MSSTP 2015, J.Maldacena Inflation related problems

1) Background [easy]

1a) Einstein equations

In this exercise we consider an inflationary model with a potential $V = \frac{1}{2}m^2\phi^2$. We consider a uniform universe with

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2$$

for which Einsteins equations read

$$3M_p^2\frac{\dot{a}^2}{a^2} = \frac{1}{2}\dot{\phi}^2 + V$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0$$

As a first exercise we shall reproduce these equations using the package diffgeo.m which you should download from
http://people.brandeis.edu/~headrick/Mathematica/
and put in same directory as your notebook.
Define `coord` and `metric` as below and then load the package

```
coord = {t, x, y, z};
metric = DiagonalMatrix[{-1, a[t]^2, a[t]^2, a[t]^2}]
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix}$$

```
<< (NotebookDirectory[] <> "diffgeo.m");
```

Evaluate the l.h.s. of the 00 component of Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{1}{M_p^2}T_{\mu\nu}$ (use `RicciTensor` and `RicciScalar`.)

Using that T_{00} is the energy of the scalar field i.e. $\frac{1}{2}\dot{\phi}^2 + V$ you should get the first equation. The other equation is the standard equation for the scalar field in the curved space

$$\nabla_\mu\nabla^\mu\phi = dV/d\phi$$

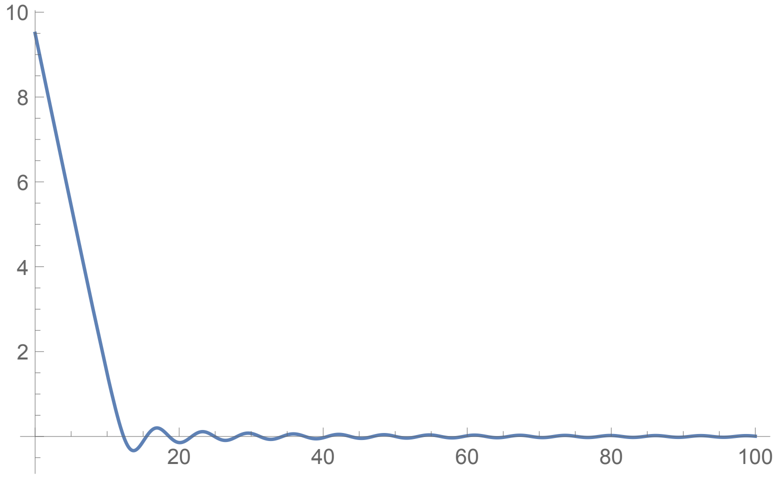
Use `covariant` (from the diffgeo package) to compute the covariant derivatives thus reproducing the second equation.

1b) Solving the equations numerically

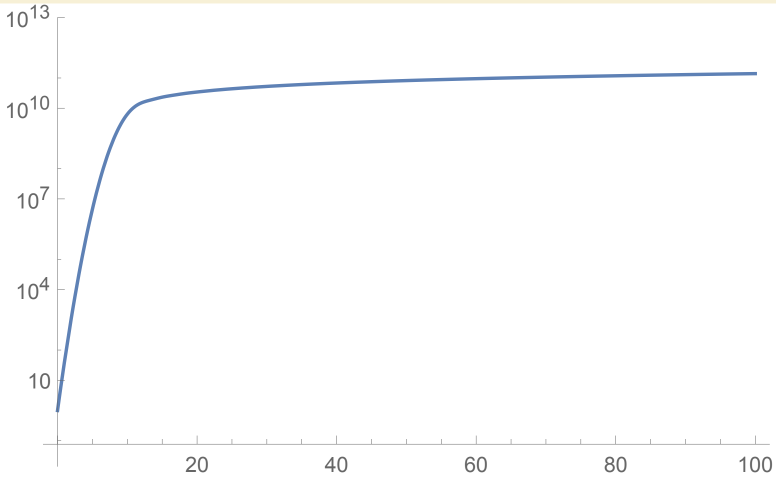
Explain first (without *Mathematica*) why we can always set $M_p = m = 1$.

Solve the equations numerically and find initial conditions that lead to a period of inflation followed by a matter dominated region. The plots below are one example of such nice initial conditions:

```
Plot[ϕ[t], {t, 0, 100}, PlotRange → All]
```



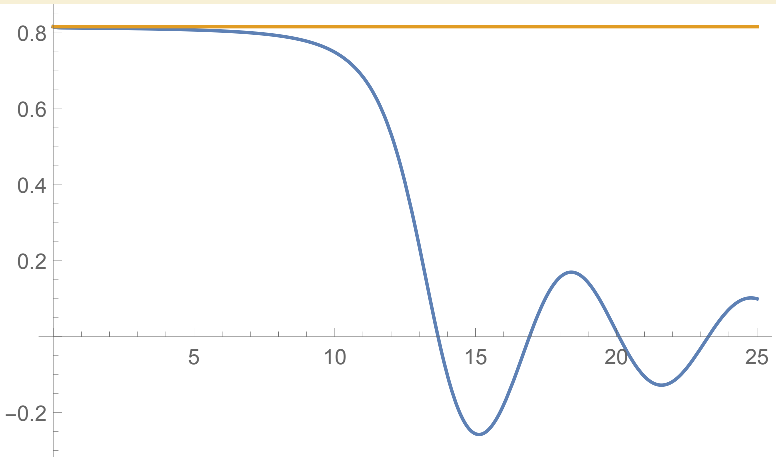

```
LogPlot[a[t], {t, 0, 100}, PlotRange → All]
```



1c) Comparison with slow roll

In the slow roll approximation we have $\ddot{\phi}$ negligible and $\dot{\phi}^2 \ll V$. Solve the equations above in this limit analytically. In particular show that $3h^2 = \frac{\phi^2}{2}$ where the Hubble $h \equiv \dot{a}/a$ in this regime. You should also find that $\dot{\phi}$ is constant. Plot the slow roll analytic solutions together with the previous numerical solutions to observe that indeed, for large enough scalar initial conditions we start in a period of slow roll. An example of one such plot could be:

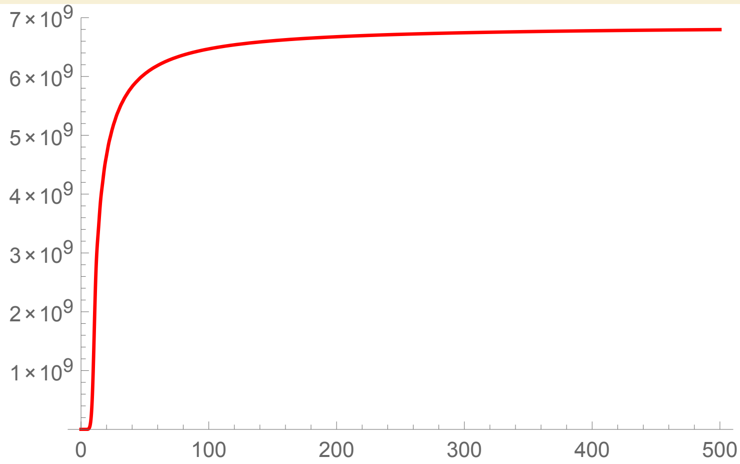
```
Plot[{-ϕ'[t], √(2/3)}, {t, 0, 25}, PlotRange → All]
```



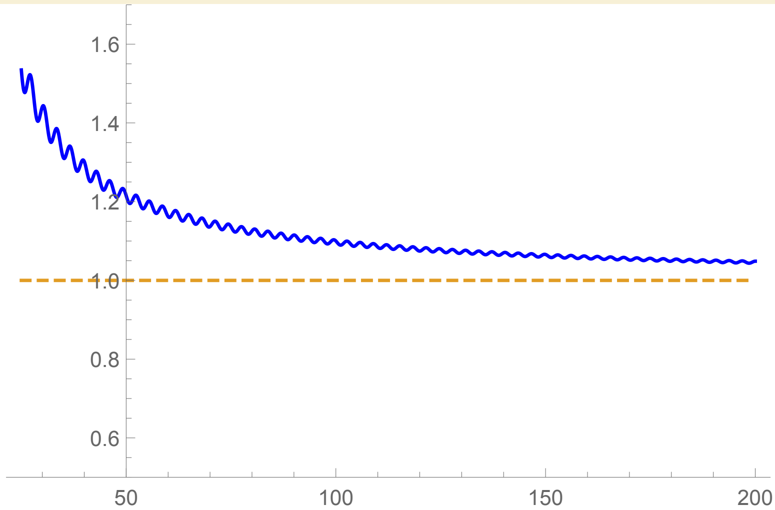
1d) Comparison with matter domination

Here we verify that at late times the universe behaves as matter dominated. For a matter dominated universe, the energy density would be given as ρ/a^3. As a consequence, the scalar factor grows as t^α. What is α? Check that the scale factor obtained numerically above indeed behaves as such power law. Plots where you should see this cleanly include:

```
Plot[a[t]/t^α, {t, 0, 500}, PlotStyle → Red, PlotRange → {0, 7 × 10^9}]
```




```
Plot[{1/α a'[t]/a[t] t, 1}, {t, 25, 200}, PlotRange → {1/2, 1.7}, PlotStyle → {Blue, Dashed},
 PlotPoints → 100]
```



2) Fluctuations [medium]

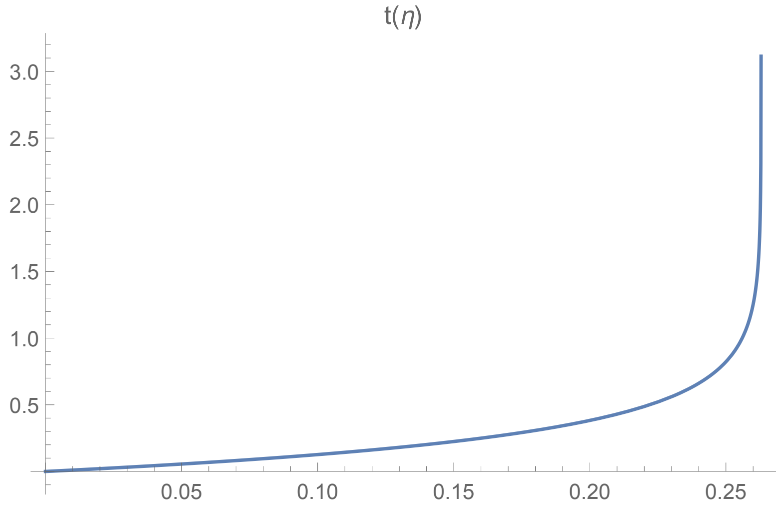
Here we study the behaviour of curvature fluctuations around the background solution studied in the previous part of the problem. (If you are curious, a step by step derivation of the curvature fluctuation action is done in another problem.) Here we assume it is known:

$$S = \frac{1}{2}\int d\eta d^3x\, a^2 \frac{\dot{\phi}^2}{h^2}\left[(\partial_\eta \zeta)^2 - (\partial_i \zeta)^2\right]$$

where η is the conformal time defined by $dt = a(t)d\eta$

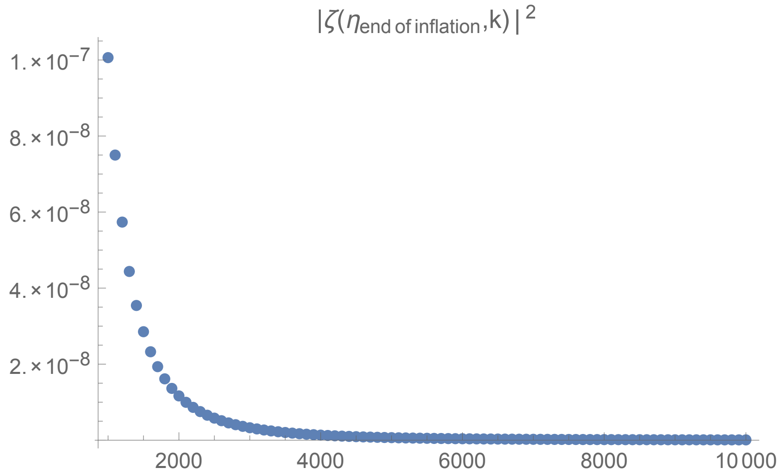
- Find the function $t(\eta)$. As a check, you should get something like this as a plot:

```
Plot[t[η], {η, 0, ηmax}, PlotRange → All, PlotLabel → "t(η)"]
```

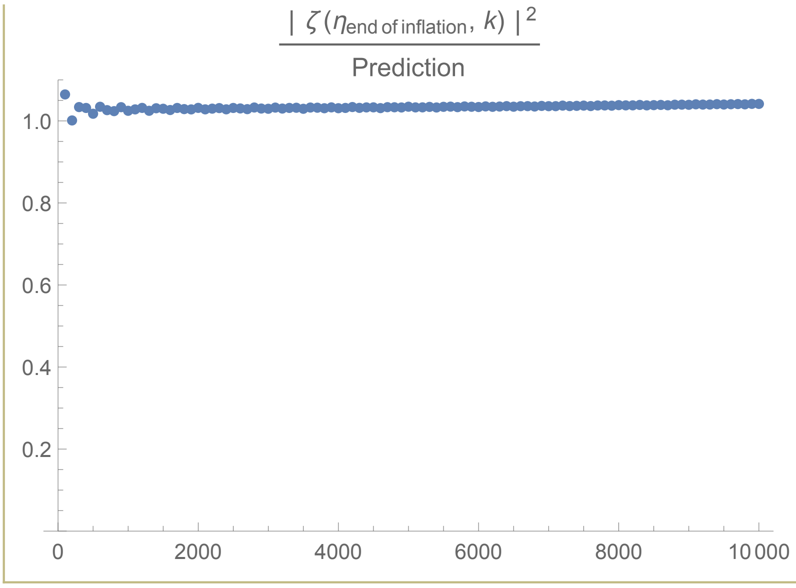


• Show that the equation for the time evolution for each of the Fourier mode in space reads $\partial_\eta(f\partial_\eta\zeta(\eta,k)) + fk^2\zeta(\eta,k) = 0$. What is f? Argue (using the Born-Sommerfeld quantization conditon for instance) that the vacuum state is described quasiclassically by $\zeta(\eta,k) \simeq \frac{e^{ik\eta}}{\sqrt{2f(\eta)k}}$. We will use this as our early time boundary condition latter on.

• Imposing $\zeta(\eta,k) \simeq \frac{e^{ik\eta}}{\sqrt{2f(\eta)k}}$ for some small η find the vacuum state $\zeta(\eta,k)$ at the end of the inflation phase. Repeat it for several different frequencies and plot the outcome. You should find a plot of the sort



• Check numerically that $|\zeta|^2 \simeq \frac{1}{2k^3}\frac{H(t_k^*)^4}{\dot\phi(t_k^*)^2}$ where t_k^* is the time when the mode k exits the horizon. This corresponds to $a(t_k^*)H(t_k^*) = k$. You should find a nice match as illustrated here:



The power spectrum (which as we observed is approximately scale invariant) will be the starting point of the next exercise. There we shall generate random distributions of curvature fluctuations and generate interesting plots resembling those observed in the CMB sky.

3) Sky [easy/medium]

Consider the fluctuations with the spectrum that you got for in the last problem. For simplicity you can also just take a scale invariant $1/k^d$ spectrum. Plot examples of random position space distributions. You can consider first a 1d example, then 2d, and finally 3d (you can plot slices of the 3d solutions). What is the typical amplitude for the fluctuations in the fluctuations in these plots?

Next you can try plotting examples of distributions that are not scale incariant but that instead have peaks at specific length scales. Note that the acoustic peaks in the CMB sky are an exmaple of such a distrubution.

A typical example of such plot would be