```
<< Local`QFTToolKit2`
Get[NotebookDirectory[] <>
   "1204.0328 ParticlePhysicsFromAlmostCommutativeSpacetime.3.out"];
$defCInv =
  {};

"Notational definitions"
"Note that in the text the symbols may reference
  different Hilbert spaces. This has caused confusion in some of the
  calculations.  To address this problem we will try to label the
  variables by subscripts to designate the applicable Hilbert space.
  NOTE: Need to do notational change for .1,.2 notebooks."

rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := ∣a∣;
it[a_] := Style[a, Italic]
iD := it[D]
iA := it[A]
iI := it["I"]
C∞ := C^"∞"
Bx_ := T[B, "d", {x}]
("∇"^S)n_ := T["∇"^S, "d", {n}]
disjointQ[b_, c_, free_List: {u, d, e, ν, L, R}] :=
  Apply[Or, Map[FreeQ[b, #] && ! FreeQ[c, #] &, free]];

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
   ""];
accumStdMdl[item_] := Block[{}, $defStdMdl = tuAppendUniq[item][$defStdMdl];
   ""];
accumCInv[item_] := Block[{}, $defCInv = tuAppendUniq[item][$defCInv];
   ""];

selectStdMdl[heads_, with_: {}, all_: Null] :=
  tuRuleSelect[$defStdMdl][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectGWS[heads_, with_: {}, all_: Null] := tuRuleSelect[$defGWS][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectDef[heads_, with_: {}, all_: Null] := tuRuleSelect[$defall][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectCInv[heads_, with_: {}, all_: Null] := tuRuleSelect[$defCInv][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;

Clear[expandDC];
expandDC[sub_: {}, scalar_: {}] :=
 tuRepeat[{sub, tuOpDistribute[Dot], tuOpSimplify[Dot, scalar],
   tuOpDistribute[CircleTimes]}, tuCircleTimesSimplify]
Clear[expandCom]
expandCom[subs_: {}][exp_] := Block[{tmp = exp},
   tmp = tmp //. tuCommutatorExpand // expandDC[];
   tmp = tmp /. toxDot //. Flatten[{subs}];
   tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
```

```
    tmp
   ];
(**)
$sgeneral := {
   T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
   T[γ, "d", {5}].T[γ, "d", {5}] → 1,
   ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
   CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
   T["∇", "d", {_}][1n_] → 0, a_ . 1n_ → a, 1n_ . a_ → a}
$sgeneral // ColumnBar

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "dd", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] :→ CommutatorP[b, a] /; OrderedQ[{b, a}],
    tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
   };
$symmetries // ColumnBar

εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1,}}},
   {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
  ]
εRule[6]

$transformVar = Map[# -> #̃ &, {B, Φ, Λ, s, g, C, F, iD}];
$transformVar // ColumnBar
```

Notational definitions

Note that in the text the symbols may reference different Hilbert spaces. This has caused confusion in some of the calculations. To address this problem we will try to label the variables by subscripts to designate the applicable Hilbert space. NOTE: Need to do notational change for .1,.2 notebooks.

$\gamma_5 \to \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$
$\gamma_5 . \gamma_5 \to 1$
$(\gamma_5)^\dagger \to \gamma_5$
$\{\gamma_5,\ \gamma^\mu\}_+ \to 0$
$\nabla_{\_}\,[1_{n\_}] \to 0$
$(a\_) . 1_{n\_} \to a$
$1_{n\_} . (a\_) \to a$

$tt : g^{\mu\_\,\nu\_} :\to \text{tuIndexSwap}[\{\mu,\ \nu\}][tt]\ /;\ \text{OrderedQ}[\{\nu,\ \mu\}]$
$tt : F^{\mu\_\,\nu\_} :\to -\text{tuIndexSwap}[\{\mu,\ \nu\}][tt]\ /;\ \text{OrderedQ}[\{\nu,\ \mu\}]$
$tt : F_{\mu\_\,\nu\_} :\to -\text{tuIndexSwap}[\{\mu,\ \nu\}][tt]\ /;\ \text{OrderedQ}[\{\nu,\ \mu\}]$
$[a\_,\ b\_]_- :\to -[b,\ a]_-\ /;\ \text{OrderedQ}[\{b,\ a\}]$
$\{a\_,\ b\_\}_+ :\to \{b,\ a\}_+\ /;\ \text{OrderedQ}[\{b,\ a\}]$
$tt : \gamma^\mu . \gamma_5 :\to \text{Reverse}[tt]$

$\{\varepsilon \to 1,\ \varepsilon' \to 1,\ \varepsilon'' \to -1\}$

$B \to \tilde{B}$
$\Phi \to \tilde{\Phi}$
$\Lambda \to \tilde{\Lambda}$
$s \to \tilde{s}$
$g \to \tilde{g}$
$C \to \tilde{C}$
$F \to \tilde{F}$
$D \to \tilde{D}$

7. Conformal invariance

7.1 Conformal Invariance

7.1.1 Conformal Transformation

```
PR["*Conformal transformation of metric: ",
 $ = {T[g̃, "dd", {μ, ν}] → Ω^2 T[g, "dd", {μ, ν}],
   T[g̃, "uu", {μ, ν}] → Ω^(-2) T[g, "uu", {μ, ν}], Ω ∈ C"∞"[M, ℝ+]},
 NL, "Transformed Scalar curvature: ",
 $e71 =
  {s̃ → Ω^-2 (s - 2 (m - 1) tuDs["∇"][tuDsu["∇"][Log[Ω], β], β] - (m - 1) (m - 2) tuDs["∇"][Log[
          Ω], β] tuDsu["∇"][Log[Ω], β]), "∇"[CG["Levi-Civita connection"]]},
 accumCInv[{$, $e71}];
]
```

*Conformal transformation of metric: $\{\tilde{g}_{\mu\nu} \to \Omega^2 g_{\mu\nu},\ \tilde{g}^{\mu\nu} \to \frac{g^{\mu\nu}}{\Omega^2},\ \Omega \in C^\infty[M, \mathbb{R}^+]\}$

Transformed Scalar curvature:

$$\{\tilde{s} \to \frac{s - 2(-1+m)\,\underline{\nabla}_\beta[\underline{\nabla}^\beta[\mathrm{Log}[\Omega]]] - (-2+m)(-1+m)\,\underline{\nabla}_\beta[\mathrm{Log}[\Omega]]\,\underline{\nabla}^\beta[\mathrm{Log}[\Omega]]}{\Omega^2},$$

∇[Levi-Civita connection]}Null

7.1.2 Conformal gravity

```
PR["*Weyl action: ",
 $sWeyl = {SW[g_] →
     xIntegral[√Det[g] T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}], x ∈ M],
   T[C, "uddd", {μ, ν, ρ, σ}][CG["Weyl tensor"]],
   T[C, "uddd", {μ, ν, ρ, σ}] -> T[C̃, "uddd", {μ, ν, ρ, σ}]},
 accumCInv[$sWeyl];
 NL, "●Proposition 7.1: Weyl action is conformally invariant for dim[M]→4. ",
 line,
 NL, "Examine ", $ = $sWeyl // tuExtractIntegrand,
 NL, "•Since ",
 $ = (T[C, "dddd", {μ, ν, ρ, σ}] → T[g, "dd", {μ, α}] T[C, "uddd", {α, ν, ρ, σ}] /.
     $transformVar),
 accumCInv[$];
 " and ", $s = {
     {selectCInv[T[g̃, "dd", {_, _}]]} // tuAddPatternVariable[{μ, ν}],
     tuRuleSelect[$sWeyl][T[C, "uddd", {μ, ν, ρ, σ}]] // First // Reverse //
      tuAddPatternVariable[{μ, ν, ρ, σ}]} // Flatten;
 $s // ColumnBar,
 Yield, $ = $ /. $s,
 Yield, $1 = $ = $ // tuMetricContractAll[g]; $1 // Framed,
 NL, "Since ",
 $s = T[g, "uu", {μ1, μ}] T[g, "uu", {ν1, ν}] T[g, "uu", {ρ1, ρ}] T[g, "uu", {σ1, σ}]
      T[C_, "dddd", {μ, ν, ρ, σ}] -> T[C, "uuuu", {μ1, ν1, ρ1, σ1}] //
   tuAddPatternVariable[{μ, ν, σ, ρ, g}],
 NL, "Apply common multiplier, definition of g, contraction,: ",
 Yield, $ =
  (T[g, "uu", {μ1, μ}] T[g, "uu", {ν1, ν}] T[g, "uu", {ρ1, ρ}] T[g, "uu", {σ1, σ}] /. g → g̃)
      # & /@ $,
 Yield, $[[1]] = $[[1]] // tuMetricContractAll[g̃]; $,
 Yield, $ = $ /. {selectCInv[T[g̃, "uu", {_, _}]] // tuAddPatternVariable[{μ, ν}]},
 Yield, $2 = $ = $ // tuMetricContractAll[g];
 $2 = $ = $ /. {μ1 → μ, ν1 → ν, ρ1 → ρ, σ1 → σ}; $2 // Framed,
 NL, "From ",
 $ = ($d = Det[g] → T[∈, "uuuu", {μ, ν, ρ, σ}] T[g, "dd", {1, μ}]
        T[g, "dd", {2, ν}] T[g, "dd", {3, ρ}] T[g, "dd", {4, σ}]) /. g → g̃,
 Yield, $ = $ /. (selectCInv[T[g̃, "dd", {_, _}]] // tuAddPatternVariable[{μ, ν}]),
 Yield, $ = $ /. Reverse[$d],
 Yield, $3 = $ = Sqrt[#] & /@ $ // PowerExpand,
 NL,
 Yield, $ = {$1, $2, $3}; $ // ColumnBar,
 NL, "Taking the product ",
 $ = $ // tuRuleOp[Times]; $ // Framed,
 accumCInv[{$, $1, $2, $3}]
]
```

*Weyl action: $\{S_W[g\_] \to \int_{x\in M} \sqrt{\mathrm{Det}[g]}\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma},\ C^{\mu}{}_{\nu\rho\sigma}$[Weyl tensor], $C^{\mu}{}_{\nu\rho\sigma} \to \tilde{C}^{\mu}{}_{\nu\rho\sigma}\}$

●Proposition 7.1: Weyl action is conformally invariant for dim[M]→4.

Examine $\{1, 2, 1\} \to \sqrt{\mathrm{Det}[g]}\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$

•Since $\tilde{C}_{\mu\nu\rho\sigma} \to \tilde{C}^{\alpha}{}_{\nu\rho\sigma}\, \tilde{g}_{\mu\alpha}$ and $\left|\begin{array}{l} \tilde{g}_{\mu\_\nu\_} \to \Omega^2 g_{\mu\nu} \\ \tilde{C}^{\mu\_}{}_{\nu\_\rho\_\sigma\_} \to C^{\mu}{}_{\nu\rho\sigma} \end{array}\right.$

→ $\tilde{C}_{\mu\nu\rho\sigma} \to \Omega^2 C^{\alpha}{}_{\nu\rho\sigma}\, g_{\mu\alpha}$

→ $\boxed{\tilde{C}_{\mu\nu\rho\sigma} \to \Omega^2 C_{\mu\nu\rho\sigma}}$

Since $C_{\mu\_\nu\_\rho\_\sigma\_}\, g\_^{\mu1\,\mu\_}\, g\_^{\nu1\,\nu\_}\, g\_^{\rho1\,\rho\_}\, g\_^{\sigma1\,\sigma\_} \to C^{\mu1\,\nu1\,\rho1\,\sigma1}$

Apply common multiplier, definition of g, contraction,:

→ $\tilde{C}_{\mu\nu\rho\sigma}\, \tilde{g}^{\mu1\,\mu}\, \tilde{g}^{\nu1\,\nu}\, \tilde{g}^{\rho1\,\rho}\, \tilde{g}^{\sigma1\,\sigma} \to \Omega^2 C_{\mu\nu\rho\sigma}\, \tilde{g}^{\mu1\,\mu}\, \tilde{g}^{\nu1\,\nu}\, \tilde{g}^{\rho1\,\rho}\, \tilde{g}^{\sigma1\,\sigma}$

→ $\tilde{C}^{\mu1\,\nu1\,\rho1\,\sigma1} \to \Omega^2 C_{\mu\nu\rho\sigma}\, \tilde{g}^{\mu1\,\mu}\, \tilde{g}^{\nu1\,\nu}\, \tilde{g}^{\rho1\,\rho}\, \tilde{g}^{\sigma1\,\sigma}$

→ $\tilde{C}^{\mu1\,\nu1\,\rho1\,\sigma1} \to \dfrac{C_{\mu\nu\rho\sigma}\, g^{\mu1\,\mu}\, g^{\nu1\,\nu}\, g^{\rho1\,\rho}\, g^{\sigma1\,\sigma}}{\Omega^6}$

→ $\boxed{\tilde{C}^{\mu\nu\rho\sigma} \to \dfrac{C^{\mu\nu\rho\sigma}}{\Omega^6}}$

From $\mathrm{Det}[\tilde{g}] \to \epsilon^{\mu\nu\rho\sigma}\, \tilde{g}_{1\mu}\, \tilde{g}_{2\nu}\, \tilde{g}_{3\rho}\, \tilde{g}_{4\sigma}$

→ $\mathrm{Det}[\tilde{g}] \to \Omega^8\, g_{1\mu}\, g_{2\nu}\, g_{3\rho}\, g_{4\sigma}\, \epsilon^{\mu\nu\rho\sigma}$

→ $\mathrm{Det}[\tilde{g}] \to \Omega^8\, \mathrm{Det}[g]$

→ $\sqrt{\mathrm{Det}[\tilde{g}]} \to \Omega^4 \sqrt{\mathrm{Det}[g]}$

→ $\left|\begin{array}{l} \tilde{C}_{\mu\nu\rho\sigma} \to \Omega^2 C_{\mu\nu\rho\sigma} \\ \tilde{C}^{\mu\nu\rho\sigma} \to \frac{C^{\mu\nu\rho\sigma}}{\Omega^6} \\ \sqrt{\mathrm{Det}[\tilde{g}]} \to \Omega^4 \sqrt{\mathrm{Det}[g]} \end{array}\right.$

Taking the product $\boxed{\sqrt{\mathrm{Det}[\tilde{g}]}\, \tilde{C}_{\mu\nu\rho\sigma}\, \tilde{C}^{\mu\nu\rho\sigma} \to \sqrt{\mathrm{Det}[g]}\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma}}$

7.2 Conformal symmetry breaking

```
PR["*For field transforming as ", {ϕ → Ω^-1 ϕ, ϕ ∈ ℝ},
  NL, "Invariant action for ϕ: ",
  $action = {S[T[g, "dd", {μ, ν}], ϕ] → xIntegral[√Det[g]
        (tuDPartial[ϕ, μ] tuDPartialu[ϕ, μ] / 2 + s ϕ^2 / 12 + λ ϕ^4), x ∈ M], s < 0, λ > 0},
  NL, "Minimum of Potential: ", $lagrange = $ = $action // tuExtractIntegrand,
  Yield, $ = V -> $[[2]] // (# /. {√_ → 1, tuDPartial[ϕ, μ] → 0}) &,
  Yield, $ = (# → 0 &) /@ (tuDPartial[#, ϕ] & /@ $),
  imply, $ = $[[2]] // tuDerivativeExpand[{s, λ}] // tuRuleSolve[#, ϕ] & // Last,
  NL, "Vacuum expectation value: v→ ", $vev = $ = Map[#^2 &, $];
  $ // Framed
 ];
```

*For field transforming as $\{\phi \to \frac{\phi}{\Omega}, \phi \in \mathbb{R}\}$

Invariant action for ϕ: $\{S[g_{\mu\nu}, \phi] \to \int_{x\in M} \sqrt{\text{Det}[g]}\, (\frac{s\phi^2}{12} + \lambda\phi^4 + \frac{1}{2}\underline{\partial}_{\mu}[\phi]\, \underline{\partial}^{\mu}[\phi]),\ s<0,\ \lambda>0\}$

Minimum of Potential: $\{1, 2, 1\} \to \sqrt{\text{Det}[g]}\, (\frac{s\phi^2}{12} + \lambda\phi^4 + \frac{1}{2}\underline{\partial}_{\mu}[\phi]\, \underline{\partial}^{\mu}[\phi])$

→ $V \to \frac{s\phi^2}{12} + \lambda\phi^4$

→ $(\underline{\partial}_{\phi}[V] \to 0) \to \underline{\partial}_{\phi}[\frac{s\phi^2}{12} + \lambda\phi^4] \to 0 \Rightarrow \phi \to \frac{i\sqrt{s}}{2\sqrt{6}\sqrt{\lambda}}$

Vacuum expectation value: v→ $\boxed{\phi^2 \to -\frac{s}{24\lambda}}$

```
PR["*If scalar curvature(s) not constant, kinetic terms of Lagrangian need to be
    consider to find vacuum expectation value: ", $L = $ = ℒ -> $lagrange[[2, 2]],
  NL, "Direct solution of the Euler-Lagrange equation to find
    extremum(apply Mathematica EulerLagrange): ",
  Yield, $ = $L /. ϕ → ϕ[x] /. tuDPartial[a_, μ_] :> D[a, x] /.
    tuDPartialu[a_, μ_] :> D[a, x],
  Yield, ($ = EulerEquations[$[[2]], ϕ[x], x]) // D2xPartialD[] //
   (# /. ϕ[x] → ϕ /. x → μ) &,
  NL, "Mathematica's solution is complex: ", "POFF",
  Yield, DSolve[$, ϕ[x], x]
 ];
```

*If scalar curvature(s) not constant, kinetic terms of Lagrangian need to be consider to find vacuum expectation value: $\mathcal{L} \to \frac{s\phi^2}{12} + \lambda\phi^4 + \frac{1}{2}\underline{\partial}_{\mu}[\phi]\, \underline{\partial}^{\mu}[\phi]$

Direct solution of the Euler-Lagrange equation to find extremum(apply Mathematica EulerLagrange):

→ $\mathcal{L} \to \frac{1}{12} s\,\phi[x]^2 + \lambda\,\phi[x]^4 + \frac{1}{2}\phi'[x]^2$

→ $\frac{s\phi}{6} + 4\lambda\phi^3 - \underline{\partial}_{\mu}[\underline{\partial}_{\mu}[\phi]] == 0$

Mathematica's solution is complex:

```
PR["*Apply conformal invariance of the Higgs potential: ",
 NL, "If we apply: ", $s = {s → Ω² s₀[CG["constant"]]},
 Yield, $ = $Lt = $L /. tuRule[$s]; CR[$, back, "ϕ is now transformed."],
 NL, "Then the vacuum expectation value (minimum potential) of: ",
 Yield, $ = V -> $[[2]] /. {tuDPartial[_, _] → 0},

 Yield, $ = (# → 0 &) /@ (tuDPartial[#, ϕ] & /@ $),
 imply, $ = $[[2]] // tuDerivativeExpand[{s₀, λ, Ω}] // tuRuleSolve[#, ϕ] & // Last,
 NL, "Vacuum expectation value: ", $vev = $ = Map[#^2 &, $] /. ϕ → v₀;
 $ // Framed, CR[" not constant?"],

 NL, "Investigate transformed: ", $s = ϕ → (v₀ + h) / Ω,
 Yield, $ = $Lt /. $s,
 NL, "Constants: ", $s = {s₀, v₀, Ω},
 Yield, $ = $ // tuDerivativeExpand[$s],
 NL, "Letting: ", $s = {tuRuleSolve[$vev, s₀], Ω → 1} // Flatten,
 Yield, $ = $ //. $s // Expand // Collect[#, λ] &; $ // Framed
]
```

*Apply conformal invariance of the Higgs potential:
If we apply: $\{s \to \Omega^2\, s_0[\text{constant}]\}$

$\to \mathcal{L} \to \lambda\,\phi^4 + \frac{1}{12}\phi^2\,\Omega^2\,s_0 + \frac{1}{2}\,\underline{\partial}_\mu[\phi]\,\underline{\partial}^\mu[\phi]$ ⟵ϕ is now transformed.

Then the vacuum expectation value (minimum potential) of:

$\to V \to \lambda\,\phi^4 + \frac{1}{12}\phi^2\,\Omega^2\,s_0$

$\to (\underline{\partial}_\phi[V] \to 0) \to \underline{\partial}_\phi[\lambda\,\phi^4 + \frac{1}{12}\phi^2\,\Omega^2\,s_0] \to 0 \;\Rightarrow\; \phi \to \frac{i\,\Omega\,\sqrt{s_0}}{2\,\sqrt{6}\,\sqrt{\lambda}}$

Vacuum expectation value: $\boxed{v_0^2 \to -\frac{\Omega^2\,s_0}{24\,\lambda}}$ not constant?

Investigate transformed: $\phi \to \frac{h + v_0}{\Omega}$

$\to \mathcal{L} \to \frac{1}{12}\,s_0\,(h + v_0)^2 + \frac{\lambda\,(h + v_0)^4}{\Omega^4} + \frac{1}{2}\,\underline{\partial}_\mu[\frac{h + v_0}{\Omega}]\,\underline{\partial}^\mu[\frac{h + v_0}{\Omega}]$

Constants: $\{s_0, v_0, \Omega\}$

$\to \mathcal{L} \to \frac{1}{12}\,s_0\,(h + v_0)^2 + \frac{\lambda\,(h + v_0)^4}{\Omega^4} + \frac{\underline{\partial}_\mu[h]\,\underline{\partial}^\mu[h]}{2\,\Omega^2}$

Letting: $\{s_0 \to -\frac{24\,\lambda\,v_0^2}{\Omega^2}, \Omega \to 1\}$

$\to \boxed{\mathcal{L} \to \lambda\,(h^4 + 4\,h^3\,v_0 + 4\,h^2\,v_0^2 - v_0^4) + \frac{1}{2}\,\underline{\partial}_{-\mu}[h]\,\underline{\partial}^\mu[h]}$

7.3 Conformal transformations of the spectral action

```
PR["*From (2.16-.17): ",
 CO[$ = {selectDef[T[γ, "d", {5}] ⊗ _] /. (εRule[KOdim] /. selectStdMdl[KOdim]) //.
      tuOpDistribute[CircleTimes] // Reverse,
    selectDef[T[γ, "u", {μ}] ⊗ _] /. (εRule[KOdim] /. selectStdMdl[KOdim]) // Reverse
   }],
```

```
NL,
$e216x7 = $ = {T[γ, "d", {5}] ⊗ Φ ->
     T[γ, "d", {5}] ⊗ 𝒟F + T[γ, "d", {5}] ⊗ ϕ + J.(T[γ, "d", {5}] ⊗ ϕ) . ct[J],
    T[γ, "u", {μ}] ⊗ T[B, "d", {μ}] -> T[γ, "u", {μ}] ⊗
      (T[A, "d", {μ}] - JF. T[A, "d", {μ}] . ct[JF])};
$ // ColumnBar,

NL, "With Conformal transformations: ",
$ct = {T[B, "d", {μ}] -> T[B, "d", {μ}], Φ → Φ / Ω, Λ → Λ / Ω, (**)
   tuDUp[iD][a_, b_] -> tuDUp[iD][a, b] / Ω^2,
   tuDDown[iD][a_, b_] -> tuDDown[iD][a, b],
   tuExtractIntegrand[$sWeyl][[2]] -> tuExtractIntegrand[$sWeyl][[2]],
   T[F, "uu", {μ, ν}] → Ω^-4 T[F, "uu", {μ, ν}],
   T[F, "dd", {μ, ν}] → T[F, "dd", {μ, ν}]
  };
$ct = Map[(#[[1]] /. $transformVar) → #[[2]] &, $ct];
$ct = Append[$ct, selectCInv[Sqrt[Det[g̃]]]];
$ct // ColumnBar,

line,
NL, "●Proposition 7.3: Conformal transform of the spectral action is: ",
NL, ℒ[T[g̃, "dd", {μ, ν}], T[B̃, "d", {μ}], Φ̃, Λ̃] ->
 ℒ[T[g, "dd", {μ, ν}], T[B, "d", {μ}], Φ, Λ] +
   N f2 Λ^2 / (Ω^2 4 π^2) tuDsu["∇"][Ω, β] tuDs["∇"][Ω, β] √Det[g]
]
PR["• Proof: Transform the ℒ terms of Proposition 3.7 ",
  tuRuleSelect[$p37][ℒ[__]][[1]],
  line,
  NL, CO[">The Transform of (3.19): "],
  CR["(ignoring topological and boundary terms in Proposition 3.7)"],
  $ = $ = $e319 = ℒM[T[g, "dd", {μ, ν}]] → f4 Λ^4 / (2 π^2) - s f2 Λ^2 / (24 π^2) -
       f[0] / (480 π^2) T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}];
  $ = Sqrt[Det[g]] # & /@ $ // Expand;
  $ = $ /. $transformVar;
  Imply, $[[2]] = $[[2]] /. tuRule[$e71] //. $ct // Expand // tuDerivativeExpand[];
  $ // ColumnSumExp, (**)

  NL, "Let: ", $s = {m → 4},
  Yield, $ = MapAt[# /. $s &, $, 2],
  NL, "Subtract 3.19 to evaluate difference: ", $s = √Det[g] # & /@ $e319,
  Yield, $ = tuRuleSubtract[{$, $s}] // Expand // Simplify;
  $ // ColumnSumExp // Framed;
  NL, "The Transformed (3.19): ",
  $a1 = $ = tuRuleSolve[$, $[[1, 2]]][[1]] // Expand; $ // Framed
 ];

PR[CO[">The Transform of (3.19) : "], $ = tuRuleSelect[$p37][ℒB[_]][[1]];
  $ = ($0 = √Det[g] # & /@ $) /. $transformVar // Expand,
  Yield, $[[2]] = $[[2]] /. $ct /. selectCInv[Sqrt[Det[_]]] // tuTrSimplify[{Ω}];
  $,
  Yield, $a2 = $ = $ /. Reverse[$0]; $ // Framed
 ];
PR[(*for Consistent notation*)
  $sDc =
   {T[iD_, "u", {μ_}][a_] → tuDUp[iD][a, μ], T[iD_, "d", {μ_}][a_] → tuDDown[iD][a, μ]};
  CO[">The Transform of: "], $ = tuRuleSelect[$p37][ℒH[__]][[1]] /. s[x] → s;
```

```
    $0 = $ = √Det[g] # & /@ $ /. $sDc;
    $ = $ /. $transformVar,
    NL, "For ", $s0 = {△[__] → 0, m → 4},

    Yield, $[[2]] = $[[2]] /. tuRule[$e71] /. selectCInv[Sqrt[Det[_]]];
    $[[2]] = $[[2]] /. $s0 //. $ct /. β → μ //. tuOpSimplify[Dot, {Ω}] //
        tuTrSimplify[{Ω}] // Expand;
    $1 = $;
    $ // ColumnSumExp,
    NL, "•Examine the difference between the transformed
       and untransformed equations, ignoring boundary terms: ",
    Yield, $ = tuRuleSubtract[{$1, $0}] /. $s0 // Expand // Simplify;
    $ // ColumnSumExp,
    Yield, $ = $ // tuDerivativeExpand[] // Simplify // tuOpSimplifyF[Dot, {Ω}];
    $ // ColumnSumExp,
    NL, "Simplify ",
    $ = $ /. Dot → Times //. tuOpSimplify[Tr, {Ω}] // ExpandAll;
    $ = $ /. Tr[a_] :→ Tr[tuIndexDummyOrdered[a]] //. tuOpDistribute[Tr] // Simplify;
    NL, "Assume differential equivalent ", $s = "∇" → iD,
    Yield, $ = $ /. $s //. tuOpSimplify[Tr, {Ω, tuDUp[_][Ω, μ], tuDDown[_][Ω, μ]}];
    $ // ColumnSumExp,

    $ = $ /. { Tr[Φ (dd : tuDUp[_])[Φ, μ]] → Tr[dd[Φ^2, μ]] / 2} // Expand;
    $ = $ // Collect[#, Sqrt[_] f[0]] &;
    $ // ColumnSumExp,
    NL, "The total differential of ",
    $s = tuDUp[iD][Tr[Φ^2] tuDDown[iD][Ω, μ] / Ω, μ];
    $s = 0 → ($s // tuDerivativeExpand[]);
    $s = tuRuleSolve[$s, $s[[2, 1, 2 ;; -1]]],
    Yield, $ = $ /. $s // Expand;
    $ = $ /. {tuDUp[iD][Tr[a_], μ_] → Tr[tuDUp[iD][a, μ]]} // tuIndexDummyOrdered,
    $a3 = $ = $[[1, 2]] → -$[[1, 1]];
    $ // ColumnSumExp
  ];
PR["• Summing these term: ", $ = {$a1, $a2, $a3} // Flatten // tuRuleAdd;
 Yield,
 $ = $ /. $s // tuIndexDummyOrdered // ColumnSumExp,
 NL, "Proves the Proposition."
]
```

*From (2.16-.17):

$\{\gamma_5 \otimes \Phi \to \gamma_5 \otimes \phi + \gamma_5 \otimes J_F . \phi . (J_F)^\dagger + \gamma_5 \otimes D_F,\ \gamma^\mu \otimes B_\mu \to \gamma^\mu \otimes (-J_F . A_{M\mu} . (J_F)^\dagger + A_{M\mu})\}$

$\gamma_5 \otimes \Phi \to \gamma_5 \otimes \phi + \gamma_5 \otimes \mathcal{D}_F + J . (\gamma_5 \otimes \phi) . J^\dagger$

$\gamma^\mu \otimes B_\mu \to \gamma^\mu \otimes (-J_F . A_\mu . (J_F)^\dagger + A_\mu)$

With Conformal transformations:

$\tilde{B}_\mu \to B_\mu$

$\tilde{\Phi} \to \frac{\Phi}{\Omega}$

$\tilde{\Lambda} \to \frac{\Lambda}{\Omega}$

$\underline{\tilde{D}}^{\underline{b}}[a\_] \to \frac{\underline{D}^{b}[a]}{\Omega^2}$

$\tilde{D}_{\underline{b}\_}[a\_] \to D_{\underline{b}}[a]$

$\sqrt{\text{Det}[\tilde{g}]}\ \tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} \to \sqrt{\text{Det}[g]}\ C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$

$\tilde{F}^{\mu\nu} \to \frac{F^{\mu\nu}}{\Omega^4}$

$\tilde{F}_{\mu\nu} \to F_{\mu\nu}$

$\sqrt{\text{Det}[\tilde{g}]} \to \Omega^4 \sqrt{\text{Det}[g]}$

●Proposition 7.3: Conformal transform of the spectral action is:

$$\mathcal{L}[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}, \tilde{\Lambda}] \to \mathcal{L}[g_{\mu\nu}, B_\mu, \Phi, \Lambda] + \frac{N\Lambda^2 \sqrt{\text{Det}[g]}\ f_2\, \underline{\nabla}_\beta[\Omega]\, \underline{\nabla}^\beta[\Omega]}{4\pi^2\Omega^2}$$

- Proof: Transform the $\mathcal{L}$ terms of Proposition 3.7

$\mathcal{L}[g_{\mu\nu}, B_\mu, \Phi] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + N\,\mathcal{L}_M[g_{\mu\nu}]$

>The Transform of (3.19):
(ignoring topological and boundary terms in Proposition 3.7)

$$\Rightarrow \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_M[\tilde{g}_{\mu\nu}] \to \sum\Big[-\frac{s\,\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2}{24\pi^2},\ \frac{\Lambda^4\sqrt{\mathrm{Det}[g]}\,f_4}{2\pi^2},\ -\frac{\sqrt{\mathrm{Det}[g]}\,f[0]\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}{480\pi^2},\ \frac{\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\,\underline{\nabla}_\beta[\Omega]\,\nabla^\beta[\underline{\Omega}]}{12\pi^2\Omega^2},\ -\frac{m\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\,\underline{\nabla}_\beta[\Omega]\,\nabla^\beta[\underline{\Omega}]}{8\pi^2\Omega^2},\ \frac{m^2\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\,\underline{\nabla}_\beta[\Omega]\,\nabla^\beta[\underline{\Omega}]}{24\pi^2\Omega^2},\ -\frac{\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\left(\frac{\underline{\nabla}_\beta[\nabla^\beta[\underline{\Omega}]]}{\Omega}-\frac{\underline{\nabla}_\beta[\Omega]\,\nabla^\beta[\underline{\Omega}]}{\Omega^2}\right)}{12\pi^2},\ \frac{m\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\left(\frac{\underline{\nabla}_\beta[\nabla^\beta[\underline{\Omega}]]}{\Omega}-\frac{\underline{\nabla}_\beta[\Omega]\,\nabla^\beta[\underline{\Omega}]}{\Omega^2}\right)}{12\pi^2} \Big]$$

Let: $\{m \to 4\}$

$$\to \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_M[\tilde{g}_{\mu\nu}] \to -\frac{s\,\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2}{24\pi^2}+\frac{\Lambda^4\sqrt{\mathrm{Det}[g]}\,f_4}{2\pi^2}-\frac{\sqrt{\mathrm{Det}[g]}\,f[0]\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}{480\pi^2}+\frac{\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\,\underline{\nabla}_\beta[\Omega]\,\nabla^\beta[\underline{\Omega}]}{4\pi^2\Omega^2}+\frac{\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\left(\frac{\underline{\nabla}_\beta[\nabla^\beta[\underline{\Omega}]]}{\Omega}-\frac{\underline{\nabla}_\beta[\Omega]\,\nabla^\beta[\underline{\Omega}]}{\Omega^2}\right)}{4\pi^2}$$

Subtract 3.19 to evaluate difference:

$$\sqrt{\mathrm{Det}[g]}\ \mathcal{L}_M[g_{\mu\nu}] \to \sqrt{\mathrm{Det}[g]}\ \left(-\frac{s\,\Lambda^2 f_2}{24\pi^2}+\frac{\Lambda^4 f_4}{2\pi^2}-\frac{f[0]\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}{480\pi^2}\right)$$

$\to$

The Transformed (3.19): $\boxed{\sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_M[\tilde{g}_{\mu\nu}] \to \sqrt{\mathrm{Det}[g]}\ \mathcal{L}_M[g_{\mu\nu}] + \frac{\Lambda^2\sqrt{\mathrm{Det}[g]}\,f_2\,\underline{\nabla}_\beta[\nabla^\beta[\underline{\Omega}]]}{4\pi^2\,\Omega}}$

>The Transform of (3.19) : $\sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_{\tilde{B}}[\tilde{B}_\mu] \to \frac{\sqrt{\mathrm{Det}[\tilde{g}]}\ f[0]\,\mathrm{Tr}[\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}]}{24\pi^2}$

$$\to \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_{\tilde{B}}[\tilde{B}_\mu] \to \frac{\sqrt{\mathrm{Det}[g]}\ f[0]\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]}{24\pi^2}$$

$$\to \boxed{\sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_{\tilde{B}}[\tilde{B}_\mu] \to \sqrt{\mathrm{Det}[g]}\ \mathcal{L}_B[B_\mu]}$$

>The Transform of: $\sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \to \sqrt{\mathrm{Det}[\tilde{g}]}$

$$\left(\frac{f[0]\,\tilde{s}\,\mathrm{Tr}[\tilde{\Phi}.\tilde{\Phi}]}{48\pi^2}-\frac{\tilde{\Lambda}^2 f_2\,\mathrm{Tr}[\tilde{\Phi}.\tilde{\Phi}]}{2\pi^2}+\frac{f[0]\,\mathrm{Tr}[\underline{\tilde{D}}_\mu[\tilde{\Phi}].\underline{\tilde{D}}^\mu[\tilde{\Phi}]]}{8\pi^2}+\frac{f[0]\,\mathrm{Tr}[\tilde{\Phi}.\tilde{\Phi}.\tilde{\Phi}.\tilde{\Phi}]}{8\pi^2}+\frac{f[0]\,\triangle[\mathrm{Tr}[\tilde{\Phi}.\tilde{\Phi}]]}{24\pi^2}\right)$$

For $\{\triangle[\_\_] \to 0,\ m \to 4\}$

$$\to \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \to \sum\left[\begin{array}{l} \frac{s\sqrt{\mathrm{Det}[g]}\ f[0]\ \mathrm{Tr}[\Phi.\Phi]}{48\pi^2} \\ -\frac{\Lambda^2\sqrt{\mathrm{Det}[g]}\ f_2\ \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} \\ \frac{\Omega^2\sqrt{\mathrm{Det}[g]}\ f[0]\ \mathrm{Tr}[D_{-\mu}[\frac{\Phi}{\Omega}].\underline{D}^\mu[\frac{\Phi}{\Omega}]]}{8\pi^2} \\ \frac{\sqrt{\mathrm{Det}[g]}\ f[0]\ \mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} \\ -\frac{\sqrt{\mathrm{Det}[g]}\ f[0]\ \mathrm{Tr}[\Phi.\Phi]\ \nabla_{-\mu}[\underline{\nabla}^\mu[\mathrm{Log}[\Omega]]]}{8\pi^2} \\ -\frac{\sqrt{\mathrm{Det}[g]}\ f[0]\ \mathrm{Tr}[\Phi.\Phi]\ \nabla_{-\mu}[\mathrm{Log}[\Omega]]\ \underline{\nabla}^\mu[\mathrm{Log}[\Omega]]}{8\pi^2} \end{array}\right]$$

•Examine the difference between the transformed and untransformed equations, ignoring boundary terms:

$$\to \sum\left[\begin{array}{l} -\sqrt{\mathrm{Det}[g]}\ \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \\ \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \end{array}\right] \to$$

$$-\frac{\sum\left[\begin{array}{l} \mathrm{Tr}[D_{-\mu}[\Phi].\underline{D}^\mu[\Phi]] \\ -\Omega^2\ \mathrm{Tr}[D_{-\mu}[\frac{\Phi}{\Omega}].\underline{D}^\mu[\frac{\Phi}{\Omega}]] \\ \mathrm{Tr}[\Phi.\Phi]\ (\nabla_{-\mu}[\underline{\nabla}^\mu[\mathrm{Log}[\Omega]]] + \nabla_{-\mu}[\mathrm{Log}[\Omega]]\ \underline{\nabla}^\mu[\mathrm{Log}[\Omega]]) \end{array}\right]\sqrt{\mathrm{Det}[g]}\ f[0]}{8\pi^2}$$

$$\to \sum\left[\begin{array}{l} -\sqrt{\mathrm{Det}[g]}\ \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \\ \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \end{array}\right] \to \frac{\sum\left[\begin{array}{l} -\Omega\ \mathrm{Tr}[D_{-\mu}[\Phi].\underline{D}^\mu[\Phi]] \\ \Omega^3\ \mathrm{Tr}[\frac{(\Omega D_{-\mu}[\Phi]-\Phi D_{-\mu}[\Omega]).(\Omega \underline{D}^\mu[\Phi]-\Phi \underline{D}^\mu[\Omega])}{\Omega^4}] \\ -\mathrm{Tr}[\Phi.\Phi]\ \nabla_{-\mu}[\underline{\nabla}^\mu[\Omega]] \end{array}\right]\sqrt{\mathrm{Det}[g]}\ f[0]}{8\pi^2\Omega}$$

Simplify
Assume differential equivalent $\nabla \to D$

$$\to \sum\left[\begin{array}{l} -\sqrt{\mathrm{Det}[g]}\ \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \\ \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \end{array}\right] \to -\frac{\sum\left[\begin{array}{l} 2\Omega\ \mathrm{Tr}[\underline{\Phi}\ D^\mu[\Phi]]\ D_{-\mu}[\Omega] \\ \Omega\ \mathrm{Tr}[\Phi^2]\ D_{-\mu}[\underline{D}^\mu[\Omega]] \\ -\mathrm{Tr}[\Phi^2]\ D_{-\mu}[\Omega]\ \underline{D}^\mu[\Omega] \end{array}\right]\sqrt{\mathrm{Det}[g]}\ f[0]}{8\pi^2\Omega^2}$$

$$\sum\left[\begin{array}{l} -\sqrt{\mathrm{Det}[g]}\ \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \\ \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \end{array}\right] \to \sum\left[\begin{array}{l} -\frac{\mathrm{Tr}[\underline{D}^\mu[\Phi^2]]\ D_{-\mu}[\Omega]}{8\pi^2\Omega} \\ -\frac{\mathrm{Tr}[\Phi^2]\ D_{-\mu}[\underline{D}^\mu[\Omega]]}{8\pi^2\Omega} \\ \frac{\mathrm{Tr}[\Phi^2]\ D_{-\mu}[\Omega]\ \underline{D}^\mu[\Omega]}{8\pi^2\Omega^2} \end{array}\right]\sqrt{\mathrm{Det}[g]}\ f[0]$$

The total differential of $\left\{\frac{\mathrm{Tr}[\Phi^2]\ \underline{D}_\mu[\Omega]\ \underline{D}^\mu[\Omega]}{\Omega^2} \to \frac{\underline{D}_\mu[\Omega]\ \underline{D}^\mu[\mathrm{Tr}[\Phi^2]] + \mathrm{Tr}[\Phi^2]\ \underline{D}^\mu[\underline{D}_\mu[\Omega]]}{\Omega}\right\}$

$$\to -\sqrt{\mathrm{Det}[g]}\ \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + \sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \to 0$$

$$\sqrt{\mathrm{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \to \sqrt{\mathrm{Det}[g]}\ \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi]$$

• Summing these term:

$$\rightarrow \sum\left[\begin{array}{l}\sqrt{\text{Det}[\tilde{g}]}\ \mathcal{L}_H[\tilde{g}_{\mu\nu}, \tilde{B}_\mu, \tilde{\Phi}] \\ \sqrt{\text{Det}[\tilde{g}]}\ \mathcal{L}_M[\tilde{g}_{\mu\nu}] \\ \sqrt{\text{Det}[\tilde{g}]}\ \mathcal{L}_{\tilde{B}}[\tilde{B}_\mu]\end{array}\right] \rightarrow \sum\left[\begin{array}{l}\sqrt{\text{Det}[g]}\ \mathcal{L}_B[B_\mu] \\ \sqrt{\text{Det}[g]}\ \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \\ \sqrt{\text{Det}[g]}\ \mathcal{L}_M[g_{\mu\nu}] \\ \frac{\Lambda^2 \sqrt{\text{Det}[g]}\ f_2 \nabla_{-\beta} [\nabla^\beta[\Omega]]}{4\pi^2 \Omega}\end{array}\right]$$

Proves the Proposition.

7.4 The Higgs mechanims revisited(GWS model with variable scalar curvature)

```
PR["● From the GWS model (5.17): ",
 $ = tuTermExtract[H][selectGWS[ℒH[__], {H'}]] /. H' → H;
 NL, "• Transform variables so EulerEquations[] can be applied: ",
 $s = {Abs[H] → h[x], Abs[tuDDown[iD̃][H, μ]] → D[h[x], x], s → s[x]},

 Yield, $ = $ /. $s,
 NL, "Apply EulerEquations: ",
 Yield, $ = EulerEquations[$, h[x], x];
 Yield, $ = $ // tuD2tuDOp[tuDDown["∂"]];
 $ = (# / 2 & /@ First[tuRuleSolve[$, tuDDown["∂"][tuDDown["∂"][h[x], x], x]]]) //
   Simplify;

 $ // ColumnSumExp, CR["←b/a term different from text p.85."],

 line,
 NL, "See if action in invarient: Apply Conformal transform: ",
 tuRuleSelect[$ct][Φ̃] ⇒ {H → H / Ω},
 Imply, $s = {h[x] → h[x] / Ω, Ω → Ω0 √((2 a f2 Λ^2 - e f[0]) / (a f[0]) - s[x] / 12 },

 Yield, $ = $ /. $s[[1]],
 Yield, $ = $ // tuDerivativeExpand[{Λ, f[0], a, f2, Ω0}],
 $ = tuRuleSolve[$, tuDDown["∂"][tuDDown["∂"][h[x], x], x]] // First // Simplify;
 $ // ColumnSumExp,
 NL, "If we substitute the above ", $s[[2]],
 and, "take as constant ", $const = {Λ, f[0], a, f2, Ω0, e},
 Yield, $ = $ /. $s[[2]] // tuDerivativeExpand[$const] // Simplify;
 $ // ColumnSumExp,
 CR[
  "? Does not seem to be simple relationship for h[x] as in text for ṽ[x]. "]

]
```

● From the GWS model (5.17):

• Transform variables so EulerEquations[] can be applied:

$\{\mathrm{Abs}[H] \to h[x],\ \mathrm{Abs}[\tilde{\underline{D}}_{\mu}[H]] \to h'[x],\ s \to s[x]\}$

$$\to \frac{b\, f[0]\, h[x]^4}{2\,\pi^2} + \frac{a\, f[0]\, h[x]^2\, s[x]}{12\,\pi^2} + \frac{h[x]^2\,(e\, f[0] - 2\, a\, \Lambda^2\, f_2)}{\pi^2} + \frac{a\, f[0]\, h'[x]^2}{2\,\pi^2}$$

Apply EulerEquations:

$\to$

$$\to \frac{1}{2}\underline{\partial}_x[\underline{\partial}_x[h[x]]] \to \sum\left[\begin{array}{l} \frac{e}{a} \\ \frac{b\, h[x]^2}{a} \\ \frac{s[x]}{12} \\ -\frac{2\,\Lambda^2\, f_2}{f[0]} \end{array}\right] h[x]$$ ←b/a term different from text p.85.

See if action in invarient: Apply Conformal transform: $\{\tilde{\Phi} \to \frac{\Phi}{\Omega}\} \Rightarrow \{H \to \frac{H}{\Omega}\}$

$$\Rightarrow \left\{h[x] \to \frac{h[x]}{\Omega},\ \Omega \to \sqrt{-\frac{s[x]}{12} + \frac{-e\, f[0] + 2\, a\, \Lambda^2\, f_2}{a\, f[0]}}\ \Omega_0\right\}$$

$$\to \frac{1}{2}\underline{\partial}_x\left[\underline{\partial}_x\left[\frac{h[x]}{\Omega}\right]\right] \to \frac{h[x]\,\left(\frac{e}{a} + \frac{b\, h[x]^2}{a\,\Omega^2} + \frac{s[x]}{12} - \frac{2\,\Lambda^2\, f_2}{f[0]}\right)}{\Omega}$$

$$\to \frac{1}{2}\left(-\frac{2\,\underline{\partial}_x[\Omega]\,\underline{\partial}_x[h[x]]}{\Omega^2} + h[x]\left(\frac{2\,\underline{\partial}_x[\Omega]^2}{\Omega^3} - \frac{\underline{\partial}_x[\underline{\partial}_x[\Omega]]}{\Omega^2}\right) + \frac{\underline{\partial}_x[\underline{\partial}_x[h[x]]]}{\Omega}\right) \to$$

$$\frac{h[x]\,\left(\frac{e}{a} + \frac{b\, h[x]^2}{a\,\Omega^2} + \frac{s[x]}{12} - \frac{2\,\Lambda^2\, f_2}{f[0]}\right)}{\Omega}\underline{\partial}_x[\underline{\partial}_x[h[x]]] \to \sum\left[\begin{array}{l} \frac{2\, b\, h[x]^3}{a\,\Omega^2} \\ \frac{2\,\partial_{-x}[\Omega]\,\partial_{-x}[h[x]]}{\Omega} \\ h[x]\,\left(\frac{2\, e}{a} + \frac{s[x]}{6} - \frac{4\,\Lambda^2\, f_2}{f[0]} - \frac{2\,\partial_{-x}[\Omega]^2}{\Omega^2} + \frac{\partial_{-x}[\partial_{-x}[\Omega]]}{\Omega}\right) \end{array}\right]$$

If we substitute the above $\Omega \to \sqrt{-\frac{s[x]}{12} + \frac{-e\, f[0] + 2\, a\, \Lambda^2\, f_2}{a\, f[0]}}\ \Omega_0$

and take as constant $\{\Lambda,\ f[0],\ a,\ f_2,\ \Omega_0,\ e\}$

$$\to \underline{\partial}_x[\underline{\partial}_x[h[x]]] \to \sum\left[\begin{array}{l} -\frac{24\, b\, f[0]\, h[x]^3}{(12\, e\, f[0] + a\, f[0]\, s[x] - 24\, a\,\Lambda^2\, f_2)\,\Omega_0^2} \\ \frac{a\, f[0]\,\partial_{-x}[h[x]]\,\partial_{-x}[s[x]]}{12\, e\, f[0] + a\, f[0]\, s[x] - 24\, a\,\Lambda^2\, f_2} \\ h[x]\,\left(\frac{2\, e}{a} + \frac{s[x]}{6} - \frac{4\,\Lambda^2\, f_2}{f[0]} - \frac{3\, a^2\, f[0]^2\,\partial_{-x}[s[x]]^2}{4\,(12\, e\, f[0] + a\, f[0]\, s[x] - 24\, a\,\Lambda^2\, f_2)^2} + \frac{a\, f[0]\,\partial_{-x}[\partial_{-x}[s[x]]]}{24\, e\, f[0] + 2\, a\, f[0]\, s[x] - 48\, a\,\Lambda^2\, f_2}\right) \end{array}\right]$$

? Does not seem to be simple relationship for h[x] as in text for $\tilde{v}[x]$.

Theorem 7.6

```
PR["Theorem 7.6: The gauge and conformal transformations: ",
  $sH = $ = {H → u[x] / Ω[x] {{v₀ + h[x]}, {0}}, h[x_] → Ω[x] Abs[H[x]] - v₀};
  $ // MatrixForms // ColumnBar,
  NL, "break gauge and conformal symmetry.  Broken bosonic action (GWS) is: ",
  $ =
   S_B → IntegralOp[{{x ∈ M}}, √Det[g] (4 f₄ Λ^4 / (π^2) - c f₂ Λ^2 / (π^2) + d f[0] / (4 π^2) -
        b π^2 v₀^4 / (2 a^2 f[0]) + (c f[0] / (24 π^2) - f₂ Λ^2 / (3 π^2)) s -
        f[0] / (40 π^2) T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}]
        + T[B, "dd", {μ, ν}] T[B, "uu", {μ, ν}] / 4
        + T[W, "udd", {a, μ, ν}] T[W, "uuu", {a, μ, ν}] / 4
        + f₂ Λ^2 / (3 π^2) tuDPartial[η, β] tuDPartialu[η, β]
        + tuDPartial[h, β] tuDPartialu[h, β] / 2
        + bπ^2 / (2 a^2 f[0]) (h^4 + 4 v₀ h^3 + 4 v₀^2 h^2)
        + g₂^2 / 4 (v₀ + h)^2 T[W, "d", {μ}] ct[T[W, "u", {μ}]]
        + g₂^2 / (8 c_w^2) (v₀ + h)^2 T[Z, "d", {μ}] T[Z, "u", {μ}])];
  $ // ColumnSumExp,
  NL, "where ", $s = {Ω → Exp[η]}
 ];
PR["● Sketch of Proof: 1. Starting with Proposition 5.7: selectGWS[prop57]",
  Grid[{{"Since ℒ_H is conformally invariant in (5.7) replace",
      $sH[[1]] /. {Ω[x] → 1, u[x] → 1} // MatrixForms},
    {"Apply ", {selectGWS[H, {a}], selectGWS[ℒ_], selectGWS[Abs[_]^2]} // ColumnBar },
    {"Impose (5.16) on (5.14) to get gauge kinetic term", selectGWS[Tr[_], {g₁}]},
    {"Kinetic term of dilation field η from ", selectGWS[ℒ_M[_], {}]}
   }, Frame → All]
 ];
```

Theorem 7.6: The gauge and conformal transformations: $\left| \begin{array}{l} H \to \begin{pmatrix} \frac{(h[x]+v_0)\,u[x]}{\Omega[x]} \\ 0 \end{pmatrix} \\ h[x\_] \to -v_0 + \mathrm{Abs}[H[x]]\,\Omega[x] \end{array} \right.$

break gauge and conformal symmetry. Broken bosonic action (GWS) is:

$$S_B \to \int_{\{x\in M\}} \left[\sum \left[\begin{array}{|l} \frac{d\, f[0]}{4\pi^2} \\ -\frac{c\,\Lambda^2 f_2}{\pi^2} \\ s\left(\frac{c\, f[0]}{24\pi^2} - \frac{\Lambda^2 f_2}{3\pi^2}\right) \\ \frac{4\Lambda^4 f_4}{\pi^2} \\ -\frac{b\,\pi^2 v_0^4}{2 a^2 f[0]} \\ \frac{b\pi^2 (h^4 + 4 h^3 v_0 + 4 h^2 v_0^2)}{2 a^2 f[0]} \\ \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ -\frac{f[0]\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}}{40\pi^2} \\ \frac{1}{4} (W^\mu)^\dagger g_2^2 (h+v_0)^2 W_\mu \\ \frac{1}{4} W^a{}_{\mu\nu} W^{a\mu\nu} \\ \frac{g_2^2 (h+v_0)^2 Z_\mu Z^\mu}{8 c_W^2} \\ \frac{1}{2} \underset{-\beta}{\partial}[h]\, \underset{-}{\partial^\beta}[h] \\ \frac{\Lambda^2 f_2\, \underset{-\beta}{\partial}[\eta]\, \underset{-}{\partial^\beta}[\eta]}{3\pi^2} \end{array} \right] \sqrt{\mathrm{Det}[g]} \right]$$

where $\{\Omega \to e^\eta\}$

- Sketch of Proof: 1. Starting with Proposition 5.7: selectGWS[prop57]

Since $\mathcal{L}_H$ is conformally invariant in (5.7) replace	$H \to \begin{pmatrix} h[x] + v_0 \\ 0 \end{pmatrix}$
Apply	$H \to \frac{\sqrt{a\, f[0]}\, H'}{\pi}$ $\mathcal{L}_{Hpot} \to \frac{b\, \pi^2 \left(-\frac{v^4}{2} + 2 v^2 h[x]^2 + 2 v\, h[x]^3 + \frac{h[x]^4}{2}\right)}{a^2 f[0]}$ $\mathrm{Abs}[\underset{-\mu}{\tilde{D}}[H]]^2 \to \frac{1}{4}\big((v + h[x])^2 g_1^2 B_\mu B^\mu - 2 (v + h[x])^2 g_1 g_2 B^\mu W_\mu{}^3 + (v + h[x])^2 g_2^2 (W_\mu{}^1 W^{\mu 1} + W_\mu{}^2 W^{\mu 2} + W_\mu{}^3 W^{\mu 3}) + 4\, \underset{-\mu}{\partial}[h[x]]\, \underset{-}{\partial^\mu}[h[x]]\big)$
Impose (5.16) on (5.14) to get gauge kinetic term	$\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3 g_1^2 B_{\mu\nu} B^{\mu\nu} + g_2^2 W_{\mu\nu}{}^a W^{\mu\nu}{}_a$
Kinetic term of dilation field η from	$\mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 f_2}{24 \pi^2} + \frac{\Lambda^4 f_4}{2 \pi^2} + \frac{f[0]\left(\frac{11 R^*.R^*}{360} - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{\Delta[s]}{30}\right)}{16 \pi^2}$

Theorem 7.8

```
PR["Theorem 7.8. The bosonic action of the Standard Model: ",
 Yield, $t78 = SB → xIntegral[
     √Det[g] (48 f4 Λ^4 / (π^2) - c f2 Λ^2 / (π^2) + df[0] / (4 π^2) - b π^2 v0^4 / (2 a^2 f[0])
        + (c f[0] / (24 π^2) - 4 f2 Λ^2 / (π^2)) s -
        3 f[0] / (10 π^2) T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}]
        + T[B, "dd", {μ, ν}] T[B, "uu", {μ, ν}] / 4
        + T[W, "udd", {a, μ, ν}] T[W, "uuu", {a, μ, ν}] / 4
        + T[G, "udd", {i, μ, ν}] T[G, "uuu", {i, μ, ν}] / 4
        + f2 Λ^2 / (π^2) tuDPartial[η, β] tuDPartialu[η, β]
        + tuDPartial[h, β] tuDPartialu[h, β] / 2
        + b π^2 / (2 a^2 f[0]) (h^4 + 4 v0 h^3 + 4 v0^2 h^2)
        + g2^2 / 4 (v0 + h)^2 T[W, "d", {μ}] ct[T[W, "u", {μ}]]
        + g2^2 / (8 cw^2) (v0 + h)^2 T[Z, "d", {μ}] T[Z, "u", {μ}]), x ∈ M
   ]
]
PR["● Sketch of proof: Start with
   Proposition 6.8 and apply same procedure as Theorem 7.6."
]
```

Theorem 7.8. The bosonic action of the Standard Model:

→ $S_B \rightarrow$

$$\int_{x\in M}\sqrt{\mathrm{Det}[g]}\,\Bigg(\frac{d f[0]}{4\pi^2}-\frac{c\Lambda^2 f_2}{\pi^2}+s\left(\frac{c f[0]}{24\pi^2}-\frac{4\Lambda^2 f_2}{\pi^2}\right)+\frac{48\Lambda^4 f_4}{\pi^2}-\frac{b\pi^2 v_0^4}{2a^2 f[0]}+\frac{b\pi^2(h^4+4h^3v_0+4h^2v_0^2)}{2a^2 f[0]}+$$
$$\frac{1}{4}B_{\mu\nu}B^{\mu\nu}-\frac{3 f[0]\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}{10\pi^2}+\frac{1}{4}G^{i}{}_{\mu\nu}G^{i\mu\nu}+\frac{1}{4}(W^{\mu})^{\dagger}g_2^2(h+v_0)^2W_{\mu}+$$
$$\frac{1}{4}W^{a}{}_{\mu\nu}W^{a\mu\nu}+\frac{g_2^2(h+v_0)^2 Z_{\mu}Z^{\mu}}{8c_w^2}+\frac{1}{2}\partial_{-\beta}[h]\,\partial^{\beta}_{-}[h]+\frac{\Lambda^2 f_2\,\partial_{-\beta}[\eta]\,\partial^{\beta}_{-}[\eta]}{\pi^2}\Bigg)$$

● Sketch of proof: Start with
Proposition 6.8 and apply same procedure as Theorem 7.6.

8. Phenomenology

8.1 Mass relations

8.1.1 Fermion masses

```
PR["Using Definition (6.9): ", $e69 = Yx_ → -I √(a f[0]) / (π v) mx,
 NL, "In (Lemma 6.6): ", $ = tuRuleSelect[$166][a],
 Yield,
 $ = $ /. $e69 // tuConjugateTransposeSimplify[{√_, v}, {v, a, f[0], √_}] // Simplify;
 Yield, $ = $ // tuTrSimplify[{v, a, f[0]}],
 NL, "(5.23): ", $e523 = {MW → v g2 / 2, MZ → v g2 / (2 cw)},
 NL, "From (6.6) ", $g2 = $s = tuRuleSolve[$e66, g2],
 Yield, $s = $e523 /. $s // First,
 Yield, $e81 = $s = tuRuleSolve[$s, f[0]] // Normal,
 Imply, $ = $ /. $s,
 Yield, $ = tuRuleSolve[$, MW] // First // #^2 & /@ # &; $ // Framed
]
```

Using Definition (6.9): $Y_{x\_} \to -\frac{i \sqrt{a\, f[0]}\; m_x}{\pi\, v}$

In (Lemma 6.6): $\{a \to \mathrm{Tr}[3\,(Y_d)^\dagger . Y_d + (Y_e)^\dagger . Y_e + 3\,(Y_u)^\dagger . Y_u + (Y_\nu)^\dagger . Y_\nu]\}$

→

→ $\{a \to \frac{a\, f[0]\, \mathrm{Tr}[3\,(m_d)^\dagger . m_d + (m_e)^\dagger . m_e + 3\,(m_u)^\dagger . m_u + (m_\nu)^\dagger . m_\nu]}{\pi^2\, v\, v^*}\}$

(5.23): $\{M_W \to \frac{v\, g_2}{2},\ M_Z \to \frac{v\, g_2}{2\, c_w}\}$

From (6.6) $\{g_2 \to -\frac{\pi}{\sqrt{2}\,\sqrt{f[0]}},\ g_2 \to \frac{\pi}{\sqrt{2}\,\sqrt{f[0]}}\}$

→ $M_W \to -\frac{\pi\, v}{2\,\sqrt{2}\,\sqrt{f[0]}}$

→ $\{f[0] \to \frac{\pi^2\, v^2}{8\, M_W^2}\}$

⇒ $\{a \to \frac{a\, v\, \mathrm{Tr}[3\,(m_d)^\dagger . m_d + (m_e)^\dagger . m_e + 3\,(m_u)^\dagger . m_u + (m_\nu)^\dagger . m_\nu]}{8\, v^*\, M_W^2}\}$

→ $\boxed{M_W^2 \to \frac{v\, \mathrm{Tr}[3\,(m_d)^\dagger . m_d + (m_e)^\dagger . m_e + 3\,(m_u)^\dagger . m_u + (m_\nu)^\dagger . m_\nu]}{8\, v^*}}$

8.1.2 The Higgs mass

```
PR["Extracting the h^2 terms from Theorem 7.8: ",
 Yield, $0 = $ = $t78 // tuExtractIntegrand // (# /. Det[g] → 1 &) // Expand;
 $ = $ // tuTermExtract[h^2 a_, {W, Z}],
 NL, "and setting it to canonical mass term: ",
 Yield, $ = $ → m_h ^2 h^2 / 2,
 NL, "Gives an expression for the Higg's mass: ",
 Yield, $ = tuRuleSolve[$, m_h] // Last,
 NL, "Using the relation (8.1): ", $e81,
 Yield, $mh = $ = $ /. $e81 /. v_0 → v // PowerExpand; $ // Framed
]
```

Extracting the h^2 terms from Theorem 7.8:

$\rightarrow \frac{2\,b\,h^2\,\pi^2\,v_0^2}{a^2\,f[0]}$

and setting it to canonical mass term:

$\rightarrow \frac{2\,b\,h^2\,\pi^2\,v_0^2}{a^2\,f[0]} \to \frac{1}{2}\,h^2\,m_h^2$

Gives an expression for the Higg's mass:

$\rightarrow m_h \to \frac{2\,\sqrt{b}\,\pi\,v_0}{a\,\sqrt{f[0]}}$

Using the relation (8.1): $\left\{f[0] \to \frac{\pi^2\,v^2}{8\,M_W^2}\right\}$

$\rightarrow \boxed{m_h \to \frac{4\,\sqrt{2}\,\sqrt{b}\,M_W}{a}}$

```
PR["From the quartic Higg's coupling constant ",
 $ = $0 // tuTermExtract[h^4 a_],
 NL, "setting it to canonical quartic interaction term ",
 $1 = $ = $ → λ h^4 / 24,
 NL, "From the relations: ",
 $ = {$1, $mh, $e81, $e523} // Flatten,
 Imply,
 $ =
  tuEliminate[$, {f[0], v, a}] /. Equal → Rule // First // tuRuleSolve[#, m_h^2] & // First;
 $ // Framed, CG["(8.6)"]
]
```

From the quartic Higg's coupling constant $\frac{b\,h^4\,\pi^2}{2\,a^2\,f[0]}$

setting it to canonical quartic interaction term $\frac{b\,h^4\,\pi^2}{2\,a^2\,f[0]} \to \frac{h^4\,\lambda}{24}$

From the relations: $\left\{\frac{b\,h^4\,\pi^2}{2\,a^2\,f[0]} \to \frac{h^4\,\lambda}{24},\ m_h \to \frac{4\,\sqrt{2}\,\sqrt{b}\,M_W}{a},\ f[0] \to \frac{\pi^2\,v^2}{8\,M_W^2},\ M_W \to \frac{v\,g_2}{2},\ M_Z \to \frac{v\,g_2}{2\,c_w}\right\}$

$\Rightarrow \boxed{m_h^2 \to \frac{4\,\lambda\,M_W^2}{3\,g_2^2}}$ (8.6)

8.1.3 The seesaw mechanism (neutrino masses)

```
PR["● Examine ν basis ", $df = selectStdMdl[iD_F];
  NL, $ = selectStdMdl[basisSM[_]],
  Yield, $ = $basisNu = $ // tuExtractPositionPattern[Tensor[ν | ν̄, _, _]];
  $ // ColumnBar,
  Yield, $0 = $ = #[[1, 2]] & /@ $; $ = Tuples[{$, $}],
  $map = MapIndexed[#1 → #2[[1]] &, $0];
  NL, "Extract ", iD_F, " elements ",
  Yield, $ = SparseArray[Map[(# /. $map) -> Extract[$df[[2]], #] &, $]] // Normal;
  $ // MatrixForm;

  NL, "For scalars ", $scal = {Y_},
  NL, "The " iD_F, yield, $ = $ // tuConjugateTransposeSimplify[{}, $scal];
  $ // MatrixForms,
  NL, "Eigenvalues of this matrix ",
  $ = Eigenvalues[$];
  Yield, $ = $ // gatherSqrt // FullSimplify // PowerExpand,
  NL, "Define ∈: ", $s = {Abs[Y_ν] -> Abs[Y_R] ∈, Abs[Y_ν Y_R] -> Abs[Y_ν] Abs[Y_R]},
  Yield, $ = $ //. $s // FullSimplify;
  Yield, $ = Series[$, {∈, 0, 2}] // Normal; $ // ColumnBar // Framed
 ];
```

● Examine ν basis

basisSM[without generations(3) and color(3 for u,d) indices] →

$\{\nu_R, e_R, \nu_L, e_L, u_R, d_R, u_L, d_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L, \bar{u}_R, \bar{d}_R, \bar{u}_L, \bar{d}_L\}$

→
{2, 1} → ν_R
{2, 3} → ν_L
{2, 9} → $\bar{\nu}_R$
{2, 11} → $\bar{\nu}_L$

→ {{1, 1}, {1, 3}, {1, 9}, {1, 11}, {3, 1}, {3, 3}, {3, 9}, {3, 11}, {9, 1}, {9, 3}, {9, 9}, {9, 11}, {11, 1}, {11, 3}, {11, 9}, {11, 11}}

Extract D_F elements

→

For scalars {Y_}

The $D_F \longrightarrow \begin{pmatrix} 0 & Y_\nu & (Y_R)^* & 0 \\ (Y_\nu)^* & 0 & 0 & 0 \\ Y_R & 0 & 0 & (Y_\nu)^* \\ 0 & 0 & Y_\nu & 0 \end{pmatrix}$

Eigenvalues of this matrix

$$\to \Big\{-\frac{\sqrt{\text{Abs}[Y_R]^2 + 2\,\text{Abs}[Y_\nu]^2 - \sqrt{\text{Abs}[Y_R]^4 + 4\,\text{Abs}[Y_R\,Y_\nu]^2}}}{\sqrt{2}},$$
$$\frac{\sqrt{\text{Abs}[Y_R]^2 + 2\,\text{Abs}[Y_\nu]^2 - \sqrt{\text{Abs}[Y_R]^4 + 4\,\text{Abs}[Y_R\,Y_\nu]^2}}}{\sqrt{2}},$$
$$-\frac{\sqrt{2\,\text{Abs}[Y_\nu]^2 + \text{Abs}[Y_R]\left(\text{Abs}[Y_R] + \sqrt{\text{Abs}[Y_R]^2 + 4\,\text{Abs}[Y_\nu]^2}\right)}}{\sqrt{2}},$$
$$\frac{\sqrt{2\,\text{Abs}[Y_\nu]^2 + \text{Abs}[Y_R]\left(\text{Abs}[Y_R] + \sqrt{\text{Abs}[Y_R]^2 + 4\,\text{Abs}[Y_\nu]^2}\right)}}{\sqrt{2}}\Big\}$$

Define ϵ: $\{\text{Abs}[Y_\nu] \to \epsilon\,\text{Abs}[Y_R], \text{Abs}[Y_R\,Y_\nu] \to \text{Abs}[Y_R]\,\text{Abs}[Y_\nu]\}$

→

→
$-\epsilon^2\,\text{Abs}[Y_R]$
$\epsilon^2\,\text{Abs}[Y_R]$
$-\text{Abs}[Y_R] - \epsilon^2\,\text{Abs}[Y_R]$
$\text{Abs}[Y_R] + \epsilon^2\,\text{Abs}[Y_R]$

■ 8.2 Renormalization group flow

```
PR["•At the GUT scale ΛGUT assume relationships: ", $ = selectStdMdl[g_^2 _, {}, all];
 $sg2 = {tuEliminate[$, {f[0], g1}], tuEliminate[$, {f[0], g2}]} /.
    T[b, "d", {i_}] → bi // Simplify,
 NL, "•Consider only 1-loop approximation.",
 NL, "•The Renormalization group equation For ", $sn = ng → 3,
 Yield,
 $g = $ = {tuDDown[d][gi, t] → -bi gi^3 / (16 π^2), bi_ → {-41 / 6, 19 / 6, 7}[[i]], t → Log[μ]};
 $ // ColumnBar]
PR["•Solve for ", gi[μ],
  NL, "• Setup for Mathematica ",
  $ = $g[[1]] /. gg : gi → gg[t] /. Rule → Equal // tuDerivOps2D,
  Yield, $ = DSolve[$, gi[t], t],
  NL, "Set C[1] at ", $s = μ → MZ,
  Yield, $ = $[[2, 1]];
  $ = $ /. $g[[-1]];
  $ = #^2 & /@ $;
  $1 = $ /. $s,
  NL, "Invert equation and apply C[1]",
  Yield, $ = 1 / # & /@ $,
  Yield,
  $gsol = $ = $ /. tuRuleSolve[$1, C[1]] // Simplify // tuOpGather[Log] // Simplify //
     (# /. Log[mm : μ | MZ] → mm &);

  $ // Framed, CG["(8.8)"]
 ];
```

•At the GUT scale Λ_{GUT} assume relationships: $\{g_2^2 == g_3^2,\ 5\, g_1^2 == 3\, g_3^2\}$
•Consider only 1-loop approximation.
•The Renormalization group equation For $n_g \to 3$

$$\to \left| \begin{array}{l} \underset{-t}{d}\,[g_i] \to -\frac{b_i\, g_i^3}{16\,\pi^2} \\ b_{i\_} \to \{-\frac{41}{6},\ \frac{19}{6},\ 7\}[[i]] \\ t \to \mathrm{Log}[\mu] \end{array} \right.$$

•Solve for $g_i[\mu]$

• Setup for Mathematica $g_i{}'[t] == -\dfrac{b_i\, g_i[t]^3}{16\,\pi^2}$

$$\to \left\{\left\{g_i[t] \to -\frac{2\sqrt{2}\,\pi}{\sqrt{-16\,\pi^2\, C[1] + t\, b_i}}\right\},\ \left\{g_i[t] \to \frac{2\sqrt{2}\,\pi}{\sqrt{-16\,\pi^2\, C[1] + t\, b_i}}\right\}\right\}$$

Set C[1] at $\mu \to M_Z$

$$\to g_i[\mathrm{Log}[M_Z]]^2 \to \frac{8\,\pi^2}{-16\,\pi^2\, C[1] + \mathrm{Log}[M_Z]\, b_i}$$

Invert equation and apply C[1]

$$\to \frac{1}{g_i[\mathrm{Log}[\mu]]^2} \to \frac{-16\,\pi^2\, C[1] + \mathrm{Log}[\mu]\, b_i}{8\,\pi^2}$$

$$\to \boxed{\frac{1}{g_i[\mu]^2} \to \frac{\mathrm{Log}[\frac{\mu}{M_Z}]\, b_i}{8\,\pi^2} + \frac{1}{g_i[M_Z]^2}} \quad (8.8)$$

```
PR["Experimental values: ",
  $se = $ = {g1[Mz] → .3575, g2[Mz] → .6519, g3[Mz] → 1.220, Mz → 91.1876[CG[GeV]]};
  $ // ColumnBar,
  NL, "Plot ", $gsol,
  NL, "For ", $s = i → 1,
  Yield, $1 = $gsol /. $s //. tuRule[$se] /. tuRule[$g] // Last,

  NL, "For ", $s = i → 2,
  Yield, $2 = $gsol /. $s //. tuRule[$se] /. tuRule[$g] // Last,

  NL, "For ", $s = i → 3,
  Yield, $3 = $gsol /. $s //. tuRule[$se] /. tuRule[$g] // Last
 ];
```

Experimental values: $g_1[M_Z] \to 0.3575$, $g_2[M_Z] \to 0.6519$, $g_3[M_Z] \to 1.22$, $M_Z \to 91.1876[\text{GeV}]$

Plot $\frac{1}{g_i[\mu]^2} \to \frac{\text{Log}[\frac{\mu}{M_Z}]\, b_i}{8\,\pi^2} + \frac{1}{g_i[M_Z]^2}$

For $i \to 1$

$\to 7.82434 - \frac{41\,\text{Log}[0.0109664\,\mu]}{48\,\pi^2}$

For $i \to 2$

$\to 2.35309 + \frac{19\,\text{Log}[0.0109664\,\mu]}{48\,\pi^2}$

For $i \to 3$

$\to 0.671862 + \frac{7\,\text{Log}[0.0109664\,\mu]}{8\,\pi^2}$

Plot of g_i vs μ

```
$ = Sqrt[1 / {$1 , $2, $3}];
LogLinearPlot[$, {μ, Exp[2], Exp[50]},
 PlotLabels → {g1, g2, g3}]
```