcal{J}_M\bullet\psi^e{}_R\cdot\psi^e{}_L)\cdot\phi_1)\,m_e\\
-((\nu_L\cdot e_R)\otimes(\mathcal{J}_M\bullet\psi^\nu{}_L\cdot\psi^e{}_R)\cdot(\phi_2)^\dagger)\,m_e\\
-((\nu_R\cdot e_L)\otimes(\mathcal{J}_M\bullet\psi^\nu{}_R\cdot\psi^e{}_L)\cdot(\phi_2)^\dagger)\,m_e\\
(e_L\cdot e_R)\otimes(\mathcal{J}_M\bullet\psi^e{}_L\cdot\psi^e{}_R)\,m_e\\
-((e_R\cdot e_L)\otimes(\mathcal{J}_M\bullet\psi^e{}_R\cdot\psi^e{}_L))\,m_e\\
(e_L\cdot\nu_R)\otimes(\mathcal{J}_M\bullet\psi^e{}_L\cdot\psi^\nu{}_R)\cdot\phi_2\,m_\nu\\
(e_R\cdot\nu_L)\otimes(\mathcal{J}_M\bullet\psi^e{}_R\cdot\psi^\nu{}_L)\cdot\phi_2\,m_\nu\\
(\nu_L\cdot\nu_R)\otimes(\mathcal{J}_M\bullet\psi^\nu{}_L\cdot\psi^\nu{}_R)\cdot\phi_1\,m_\nu\\
-((\nu_R\cdot\nu_L)\otimes(\mathcal{J}_M\bullet\psi^\nu{}_R\cdot\psi^\nu{}_L)\cdot(\phi_1)^\dagger)\,m_\nu\\
(\nu_L\cdot\nu_R)\otimes(\mathcal{J}_M\bullet\psi^\nu{}_L\cdot\psi^\nu{}_R)\,m_\nu\\
-((\nu_R\cdot\nu_L)\otimes(\mathcal{J}_M\bullet\psi^\nu{}_R\cdot\psi^\nu{}_L))\,m_\nu
\end{array}\right]$

Ignore finite space.

Select and combine terms with $\mathcal{J}_M\bullet\psi-_L$

→ $\sum\left[\begin{array}{|l}
\mathcal{J}_M\bullet\psi^e{}_L\cdot((1+(\phi_1)^\dagger)\,m_e\,\psi^e{}_R+m_\nu\,\phi_2\,\psi^\nu{}_R)\\
\mathcal{J}_M\bullet\psi^\nu{}_L\cdot(-(\phi_2)^\dagger\,m_e\,\psi^e{}_R+m_\nu\,(1+\phi_1)\,\psi^\nu{}_R)
\end{array}\right]$

This is the same as above. Does this mean
we can ignore finite space in this part of the calculation?

```
PR["Redo calculation without imposing orthogonality of finite space.",
 $ = $00[[1, 1]];
 $ = $ /. (a_ ⊗ b_) → b //. tuOpSimplify[CenterDot, {ϕ_, ct[ϕ_]}];
 $ // ColumnSumExp;
 NL, "Ignore Higgs terms that we did previously.",
 $ = Select[$, ! FreeQ[#, D | Λ | Q] &];
 $ = $ /. (a_ · b_) c__ → a · (c b);
 $ // ColumnSumExp,
 NL, "Same results as Proposition 6.2.",
 CR["Apparently the orthogonality of the
    finite space does not affect the action of the M-space.??"]
]
```

Redo calculation without imposing orthogonality of finite space.

Ignore Higgs terms that we did previously. $\sum\left[\begin{array}{l} \mathcal{J}_M \cdot \psi^e{}_L \cdot (\mathbb{i}^t (\not{D}) \cdot \psi^e{}_L) \\ \mathcal{J}_M \cdot \psi^e{}_L \cdot (\mathbb{i}^t \gamma^\mu \cdot \psi^\nu{}_L Q_\mu{}^{21}) \\ \mathcal{J}_M \cdot \psi^e{}_L \cdot (\mathbb{i}^t \gamma^\mu \cdot \psi^e{}_L Q_\mu{}^{22}) \\ \mathcal{J}_M \cdot \psi^e{}_L \cdot (-\mathbb{i}^t \gamma^\mu \cdot \psi^e{}_L \Lambda_\mu) \\ \mathcal{J}_M \cdot \psi^e{}_R \cdot (\mathbb{i}^t (\not{D}) \cdot \psi^e{}_R) \\ \mathcal{J}_M \cdot \psi^e{}_R \cdot (-2\, \mathbb{i}^t \gamma^\mu \cdot \psi^e{}_R \Lambda_\mu) \\ \mathcal{J}_M \cdot \psi^\nu{}_L \cdot (\mathbb{i}^t (\not{D}) \cdot \psi^\nu{}_L) \\ \mathcal{J}_M \cdot \psi^\nu{}_L \cdot (\mathbb{i}^t \gamma^\mu \cdot \psi^\nu{}_L Q_\mu{}^{11}) \\ \mathcal{J}_M \cdot \psi^\nu{}_L \cdot (\mathbb{i}^t \gamma^\mu \cdot \psi^e{}_L Q_\mu{}^{12}) \\ \mathcal{J}_M \cdot \psi^\nu{}_L \cdot (-\mathbb{i}^t \gamma^\mu \cdot \psi^\nu{}_L \Lambda_\mu) \\ \mathcal{J}_M \cdot \psi^\nu{}_R \cdot (\mathbb{i}^t (\not{D}) \cdot \psi^\nu{}_R) \end{array}\right]$

Same results as Proposition 6.2.
Apparently the orthogonality of the finite space does
not affect the action of the M-space.??

6.1 Majorana masses

```
PR["Accomodate Majorana masses by
   doubling the Hilbert space and adding real structure to: ",
 $ = {𝓗F, tuRuleSelect[$dEW][𝓗F[basis]][[1, 2]]},
 Yield, $ = {𝓗̄F, OverBar[#] & /@ tuRuleSelect[$dEW][𝓗F[basis]][[1, 2]]},
 NL, "Interpret as anti-particles.",
 NL, "•The new space: ",
 $ewm = $ = {𝓗̂F → 𝓗F ⊕ 𝓗̄F, 𝒟̂F[CG["mass matrix, Krein-self-adjoint"]],
    𝒥̂F[CG["fundametal symmetry"]], Γ̂F[CG["grading"]], ĴF[CG["real structure"]],
    𝒟̂F → {{𝒟F, -ct[𝒟M]}, {𝒟M, Conjugate[𝒟F]}},
    𝒥̂F → {{1, 0}, {0, -1}},
    Γ̂F → {{ΓF, 0}, {0, -ΓF}},
    ĴF → {{0, CC}, {CC, 0}},
    {𝒟M[𝓗F] -> 𝓗̄F, 𝒟M[CG["Part of 𝒟A that mixes 𝓗F and 𝓗̄F."]]},
    𝒟M · νR → I mR ν̄R,
    mR[CG["Majorana mass"]] ∈ ℝ,
    𝒟M · (eR | νL | eL) → 0,
    CommutatorP[ĴF, 𝒥̂F | Γ̂F] → 0,
    π̂ · 𝒜F → ℬ[𝓗̂F],
    π̂^op · 𝒜F^op → ℬ[𝓗̂F],
    π̂[a] → π[a] ⊕ π^op[a^T],
    π̂^op[a] → ĴF · π̂[ct[a]] · ĴF,
    F̂EW → {𝒜F ⊙ 𝒜F^op, 𝓗̂F, 𝒟̂F, 𝒥̂F}

   }; $ // Column
]
```

Accomodate Majorana masses by doubling the Hilbert space and adding real structure to: $\{\mathcal{H}_F, \{\nu_R, e_R, \nu_L, e_L\}\}$

→ $\{\mathcal{H}_{\bar{F}}, \{\bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}\}$

Interpret as anti-particles.

•The new space:

$\hat{\mathcal{H}}_F \to \mathcal{H}_F \oplus \mathcal{H}_{\bar{F}}$

$\hat{\mathcal{D}}_F$[mass matrix, Krein-self-adjoint]

$\hat{\mathcal{J}}_F$[fundametal symmetry]

$\hat{\Gamma}_F$[grading]

$\hat{J}_F$[real structure]

$\hat{\mathcal{D}}_F \to \{\{\mathcal{D}_F, -(\mathcal{D}_M)^\dagger\}, \{\mathcal{D}_M, (\mathcal{D}_F)^*\}\}$

$\hat{\mathcal{J}}_F \to \{\{1, 0\}, \{0, -1\}\}$

$\hat{\Gamma}_F \to \{\{\Gamma_F, 0\}, \{0, -\Gamma_F\}\}$

$\hat{J}_F \to \{\{0, CC\}, \{CC, 0\}\}$

$\{\mathcal{D}_M[\mathcal{H}_F] \to \mathcal{H}_{\bar{F}}, \mathcal{D}_M$[Part of $\mathcal{D}_A$ that mixes $\mathcal{H}_F$ and $\mathcal{H}_{\bar{F}}$.]}

$\mathcal{D}_M \cdot \nu_R \to i\, \bar{\nu}_R\, m_R$

m_R[Majorana mass] $\in \mathbb{R}$

$\mathcal{D}_M \cdot (e_R \mid \nu_L \mid e_L) \to 0$

$\{\hat{J}_F, \hat{\mathcal{J}}_F \mid \hat{\Gamma}_F\} \to 0$

$\hat{\pi} \cdot \mathcal{A}_F \to \mathcal{B}[\hat{\mathcal{H}}_F]$

$\hat{\pi}^{op} \cdot \mathcal{A}_F^{op} \to \mathcal{B}[\hat{\mathcal{H}}_F]$

$\hat{\pi}[a] \to \pi[a] \oplus \pi^{op}[a^T]$

$\hat{\pi}^{op}[a] \to \hat{J}_F \cdot \hat{\pi}[a^\dagger] \cdot \hat{J}_F$

$\hat{F}_{EW} \to \{\mathcal{A}_F \odot \mathcal{A}_F^{op}, \hat{\mathcal{H}}_F, \hat{\mathcal{D}}_F, \hat{\mathcal{J}}_F\}$

```
PR["A 4-dimensional Lorentzian spin manifold: ",
  $ = {M[CG["Krein spectral triple, J_M provides real structure"]],
    J_M[CG["Charge conjugation"]],
    CommutatorM[J_M, slash[D]] → 0,
    CommutatorP[J_M, Γ_M] → 0,
    J_M · J_M → -1,
    𝒥_M → I^(t (t - 1) / 2) γ[e_1] · … · γ[e_t]
   };
  $ // Column
 ];
PR["Consider ACM: ",
 $acmFEWM = $ = {F̂_EW × M, {J -> Ĵ_F ⊗ J_M, J[CG["real structure"]]},
    CO["To reduce the doubling of DOF:"],
    ForAll[η, {η ∈ ℋ^"0", Γ · η → η, J · η → η, J · J → 1, ξ ∈ (ℋ_F ⊗ L^2[S])^"0"}, η → ξ + J · ξ],
    CO["Fermionic action:"],
    BraKet[𝒥 · η, 𝒟_A · η]_𝒥 -> BraKet[𝒥 · ξ, 𝒟_A · ξ]_𝒥 +
      BraKet[𝒥 · J · ξ, 𝒟_A · ξ]_𝒥 + BraKet[𝒥 · ξ, 𝒟_A · J · ξ]_𝒥 + BraKet[𝒥 · J · ξ, 𝒟_A · J · ξ]_𝒥,
    CommutatorM[J, 𝒟_A | 𝒥] → 0,
    BraKet[𝒥 · J · ξ, 𝒟_A · J · ξ]_𝒥 -> BraKet[𝒥 · ξ, 𝒟_A · ξ]_𝒥
   }; $ // Column,
 NL, "•For: ", $x = (ξ^∨)_R → ν_R ⊗ (ψ^∨)_R,
 Yield, $0 = $ = J · (ξ^∨)_R,
 Yield, $ = $ /. tuRuleSelect[$acmFEWM][J] /. $x // tuCircleTimesGather[],
 CR["Minus sign in text."],
 NL, "Since ", $s = tuRuleSelect[$ewm][Ĵ_F] // First,
 Yield, $s = $s[[1]] · a_ :> Conjugate[a],
 Yield, $ = $ /. $s; $jx = $0 → $; $jx // Framed, CR["Minus sign in text."],
 NL, "•For ", $0 = 𝒥 · # & /@ ($0 → $),
 NL, "Use ", $s = 𝒥 → 𝒥̂_F ⊗ 𝒥_M,
 Yield, $ = $0[[2]] /. $s,
 Yield, $ = $ // tuCircleTimesGather[],
 NL, "Since ", $s = tuRuleSelect[$ewm][𝒥̂_F] // First,
 Imply, $jJx = $ = $0[[1]] → ($ /. 𝒥̂_F · Conjugate[ν_R] → -Conjugate[ν_R]);
 $ // Framed, CR["Plus sign in text."]
]
PR["For ", $ = $0 = (I 𝒟_M ⊗ 1) · $x[[1]],
  Yield, $ = $ /. $x,
  Yield, $ = $ /. tuOpSimplify[CenterDot] // tuCircleTimesGather[],
  Yield, $ = $0 → ($ /. tuRuleSelect[$ewm][𝒟_M · ν_R] //. tuOpSimplify[CenterDot] //.
      tuOpSimplify[CircleTimes, {m_R}]);
  $ // Framed,
  NL, "For ", $ = $0 = (-I ct[𝒟_M] ⊗ 1) · J · $x[[1]],
  Yield, $ = $ /. $jx,
  Yield, $ =
   $ /. tuOpSimplify[CenterDot] // tuCircleTimesGather[] // tuOpSimplifyF[CenterDot],
  Yield, $ = $0 → ($ /. ct[𝒟_M] · Conjugate[ν_R] → -I ν_R m_R //.
      tuOpSimplify[CircleTimes, {m_R}]);
  $ // Framed, CR["Plus sign in text."]
 ];
```

A 4-dimensional Lorentzian spin manifold:

M[Krein spectral triple, J_M provides real structure]

J_M[Charge conjugation]

$[J_M, \not{D}] \to 0$

$\{J_M, \Gamma_M\} \to 0$

$J_M \cdot J_M \to -1$

$\mathcal{J}_M \to (-1)^{\frac{1}{4}(-1+t)\, t}\, \gamma[e_1] \cdot \ldots \cdot \gamma[e_t]$

Consider ACM:

$\hat{F}_{EW} \times M$

$\{J \to \hat{J}_F \otimes J_M,\ J[\text{real structure}]\}$

To reduce the doubling of DOF:

$\forall_{\eta, \{\eta \in \mathcal{H}^0, \Gamma \cdot \eta \to \eta, J \cdot \eta \to \eta, J \cdot J \to 1, \xi \in (\mathcal{H}_F \otimes L^2[S])^0\}} (\eta \to \xi + J \cdot \xi)$

Fermionic action:

$\langle \mathcal{J} \cdot \eta \mid \mathcal{D}_A \cdot \eta \rangle_{\mathcal{J}} \to \langle \mathcal{J} \cdot \xi \mid \mathcal{D}_A \cdot \xi \rangle_{\mathcal{J}} + \langle \mathcal{J} \cdot \xi \mid \mathcal{D}_A \cdot J \cdot \xi \rangle_{\mathcal{J}} + \langle \mathcal{J} \cdot J \cdot \xi \mid \mathcal{D}_A \cdot \xi \rangle_{\mathcal{J}} + \langle \mathcal{J} \cdot J \cdot \xi \mid \mathcal{D}_A \cdot J \cdot \xi \rangle_{\mathcal{J}}$

$[J, \mathcal{D}_A \mid \mathcal{J}] \to 0$

$\langle \mathcal{J} \cdot J \cdot \xi \mid \mathcal{D}_A \cdot J \cdot \xi \rangle_{\mathcal{J}} \to \langle \mathcal{J} \cdot \xi \mid \mathcal{D}_A \cdot \xi \rangle_{\mathcal{J}}$

•For: $\xi^{\vee}{}_R \to \nu_R \otimes \psi^{\vee}{}_R$

→ $J \cdot \xi^{\vee}{}_R$

→ $(\hat{J}_F \cdot \nu_R) \otimes (J_M \cdot \psi^{\vee}{}_R)$ Minus sign in text.

Since $\hat{J}_F \to \{\{0, CC\}, \{CC, 0\}\}$

→ $\hat{J}_F \cdot a\_ :\to a^*$

→ $\boxed{J \cdot \xi^{\vee}{}_R \to (\nu_R)^* \otimes (J_M \cdot \psi^{\vee}{}_R)}$ Minus sign in text.

•For $\mathcal{J} \cdot J \cdot \xi^{\vee}{}_R \to \mathcal{J} \cdot (\nu_R)^* \otimes (J_M \cdot \psi^{\vee}{}_R)$

Use $\mathcal{J} \to \hat{\mathcal{J}}_F \otimes \mathcal{J}_M$

→ $\hat{\mathcal{J}}_F \otimes \mathcal{J}_M \cdot (\nu_R)^* \otimes (J_M \cdot \psi^{\vee}{}_R)$

→ $(\hat{\mathcal{J}}_F \cdot (\nu_R)^*) \otimes (\mathcal{J}_M \cdot J_M \cdot \psi^{\vee}{}_R)$

Since $\hat{\mathcal{J}}_F \to \{\{1, 0\}, \{0, -1\}\}$

⇒ $\boxed{\mathcal{J} \cdot J \cdot \xi^{\vee}{}_R \to -(\nu_R)^* \otimes (\mathcal{J}_M \cdot J_M \cdot \psi^{\vee}{}_R)}$ Plus sign in text.

For $(i\, \mathcal{D}_M \otimes 1) \cdot \xi^{\vee}{}_R$

→ $(i\, \mathcal{D}_M \otimes 1) \cdot \nu_R \otimes \psi^{\vee}{}_R$

→ $i\, (\mathcal{D}_M \cdot \nu_R) \otimes (1 \cdot \psi^{\vee}{}_R)$

→ $\boxed{(i\, \mathcal{D}_M \otimes 1) \cdot \xi^{\vee}{}_R \to -(\overline{\nu_R} \otimes \psi^{\vee}{}_R)\, m_R}$

For $(-i\, (\mathcal{D}_M)^{\dagger} \otimes 1) \cdot J \cdot \xi^{\vee}{}_R$

→ $(-i\, (\mathcal{D}_M)^{\dagger} \otimes 1) \cdot (\nu_R)^* \otimes (J_M \cdot \psi^{\vee}{}_R)$

→ $-i\, ((\mathcal{D}_M)^{\dagger} \cdot (\nu_R)^*) \otimes (J_M \cdot \psi^{\vee}{}_R)$

→ $\boxed{(-i\, (\mathcal{D}_M)^{\dagger} \otimes 1) \cdot J \cdot \xi^{\vee}{}_R \to -(\nu_R \otimes (J_M \cdot \psi^{\vee}{}_R))\, m_R}$ Plus sign in text.

```
PR["The components of the action: ",
 $0 = tuRuleSelect[$acmFEWM][BraKet[_, _]𝒥][[1, 2, {2, 3}]],
 NL, CR["Is text only considering ξ→", $x, "?"],
 Yield, $ = $0[[1]],
 yield, $ = $ /. ξ -> $x[[1]],
 Yield, $1 = $ /. $jx,
 aside,
 NL, "It appears that text uses:
For: ", {$$ = $ = 𝒥 · # & /@ $x, $s = 𝒥 → 1 ⊗ 𝒥M},
 Imply, $[[2]] = $[[2]] /. $s,
 yield, $[[2]] = $[[2]] // tuCircleTimesGather[] // tuOpSimplifyF[CenterDot];
 ($jxR = $) // Framed,
 NL, "For: ",
 {$$ = $ = 𝒥 · J · # & /@ $x, $s = 𝒥 → 1 ⊗ 𝒥M},
 NL, "we use: ", $jJx,
 asideout,
 Yield, $ = $1 /. $jxR
]
$acmFEWM;
tuRuleSelect[$acmFEWM][J];
tuRuleSelect[$p62][𝒟][[1]];
```

The components of the action: $\langle \mathcal{J}\cdot\xi \mid \mathcal{D}_A\cdot J\cdot\xi\rangle_{\mathcal{J}} + \langle \mathcal{J}\cdot J\cdot\xi \mid \mathcal{D}_A\cdot\xi\rangle_{\mathcal{J}}$

Is text only considering $\xi \to \xi^{\vee}{}_R \to \nu_R \otimes \psi^{\vee}{}_R$?

$\rightarrow \langle \mathcal{J}\cdot\xi \mid \mathcal{D}_A\cdot J\cdot\xi\rangle_{\mathcal{J}} \longrightarrow \langle \mathcal{J}\cdot\xi^{\vee}{}_R \mid \mathcal{D}_A\cdot J\cdot\xi^{\vee}{}_R\rangle_{\mathcal{J}}$

$\rightarrow \langle \mathcal{J}\cdot\xi^{\vee}{}_R \mid \mathcal{D}_A\cdot(\nu_R)^*\otimes(J_M\cdot\psi^{\vee}{}_R)\rangle_{\mathcal{J}}$

←←←←←←←←Side Note

It appears that text uses:

For: $\{\mathcal{J}\cdot\xi^{\vee}{}_R \to \mathcal{J}\cdot\nu_R\otimes\psi^{\vee}{}_R,\ \mathcal{J}\to 1\otimes\mathcal{J}_M\}$

$\Rightarrow 1\otimes\mathcal{J}_M\cdot\nu_R\otimes\psi^{\vee}{}_R \longrightarrow \boxed{\mathcal{J}\cdot\xi^{\vee}{}_R \to \nu_R\otimes(\mathcal{J}_M\cdot\psi^{\vee}{}_R)}$

For: $\{\mathcal{J}\cdot J\cdot\xi^{\vee}{}_R \to \mathcal{J}\cdot J\cdot\nu_R\otimes\psi^{\vee}{}_R,\ \mathcal{J}\to 1\otimes\mathcal{J}_M\}$

we use: $\mathcal{J}\cdot J\cdot\xi^{\vee}{}_R \to -(\nu_R)^*\otimes(\mathcal{J}_M\cdot J_M\cdot\psi^{\vee}{}_R)$

→→→→→→→→

$\rightarrow \langle \nu_R\otimes(\mathcal{J}_M\cdot\psi^{\vee}{}_R) \mid \mathcal{D}_A\cdot(\nu_R)^*\otimes(J_M\cdot\psi^{\vee}{}_R)\rangle_{\mathcal{J}}$

7 The Standard Model

```
PR["The algebra: ",
 $ = { 𝒜F → ℂ ⊕ ℍ ⊕ M3[ℂ],
   ℋF → (ℋR ⊕ ℋL) ⊗ ℂ^3 [CG["3-generations"]],
   ℋR ∈ ℂ^2 ⊕ {ℂ^2 ⊗ ℂ^3}, CG["R→right handed particles"],
   ℋL ∈ ℂ^2 ⊕ {ℂ^2 ⊗ ℂ^3}, CG["L→left handed particles"],
   ℂ^2 ⊕ {ℂ^2 ⊗ ℂ^3} -> ℂ^2 [CG[ν, e]] ⊕ {ℂ^2 ⊗ ℂ^3}[CG[{u^c, d^c}, c → {r, g, b}]]
  }; $ // ColumnBar,
 NL, "The representation(commuting):",
 $ = {π[𝒜F] → ℬ[(ℋR ⊕ ℋL) ⊗ ℂ^3],
   π^op[𝒜F^op] → ℬ[(ℋR ⊕ ℋL) ⊗ ℂ^3],
   π[λ, q, b] → {{qλ ⊕ {qλ ⊗ 13}} ⊕ {qλ ⊕ {qλ ⊗ 13}}} ⊗ 13,
   π^op[λ, q, b] → {{(λ 12) ⊕ {12 ⊗ b^T}} ⊕ {(λ 12) ⊕ {12 ⊗ b^T}}} ⊗ 13,
   π̃ → π ⊗ π^op,
   π̃[𝒜F ⊙ 𝒜F^op] → ℬ[ℋF],
   π̃[{λ, q, b} ⊗ {λ', q', b'}^op] → {{(λ' qλ) ⊕ {qλ ⊗ b'^T}} ⊕ {(λ' q) ⊕ {q ⊗ b'^T}}} ⊗ 13
  }; $ // ColumnBar,
 NL, "The even finite space: ",
 $ = {FSM → {𝒜F, ℋF, 𝒟F, 𝒥F → 1},
   𝒟F → {{0, 0, -I Yν, 0}, {0, 0, 0, -I Ye}, {I Yν, 0, 0, 0}, {0, I Ye, 0, 0}} ⊕
     {{{0, 0, -I Yu, 0}, {0, 0, 0, -I Yd}, {I Yu, 0, 0, 0}, {0, I Yd, 0, 0}} ⊗ 13},
   Y_[CG["3×3 matrix"]]
  }; $ // MatrixForms // ColumnBar,
 NL, "Gauge group: ",
 $ = {𝒢[FSM] → Mod[U[1] × SU[2] × U[3], ℤ2],
   CG["Does not match Standard Model so impose inimodularity condition"],
   Det[ρ[u]]ℋF → 1,
   CG["⇒ subset"],
   S𝒢[FSM] → {ρ[u] ∈ 𝒢[FSM], u → {λ, q, b} ∈ 𝒰[𝒜F], (λ Det[b])^12 → 1}

  }; $ // ColumnBar
]
```

The algebra:
$\mathcal{A}_F \to \mathbb{C}\oplus\mathbb{H}\oplus M_3[\mathbb{C}]$
$\mathcal{H}_F \to (\mathcal{H}_R\oplus\mathcal{H}_L)\otimes\mathbb{C}^3$[3-generations]
$\mathcal{H}_R \in \mathbb{C}^2\oplus\{\mathbb{C}^2\otimes\mathbb{C}^3\}$
R→right handed particles
$\mathcal{H}_L \in \mathbb{C}^2\oplus\{\mathbb{C}^2\otimes\mathbb{C}^3\}$
L→left handed particles
$\mathbb{C}^2\oplus\{\mathbb{C}^2\otimes\mathbb{C}^3\}\to\mathbb{C}^2[\nu, e]\oplus\{\mathbb{C}^2\otimes\mathbb{C}^3\}[\{u^c, d^c\}, c\to\{r, g, b\}]$

The representation(commuting):
$\pi[\mathcal{A}_F]\to\mathcal{B}[(\mathcal{H}_R\oplus\mathcal{H}_L)\otimes\mathbb{C}^3]$
$\pi^{op}[\mathcal{A}_F^{op}]\to\mathcal{B}[(\mathcal{H}_R\oplus\mathcal{H}_L)\otimes\mathbb{C}^3]$
$\pi[\lambda, q, b]\to\{\{q_\lambda\oplus\{q_\lambda\otimes 1_3\}\}\oplus\{q_\lambda\oplus\{q_\lambda\otimes 1_3\}\}\}\otimes 1_3$
$\pi^{op}[\lambda, q, b]\to\{\{\lambda\, 1_2\oplus\{1_2\otimes b^T\}\}\oplus\{\lambda\, 1_2\oplus\{1_2\otimes b^T\}\}\}\otimes 1_3$
$\tilde{\pi}\to\pi\otimes\pi^{op}$
$\tilde{\pi}[\mathcal{A}_F\odot\mathcal{A}_F^{op}]\to\mathcal{B}[\mathcal{H}_F]$
$\tilde{\pi}[\{\lambda, q, b\}\otimes\{(\lambda')^{op}, (q')^{op}, (b')^{op}\}]\to\{\{q_\lambda\,\lambda'\oplus\{q_\lambda\otimes b'^T\}\}\oplus\{q\,\lambda'\oplus\{q\otimes b'^T\}\}\}\otimes 1_3$

The even finite space:
$F_{SM}\to\{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F\to 1\}$
$$\mathcal{D}_F\to\left(\begin{array}{cccc}0&0&-i\,Y_\nu&0\\0&0&0&-i\,Y_e\\i\,Y_\nu&0&0&0\\0&i\,Y_e&0&0\end{array}\right)\oplus\left\{\left(\begin{array}{cccc}0&0&-i\,Y_u&0\\0&0&0&-i\,Y_d\\i\,Y_u&0&0&0\\0&i\,Y_d&0&0\end{array}\right)\otimes 1_3\right\}$$
Y_[3×3 matrix]

Gauge group:
$\mathcal{G}[F_{SM}]\to \text{Mod}[U[1]\times SU[2]\times U[3], \mathbb{Z}_2]$
Does not match Standard Model so impose inimodularity condition
$\text{Det}[\rho[u]]_{\mathcal{H}_F}\to 1$
⇒ subset
$S\mathcal{G}[F_{SM}]\to\{\rho[u]\in\mathcal{G}[F_{SM}], u\to\{\lambda, q, b\}\in\mathcal{U}[\mathcal{A}_F], \lambda^{12}\,\text{Det}[b]^{12}\to 1\}$

```
PR["●Proposition 7.1: The fluctuation of 𝒟 by ", A ∈ Pert[Cc"∞"[M, 𝒜F]], " is ",
 $ = {𝒟A → 𝒟 + η𝒟[A],
   𝒟A → 1 ⊗ (It slash[D]) + T[A, "d", {μ}] ⊗ (It T[γ, "u", {μ}]) + {I 𝒟F + ϕ} ⊗ 11,
   ϕ → {{0, 0, Yν Conjugate[ϕ1], Yν Conjugate[ϕ2]},
       {0, 0, -Ye ϕ2, -Ye ϕ1}, {-Yν ϕ1, -Ye Conjugate[ϕ2], 0, 0},
       {-Yν ϕ2, -Ye Conjugate[ϕ1], 0, 0}} ⊕ {{{0, 0, Yu Conjugate[ϕ1], Yu Conjugate[ϕ2]},
         {0, 0, -Yd ϕ2, -Yd ϕ1}, {-Yu ϕ1, -Yd Conjugate[ϕ2], 0, 0},
         {-Yu ϕ2, -Yd Conjugate[ϕ1], 0, 0}} ⊗ 13},
   {T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}]}[CG["gauge fields"]] ∈
    Cc"∞"[M, I ℝ ⊕ su[2] ⊕ su[3]],
   {ϕ1, ϕ2}[CG["Higgs fields"]] ∈ Cc"∞"[M, ℂ2],
   CO["arbitrary vector "],
   {ξ ∈ ℋ"0" → {ℋL ⊗ L2[S]"0" ⊕ ℋR ⊗ L2[S]"1"}} ⟸ {ψν, ψe, ψu, ψd}
  }; $ // MatrixForms // ColumnBar
]
```

●Proposition 7.1: The fluctuation of $\mathcal{D}$ by $A\in \text{Pert}[C_c^\infty[M, \mathcal{A}_F]]$ is
$\mathcal{D}_A\to\mathcal{D}+\eta_{\mathcal{D}}[A]$
$\mathcal{D}_A\to 1\otimes(i^t\,(\not{D}))+\{\phi+i\,\mathcal{D}_F\}\otimes 1_1+A_\mu\otimes(i^t\,\gamma^\mu)$
$$\phi\to\left(\begin{array}{cccc}0&0&(\phi_1)^*\,Y_\nu&(\phi_2)^*\,Y_\nu\\0&0&-Y_e\,\phi_2&-Y_e\,\phi_1\\-Y_\nu\,\phi_1&-(\phi_2)^*\,Y_e&0&0\\-Y_\nu\,\phi_2&-(\phi_1)^*\,Y_e&0&0\end{array}\right)\oplus\left\{\left(\begin{array}{cccc}0&0&(\phi_1)^*\,Y_u&(\phi_2)^*\,Y_u\\0&0&-Y_d\,\phi_2&-Y_d\,\phi_1\\-Y_u\,\phi_1&-(\phi_2)^*\,Y_d&0&0\\-Y_u\,\phi_2&-(\phi_1)^*\,Y_d&0&0\end{array}\right)\otimes 1_3\right\}$$
$\{\Lambda_\mu, Q_\mu, V_\mu\}$[gauge fields] $\in C_c^\infty[M, i\,\mathbb{R}\oplus su[2]\oplus su[3]]$
$\{\phi_1, \phi_2\}$[Higgs fields] $\in C_c^\infty[M, \mathbb{C}^2]$
arbitrary vector
$\{\xi\in\mathcal{H}^0\to\{\mathcal{H}_L\otimes L^2[S]^0\oplus\mathcal{H}_R\otimes L^2[S]^1\}\}\Longleftarrow\{\psi^\nu, \psi^e, \psi^u, \psi^d\}$

```
PR["●Proposition 7.2: The Krein action for F_sm×M is given by: ",
 $ = {S_SM[Ψ, A] → Inactive[Plus][
      BraKet[Ψ^l, I^t slash[D] · Ψ^l],
      BraKet[Ψ^q, I^t slash[D] · Ψ^q],
      BraKet[ψ_R^e, -2 I^t T[γ, "u", {μ}] · T[Λ, "d", {μ}] · ψ_R^e ],
      BraKet[ψ_R^u, 4 / 3 I^t T[γ, "u", {μ}] · T[Λ, "d", {μ}] · ψ_R^u ],
      BraKet[ψ_R^d, -2 / 3 I^t T[γ, "u", {μ}] · T[Λ, "d", {μ}] · ψ_R^d ],
      BraKet[Ψ_L^l, I^t T[γ, "u", {μ}] · (T[Q, "d", {μ}] - T[Λ, "d", {μ}]) · Ψ_L^l ],
      BraKet[Ψ_L^q, I^t T[γ, "u", {μ}] · (T[Q, "d", {μ}] - T[Λ, "d", {μ}]) · Ψ_L^q ],
      CR[BraKet[Ψ^q, T[Λ, "d", {μ}] · Ψ^q ]],
      BraKet[Ψ_R^l, Φ^l · Ψ_L^l ],
      BraKet[Ψ_L^l, ct[Φ^l] · Ψ_R^l ],
      BraKet[Ψ_R^q, Φ^q · Ψ_L^q ],
      BraKet[Ψ_L^q, ct[Φ^q] · Ψ_R^q ]],
    Φ^l → {{-Y_ν · Conjugate[ϕ1 + 1], -Y_ν · Conjugate[ϕ2]}, {Y_e · ϕ2, -Y_e · (ϕ1 + 1)}},
    Φ^q → {{-Y_u · Conjugate[ϕ1 + 1], -Y_u · Conjugate[ϕ2]}, {Y_d · ϕ2, -Y_d · (ϕ1 + 1)}}
   };
 $ // MatrixForms // ColumnSumExp // Column

]
```

●Proposition 7.2: The Krein action for $F_{sm\times}M$ is given by:

$$S_{SM}[\Psi, A] \to \sum\left[\begin{array}{|l}
\langle \Psi^l \mid i^t\,(\not{D})\cdot\Psi^l\rangle \\
\langle \Psi^q \mid i^t\,(\not{D})\cdot\Psi^q\rangle \\
\langle \psi_R^e \mid -2\, i^t\,\gamma^\mu\cdot\Lambda_\mu\cdot\psi_R^e\rangle \\
\langle \psi_R^u \mid \frac{4}{3}\, i^t\,\gamma^\mu\cdot\Lambda_\mu\cdot\psi_R^u\rangle \\
\langle \psi_R^d \mid -\frac{2}{3}\, i^t\,\gamma^\mu\cdot\Lambda_\mu\cdot\psi_R^d\rangle \\
\langle \Psi_L^l \mid i^t\,\gamma^\mu\cdot(Q_\mu-\Lambda_\mu)\cdot\Psi_L^l\rangle \\
\langle \Psi_L^q \mid i^t\,\gamma^\mu\cdot(Q_\mu-\Lambda_\mu)\cdot\Psi_L^q\rangle \\
\color{red}{\langle \Psi^q \mid \Lambda_\mu\cdot\Psi^q\rangle} \\
\langle \Psi_R^l \mid \Phi^l\cdot\Psi_L^l\rangle \\
\langle \Psi_L^l \mid (\Phi^l)^\dagger\cdot\Psi_R^l\rangle \\
\langle \Psi_R^q \mid \Phi^q\cdot\Psi_L^q\rangle \\
\langle \Psi_L^q \mid (\Phi^q)^\dagger\cdot\Psi_R^q\rangle
\end{array}\right]$$

$$\Phi^l \to \begin{pmatrix} -Y_\nu\cdot\sum\left[\begin{array}{|l}1\\(\phi_1)^*\end{array}\right] & -Y_\nu\cdot(\phi_2)^* \\ Y_e\cdot\phi_2 & -Y_e\cdot\sum\left[\begin{array}{|l}1\\\phi_1\end{array}\right]\end{pmatrix}$$

$$\Phi^q \to \begin{pmatrix} -Y_u\cdot\sum\left[\begin{array}{|l}1\\(\phi_1)^*\end{array}\right] & -Y_u\cdot(\phi_2)^* \\ Y_d\cdot\phi_2 & -Y_d\cdot\sum\left[\begin{array}{|l}1\\\phi_1\end{array}\right]\end{pmatrix}$$