2 Scaling operators and entanglement (medium/hard)

This problem set will first compute expectation values for various two point functions at various (easy) distances explicitly. This will lead to an intuitive and easy way to find the scaling dimensions of the CFT describing the MERA. Somewhat similarly, we then continue computing reduced density matrices, which will allow us to find the entanglement entropy, and even more efficiently the 2nd Renyi entropy, which gives the central charge of the CFT.

1. Identify and contract the networks corresponding to $\langle \sigma_{x,1}\sigma_{x,4}\rangle$ and $\langle \sigma_{x,1}\sigma_{x,10}\rangle$, where $\sigma_{x,i}$represents the first Pauli matrix at site i. Note that in this translationally invariant setting[3] you may chose the location of the first lattice point, and there are choices where the computation becomes particularly simple.

2. In a CFT scaling operators obey the following equation

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim (x-y)^{-2\Delta}, \tag{2}$$

   with Δ the scaling dimension of $\mathcal{O}$. Also compute $\langle \sigma_{x,1}\sigma_{x,28}\rangle$ and check if σ_x is a scaling operator.

3. As seen above, every tripling of the distance of the two point correlator results in the addition of a 4 isometries, which is a linear operation on the operators. For scaling operators this will simply scale the result according to (2). Scaling operators are hence eigenvectors of the previous linear operator, with its eigenvalues related to the scaling dimension Δ. Write down the tensor network of this linear operator, compute its spectrum, and thereby derive the few smallest scaling dimensions in the theory.

4. In a similar way it is possible to obtain the equivalent reduced density matrix $\tilde{\rho}$, by tracing out all but L lattice sites and applying an isometric transformation (keeping the spectrum fixed). Draw and compute the reduced density matrices for $L = 2$ and $L = 8$, which are again particularly simple. Compute the spectrum of these matrices, and use these to find the entanglement entropy ($S_{EE} = -\text{tr}(\rho \log \rho)$) and the second Renyi entropy ($S_2 = \text{tr}(\rho^2)$).

5. The second Renyi entropy can of course also computed directly, without computing the spectrum. This is more efficient and it makes it possible to compute the Renyi entropy for any length of the form $L = 3^n - 1$ (this unfortunately does not work for the entanglement entropy). Compute a few of these entropies (the entropies for $n > 2$ are optional), and then extract the central charge using the identity:

$$S_2 = \frac{c}{4}\log(L) + \mathcal{O}(1) \tag{3}$$

[3]In fact, a MERA is not manifestly translationally invariant (it is broken when coarse graining). Nevertheless, for physical MERAs you recover translational invariance (approximately) when computing observables. It is a good exercise, albeit somewhat non-trivial, to also compute the two-point function at more complicated lattice locations.