Quark–anti-quark potential in $N = 4$ SYM

by N.Gromov

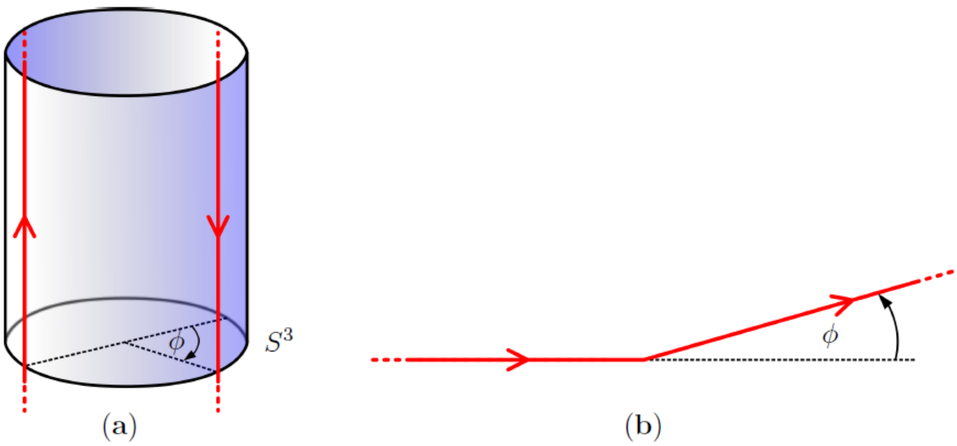


Figure 1: (**a**) *A quark–anti-quark pair sitting at two points on S^3 at a relative angle $\pi - \phi$. The quark–anti-quark lines are extended along the (Euclidian) time direction.* (**b**) *Under the cylinder to plane conformal map, the quark and anti-quark lines in* (**a**) *are mapped to the two half lines of a Wilson line with a cusp angle ϕ.*

The most natural quarks in $\mathcal{N} = 4$ SYM are infinitely massive W-bosons on the boundary of the Coulomb branch. These are locally supersymmetric quarks probes that also couple to a scalar. As we have six scalars, this coupling selects a point $\vec{n}$ in S^5. As a result, one of the new key features of the cusp TBA system is that it is parameterized by two continuous parameters. That is, Γ_{cusp} is a function of two angles ϕ and θ [18]. The angle ϕ is the geometrical angle between the two lines, see figure 1. The second angle θ, is the angle on S^5 between the quark and anti-quark points $\cos\theta = \vec{n}_q \cdot \vec{n}_{\bar{q}}$. The corresponding cusped Wilson loop is

$$W_0 = \mathrm{P}\exp\int_{-\infty}^{0} dt\,\left[iA\cdot\dot{x}_q + \vec{\Phi}\cdot\vec{n}_q\,|\dot{x}_q|\right] \times \mathrm{P}\exp\int_{0}^{\infty} dt\,\left[iA\cdot\dot{x}_{\bar{q}} + \vec{\Phi}\cdot\vec{n}_{\bar{q}}\,|\dot{x}_{\bar{q}}|\right] \qquad (2)$$

where $\vec{\Phi}$ is a vectors made of the six scalars of $\mathcal{N} = 4$ SYM. Here, $x_q(t)$ and $x_{\bar{q}}(t)$ are two straight lines representing the quark and anti-quark trajectories. They connect the origin and infinity such that $\dot{x}_q \cdot \dot{x}_{\bar{q}}/(|\dot{x}_q||\dot{x}_{\bar{q}}|) = \cos\phi$.

We also consider a generalization of this observable with L scalar fields inserted at the cusp.

Solving Pμ-system for any coupling

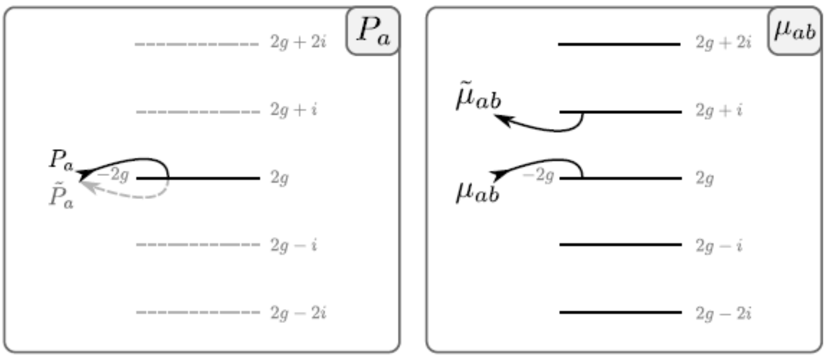


FIG. 2. Cut structure of $\mathbf{P}$ and μ

For this excersice we take near-BPS limit $\phi \sim \theta \sim 0$. This observable can be studied using exactly the same system of $\mathbf{P}\mu$ equations as for the local operators. The difference appear at the level of asymptotics.

$$\tilde{\mathbf{P}}_a = -\mu_{ab}\chi^{bc}\mathbf{P}_c \;\;,\;\; \tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \;\;,\;\; \mu\chi\mu\chi = 1$$

where χ is a constant matrix

```
χ = ( □   □   □  -1
      □   □   1   □
      □  -1   □   □
      1   □   □   □ ) /. □ → 0;
```

The information about the state comes from asymptotics:

$$\mathbf{P}_a \simeq (A_1 u^{-L}, A_2 u^{-L-1}, A_3 u^{L+1}, A_4 u^L)$$

where

$$A_1 A_4 = A_2 A_3 = i\phi^2 \;\;,\;\; \gamma = -\lim_{u\to\infty} iu^2(\mathbf{P}_1\mathbf{P}_4 - \mathbf{P}_2\mathbf{P}_3)$$

Pμ-system in the near BPS limit

In the near-BPS limit $\mathbf{P} \to 0$ which leads to the main simplification. We see that $\tilde{\mu} - \mu$ is small and thus μ does not have cuts and is simply an analytic periodic antysymmetric matrix. It can be written in the form

```
μ = (  0    μ1   μ2   μ3
      -μ1   0    μ3   μ4
      -μ2  -μ3   0    μ5
      -μ3  -μ4  -μ5   0  );
```

Where the special feature of this observable is that μ_1 is

```
μ1 = C1 Sinh[2 π u];
```

the freedom in redefining $\mathbf{P}_a$ allows to set $\mu_2 = \mu_4 = 0$. Show that $\mu\chi\mu\chi = 1_{4\times4}$ implies

```
μ = {{0, C1 Sinh[2 π u], 0, -1}, {-C1 Sinh[2 π u], 0, -1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}
```

$$\begin{pmatrix} 0 & C1\sinh(2\pi u) & 0 & -1 \\ -C1\sinh(2\pi u) & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

After that the system reduces to the following Riemann-Hilbert problem

$$\begin{aligned} &\tilde{\mathbf{P}}_1 - \mathbf{P}_1 = -C\sinh(2\pi u)\mathbf{P}_3 \;\;,\;\; \tilde{\mathbf{P}}_3 + \mathbf{P}_3 = 0 \\ &\tilde{\mathbf{P}}_2 + \mathbf{P}_2 = -C\sinh(2\pi u)\mathbf{P}_4 \;\;,\;\; \tilde{\mathbf{P}}_4 - \mathbf{P}_4 = 0 \end{aligned}$$

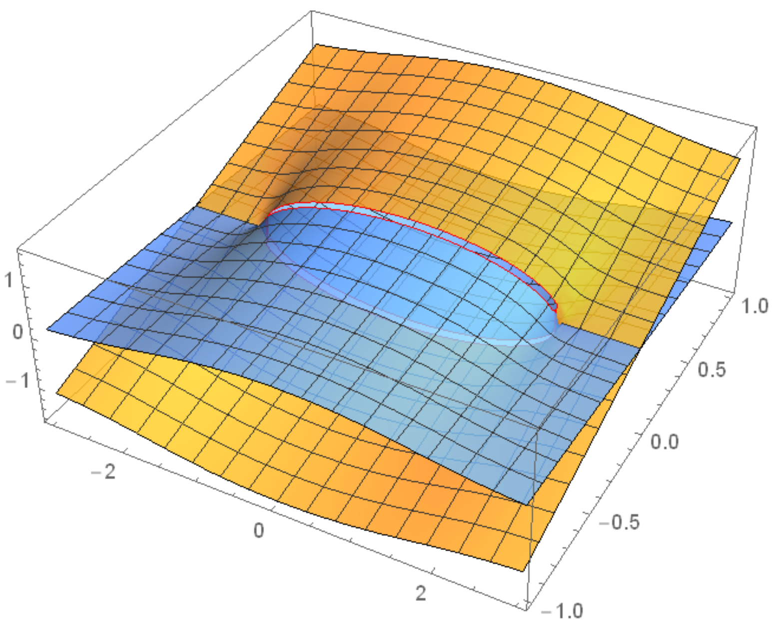
Solving Riemann-Hilbert problem

As we know on the main sheet $\mathbf{P}_a$ has only one cut. In the near BPS limit on the next sheet there is again only one cut $[-2g, 2g]$, because μ is almost trivial. Thus $\mathbf{P}_a$ is simply a double valued function, leaving on the Riemann surface which is equivalent to a sphere. We use this to retionalize $\mathbf{P}_a$. The map which maps the complex plain into double covered complex plain is so-called Zhukovsky map $x + 1/x = u/g$

• One can write solution $x(u)$ of this equation such that the Plot3D of its imaginary part (and of the imaginary part of its analytical continuation $1/x(u)$) gives the following nice picture (for $g = 1$ for example). Check that the $x(u)$, which has only one cut $[-2g, 2g]$ is

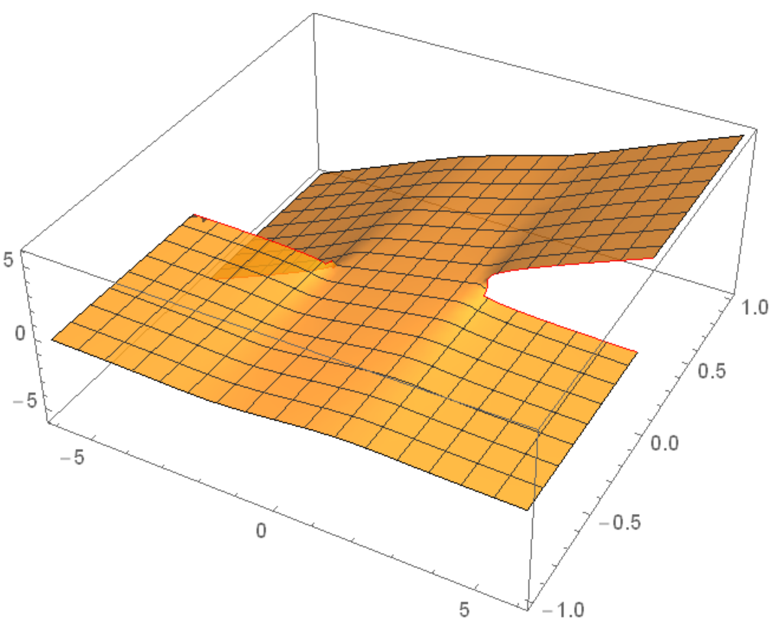
$$X[u\_] = \frac{\sqrt{u-2g}\,\sqrt{u+2g}+u}{2g};$$

```
Plot3D[{Im[X[u] /. g → 1 /. u → a + I b], Im[1 / X[u] /. g → 1 /. u → a + I b]},
  {a, -3, 3}, {b, -1, 1}, ExclusionsStyle → {None, Red},
  Exclusions → {{b == 0, Abs[a] < 2}}, PlotStyle → Opacity[0.8]] //
 Rasterize[#, RasterSize → 650] &
```



• For extra points find another solution Xmirror[u] which instead has the cuts going to infinity

```
Plot3D[{Re[Xmirror[u] /. g → 1 /. u → a + I b]}, {a, -6, 6}, {b, -1, 1},
  ExclusionsStyle → {None, Red}, Exclusions → {{b == 0, Abs[a] > 2}},
  PlotStyle → Opacity[0.8]] // Rasterize[#, RasterSize → 650] &
```



• In terms of the new variable x functions $\mathbf{P}$ are an analytic function on the complex plain with only a possible singularity at $x=0$ or $x=\infty$. We just have to find the expansion coefficients

$$\mathbf{P}_a(x)=\sum_{n=-\infty}^{N_a} c_{a,n}x^n \quad , \quad N_1=-L,\ N_2=-L-1,\ N_3=L+1,\ N_4=L$$

from the equation

$$\mathbf{P}_1(1/x)-\mathbf{P}_1(x)=-C\sinh(2\pi g(x+1/x))\mathbf{P}_3(x) \quad , \quad \mathbf{P}_3(1/x)+\mathbf{P}_3(x)=0$$

$$\mathbf{P}_2(1/x)+\mathbf{P}_2(x)=-C\sinh(2\pi g(x+1/x))\mathbf{P}_4(x) \quad , \quad \mathbf{P}_4(1/x)-\mathbf{P}_4(x)=0$$

→ Check numerically with very heigh precision that for some $g\sim 1$ and $x\sim 1$ that

$$\sinh(2\pi g(x+1/x))=\sum_{n=-\infty}^{\infty} I_{2n+1}(4\pi g)x^{2n+1}$$

where I_n is `BesselI` in `Mathematica`

$-6.4991651618287995642057837803991557950 59\times 10^{-80}$

→ For $L=0$, $L=2$, $L=4$ and $L=6$ find $\mathbf{P}_a$ by truncating infinite sums in x^n at some large number and by requiring that most of the terms cancel. Fix asymptotics and compute the energy γ

```
(*For L=0 you find*)
Print["γ=", SolvePμ[0] // FullSimplify]
Print["P3=", p3]
Print["P4=", p4]
```

$$\gamma=\frac{g\,\phi^2\, I_2(4\,g\,\pi)}{\pi\, I_1(4\,g\,\pi)}$$

$$\text{P3}=\frac{c(3,-1)}{x}-x\,c(3,-1)$$

$$\text{P4}=\frac{i\,\phi^2}{2\,C\,I_1(4\,g\,\pi)\,c(3,-1)}$$

```
(*For L=2 you find*)
Print["γ=", SolvePμ[2] // FullSimplify]
Print["P3=", p3]
Print["P4=", p4]
```

$$\gamma = \frac{\phi^2 \left(-\frac{12 g \pi I_1(4 g \pi)}{I_2(4 g \pi)} + \frac{3\left((8\pi^2 g^2+9) I_2(4 g \pi) - 9 g \pi I_1(4 g \pi)\right) I_1(4 g \pi)}{3 g \pi I_1(4 g \pi)^2 - 3 I_2(4 g \pi) I_1(4 g \pi) - g \pi I_2(4 g \pi)^2} + 6 + \frac{2 g \pi I_2(4 g \pi)}{I_1(4 g \pi)} \right)}{2\pi^2}$$

$$\begin{aligned}\mathrm{P3} = &-\frac{i \phi^2 I_1(4 g \pi) x^3}{2 C \left(I_1(4 g \pi)^2 + (I_5(4 g \pi) - I_3(4 g \pi)) I_1(4 g \pi) - I_3(4 g \pi)^2\right) c(4, -2)} + \\ &\frac{i \phi^2 \left(I_3(4 g \pi)^2 + I_1(4 g \pi)(I_3(4 g \pi) - 2 I_5(4 g \pi))\right) x}{2 C (I_1(4 g \pi) - I_3(4 g \pi)) \left(I_1(4 g \pi)^2 + (I_5(4 g \pi) - I_3(4 g \pi)) I_1(4 g \pi) - I_3(4 g \pi)^2\right) c(4, -2)} - \\ &\frac{i \phi^2 \left(I_3(4 g \pi)^2 + I_1(4 g \pi)(I_3(4 g \pi) - 2 I_5(4 g \pi))\right)}{2 C (I_1(4 g \pi) - I_3(4 g \pi)) \left(I_1(4 g \pi)^2 + (I_5(4 g \pi) - I_3(4 g \pi)) I_1(4 g \pi) - I_3(4 g \pi)^2\right) c(4, -2)\, x} + \\ &\frac{i \phi^2 I_1(4 g \pi)}{2 C \left(I_1(4 g \pi)^2 + (I_5(4 g \pi) - I_3(4 g \pi)) I_1(4 g \pi) - I_3(4 g \pi)^2\right) c(4, -2)\, x^3}\end{aligned}$$

$$\mathrm{P4} = c(4, -2)\, x^2 - \frac{(I_1(4 g \pi) + I_3(4 g \pi))\, c(4, -2)}{I_1(4 g \pi)} + \frac{c(4, -2)}{x^2}$$

$\rightarrow$ Compare your result for Δ with the result of 1207.5489 eq.125 i.e.

$$\gamma = \phi^2 g^2 \left(-\frac{\det \mathcal{M}_{L+2}^{(2,1)}}{\det \mathcal{M}_{L+2}^{(1,1)}} + 2\frac{\det \mathcal{M}_{L+1}^{(2,1)}}{\det \mathcal{M}_{L+1}^{(1,1)}} - \frac{\det \mathcal{M}_{L}^{(2,1)}}{\det \mathcal{M}_{L}^{(1,1)}} \right)$$

where $\mathcal{M}_L^{(a,b)}$ is the matrix obtained by deleting the a^{th} row and b^{th} column of $\mathcal{M}_L$

$$\mathcal{M}_L = \begin{pmatrix} I_{-1} & I_1 & \dots & I_{2L-3} & I_{2L-1} \\ I_{-3} & I_{-1} & \dots & I_{2L-5} & I_{2L-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{1-2L} & I_{3-2L} & \dots & I_{-1} & I_1 \\ I_{-1-2L} & I_{1-2L} & \dots & I_{-3} & I_{-1} \end{pmatrix}$$

Do the comparison perturbatively in g first. For $L = 2$ you get

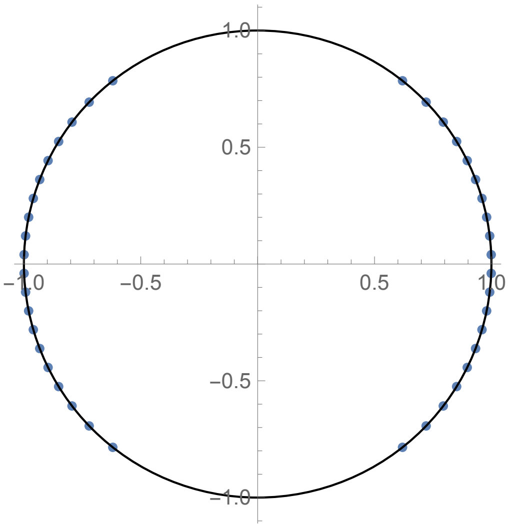
$$\boxed{\frac{14}{45}\pi^4 g^6 \phi^2 - \frac{40}{63} g^8 \left(\pi^6 \phi^2\right) + \frac{1934\, \pi^8 g^{10} \phi^2}{2025} - \frac{18\,352\, g^{12} \left(\pi^{10} \phi^2\right)}{14\,175} + \frac{119\,828\, \pi^{12} g^{14} \phi^2}{70\,875} + O(g^{15})}$$

$\rightarrow$ Show that $\mathbf{P}_4$ is proportional to

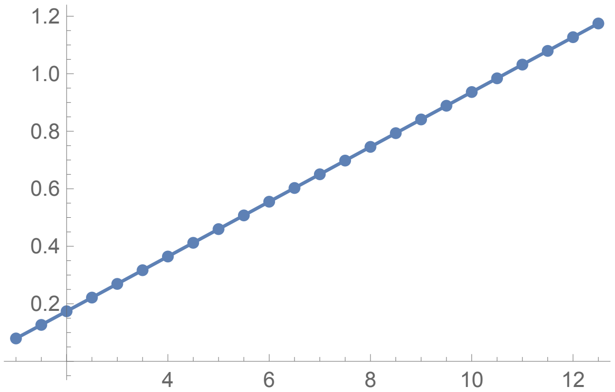
$$\begin{vmatrix} I_{-1} & I_1 & \dots & I_{2L-3} & I_{2L-1} \\ I_{-3} & I_{-1} & \dots & I_{2L-5} & I_{2L-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{1-2L} & I_{3-2L} & \dots & I_{-1} & I_1 \\ 1/x^L & 1/x^{L-2} & \dots & x^{L-2} & x^L \end{vmatrix}$$

$\rightarrow$ Find zeros of $\mathbf{P}_4$ for $L = 20$ and $g = 5$ numerically. Make the following plot

Solve::ratnz : Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result. >>



• Find classical limit. For that compute numerically γ with heigh precision for $L = 2, 4, ..., 50$ keeping $g = L/4$. Make a fit with inverse powers of g and extrapolate the result to infinity



$\rightarrow$ Compare the leading linear coefficient with the classical string prediction, given in the parametric form

$$L = 4g(K(\omega) - E(\omega)) \quad , \quad \gamma = \phi^2 g \frac{1-\omega}{2E(\omega)}$$

where K,E are the complite elliptic integrals `EllipticE` and `EllipicK`

```
g (1 - ω)/(2 EllipticE[ω]) /. Ω[4]
```

0.095365567022833140305292863539 g

```
Coefficient[fit, g] g
```

0.095365567022833140305292823027841460684589952002185619214700441542647992671088393842903008019582⋱
814114735783451710259112074411828814391776158383555464391283138825347649165504463914140 g

```
% - %%
```

$-4.0511 \times 10^{-26}\, g$

you should be able to get easily at least 20 digits