```
<< Local`QFTToolKit`

$def = {};
ct[a_] := ConjugateTranspose[a];
PR[CO[
  "We use equivalence symbol for isomorphism, and Mod[] symbol for quotient group."]
]
```

We use equivalence symbol for isomorphism, and Mod[] symbol for quotient group.

```
PR["● M→4-d manifold with canonical triple ", {C"∞"[M], L^2[M, S], slash[D]},
 NL, "The connection: ", $connection = "∇"^S[S[]],
 NL, "Dirac operator: ",
 {slash[D][ψ_] → -I T[γ, "u", {μ}].T["∇"^S, "d", {μ}][ψ], ψ ∈ Γ[M, S],
   T["∇"^S, "d", {μ}][f ψ] → f "∇"^S[ψ] + tuPartialD[f, μ] ψ,
   CommutatorM[slash[D], f].ψ → -I T[γ, "u", {μ}]. tuPartialD[f, μ].ψ
  } // Column,
 NL, "Have ℤ2-grading(chirality): ",
 {T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}], T[γ, "d", {5}].T[γ, "d", {5}] → 1,
   ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
   T[γ, "d", {5}][L^2[M, S]] -> L^2[M, S]^+ ⊕ L^2[M, S]^-} // Column,
 NL, "Charge conjugation: ", J_M → {J_M.J_M → -1, CommutatorM[J_M, slash[D]] → 0,
    CommutatorM[J_M, T[γ, "d", {5}]] → 0} // ColumnForms
]
```

● M→4-d manifold with canonical triple $\{C^\infty[M], L^2[M, S], \not{D}\}$

The connection: $\nabla^S[S[]]$

Dirac operator:
$$(\not{D})[\psi\_] \to -i\, \gamma^\mu . \nabla^S{}_\mu[\psi]$$
$$\psi \in \Gamma[M, S]$$
$$\nabla^S{}_\mu[f\, \psi] \to f\, \nabla^S[\psi] + \psi\, \underset{-\mu}{\partial}[f]$$
$$[\not{D}, f].\psi \to -i\, \gamma^\mu . \underset{-\mu}{\partial}[f].\psi$$

Have $\mathbb{Z}_2$-grading(chirality):
$$\gamma_5 \to \gamma^1\, \gamma^2\, \gamma^3\, \gamma^4$$
$$\gamma_5 . \gamma_5 \to 1$$
$$(\gamma_5)^\dagger \to \gamma_5$$
$$\gamma_5[L^2[M, S]] \to L^2[M, S]^+ \oplus L^2[M, S]^-$$

Charge conjugation: $J_M \to$
$$J_M . J_M \to -1$$
$$[J_M, \not{D}] \to 0$$
$$[J_M, \gamma_5] \to 0$$

```
PR["● F→finite space triple: ", F → {𝒜F, ℋF, 𝒟F},
 " where ", {𝒜F → MN[ℂ], ℋF → "N-dim complex Hilbert space",
   𝒟F -> "hermitian MN[ℂ]", MN[ℂ] → "NxN matrix"} // Column,
 NL, "•ℋF is ℤ2 graded (even) if ∃ a grading operator: ",
 γF ∋ {ConjugateTranspose[γF] → γF, γF γF → 1, γF[ℋF] -> ℋF+ ⊕ ℋF-,
     {γF[ψ ∈ ℋF] → ±ψ},
     CommutatorM[γF, a ∈ AF] → 0,
     CommutatorP[γF, 𝒟F] → 0
    } // ColumnForms
]
```

● F→finite space triple: $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F\}$ where $\begin{array}{l} \mathcal{A}_F \to M_N[\mathbb{C}] \\ \mathcal{H}_F \to \text{N-dim complex Hilbert space} \\ \mathcal{D}_F \to \text{hermitian } M_N[\mathbb{C}] \\ M_N[\mathbb{C}] \to \text{NxN matrix} \end{array}$

•$\mathcal{H}_F$ is $\mathbb{Z}_2$ graded (even) if ∃ a grading operator: $\gamma_F \ni \begin{array}{l} (\gamma_F)^\dagger \to \gamma_F \\ \gamma_F^2 \to 1 \\ \gamma_F[\mathcal{H}_F] \to (\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^- \\ \{\gamma_F[\psi \in \mathcal{H}_F] \to \pm\psi\} \\ [\gamma_F, a \in A_F] \to 0 \\ \{\gamma_F, \mathcal{D}_F\} \to 0 \end{array}$

```
εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1, }}},
   {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
  ];
PR["Almost-commutative spin manifold: ",
 $ = M × F → {C"∞"[M, 𝒜F], L^2[M, S] ⊗ ℋF, 𝒟 -> slash[𝒟] ⊗ 1N + T[γ, "d", {5}] ⊗ 𝒟F};
 ColumnForms[$],
 NL, "with grading: ", γ -> T[γ, "d", {5}] ⊗ γF,
 NL, "•Distance: ", d𝒟[x, y] → sup[‖a[x] - a[y]‖ | a ∈ 𝒜 && ‖CommutatorM[𝒟, a]‖ ≤ 1],
 NL, "●Charge conjugation for F: even space F is real if ∃ ",
 $J = JF[ℋF] ∋ { JF.JF -> ε, JF.𝒟F -> ε'.𝒟F.JF, JF.γF -> ε''.γF.JF};
 ColumnForms[$J],
 NL, "where the routine εRule[KOdim_] is provided ",
 CR[" What is the meaning of ε's?"],
 NL, "•", $ = ForAll[{a, b}, a | b ∈ 𝒜F,
   {CommutatorM[a, b"0"] → 0, b"0" → JF.ConjugateTranspose[b].ConjugateTranspose[JF]}],
 $def = $def // tuAppendUniq[$];
 NL, "•", $ = ForAll[{a, b}, a | b ∈ 𝒜F, {CommutatorM[CommutatorM[𝒟F, a], b"0"] → 0,
     b"0" → JF.ConjugateTranspose[b].ConjugateTranspose[JF]}],
 $def = $def // tuAppendUniq[$];
]
```

Almost-commutative spin manifold: $M \times F \to \begin{array}{l} C^\infty[M, \mathcal{A}_F] \\ L^2[M, S] \otimes \mathcal{H}_F \\ \mathcal{D} \to (\not{D}) \otimes 1_N + \gamma_5 \otimes \mathcal{D}_F \end{array}$

with grading: $\gamma \to \gamma_5 \otimes \gamma_F$
•Distance: $d_{\mathcal{D}}[x, y] \to \sup[\|a[x] - a[y]\| \mid a \in \mathcal{A} \,\&\&\, \|[\mathcal{D}, a]\| \le 1]$

●Charge conjugation for F: even space F is real if ∃ $J_F[\mathcal{H}_F] \ni \begin{array}{l} J_F.J_F \to \varepsilon \\ J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F \\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F \end{array}$

where the routine εRule[KOdim_] is provided What is the meaning of ε's?
•$\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$
•$\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[[\mathcal{D}_F, a], b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$Null

```
PR["●Lemma2.7. Definition 2.5: ", $J[[2]],
 NL, "Where γF decomposes ", $h = ℋ → Table[ℋi,j, {i, 2}, {j, 2}];
 MatrixForms[$h],
```

```
 " into ", ℋ → ℋ⁺ ⊕ ℋ⁻ , " i.e. ", $gh = γF.ℋ → {{ℋ⁺, 0}, {0, ℋ⁻}};
 MatrixForms[$gh],
 $gh0 = $gh /. {ℋ⁺ -> ℋ1,1, ℋ⁻ -> ℋ2,2};
 Yield, $gh1 = γF.{{a_, b_}, {c_, d_}} -> DiagonalMatrix[{a, d}];
 MatrixForms[$gh1],
 NL, "Represent ", $j = JF → Table[ji,j, {i, 2}, {j, 2}];
 MatrixForms[$j], " of the same dimensions.",
 NL, "•For: ",
 $JF = {JF → U.cc, U.ConjugateTranspose[U] → 1N, U ∈ U[ℋ"±"], cc → Conjugate},
 NL, "where: ",
 $cc = {ConjugateTranspose[cc] → cc,
   Conjugate[cc] → cc, cc . cc → 1, cc.a_ :→ Conjugate[a].cc},
 Imply, $0 = $ = JF.ConjugateTranspose[JF],
 yield, $ = $0 → ($ /. $JF[[1]] // tuRepeat[$cc, ConjugateCTSimplify1[{cc}]]);
 Framed[$];
 $ = $ /. $JF[[2]]; Framed[$],
 Yield, $ = $ /. ConjugateTranspose → SuperDagger /. Dot → xDot /. $j /.
   SuperDagger[a_] :→ Map[Thread[SuperDagger[#]] &, Transpose[a]] /; MatrixQ[a];
 MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ], CK
]
PR[
 line, "•For ", $s = n → 0; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0;
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] → γF.xDot[a];
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$];
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$];
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. 1 → 1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "•Then we have: ", $ = {$JJ1, $JJ, $Jg}; ColumnForms[$],
 Yield, $ = $ /. j1,2 | j2,1 → 0 // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply, {ConjugateTranspose[j1,1] -> j1,1, ConjugateTranspose[j2,2] -> j2,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 2; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
    γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
```

```
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {H1,2 | H2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply,
 {ConjugateTranspose[j1,2] → -j2,1, ConjugateTranspose[j2,1] → -j1,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 4; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . H & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
    γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {H1,2 | H2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,2 | j2,1 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply,
 {ConjugateTranspose[j1,1] → -j1,1, ConjugateTranspose[j2,2] → -j2,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 6; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . H & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
    γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
```

```
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. 1 → 1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply, {ConjugateTranspose[j1,2] → j2,1, ConjugateTranspose[j2,1] → j1,2} // FramedColumn
]
```

●Lemma2.7. Definition 2.5: $\{J_F.J_F \to \varepsilon,\ J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F,\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F\}$

Where γ_F decomposes $\mathcal{H} \to \begin{pmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} \end{pmatrix}$ into $\mathcal{H} \to \mathcal{H}^+ \oplus \mathcal{H}^-$ i.e. $\gamma_F.\mathcal{H} \to \begin{pmatrix} \mathcal{H}^+ & 0 \\ 0 & \mathcal{H}^- \end{pmatrix}$

$\to\ \gamma_F.\begin{pmatrix} a\_ & b\_ \\ c\_ & d\_ \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Represent $J_F \to \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}$ of the same dimensions.

•For: $\{J_F \to U.cc,\ U.U^\dagger \to 1_N,\ U \in U[\mathcal{H}^\pm],\ cc \to \text{Conjugate}\}$

where: $\{cc^\dagger \to cc,\ cc^* \to cc,\ cc.cc \to 1,\ cc.(a\_) \mapsto a^*.cc\}$

$\Rightarrow\ J_F.(J_F)^\dagger \longrightarrow \boxed{J_F.(J_F)^\dagger \to 1_N}$

$\to\ \text{xDot}\left[\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}, \begin{pmatrix} (j_{1,1})^\dagger & (j_{2,1})^\dagger \\ (j_{1,2})^\dagger & (j_{2,2})^\dagger \end{pmatrix}\right] \to 1_N$

$\to\ \begin{pmatrix} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger & j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger & j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \end{pmatrix} \to 1_N$

$\to\ \{\{j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger,\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger\},$
$\{j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger,\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger\}\} \to \{\{1_{N^+}, 0\}, \{0, 1_{N^-}\}\}$

$$\to\ \boxed{\begin{array}{l} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \to 0 \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger \to 0 \\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \to 1_{N^-} \end{array}} \longleftarrow \text{CHECK}$$

•For $\boxed{n \to 0}$ $\longrightarrow$ $\begin{matrix} J_F . J_F \to 1 \\ J_F . \gamma_F \to \gamma_F . J_F \end{matrix}$

$\to$ $\boxed{J_F . \gamma_F \to \gamma_F . J_F}$ $\longrightarrow$ $J_F . \gamma_F . \mathcal{H} \to \gamma_F . J_F . \mathcal{H}$

....

$\to \begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

$\to \boxed{\begin{matrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{matrix}}$ $\longleftarrow$CHECK

•For $\boxed{J_F . J_F \to 1_N}$

$\to$ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] $\to 1_N$

$\to \begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to 1_N$

$\to$

$\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1},\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\},\ \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1},\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to \{\{1_{N^+}, 0\}, \{0, 1_{N^-}\}\}$

$\to \boxed{\begin{matrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \end{matrix}}$ $\longleftarrow$CHECK

•Then we have:

$\left\{ \begin{matrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{matrix} \right\}$

$\to \left\{ \begin{matrix} j_{1,1} . j_{1,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . j_{2,2} \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} . (j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{matrix} \right\}$

$\Rightarrow \boxed{\begin{matrix} (j_{1,1})^\dagger \to j_{1,1} \\ (j_{2,2})^\dagger \to j_{2,2} \end{matrix}}$

•For $\boxed{n \to 2}$ $\longrightarrow$ $J_F . J_F \to -1$, $J_F . \gamma_F \to -\gamma_F . J_F$

$\to$ $\boxed{J_F . \gamma_F \to -\gamma_F . J_F}$ $\longrightarrow$ $J_F . \gamma_F . \mathcal{H} \to (-\gamma_F . J_F) . \mathcal{H}$

....

$$\to \begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$$

$$\to \boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{array}} \longleftarrow \text{CHECK}$$

•For $\boxed{J_F . J_F \to -1_N}$

$\to$ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] $\to -1_N$

$$\to \begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to -1_N$$

$\to$ {{$j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1}$, $j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}$}, {$j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1}$, $j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}$}} $\to$ {{-1_{N^+}, 0}, {0, -1_{N^-}}}

$$\to \boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}} \longleftarrow \text{CHECK}$$

with: {$\mathcal{H}_{1,2}$ | $\mathcal{H}_{2,1} \to 0$}

•All conditions:

$$\left\{ \begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}, \begin{array}{l} j_{1,1} . (j_{1,1})^{\dagger} + j_{1,2} . (j_{1,2})^{\dagger} \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^{\dagger} + j_{1,2} . (j_{2,2})^{\dagger} \to 0 \\ j_{2,1} . (j_{1,1})^{\dagger} + j_{2,2} . (j_{1,2})^{\dagger} \to 0 \\ j_{2,1} . (j_{2,1})^{\dagger} + j_{2,2} . (j_{2,2})^{\dagger} \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,2} . \mathcal{H}_{2,2} \end{array} \right\}$$

$\Rightarrow$ $\boxed{j_{1,1} \mid j_{2,2} \to 0}$

$$\to \left\{ \begin{array}{l} j_{1,2} . j_{2,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . j_{1,2} \to -1_{N^-} \end{array}, \begin{array}{l} j_{1,2} . (j_{1,2})^{\dagger} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . (j_{2,1})^{\dagger} \to 1_{N^-} \end{array}, \begin{array}{l} 0 \to 0 \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{array} \right\}$$

$$\Rightarrow \boxed{\begin{array}{l} (j_{1,2})^{\dagger} \to -j_{2,1} \\ (j_{2,1})^{\dagger} \to -j_{1,2} \end{array}}$$

•For $\boxed{n \to 4}$ $\longrightarrow$ $J_F . J_F \to -1$
$J_F . \gamma_F \to \gamma_F . J_F$

$\to$ $\boxed{J_F . \gamma_F \to \gamma_F . J_F}$ $\longrightarrow$ $J_F . \gamma_F . \mathcal{H} \to \gamma_F . J_F . \mathcal{H}$

....

$$\to \begin{pmatrix} j_{1,1}.\mathcal{H}_{1,1} \to j_{1,1}.\mathcal{H}_{1,1} + j_{1,2}.\mathcal{H}_{2,1} & j_{1,2}.\mathcal{H}_{2,2} \to 0 \\ j_{2,1}.\mathcal{H}_{1,1} \to 0 & j_{2,2}.\mathcal{H}_{2,2} \to j_{2,1}.\mathcal{H}_{1,2} + j_{2,2}.\mathcal{H}_{2,2} \end{pmatrix}$$

$$\to \boxed{\begin{array}{l} j_{1,1}.\mathcal{H}_{1,1} \to j_{1,1}.\mathcal{H}_{1,1} + j_{1,2}.\mathcal{H}_{2,1} \\ j_{1,2}.\mathcal{H}_{2,2} \to 0 \\ j_{2,1}.\mathcal{H}_{1,1} \to 0 \\ j_{2,2}.\mathcal{H}_{2,2} \to j_{2,1}.\mathcal{H}_{1,2} + j_{2,2}.\mathcal{H}_{2,2} \end{array}} \longleftarrow \text{CHECK}$$

•For $\boxed{J_F . J_F \to -1_N}$

$\to$ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] $\to -1_N$

$$\to \begin{pmatrix} j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1} & j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2} \\ j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1} & j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2} \end{pmatrix} \to -1_N$$

$\to$ {{$j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1}$, $j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2}$}, {$j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1}$, $j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2}$}} $\to$
{{-1_{N^+}, 0}, {0, -1_{N^-}}}

$$\to \boxed{\begin{array}{l} j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1} \to -1_{N^+} \\ j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2} \to 0 \\ j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1} \to 0 \\ j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2} \to -1_{N^-} \end{array}} \longleftarrow \text{CHECK}$$

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions:
$$\left\{ \begin{array}{l} j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1} \to -1_{N^+} \\ j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2} \to 0 \\ j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1} \to 0 \\ j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2} \to -1_{N^-} \end{array}, \begin{array}{l} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \to 0 \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger \to 0 \\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,1}.\mathcal{H}_{1,1} \to j_{1,1}.\mathcal{H}_{1,1} \\ j_{1,2}.\mathcal{H}_{2,2} \to 0 \\ j_{2,1}.\mathcal{H}_{1,1} \to 0 \\ j_{2,2}.\mathcal{H}_{2,2} \to j_{2,2}.\mathcal{H}_{2,2} \end{array} \right\}$$

$\Rightarrow$ $\boxed{j_{1,2} \mid j_{2,1} \to 0}$

$$\to \left\{ \begin{array}{l} j_{1,1}.j_{1,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2}.j_{2,2} \to -1_{N^-} \end{array}, \begin{array}{l} j_{1,1}.(j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2}.(j_{2,2})^\dagger \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,1}.\mathcal{H}_{1,1} \to j_{1,1}.\mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2}.\mathcal{H}_{2,2} \to j_{2,2}.\mathcal{H}_{2,2} \end{array} \right\}$$

$$\Rightarrow \boxed{\begin{array}{l} (j_{1,1})^\dagger \to -j_{1,1} \\ (j_{2,2})^\dagger \to -j_{2,2} \end{array}}$$

•For $\boxed{n \to 6}$ $\longrightarrow$ $J_F . J_F \to 1$
$J_F . \gamma_F \to -\gamma_F . J_F$

$\to$ $\boxed{J_F . \gamma_F \to -\gamma_F . J_F}$ $\longrightarrow$ $J_F . \gamma_F . \mathcal{H} \to (-\gamma_F . J_F) . \mathcal{H}$

....

$$\to \begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$$

$$\to \boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{array}} \longleftarrow \text{CHECK}$$

•For $\boxed{J_F . J_F \to 1_N}$

$\to$ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] $\to 1_N$

$$\to \begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to 1_N$$

$\to$
$\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1},\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\},\ \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1},\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to \{\{1_{N^+}, 0\}, \{0, 1_{N^-}\}\}$

$$\to \boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \end{array}} \longleftarrow \text{CHECK}$$

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

$$\text{•All conditions: } \left\{ \begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to 1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,2} . \mathcal{H}_{2,2} \end{array} \right\}$$

$\Rightarrow$ $\boxed{j_{1,1} \mid j_{2,2} \to 0}$

$$\to \left\{ \begin{array}{l} j_{1,2} . j_{2,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . j_{1,2} \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . (j_{2,1})^\dagger \to 1_{N^-} \end{array}, \begin{array}{l} 0 \to 0 \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{array} \right\}$$

$$\Rightarrow \boxed{\begin{array}{l} (j_{1,2})^\dagger \to j_{2,1} \\ (j_{2,1})^\dagger \to j_{1,2} \end{array}}$$

Commutative Subalgebras

```
PR["● Define subalgebra of 𝒜: ",
 $sAt = 𝒜̃J → {a ∈ 𝒜, a.J → J.ConjugateTranspose[a], a^"0" → a},
 NL, "•Unitary group: ",
 U[𝒜] → {u ∈ 𝒜, u.ConjugateTranspose[u] | ConjugateTranspose[u].u → 1N},
 Imply, ForAll[x ∈ M,
  u[x].ConjugateTranspose[u[x]] | ConjugateTranspose[u[x]].u[x] → 1N],
 Imply, u ∈ U[𝒜] ⇔ u[x] ∈ U[𝒜F],
 NL, "•Lie algebra: ", u[𝒜] → {X ∈ 𝒜, ConjugateTranspose[X] → -X} → C^"∞"[M, u[𝒜F]],
 NL, "•Special unitary group: ", SU[𝒜F] → {𝓊 ∈ U[𝒜F], Det[u] → 1},
 NL, "•Lie algebra SU[𝒜F]: ", su[𝒜F] → {X ∈ 𝒜F, ConjugateTranspose[X] → -X, Tr[X] → 0},
 line,
 "●Adjoint action. space: ", $F = F → Table[Subscript[i, F], {i, {𝒜, ℋ, 𝒟, γ, J}}],
 NL, "Define: for ", ξ ∈ $F[[2, 2]],
 Yield, $ = {Ad[U[𝒜F]] → Endo[$F[[2, 2]]], ad[u[$F[[2, 1]]]] → Endo[$F[[2, 2]]]};
 Column[$],
 yield, $ = {Ad[u][ξ] → u.ξ.ConjugateTranspose[u] → u.ConjugateTranspose[u]^"0".ξ,
   ad[A][ξ] → A.ξ - ξ.A → (A - A^"0").ξ}; Column[$]
]
```

● **Define subalgebra of** $\mathcal{A}$: $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A},\ a.J \to J.a^{\dagger},\ a^0 \to a\}$

•**Unitary group:** $U[\mathcal{A}] \to \{u \in \mathcal{A},\ u.u^{\dagger} \mid u^{\dagger}.u \to 1_N\}$

$\Rightarrow\ \forall_{x\in M}\,(u[x].u[x]^{\dagger} \mid u[x]^{\dagger}.u[x] \to 1_N)$

$\Rightarrow\ u \in U[\mathcal{A}] \Leftrightarrow u[x] \in U[\mathcal{A}_F]$

•**Lie algebra:** $\mathfrak{u}[\mathcal{A}] \to \{X \in \mathcal{A},\ X^{\dagger} \to -X\} \to C^{\infty}[M, \mathfrak{u}[\mathcal{A}_F]]$

•**Special unitary group:** $SU[\mathcal{A}_F] \to \{\upsilon \in U[\mathcal{A}_F],\ \mathrm{Det}[u] \to 1\}$

•**Lie algebra SU[$\mathcal{A}_F$]:** $\mathfrak{su}[\mathcal{A}_F] \to \{X \in \mathcal{A}_F,\ X^{\dagger} \to -X,\ \mathrm{Tr}[X] \to 0\}$

●**Adjoint action. space:** $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F, J_F\}$

Define: for $\xi \in \mathcal{H}_F$

$$\to \begin{array}{l} \mathrm{Ad}[U[\mathcal{A}_F]] \to \mathrm{Endo}[\mathcal{H}_F] \\ \mathrm{ad}[\mathfrak{u}[\mathcal{A}_F]] \to \mathrm{Endo}[\mathcal{H}_F] \end{array} \longrightarrow \begin{array}{l} \mathrm{Ad}[u][\xi] \to u.\xi.u^{\dagger} \to u.u^{\dagger 0}.\xi \\ \mathrm{ad}[A][\xi] \to A.\xi - \xi.A \to (A - A^0).\xi \end{array}$$

```
PR["●Gauge symmetry. ", {ϕ[M] → M, "diffeomorphism of C^∞[M]"},
 NL, "define automorphism: ", {αϕ[f] → f.inv[ϕ], f ∈ (C^"∞") [M]},
 NL, "define diffeomorphism: ", Diff[M×F] → Aut[(C^"∞") [M, 𝒜F]],
 Imply, {a ∈ (C^"∞") [M, 𝒜F], αϕ[a] → a.inv[ϕ], αϕ[a][x] → a.inv[ϕ][x]} // Column,
 NL, "•Define for ", Inn[a] ->
   {u ∈ (C^"∞") [M, U[𝒜F]], αu[a] → u.a.ConjugateTranspose[u] → Inn[𝒜]} // ColumnForms,
 NL, "•Define outer automorphism: ", Out[𝒜] → Mod[Aut[𝒜], Inn[𝒜]],
 NL, "•Define kernel: ", Ker[ϕ] → {ϕ[U[𝒜]] → Inn[𝒜], ϕ[u → αu],
    u ∈ U[𝒜], ForAll[a ∈ 𝒜, u.a.ConjugateTranspose[u] → a]} // ColumnForms
]
```

●**Gauge symmetry.** $\{\phi[M] \to M,$ diffeomorphism of $C^{\infty}[M]\}$

define automorphism: $\{\alpha_{\phi}[f] \to f.\phi^{-1},\ f \in C^{\infty}[M]\}$

define diffeomorphism: $\mathrm{Diff}[M \times F] \to \mathrm{Aut}[C^{\infty}[M, \mathcal{A}_F]]$

$$\Rightarrow \begin{array}{l} a \in C^{\infty}[M, \mathcal{A}_F] \\ \alpha_{\phi}[a] \to a.\phi^{-1} \\ \alpha_{\phi}[a][x] \to a.\phi^{-1}[x] \end{array}$$

•**Define for** $\mathrm{Inn}[a] \to \begin{array}{l} u \in C^{\infty}[M, U[\mathcal{A}_F]] \\ \alpha_u[a] \to u.a.u^{\dagger} \to \mathrm{Inn}[\mathcal{A}] \end{array}$

•**Define outer automorphism:** $\mathrm{Out}[\mathcal{A}] \to \mathrm{Mod}[\mathrm{Aut}[\mathcal{A}], \mathrm{Inn}[\mathcal{A}]]$

•**Define kernel:** $\mathrm{Ker}[\phi] \to \begin{array}{l} \phi[U[\mathcal{A}]] \to \mathrm{Inn}[\mathcal{A}] \\ \phi[u \to \alpha_u] \\ u \in U[\mathcal{A}] \\ \forall_{a\in\mathcal{A}}\,(u.a.u^{\dagger} \to a) \end{array}$

```
PR["●Unitary transform. Given a triple: ", {𝒜, ℋ, 𝒟},
  " the representation π of 𝒜 on ℋ: ", π[a][ℋ] ,
  NL, "•Define unitary transform: ",
  $0 = U -> {U[ℋ] → ℋ, {𝒜, ℋ, 𝒟} -> {𝒜, ℋ, U. 𝒟.ConjugateTranspose[U]},
     (a ∈ 𝒜) -> U. π[a].ConjugateTranspose[U],
     γ -> U. γ.ConjugateTranspose[U], J -> U. J.ConjugateTranspose[U]};
  ColumnForms[$0],
  NL, "•EG1. ", {U → π[u], u ∈ U[𝒜]},
  NL, "•EG2. (adjoint action) ", $s = {U → Ad[u] → u.J.u.ConjugateTranspose[J]},
  Yield, $ = U.π[a].ConjugateTranspose[U], "POFF",
  Yield, $ = $ /. ($s[[1, 1]] -> $s[[1, 2, 2]] /. u → π[u]) // ConjugateCTSimplify1[{}],
  Yield, $ = $ /. aa_ . bb_ . π[a] → aa . π[a].bb, (*could be more specific*)
  Yield, $ = $ // tuRepeat[{ConjugateTranspose[J_] . J_ → 1,
       J_ .ConjugateTranspose[J_] → 1}, tuDotSimplify[]],
  Yield, $ = $ /. π[a_].π[b_]. ConjugateTranspose[π[c_]] →
     π[a. b. ConjugateTranspose[c]], "PONdd",
  Yield, $ = $ /. u_ .a_ .ConjugateTranspose[u_] → αu[a]
 ];
```

```
●Unitary transform. Given a triple:
 {𝒜, ℋ, 𝒟} the representation π of 𝒜 on ℋ: π[a][ℋ]
                                       U[ℋ] → ℋ
                                       {𝒜, ℋ, 𝒟} → {𝒜, ℋ, U.𝒟.U†}
•Define unitary transform: U →  a ∈ 𝒜 → U.π[a].U†
                                       γ → U.γ.U†
                                       J → U.J.U†
•EG1. {U → π[u], u ∈ U[𝒜]}
•EG2. (adjoint action) {U → Ad[u] → u.J.u.J†}
→ U.π[a].U†
.......
→ π[αu[a]]
```

```
PR["•Define Gauge group: ", 𝒢[M×F] → {u.J.u.ct[J], u ∈ U[𝒜]},
 NL, "Consider: ", {Ad[U[𝒜]] → 𝒢[M×F], Ad[u] → u.ct[u]"0"} // Column,
 Imply, Ker[Ad] → {u ∈ U[𝒜], (u.J.u.ct[J] → 1) ⇒ (u.J → ct[J].u)},
 NL, "•Define finite gauge group for finite space F: ",
 𝒢[F] → {ℋF → U[(𝒜̃F)JF], hF → u[(𝒜̃F)JF]} // ColumnForms,
 NL, "•Proposition 2.13. ",
 e213 = {𝒢[F] ≃ Mod[SU[𝒜F], SHF], 𝒜F → "complex algebra", SHF → {g ∈ HF, Det[g] → 1}};
 Column[e213],
 NL, "●Proof 2.13: ",
 NL, "•define UH-equivalence: ", $su = u_ ⇔ u_ . h_ -> ForAll[h, h ∈ HF, (u | u.h ∈ U[𝒜F])],
 Yield, $G = {𝒢[F] ≃ Mod[U[𝒜F], HF]} → {u ⇔ u.h},
 Yield, $ = $G /. $su,
 NL, "•define SUSH equivalence: ",
 $su = su_ ⇔ su_ . g_ -> ForAll[g, g ∈ SHF, (su | su.g ∈ SU[𝒜F])],
 Yield, $SU = {Mod[SU[𝒜F], SHF]} → {su ⇔ su.g},
 Yield, $0 = $SU /. $su,
 NL, "(1)•Is SHF a normal subgroup of SU[𝒜F]?: ",
 $ = ForAll[{g, v}, g ∈ SHF && v ∈ SU[𝒜F], (v.g. inv[v]) ∈ SHF],

 NL, "•Evaluate: ", $ = Det[$0 = v.g. inv[v] ∈ HF],
 yield, $ = $ /. a_ ∈ b_ → a,
 yield, $ = Thread[$, Dot] /. Det[inv[a_]] → 1 / Det[a] /. Dot → Times,
 NL, "Since: ", g ∈ SHF,
 imply, $s = Det[g] → 1,
 imply, $0 ∈ SHF,
 imply, "SHF Normal Subgroup of SU[𝒜F]" // Framed
]
```

•Define Gauge group: $\mathcal{G}[M\times F]\to\{u.J.u.J^{\dagger},\ u\in U[\mathcal{A}]\}$

Consider: $\begin{array}{l} Ad[U[\mathcal{A}]]\to\mathcal{G}[M\times F] \\ Ad[u]\to u.u^{\dagger 0}\end{array}$

$\Rightarrow$ $Ker[Ad]\to\{u\in U[\mathcal{A}],\ (u.J.u.J^{\dagger}\to 1)\Rightarrow(u.J\to J^{\dagger}.u)\}$

•Define finite gauge group for finite space F: $\mathcal{G}[F]\to\begin{array}{l}\mathcal{H}_F\to U[\tilde{\mathcal{A}}_{F J_F}] \\ h_F\to u[\tilde{\mathcal{A}}_{F J_F}]\end{array}$

•Proposition 2.13. $\begin{array}{l}\mathcal{G}[F]\simeq Mod[SU[\mathcal{A}_F],\ SH_F] \\ \mathcal{A}_F\to \text{complex algebra} \\ SH_F\to\{g\in H_F,\ Det[g]\to 1\}\end{array}$

●Proof 2.13:

•define UH-equivalence: $(u\_).(h\_)\Leftrightarrow u\_\to\forall_{h,h\in H_F}(u\mid u.h\in U[\mathcal{A}_F])$

$\to$ $\{\mathcal{G}[F]\simeq Mod[U[\mathcal{A}_F],\ H_F]\}\to\{u\Leftrightarrow u.h\}$

$\to$ $\{\mathcal{G}[F]\simeq Mod[U[\mathcal{A}_F],\ H_F]\}\to\{\forall_{h,h\in H_F}(u\mid u.h\in U[\mathcal{A}_F])\}$

•define SUSH equivalence: $(su\_).(g\_)\Leftrightarrow su\_\to\forall_{g,g\in SH_F}(su\mid su.g\in SU[\mathcal{A}_F])$

$\to$ $\{Mod[SU[\mathcal{A}_F],\ SH_F]\}\to\{su\Leftrightarrow su.g\}$

$\to$ $\{Mod[SU[\mathcal{A}_F],\ SH_F]\}\to\{\forall_{g,g\in SH_F}(su\mid su.g\in SU[\mathcal{A}_F])\}$

(1)•Is SH_F a normal subgroup of $SU[\mathcal{A}_F]$?: $\forall_{\{g,v\},g\in SH_F\&\&v\in SU[\mathcal{A}_F]}\ v.g.v^{-1}\in SH_F$

•Evaluate: $Det[v.g.v^{-1}\in H_F]\longrightarrow Det[v.g.v^{-1}]\longrightarrow Det[g]$

Since: $g\in SH_F\Rightarrow Det[g]\to 1\Rightarrow(v.g.v^{-1}\in H_F)\in SH_F\Rightarrow$ $\boxed{SH_F\text{ Normal Subgroup of }SU[\mathcal{A}_F]}$

```
PR["•Property of unitary matrix u: ",
 {Abs[Det[u]] → 1,
   {"Eigenvalues of u", λu ∈ U[1],
    Exists[{u, u'}, u ∈ U[𝒜F] && u' ∈ U[N], u'.u.ct[u'] -> λu 1N]}} // FramedColumn,
 Imply, Exists[λu, λu ∈ U[1] && λu^N → Det[u] && N → dim[ℋF] && U[1] ≤ U[𝒜F]],
 Imply, $ = ($0 = inv[λu].u ∈ SU[𝒜F]) ⟸ {$ = Det[$0[[1]]], $ = Thread[$, Dot],
     $ = $ /. Det[inv[λu]] → λu^(-N), $ = $ /. Det[u] → λu^N, SU[𝒜F]} // ColumnForms,

 NL, "■define group homomorphism from UH->SUSH: ",
 $ph = {φ[$G[[1, 1]]] → Mod[SU[𝒜F], SHF], φ[{u}] → {inv[λu].u}};
 Column[$ph],
 NL, "□Check if φ is independent of representative ", λu,
 NL, "•suppose: ", Implies[Exists[λu', (λu')^N → Det[u]],
  inv[λu].λu' ∈ μN["multiplicative group Nth root of unity"]],
 NL, "•", Implies[Implies[Implies[U[1] ≤ HF, μN ≤ SHF], {inv[λu].u} == {inv[λu'].u}],
  Framed[φ["independent of λu"]]],
 NL, "□Check if φ is independent of representative ", u ∈ U[𝒜F],
 NL, "?: ", $0 = $ = ForAll[u, u ∈ HF, φ[{u}]],
 Yield, $ = $ /. $ph, "POFF",
 NL, "For ", $s = (g -> inv[λh].h) ∈ SHF,
 Yield, $ = $ /. dd : HoldPattern[Dot[a_]] → dd .g,
 Yield, $ = $ /. $s[[1]],
 Yield, $ = $ /. dd : HoldPattern[Dot[_]] :→ tuDotTermLeft[inv[_], {inv[λu]}][dd],
 Yield, $ = $ /. inv[a_]. inv[b_] → inv[b.a],
 Yield, $[[3]] = φ[{u.h}]; $, "PONdd",
 yield, $[[3]] == $0[[3]] // Framed,
 NL, "•Suppose ", $ = ForAll[{u1, u2}, {u1 | u2 ∈ U[𝒜F]}, φ[{u1}] == φ[{u2}]],
 Yield, $ = $ /. φ[{a_}] → {inv[λa].a.ga} /. gu1 → 1 /. gu2 → (g ∈ SHF),
 Yield, $ = $ /. HoldPattern[Dot[a_]] → Dot[λu1, a],
 Yield, $ = $ /. a_ . inv[a_] → 1 /. g ∈ SHF → g // tuDotSimplify[],
 Yield, $3 = $ = $[[3]],
 " for some: ", $ = $[[2]] // First // DeleteCases[#, u2] & // tuDotSimplify[];
 $ ∈ SHF,
 imply, "φ is injective.",
 Imply, $ = $3 /. Thread[Apply[List, $] → 1] // tuDotSimplify[]; Framed[$]
]
```

•Property of unitary matrix u:

$$\boxed{\begin{array}{l}\mathrm{Abs}[\mathrm{Det}[u]]\to 1\\ \{\text{Eigenvalues of } u,\ \lambda_u\in U[1],\ \exists_{\{u,u'\},u\in U[\mathcal{A}_F]\&\&u'\in U[N]}\ (u'.u.(u')^{\dagger}\to 1_N\,\lambda_u)\}\end{array}}$$

$\Rightarrow\ \exists_{\lambda_u}\ (\lambda_u\in U[1]\ \&\&\ \lambda_u^N\to \mathrm{Det}[u]\ \&\&\ N\to \dim[\mathcal{H}_F]\ \&\&\ U[1]\le U[\mathcal{A}_F])$

$$\Rightarrow\ (\lambda_u^{-1}.u\in SU[\mathcal{A}_F])\Longleftarrow \begin{array}{l}\mathrm{Det}[\lambda_u^{-1}.u]\\ \mathrm{Det}[\lambda_u^{-1}].\mathrm{Det}[u]\\ \lambda_u^{-N}.\mathrm{Det}[u]\\ \lambda_u^{-N}.\lambda_u^{N}\\ SU[\mathcal{A}_F]\end{array}$$

■define group homomorphism from UH->SUSH: $\begin{array}{l}\varphi[\mathcal{G}[F]\simeq \mathrm{Mod}[U[\mathcal{A}_F],H_F]]\to \mathrm{Mod}[SU[\mathcal{A}_F],SH_F]\\ \varphi[\{u\}]\to\{\lambda_u^{-1}.u\}\end{array}$

□Check if φ is independent of representative λ_u

•suppose: $\exists_{\lambda_u'}\ ((\lambda_u')^N\to \mathrm{Det}[u])\Rightarrow \lambda_u^{-1}.\lambda_u'\in\mu_N$[multiplicative group Nth root of unity]

•$((U[1]\le H_F\Rightarrow \mu_N\le SH_F)\Rightarrow\{\lambda_u^{-1}.u\}==\{(\lambda_u')^{-1}.u\})\Rightarrow \boxed{\varphi[\text{independent of }\lambda_u]}$

□Check if φ is independent of representative $u\in U[\mathcal{A}_F]$

?: $\forall_{u,u\in H_F}\ \varphi[\{u\}]$

$\to\ \forall_{u,u\in H_F}\ \{\lambda_u^{-1}.u\}$

.... $\longrightarrow\ \boxed{\varphi[\{u.h\}]==\varphi[\{u\}]}$

•Suppose $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \varphi[\{u_1\}]==\varphi[\{u_2\}]$

$\to\ \forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \{\lambda_{u_1}^{-1}.u_1.1\}==\{\lambda_{u_2}^{-1}.u_2.(g\in SH_F)\}$

$\to\ \forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \{\lambda_{u_1}.\lambda_{u_1}^{-1}.u_1.1\}==\{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.(g\in SH_F)\}$

$\to\ \forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}}\ \{u_1\}==\{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$

$\to\ \{u_1\}==\{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$ for some: $\lambda_{u_1}.\lambda_{u_2}^{-1}.g\in SH_F\ \Rightarrow\ \varphi$ is injective.

$\Rightarrow\ \boxed{\{u_1\}==\{u_2\}}$

```
PR["●Full symmetry group. ",
 NL, "•Homomorphic action θ of a group H on group N: ", θ[H] → Aut[N],
 NL, "•semi-direct product ", $ = N ⊳ H → {{n, h}, n ∈ N && h ∈ H},
 NL, "Properties: ", $sdg = {
   {"product", {n_, h_} · {n1_, h1_} → {n . θ[h]. n1, h . h1}},
   {"unit", {1, 1}},
   {"inverse", invSDG[{n_, h_}] → {θ[inv[h]].inv[n], inv[h]}
   }}; FramedColumn[$sdg],
 "POFF",
 NL, "•Check inverse: ",
 NL, "Let: ", $n = {n, h},
 and, "inverse: ", $i = invSDG[$n],
 NL, "For ", $ = $n · $i,
 Yield, $ = $ /. $sdg[[3, 2]] /. $sdg[[1, 2]],
 NL, "If ", $s = {inv[a_] .a_ → 1, a_ .inv[a_] → 1, θ[a_]. θ[inv[a_]] → 1,
   θ[a_] . n1_ . θ[a_] . n2_ → θ[a].n1.n2, (*homomorphic property*)
   {θ[a_], b_} -> {1, b}(* Is θ[h].1→1? *)
  },
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], OK,
 NL, "For ", $ = $i · $n,
 Yield, $ = $ /. $sdg[[3, 2]] /. $sdg[[1, 2]],
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], OK,
 "PONdd",
 NL, "•Invariance under Diff[M]: ", Exists[θ, θ → "homomorphism",
  {θ[Diff[M]] → Aut[𝒢[M×F]] ↦ θ[ϕ].U → U∘inv[ϕ], ϕ ∈ Diff[M], U ∈ 𝒢[M×F]}],
 Yield, "Full symmetry group: ", 𝒢[M×F] ⊳ Diff[M]
]
```

●Full symmetry group.
•Homomorphic action θ of a group H on group N: θ[H] → Aut[N]
•semi-direct product N ⊳ H → {{n, h}, n ∈ N && h ∈ H}

Properties:

{product, {n_, h_} · {n1_, h1_} → {n.θ[h].n1, h.h1}}
{unit, {1, 1}}
{inverse, invSDG[{n_, h_}] → {θ[h^{-1}].n^{-1}, h^{-1}}}

....
•Invariance under Diff[M]:
$\exists_{\theta,\theta\to\text{homomorphism}}$ {θ[Diff[M]] → Aut[𝒢[M×F]] ↦ θ[ϕ].U → U∘ϕ^{-1}, ϕ ∈ Diff[M], U ∈ 𝒢[M×F]}
→ Full symmetry group: 𝒢[M×F] ⊳ Diff[M]

```
PR["●Principal bundles. ",
 NL, "Let ", $ = {{G → "Lie group", P → "principal G-bundle"} ↦ (π[P] → M),
   Aut[P] -> {f[P] → P, ForAll[{p, g}, p ∈ P && g ∈ G, f[p.g] → f[p].g]},
   Implies[f, Exists[f̄, {(f̄[M] → M) ↦ (f̄[π[p]] → π[f[p]]), f̄ → "diffeomorphism"}]]
  }; Column[$],
 NL, "•Gauge transformation of P: ",
 𝒢[P] → ForAll[g, g ∈ Aut[P], {ḡ == Id_M, π[g[p]] → π[p]}],
 NL, "?Is 𝒢[P] a normal subgroup: ",
 NL, "Since ", $ = f̄[π[p]] → π[f[p]],
 Yield, $ = $ /. f → f∘g∘inv[f],
 NL, "Since: ", $s = {(c__∘a_∘b_)[p_] → (c∘a)[b[p]], (a_ ∘ b_)[p_] → a[b[p]]},
 Yield, $ = MapAt[# //. $s &, $, 2],
 NL, "Using: ", $s = {π[f_[p_]] -> f̄[π[p]], a_[b_[π[p]]] :→ Flatten[a∘b][π[p]]},
 Yield, $ = MapAt[# //. $s &, $, 2]; Framed[Head /@ $],
 NL, "For ", $s = {ḡ → Id_M, f_∘Id_M∘f1_ → f∘f1, f̄_∘inv[f̄_] → Id_M},
 Yield, $ = $ //. $s; $ = Head /@ $,
 imply, $ = $[[1, 1]] ∈ 𝒢[P]; Framed[$ ≤ Aut[P]],
 NL, "Quotient: ", Quotient[Aut[P], 𝒢[P]] ≃ Diff[M]

]
```

●Principal bundles.

Let $\begin{array}{l} \{G \to \text{Lie group},\ P \to \text{principal G-bundle}\} \mapsto (\pi[P] \to M) \\ \text{Aut}[P] \to \{f[P] \to P,\ \forall_{\{p,g\},p\in P\&\&g\in G}\,(f[p.g] \to f[p].g)\} \\ f \Rightarrow \exists_{\bar f}\,\{(\bar f[M] \to M) \mapsto (\bar f[\pi[p]] \to \pi[f[p]]),\ \bar f \to \text{diffeomorphism}\} \end{array}$

•Gauge transformation of P: $\mathcal{G}[P] \to \forall_{g,g\in \text{Aut}[P]}\,\{\bar g == \text{Id}_M,\ \pi[g[p]] \to \pi[p]\}$

?Is $\mathcal{G}[P]$ a normal subgroup:

Since $\bar f[\pi[p]] \to \pi[f[p]]$

→ $\overline{f\circ g\circ f^{-1}}[\pi[p]] \to \pi[(f\circ g\circ f^{-1})[p]]$

Since: $\{(c\_\_\circ a\_\circ b\_)[p\_] \to (c\circ a)[b[p]],\ (a\_\circ b\_)[p\_] \to a[b[p]]\}$

→ $\overline{f\circ g\circ f^{-1}}[\pi[p]] \to \pi[f[g[f^{-1}[p]]]]$

Using: $\{\pi[f\_[p\_]] \to \bar f[\pi[p]],\ a\_[b\_[\pi[p]]] :\to \text{Flatten}[a\circ b][\pi[p]]\}$

→ $\boxed{\overline{f\circ g\circ f^{-1}} \to \bar f\circ \bar g\circ \overline{f^{-1}}}$

For $\{\bar g \to \text{Id}_M,\ f\_\circ \text{Id}_M\circ f1\_ \to f\circ f1,\ \bar f\_\circ \overline{f\_}^{-1} \to \text{Id}_M\}$

→ $\overline{f\circ g\circ f^{-1}} \to \text{Id}_M$ ⇒ $\boxed{(f\circ g\circ f^{-1} \in \mathcal{G}[P]) \le \text{Aut}[P]}$

Quotient: $\text{Quotient}[\text{Aut}[P],\ \mathcal{G}[P]] \simeq \text{Diff}[M]$

Inner fluctuations

```
PR["●For a Real ACM: ", M×F → {𝒜, ℋ, 𝒟, J},
  NL, "•Define: ", $O = Ω𝒟^"1" → {xSum[aj. CommutatorM[𝒟, bj], {j}], aj | bj ∈ 𝒜},
  NL, "•inner fluctuations: ",
  𝒜f -> {ForAll[𝒜, 𝒜 ∈ $O[[1]], ConjugateTranspose[𝒜] == 𝒜]},
  NL, "•fluctuated Dirac operator: ", $DA = 𝒟𝒜 → 𝒟 + 𝒜f + ε'.J.𝒜f.ConjugateTranspose[J],
  NL, "■Calculate on inner fluctuations: ",
  NL, $A = $0 = {𝒜 → a.CommutatorM[slash[𝒟], b],
     a | b ∈ C^"∞"[M], slash[𝒟] → -I T[γ, "u", {μ}] tuDs["∇"^"S"][_, μ]},
  Yield, $ = $0[[1]] /. $0[[-1]] /. CommutatorM → MCommutator //
    tuDotSimplify[{T[γ, "u", {μ}]}],
  yield, $0 = $ = $ /. tuDs["∇"^"S"][_, μ].b -> tuDs["∇"^"S"][b, μ] + b.tuDs["∇"^"S"][_, μ] //
     tuDotSimplify[{T[γ, "u", {μ}]}],
  NL, "Define ", $Am = $ = I T[𝒜, "d", {μ}] -> $[[2]] /. T[γ, "u", {μ}] → I;
  $ = -I # & /@ $;
  Framed[$ ∈ Real[C^"∞"[M]]],
  NL, "Proof:",
  "POFF",
  NL, $0;
  $1 = ConjugateTranspose /@ $0 // ConjugateCTSimplify1[{}, {}, {T[γ, "u", {μ}]}];
  $2 = 𝒜 → ConjugateTranspose[𝒜];
  $ = {$0, $1, $2},
  Yield, $ = tuEliminate[$, {𝒜}],
  yield, $ = Implies[$[[-1]], $[[-1, 2]] ∈ Reals] /. T[γ, "u", {μ}] → 1;
  Framed[$],
  "PONdd",
  NL, "For ", $ = slash[𝒟]𝒜 -> slash[𝒟] + 𝒜 + JM.𝒜.ConjugateTranspose[JM],
  NL, "Since: ", $s = {jj : JM.𝒜 → -Reverse[jj], JM.ConjugateTranspose[JM] → 1},
  imply, $ = slash[𝒟]𝒜 -> slash[𝒟] + 𝒜 + JM.𝒜.ConjugateTranspose[JM]
    // tuRepeat[$s, tuDotSimplify[]]
];
```

●For a Real ACM: $M\times F\to\{\mathcal{A},\mathcal{H},\mathcal{D},J\}$

•Define: $\Omega^1_{\mathcal{D}}\to\{\sum_{\{j\}}[a_j.[\mathcal{D},b_j]],\ a_j\mid b_j\in\mathcal{A}\}$

•inner fluctuations: $\mathcal{A}_f\to\{\forall_{\mathcal{A},\mathcal{A}\in\Omega^1_{\mathcal{D}}}\ \mathcal{A}^\dagger==\mathcal{A}\}$

•fluctuated Dirac operator: $\mathcal{D}_{\mathcal{A}}\to\mathcal{D}+\varepsilon'.J.\mathcal{A}_f.J^\dagger+\mathcal{A}_f$

■Calculate on inner fluctuations:

$\{\mathcal{A}\to a.[\not{\mathcal{D}},b],\ a\mid b\in C^\infty[M],\ \not{\mathcal{D}}\to -\mathrm{i}\,\gamma^\mu\,\underline{\nabla}^S_{\ \mu}[\_]\}$

$\to\ \mathcal{A}\to\mathrm{i}\,a.b.\underline{\nabla}^S_{\ \mu}[\_]\,\gamma^\mu-\mathrm{i}\,a.\underline{\nabla}^S_{\ \mu}[\_].b\,\gamma^\mu\ \longrightarrow\ \mathcal{A}\to-\mathrm{i}\,a.\underline{\nabla}^S_{\ \mu}[b]\,\gamma^\mu$

Define $\boxed{(\mathcal{A}_\mu\to-\mathrm{i}\,a.\underline{\nabla}^S_{\ \mu}[b])\in\mathrm{Real}[C^\infty[M]]}$

Proof:

....

For $\not{\mathcal{D}}_{\mathcal{A}}\to\mathcal{A}+J_M.\mathcal{A}.(J_M)^\dagger+\not{\mathcal{D}}$

Since: $\{jj: J_M.\mathcal{A}\to-\mathrm{Reverse}[jj],\ J_M.(J_M)^\dagger\to 1\}\ \Rightarrow\ \not{\mathcal{D}}_{\mathcal{A}}\to\not{\mathcal{D}}$

```
PR["●Inner fluctuations. ",
 NL, "•Dirac operator: ", $d = 𝒟 → slash[𝒟] ⊗ 1N + T[γ, "d", {5}] ⊗ 𝒟F,
 NL, "•Examine: ", $ = $A[[1]] /. slash[𝒟] → 𝒟; Framed[$],
 yield, $0 = $ /. $d // simpleCommutator[{}] // tuDotSimplify[],

 NL, "Evaluate[1]: ", $1 = $ = $0[[2, 1]],
 yield, $ = $ /. commutatorDot // tuDotSimplify[],
 NL, "Use: ", $s = {(slash[𝒟] ⊗ 1N).b -> slash[𝒟] ⊗ b + b. (slash[𝒟] ⊗ 1N)},
 Yield, $ = $ /. $s // tuDotSimplify[],
 NL, "Use: ", $slashD =
  $s = $sD = {$A[[-1]], a_ .((c_ tuDs["∇"S][_, μ]) ⊗ b_) → c ⊗ ( a.tuDs["∇"S][b, μ]),
     (-I a_) ⊗ b_ → a ⊗ (-I b)},
 Yield, $1 = $1 → ($ //. $s); Framed[$1], CK,

 NL, "Evaluate[2]: ", $2 = $ = $0[[2, 2]],
 NL, "Since: ", CommutatorM[T[γ, "d", {5}], b] → 0,
 NL, "Use: ", $s = {$[[2]] → ($[[2]] /. CommutatorM[a_ ⊗ b_, c_] → a ⊗ CommutatorM[b, c]),
   a_ . ((tt : T[γ, "d", {5}]) ⊗ b_) → tt ⊗ (a.b)},
 Yield, $ = $ /. $s /. $s ; Framed[$2 = $2 -> $],
 yield, "define: ", Framed[$2a = $[[2]] → ϕ],
 NL, "with ", Reverse[$Am],
 Imply, $A1 = $ = $0 /. $2 /. $1 /. $2a /. Reverse[$Am] // Simplify; Framed[$]
]
```

●Inner fluctuations.

•Dirac operator: $\mathcal{D} \to (\not{D}) \otimes 1_N + \gamma_5 \otimes \mathcal{D}_F$

•Examine: $\boxed{\mathcal{A} \to a.[\mathcal{D}, b]}$ $\longrightarrow$ $\mathcal{A} \to a.[(\not{D}) \otimes 1_N, b] + a.[\gamma_5 \otimes \mathcal{D}_F, b]$

Evaluate[1]: $a.[(\not{D}) \otimes 1_N, b]$ $\longrightarrow$ $-a.b.((\not{D}) \otimes 1_N) + a.((\not{D}) \otimes 1_N).b$

Use: $\{((\not{D}) \otimes 1_N).b \to (\not{D}) \otimes b + b.((\not{D}) \otimes 1_N)\}$

$\to$ $a.((\not{D}) \otimes b)$

Use: $\{\not{D} \to -i\, \gamma^\mu\, \underline{\nabla}^S_\mu[\_],\ (a\_).((c\_\ \underline{\nabla}^S_\mu[\_]) \otimes b\_) \to c \otimes a.\underline{\nabla}^S_\mu[b],\ (-i\, a\_) \otimes b\_ \to a \otimes (-i\, b)\}$

$\to$ $\boxed{a.[(\not{D}) \otimes 1_N, b] \to \gamma^\mu \otimes (-i\, a.\underline{\nabla}^S_\mu[b])}$ ⟵CHECK

Evaluate[2]: $a.[\gamma_5 \otimes \mathcal{D}_F, b]$

Since: $[\gamma_5, b] \to 0$

Use: $\{[\gamma_5 \otimes \mathcal{D}_F, b] \to \gamma_5 \otimes [\mathcal{D}_F, b],\ (a\_).((tt : \gamma_5) \otimes b\_) \to tt \otimes a.b\}$

$\to$ $\boxed{a.[\gamma_5 \otimes \mathcal{D}_F, b] \to \gamma_5 \otimes a.[\mathcal{D}_F, b]}$ $\longrightarrow$ define: $\boxed{a.[\mathcal{D}_F, b] \to \phi}$

with $a.\underline{\nabla}^S_\mu[b] \to i\, \mathcal{A}_\mu$

$\Rightarrow$ $\boxed{\mathcal{A} \to \gamma_5 \otimes \phi + \gamma^\mu \otimes \mathcal{A}_\mu}$

```
PR["•Fluctuated Dirac operator: ", $ = $DA,
 Yield, $ = $ /. 𝒜f → 𝒜;
 Yield, $0 = MapAt[# /. $A1 &, $, 2] // tuDotSimplify[],

 NL, "■Examine[𝒜]: ", $ = Select[$0[[2]], ! FreeQ[#, 𝒜] &],
 NL, "J Anticommutes: ",
 $s = e_ .J.(T[γ, "u", {μ}] ⊗ a_).b_ → -T[γ, "u", {μ}] ⊗ (e.J. a.b),
 Yield, $ = $ /. $s,
 yield, $ = $ /. a_ ⊗ b_ + (-a_ ⊗ c_) -> a ⊗ (b - c); Framed[$],
 NL, "Define ", $e216B = e216 = {Bμ -> $[[2]], Bμ ∈ Γ[End["E"]]};
 Framed[$e216B], CG[" (2.16)"],
 NL, "Define twisted connection: ",
 $ = T["∇""E", "d", {μ}] -> T["∇"S, "d", {μ}] ⊗ Id + I Id ⊗ Bμ;
 Framed[$],
 Yield, $ = -I T[γ, "u", {μ}].# & /@ $ // tuDotSimplify[],
 $ = $ /. T[γ, "u", {μ}].(Id ⊗ b_) -> T[γ, "u", {μ}] ⊗ b;
 Yield, $ = $ /. -I a_ . (b_ ⊗ c_) → (-I a b) ⊗ c,
 NL, "Using: ", $s = (I # & /@ Reverse[$A[[-1]]] /. tuDDown[a_][_, m_] → T[a, "d", {m}]),
 Yield, e216a = $ /. $s; Framed[$],

 NL, "■Examine[ϕ]: ",
 NL, "Define ", Φ ∈ Γ[End["E"]] ∍
  ($ = T[γ, "d", {5}] ⊗ Φ -> Select[$0[[2]], ! FreeQ[#, ϕ] &] + T[γ, "d", {5}] ⊗ 𝒟F),
 Imply, e218 = 𝒟A → e216a[[1]] + $[[1]]; Framed[e218]
]
```

•Fluctuated Dirac operator: $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \varepsilon'.J.\mathcal{A}_f.J^{\dagger} + \mathcal{A}_f$

→

→ $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \gamma_5 \otimes \phi + \gamma^{\mu} \otimes \mathcal{A}_{\mu} + \varepsilon'.J.(\gamma_5 \otimes \phi).J^{\dagger} + \varepsilon'.J.(\gamma^{\mu} \otimes \mathcal{A}_{\mu}).J^{\dagger}$

■Examine[𝒜]: $\gamma^{\mu} \otimes \mathcal{A}_{\mu} + \varepsilon'.J.(\gamma^{\mu} \otimes \mathcal{A}_{\mu}).J^{\dagger}$

J Anticommutes: $(e\_).J.(\gamma^{\mu} \otimes a\_).(b\_) \to -\gamma^{\mu} \otimes e.J.a.b$

→ $-\gamma^{\mu} \otimes \varepsilon'.J.\mathcal{A}_{\mu}.J^{\dagger} + \gamma^{\mu} \otimes \mathcal{A}_{\mu}$ ⟶ $\boxed{\gamma^{\mu} \otimes (-\varepsilon'.J.\mathcal{A}_{\mu}.J^{\dagger} + \mathcal{A}_{\mu})}$

Define $\boxed{\{B_{\mu} \to -\varepsilon'.J.\mathcal{A}_{\mu}.J^{\dagger} + \mathcal{A}_{\mu},\ B_{\mu} \in \Gamma[\mathrm{End}[E]]\}}$ (2.16)

Define twisted connection: $\boxed{\nabla^{E}{}_{\mu} \to i\ \mathrm{Id} \otimes B_{\mu} + \nabla^{S}{}_{\mu} \otimes \mathrm{Id}}$

→ $-i\ \gamma^{\mu}.\nabla^{E}{}_{\mu} \to \gamma^{\mu}.(\mathrm{Id} \otimes B_{\mu}) - i\ \gamma^{\mu}.(\nabla^{S}{}_{\mu} \otimes \mathrm{Id})$

→ $-i\ \gamma^{\mu}.\nabla^{E}{}_{\mu} \to \gamma^{\mu} \otimes B_{\mu} + (-i\ \nabla^{S}{}_{\mu}\ \gamma^{\mu}) \otimes \mathrm{Id}$

Using: $\nabla^{S}{}_{\mu}\ \gamma^{\mu} \to i\ (\not{D})$

→ $\boxed{-i\ \gamma^{\mu}.\nabla^{E}{}_{\mu} \to \gamma^{\mu} \otimes B_{\mu} + (-i\ \nabla^{S}{}_{\mu}\ \gamma^{\mu}) \otimes \mathrm{Id}}$

■Examine[ϕ]:

Define $\Phi \in \Gamma[\mathrm{End}[E]] \ni (\gamma_5 \otimes \Phi \to \gamma_5 \otimes \phi + \gamma_5 \otimes \mathcal{D}_F + \varepsilon'.J.(\gamma_5 \otimes \phi).J^{\dagger})$

⇒ $\boxed{\mathcal{D}_A \to \gamma_5 \otimes \Phi - i\ \gamma^{\mu}.\nabla^{E}{}_{\mu}}$

```
elementQ[a_, h_List] := tuMemberQ[a, h];
$hermitian = {𝒜μ};
$Iu = {I 𝒜μ};
PR["Since: ",
 $ = Implies[Inactive[elementQ[𝒜μ, $hermitian]], ConjugateTranspose[𝒜μ] == 𝒜μ],
 imply, $ = -I # & /@ Activate[$] /. -I ConjugateTranspose[a_] → SuperDagger[I a],
 imply, Framed[I $[[2]] ∈ I u],
 NL, "For ", I 𝔤[F] → I Mod[u[F], h[F]],
 imply, e219 = 𝒜μ ∈ C"∞"[M, I 𝔤[F]]
]
```

Since: Inactive[elementQ[$\mathcal{A}_\mu$, $hermitian]] ⇒ $(\mathcal{A}_\mu)^\dagger == \mathcal{A}_\mu$ ⇒ $(i\,\mathcal{A}_\mu)^\dagger == -i\,\mathcal{A}_\mu$ ⇒ $\boxed{\mathcal{A}_\mu \in i\,u}$

For $i\,\mathfrak{g}[F] \to i$ Mod[u[F], h[F]] ⇒ $\mathcal{A}_\mu \in C^\infty[M, i\,\mathfrak{g}[F]]$

```
PR["Gauge transformation on fluctuating Dirac operator. ",
  Yield, $00 = $0 = 𝒟𝒜 → 𝒟 + 𝒜 + ε'.J.𝒜.ConjugateTranspose[J],
  NL, "Expanding Rules: ",
  $s0 = {U → u.J.u.ConjugateTranspose[J], CommutatorM[a, b"0"] → 0,
    CommutatorM[𝒜, J.u.ConjugateTranspose[J]] → 0,
    CommutatorM[CommutatorM[𝒟, a], b"0"] → 0,
    J.𝒟 → ε'.𝒟.J, b"0" → J.ConjugateTranspose[b].ConjugateTranspose[J],
    J_ .ConjugateTranspose[J_] :> 1 /; MemberQ[{J, u}, J],
    ConjugateTranspose[J_].J_ :> 1 /; MemberQ[{J, u}, J],
    ε^2 → 1};
  Yield, $s0x =
   $s0 /. CommutatorM → MCommutator // tuDotSimplify[{ε'}] // tuRuleEliminate[{b"0"}];
  FramedColumn[$s0x],
  NL, "Evaluate: ",
  $0a = $ = U.#.ConjugateTranspose[U] & /@ $0 // tuDotSimplify[{ε', ε}],

  Yield,
  $1 = $ = $[[2]] // tuRepeat[$s0x, tuDotSimplify[]] // ConjugateCTSimplify1[{ε', ε}];
  $1 = $1 // tuRepeat[$s0x, tuDotSimplify[]], (*Need to repeat for some reason*)
  NL, "From commutation rules: ",
  $s = tuRuleSolve[$s0x[[5]], Dot[𝒟, J]],

  NL, "■Simplify the term: ",
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  yield, $ = $ /. $s0x[[7]] // tuDotSimplify[{ε', ε}],
  NL, "From ", $s = u.CommutatorM[𝒟, ConjugateTranspose[u]] ->
    u.MCommutator[𝒟, ConjugateTranspose[u]],
  $s = $s // tuDotSimplify[];
  yield, $s = $s /. $s0 // tuDotSimplify[],
  yield, $s = tuRuleEliminate[{u.𝒟.ConjugateTranspose[u]}][{$s}];
  Framed[$s],
  Imply, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  Yield, $ = $ //. $s0 // tuDotSimplify[{ε', ε}],
  yield, $1a = $ = $ /. $s; Framed[$], CK
 ];
PR[
  "■Simplify the term: ",
  Yield, $0 = $ = $1[[1]]; Framed[$],
  NL, "Use: ", $s = tuRuleSolve[$s0x /. u → ConjugateTranspose[u], 𝒜._],
  Yield, $ = $ /. $s // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1b = $]
 ];
```

```
$s0x /. xu → ConjugateTranspose[u];
PR[
 "■Simplify the term: ",
 Yield, $0 = $ = $1[[3]]; Framed[$],
 NL, "Append 1→ ", $s = J.ConjugateTranspose[J],
 imply, $ = $. $s // tuDotSimplify[{ε', ε}],
 NL, "Use ",
 $s = tuRuleSolve[$s0x /. u → ConjugateTranspose[u], 𝒜._],
 " with ConjugateTranspose: ", $sa = aa : a | J → ConjugateTranspose[aa],
 Yield, $s = $s /. ConditionalExpression[a_, b_] → a /. $sa //
   tuAddPatternVariable[{a, b}],
 NL, "The Rule applies to: ", $sa = 𝒜 → u.𝒜.ConjugateTranspose[u],
 yield, $s = $s /. $sa,
 Imply, $ = $ /. $s,
 yield, $ = $ // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1c = $]
]
PR["■Check if equal to (2.20). Our calculation: ",
 $ = $0a[[1]] -> $1a + $1b + $1c; Framed[$],
 NL, "Evaluate (2.20) with ", $ = $00 /. 𝒜 → 𝒜^u, CK,
 Yield, $[[2]] =
  $[[2]] /. 𝒜^u → u.𝒜.ConjugateTranspose[u] + u.CommutatorM[𝒟, ConjugateTranspose[u]] //
   tuDotSimplify[{ε'}];
 Framed[$],
 NL, CR["Almost equal."]

]
```

Gauge transformation on fluctuating Dirac operator.

$\rightarrow\ \mathcal{D}_{\mathcal{A}} \to \mathcal{A} + \mathcal{D} + \varepsilon'.J.\mathcal{A}.J^{\dagger}$

Expanding Rules:

$$\rightarrow\ \boxed{\begin{array}{l} U \to u.J.u.J^{\dagger} \\ a.J.b^{\dagger}.J^{\dagger} - J.b^{\dagger}.J^{\dagger}.a \to 0 \\ -J.u.J^{\dagger}.\mathcal{A} + \mathcal{A}.J.u.J^{\dagger} \to 0 \\ -a.\mathcal{D}.J.b^{\dagger}.J^{\dagger} + J.b^{\dagger}.J^{\dagger}.a.\mathcal{D} - J.b^{\dagger}.J^{\dagger}.\mathcal{D}.a + \mathcal{D}.a.J.b^{\dagger}.J^{\dagger} \to 0 \\ J.\mathcal{D} \to \mathcal{D}.J\ \varepsilon' \\ (J\_).J\_^{\dagger} :\to 1\ /;\ \mathrm{MemberQ}[\{J, u\}, J] \\ J\_^{\dagger}.(J\_) :\to 1\ /;\ \mathrm{MemberQ}[\{J, u\}, J] \\ \varepsilon^2 \to 1 \end{array}}$$

Evaluate: $U.\mathcal{D}_{\mathcal{A}}.U^{\dagger} \to U.\mathcal{A}.U^{\dagger} + U.\mathcal{D}.U^{\dagger} + U.J.\mathcal{A}.J^{\dagger}.U^{\dagger}\ \varepsilon'$

$\rightarrow\ u.J.u.J^{\dagger}.\mathcal{A}.J.u^{\dagger}.J^{\dagger}.u^{\dagger} + u.J.u.J^{\dagger}.\mathcal{D}.J.u^{\dagger}.J^{\dagger}.u^{\dagger} + u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}\ \varepsilon'$

From commutation rules: $\{\mathcal{D}.J \to \frac{J.\mathcal{D}}{\varepsilon'}\}$

■Simplify the term:

$$\rightarrow\ \boxed{u.J.u.J^{\dagger}.\mathcal{D}.J.u^{\dagger}.J^{\dagger}.u^{\dagger}} \longrightarrow \frac{u.J.u.J^{\dagger}.J.\mathcal{D}.u^{\dagger}.J^{\dagger}.u^{\dagger}}{\varepsilon'} \longrightarrow \frac{u.J.u.\mathcal{D}.u^{\dagger}.J^{\dagger}.u^{\dagger}}{\varepsilon'}$$

From $u.[\mathcal{D}, u^{\dagger}] \to u.(\mathcal{D}.u^{\dagger} - u^{\dagger}.\mathcal{D}) \longrightarrow u.[\mathcal{D}, u^{\dagger}] \to -\mathcal{D} + u.\mathcal{D}.u^{\dagger} \longrightarrow \boxed{\{u.\mathcal{D}.u^{\dagger} \to \mathcal{D} + u.[\mathcal{D}, u^{\dagger}]\}}$

$$\Rightarrow\ \frac{u.J.\mathcal{D}.J^{\dagger}.u^{\dagger}}{\varepsilon'} + \frac{u.J.u.[\mathcal{D}, u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'}$$

$$\rightarrow\ u.\mathcal{D}.u^{\dagger} + \frac{u.J.u.[\mathcal{D}, u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'} \longrightarrow \boxed{\mathcal{D} + u.[\mathcal{D}, u^{\dagger}] + \frac{u.J.u.[\mathcal{D}, u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'}} \longleftarrow \text{CHECK}$$

■Simplify the term:

→ $\boxed{u.J.u.J^{\dagger}.\mathcal{A}.J.u^{\dagger}.J^{\dagger}.u^{\dagger}}$

Use: $\{\mathcal{A}.J.u^{\dagger}.J^{\dagger} \to J.u^{\dagger}.J^{\dagger}.\mathcal{A}\}$

→ $\boxed{u.\mathcal{A}.u^{\dagger}}$

■Simplify the term:

→ $\boxed{u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}\ \varepsilon'}$

Append 1→ $J.J^{\dagger}$ ⇒ $u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}.J.J^{\dagger}\ \varepsilon'$

Use $\{\mathcal{A}.J.u^{\dagger}.J^{\dagger} \to J.u^{\dagger}.J^{\dagger}.\mathcal{A}\}$ with ConjugateTranspose: $aa : a \mid J \to aa^{\dagger}$

→ $\{\mathcal{A}.J^{\dagger}.u^{\dagger}.J \to J^{\dagger}.u^{\dagger}.J.\mathcal{A}\}$

The Rule applies to: $\mathcal{A} \to u.\mathcal{A}.u^{\dagger} \longrightarrow \{u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}.J \to J^{\dagger}.u^{\dagger}.J.u.\mathcal{A}.u^{\dagger}\}$

⇒ $u.J.J^{\dagger}.u^{\dagger}.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon' \longrightarrow \boxed{J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$

■Check if equal to (2.20). Our calculation:

$$\boxed{U.\mathcal{D}_{\mathcal{A}}.U^{\dagger} \to \mathcal{D} + u.[\mathcal{D},\ u^{\dagger}] + u.\mathcal{A}.u^{\dagger} + \frac{u.J.u.[\mathcal{D},\ u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'} + J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$$

Evaluate (2.20) with $\mathcal{D}_{\mathcal{A}^u} \to \mathcal{A}^u + \mathcal{D} + \varepsilon'.J.\mathcal{A}^u.J^{\dagger}$⟵CHECK

→ $\boxed{\mathcal{D}_{\mathcal{A}^u} \to \mathcal{D} + u.[\mathcal{D},\ u^{\dagger}] + u.\mathcal{A}.u^{\dagger} + J.u.[\mathcal{D},\ u^{\dagger}].J^{\dagger}\ \varepsilon' + J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$

Almost equal.

```
PR["●Define bilinear form: ", $0 = $ = ᑌ𝒟[ξ, ξp] → BraKet[J.ξ, 𝒟.ξp](*⟨J.ξ,𝒟.ξp⟩*),
  Yield, $ = $ /. dd : 𝒟.ξp → -J.J.dd //. simpleBraKet[],
  Yield, $ = $ /. BraKet[J.a_, J.b_] → BraKet[b, a] /. J.𝒟 → 𝒟.J,
  Yield, $ = $ /. BraKet[𝒟.a_, b_] → BraKet[a, 𝒟.b](*𝒟 is Hermitian*),
  Yield, $s = Reverse[$0] // tuAddPatternVariable[{ξp, ξ}],
  Yield, $ = $ /. $s; Framed[$]
 ];
```

●Define bilinear form: $ᑌ_{\mathcal{D}}[\xi,\ \xi p] \to \langle J.\xi \mid \mathcal{D}.\xi p\rangle$

→ $ᑌ_{\mathcal{D}}[\xi,\ \xi p] \to -\langle J.\xi \mid J.J.\mathcal{D}.\xi p\rangle$

→ $ᑌ_{\mathcal{D}}[\xi,\ \xi p] \to -\langle \mathcal{D}.J.\xi p \mid \xi\rangle$

→ $ᑌ_{\mathcal{D}}[\xi,\ \xi p] \to -\langle J.\xi p \mid \mathcal{D}.\xi\rangle$

→ $\langle J.(\xi\_) \mid \mathcal{D}.(\xi p\_)\rangle \to ᑌ_{\mathcal{D}}[\xi,\ \xi p]$

→ $\boxed{ᑌ_{\mathcal{D}}[\xi,\ \xi p] \to -ᑌ_{\mathcal{D}}[\xi p,\ \xi]}$

```
PR["●Define classical fermions: ", (ℋ⁺)cl → {ξ̃ → Grassmann, ξ ∈ ℋ⁺},
  NL, "●Define action functional: ", $S = S → Sb + Sf → Tr[f[𝒟𝒜 / Λ]] + BraKet[J.ξ̃, 𝒟𝒜.ξ̃] / 2
 ];
```

●Define classical fermions: $\mathcal{H}^{+}{}_{cl} \to \{\tilde{\xi} \to \text{Grassmann},\ \xi \in \mathcal{H}^{+}\}$

●Define action functional: $S \to S_b + S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle + Tr[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

```
PR["●INvariance of action functional under ",
 $s = {𝒟𝒜 → U.𝒟𝒜.ConjugateTranspose[U], xx : ξ̃ → U.xx},
 NL, "■Boson ", $0 = $ = tuExtractPattern[Tr[_]][$S] // First,
 yield, $ = $ /. $s,
 yield, xSum[f[λn / Λ], n], CG[" Invariant"],
 NL, "■Fermion ", $0 = $ = tuExtractPattern[BraKet[_, _]][$S] // First,
 Yield, $ = $ /. $s,
 NL, "Apply ",
 $s = {J.U → U.J, ConjugateTranspose[u_].u_ → 1, BraKet[U . a_, U.b_] -> BraKet[a, b]},
 Yield, $ = $ //. $s // tuDotSimplify[], CG[" Invariant"]

]
```

●INvariance of action functional under $\{\mathcal{D}_{\mathcal{A}} \to U.\mathcal{D}_{\mathcal{A}}.U^{\dagger},\ xx : \tilde{\xi} \to U.xx\}$

■Boson $\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \longrightarrow \mathrm{Tr}[f[\frac{U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}}{\Lambda}]] \longrightarrow \sum_{n}[f[\frac{\lambda_n}{\Lambda}]]$ Invariant

■Fermion $\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$

$\to \langle J.U.\tilde{\xi} \mid U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}.U.\tilde{\xi}\rangle$

Apply $\{J.U \to U.J,\ u\_^{\dagger}.(u\_) \to 1,\ \langle U.(a\_) \mid U.(b\_)\rangle \to \langle a \mid b\rangle\}$

$\to \langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$ Invariant

```
PR["●Theorem 2.19. A real even almost-commutative manifold M×F describes
    a gauge theory on M with gauge group 𝒢[M×F]->C∞[M,𝒢[F]]. ",
  NL, "•Sketch of Proof: ",
  $t219 = $ = {{"(2.19)" -> I 𝒜μ[x] ∈ 𝔤[F] → Mod[𝒰[𝒜F], 𝒽F],
      "Total algebra" → 𝒜 → C"∞"[M, 𝒜F] → xSum[section[i, Γ[M×𝒜F]], {i}],
      ω["𝔤[F]-valued 1-form"] → I T[𝒜, "d", {μ}] . DifForm[T[x, "u", {μ}]],
      P["Principal bundle"] → M×𝒢[F],
      "(2.22)" → ω["connection form on P"],
      "group of gauge transform"[P] -> C"∞"[M, 𝒢[F]],
      "(2.12)" ⇒ "group of gauge transform"[P] == 𝒢[M×F],
      "(2.11)" ⇒ 𝒢[M×F] -> {U → u.J.u.ConjugateTranspose[J], u ∈ U[𝒜]},
      rep[𝒜F[ℋF]] → rep[𝒢[ℋF]]
        ⇒ M×ℋF ↔ "vector bundle of principal bundle"[P → M×𝒢[F]]
     }}; Grid[Transpose[$], Frame → All],
  NL, "Note: ", {("E" → M×ℋF) ↔
     (P["Principal bundle"] → M×𝒢[F])⟹"action of gauge group on fermions",
    ℋ["ACM"] → L2[M, S]⊗ℋF → L2[M, S⊗"E"],
    "⇒ particle fields"⟶section[S⊗"E"]} // Column
 ];
```

●Theorem 2.19. A real even almost-commutative manifold M×F describes a gauge theory on M with gauge group $\mathcal{G}[M\times F]$->$C^\infty[M,\mathcal{G}[F]]$.
•Sketch of Proof:

$(2.19)\to i\,\mathcal{A}_\mu[x]\in\mathfrak{g}[F]\to \mathrm{Mod}[\mathcal{U}[\mathcal{A}_F],\,h_F]$
Total algebra $\to\mathcal{A}\to C^\infty[M,\,\mathcal{A}_F]\to\sum_{\{i\}}[\mathrm{section}[i,\,\Gamma[M\times\mathcal{A}_F]]]$
$\omega[\mathfrak{g}[F]$-valued 1-form$]\to i\,\mathcal{A}_\mu.\mathrm{d}[x^\mu]$
P[Principal bundle] $\to M\times\mathcal{G}[F]$
(2.22) $\to\omega$[connection form on P]
group of gauge transform[P] $\to C^\infty[M,\,\mathcal{G}[F]]$
(2.12) $\Rightarrow$ group of gauge transform[P] $==\mathcal{G}[M\times F]$
(2.11) $\Rightarrow\mathcal{G}[M\times F]\to\{U\to u.J.u.J^\dagger,\ u\in U[\mathcal{A}]\}$
$\mathrm{rep}[\mathcal{A}_F[\mathcal{H}_F]]\to\mathrm{rep}[\mathcal{G}[\mathcal{H}_F]]\Rightarrow M\times\mathcal{H}_F\leftrightarrow$ vector bundle of principal bundle$[P\to M\times\mathcal{G}[F]]$

Note:
$(E\to M\times\mathcal{H}_F)\leftrightarrow$ (P[Principal bundle] $\to M\times\mathcal{G}[F]$) $\Longrightarrow$ action of gauge group on fermions
$\mathcal{H}[\mathrm{ACM}]\to L^2[M,\,S]\otimes\mathcal{H}_F\to L^2[M,\,S\otimes E]$
$\Rightarrow$ particle fields $\longrightarrow$ section[S⊗E]

The Spectral Action

```
PR["●Lichnerowicz formula.",
 NL, "•vector bundle ", "E" → M,
 NL, "•Laplacian ", △"E"["∇""E"["connection on E"]],
 NL, "•generalized Laplacian ", H → {△"E" - F, F ∈ Γ[End["E"]]},
 NL, "•generalized Dirac operator[ℤ2graded vector bundle E]",
 yield, (𝒟[Γ[M, "E""±"]] -> Γ[M, "E""∓"]), imply, 𝒟.𝒟 → H
]
```

●Lichnerowicz formula.
•vector bundle $E\to M$
•Laplacian $\Delta^E[\nabla^E[\text{connection on E}]]$
•generalized Laplacian $H\to\{-F+\Delta^E,\ F\in\Gamma[\mathrm{End}[E]]\}$
•generalized Dirac operator[$\mathbb{Z}_2$graded vector bundle E] $\longrightarrow\mathcal{D}[\Gamma[M,\,E^\pm]]\to\Gamma[M,\,E^\mp]\ \Rightarrow\ \mathcal{D}.\mathcal{D}\to H$

```
PR["■Show ", $ = 𝒟𝒜 -> "generalized Dirac operator" → 𝒟𝒜.𝒟𝒜 → H,
  NL, "•compute ", $[[2, 2, 1]],
  " where ",
  $sDA = $s0 = $s = {𝒟𝒜 -> -I T[γ, "u", {μ}].T["∇""E", "d", {μ}] + T[γ, "d", {5}] ⊗ Φ,
      T["∇""E", "d", {μ_}] -> T["∇""S, "d", {μ}] ⊗ 1ℋF + I 1N ⊗ Bμ,
      T["∇""E", "d", {μ}][S ⊗ "E"],
      Φ ∈ Γ[End["E"]] -> "Higg's field"
     }; Column[$s],
  NL, "•Define ",
  $d = {T[𝒟, "d", {μ}][a_] → ad[T["∇""E", "d", {μ}]][a], ad[aa_][bb_] → aa.bb - bb.aa},
  "xPOFF",
  Yield, $ = $0 = T[𝒟, "d", {μ}][Φ],
  Yield, $ = $ /. $d,
  Yield, $ = $ /. $d,
  Yield, $ = $ /. $s[[1 ;; 2]],
  Yield, $ = $ // tuDotSimplify[], "PONdd",
  NL, "Using ", $s = {(op_ ⊗ 1ℋF).ph_ → op[ph] ⊗ 1ℋF + ph.(op ⊗ 1ℋF), (1N ⊗ op_).ph_ → 1N ⊗ op.ph,
    ph_ .(1N ⊗ op_) → 1N ⊗ ph.op, ca_ 1N ⊗ a_ + cb_ 1N ⊗ b_ -> 1N ⊗ (ca a + cb b)};
  Column[$s],
  Yield, $ = $0 -> $ //. $s // Simplify,
  Yield, $ = $ /. a_ ⊗ 1ℋF → 1N ⊗ a //
    tuRepeat[{}, (Expand[tuDotSimplify[][#]] //. tuOpDistribute[CircleTimes] //.
        tuOpSimplify[CircleTimes]) &];
  Framed[$D1 = $]
 ];
```

■Show $\mathcal{D}_{\mathcal{A}} \to$ generalized Dirac operator $\to \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to H$

•compute $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}}$ where

$$\mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \Phi - i\,\gamma^{\mu}.\nabla^{E}{}_{\mu}$$
$$\nabla^{E}{}_{\mu\_} \to i\,1_N \otimes B_\mu + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}$$
$$\nabla^{E}{}_{\mu}[S \otimes E]$$
$$\Phi \in \Gamma[\mathrm{End}[E]] \to \text{Higg's field}$$

•Define $\{\mathcal{D}_\mu[a_] \to \mathrm{ad}[\nabla^{E}{}_{\mu}][a],\ \mathrm{ad}[aa_][bb_] \to aa.bb - bb.aa\}$xPOFF

$\to \mathcal{D}_\mu[\Phi]$

$\to \mathrm{ad}[\nabla^{E}{}_{\mu}][\Phi]$

$\to -\Phi.\nabla^{E}{}_{\mu} + \nabla^{E}{}_{\mu}.\Phi$

$\to -\Phi.(i\,1_N \otimes B_\mu + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}) + (i\,1_N \otimes B_\mu + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).\Phi$

$\to -i\,\Phi.(1_N \otimes B_\mu) - \Phi.(\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}) + i\,(1_N \otimes B_\mu).\Phi + (\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).\Phi$PONdd

Using

$$(op\_ \otimes 1_{\mathcal{H}_F}).(ph\_) \to op[ph] \otimes 1_{\mathcal{H}_F} + ph.(op \otimes 1_{\mathcal{H}_F})$$
$$(1_N \otimes op\_).(ph\_) \to 1_N \otimes op.ph$$
$$(ph\_).(1_N \otimes op\_) \to 1_N \otimes ph.op$$
$$1_N \otimes a\_\ ca\_ + 1_N \otimes b\_\ cb\_ \to 1_N \otimes (a\,ca + b\,cb)$$

$\to \mathcal{D}_\mu[\Phi] \to 1_N \otimes (-i\,(\Phi.B_\mu - B_\mu.\Phi)) + \nabla^{S}{}_{\mu}[\Phi] \otimes 1_{\mathcal{H}_F}$

$\to \boxed{\mathcal{D}_\mu[\Phi] \to -i\,1_N \otimes \Phi.B_\mu + i\,1_N \otimes B_\mu.\Phi + 1_N \otimes \nabla^{S}{}_{\mu}[\Phi]}$

```
PR["•Define curvature of Bμ: ",
  $F = T[F, "dd", {μ, ν}] → tuDPartial[Bν, μ] - tuDPartial[Bμ, ν] + I CommutatorM[Bμ, Bν],
  NL, "•Define curvature of ", "∇""E", ": ",
  $O = {Ω"E"[X, Y] -> T["∇""E", "d", {X}].T["∇""E", "d", {Y}] - T["∇""E", "d", {Y}].T["∇""E",
         "d", {X}] - T["∇""E", "d", {CommutatorM[X, Y]}], {X, Y} → "vector fields"},
  NL, CO["▪For local coordinates: "], CommutatorM[tuDPartial[_, μ],
    tuDPartial[_, ν]] → 0,
  NL, "define ", {tuDPartial[_, μ] → X, tuDPartial[_, ν] → Y},
  Yield, $s = {CommutatorM[X, Y] → 0, X -> μ, Y → ν, T["∇""E", "d", {0}] → 0},
  Imply, e33 = $ = $O[[1]] //. $s,
  Yield, $ = $ /. $sDA[[1 ;; 2]],
  Yield, $ = $ // tuDotSimplify[],
  NL, "Using: ", $scc = $s = {
     (a_ ⊗ b_).(c_ ⊗ d_) :→ a . c ⊗ b . d +
       If[! FreeQ [a, "∇"] && ! FreeQ [d, B | Φ], c ⊗ a[d] , 0] +
       If[! FreeQ [b, "∇"] && ! FreeQ [d, B | Φ], a ⊗ b[d], 0],
     1N_ . a_ → a, a_ . 1N_ → a , (a_ ⊗ 1HF) - (b_ ⊗ 1HF) → (a - b) ⊗ 1HF,
     (1N_ ⊗ a_) - (1N_ ⊗ b_) → 1N ⊗ (a - b)};
  ColumnSumExp[$s],
  Yield, $ = $ //. $s // Simplify // Expand; $ // ColumnSumExp // Framed,
  NL, "Use ", $s = {I 1N ⊗ a_ - I 1N ⊗ b_ -> 1N ⊗ ( I a - I b),
    1N ⊗ a_ + 1N ⊗ b_ -> 1N ⊗ ( a + b), T["∇""S", "d", {a_}][b_] → tuDPartial[b, a]
   }; Column[$s],
  Yield, $ = $ //. $s,
  NL, "Apply (3.2) ",
  $s = tuRuleSolve[$F, CommutatorM[_, _]] /. CommutatorM → MCommutator // First //
    Map[-# &, #] &,
  NL, "Define ", $s1 = $O[[1]] //. {"E" → S, CommutatorM[X, Y] → 0,
     X -> μ, Y → ν, T["∇""S", "d", {0}] → 0}, CK,
  Yield, $s34 = e34 = $ = $ /. $s /. Reverse[$s1] // Simplify;
  Framed[$], CG[" (3.4)"]
 ];
```

•Define curvature of B_μ: $F_{\mu\nu} \to i\,[B_\mu, B_\nu] - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

•Define curvature of ∇^E: $\{\Omega^E[X, Y] \to \nabla^E{}_X.\nabla^E{}_Y - \nabla^E{}_Y.\nabla^E{}_X - \nabla^E{}_{[X,Y]},\ \{X, Y\} \to \text{vector fields}\}$

▪For local coordinates: $[\underline{\partial}_\mu[\_], \underline{\partial}_\nu[\_]] \to 0$

define $\{\underline{\partial}_\mu[\_] \to X,\ \underline{\partial}_\nu[\_] \to Y\}$

$\to \{[X, Y] \to 0,\ X \to \mu,\ Y \to \nu,\ \nabla^E{}_0 \to 0\}$

$\Rightarrow \Omega^E[\mu, \nu] \to \nabla^E{}_\mu.\nabla^E{}_\nu - \nabla^E{}_\nu.\nabla^E{}_\mu$

$\to \Omega^E[\mu, \nu] \to (i\,1_N\otimes B_\mu + \nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(i\,1_N\otimes B_\nu + \nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) - (i\,1_N\otimes B_\nu + \nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}).(i\,1_N\otimes B_\mu + \nabla^S{}_\mu\otimes 1_{\mathcal{H}_F})$

$\to \Omega^E[\mu, \nu] \to -(1_N\otimes B_\mu).(1_N\otimes B_\nu) + i\,(1_N\otimes B_\mu).(\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) + (1_N\otimes B_\nu).(1_N\otimes B_\mu) - i\,(1_N\otimes B_\nu).(\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}) + i\,(\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(1_N\otimes B_\nu) + (\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) - i\,(\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}).(1_N\otimes B_\mu) - (\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F})$

Using: $\{(a\_\otimes b\_).(c\_\otimes d\_) :\to \sum\left[\begin{array}{l} a.c\otimes b.d \\ \text{If}[!\text{FreeQ}[a, \nabla]\ \&\&\ !\text{FreeQ}[d, B \mid \Phi], c\otimes a[d], 0], \\ \text{If}[!\text{FreeQ}[b, \nabla]\ \&\&\ !\text{FreeQ}[d, B \mid \Phi], a\otimes b[d], 0] \end{array}\right],$

$1_{N}\_.(a\_) \to a,\ (a\_).1_{N}\_ \to a,\ \sum\left[\begin{array}{l} a\_\otimes 1_{\mathcal{H}_F} \\ -(b\_\otimes 1_{\mathcal{H}_F}) \end{array}\right] \to \sum\left[\begin{array}{l} a \\ -b \end{array}\right]\otimes 1_{\mathcal{H}_F},\ \sum\left[\begin{array}{l} 1_N\_\otimes a\_ \\ -(1_N\_\otimes b\_) \end{array}\right] \to 1_N\otimes\sum\left[\begin{array}{l} a \\ -b \end{array}\right]\}$

$$\to \boxed{\Omega^E[\mu, \nu] \to \sum\left[\begin{array}{l} (\nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu)\otimes 1_{\mathcal{H}_F} \\ 1_N\otimes(-B_\mu.B_\nu + B_\nu.B_\mu) \\ i\,1_N\otimes\nabla^S{}_\mu[B_\nu] \\ -i\,1_N\otimes\nabla^S{}_\nu[B_\mu] \end{array}\right]}$$

Use $\begin{array}{l} i\,1_N\otimes a\_ - i\,1_N\otimes b\_ \to 1_N\otimes(i\,a - i\,b) \\ 1_N\otimes a\_ + 1_N\otimes b\_ \to 1_N\otimes(a + b) \\ \nabla^S{}_{a\_}[b\_] \to \underset{-a}{\partial}[b] \end{array}$

$\to \Omega^E[\mu, \nu] \to (\nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu)\otimes 1_{\mathcal{H}_F} + 1_N\otimes(-B_\mu.B_\nu + B_\nu.B_\mu - i\,\underline{\partial}_\nu[B_\mu] + i\,\underline{\partial}_\mu[B_\nu])$

Apply (3.2) $-B_\mu.B_\nu + B_\nu.B_\mu \to i\,(F_{\mu\nu} + \underline{\partial}_\nu[B_\mu] - \underline{\partial}_\mu[B_\nu])$

Define $\Omega^S[\mu, \nu] \to \nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu$ ⟵CHECK

$$\to \boxed{\Omega^E[\mu, \nu] \to 1_N\otimes(i\,F_{\mu\nu}) + \Omega^S[\mu, \nu]\otimes 1_{\mathcal{H}_F}} \quad (3.4)$$

```
$d;
PR["•Calculate ", $0 = $ = CommutatorM[T[𝒟, "d", {μ}], T[𝒟, "d", {ν}]].Φ,
  NL, "From the definition: ", $d,
  Yield, $ = $ /. CommutatorM → MCommutator // tuDotSimplify[],
  yield, $ = $ //. a_ . b_ → a[b],
  Yield, $ = $ //. $d,
  Yield, $ = $ // tuDotSimplify[],
  NL, "Use ", $s =
   {a_ .Φ - b_ .Φ → (a - b).Φ, Φ.a_ - Φ. b_ → Φ.(a - b), a_ .b_ - b_ .a_ → CommutatorM[a, b],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}]},
  Yield, $ = $ // tuRepeat[$s, tuDotSimplify[]]; Framed[$0 → $],
  NL, "From ", $s1 = e33,
  yield, $s1 = $s1 /. $s // Reverse // tuAddPatternVariable[{μ, ν}],
  Imply, $ = $ /. $s1; Framed[$0 → $],
  yield, $ = $ /. CommutatorM → MCommutator /.
    ($d[[2]] // RemovePatterns // Reverse // tuAddPatternVariable[{aa, bb}]);
  Framed[$0 → $]
 ];
```

•Calculate $[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi$

From the definition: $\{\mathcal{D}_\mu[a\_] \to ad[\nabla^E{}_\mu][a],\ ad[aa\_][bb\_] \to aa.bb - bb.aa\}$

$\to\ \mathcal{D}_\mu.\mathcal{D}_\nu.\Phi - \mathcal{D}_\nu.\mathcal{D}_\mu.\Phi \longrightarrow \mathcal{D}_\mu[\mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[\mathcal{D}_\mu[\Phi]]$

$\to\ \nabla^E{}_\mu.\mathcal{D}_\nu[\Phi] - \mathcal{D}_\nu[\Phi].\nabla^E{}_\mu - \mathcal{D}_\nu[-\Phi.\nabla^E{}_\mu + \nabla^E{}_\mu.\Phi]$

$\to\ \nabla^E{}_\mu.\mathcal{D}_\nu[\Phi] - \mathcal{D}_\nu[\Phi].\nabla^E{}_\mu - \mathcal{D}_\nu[-\Phi.\nabla^E{}_\mu + \nabla^E{}_\mu.\Phi]$

Use $\{(a\_).\Phi - (b\_).\Phi \to (a-b).\Phi,\ \Phi.(a\_) - \Phi.(b\_) \to \Phi.(a-b),$
$(a\_).(b\_) - (b\_).(a\_) \to [a, b],\ [a\_, b\_] :\to -[b, a] /; OrderedQ[\{b, a\}]\}$

$\to\ \boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi \to [\nabla^E{}_\mu, \mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[-[\Phi, \nabla^E{}_\mu]]}$

From $\Omega^E[\mu, \nu] \to \nabla^E{}_\mu.\nabla^E{}_\nu - \nabla^E{}_\nu.\nabla^E{}_\mu \longrightarrow [\nabla^E{}_{\mu\_}, \nabla^E{}_{\nu\_}] \to \Omega^E[\mu, \nu]$

$\Rightarrow\ \boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi \to [\nabla^E{}_\mu, \mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[-[\Phi, \nabla^E{}_\mu]]} \longrightarrow \boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi \to ad[\nabla^E{}_\mu][\mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[-\Phi.\nabla^E{}_\mu + \nabla^E{}_\mu.\Phi]}$

```
$scc;
PR["Local Laplacian: ",
  $0 = $ = △"E" → -T[g, "uu", {μ, ν}].(T["∇""E", "d", {μ}].T["∇""E", "d", {ν}] -
         T[Γ, "udd", {ρ, μ, ν}].T["∇""E", "d", {ρ}]),
  NL, "Use definition ", $s = $sDA[[2]],
  Yield, $ = $ /. $s // tuDotSimplify[],
  Yield, $ = $ //. $scc /. {a_ . (b_ ⊗ c : 1_) → (a.b) ⊗ c};
  ColumnSumExp[$] // Framed,
  NL, "Define ", $s = $0 /. "E" → S,
  yield, $s = Map[# ⊗ 1_ℋF &, $s] // tuDotSimplify[];
  $s = $s /. (a_ + b_) ⊗ c_ → a ⊗ c + b ⊗ c /. tuOpSimplify[CircleTimes] // Reverse,
  Imply, $ =
   $ /. $s /. a_ ( tt : T[g, "uu", {μ, ν}]) .b_ → tt.(a b) //. ( tt : T[g, "uu", {μ, ν}]).(a_) +
       ( tt : T[g, "uu", {μ, ν}]) .b_ → tt.(a + b) // ExpandAll;
  ColumnSumExp[$],
  NL, "Use ", $s = {a_ . (1_N ⊗ c_) → a ⊗ c, a_ ⊗ B_μ :> (a /. ν → μ) ⊗ B_ν},
  $ = $ /. $s; Framed[e35 = $], CG[" (3.5)"]
 ];
```

Local Laplacian: $\triangle^{E} \to -g^{\mu\nu}.(\nabla^{E}{}_{\mu}.\nabla^{E}{}_{\nu} - \Gamma^{\rho}{}_{\mu\nu}.\nabla^{E}{}_{\rho})$

Use definition $\nabla^{E}{}_{\mu\_} \to i\, 1_N \otimes B_\mu + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}$

$\to \triangle^{E} \to g^{\mu\nu}.(1_N \otimes B_\mu).(1_N \otimes B_\nu) - i\, g^{\mu\nu}.(1_N \otimes B_\mu).(\nabla^{S}{}_{\nu} \otimes 1_{\mathcal{H}_F}) - i\, g^{\mu\nu}.(\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).(1_N \otimes B_\nu) - g^{\mu\nu}.(\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).(\nabla^{S}{}_{\nu} \otimes 1_{\mathcal{H}_F}) + i\, g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}.(1_N \otimes B_\rho) + g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}.(\nabla^{S}{}_{\rho} \otimes 1_{\mathcal{H}_F})$

$$\to \boxed{\triangle^{E} \to \sum\left[\begin{array}{l} -(g^{\mu\nu}.\nabla^{S}{}_{\mu}.\nabla^{S}{}_{\nu} \otimes 1_{\mathcal{H}_F}) \\ g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}.\nabla^{S}{}_{\rho} \otimes 1_{\mathcal{H}_F} \\ g^{\mu\nu}.(1_N \otimes B_\mu.B_\nu) \\ -i\, g^{\mu\nu}.(1_N \otimes \nabla^{S}{}_{\mu}[B_\nu] + \nabla^{S}{}_{\mu} \otimes B_\nu) \\ -i\, g^{\mu\nu}.(\nabla^{S}{}_{\nu} \otimes B_\mu) \\ i\, g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}.(1_N \otimes B_\rho) \end{array}\right]}$$

Define $\triangle^{S} \to -g^{\mu\nu}.(\nabla^{S}{}_{\mu}.\nabla^{S}{}_{\nu} - \Gamma^{\rho}{}_{\mu\nu}.\nabla^{S}{}_{\rho}) \longrightarrow -(g^{\mu\nu}.\nabla^{S}{}_{\mu}.\nabla^{S}{}_{\nu} \otimes 1_{\mathcal{H}_F}) + g^{\mu\nu}.\Gamma^{\rho}{}_{\mu\nu}.\nabla^{S}{}_{\rho} \otimes 1_{\mathcal{H}_F} \to \triangle^{S} \otimes 1_{\mathcal{H}_F}$

$\Rightarrow \triangle^{E} \to \sum\left[\begin{array}{l} \triangle^{S} \otimes 1_{\mathcal{H}_F} \\ g^{\mu\nu}.(1_N \otimes B_\mu.B_\nu - i\, 1_N \otimes \nabla^{S}{}_{\mu}[B_\nu] - i\, \nabla^{S}{}_{\mu} \otimes B_\nu - i\, \nabla^{S}{}_{\nu} \otimes B_\mu + i\, \Gamma^{\rho}{}_{\mu\nu}.(1_N \otimes B_\rho)) \end{array}\right]$

Use $\{(a\_).(1_N \otimes c\_) \to a \otimes c,\ a\_ \otimes B_\mu :\to (a /. \nu \to \mu) \otimes B_\nu\}$

$$\boxed{\triangle^{E} \to \triangle^{S} \otimes 1_{\mathcal{H}_F} + g^{\mu\nu}.(1_N \otimes B_\mu.B_\nu - i\, 1_N \otimes \nabla^{S}{}_{\mu}[B_\nu] - 2\, i\, \nabla^{S}{}_{\mu} \otimes B_\nu + i\, \Gamma^{\rho}{}_{\mu\nu} \otimes B_\rho)} \quad (3.5)$$

```
$sDA;
$sD;
$scc;
PR["•Given the Lichnerowicz formula: ",
  $L = slash[𝒟].slash[𝒟] → △^S + s / 4,
  NL, "Show(prop.3.1) ", $31 = $0 = $ =
     {𝒟𝒜.𝒟𝒜 → △^"E" - Q, Q → -( s ⊗ 1_ℋF) / 4 - 1_N ⊗ (Φ . Φ) + I / 2 (T[γ, "u", {μ}].T[γ, "u", {ν}]) ⊗
          T[F, "dd", {μ, ν}] - I T[γ, "u", {μ}]. T[γ, "d", {5}] ⊗ T[𝒟, "d", {μ}].Φ},
  Yield, $ = $0[[1, 1]], CK,
  Yield, xtmp = $ = $ //. $sDA[[1 ;; 2]] /. a_ . b_ :→ a. (b /. μ → ν), CK, (***)
  NL, "Use ",
  $s = $ss = {
     T[γ, "d", {5}].T[γ, "d", {5}] → 1_N,
     (tt : T[γ, "u", {μ_}]).(1_N ⊗ b_) :→ (tt ⊗ b) /; ! FreeQ[b, μ],
     (a_ ⊗ b_).(c_ ⊗ d_) :→ a . c ⊗ b . d +
       If[! FreeQ [a, "∇"] && ! FreeQ [d, B], c ⊗ a[d] , 0] +
       If[! FreeQ [b, "∇"] && ! FreeQ [d, B], a ⊗ b[d], 0],
     (tt : T[γ, "u", {μ_}]).(a_ ⊗ b_) :→ (tt.a ⊗ b) /; ! FreeQ[a, "∇"],
     (tt : T[γ, "u", {μ_}]). Shortest[a_] .b_ :→
      I slash[𝒟].b /; ! FreeQ[a, "∇" && ! (FreeQ[a, μ])],
     (tt : T[γ, "u", {μ_}]) . a_ .b_ :→ I slash[𝒟] /;
       ! FreeQ[a, "∇" && ! (FreeQ[a, μ] && FreeQ[b, "∇"])],
     b__ .(tt : T[γ, "u", {μ_}]). a_ :→ I b.slash[𝒟] /; ! FreeQ[a, "∇" && ! (FreeQ[a, μ])],
     1_N_ . a_ → a, a_ . 1_N_ → a ,
     (a_ ⊗ 1_N_) - (b_ ⊗ 1_N_) → (a - b) ⊗ 1_N, (1_N_ ⊗ a_) - (1_N_ ⊗ b_) → 1_N ⊗ (a - b)};
 Column[$s],
 $ = $ // tuRepeat[$s, tuDotSimplify[]];
 $pass = $ = $ /. tuOpSimplify[CircleTimes]; ColumnSumExp[$] // Framed
 ];
```

•Given the Lichnerowicz formula: $(\not{D}).(\not{D}) \to \frac{s}{4} + \triangle^S$

Show(prop.3.1) $\{\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to -Q + \triangle^E,\ Q \to -\frac{1}{4} s \otimes 1_{\mathcal{H}_F} - i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi\}$

→ $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}}$ ⟵CHECK

→ $(\gamma_5 \otimes \Phi - i\,\gamma^\mu.(i\,1_N \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})).(\gamma_5 \otimes \Phi - i\,\gamma^\nu.(i\,1_N \otimes B_\nu + \nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}))$ ⟵CHECK

Use

```
γ5.γ5 → 1N
(tt : γ^μ_).(1N⊗b_) :→ tt⊗b /; ! FreeQ[b, μ]
(a_⊗b_).(c_⊗d_) :→ a.c⊗b.d +
   If[! FreeQ[a, ∇] && ! FreeQ[d, B], c⊗a[d], 0] + If[! FreeQ[b, ∇] && ! FreeQ[d, B], a⊗b[d], 0]
(tt : γ^μ_).(a_⊗b_) :→ tt.a⊗b /; ! FreeQ[a, ∇]
(tt : γ^μ_).Shortest[a_].(b_) :→ i (D̸).b /; ! FreeQ[a, ∇ && ! FreeQ[a, μ]]
(tt : γ^μ_).(a_).(b_) :→ i (D̸) /; ! FreeQ[a, ∇ && ! (FreeQ[a, μ] && FreeQ[b, ∇])]
(b__).(tt : γ^μ_).(a_) :→ i b.(D̸) /; ! FreeQ[a, ∇ && ! FreeQ[a, μ]]
1N_.(a_) → a
(a_).1N_ → a
a_⊗1N_ - b_⊗1N_ → (a - b)⊗1N
1N_⊗a_ - 1N_⊗b_ → 1N⊗(a - b)
```

$$\boxed{\sum\left[\begin{array}{l} (\not{D}).\gamma_5 \otimes \Phi \\ (\not{D}).\gamma^\nu \otimes B_\nu \\ \gamma_5.(\not{D}) \otimes \Phi \\ \gamma_5.\gamma^\nu \otimes \Phi.B_\nu \\ \gamma^\mu.(\not{D}) \otimes B_\mu \\ \gamma^\mu.\gamma_5 \otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\nu \otimes B_\mu.B_\nu \\ 1_N \otimes \Phi.\Phi \\ -i\,(\gamma^\mu.\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(\not{D}) \\ -i\,\gamma^\mu.(\gamma^\nu \otimes \nabla^S{}_\mu[B_\nu]) \end{array}\right]}$$

```
PR["•Examine different terms of: ", $0 = $pass; ColumnSumExp[$0],
  NL, "•1: ", $ = $0[[1]] → "Lichnerowicz formula" → Framed[$p[1] = $L[[2]] ⊗ 1HF],
  NL, "•2,4: ", $ = $0[[{2, 4}]],
  NL, "Use ", CommutatorM[T[γ, "d", {5}], slash[D]] → 0,
  imply, $ → Framed[0],
  CO[back, "Liebnitz like rule accounted for by[[10]] ", $p[6] = $0[[10]]],

  NL, "•3,6,10: ", $p[2] = $ = $0[[{3, 6, 10}]]; Framed[$], CK,
  NL, "•5,7: ", $ = $0[[{5, 7}]],
  NL, "Use ", $s = CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  yield, $s = $s /. CommutatorP → ACommutator,
  yield, $s = -$s[[1, 2]] + # & /@ $s // tuAddPatternVariable[{μ}],
  Imply, $ = $ /. $s /. tuOpSimplify[CircleTimes] /. ν → μ,
  yield, $ = $ /. (a_ ⊗ b_) - (a_ ⊗ c_) → a ⊗ (b - c); Framed[$p[3] = $],
  NL, "•8: ", $ = $0[[8]],
  NL, "Use symmetic and antisymmetric form: ",
  $s = $[[2]] → 1 / 2 (Apply[CommutatorP, $[[2]]] + Apply[CommutatorM, $[[2]]]),
  Yield, $ = $ /. $s /. tuOpSimplify[CircleTimes],
  Yield, $ = $ /. a_ ⊗ (b_ + c_) -> a ⊗ (b) + a ⊗ (c); Framed[$p[4] = $],
  NL, "•9: ", $ = $0[[9]]; Framed[$p[5] = $],
  NL, "●All terms: ", $pass1 = Sum[$p[i], {i, 6}]; ColumnSumExp[$pass1]
 ];
```

•Examine different terms of: $\sum\left[\begin{array}{l} (\not{D}).\gamma_5\otimes\Phi \\ (\not{D}).\gamma^\nu\otimes B_\nu \\ \gamma_5.(\not{D})\otimes\Phi \\ \gamma_5.\gamma^\nu\otimes\Phi.B_\nu \\ \gamma^\mu.(\not{D})\otimes B_\mu \\ \gamma^\mu.\gamma_5\otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu \\ 1_N\otimes\Phi.\Phi \\ -i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D}) \\ -i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu]) \end{array}\right]$

•1: $(\not{D}).\gamma_5\otimes\Phi\to$ Lichnerowicz formula $\to\boxed{(\frac{s}{4}+\triangle^S)\otimes 1_{\mathcal{H}_F}}$

•2,4: $(\not{D}).\gamma^\nu\otimes B_\nu+\gamma_5.\gamma^\nu\otimes\Phi.B_\nu$

Use $[\gamma_5,\not{D}]\to 0\;\Rightarrow\;(\not{D}).\gamma^\nu\otimes B_\nu+\gamma_5.\gamma^\nu\otimes\Phi.B_\nu\to\boxed{0}\longleftarrow$

Liebnitz like rule accounted for by[[10]] $-i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu])$

•3,6,10: $\boxed{\gamma_5.(\not{D})\otimes\Phi+\gamma^\mu.\gamma_5\otimes B_\mu.\Phi-i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu])}\longleftarrow$CHECK

•5,7: $\gamma^\mu.(\not{D})\otimes B_\mu+\gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu$

Use $\{\gamma_5,\gamma^\mu\}\to 0\longrightarrow\gamma_5.\gamma^\mu+\gamma^\mu.\gamma_5\to 0\longrightarrow\gamma_5.\gamma^\mu-\to-\gamma^\mu.\gamma_5$

$\Rightarrow\;\gamma^\mu.(\not{D})\otimes B_\mu+\gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu\longrightarrow\boxed{\gamma^\mu.(\not{D})\otimes B_\mu+\gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu}$

•8: $1_N\otimes\Phi.\Phi$

Use symmetric and antisymmetric form: $\Phi.\Phi\to\frac{1}{2}([\Phi,\Phi]+\{\Phi,\Phi\})$

$\to\frac{1}{2}1_N\otimes([\Phi,\Phi]+\{\Phi,\Phi\})$

$\to\boxed{\frac{1}{2}(1_N\otimes[\Phi,\Phi]+1_N\otimes\{\Phi,\Phi\})}$

•9: $\boxed{-i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D})}$

●All terms: $\sum\left[\begin{array}{l} (\frac{s}{4}+\triangle^S)\otimes 1_{\mathcal{H}_F} \\ \gamma_5.(\not{D})\otimes\Phi \\ \gamma^\mu.(\not{D})\otimes B_\mu \\ \gamma^\mu.\gamma_5\otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu \\ \frac{1}{2}(1_N\otimes[\Phi,\Phi]+1_N\otimes\{\Phi,\Phi\}) \\ -i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D}) \\ -2\,i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu]) \end{array}\right]$

```
PR["●Manipulate (3.5) to apply to this form: ", $35 = $ = e35,
  NL, "Use ",
  $s = $s2 = {T[g, "uu", {μ, ν}] →
      1 / 2 (T[γ, "u", {μ}].T[γ, "u", {ν}] + T[γ, "u", {ν}].T[γ, "u", {μ}]),
     a_ .(1N ⊗ c_) → (a) ⊗ c, T["∇"S, "d", {a_}][b_] → tuDPartial[b, a]},
  Imply, $ = $ // tuRepeat[Join[$s, $ss], tuDotSimplify[]];
  ColumnSumExp[$];
  Yield, $ = $ //. $s /. tuOpSimplify[CircleTimes]; ColumnSumExp[$],
  NL, "■Evaluate parts of RHS: ", $1 = $[[2]];

  NL, CB["▪", $i = {3, 6}, ": "], $ = $1[[$i]],
  Yield, $ = MapAt[Swap[{μ, ν}][#] &, $, 2] /. a_ (b_ ⊗ c_) + a_ (b_ ⊗ c1_) → a (b ⊗ (c + c1)),
  NL, "From definition: ", $F,
  yield, $s = Map[# - $F[[2, {1, 2}]] &, $F] // Reverse,
  Yield, $p[1] = {#} & /@ $i -> $ /. $s //. tuOpDistribute[CircleTimes] /.
     tuOpSimplify[CircleTimes] // Expand;
  Framed[$p[1]], "POFF",
  $i1 = $i;
  NL, "▪{}: ", Delete[$1, ({#} & /@ $i1)] // ColumnSumExp, CK, "PON",

  NL, CB["▪", $i = {2, 5}, ": "], $ = $1[[$i]],
  yield, $ = MapAt[Swap[{μ, ν}][#] &, $, 2] /. a_ (b_ ⊗ c_) + a_ (b_ ⊗ c1_) → a (b ⊗ (c + c1)),
  NL, "Use ", $s = ACommutator[a_, b_] -> CommutatorP[a, b],
  Yield, $p[2] = {#} & /@ $i -> $ /. $s //. tuOpDistribute[CircleTimes] /.
     tuOpSimplify[CircleTimes] // Expand;
  Framed[$p[2]], "POFF",
  $i1 = Join[$i1, $i];
  NL, "▪{}: ", $3 = Delete[$1, ({#} & /@ $i1)]; ColumnSumExp[$3], "PON",
  Yield, $s = {$p[1], $p[2]},
  Imply, $35 = $35[[1]] → ($3 + Apply[Plus, #[[2]] & /@ $s]);
  ColumnSumExp[$35]
 ];
```

●Manipulate (3.5) to apply to this form:

$\triangle^E \to \triangle^S \otimes 1_{\mathcal{H}_F} + g^{\mu\nu}.(1_N \otimes B_\mu . B_\nu - i\, 1_N \otimes \nabla^S{}_\mu[B_\nu] - 2\, i\, \nabla^S{}_\mu \otimes B_\nu + i\, \Gamma^\rho{}_{\mu\nu} \otimes B_\rho)$

Use $\{g^{\mu\nu} \to \frac{1}{2}(\gamma^\mu.\gamma^\nu + \gamma^\nu.\gamma^\mu),\ (a\_).(1_N \otimes c\_) \to a \otimes c,\ \nabla^S{}_{a\_}[b\_] \to \underline{\partial}_a[b]\}$

⇒

$$\to \triangle^E \to \sum\left[\begin{array}{l} \triangle^S \otimes 1_{\mathcal{H}_F} \\ \frac{1}{2}\gamma^\mu.\gamma^\nu \otimes B_\mu.B_\nu \\ -\frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes \underline{\partial}_\mu[B_\nu] \\ \gamma^\nu.(\not{D}) \otimes B_\nu \\ \frac{1}{2}\gamma^\nu.\gamma^\mu \otimes B_\mu.B_\nu \\ -\frac{1}{2} i\, \gamma^\nu.\gamma^\mu \otimes \underline{\partial}_\mu[B_\nu] \\ -i\, \gamma^\mu.\gamma^\nu.\nabla^S{}_\mu \otimes B_\nu \\ \frac{1}{2} i\, \gamma^\mu.\gamma^\nu.(\Gamma^\rho{}_{\mu\nu} \otimes B_\rho) \\ \frac{1}{2} i\, \gamma^\nu.\gamma^\mu.(\Gamma^\rho{}_{\mu\nu} \otimes B_\rho) \end{array}\right]$$

■Evaluate parts of RHS:

■{3, 6}: $-\frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes \underline{\partial}_\mu[B_\nu] - \frac{1}{2} i\, \gamma^\nu.\gamma^\mu \otimes \underline{\partial}_\mu[B_\nu]$

$\to -\frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes (\underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu])$

From definition: $F_{\mu\nu} \to i\,[B_\mu, B_\nu] - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu] \longrightarrow \underline{\partial}_\mu[B_\nu] \to -i\,[B_\mu, B_\nu] + F_{\mu\nu} + \underline{\partial}_\nu[B_\mu]$

$\to \boxed{\{\{3\},\{6\}\} \to -\frac{1}{2}\gamma^\mu.\gamma^\nu \otimes [B_\mu, B_\nu] - \frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - i\, \gamma^\mu.\gamma^\nu \otimes \underline{\partial}_\nu[B_\mu]}$

■{2, 5}: $\frac{1}{2}\gamma^\mu.\gamma^\nu \otimes B_\mu.B_\nu + \frac{1}{2}\gamma^\nu.\gamma^\mu \otimes B_\mu.B_\nu \longrightarrow \frac{1}{2}\gamma^\mu.\gamma^\nu \otimes (B_\mu.B_\nu + B_\nu.B_\mu)$

Use $(a\_).(b\_) + (b\_).(a\_) \to \{a, b\}$

$\to \boxed{\{\{2\},\{5\}\} \to \frac{1}{2}\gamma^\mu.\gamma^\nu \otimes \{B_\mu, B_\nu\}}$

→

$\{\{\{3\},\{6\}\} \to -\frac{1}{2}\gamma^\mu.\gamma^\nu \otimes [B_\mu, B_\nu] - \frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - i\, \gamma^\mu.\gamma^\nu \otimes \underline{\partial}_\nu[B_\mu],\ \{\{2\},\{5\}\} \to \frac{1}{2}\gamma^\mu.\gamma^\nu \otimes \{B_\mu, B_\nu\}\}$

$$\Rightarrow \triangle^E \to \sum\left[\begin{array}{l} \triangle^S \otimes 1_{\mathcal{H}_F} \\ -\frac{1}{2}\gamma^\mu.\gamma^\nu \otimes [B_\mu, B_\nu] \\ \frac{1}{2}\gamma^\mu.\gamma^\nu \otimes \{B_\mu, B_\nu\} \\ -\frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} \\ -i\, \gamma^\mu.\gamma^\nu \otimes \underline{\partial}_\nu[B_\mu] \\ \gamma^\nu.(\not{D}) \otimes B_\nu \\ -i\, \gamma^\mu.\gamma^\nu.\nabla^S{}_\mu \otimes B_\nu \\ \frac{1}{2} i\, \gamma^\mu.\gamma^\nu.(\Gamma^\rho{}_{\mu\nu} \otimes B_\rho) \\ \frac{1}{2} i\, \gamma^\nu.\gamma^\mu.(\Gamma^\rho{}_{\mu\nu} \otimes B_\rho) \end{array}\right]$$

```
PR["Simplifying ", $ =
    $31[[1, 1]] -> $pass1 /. tuOpDistribute[CircleTimes] /. tuOpSimplify[CircleTimes],
  $ = $ /. tuRuleSolve[$35, △S ⊗ 1ℋF] // Simplify; ColumnSumExp[$],
  Yield, $ = $ //. {a_ . (b_ ⊗ c_) → (a. b) ⊗ c, a_ (b_ ⊗ c_) + a_ (b1_ ⊗ c_) → a (b + b1) ⊗ c,
        a_ .b_ + a1_ . b_ → (a + a1).b, aa : a_ ⊗ Bν :→ (aa /. ν → μ) /; FreeQ[aa, μ],
        T["∇"S, "d", {a_}][b_] → tuDPartial[b, a],
        T[γ, "u", {μ}]. a_ . T["∇"S, "d", {μ}] → I slash[𝒟].a,
         b_ ⊗ c_ - b_ ⊗ d_ → b ⊗ (c - d),
        b_ ⊗ c_ - I b_ ⊗ d_ → b ⊗ (c - I d),
        b_ ⊗ c_ - a1_ b_ ⊗ d_ → b ⊗ (c - a1 d),
        Reverse[2 T[g, "uu", {μ, ν}] →
          (T[γ, "u", {μ}].T[γ, "u", {ν}] + T[γ, "u", {ν}].T[γ, "u", {μ}])]
      } /. tuOpSimplify[CircleTimes] /. CommutatorM → MCommutator // tuDotSimplify[];
  ColumnSumExp[$],
  NL, "Using ",
  $s = {-I # & /@ $D1 /. tuOpSimplify[CircleTimes] //.
        { 1_ ⊗ a_ → a, a_ ⊗ 1_ → a, T["∇"S, "d", {a_}][b_] → tuDPartial[b, a]
        } // Simplify // Reverse // tuAddPatternVariable[Φ],
    T[g, "uu", {μ, ν}].T[Γ, "udd", {ρ, μ, ν}] → 0, I b_ ⊗ c_ - I b_ ⊗ d_ → I b ⊗ (c - d),
    0 ⊗ _ → 0
   },
  Yield, $ = $ //. $s /. tuOpSimplify[CircleTimes] //. $s;
  Yield, $ = $ /. a_ . b_ - b_ . a_ → CommutatorM[a, b] /.
     tuRuleSolve[$F, CommutatorM[_, _]] // ExpandAll,
  Yield, $ = $ //. tuOpDistribute[CircleTimes] //. tuOpSimplify[CircleTimes],
  ColumnSumExp[$] // Framed,
  CG[" QED"]
 ];
```

Simplifying $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \frac{s\otimes 1_{\mathcal{H}_F}}{4} + \triangle^S\otimes 1_{\mathcal{H}_F} + \gamma_5.(\not{D})\otimes\Phi + \gamma^\mu.(\not{D})\otimes B_\mu + \gamma^\mu.\gamma_5\otimes B_\mu.\Phi + \gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu +$
$\frac{1}{2}(1_N\otimes[\Phi,\Phi] + 1_N\otimes\{\Phi,\Phi\}) - i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D}) - 2\,i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu])\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \frac{1}{4}\sum[$

$s\otimes 1_{\mathcal{H}_F}$

$2\,(2\,\triangle^E + 2\,\gamma_5.(\not{D})\otimes\Phi + 2\,\gamma^\mu.(\not{D})\otimes B_\mu + 2\,\gamma^\mu.\gamma_5\otimes B_\mu.\Phi + 2\,\gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu + \gamma^\mu.\gamma^\nu\otimes[B_\mu, B_\nu] - \gamma^\mu.\gamma^\nu\otimes\{B_\mu, B_\nu\} +$
$i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} + 2\,i\,\gamma^\mu.\gamma^\nu\otimes\underset{-\nu}{\partial}[B_\mu] - 2\,\gamma^\nu.(\not{D})\otimes B_\nu + 2\,i\,\gamma^\mu.\gamma^\nu.\nabla^S{}_\mu\otimes B_\nu + 1_N\otimes[\Phi,\Phi] + 1_N\otimes\{\Phi,\Phi\} -$]
$2\,i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D}) - 4\,i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu]) - i\,\gamma^\mu.\gamma^\nu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho) - i\,\gamma^\nu.\gamma^\mu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho))$

$\rightarrow \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^E \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -((\not{D}).\gamma^\mu\otimes B_\mu) \\ -\frac{1}{2}\,i\,(2\,g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu})\otimes B_\rho \\ \gamma_5.(\not{D})\otimes\Phi \\ \gamma^\mu.\gamma_5\otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu \\ \frac{1}{2}\gamma^\mu.\gamma^\nu\otimes(-\{B_\mu, B_\nu\} + B_\mu.B_\nu - B_\nu.B_\mu) \\ \frac{1}{2}\,i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ i\,\gamma^\mu.\gamma^\nu\otimes\underset{-\nu}{\partial}[B_\mu] \\ -2\,i\,\gamma^\mu.\gamma^\nu\otimes\underset{-\mu}{\partial}[B_\nu] \\ \frac{1_N\otimes 0}{2} \\ \frac{1}{2}1_N\otimes\{\Phi,\Phi\} \\ -i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D}) \end{array}\right]$

Using

$\{-(\bar{\Phi}\_).B_\mu + B_\mu.(\Phi\_) - i\,\underline{\partial}_\mu[\Phi\_] \to -i\,\mathcal{D}_\mu[\Phi],\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu} \to 0,\ i\,b\_\otimes c\_ - i\,b\_\otimes d\_ \to i\,b\otimes(c-d),\ 0\otimes\_ \to 0\}$

$\to$

$\to$

$$\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \triangle^E + \frac{s\otimes 1_{\mathcal{H}_F}}{4} - (\not{D}).\gamma^\mu\otimes B_\mu + \gamma_5.(\not{D})\otimes\Phi + \gamma^\mu.\gamma_5\otimes B_\mu.\Phi + \gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu + \frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} + i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] + \frac{1}{2}\gamma^\mu.\gamma^\nu\otimes(-\{B_\mu, B_\nu\} - i\,F_{\mu\nu} - i\,\underline{\partial}_\nu[B_\mu] + i\,\underline{\partial}_\mu[B_\nu]) - 2\,i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] + \frac{1}{2}1_N\otimes\{\Phi,\Phi\} - i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D})$$

$$\to \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \triangle^E + \frac{s\otimes 1_{\mathcal{H}_F}}{4} - (\not{D}).\gamma^\mu\otimes B_\mu + \gamma_5.(\not{D})\otimes\Phi + \gamma^\mu.\gamma_5\otimes B_\mu.\Phi + \gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu + \frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} + i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] + \frac{1}{2}(-(\gamma^\mu.\gamma^\nu\otimes\{B_\mu, B_\nu\}) - i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] + i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu]) - 2\,i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] + \frac{1}{2}1_N\otimes\{\Phi,\Phi\} - i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D})$$

$$\boxed{\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum\left[\begin{array}{l} \triangle^E \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -((\not{D}).\gamma^\mu\otimes B_\mu) \\ \gamma_5.(\not{D})\otimes\Phi \\ \gamma^\mu.\gamma_5\otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\mu\otimes B_\mu.B_\mu \\ \frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] \\ \frac{1}{2}(-(\gamma^\mu.\gamma^\nu\otimes\{B_\mu, B_\nu\}) - i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] + i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu]) \\ -2\,i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] \\ \frac{1}{2}1_N\otimes\{\Phi,\Phi\} \\ -i\,(\gamma^\mu.\nabla^S{}_\mu\otimes 1_{\mathcal{H}_F}).(\not{D}) \end{array}\right]}$$ QED

Alternative calculation UNFINISHED

```
$s = Join[$scc, {(tt : T[γ, "u", {μ_}]).(a_ ⊗ b_) -> (tt.a ⊗ b),
    T[γ, "d", {5}] . T[γ, "d", {5}] → 1N,
    gg : T[γ, "u", {ν_}].T[γ, "d", {5}] :> -Reverse[gg]
   }]; Column[$s];
Bi_ := T[B, "d", {i}];
xtmp // ColumnSumExp;
$ = xtmp //
    tuRepeat[$s, (Simplify[tuDotSimplify[][#]] //. tuOpSimplify[CircleTimes]) &];

$s1 = {aa : a_ ⊗ b_ ⧴ (aa /. ν :> μ) /; FreeQ[aa, μ], (*
    (aa_⊗a_)-(aa_⊗b_)→aa⊗(a-b),*)(*
    a_⊗((gg:Tensor[γ,_,_]). b_)[c_]:>(a.gg.b)⊗c/;¬FreeQ[b,"∇"],*)
    a_ ⊗ ((gg : Tensor[γ, _, _]). b_)[c_] :> (a.gg) ⊗ b[c] /; ¬ FreeQ[b, "∇"],
    Bμ.Bν → (CommutatorM[Bμ, Bν] + CommutatorP[Bμ, Bν]) / 2,
    (T[γ, "u", {μ_}].T[γ, "u", {ν_}]) ⊗ CommutatorP[a_, b_] →
     2 T[g, "uu", {μ, ν}] 1N ⊗ a.b
   };
FramedColumn[$s1]
$ = $ //. $s1;
ColumnSumExp[$];
$ = $ // tuRepeat[{}, (Expand[tuDotSimplify[][#]] //. tuOpDistribute[CircleTimes] //.
        tuOpSimplify[CircleTimes]) &];
ColumnSumExp[$];
$ = $ /. Join[{(T[γ, "u", {μ_}] . T[γ, "u", {ν_}]) ⊗ CommutatorP[a_, b_] →
        2 × 1N ⊗ (a.b T[g, "uu", {μ, ν}])},
      tuRuleSolve[$F, CommutatorM[_, _]]] // ContractUpDn[g];
$ = $ /. {a_ .(tt : T["∇"^S, "d", {μ}]).b ⊗ Φ → a.b ⊗ tt[Φ]};
ColumnSumExp[$]


$d1 = Map[T[γ, "d", {5}] . T[γ, "u", {μ}].# &, $D1] // tuRepeat[{a_ .(1N ⊗ b_) → a ⊗ b},
    (Expand[tuDotSimplify[][#]] //. tuOpSimplify[CircleTimes]) &]
$d1 = tuRuleSolve[$d1, $d1[[2, -1]]] // Expand // First
$ = $ /. $d1 // Expand;


$ = $ /. tt : T["∇"^S, "d", {μ}].T[γ, "d", {5}] ⧴ Reverse[tt];
$ = $ /. tuRuleSolve[$x = tuIndicesLower[5][ps371], $x[[1, 2]]] //
    tuRepeat[{}, (Expand[tuDotSimplify[][#]] //. tuOpDistribute[CircleTimes] //.
        tuOpSimplify[CircleTimes]) &];
$ = $ /. a_ . T["∇"^S, "d", {μ_}] . b___ ⊗ c_ :> a . b ⊗ tuDDown["∂"][c, μ] /; FreeQ[c, 1] /.
   T["∇"^S, "d", {μ_}][a_] → tuDDown["∂"][a, μ]
ColumnSumExp[$]


$ = Select[$, ! FreeQ[#, B] &]
$s = {tt : a_ ⊗ tuDDown["∂"][T[B, "d", {μ}], ν] ⧴ tuIndexSwap[{μ, ν}][tt],
   XX a1_ (aa_ ⊗ a_) + b1_ (aa_ ⊗ b_) :> aa ⊗ (a1 a + b1 b) /; ! FreeQ[a, B] && ! FreeQ[b, B]};
Column[$s]
$ = $ //. $s ;
ColumnSumExp[$]
```

```
aa : a_ ⊗ b_ :→ (aa /. ν :→ μ) /; FreeQ[aa, μ]
a_ ⊗ ((gg : Tensor[γ, _, _]).(b_))[c_] :→ a.gg ⊗ b[c] /; ! FreeQ[b, ∇]
B_μ.B_ν → 1/2 ([B_μ, B_ν] + {B_μ, B_ν})
γ^μ_.γ^ν_ ⊗ {a_, b_} → 2 × 1_N ⊗ a.b g^μν
```

$$\sum\left[\begin{array}{l} \gamma_5.\gamma^{\mu}\otimes\Phi.B_{\mu} \\ -(\gamma_5.\gamma^{\mu}\otimes B_{\mu}.\Phi) \\ \mathbb{i}\,\gamma_5.\gamma^{\mu}\otimes\nabla^{S}{}_{\mu}[\Phi] \\ \frac{1}{2}\gamma^{\mu}.\gamma^{\nu}\otimes(-\mathbb{i}\,(F_{\mu\nu}+\underset{-\nu}{\partial}[B_{\mu}]-\underset{-\mu}{\partial}[B_{\nu}])) \\ -\mathbb{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\nabla^{S}{}_{\mu}[B_{\nu}] \\ -\mathbb{i}\,\gamma_5.\gamma^{\mu}.\nabla^{S}{}_{\mu}\otimes\Phi \\ -\mathbb{i}\,\gamma^{\mu}.\nabla^{S}{}_{\mu}.\gamma_5\otimes\Phi \\ -\mathbb{i}\,\gamma^{\mu}.\nabla^{S}{}_{\mu}.\gamma^{\nu}\otimes B_{\nu} \\ -\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes B_{\mu} \\ -(\gamma^{\mu}.\nabla^{S}{}_{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes 1_{\mathcal{H}_F}) \\ 1_N\otimes\Phi.\Phi \\ 1_N\otimes B^{\nu}.B_{\nu} \end{array}\right]$$

$$\gamma_5.\gamma^{\mu}.\mathcal{D}_{\mu}[\Phi]\to -\mathbb{i}\,\gamma_5.\gamma^{\mu}\otimes\Phi.B_{\mu}+\mathbb{i}\,\gamma_5.\gamma^{\mu}\otimes B_{\mu}.\Phi+\gamma_5.\gamma^{\mu}\otimes\nabla^{S}{}_{\mu}[\Phi]$$

$$\gamma_5.\gamma^{\mu}\otimes\nabla^{S}{}_{\mu}[\Phi]\to \mathbb{i}\,\gamma_5.\gamma^{\mu}\otimes\Phi.B_{\mu}-\mathbb{i}\,\gamma_5.\gamma^{\mu}\otimes B_{\mu}.\Phi+\gamma_5.\gamma^{\mu}.\mathcal{D}_{\mu}[\Phi]$$

$$-\frac{1}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}-\frac{3}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\nu}[B_{\mu}]-\frac{1}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\mu}[B_{\nu}]- \mathbb{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[B_{\nu}]-\gamma^{\mu}.\nabla^{S}{}_{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes 1_{\mathcal{H}_F}+1_N\otimes\Phi.\Phi+1_N\otimes B^{\nu}.B_{\nu}+\mathbb{i}\,\gamma_5.\gamma^{\mu}.\mathcal{D}_{\mu}[\Phi]$$

$$\sum\left[\begin{array}{l} -\frac{1}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} \\ -\frac{3}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes\underset{-\nu}{\partial}[B_{\mu}] \\ -\frac{1}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes\underset{-\mu}{\partial}[B_{\nu}] \\ -\mathbb{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[B_{\nu}] \\ -(\gamma^{\mu}.\nabla^{S}{}_{\mu}.\gamma^{\nu}.\nabla^{S}{}_{\nu}\otimes 1_{\mathcal{H}_F}) \\ 1_N\otimes\Phi.\Phi \\ 1_N\otimes B^{\nu}.B_{\nu} \\ \mathbb{i}\,\gamma_5.\gamma^{\mu}.\mathcal{D}_{\mu}[\Phi] \end{array}\right]$$

$$-\frac{3}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\nu}[B_{\mu}]-\frac{1}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes\underline{\partial}_{\mu}[B_{\nu}]-\mathbb{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underline{\partial}_{\mu}[B_{\nu}]+1_N\otimes B^{\nu}.B_{\nu}$$

```
tt : a_ ⊗ ∂_{-ν} [B_μ] :→ tuIndexSwap[{μ, ν}][tt]
XX aa_ ⊗ a_ a1_ + aa_ ⊗ b_ b1_ :→ aa ⊗ (a1 a + b1 b) /; ! FreeQ[a, B] && ! FreeQ[b, B]
```

$$\sum\left[\begin{array}{l} -\frac{1}{2}\mathbb{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes\underset{-\mu}{\partial}[B_{\nu}] \\ -\frac{5}{2}\mathbb{i}\,\gamma^{\nu}.\gamma^{\mu}\otimes\underset{-\mu}{\partial}[B_{\nu}] \\ 1_N\otimes B^{\nu}.B_{\nu} \end{array}\right]$$

Heat expansion

```
PR["●Theorem 3.2. ",
  $t32 = {Tr[Exp[-t H]] ~ xSum[t^((k - n) / 2) ak[H], {k ≥ 0}],
    H → "Laplacian"["E"],
    n → dim[M],
    ak[H] → IntegralOp[{{M}}, ak[x, H] √Det[g] ]
   }; Column[$t32]
 ];
```

●Theorem 3.2.

$\mathrm{Tr}[e^{-H t}] \sim \sum_{\{k \geq 0\}} [t^{\frac{k-n}{2}} a_k[H]]$

$H \to \mathrm{Laplacian}[E]$

$n \to \dim[M]$

$a_k[H] \to \int_{\{M\}} [\sqrt{\mathrm{Det}[g]}\; a_k[x, H]]$

```
PR["●Theorem 3.3. ",
  $t33 = {a0[x, H] → (4 π)^(-n / 2) Tr"E"x[1N],
    a2[x, H] -> (4 π)^(-n / 2) Tr"E"x[s / 6 1N + F],
    a4[x, H] -> (4 π)^(-n / 2) (1 / 360)
      Tr"E"x[(-12 Δ[s] + 5 s.s - 2 T[R, "dd", {μ, ν}].T[R, "uu", {μ, ν}] +
         2 T[R, "dddd", {μ, ν, ρ, σ}]. T[R, "uuuu", {μ, ν, ρ, σ}] + 60 s. F +
         180 F . F - 60 Δ[F] + 30 T[Ω"E", "dd", {μ, ν}]. T[Ω"E", "uu", {μ, ν}])],
    s → "scalar curvature of ∇",
    Δ → "scalar Laplacian",
    T[Ω"E", "dd", {μ, ν}] → "curvature of connnection ∇E"
   }; Column[$t33]
 ];
```

●Theorem 3.3.

$a_0[x, H] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[1_N]$

$a_2[x, H] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[F + \frac{s\, 1_N}{6}]$

$a_4[x, H] \to$

$\frac{1}{45} 2^{-3-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[180\, F.F + 60\, s.F + 5\, s.s - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \Delta[F] - 12\, \Delta[s]]$

$s \to$ scalar curvature of ∇

$\Delta \to$ scalar Laplacian

$\Omega^E{}_{\mu\nu} \to$ curvature of connnection ∇^E

```
PR["●Proposition 3.4. ",
  $t34 =
   {Tr[f[𝒟𝒜 / Λ]] ~ a4[𝒟𝒜^2] f[0] + 2 xSum[f4-k Λ^(4-k) ak[𝒟𝒜^2] / Γ[(4 - k) / 2], {k, 0, 4, even}],
    fi → IntegralOp[{{v}}, v^(j-1) f[v]]},
  Yield, $t34 = $t34 /. {k, 0, 4, even} → {k, {0, 2}} /. xSum → Sum
 ];
```

●Proposition 3.4. $\{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \sim 2 \sum_{\{k,0,4,\mathrm{even}\}} [\frac{\Lambda^{4-k} f_{4-k}\, a_k[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[\frac{4-k}{2}]}] + f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\; f_i \to \int_{\{v\}} [v^{-1+j} f[v]]\}$

$\to \{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \sim 2 (\frac{\Lambda^4 f_4\, a_0[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2\, a_2[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[1]}) + f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\; f_i \to \int_{\{v\}} [v^{-1+j} f[v]]\}$

```
PR["●Proposition 3.5. For canonical triple ", {C"∞"[M], L²[M, S], slash[𝒟]},
  Yield,
  $p35 = $ = {Tr[f[slash[𝒟] / Λ]] ~ IntegralOp[{{x⁴}}, ℒM[T[g, "dd", {μ, ν}]]],
     ℒM[T[g, "dd", {μ, ν}]] → f₄ Λ⁴ / (2 π²) - f₂ Λ²
         / (24 π²) + f[0] / (16 π²) (△[s] / 30 -
          T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] / 20 + 11 / 360 R*.R*)};
  Column[$],
  NL, CO["Sketch proof: with ",
   $s0 = {m → dim[M], dim[M] → 4, Tr"E"x[1N] → dim[S], dim[S] → 2^(m/2)}],
  NL, "■Evaluate terms in T.3.4. ", $t34s = $t34 /. 𝒟𝒜 → slash[𝒟],
  NL, "■ ", $0 = $ = tuExtractPattern[a₀[_]][$t34s[[1, 2]]] // First,
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {M} → {x, x ∈ M} /. g → g[x],
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[1]]],
  Yield, $a0 = $0 -> $ /. $t32[[3 ;; -1]] //. $s0 // tuSimpleIntegralOp;
  Framed[$a0],

  NL, "■ ", $0 = $ = tuExtractPattern[a₂[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = F → -s / 4 1N,
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {{M} → {x, x ∈ M}, g → g[x]},
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[2]]] /. $sF,
  Yield, $ = $ //. tuOpSimplify[Tr"E"x, {s}] /. s → s[x],
  Yield, $a2 = $0 -> $ /. $t32[[3 ;; -1]] //. $s0 // tuSimpleIntegralOp;
  Framed[$a2],

  NL, "■ ", $0 = $ = tuExtractPattern[a₄[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = {s → s . 1N, F → -s / 4 1N, Ω"E" → ΩS},
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {{M} → {x, x ∈ M}, g → g[x]},
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[3]]] /. $sF, "POFF",
  Yield, $ = $ // tuDotSimplify[{s}],
  Yield, $ = $ //. tuOpSimplify[△, {1N}] /. 1N_ . 1N_ → 1N,
  Yield, $ = $ //. tuOpSimplify[Tr"E"x, {s}] /. s → s[x],
  Yield, $ = $0 -> $ /. $t32[[3 ;; -1]] //. $s0 // tuSimpleIntegralOp, "PONdd",
  Yield, $ = $ //. tuOpDistribute[ Tr"E"x ],
  Yield, $ = $ //. tuOpSimplify[Tr"E"x, {s[x], △[_]}] //. $s0 // Simplify;
  Framed[$a4b = $]
 ];
```

●Proposition 3.5. For canonical triple $\{C^\infty[M], L^2[M, S], D\!\!\!/\}$

$$\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim \int_{\{x^4\}}[\mathcal{L}_M[g_{\mu\nu}]]$$

$$\to \mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} + \frac{f[0]\,(\frac{11 R^*.R^*}{360} - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{\triangle[s]}{30})}{16\pi^2}$$

Sketch proof: with $\{m \to \dim[M], \dim[M] \to 4, \mathrm{Tr}_{E_x}[1_N] \to \dim[S], \dim[S] \to 2^{m/2}\}$

■Evaluate terms in T.3.4.

$$\{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim 2\,(\frac{\Lambda^4 f_4\, a_0[(D\!\!\!/)^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2\, a_2[(D\!\!\!/)^2]}{\Gamma[1]}) + f[0]\, a_4[(D\!\!\!/)^2],\ f_i \to \int_{\{v\}}[v^{-1+j} f[v]]\}$$

■ $a_0[(D\!\!\!/)^2]$

$$\to \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ a_0[x, (D\!\!\!/)^2]]$$

$$\to \int_{\{x,x\in M\}}[2^{-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[1_N]]$$

$$\to \boxed{a_0[(D\!\!\!/)^2] \to \frac{\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\,]}{4\pi^2}}$$

■ $a_2[(D\!\!\!/)^2]$ using $F \to -\frac{s\, 1_N}{4}$

$$\to \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ a_2[x, (D\!\!\!/)^2]]$$

$$\to \int_{\{x,x\in M\}}[2^{-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[-\frac{s\, 1_N}{12}]]$$

$$\to \int_{\{x,x\in M\}}[-\frac{1}{3} 2^{-2-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[s[x]\, 1_N]]$$

$$\to \boxed{a_2[(D\!\!\!/)^2] \to -\frac{\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[s[x]\, 1_N]]}{192\pi^2}}$$

■ $a_4[(D\!\!\!/)^2]$ using $\{s \to s.1_N,\ F \to -\frac{s\, 1_N}{4},\ \Omega^E \to \Omega^S\}$

$$\to \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ a_4[x, (D\!\!\!/)^2]]$$

$$\to \int_{\{x,x\in M\}}[\frac{1}{45} 2^{-3-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[180\,(-\frac{s\, 1_N}{4}).(-\frac{s\, 1_N}{4}) - 2 R_{\mu\nu}.R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu} + 60\, s.1_N.(-\frac{s\, 1_N}{4}) + 5\, s.1_N.s.1_N - 12\,\triangle[s.1_N] - 60\,\triangle[-\frac{s\, 1_N}{4}]]]$$

....

$$\to a_4[(D\!\!\!/)^2] \to \frac{1}{5760\pi^2}\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ (\mathrm{Tr}_{E_x}[-2 R_{\mu\nu}.R^{\mu\nu}] + \mathrm{Tr}_{E_x}[2 R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + \mathrm{Tr}_{E_x}[30\,\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] + \mathrm{Tr}_{E_x}[\frac{5}{4} s[x]^2\, 1_N] + \mathrm{Tr}_{E_x}[3\,\triangle[s[x]\, 1_N]])]$$

$$\to \boxed{a_4[(D\!\!\!/)^2] \to \frac{1}{5760\pi^2}\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ (-2\,\mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\,\mathrm{Tr}_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] + \frac{5}{4}\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] + 3\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]])]}$$

```
PR["From (3.14): ", $s = e314 =
    T[Ω^S, "dd", {μ, ν}] → 1 / 4 T[R, "dddd", {μ, ν, ρ, σ}] T[γ, "u", {ρ}].T[γ, "u", {σ}],
  yield, $s314 = {e314, e314 /. ρ → ρ1 /. σ → σ1 // tuIndicesRaise[{μ, ν}]} //
    tuAddPatternVariable[{μ, ν}],
  NL, "Evaluate ", $ = $a4b // tuExtractPattern[T[Ω^S, "dd", {μ, ν}].T[Ω^S, "uu", {μ, ν}]] //
    First;
  $TO = $ = Tr[$],
  Yield, $ = $ /. $s314 // tuDotSimplify[{Tensor[R, __]}],
  Yield, $ = $ //. tuOpSimplify[Tr, {Tensor[R, _, _]}] /. subTraceGamma0,
  Yield, $ = $ // Expand // ContractUpDn[g],
  NL, "Use ", $s = {T[R, "ddud", {μ_, ν_, ρ_, ρ_}] → 0,
    T[R, "dduu", {μ, ν, ρ1, σ1}] → -T[R, "dduu", {μ, ν, σ1, ρ1}]},
  Yield, $TO = $TO -> $ /. $s /. Tr -> Tr_"E"x; Framed[$TO],
  Imply, $ = $a4b /. $TO; Framed[$],
  NL, "Remaining Dot[] are scalars: ",
  Yield, $ = $ /. dd : HoldPattern[Dot[_]] → 1_N dd /.
     tuOpSimplify[Tr_"E"x, {HoldPattern[Dot[_]]}] //. $s0,
  Yield, $ = UpDownIndexSwap[{ρ1, σ1}][$] /. ρ1 → ρ /. σ1 → σ /.
      tt : T[R, "dddd", {_, _, _, _}] :→ tuTensorAntiSymmetricOrdered[tt, {3, 4}] /. Dot →
      Times // Simplify;
  Framed[$a4c = $]
 ];
PR["▪Transform using: ",
 NL, "•Weyl tensor: ", T[C, "dddd", {μ, ν, ρ, σ}],
 Yield, $ = T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] ->
   T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}] -
    2 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] + s[x]^2 / 3,
 NL, "•Pontryagin class ",
 $1 = R*.R* → s[x]^2 - 4 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] +
    T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
 NL, "In integrand ",
 $2 = $a4c // tuExtractIntegrand;
 $2a = $2[[1, 2, 2]];
 $2a = test → $2a,
 $ = {$, $1, $2a};
 Imply,
 $ = tuEliminate[$, {T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
     T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}]}];
 $ = tuRuleSolve[$, test],
 $2[[1, 2, 2]] = $[[1, 2]]; $2,
 Yield, $a4d = tuReplacePart[$a4c, $2]; Framed[$], CG[" QED"]
]
```

From (3.14): $\Omega^S{}_{\mu\nu} \to \frac{1}{4}\gamma^\rho.\gamma^\sigma R_{\mu\nu\rho\sigma} \longrightarrow \{\Omega^S{}_{\mu\_\nu\_} \to \frac{1}{4}\gamma^\rho.\gamma^\sigma R_{\mu\nu\rho\sigma},\ \Omega^{S\mu\_\nu\_} \to \frac{1}{4}\gamma^{\rho 1}.\gamma^{\sigma 1} R^{\mu\nu}{}_{\rho 1\sigma 1}\}$

Evaluate $Tr[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}]$

$\to Tr[\frac{1}{16}\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}]$

$\to \frac{1}{16} Tr[\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}]$

$\to \frac{1}{16} Tr[\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}]$

Use $\{R_{\mu\_\nu\_}{}^{\rho}{}_{\rho\_} \to 0,\ R_{\mu\nu}{}^{\rho 1\sigma 1} \to -R_{\mu\nu}{}^{\sigma 1\rho 1}\}$

$\to \boxed{Tr_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] \to \frac{1}{16} Tr_{E_x}[\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}]}$

$$\Rightarrow \boxed{a_4[(D\!\!\!/)^2] \to \frac{1}{5760\pi^2}\int_{\{x,x\in M\}}[\sqrt{Det[g[x]]}\ (-2\,Tr_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\,Tr_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + \frac{5}{4}Tr_{E_x}[s[x]^2\,1_N] + \frac{15}{8}Tr_{E_x}[\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}] + 3\,Tr_{E_x}[\triangle[s[x]\,1_N]])]}$$

Remaining Dot[] are scalars:

$\to$

$$a_4[(D\!\!\!/)^2] \to \frac{1}{5760\pi^2}\int_{\{x,x\in M\}}[\sqrt{Det[g[x]]}\ (-2\,Tr_{E_x}[R_{\mu\nu}.R^{\mu\nu}\,1_N] + 2\,Tr_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}\,1_N] + \frac{5}{4}Tr_{E_x}[s[x]^2\,1_N] + \frac{15}{8}Tr_{E_x}[\gamma^\rho.\gamma^\sigma.\gamma^{\rho 1}.\gamma^{\sigma 1}\,1_N\,R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho 1\sigma 1}] + 3\,Tr_{E_x}[\triangle[s[x]\,1_N]])]$$

$$\to \boxed{a_4[(D\!\!\!/)^2] \to \frac{1}{5760\pi^2}\int_{\{x,x\in M\}}[\sqrt{Det[g[x]]}\ (\frac{5}{4}Tr_{E_x}[s[x]^2\,1_N] - 2\,Tr_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 2\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + \frac{15}{8}Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 3\,Tr_{E_x}[\triangle[s[x]\,1_N]])]}$$

▪Transform using:

•Weyl tensor: $C_{\mu\nu\rho\sigma}$

$\to C_{\mu\nu\rho\sigma}\,C^{\mu\nu\rho\sigma} \to \frac{s[x]^2}{3} - 2\,R_{\mu\nu}\,R^{\mu\nu} + R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}$

•Pontryagin class $R^*.R^* \to s[x]^2 - 4\,R_{\mu\nu}\,R^{\mu\nu} + R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}$

In integrand $test \to \frac{5}{4}Tr_{E_x}[s[x]^2\,1_N] - 2\,Tr_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 2\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + \frac{15}{8}Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 3\,Tr_{E_x}[\triangle[s[x]\,1_N]]$

$\Rightarrow \{test \to \frac{1}{8}(10\,Tr_{E_x}[s[x]^2\,1_N] - 16\,Tr_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 16\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + 15\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 24\,Tr_{E_x}[\triangle[s[x]\,1_N]])\}$

$\{\{2, 3, 2\} \to \frac{1}{8}\sqrt{Det[g[x]]}\ (10\,Tr_{E_x}[s[x]^2\,1_N] - 16\,Tr_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 16\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + 15\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 24\,Tr_{E_x}[\triangle[s[x]\,1_N]])\}$

$$\to \boxed{\{test \to \frac{1}{8}(10\,Tr_{E_x}[s[x]^2\,1_N] - 16\,Tr_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 16\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + 15\,Tr_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 24\,Tr_{E_x}[\triangle[s[x]\,1_N]])\}}$$ QED

```
PR["•NOTE: In 4-dim compact orientable manifold M without boundary ",
  Yield,
  {IntegralOp[{{M}}, R*.R* νg] → 8 π^2 χ[M], χ[M] → "Euler Characteristic"} // Column,
  imply, "Topological term",
  yield, "Constant",
  yield, "Ignore",
  NL, "With no boundaries the ", △[s[x]], " term does not contribute."
 ];
```

•NOTE: In 4-dim compact orientable manifold M without boundary

$\rightarrow$ $\int_{\{M\}}[R^*.R^*\,\nu_g] \rightarrow 8\pi^2\chi[M]$ $\Rightarrow$ Topological term $\longrightarrow$ Constant $\longrightarrow$ Ignore

$\chi[M] \rightarrow$ Euler Characteristic

With no boundaries the $\triangle[s[x]]$ term does not contribute.

```
PR[imply, "Proposition 3.5 ",
 $ = $t34s /. {$a0, $a2, $a4d} /. {R*.R* → 0, △[s[x]] → 0} /. tt : Tensor[C, _, _] → tt[x],
 Yield, $t34s1 = $ // gatherIntegralOp // Simplify,
 NL, "•Compare with (3.19). The integrand: ", $ = $t34s1[[1, 2]] // tuExtractIntegrand,
 Yield, $ = $ /. Γ → Gamma /. √_ → 1 // Expand,
 Yield, $LM = ℒM[T[g, "dd", {μ, ν}]] -> $[[1, 2]], CG[" Agrees."]
]
```

$\Rightarrow$ Proposition 3.5

$$\{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim \frac{1}{5760\,\pi^2} f[0] \int_{\{x,x\in M\}}[\frac{1}{8}\sqrt{\mathrm{Det}[g[x]]}\,(10\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N] - 16\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu} R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] + 15\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\gamma^\sigma] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]])] + 2\,(-\frac{\Lambda^2 f_2 \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\,\mathrm{Tr}_{E_x}[s[x]\,1_N]]}{192\,\pi^2\,\Gamma[1]} + \frac{\Lambda^4 f_4 \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\,]}{4\,\pi^2\,\Gamma[2]}),\ f_i \rightarrow \int_{\{v\}}[v^{-1+j} f[v]]\}$$

$$\rightarrow \{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim \int_{\{x,x\in M\}}[\frac{1}{46\,080\,\pi^2\,\Gamma[1]\,\Gamma[2]}\sqrt{\mathrm{Det}[g[x]]}\,(23\,040\,\Lambda^4 f_4\,\Gamma[1] + \Gamma[2]\,(-480\,\Lambda^2 f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N] + f[0]\,\Gamma[1]\,(10\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N] - 16\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu} R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] + 15\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\gamma^\sigma] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]]))))],\ f_i \rightarrow \int_{\{v\}}[v^{-1+j} f[v]]\}$$

•Compare with (3.19). The integrand: $\{\{2\} \rightarrow \frac{1}{46\,080\,\pi^2\,\Gamma[1]\,\Gamma[2]}\sqrt{\mathrm{Det}[g[x]]}\,(23\,040\,\Lambda^4 f_4\,\Gamma[1] + \Gamma[2]\,(-480\,\Lambda^2 f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N] + f[0]\,\Gamma[1]\,(10\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N] - 16\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu} R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] + 15\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\gamma^\sigma] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]]))))\}$

$$\rightarrow \{\{2\} \rightarrow \frac{\Lambda^4 f_4}{2\,\pi^2} - \frac{\Lambda^2 f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N]}{96\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N]}{4608\,\pi^2} - \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu} R^{\mu\nu}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\gamma^\sigma]}{3072\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]]}{1920\,\pi^2}\}$$

$$\rightarrow \mathcal{L}_M[g_{\mu\nu}] \rightarrow \frac{\Lambda^4 f_4}{2\,\pi^2} - \frac{\Lambda^2 f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N]}{96\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N]}{4608\,\pi^2} - \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu} R^{\mu\nu}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\gamma^\sigma]}{3072\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]]}{1920\,\pi^2}$$ Agrees.

```
PR[CO["p.35"],
  "●Proposition 3.7. The spectral action of the fluctuated Dirac operator is ",
  $p37 = $ = {Tr[f[𝒟_A / Λ]] ~ IntegralOp[{{x, x ∈ M}},
        √Det[g[x]] ℒ[T[g, "dd", {μ, ν}], B_μ, Φ]],
     ℒ[T[g, "dd", {μ, ν}], B_μ, Φ] → N ℒ_M[T[g, "dd", {μ, ν}]] +
        ℒ_B[B_μ] + ℒ_H[T[g, "dd", {μ, ν}], B_μ, Φ],
     $LM,
     N → dim[ℋ_F],
     ℒ_B[B_μ] → f[0] / (24 π^2) Tr[T[F, "dd", {μ, ν}] T[F, "uu", {μ, ν}]],
     ℒ_B[B_μ] → "Kinetic term gauge fields",
     ℒ_H[T[g, "dd", {μ, ν}], B_μ, Φ] →
      -2 f_2 Λ^2 / (4 π^2) Tr[Φ.Φ] + f[0] / (8 π^2) Tr[Φ.Φ.Φ.Φ] + f[0] / (24 π^2) △[Tr[Φ.Φ]] +
        f[0] / (48 π^2) s[x] Tr[Φ.Φ] + f[0] / (8 π^2) Tr[tuDs[𝒟][Φ, μ].tuDsu[𝒟][Φ, μ]],
     ℒ_H[T[g, "dd", {μ, ν}], B_μ, Φ] → "Higgs lagrangian",
     N → Tr[1_{ℋ_F}]
    }; FramedColumn[$]
 ];
```

p.35●Proposition 3.7. The spectral action of the fluctuated Dirac operator is

$$\mathrm{Tr}[f[\tfrac{\mathcal{D}_A}{\Lambda}]] \sim \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ \mathcal{L}[g_{\mu\nu}, B_\mu, \Phi]]$$

$$\mathcal{L}[g_{\mu\nu}, B_\mu, \Phi] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + N\,\mathcal{L}_M[g_{\mu\nu}]$$

$$\mathcal{L}_M[g_{\mu\nu}] \to \frac{\Lambda^4 f_4}{2\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}_{E_x}[s[x]\,1_N]}{96\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N]}{4608\pi^2} - \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu} R^{\mu\nu}]}{2880\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}]}{2880\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\gamma^\sigma]}{3072\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]]}{1920\pi^2}$$

$N \to \mathrm{dim}[\mathcal{H}_F]$

$$\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$$

$\mathcal{L}_B[B_\mu] \to$ Kinetic term gauge fields

$$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\,s[x]\,\mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2\,\mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\,\mathrm{Tr}[\underset{-\mu}{\mathcal{D}}[\Phi].\overset{-}{\mathcal{D}^\mu}[\Phi]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\,\triangle[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$$

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$ Higgs lagrangian

$N \to \mathrm{Tr}[1_{\mathcal{H}_F}]$

```
PR["●For the formulas from Theorem 3.3 ", $ = $t33[[1 ;; 3]],
  NL, "let ",
  $s = {F → Q, H → 𝒟_A},
  " ", "explicit tensor notation. ", H → S × ℋ_F,
  Yield,
  $t33a = {{$ /. $s, $31[[-1]]} /. (tt : Tr_)[1_N] :→ tt[1_N ⊗ 1_{ℋ_F}] /. s 1_N → s /. s ⊗ 1_{ℋ_F} → s /.
        s → (s 1_N ⊗ 1_{ℋ_F}) /. 1_{Nx} → 1_N ⊗ 1_ℋ,
          1_N → "Identity for spinor field"} // Flatten; FramedColumn[$t33a]
 ];
```

●For the formulas from Theorem 3.3

$$\{a_0[x, H] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[1_N],\ a_2[x, H] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[F + \frac{s\,1_N}{6}],\ a_4[x, H] \to \frac{1}{45}2^{-3-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[180\,F.F + 60\,s.F + 5\,s.s - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[F] - 12\,\triangle[s]]\}$$

let $\{F \to Q, H \to \mathcal{D}_A\}$ explicit tensor notation. $H \to S \times \mathcal{H}_F$

→

$$a_0[x, \mathcal{D}_A] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_2[x, \mathcal{D}_A] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[Q + \tfrac{1}{6} s\,1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_4[x, \mathcal{D}_A] \to \frac{1}{45}2^{-3-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[180\,Q.Q + 60\,(s\,1_N \otimes 1_{\mathcal{H}_F}).Q + 5\,(s\,1_N \otimes 1_{\mathcal{H}_F}).(s\,1_N \otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s\,1_N \otimes 1_{\mathcal{H}_F}]]$$

$$Q \to -\mathrm{i}\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}$$

```
PR["●Compute the a[] terms of ", $t34[[1, 1]], (*
  " relative to ",$p35[[1,1]],*)
  NL, "for ", $s04 = Join[$s0, {Tr[1_N] → dim[S], n → dim[M]}],
  Yield, $t33a // FramedColumn
 ];
```

●Compute the a[] terms of $\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

for $\{((a\_).(b\_))^* \to a^*.b^*,\ a\_^{\dagger *} \to a^T,\ \mathrm{Tr}[1_N] \to \dim[S],\ n \to \dim[M]\}$

$$\to \boxed{\begin{array}{l} a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}] \\ a_2[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[Q + \frac{1}{6}\,s\,1_N \otimes 1_{\mathcal{H}_F}] \\ a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45}\,2^{-3-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[180\,Q.Q + 60\,(s\,1_N \otimes 1_{\mathcal{H}_F}).Q + \\ \quad 5\,(s\,1_N \otimes 1_{\mathcal{H}_F}).(s\,1_N \otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s\,1_N \otimes 1_{\mathcal{H}_F}]] \\ Q \to -\mathbb{i}\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2}\,\mathbb{i}\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4}\,s\,1_N \otimes 1_{\mathcal{H}_F} \end{array}}$$

```
PR["■For ", $ = $t33a[[1]],
  NL, "■For : ", $ = $t33a[[1]] /. Tr_ → Tr /. $t32[[3]] /. $s0,
  Yield,
  $ = $ /. tuOpDistribute[Tr, CircleTimes] //. tuOpSimplify[CircleTimes, {Tr[_]}],
  " ", "Recall ", $s = $t33[[1]] //. Join[{H → slash[𝒟], Tr_ → Tr}, $s04[[{2, -1}]]],
  Imply, $a0a = tuRuleEliminate[{Tr[1N]}][{$s, $}] // First; Framed[$a0a],

  NL, "■For : ", $ = $t33a[[2]] /. Tr_ → Tr /. $t32[[3]] /. $s0,
  Yield, $ = $ /. $t33a[[4]] //. tuOpDistribute[Tr] //. tuOpSimplify[Tr, {s}] /.
     tuOpDistribute[Tr, CircleTimes] //. tuOpSimplify[CircleTimes, {Tr[_]}],
  NL, "• ", T[F, "dd", {μ, ν}], " is anti-symmetric: ",
  $s = Tr[T[γ, "u", {μ}].T[γ, "u", {ν}]] Tr[T[F, "dd", {μ, ν}]] → 0,
  and,
  $sg = T[γ, "d", {5}] -> T[γ, "u", {5}],
  Yield, $ = $ /. $s /. $sg /. simpleGamma,
  NL, "Recall ",
  $s = $t33[[2]] //. Join[{H → slash[𝒟], Tr_ → Tr, $sF[[2]]}, $s04[[{2, -1}]]] //.
    tuOpSimplify[Tr, {s}],
  Imply, $a2a = $ /. tuRuleSolve[$s, {s Tr[1N]}] // Expand; Framed[$a2a]
 ];
```

■For $a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\, \pi^{-n/2}\, \mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$

■For : $a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[1_N \otimes 1_{\mathcal{H}_F}]$

→ $a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[1_{\mathcal{H}_F}]$ Recall $a_0[x, \not{D}] \to 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[1_N]$

⇒ $\boxed{\{a_0[x, \not{D}] \to 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[1_N],\ a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[1_{\mathcal{H}_F}]\}}$

■For : $a_2[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[Q + \frac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]$

→ $a_2[x, \mathcal{D}_{\mathcal{A}}] \to$
$2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, (-i\, \mathrm{Tr}[\gamma^\mu.\gamma_5] \otimes \mathrm{Tr}[\mathcal{D}_\mu.\Phi] + \frac{1}{2} i\, \mathrm{Tr}[\gamma^\mu.\gamma^\nu] \otimes \mathrm{Tr}[F_{\mu\nu}] - \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[\Phi.\Phi] - \frac{1}{12} \mathrm{Tr}[s\, 1_N \otimes 1_{\mathcal{H}_F}])$

• $F_{\mu\nu}$ is anti-symmetric: $\mathrm{Tr}[\gamma^\mu.\gamma^\nu]\, \mathrm{Tr}[F_{\mu\nu}] \to 0$ and $\gamma_5 \to \gamma^5$

→ $a_2[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, (-i\, 0 \otimes \mathrm{Tr}[\mathcal{D}_\mu.\Phi] + \frac{1}{2} i\, (4 g^{\mu\nu}) \otimes \mathrm{Tr}[F_{\mu\nu}] - \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[\Phi.\Phi] - \frac{1}{12} \mathrm{Tr}[s\, 1_N \otimes 1_{\mathcal{H}_F}])$

Recall $a_2[x, \not{D}] \to -\frac{1}{3} 2^{-2-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[s\, 1_N]$

⇒ $\boxed{a_2[x, \mathcal{D}_{\mathcal{A}}] \to -i\, 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, 0 \otimes \mathrm{Tr}[\mathcal{D}_\mu.\Phi] + i\, 2^{-1-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, (4 g^{\mu\nu}) \otimes \mathrm{Tr}[F_{\mu\nu}] - 2^{-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[\Phi.\Phi] - \frac{1}{3} 2^{-2-\dim[M]}\, \pi^{-\frac{\dim[M]}{2}}\, \mathrm{Tr}[s\, 1_N \otimes 1_{\mathcal{H}_F}]}$

```
PR["■For: ", $ = $t33a[[3]] /. Tr_ → Tr /. s → s ⊗ 1ℋF /. $t32[[3]] /. $s0;
  Framed[$],
  NL, "Let: ", $sQ = {Map[#.(# /. {μ → μ1, ν → ν1}) &, $t33a[[4]]], $t33a[[4]]};
  $sQ, CK
 ];

PR["■For: ", $ = $t33a[[3]] /. Tr_ → Tr /. $t32[[3]] /. $s0 /.
    {tt : Tensor[R, _, _].Tensor[R, _, _] → tt 1N ⊗ 1ℋF},
  NL, "Scalars: ", $scal = {s, △[s], Tensor[R, _, _]},
  NL, "Use: ", $s = Join[($sQ /. s → s 1N), {$s34}, $s314];
  FramedColumn[$s],
  Yield, $ = $ //. $s; ColumnSumExp[$];
  Yield, $ = $ // tuDotSimplify[]; ColumnSumExp[$];
```

```
Yield,
$ = $ //. tuOpDistribute[△] //. tuOpSimplify[△] //. {△[ a_ ⊗ b_ ] → △[a] ⊗ b + a ⊗ △[b],
   △[ s_  a_ ⊗ b_ ] → △[s a] ⊗ b + s a ⊗ △[b], △[a_  b_ ] → △[a] b + a △[b]};
$sT = {tuOpDistribute[Tr], tuOpSimplify[Tr, $scal], tuOpDistribute[CircleTimes],
   tuOpSimplify[CircleTimes, $scal], tuOpSimplify[Dot, $scal]} // Flatten;
Yield, $ = $ // tuRepeat[$sT]; ColumnSumExp[$];
NL, "Use: ", $sX = { (a_ ⊗ b_) .( c_ ⊗ d_) → a.c ⊗ b.d,
  1_n_ . a_ → a, a_ . 1_n_ → a,
  ((SS : s | s^_) a_) ⊗ b_ → SS (a ⊗ b)},
Yield, $ = $ /. $sX // tuRepeat[$sT]; ColumnSumExp[$];
NL, "Use: ", $s = {△[1_] → 0, △[Tensor[γ, a_, b_].Tensor[γ, c_, d_]] → 0,
   1_.a_ → a, a_ . 1_ → a, {T[γ, "d", {5}] -> T[γ, "u", {5}],
    T[γ, "u", {5}].T[γ, "u", {a_}].T[γ, "u", {5}] :→ -T[γ, "u", {a}]}
  } // Flatten,
Yield, $ = $ //. tuOpDistribute[Tr, CircleTimes] //
    tuRepeat[Flatten[Join[$s, simpleGamma, $sT]]] //
   (# //. tuOpSimplify[Dot, {Tensor[R, _, _]}] &) // Expand;
ColumnSumExp[$];

$s = {a_  ⊗ b_ :→ 0 /; ($$ = ExtractPattern[T[g, "uu", {μ_, ν_}]][a] // First;
      $$ = ($$ /. g → F) // UpDownIndexSwap[1, 1] // UpDownIndexSwap[2, 2];
      ! FreeQ[b, $$]),
   aa : a_  ⊗ b_ :→ (aa //. μ1 → ν) /; FreeQ[aa, ν],
   aa : a_  ⊗ b_ :→ (aa //. μ1 → μ) /; FreeQ[aa, μ],
   aa : a_  ⊗ b_ :→ (aa //. ν1 → μ) /; FreeQ[aa, μ],
   (g_  gg : T[g, "uu", {μ_, ν_}]) ⊗ Tr[a_] → g ⊗ Tr[ gg a],
   (  gg : T[g, "uu", {μ_, ν_}]) ⊗ Tr[a_] → Tr[1_N] ⊗ Tr[ gg a]
  } // Flatten;
Yield, $ = $ //. $s // tuMetricContractAll[g] // OrderTensorDummyIndices;

NL, "Manipulate indices: ",
$s = {aa : (a_  ⊗ b_) | (a_ b_) :→ (aa //. ν1 → μ) /; FreeQ[aa, μ],
  aa : (a_  ⊗ b_) | (a_ b_) :→ (aa //. μ1 → ν) /; FreeQ[aa, ν],
  aa : (a_  ⊗ b_) | (a_ b_) :→ (aa //. ρ1 → ρ) /; FreeQ[aa, ρ],
  aa : (a_  ⊗ b_) | (a_ b_) :→ (aa //. σ1 → σ) /; FreeQ[aa, σ],
  aa : (a_  ⊗ b_) | (a_ b_) :→ (aa //. σ1 → ρ) /; FreeQ[aa, ρ]
 },
and, $sR = {
  T[R, "dddu", {μ_, ν_, ρ_, ρ_}] → 0,
  T[R, "dudu", {μ_, ν_, ρ_, ρ_}] → 0,
  aa : (a_  ⊗ b_) | (a_ b_) :→ (aa //. σ1 → ρ) /; FreeQ[aa, ρ]
 },
and, $sR1 = {tt : Tensor[R, a_, b_] Tensor[R, c_, d_] :→ UpDownIndexSwap[μ][tt],
  T[R, "dddu", {μ_, ν_, ρ_, ρ_}] → 0,
  T[R, "uuuu", {μ, ν, σ, ρ}] → -T[R, "uuuu", {μ, ν, ρ, σ}]},
and, $sR2 = {T[R, "uuuu", {μ, ν, σ, ρ}] → -T[R, "uuuu", {μ, ν, ρ, σ}],
  T[F, "dd", {ν, μ}] → -T[F, "dd", {μ, ν}], T[F, "uu", {ν, μ}] → -T[F, "uu", {μ, ν}]},
Yield, $ = $ //. $s //. $sR /. $sR1 /. $sR2 //. tuOpSimplify[Dot] //. tuOpSimplify[Tr] //.
  tuOpSimplify[CircleTimes];
NL, "Apply factor to compare with p.37: ",
$ = (4 π)^2 360 # & /@ $ /. a_ ⊗ b_ → a b /. Tr[1_N] → 4 // Expand;
ColumnSumExp[$],
CR["The coefficients 1320 and 2880 do not match."]
];
```

■For:

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45} 2^{-3-\dim[M]} \pi^{-\frac{\dim[M]}{2}} \mathrm{Tr}[180\, Q.Q + 60\, (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).Q + 5\, (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}) - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[Q] - 12\, \triangle[s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

Let: $\{Q.Q \to (-i\, \gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}).$
$(-i\, \gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2} i\, \gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\, \nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}),$
$Q \to -i\, \gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}\}$ ⟵CHECK

■For: $a_4[x, \mathcal{D}_{\mathcal{A}}] \to$
$$\frac{1}{45} 2^{-3-\dim[M]} \pi^{-\frac{\dim[M]}{2}} \mathrm{Tr}[180\, Q.Q + 60\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).Q + 5\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) - 2 \times 1_N \otimes 1_{\mathcal{H}_F}\, R_{\mu\nu}.R^{\mu\nu} + 2 \times 1_N \otimes 1_{\mathcal{H}_F}\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[Q] - 12\, \triangle[s\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

Scalars: {s, △[s], Tensor[R, _, _]}

Use:

$Q.Q \to (-i\, \gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}\, 1_N).$
$(-i\, \gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2} i\, \gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\, \nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}\, 1_N)$
$Q \to -i\, \gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}\, 1_N$
$\Omega^E[\mu, \nu] \to 1_N \otimes (i\, F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}$
$\Omega^S{}_{\mu\_\,\nu\_} \to \frac{1}{4} \gamma^\rho.\gamma^\sigma\, R_{\mu\nu\rho\sigma}$
$\Omega^{S\mu\_\,\nu\_} \to \frac{1}{4} \gamma^{\rho 1}.\gamma^{\sigma 1}\, R^{\mu\nu}{}_{\rho 1\, \sigma 1}$

→
→
→
→

Use: $\{(a\_ \otimes b\_).(c\_ \otimes d\_) \to a.c \otimes b.d,\ 1_{n\_}.(a\_) \to a,\ (a\_).1_{n\_} \to a,\ (a\_\, (SS : s \mid s-)) \otimes b\_ \to SS\, a \otimes b\}$

→

Use: $\{\triangle[1\_] \to 0,\ \triangle[\mathrm{Tensor}[\gamma, a\_, b\_].\mathrm{Tensor}[\gamma, c\_, d\_]] \to 0,$
$1\_.(a\_) \to a,\ (a\_).1\_ \to a,\ \gamma_5 \to \gamma^5,\ \gamma^5.\gamma^{a}{}_{-}.\gamma^5 :\to -T[\gamma, u, \{a\}]\}$

→
→

Manipulate indices: {aa : a_⊗b_ | a_ b_ :→ (aa //. ν1 → μ) /; FreeQ[aa, μ],
aa : a_⊗b_ | a_ b_ :→ (aa //. μ1 → ν) /; FreeQ[aa, ν], aa : a_⊗b_ | a_ b_ :→ (aa //. ρ1 → ρ) /; FreeQ[aa, ρ],
aa : a_⊗b_ | a_ b_ :→ (aa //. σ1 → σ) /; FreeQ[aa, σ], aa : a_⊗b_ | a_ b_ :→ (aa //. σ1 → ρ) /; FreeQ[aa, ρ]}
and $\{R_{\mu\_\,\nu\_\,\rho\_}{}^{\rho\_} \to 0,\ R_{\mu\_}{}^{\nu\_}{}_{\rho\_}{}^{\rho\_} \to 0$, aa : a_⊗b_ | a_ b_ :→ (aa //. σ1 → ρ) /; FreeQ[aa, ρ]} and
{tt : Tensor[R, a_, b_] Tensor[R, c_, d_] :→ UpDownIndexSwap[μ][tt], $R_{\mu\_\,\nu\_\,\rho\_}{}^{\rho\_} \to 0,\ R^{\mu\nu\sigma\rho} \to -R^{\mu\nu\rho\sigma}\}$
and $\{R^{\mu\nu\sigma\rho} \to -R^{\mu\nu\rho\sigma},\ F_{\nu\mu} \to -F_{\mu\nu},\ F^{\nu\mu} \to -F^{\mu\nu}\}$

→

Apply factor to compare with p.37:

$$5760\,\pi^2\, a_4[x, \mathcal{D}_{\mathcal{A}}] \to \sum\Big[\begin{array}{l}
-15\ 2^{8-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[\Phi.\Phi] \\
45\ i\ 2^{4-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[(\mathcal{D}^{\mu}.\Phi\ \gamma_{\mu}.\gamma^5).(1_N^2\ 1_{\mathcal{H}_F})] \\
45\times 2^{4-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[(\Phi.\Phi\ 1_N).(1_N^2\ 1_{\mathcal{H}_F})] \\
-15\ 2^{4-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s^2\ \text{Tr}[(1_N\ 1_{\mathcal{H}_F}).(1_N^2\ 1_{\mathcal{H}_F})] \\
45\ i\ 2^{4-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[(1_N^2\ 1_{\mathcal{H}_F}).(\mathcal{D}^{\nu}.\Phi\ \gamma_{\nu}.\gamma^5)] \\
45\times 2^{4-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[(1_N^2\ 1_{\mathcal{H}_F}).(\Phi.\Phi\ 1_N)] \\
45\times 2^{2-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s^2\ \text{Tr}[(1_N^2\ 1_{\mathcal{H}_F}).(1_N^2\ 1_{\mathcal{H}_F})] \\
-45\ i\ 2^{3-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[(1_N^2\ 1_{\mathcal{H}_F}).(-\gamma_{\nu}.\gamma_{\mu}\ F^{\mu\nu})] \\
45\times 2^{9-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ \text{Tr}[F_{\mu\nu}.F^{\mu\nu}] \\
-45\ i\ 2^{3-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[(\gamma_{\mu}.\gamma_{\nu}\ F^{\mu\nu}).(1_N^2\ 1_{\mathcal{H}_F})] \\
15\times 2^{5-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ \text{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] \\
45\times 2^{8-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ \text{Tr}[\Phi.\Phi.\Phi.\Phi] \\
45\times 2^{10-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ \text{Tr}[\mathcal{D}_{\nu}.\Phi.\mathcal{D}^{\nu}.\Phi] \\
5\times 2^{6-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s^2\ \text{Tr}[1_{\mathcal{H}_F}] \\
-2^{7-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ R_{\mu\nu}\ R^{\mu\nu}\ \text{Tr}[1_{\mathcal{H}_F}] \\
2^{7-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ R_{\mu\nu\rho\sigma}\ R^{\mu\nu\rho\sigma}\ \text{Tr}[1_{\mathcal{H}_F}] \\
15\times 2^{8-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ \text{Tr}[\triangle[\Phi.\Phi]] \\
15\times 2^{4-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ s\ \text{Tr}[1_{\mathcal{H}_F}]\ \text{Tr}[\triangle[1_N^2]] \\
-3\ 2^{8-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ \text{Tr}[1_{\mathcal{H}_F}]\ \triangle[s] \\
15\times 2^{4-\text{dim}[M]}\ \pi^{2-\frac{\text{dim}[M]}{2}}\ \text{Tr}[1_N^2]\ \text{Tr}[1_{\mathcal{H}_F}]\ \triangle[s]
\end{array}\Big]$$

The coefficients 1320 and 2880 do not match.

```
PR[aside,
  NL, "•Evaluate: ", $ = $sQ[[1]] // tuDotSimplify[],
  NL, CO["Is there a Logical order to the operatoins? "],
  Yield, $ = $ /. s → s 1N,
  $sX = { (a_ ⊗ b_) .( c_ ⊗ d_) → a.c ⊗ b.d,
    1n_ . a_ → a, a_ . 1n_ → a,
    ((SS : s | s^_) a_) ⊗ b_ → SS (a ⊗ b)};
  $ = $ // tuRepeat[$sX, tuDotSimplify[{s}]] ;
  $ = Tr[#] & /@ $ //. tuTrSimplify[{s}];
  $[[2]] = $[[2]] // tuDistributeOp[Tr[_], CircleTimes];

  $ = $ //. {T[γ, "d", {5}] -> T[γ, "u", {5}],
     T[γ, "u", {5}].T[γ, "u", {a_}].T[γ, "u", {5}] :→ -T[γ, "u", {a}]};
  $ = $ //. simpleGamma /. 0 ⊗ a_ → 0 //. tuOpSimplify[CircleTimes] //.
    tuOpDistribute[CircleTimes ];
  $ = $ //. ( g_ T[g, "uu", {a_, b_}]) ⊗ Tr[ c_ ] :→ 0 /; ! FreeQ[c, T[F, "dd", {a, b}]] /.
       g_ T[g, "uu", {a_, b_}] ⊗ Tr[ c_ ] :→ 0 /; ! FreeQ[c, T[F, "dd", {a, b}]] /.
      tuTrSimplify[] /. tuOpSimplify[CircleTimes] /. simpleGamma;
  $ = $ /. (gg : Tensor[g, _, _] g_ ) ⊗ Tr[a_] → 1N ⊗ Tr[gg a] // ContractUpDn[g];
  $ = $ /. T[F, "uu", {a_, b_}] :→ -T[F, "uu", {b, a}] /; OrderedQ[{b, a}] /.
     tuOpSimplify[CircleTimes] // tuDotSimplify[];
  $ = $ /. tuTrSimplify[] /. tuOpSimplify[CircleTimes];
  ColumnSumExp[$sQQ = $] // Framed, OK
 ];
```

←←←←←←←←Side Note
•Evaluate:

$$Q.Q \to -(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) + \frac{1}{2}(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + \frac{1}{2}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{4}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) - \frac{1}{2}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(1_N\otimes\Phi.\Phi) - \frac{1}{8}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(s\,1_N\otimes 1_{\mathcal{H}_F}) + i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{2}i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + (1_N\otimes\Phi.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}(1_N\otimes\Phi.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + \frac{1}{4}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{8}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + \frac{1}{4}(s\,1_N\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi) + \frac{1}{16}(s\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F})$$

Is there a Logical order to the operatoins?

→ $$Q.Q \to -(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) + \frac{1}{2}(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N) + \frac{1}{2}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{4}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) - \frac{1}{2}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(1_N\otimes\Phi.\Phi) - \frac{1}{8}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N) + i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{2}i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + (1_N\otimes\Phi.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}(1_N\otimes\Phi.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N) + \frac{1}{4}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{8}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + \frac{1}{4}(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(1_N\otimes\Phi.\Phi) + \frac{1}{16}(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N)$$

$$\boxed{Tr[Q.Q] \to \sum\left[\begin{array}{l} 2\times 1_N\otimes Tr[F^{\mu 1\,\nu 1}.F_{\mu 1\,\nu 1}] \\ 4\times 1_N\otimes Tr[\mathcal{D}^{\mu 1}.\Phi.\mathcal{D}_{\mu 1}.\Phi] \\ \frac{1}{2}s\,Tr[1_N]\otimes Tr[\Phi.\Phi] \\ Tr[1_N]\otimes Tr[\Phi.\Phi.\Phi.\Phi] \\ \frac{1}{16}s^2\,Tr[1_N]\otimes Tr[1_{\mathcal{H}_F}] \end{array}\right]}$$ OK

```
PR["■For: ", $ = $t33a[[3]] /. Tr_ → Tr /. s → s ⊗ 1_{H_F} /. $t32[[3]] /. $s0;
  Framed[$],
  NL, "Let: ", $sQ = {Map[#.(# /. {μ → μ1, ν → ν1}) &, $t33a[[4]]], $t33a[[4]]};
  $sQ, CK,

  Yield, $ = $ /. $sQ // tuDotSimplify[]; Framed[$],
  NL, "Apply (3.4): ", $s34,
  Yield, $ = $ /. $s34; Framed[$], CK,
  "POFF",
  Yield, $ = $ /. (a_ ⊗ b_).(c_ ⊗ d_) → (a.c) ⊗ (b.d) /. s → s 1_N /. 1_N_ . 1_N_ → 1_N //
    tuDotSimplify[{s}], CK, "POFF",
  Yield, $ = $ /. 1_N_ . 1_N_ → 1_N //. tuOpSimplify[CircleTimes, {s}] // tuDotSimplify[{s}],
  ColumnSumExp[$],
  NL, "Simplify indices: ",
  $ = $ /. {aa : a_ ⊗ b_ :→ (aa /. μ1 → μ /. ν1 → ν) /; FreeQ[aa, μ | ν]};
  ColumnSumExp[$];
  NL, "Simpliy 1_ with γ's Φ's: ",
  Yield, $ = $ /. HoldPattern[Dot[a__]] :→ Apply[Dot, Select[{a}, # =!= 1_N &]] /;
       ¬ FreeQ[{a}, 1_N] && ¬ FreeQ[{a}, γ] /. HoldPattern[Dot[a__]] :→
     Apply[Dot, Select[{a}, # =!= 1_{H_F} &]] /; ¬ FreeQ[{a}, 1_{H_F}] && ¬ FreeQ[{a}, Φ | F];
  ColumnSumExp[$];
  Yield, $ = $ /. tt : △[_] :> Distribute[tt] //. tuOpSimplify[△, {}] // Simplify;
  Yield, $ = $ /. tt : Tr[_] :→ Distribute[tt] //. tuOpSimplify[Tr, {s}] // Simplify;
  ColumnSumExp[$],
  Yield, $ = $ /. tt : Tr[a__] :> Distribute[tt, CircleTimes] /; Head[a] === CircleTimes //
    simpleTrGamma1[{}];
  Yield, $ = $ //. tuOpDistribute[CircleTimes] /. tuOpSimplify[CircleTimes] /.
     0 ⊗ a_ → 0 //. tuOpSimplify[CircleTimes, {s}], "PON",

  NL, "• ", T[F, "dd", {μ, ν}], " is anti-symmetric ",
  Yield, $s = {T[g, "uu", {μ, ν}] ⊗ Tr[b___ . T[F, "dd", {μ, ν}]. a___] → 0,
    T[g, "uu", {μ, ν}] ⊗ Tr[ T[F, "dd", {μ, ν}]] → 0,

    (a_ T[g, "uu", {μ, ν}]) ⊗ Tr[ T[F, "dd", {μ, ν}] . b_] → 0,
    (g_ gg : T[g, "uu", {μ_, ν_}]) ⊗ Tr[a_] → g ⊗ Tr[ gg a],
    ( gg : T[g, "uu", {μ_, ν_}]) ⊗ Tr[a_] → Tr[1_N] ⊗ Tr[ gg a],
    CircleTimes[a__] :→ 0 /; ¬ FreeQ[{a}, 0]
   }; Column[$s],
  Yield, $ = $ //. $s // tuMetricContractAll[g] // OrderTensorDummyIndices;
  Yield, (*simplify F.F*)
  $pass2 = $ = $ /. tt : Tensor[F, _, _] .Tensor[F, _, _] :→ tt /. ν → μ1 /.
        T[F, "dd", {ν1, μ1}] → -T[F, "dd", {μ1, ν1}] /. tuOpSimplify[Dot] /.
      tuOpSimplify[Tr] /. tuOpSimplify[CircleTimes];
  ColumnSumExp[$pass3 = $]
 ];
```

■For:

$$\boxed{\begin{array}{l} a_4[x, \mathcal{D}_{\mathcal{A}}] \to \\ \dfrac{1}{45} 2^{-3-\dim[M]} \pi^{-\frac{\dim[M]}{2}} \mathrm{Tr}[180\, Q.Q + 60\, (s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}).Q + 5\, (s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}).(s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}) - 2\, R_{\mu\nu}.R^{\mu\nu} + \\ \quad 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[Q] - 12\, \triangle[s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}]] \end{array}}$$

Let: $\{Q.Q \to (-i\, \gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi + \frac{1}{2} i\, \gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}).$

$(-i\, \gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2} i\, \gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\, \nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}),$

$Q \to -\mathrm{i}\, \gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi + \frac{1}{2}\mathrm{i}\, \gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}\}$ ⟵CHECK

→

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45} 2^{-3-\dim[M]} \pi^{-\frac{\dim[M]}{2}} \mathrm{Tr}[-180 (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) + 90 (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + 180\,\mathrm{i} (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(1_N \otimes \Phi.\Phi) + 45\,\mathrm{i} (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 90 (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) - 45 (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) - 90\,\mathrm{i} (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(1_N \otimes \Phi.\Phi) - \frac{45}{2}\mathrm{i} (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 180\,\mathrm{i} (1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) - 90\,\mathrm{i} (1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + 180 (1_N \otimes \Phi.\Phi).(1_N \otimes \Phi.\Phi) + 45 (1_N \otimes \Phi.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 45\,\mathrm{i} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) - \frac{45}{2}\mathrm{i} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + 45 (s\, 1_N \otimes 1_{\mathcal{H}_F}).(1_N \otimes \Phi.\Phi) + \frac{45}{4} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) - 60\,\mathrm{i} (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi) + 30\,\mathrm{i} (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}) - 60 (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(1_N \otimes \Phi.\Phi) - 15 (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 5 (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}) - 2 R_{\mu\nu}.R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu} - 12 \triangle[s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}] - 60 \triangle[-\mathrm{i}\, \gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi + \frac{1}{2}\mathrm{i}\, \gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

Apply (3.4): $\Omega^{E}[\mu, \nu] \to 1_N \otimes (\mathrm{i}\, F_{\mu\nu}) + \Omega^{S}[\mu, \nu] \otimes 1_{\mathcal{H}_F}$

→

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45} 2^{-3-\dim[M]} \pi^{-\frac{\dim[M]}{2}} \mathrm{Tr}[-180 (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) + 90 (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + 180\,\mathrm{i} (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(1_N \otimes \Phi.\Phi) + 45\,\mathrm{i} (\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 90 (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) - 45 (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) - 90\,\mathrm{i} (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(1_N \otimes \Phi.\Phi) - \frac{45}{2}\mathrm{i} (\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 180\,\mathrm{i} (1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) - 90\,\mathrm{i} (1_N \otimes \Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + 180 (1_N \otimes \Phi.\Phi).(1_N \otimes \Phi.\Phi) + 45 (1_N \otimes \Phi.\Phi).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 45\,\mathrm{i} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi) - \frac{45}{2}\mathrm{i} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1}) + 45 (s\, 1_N \otimes 1_{\mathcal{H}_F}).(1_N \otimes \Phi.\Phi) + \frac{45}{4} (s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) - 60\,\mathrm{i} (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi) + 30\,\mathrm{i} (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu}) - 60 (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(1_N \otimes \Phi.\Phi) - 15 (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) + 5 (s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}).(s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}) - 2 R_{\mu\nu}.R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu} - 12 \triangle[s \otimes 1_{\mathcal{H}_F}\, 1_N \otimes 1_{\mathcal{H}_F}] - 60 \triangle[-\mathrm{i}\, \gamma^{\mu}.\gamma_5 \otimes \mathcal{D}_{\mu}.\Phi + \frac{1}{2}\mathrm{i}\, \gamma^{\mu}.\gamma^{\nu} \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

⟵CHECK

- $F_{\mu\nu}$ is anti-symmetric

→

$g^{\mu\nu} \otimes \mathrm{Tr}[(b\_\_\_).F_{\mu\nu}.(a\_\_\_)] \to 0$
$g^{\mu\nu} \otimes \mathrm{Tr}[F_{\mu\nu}] \to 0$
$(a\_\, g^{\mu\nu}) \otimes \mathrm{Tr}[F_{\mu\nu}.(b\_)] \to 0$
$(g\_\, (gg : g^{\mu\_\,\nu\_})) \otimes \mathrm{Tr}[a\_] \to g \otimes \mathrm{Tr}[a\, gg]$
$(gg : g^{\mu\_\,\nu\_}) \otimes \mathrm{Tr}[a\_] \to \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[a\, gg]$
$\otimes a\_\_ :\to 0 /;\ !\mathrm{FreeQ}[\{a\}, 0]$

→

$$2^{-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[F_{\mu1\,\nu1}.F^{\mu1\,\nu1}]$$
$$2^{-1-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]$$
$$2^{1-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[\mathcal{D}_{\mu1}.\Phi.\mathcal{D}^{\mu1}.\Phi]$$
$$-\mathbb{i}\,2^{-3-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s\,(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi).(1_N\otimes 1_{\mathcal{H}_F})]$$
$$-\mathbb{i}\,2^{-4-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s\,(\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1}).(1_N\otimes 1_{\mathcal{H}_F})]$$
$$-\mathbb{i}\,2^{-3-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}).(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi)]$$
$$-\mathbb{i}\,2^{-4-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}).(\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1})]$$
$$\frac{1}{3}\,\mathbb{i}\,2^{-1-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi)]$$
$$\frac{1}{3}\,\mathbb{i}\,2^{-2-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1})]$$
$$-\frac{1}{3}\,2^{-1-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi)]$$
$$\frac{1}{9}\,2^{-3-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F})]$$

$\rightarrow$ $a_4[x, \mathcal{D}_{\mathcal{A}}] \to \sum[$

$$-\frac{1}{45}\,2^{-2-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[R_{\mu\,\mu1}.R^{\mu\,\mu1}]$$
$$\frac{1}{45}\,2^{-2-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[R_{\mu\,\mu1\,\rho\,\sigma}.R^{\mu\,\mu1\,\rho\,\sigma}]$$
$$\frac{1}{3}\,2^{-2-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[\Omega^{E}{}_{\mu\,\mu1}.\Omega^{E\,\mu\,\mu1}]$$
$$2^{-3-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s\,(1_N\otimes\Phi.\Phi).(1_N\otimes 1_{\mathcal{H}_F}).1_N]$$
$$2^{-3-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}).1_N.(1_N\otimes\Phi.\Phi)]$$
$$-\frac{1}{3}\,2^{-3-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(1_N\otimes 1_{\mathcal{H}_F}).1_N]$$
$$2^{-5-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[s^2\,(1_N\otimes 1_{\mathcal{H}_F}).1_N.(1_N\otimes 1_{\mathcal{H}_F}).1_N]$$
$$\frac{1}{3}\,\mathbb{i}\,2^{-1-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[\triangle[-(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi)]]$$
$$-\frac{1}{3}\,\mathbb{i}\,2^{-2-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[\triangle[\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1}]]$$
$$\frac{1}{3}\,2^{-1-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[\triangle[1_N\otimes\Phi.\Phi]]$$
$$-\frac{1}{15}\,2^{-1-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[\triangle[1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}]]$$
$$\frac{1}{3}\,2^{-3-\dim[M]}\,\pi^{-\frac{\dim[M]}{2}}\,\mathrm{Tr}[\triangle[s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N]]$$

]

```
PR["•Compare with p37: ", "POFF",
 $ = $pass3,
 Yield, $ = (4 π)^2 360 # & /@ $ // Expand; ColumnSumExp[$],
 Yield, $ = $ //. {(a_ ⊗ b_ ).(c_ ⊗ d_ ) :> a .c ⊗ b . d, 1n_ . a_ | a_ . 1n_ → a},
 Yield, $ = $ /. Tr[a_] :> Tr[a /. μ1 :> μ /; FreeQ[a, μ]] /.
     Tr[a_] :> Tr[a /. ν1 :> ν /; FreeQ[a, ν]] /.
   Tr[a_] :> Tr[a /. μ1 :> ν /; FreeQ[a, ν]],
 Yield, $ = $ /. aa : a_ ⊗ T[F, "dd", {μ, ν}] :> UpDownIndexSwap[{μ, ν}][aa],
 NL, "Let ", $s = {△[a_ ⊗ b_] → △[a] ⊗ b + a ⊗ △[b], △[a_ . b_] → △[a]. b + a.△[b],
   △[a_ b_] → △[a] b + a △[b], a_ ⊗ b_ :> 0 /; ! FreeQ[{a, b}, 0], △[] → 0,
   △[a_] :> 0 /; MatchQ[a, 1n_]},
 Yield,
 $ = $ // tuRepeat[$s, (# //. tuOpSimplify[△, {Tensor[γ, _, _]}] & // tuDotSimplify[])],
 Yield, $ = $ /. tuTrExpand /. Tr[a_ (b_ ⊗ c_)] → a Tr[b] ⊗ Tr[c] /.
     Tr[(b_ ⊗ c_)] → Tr[b] ⊗ Tr[c] /. simpleGamma //
   tuRepeat[$s, (# //. tuOpSimplify[CircleTimes] & // tuDotSimplify[])],
 Yield, $ = $ /. T[g, "dd", {μ, ν}] ⊗ a_ :> 0 /; ! FreeQ[a, F],
 "PON",
 ColumnSumExp[$]
]
```

•Compare with p37:

$$
5760\,\pi^2\, a_4[x, \mathcal{D}_A] \to \sum\left[\begin{array}{l}
-15\, i\, 2^{6-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[\gamma_5.\triangle[\gamma_\mu] + \triangle[\gamma_5].\gamma_\mu] \otimes Tr[\mathcal{D}^\mu.\Phi] \\
-15\, i\, 2^{5-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[\gamma_\mu.\triangle[\gamma_\nu] + \triangle[\gamma_\mu].\gamma_\nu] \otimes Tr[F^{\mu\nu}] \\
45 \times 2^{4-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, s\, Tr[1_N] \otimes Tr[\Phi.\Phi] \\
45 \times 2^{7-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[1_N] \otimes Tr[F_{\mu\nu}.F^{\mu\nu}] \\
15 \times 2^{6-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[1_N] \otimes Tr[\Phi.\triangle[\Phi] + \triangle[\Phi].\Phi] \\
45 \times 2^{6-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[1_N] \otimes Tr[\Phi.\Phi.\Phi.\Phi] \\
45 \times 2^{8-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[1_N] \otimes Tr[\mathcal{D}_\mu.\Phi.\mathcal{D}^\mu.\Phi] \\
-3\; 2^{6-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[1_N] \otimes Tr[1_{\mathcal{H}_F}]\, (1_N\, \triangle[s]) \otimes 1_{\mathcal{H}_F} \\
15\, i\, 2^{6-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[(1_N \otimes 1_{\mathcal{H}_F}\, (s\, 1_N) \otimes 1_{\mathcal{H}_F}).(\gamma_5.\gamma_\mu \otimes \mathcal{D}^\mu.\Phi)] \\
15\, i\, 2^{5-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[(1_N \otimes 1_{\mathcal{H}_F}\, (s\, 1_N) \otimes 1_{\mathcal{H}_F}).(\gamma_\mu.\gamma_\nu \otimes F^{\mu\nu})] \\
-15\; 2^{6-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[(1_N \otimes 1_{\mathcal{H}_F}\, (s\, 1_N) \otimes 1_{\mathcal{H}_F}).(1_N \otimes \Phi.\Phi)] \\
-15\; 2^{4-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[s\, (1_N \otimes 1_{\mathcal{H}_F}\, (s\, 1_N) \otimes 1_{\mathcal{H}_F}).(1_N \otimes 1_{\mathcal{H}_F})] \\
5 \times 2^{4-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[(1_N \otimes 1_{\mathcal{H}_F}\, (s\, 1_N) \otimes 1_{\mathcal{H}_F}).(1_N \otimes 1_{\mathcal{H}_F}\, (s\, 1_N) \otimes 1_{\mathcal{H}_F})] \\
-2^{5-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[R_{\mu\nu}.R^{\mu\nu}] \\
2^{5-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] \\
15 \times 2^{5-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu}] \\
45 \times 2^{4-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[s\, (1_N \otimes 1_{\mathcal{H}_F}).1_N.(1_N \otimes \Phi.\Phi)] \\
45 \times 2^{2-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[s^2\, (1_N \otimes 1_{\mathcal{H}_F}).1_N.(1_N \otimes 1_{\mathcal{H}_F})] \\
15 \times 2^{4-\dim[M]}\, \pi^{2-\frac{\dim[M]}{2}}\, Tr[1_N] \otimes Tr[1_{\mathcal{H}_F}]\; 1_N\, \triangle[s]
\end{array}\right]
$$

4. Electrodynamics p.38

```
PR["●EG: Two point space.", {X -> {x, y}, C[X] → ℂ², C → "complex functions"},
  NL, "•Construct even finite space ",
```

```
{Fx → {C[X], ℋF, 𝒟F, γF}, dim[ℋF] ≥ 2, γF → "ℤ²grading" },
NL, "Let ", ℋF → ℂ²,
Yield, γF ⇒ ℋF → {ℋF⁺ ⊕ ℋF⁻ → ℂ ⊕ ℂ, ℋF"±" → {ψ ∈ ℋF | γF.ψ → ±ψ}},
imply, $ = γF → {{1, 0}, {0, -1}}; MatrixForms[$],
NL, "Since ", $sD0 = {CommutatorM[γF, a] → 0,
  CommutatorP[𝒟F, γF] → 0, 𝒟F → "offDiagonal", 𝒟F → {{0, du}, {dl, 0}}},
Imply, {a.ψ → Inactive[Dot][{{a₊, 0}, {0, a₋}}, {{ψ₊}, {ψ₋}}], a ∈ 𝒜F, ψ ∈ ℋF} //
 MatrixForms,
Imply, Fx → {{𝒜F, ℋF, 𝒟F, γF} → {ℂ², ℂ², {{0, t}, {t̄, 0}}, {{1, 0}, {0, -1}}}, t ∈ ℂ} //
 MatrixForms,
NL, "■Prop.4.1. A real structure ", $ = JF ⇒ {𝒟F → 0},
NL, "Determine 𝒟F for even KO dimensions by requiring: ",
$c = $ = Join[$J[[2]], $def]; Column[$],

NL, "▪KOdim->0: ", $sj = {JF → {{j₊, 0}, {0, j₋}}.cc, j"±" ∈ U[1]},
NL, "for ", $sa = ab : a | b -> {{ab₊, 0}, {0, ab₋}}; MatrixForms[$sa],
NL, "•Compute ", $0 = $ = tuExtractPattern[b"0" → _][$c] // First,
yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];
yield, $[[2]] = $[[2]] // tuRepeat[$cc, ConjugateCTSimplify1[{}]];
MatrixForms[$],
yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
MatrixForms[$sb = $] // Framed,
NL, "Diagonal", imply, $c[[4]] // Framed,

Imply, $c[[5]],
NL, "•Evaluate: ", $ = $c[[5, -1, 1]],
$sa = ab : a | xb -> {{ab₊, 0}, {0, ab₋}};
Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
yield, $ = $ //. CommutatorM → MCommutator //
  tuRepeat[Join[tuOpSimplify[Dot], {tuOpDistribute[Dot]}]];
yield, $ = $ /. $sD0[[-1]] // Simplify;
MatrixForms[$],
$x = tuExtractPattern[du _][$][[1]] / du;
yield, $ = $x.(# / $x) & /@ $ /. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
imply, Framed[𝒟F → 0]
];
PR[
NL, "▪KOdim->0: ", $sj = {JF → {{0, j}, {-j, 0}}.cc, j ∈ U[1]},
NL, "for ", $sa = ab : a | b -> {{ab₊, 0}, {0, ab₋}}; MatrixForms[$sa],
NL, "Compute ", $0 = $ = tuExtractPattern[b"0" → _][$c] // First,
yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];

yield, $[[2]] = $[[2]] // tuRepeat[$cc, ConjugateCTSimplify1[{}]];
MatrixForms[$sb = $],
yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
MatrixForms[$sb = $],
NL, "Diagonal", imply, $c[[4]] // Framed,

NL, "•Evaluate: ", $ = $c[[5, -1, 1]],
$sa = ab : a | xb -> {{ab₊, 0}, {0, ab₋}};
Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
yield, $ = $ //. CommutatorM → MCommutator //
  tuRepeat[Join[tuOpSimplify[Dot], {tuOpDistribute[Dot]}]];
yield, $ = $ /. $sD0[[-1]] // Simplify;
MatrixForms[$],
$x = tuExtractPattern[du _][$][[1]] / du;
yield, $ = $x.(# / $x) & /@ $ //. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
```

```
imply, Framed[𝒟F → 0]
];
```

●EG: Two point space. $\{X \to \{x, y\}, C[X] \to \mathbb{C}^2, C \to \text{complex functions}\}$

•Construct even finite space $\{F_X \to \{C[X], \mathcal{H}_F, \mathcal{D}_F, \gamma_F\}, \dim[\mathcal{H}_F] \geq 2, \gamma_F \to \mathbb{Z}^2\text{grading}\}$

Let $\mathcal{H}_F \to \mathbb{C}^2$

$\to\ \gamma_F \Rightarrow \mathcal{H}_F \to \{(\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^- \to \mathbb{C}\oplus\mathbb{C}, \mathcal{H}_F^{\pm} \to \{\psi \in \mathcal{H}_F \mid \gamma_F.\psi \to \pm\psi\}\} \Rightarrow \gamma_F \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Since $\{[\gamma_F, a] \to 0, \{\mathcal{D}_F, \gamma_F\} \to 0, \mathcal{D}_F \to \text{offDiagonal}, \mathcal{D}_F \to \{\{0, du\}, \{dl, 0\}\}\}$

$\Rightarrow\ \{a.\psi \to \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \circ \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, a \in \mathcal{A}_F, \psi \in \mathcal{H}_F\}$

$\Rightarrow\ F_X \to \{\{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F\} \to \{\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}, t \in \mathbb{C}\}$

■Prop.4.1. A real structure $J_F \Rightarrow \{\mathcal{D}_F \to 0\}$

Determine $\mathcal{D}_F$ for even KO dimensions by requiring:

$J_F.J_F \to \varepsilon$
$J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F$
$J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F$
$\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$
$\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[[\mathcal{D}_F, a], b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$

▪KOdim->0: $\{J_F \to \{\{j_+, 0\}, \{0, j_-\}\}.cc, j_\pm \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

•Compute $b^0 \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow\ \longrightarrow b^0 \to \begin{pmatrix} (j_+)^* b_+ j_+ & 0 \\ 0 & (j_-)^* b_- j_- \end{pmatrix} \longrightarrow \boxed{b^0 \to \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}}$

Diagonal $\Rightarrow$ $\boxed{\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}}$

$\Rightarrow\ \forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[[\mathcal{D}_F, a], b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$

•Evaluate: $[[\mathcal{D}_F, a], b^0] \to 0$

$\to\ [[\mathcal{D}_F, \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}], \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}] \to 0 \longrightarrow\ \longrightarrow$

$\begin{pmatrix} 0 & du\,(a_- - a_+)\,(b_- - b_+) \\ dl\,(a_- - a_+)\,(b_- - b_+) & 0 \end{pmatrix} \to 0 \longrightarrow ((a_- - a_+)\,(b_- - b_+)).\mathcal{D}_F \to 0 \Rightarrow \boxed{\mathcal{D}_F \to 0}$

▪KOdim->0: $\{J_F \to \{\{0, j\}, \{-j, 0\}\}.cc, j \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

Compute $b^0 \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow\ \longrightarrow b^0 \to \begin{pmatrix} j\,j^*\,b_- & 0 \\ 0 & j\,j^*\,b_+ \end{pmatrix} \longrightarrow b^0 \to \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}$

Diagonal $\Rightarrow$ $\boxed{\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}}$

•Evaluate: $[[\mathcal{D}_F, a], b^0] \to 0$

$\to\ [[\mathcal{D}_F, \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}], \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}] \to 0 \longrightarrow\ \longrightarrow$

$\begin{pmatrix} 0 & -du\,(a_- - a_+)\,(b_- - b_+) \\ -dl\,(a_- - a_+)\,(b_- - b_+) & 0 \end{pmatrix} \to 0 \longrightarrow -((a_- - a_+)\,(b_- - b_+)).\mathcal{D}_F \to 0 \Rightarrow \boxed{\mathcal{D}_F \to 0}$

```
PR["From M, 4-dim Riemann spin manifold and FX two-point space, form ",
 M×FX → {C"∞"[M, ℂ^2], L^2[M, S]⊗ℂ^2, slash[𝒟]⊗𝟙, γ5⊗γF, JM⊗JF}
]
```

From M, 4-dim Riemann spin manifold and F_X two-point space, form

$M \times F_X \to \{C^\infty[M, \mathbb{C}^2], L^2[M, S]\otimes\mathbb{C}^2, (\not{\mathcal{D}})\otimes\mathbb{1}, \gamma_5\otimes\gamma_F, J_M\otimes J_F\}$

```
PR["●U[1] gauge theory ",
 NL, "gauge group ", 𝒢[𝒜] → Mod[U[𝒜], U[$sAt[[1]]]],
 NL, "where ", {$t219[[1, -2]], U[𝒜] ≠ U[$sAt[[1]]], $sAt} // Column,
 Imply, "KOdim[J_F]" → {2, 6},
 ", i.e., off diagonal. only KOdim→6 for Standard Model used in this case. ",
 Imply, "Can use Def.2.17 for action functional ",
 $d217 = {S → S_b + S_f, S_b → Tr[f[𝒟_𝒜 / Λ]], S_f → 1 / 2 BraKet[J.ξ̃, 𝒟_𝒜.ξ̃],
   ξ̃ ∈ ℋ_cl^+, ℋ_cl^+ → {ξ̃ | ξ ∈ ℋ^+}, ξ̃ → "GrassmannVariable"};
 Column[$d217],
 NL, "•Consider ", $Fx = F_X → {ℂ^2, ℂ^2, 0, γ_F → {{1, 0}, {0, -1}}, J_F → {{0, C}, {C, 0}}};
 MatrixForms[$Fx]
]
```

●U[1] gauge theory

gauge group $\mathcal{G}[\mathcal{A}]\to \text{Mod}[U[\mathcal{A}], U[\tilde{\mathcal{A}}_J]]$

where $(2.11)\Rightarrow \mathcal{G}[M\times F]\to \{U\to u.J.u.J^\dagger,\ u\in U[\mathcal{A}]\}$
$U[\mathcal{A}]\neq U[\tilde{\mathcal{A}}_J]$
$\tilde{\mathcal{A}}_J\to \{a\in \mathcal{A},\ a.J\to J.a^\dagger,\ a^0\to a\}$

$\Rightarrow$ KOdim$[J_F]\to\{2, 6\}$

, i.e., off diagonal. only KOdim→6 for Standard Model used in this case.

$\Rightarrow$ Can use Def.2.17 for action functional

$$\begin{array}{l} S\to S_b+S_f \\ S_b\to \text{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \\ S_f\to \frac{1}{2}\langle J.\tilde{\xi}\mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle \\ \tilde{\xi}\in(\mathcal{H}_{cl})^+ \\ (\mathcal{H}_{cl})^+\to\{\tilde{\xi}\mid \xi\in\mathcal{H}^+\} \\ \tilde{\xi}\to \text{GrassmannVariable} \end{array}$$

•Consider $F_X\to\{\mathbb{C}^2, \mathbb{C}^2, 0, \gamma_F\to\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, J_F\to\begin{pmatrix}0 & C\\ C & 0\end{pmatrix}\}$

```
PR["Prop.4.2. The gauge group ", 𝒢[𝒜F] → U[1],
  NL, "Note: ", U[𝒜F] → U[1] × U[1],
  NL, "From ", $sAt,
  yield, $ = ForAll[a,
    a ∈ ℂ^2 && a ∈ ($sAtj = ($sAt[[1]] /. J → F)_JF), (JF.ConjugateTranspose[a].JF → a)],
  NL, "Compute ", $ = tuExtractPattern[Rule[__]][$][[1]],
  yield, $ = $ /. $Fx[[2, -2 ;; -1]]; MatrixForms[$],
  NL, "Let ", $sCC = $s = {a -> DiagonalMatrix[{a1, a2}],
     C.a_ :> Conjugate[a].C /; FreeQ[a, C], Conjugate[C] → C, C.C → 1},
  Yield, $ = $ /. Dot → xDot /. $s // OrderedxDotMultiplyAll[];
  MatrixForms[$],
  yield, $ = $ // tuRepeat[$s, ConjugateCTSimplify1[{}]];
  MatrixForms[$] // Framed,
  Imply, a1 → a2, imply, a → "diagonal",
  imply, $pass4 = $ = $sAtj ≃ ℂ,
  imply, (U[$[[1]]] → U[1]) ⊂ U[𝒜F]
 ];
```

Prop.4.2. The gauge group $\mathcal{G}[\mathcal{A}_F] \to U[1]$

Note: $U[\mathcal{A}_F] \to U[1] \times U[1]$

From $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A},\ a.J \to J.a^\dagger,\ a^0 \to a\} \longrightarrow \forall_{a,a\in\mathbb{C}^2\&\&a\in\tilde{\mathcal{A}}_{F J_F}} (J_F.a^\dagger.J_F \to a)$

Compute $J_F.a^\dagger.J_F \to a \longrightarrow \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}.a^\dagger.\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \to a$

Let $\{a \to \{\{a1, 0\}, \{0, a2\}\},\ C.(a\_) :\!\to a^*.C /; FreeQ[a, C],\ C^* \to C,\ C.C \to 1\}$

$\to \begin{pmatrix} C.a2^*.C & 0 \\ 0 & C.a1^*.C \end{pmatrix} \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix} a2 & 0 \\ 0 & a1 \end{pmatrix} \to \begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix}}$

$\Rightarrow a1 \to a2 \Rightarrow a \to \text{diagonal} \Rightarrow \tilde{\mathcal{A}}_{F J_F} \simeq \mathbb{C} \Rightarrow (U[\tilde{\mathcal{A}}_{F J_F}] \to U[1]) \subset U[\mathcal{A}_F]$

```
PR["■Determine Bμ. Since ", $pass4,
  yield, (hF → u[$sAtj]) ≃ I ℝ,
  NL, "Gauge field: ",
  Aμ[x] ∈ (I gF -> I Mod[u[($a = $sAt[[1]] /. J → F)], I ℝ]) → (Isu[$a] ≃ ℝ),
  NL, "Arbitrary hermitian field ",
  $sA = {Aμ → -I a tuDPartial[b, μ], Aμ → {{T[X"1", "d", {μ}], 0}, {0, T[X^2, "d", {μ}]}},
    {T[X"1", "d", {μ}], T[X^2, "d", {μ}]} ∈ C"∞"[M, ℝ], C.tt : T[X"1"|2, "d", {μ}] → tt.C},
  NL, "Since ", Aμ , " is always in form ", $ = Bμ -> Aμ - JF.Aμ.inv[JF],
  Yield, $ = $ /. $Fx[[2, -1]] /. inv[cc : 0 | C] → cc /. Dot → xDot /.
       dd : xDot[__] :→ (dd /. $sA[[2]] //. $sA[[-1]]) /.
      Plus → xPlus /. $sA[[2]] // OrderedxDotMultiplyAll[];
  Yield, $ = $ /. xPlus → Plus /. $sA[[-1]] /. $sCC /. tuOpSimplify[Dot];
  MatrixForms[$B = $],
  " define ", $ = $ -> {{T[Y, "d", {μ}], 0}, {0, -T[Y, "d", {μ}]}};
  $ = Flatten /@ ($[[1, 2]] -> $[[-1]]);
  $sb = Thread[$] // DeleteCases[#, 0 → 0] & // First,
  imply, $B = $B /. {$sb, -1 # & /@ $sb};
  MatrixForms[$B -> T[Y, "d", {μ}] ⊗ γF] // Framed
 ];
```

■Determine B_μ. Since $\tilde{\mathcal{A}}_{F J_F} \simeq \mathbb{C} \longrightarrow (h_F \to u[\tilde{\mathcal{A}}_{F J_F}]) \simeq i\,\mathbb{R}$

Gauge field: $A_\mu[x] \in (i\, g_F \to i\, Mod[u[\tilde{\mathcal{A}}_F], i\,\mathbb{R}]) \to Isu[\tilde{\mathcal{A}}_F] \simeq \mathbb{R}$

Arbitrary hermitian field

$\{A_\mu \to -i\, a\, \underline{\partial}_\mu[b],\ A_\mu \to \{\{X^1{}_\mu, 0\}, \{0, X^2{}_\mu\}\},\ \{X^1{}_\mu, X^2{}_\mu\} \in C^\infty[M, \mathbb{R}],\ C.(tt : X^{1|2}{}_\mu) \to tt.C\}$

Since A_μ is always in form $B_\mu \to -J_F.A_\mu.J_F^{-1} + A_\mu$

$\to$

$\to B_\mu \to \begin{pmatrix} -X^2{}_\mu + X^1{}_\mu & 0 \\ 0 & X^2{}_\mu - X^1{}_\mu \end{pmatrix}$ define $-X^2{}_\mu + X^1{}_\mu \to Y_\mu \Rightarrow \boxed{\left(B_\mu \to \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix}\right) \to Y_\mu \otimes \gamma_F}$

```
PR["●Prop.4.3. Inner fluctuations for
   ACM M×F_X are parameterized by a U[1]-gauge field Y_μ ",
 Yield, 𝒟 ↦ 𝒟' → 𝒟 + T[γ, "u", {μ}].T[Y, "d", {μ}] ⊗ γ_F,
 NL, "The action of gauge group ", 𝒢[𝒜] ≃ C"∞"[M, U[1]][𝒟'],
 Yield,
 {T[Y, "d", {μ}] ↦ T[Y, "d", {μ}] - I u.tuDPartial[ConjugateTranspose[u], μ], u ∈ 𝒢[𝒜]}
]
```

●Prop.4.3. Inner fluctuations
   for ACM M×F_X are parameterized by a U[1]-gauge field Y_μ
→ $\mathcal{D} \mapsto \mathcal{D}' \to \mathcal{D} + \gamma^\mu . Y_\mu \otimes \gamma_F$
The action of gauge group $\mathcal{G}[\mathcal{A}] \simeq C^\infty[M, U[1]][\mathcal{D}']$
→ $\{Y_\mu \mapsto -i\, u.\underline{\partial}_\mu[u^\dagger] + Y_\mu,\ u \in \mathcal{G}[\mathcal{A}]\}$

```
PR["■Two modifications needed for E-M: ", {𝒟_F → ! 0, S_f → "2 independent spinors"},
 NL, "•Let ", {{e, ē} → "basis of ℋ_F",
   e → "basis of ℋ_F^+",
   ē → "basis of ℋ_F^-",
   J_F.e -> ē,
   J_F.ē → e,
   γ_F.e → e,
   γ_F.ē → -ē
  } // Column,
 imply,
 $H = {ℋ → L^2[M, S] ⊗ ℋ_F, L^2[M, S] -> L^2[M, S]^+ ⊕ L^2[M, S]^-,
   ℋ^+ → "positiveEigenSpace of γ→γ5⊗γ_F",
   ℋ^+ -> L^2[M, S]^+ ⊗ ℋ_F^+ ⊕ L^2[M, S]^- ⊗ ℋ_F^-,
   ξ ∈ ℋ^+,
   ξ → ψ_L ⊗ e + ψ_R ⊗ ē,
   ψ_L ∈ L^2[M, S]^+,
   ψ_R ∈ L^2[M, S]^-
  }; Column[$H],
 NL, "•Doubling space ", C"∞"[M, ℂ^2] ↔ (N → M × X ≃ M ⊔ M),
 NL, "Let ",
 $se = {{e_R, e_L, ē_R, ē_L} → basis [ℋ_F → ℂ^4], γ_F.e_L → e_L, γ_F.e_R → -e_R, J_F.e_R → -ē_L, J_F.e_L → -ē_R,
   KOdim → 6, J_F.J_F → 𝕀, J_F.γ_F → -γ_F.J_F}; Column[$se],
 NL, "Chirality ", {J_F.γ_F.e_L → -γ_F.J_F.e_L, J_F.γ_F.e_R → -γ_F.J_F.e_R} //
   tuRepeat[Join[$se, tuOpSimplify[Dot]]] // Column,
 Imply, $sgj = {γ_F → DiagonalMatrix[{-1, 1, 1, -1}],
    J_F → SparseArray[{Band[{1, 3}] → C, Band[{3, 1}] → C}, {4, 4}]} // Normal;
 MatrixForms[$sgj],
 NL, "•The elements ",
 $sa = {a ∈ (𝒜_F → ℂ^2), a[{e_R, e_L, ē_R, ē_L}] → DiagonalMatrix[{a_1, a_1, a_2, a_2}]};
 MatrixForms[$sa]
]
PR["■Prop.4.5. ", F_ED → {ℂ^2, ℂ^4, 0, γ_F, J_F}, " is a real even finite space of KOdim→6."
]
```

■Two modifications needed for E-M: $\{\mathcal{D}_F \to ! \, 0, S_f \to 2$ independent spinors$\}$

•Let
$$\begin{matrix} \{e, \bar{e}\} \to \text{basis of } \mathcal{H}_F \\ e \to \text{basis of } \mathcal{H}_F{}^+ \\ \bar{e} \to \text{basis of } \mathcal{H}_F{}^- \\ J_F.e \to \bar{e} \\ J_F.\bar{e} \to e \\ \gamma_F.e \to e \\ \gamma_F.\bar{e} \to -\bar{e} \end{matrix} \Rightarrow \begin{matrix} \mathcal{H} \to L^2[M, S] \otimes \mathcal{H}_F \\ L^2[M, S] \to L^2[M, S]^+ \oplus L^2[M, S]^- \\ \mathcal{H}^+ \to \text{positiveEigenSpace of } \gamma \to \gamma_5 \otimes \gamma_F \\ \mathcal{H}^+ \to L^2[M, S]^+ \otimes (\mathcal{H}_F)^+ \oplus L^2[M, S]^- \otimes (\mathcal{H}_F)^- \\ \xi \in \mathcal{H}^+ \\ \xi \to \psi_L \otimes e + \psi_R \otimes \bar{e} \\ \psi_L \in L^2[M, S]^+ \\ \psi_R \in L^2[M, S]^- \end{matrix}$$

•Doubling space $C^\infty[M, \mathbb{C}^2] \leftrightarrow (N \to M \times X \simeq M \vee M)$

Let
$$\begin{matrix} \{e_R, e_L, \bar{e}_R, \bar{e}_L\} \to \text{basis}[\mathcal{H}_F \to \mathbb{C}^4] \\ \gamma_F.e_L \to e_L \\ \gamma_F.e_R \to -e_R \\ J_F.e_R \to -\bar{e}_L \\ J_F.e_L \to -\bar{e}_R \\ \text{KOdim} \to 6 \\ J_F.J_F \to \mathbb{I} \\ J_F.\gamma_F \to -\gamma_F.J_F \end{matrix}$$

Chirality
$$\begin{matrix} -\bar{e}_R \to \gamma_F.\bar{e}_R \\ \bar{e}_L \to \gamma_F.\bar{e}_L \end{matrix}$$

$$\Rightarrow \{\gamma_F \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, J_F \to \begin{pmatrix} 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \\ C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \end{pmatrix}\}$$

•The elements $\{a \in (\mathcal{A}_F \to \mathbb{C}^2), a[\{e_R, e_L, \bar{e}_R, \bar{e}_L\}] \to \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}\}$

■Prop.4.5. $F_{ED} \to \{\mathbb{C}^2, \mathbb{C}^4, 0, \gamma_F, J_F\}$ is a real even finite space of KOdim→6.

```
PR["■Add non-trivial Dirac operator.
Since ",
  $ = 𝒟F.γF → -γF.𝒟F, "POFF",
  NL, "𝒟F Hermitian condition: ",
  $d = Table[d[i, j], {i, 4}, {j, 4}]; MatrixForms[$d],
  $ct = ct[$d]; MatrixForms[$ct],
  $ct = $d → $ct //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates,
  $ct = Select[$ct, OrderedQ[{#[[1, 2]], #[[1, 1]]}] &],
  $d = 𝒟F → $d;
  Yield, $ = $ /. $d /. $sgj; MatrixForms[$],
  Yield, $ = $ //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates,
  Yield, $s = tuRuleSolve[$, Flatten[$d[[2]]]],
  "PON",
  imply, $d0 = $d = $d /. $s /. $ct; MatrixForms[$d],

  NL, "Since ", $ = 𝒟F.JF → JF.𝒟F, "POFF",
  Yield, $ = $ /. Dot → xDot /. $d /. $sgj // OrderedxDotMultiplyAll[];
  MatrixForms[$],
  Yield, $ = $ /. C.d_ → Conjugate[d].C; MatrixForms[$],
  Yield, $ = $ //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates;
  Yield, $ = $ /. a_ .C → a; "PON",
  Imply, $d = $d /. $; MatrixForms[$d],
  NL, "Comparing these: ",
  $ = $d[[2]] → $d0[[2]] //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates;

  Yield, $ = $ /. List → And /. Rule → Equal,
  Yield, $ = Reduce[$, {d[1, 2]}, Complexes] /. And → List /. Equal → Rule,
  Imply, $d = $d /. $; MatrixForms[$d] // Framed,

  NL, "•Order one condition: ",
  $Da = $ = CommutatorM[𝒟F, a]; Framed[$],
  Yield, $ = $ /. $d /. a → $sa[[-1, -1]] /. CommutatorM → MCommutator // Simplify;
  MatrixForms[$],
  NL, "Simplifying ",
  yield, $s = Flatten[$] /. List → Plus // Simplify;
  $s = Apply[List, $s, {0}];
  yield, $1 = $Da -> $s[[2]]. ($ / $s[[2]]) // Simplify;
  MatrixForms[$1] // Framed
 ];
```

■Add non-trivial Dirac operator.

Since $\mathcal{D}_F.\gamma_F \to -\gamma_F.\mathcal{D}_F \Rightarrow \mathcal{D}_F \to \begin{pmatrix} 0 & d[1,2] & d[1,3] & 0 \\ d[1,2]^* & 0 & 0 & d[2,4] \\ d[1,3]^* & 0 & 0 & d[3,4] \\ 0 & d[2,4]^* & d[3,4]^* & 0 \end{pmatrix}$

Since $\mathcal{D}_F.J_F \to J_F.\mathcal{D}_F$

$\Rightarrow \mathcal{D}_F \to \begin{pmatrix} 0 & d[3,4]^* & d[1,3] & 0 \\ d[3,4] & 0 & 0 & d[2,4] \\ d[1,3]^* & 0 & 0 & d[1,2]^* \\ 0 & d[2,4]^* & d[1,2] & 0 \end{pmatrix}$

Comparing these:

$\to d[3,4]^* == d[1,2]$ && $d[3,4] == d[1,2]^*$ && $d[1,2]^* == d[3,4]$ && $d[1,2] == d[3,4]^*$

$\to d[1,2] \to d[3,4]^*$

$\Rightarrow \boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d[3,4]^* & d[1,3] & 0 \\ d[3,4] & 0 & 0 & d[2,4] \\ d[1,3]^* & 0 & 0 & d[3,4] \\ 0 & d[2,4]^* & d[3,4]^* & 0 \end{pmatrix}}$

•Order one condition: $\boxed{[\mathcal{D}_F, a]}$

$\to \begin{pmatrix} 0 & 0 & d[1,3](-a_1+a_2) & 0 \\ 0 & 0 & 0 & d[2,4](-a_1+a_2) \\ d[1,3]^*(a_1-a_2) & 0 & 0 & 0 \\ 0 & d[2,4]^*(a_1-a_2) & 0 & 0 \end{pmatrix}$

Simplifying $\longrightarrow$ $\longrightarrow$ $\boxed{[\mathcal{D}_F, a] \to (a_1-a_2).\begin{pmatrix} 0 & 0 & -d[1,3] & 0 \\ 0 & 0 & 0 & -d[2,4] \\ d[1,3]^* & 0 & 0 & 0 \\ 0 & d[2,4]^* & 0 & 0 \end{pmatrix}}$

```
PR["The condition ", $ = $c[[5, -1]],
 Yield, $ = $[[1]] /. $1 /. $[[2]] /. ($saa = (a₁ - a₂) → a1m2);
 Yield, $ = $ /. b -> DiagonalMatrix[{b₁, b₁, b₂, b₂}] /. Dot → xDot /. $sgj //
   OrderedxDotMultiplyAll[];
 Yield, $ = $ /. C.d_ → Conjugate[d].C // ConjugateCTSimplify1[{}];
 Yield, $ = $ /. C.C → 1 /. tuOpSimplify[Dot] /. CommutatorM → MCommutator /.
    tuOpSimplify[Dot, {a1m2}] // Simplify;
 MatrixForms[$],

 NL, "Move common factors outside ",
 Yield, $s = Flatten[$[[1]]],
 Yield, $s = $s /. List → Plus // Simplify,
 Yield, $s = Apply[List, $s, {0}],
 Yield, $2 = ($s[[1]] $s[[3]]).($[[1]] / ($s[[1]] $s[[3]])) // Simplify;
 MatrixForms[($2 → 0) /. Reverse[$saa]] // Framed,
 NL, "Since a's and b's arbitrary ",
 imply, $ = (tuExtractPattern[List[__]][$2] // Flatten // DeleteDuplicates) → 0,
 Yield, $s = Thread[$]; FramedColumn[$s],
 imply, $d = $d /. $s; MatrixForms[$d] // Framed,
 " relabel ", $Dd = $d /. d[3, 4] → Conjugate[d];
 MatrixForms[$Dd] // Framed
]
```

The condition $\{[[\mathcal{D}_F, a], b^0] \to 0,\ b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$

$\to$

$\to$

$\to$

$$\to \text{a1m2}.\begin{pmatrix} 0 & 0 & -d[1,3]\,b_1 & 0 \\ 0 & 0 & 0 & -d[2,4]\,b_1 \\ d[1,3]^*\,b_2 & 0 & 0 & 0 \\ 0 & d[2,4]^*\,b_2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1 \end{pmatrix}.\text{a1m2}.\begin{pmatrix} 0 & 0 & -d[1,3] & 0 \\ 0 & 0 & 0 & -d[2,4] \\ d[1,3]^* & 0 & 0 & 0 \\ 0 & d[2,4]^* & 0 & 0 \end{pmatrix} \to 0$$

Move common factors outside

$\to$

$\text{a1m2}.\{\{0, 0, -d[1,3]\,b_1, 0\}, \{0, 0, 0, -d[2,4]\,b_1\}, \{d[1,3]^*\,b_2, 0, 0, 0\}, \{0, d[2,4]^*\,b_2, 0, 0\}\} - \{\{b_2, 0, 0, 0\}, \{0, b_2, 0, 0\}, \{0, 0, b_1, 0\}, \{0, 0, 0, b_1\}\}.\text{a1m2}.\{\{0, 0, -d[1,3], 0\}, \{0, 0, 0, -d[2,4]\}, \{d[1,3]^*, 0, 0, 0\}, \{0, d[2,4]^*, 0, 0\}\}$

$\to \text{a1m2}.(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^* + d[2,4]^*)\,b_2) - (2\,(b_1+b_2)).\text{a1m2}.(d[1,3]^* + d[2,4]^* - d[1,3] - d[2,4])$

$\to \{\text{a1m2}.(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^* + d[2,4]^*)\,b_2),\ -(2\,(b_1+b_2)).\text{a1m2}.(d[1,3]^* + d[2,4]^* - d[1,3] - d[2,4])\}$

$$\to \boxed{\begin{array}{l} ((a_1-a_2).(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^*+d[2,4]^*)\,b_2) \\ \quad \{(a_1-a_2).(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^*+d[2,4]^*)\,b_2), \\ \quad -(2\,(b_1+b_2)).(a_1-a_2).(d[1,3]^*+d[2,4]^*-d[1,3]-d[2,4])\}[\![3]\!]). \\ \quad (((a_1-a_2).\begin{pmatrix} 0 & 0 & -d[1,3]\,b_1 & 0 \\ 0 & 0 & 0 & -d[2,4]\,b_1 \\ d[1,3]^*\,b_2 & 0 & 0 & 0 \\ 0 & d[2,4]^*\,b_2 & 0 & 0 \end{pmatrix} - \\ \quad \begin{pmatrix} b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1 \end{pmatrix}.(a_1-a_2).\begin{pmatrix} 0 & 0 & -d[1,3] & 0 \\ 0 & 0 & 0 & -d[2,4] \\ d[1,3]^* & 0 & 0 & 0 \\ 0 & d[2,4]^* & 0 & 0 \end{pmatrix})/ \\ \quad ((a_1-a_2).(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^*+d[2,4]^*)\,b_2) \\ \quad \{(a_1-a_2).(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^*+d[2,4]^*)\,b_2), \\ \quad -(2\,(b_1+b_2)).(a_1-a_2).(d[1,3]^*+d[2,4]^*-d[1,3]-d[2,4])\}[\![3]\!])) \to 0 \end{array}}$$

Since a's and b's arbitrary $\Rightarrow \{\text{a1m2}.(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^* + d[2,4]^*)\,b_2),\ -(2\,(b_1+b_2)).\text{a1m2}.(d[1,3]^* + d[2,4]^* - d[1,3] - d[2,4]),\ 0,\ -d[1,3]\,b_1,\ -d[2,4]\,b_1,\ d[1,3]^*\,b_2,\ d[2,4]^*\,b_2,\ b_2,\ b_1,\ -d[1,3],\ -d[2,4],\ d[1,3]^*,\ d[2,4]^*\} \to 0$

$$\to \boxed{\begin{array}{l} \text{a1m2}.(-(d[1,3]+d[2,4])\,b_1 + (d[1,3]^*+d[2,4]^*)\,b_2) \to 0 \\ -(2\,(b_1+b_2)).\text{a1m2}.(d[1,3]^*+d[2,4]^*-d[1,3]-d[2,4]) \to 0 \\ 0 \to 0 \\ -d[1,3]\,b_1 \to 0 \\ -d[2,4]\,b_1 \to 0 \\ d[1,3]^*\,b_2 \to 0 \\ d[2,4]^*\,b_2 \to 0 \\ b_2 \to 0 \\ b_1 \to 0 \\ -d[1,3] \to 0 \\ -d[2,4] \to 0 \\ d[1,3]^* \to 0 \\ d[2,4]^* \to 0 \end{array}} \Rightarrow$$

$$\boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d[3,4]^* & d[1,3] & 0 \\ d[3,4] & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d[3,4] \\ 0 & 0 & d[3,4]^* & 0 \end{pmatrix}} \ \text{relabel}\ \boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}}$$

```
PR["●Then ", $MF = M×F_X → {C"∞"[M, ℂ^2], L^2[M, S]⊗ℂ^2, slash[𝒟]⊗𝟙, γ5⊗γF, JM⊗JF};
 ColumnForms[$MF],
 " becomes ",
 M×F_ED → {C"∞"[M, ℂ^2], L^2[M, S]⊗ℂ^4, slash[𝒟]⊗𝟙 + T[γ, "d", {5}]⊗𝒟F, γ5⊗γF, JM⊗JF} //
  ColumnForms,
 NL, "Decompose ", {𝒜 ← C"∞"[M, ℂ^2] -> C"∞"[M, ℂ] ⊕ C"∞"[M, ℂ],
    (ℋ ← L^2[M, S]⊗ℂ^4) -> L^2[M, S]⊗ℋe ⊕ L^2[M, S]⊗ℋe,
    a ∈ 𝒜 → $sa[[2]]
   } // Column // MatrixForms,
 NL, "Gauge group ", 𝒢[𝒜F] ≃ U[1],
 Yield, $B = {T[B, "d", {μ}] →
    DiagonalMatrix[{T[Y, "d", {μ}], T[Y, "d", {μ}], -T[Y, "d", {μ}], -T[Y, "d", {μ}]}],
   T[Y, "d", {μ}][x] ∈ ℝ}; MatrixForms[$B]
]
```

●Then $M\times F_X \to \begin{array}{l} C^\infty[M,\ \mathbb{C}^2] \\ L^2[M,\ S]\otimes\mathbb{C}^2 \\ (\not{\mathcal{D}})\otimes\mathbb{1} \\ \gamma_5\otimes\gamma_F \\ J_M\otimes J_F \end{array}$ becomes $M\times F_{ED} \to \begin{array}{l} C^\infty[M,\ \mathbb{C}^2] \\ L^2[M,\ S]\otimes\mathbb{C}^4 \\ (\not{\mathcal{D}})\otimes\mathbb{1}+\gamma_5\otimes\mathcal{D}_F \\ \gamma_5\otimes\gamma_F \\ J_M\otimes J_F \end{array}$

Decompose $\begin{array}{l} \mathcal{A}\leftarrow C^\infty[M,\ \mathbb{C}^2]\to C^\infty[M,\ \mathbb{C}]\oplus C^\infty[M,\ \mathbb{C}] \\ \mathcal{H}\leftarrow L^2[M,\ S]\otimes\mathbb{C}^4\to L^2[M,\ S]\otimes\mathcal{H}_e\oplus L^2[M,\ S]\otimes\mathcal{H}_{\bar{e}} \\ a\in\mathcal{A}\to a[\{e_R,\ e_L,\ \bar{e}_R,\ \bar{e}_L\}]\to \left(\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{array}\right) \end{array}$

Gauge group $\mathcal{G}[\mathcal{A}_F]\simeq U[1]$

$\to \left\{B_\mu \to \left(\begin{array}{cccc} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & -Y_\mu & 0 \\ 0 & 0 & 0 & -Y_\mu \end{array}\right),\ Y_\mu[x]\in\mathbb{R}\right\}$

4.2.4 Lagrangian

Spectral Action

```
PR["Insert ", $sΦ = $s = {Φ → 𝒟F, N → dim[ℋF], dim[ℋF] → 4, Tr[1_ℋF] → N},
 and, $ = { $B[[1]], $Dd}; MatrixForms[$],
 NL, "into Prop.3.7 Lagrangian ", $ = $p37[[{2, 3, 5, 7}]] /. $s;
 Column[$0 = $],
 NL, "•Evaluate term ", $ = $0[[3]],
 " where ", $s =
  {$$ = T[F, "dd", {μ, ν}] → tuDPartial[T[Y, "d", {ν}], μ] - tuDPartial[T[Y, "d", {μ}], ν],
   tuIndicesRaise[{μ, ν}] [$$]},
 Imply, $ = $ /. $s; Framed[$],
 NL, "•Evaluate term ", $ = $0[[4]],
 Yield, $[[2]] = $[[2]] /. $Dd; MatrixForms[$],
 NL, "Evaluate Tr[]'s ",
 $1 = $ // tuExtractPositionPattern[Tr[_]];
 Yield, $1 = $1 /. tt : (T[𝒟, "d", {μ}] | T[𝒟, "u", {μ}])[a_] :→ Thread[tt] /.
    tt : (T[𝒟, "d", {μ}] | T[𝒟, "u", {μ}])[a_] :→ Thread[tt] /.
   (T[𝒟, "d", {μ}] | T[𝒟, "u", {μ}])[0] → 0,
 Yield, $ = tuReplacePart[$, $1]; Framed[$]
]
```

Insert $\{\Phi \to \mathcal{D}_F,\ N \to \dim[\mathcal{H}_F],\ \dim[\mathcal{H}_F] \to 4,\ \mathrm{Tr}[1_{\mathcal{H}_F}] \to N\}$

and $\left\{B_\mu \to \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & -Y_\mu & 0 \\ 0 & 0 & 0 & -Y_\mu \end{pmatrix},\ \mathcal{D}_F \to \begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}\right\}$

into Prop.3.7 Lagrangian

$$\mathcal{L}[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] + \dim[\mathcal{H}_F]\, \mathcal{L}_M[g_{\mu\nu}]$$

$$\mathcal{L}_M[g_{\mu\nu}] \to \frac{\Lambda^4 f_4}{2\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}_{E_x}[s[x]\, 1_{\dim[\mathcal{H}_F]}]}{96\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[s[x]^2\, 1_{\dim[\mathcal{H}_F]}]}{4608\pi^2} - \frac{f[0]\, \mathrm{Tr}_{E_x}[1_{\dim[\mathcal{H}_F]}\, R_{\mu\nu} R^{\mu\nu}]}{2880\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[1_{\dim[\mathcal{H}_F]}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}]}{2880\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[1_{\dim[\mathcal{H}_F]}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \gamma_\rho \gamma_\sigma \gamma^\rho \gamma^\sigma]}{3072\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_{\dim[\mathcal{H}_F]}]]}{1920\pi^2}$$

$$\mathcal{L}_B[B_\mu] \to \frac{f[0]\, \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$$

$$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F]}{2\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_\mu[\mathcal{D}_F] . \mathcal{D}^\mu[\mathcal{D}_F]]}{8\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F . \mathcal{D}_F . \mathcal{D}_F]}{8\pi^2} + \frac{f[0]\, \triangle[\mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F]]}{24\pi^2}$$

•Evaluate term $\mathcal{L}_B[B_\mu] \to \frac{f[0]\, \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$ where $\{F_{\mu\nu} \to -\underline{\partial}_\nu[Y_\mu] + \underline{\partial}_\mu[Y_\nu],\ F^{\mu\nu} \to -\underline{\partial}^\nu[Y^\mu] + \underline{\partial}^\mu[Y^\nu]\}$

$$\Rightarrow \boxed{\mathcal{L}_B[B_\mu] \to \frac{f[0]\, \mathrm{Tr}[(-\underline{\partial}_\nu[Y_\mu] + \underline{\partial}_\mu[Y_\nu])\, (-\underline{\partial}^\nu[Y^\mu] + \underline{\partial}^\mu[Y^\nu])]}{24\pi^2}}$$

•Evaluate term $\mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F]}{2\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_\mu[\mathcal{D}_F] . \mathcal{D}^\mu[\mathcal{D}_F]]}{8\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F . \mathcal{D}_F . \mathcal{D}_F]}{8\pi^2} + \frac{f[0]\, \triangle[\mathrm{Tr}[\mathcal{D}_F . \mathcal{D}_F]]}{24\pi^2}$

$$\to \mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{d^2 d^{*2} f[0]}{2\pi^2} + \frac{d\, d^*\, f[0]\, s[x]}{12\pi^2} - \frac{2\, d\, \Lambda^2 d^* f_2}{\pi^2} + \frac{f[0]\, \mathrm{Tr}\left[\mathcal{D}_\mu\left[\begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}\right] . \mathcal{D}^\mu\left[\begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}\right]\right]}{8\pi^2} + \frac{f[0]\, \triangle[4\, d\, d^*]}{24\pi^2}$$

Evaluate Tr[]'s

$$\to \{\{2, 4, 4\} \to 2\, \mathcal{D}_\mu[d]\, \mathcal{D}^\mu[d] + 2\, \mathcal{D}_\mu[d^*]\, \mathcal{D}^\mu[d^*] + \mathcal{D}_\mu[d[1,3]]\, \mathcal{D}^\mu[d[1,3]] + \mathcal{D}_\mu[d[2,4]]\, \mathcal{D}^\mu[d[2,4]]\}$$

$$\to \boxed{\mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{d^2 d^{*2} f[0]}{2\pi^2} + \frac{d\, d^*\, f[0]\, s[x]}{12\pi^2} - \frac{2\, d\, \Lambda^2 d^* f_2}{\pi^2} + \frac{f[0]\, \triangle[4\, d\, d^*]}{24\pi^2} + \frac{f[0]\, (2\, \mathcal{D}_\mu[d]\, \mathcal{D}^\mu[d] + 2\, \mathcal{D}_\mu[d^*]\, \mathcal{D}^\mu[d^*] + \mathcal{D}_\mu[d[1,3]]\, \mathcal{D}^\mu[d[1,3]] + \mathcal{D}_\mu[d[2,4]]\, \mathcal{D}^\mu[d[2,4]])}{8\pi^2}}$$

4.2.5 Fermionic action

```
PR[$H // Column,
 NL, "Basis ", $sa[[2, 1, 1]],
 Yield, $H[[4]],
 NL, "Spanning basis ", {ℋF+[{eL, e̅R}], ℋF-[{eR, e̅L}]},
 NL, "Arbitrary vector ",
 $sξ = {ξ -> χR ⊗ eR + χL ⊗ eL + ψL ⊗ e̅R + ψR ⊗ e̅L, {χL, ψL} ∈ L²[M, S]+, {χR, ψR} ∈ L²[M, S]-};

 Column[$sξ],
 NL, "Then fermionic action for ", $MF,
 Yield,
 $Sf = $ = Sf → -I BraKet[JM.χ̃, T[γ, "u", {μ}].(T["∇"S, "d", {μ}] - I T[Y, "d", {μ}]).ψ̃] +
     BraKet[JM.χ̃L, ct[d].ψ̃L] - BraKet[JM.χ̃R, d.ψ̃R];
 Framed[$], CO["Prop.4.7"],
 NL, "where the ~ means ", $sAt, CK,
 NL, "■Proof: ",
 NL, "The fluctuated Dirac operator ",
 Yield, $sDA1 = $ = $sDA[[1]] /. $sDA[[2]] /. $sΦ, "POFF",
 Yield, $ = $ /. tuOpDistribute[dotOps] //. tuOpSimplify[Dot] // Expand,
 Yield, $ = $ //. {a_ . (b_ ⊗ c_) → (a.b) ⊗ c, a_ . 1_ → a}, "PON",
 NL, "Since ", $s = $slashD[[1]] /. a_ tuDDown[tt : _][_, i_] ↦ a .
      T[tt, "d", {i}] //. tuOpSimplify[Dot],
 $slashd = $s = tuRuleSolve[$s, Dot[_, _]];
 yield, $ = $ /. Reverse[$s] //. tuOpSimplify[CircleTimes] //. tuOpSimplify[Dot];
 Framed[$sDA0 = $], CO["p.48"],
 NL, "■Using ", $sCT = {J -> $MF[[2, -1]]}, and,
 $s = Map[$Dd[[1]].# &, $sa[[2, 1, 1]]];
 $s = $s → ($Dd[[2]].Transpose[{$sa[[2, 1, 1]]}] // Transpose // Flatten) // Thread;
 $s1 = $B[[1, 2]].Transpose[{$sa[[2, 1, 1]]}] // Flatten;
 $s1 = Map[$B[[1, 1]].# &, $sa[[2, 1, 1]]] → $s1 // Thread;
 Yield,
 $s0J = {JF.ei_ -> e̅i, JF.e̅i_ → ei, γF.ei_ → ei, γF.e̅i_ → -e̅i, $s, $s1
   } // Flatten,

 NL, "Compute ",
 NL, "•", $ = J.ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$],
 NL, "•", $ = $sDA0[[2, 1]].ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$],
 NL, "•", $ = $sDA0[[2, 2]].ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$],
 NL, "•", $ = $sDA0[[2, 3]].ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$]

]
```

$\mathcal{H} \to L^2[M, S] \otimes \mathcal{H}_F$
$L^2[M, S] \to L^2[M, S]^+ \oplus L^2[M, S]^-$
$\mathcal{H}^+ \to$ positiveEigenSpace of $\gamma \to \gamma_5 \otimes \gamma_F$
$\mathcal{H}^+ \to L^2[M, S]^+ \otimes (\mathcal{H}_F)^+ \oplus L^2[M, S]^- \otimes (\mathcal{H}_F)^-$
$\xi \in \mathcal{H}^+$
$\xi \to \psi_L \otimes e + \psi_R \otimes \bar{e}$
$\psi_L \in L^2[M, S]^+$
$\psi_R \in L^2[M, S]^-$

Basis $\{e_R, e_L, \bar{e}_R, \bar{e}_L\}$
$\to \mathcal{H}^+ \to L^2[M, S]^+ \otimes (\mathcal{H}_F)^+ \oplus L^2[M, S]^- \otimes (\mathcal{H}_F)^-$
Spanning basis $\{(\mathcal{H}_F)^+[\{e_L, \bar{e}_R\}], (\mathcal{H}_F)^-[\{e_R, \bar{e}_L\}]\}$
$\xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \bar{e}_R + \psi_R \otimes \bar{e}_L$
Arbitrary vector $\{\chi_L, \psi_L\} \in L^2[M, S]^+$
$\{\chi_R, \psi_R\} \in L^2[M, S]^-$
Then fermionic action for $M \times F_X \to \{C^\infty[M, \mathbb{C}^2], L^2[M, S] \otimes \mathbb{C}^2, (D\!\!\!/) \otimes \mathbb{1}, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\}$

$\to \boxed{S_f \to -i \langle J_M . \tilde{\chi} \mid \gamma^\mu . (\nabla^S{}_\mu - i Y_\mu) . \tilde{\psi} \rangle + \langle J_M . \tilde{\chi}_L \mid d^\dagger . \tilde{\psi}_L \rangle - \langle J_M . \tilde{\chi}_R \mid d . \tilde{\psi}_R \rangle}$ Prop.4.7

where the ~ means $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A}, a.J \to J.a^\dagger, a^0 \to a\}$ ⟵CHECK
▪Proof:
The fluctuated Dirac operator
$\to \mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \mathcal{D}_F - i \gamma^\mu . (i\, 1_{\dim[\mathcal{H}_F]} \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})$

Since $D\!\!\!/ \to -i \gamma^\mu . \nabla^S{}_\mu \longrightarrow \boxed{\mathcal{D}_{\mathcal{A}} \to (D\!\!\!/) \otimes 1_{\mathcal{H}_F} + \gamma_5 \otimes \mathcal{D}_F + \gamma^\mu \otimes B_\mu}$ p.48

▪Using $\{J \to J_M \otimes J_F\}$ and
$\to \{J_F . e_{i\_} \to \bar{e}_i, J_F . \bar{e}_{i\_} \to e_i, \gamma_F . e_{i\_} \to e_i, \gamma_F . \bar{e}_{i\_} \to -\bar{e}_i, \mathcal{D}_F . e_R \to d[1, 3]\, \bar{e}_R + d\, e_L, \mathcal{D}_F . e_L \to d[2, 4]\, \bar{e}_L + d^*\, e_R,$
$\mathcal{D}_F . \bar{e}_R \to d^*\, \bar{e}_L, \mathcal{D}_F . \bar{e}_L \to d\, \bar{e}_R, B_\mu . e_R \to e_R Y_\mu, B_\mu . e_L \to e_L Y_\mu, B_\mu . \bar{e}_R \to -\bar{e}_R Y_\mu, B_\mu . \bar{e}_L \to -\bar{e}_L Y_\mu\}$
Compute

•$J.\xi \longrightarrow \boxed{J_M . \chi_L \otimes \bar{e}_L + J_M . \chi_R \otimes \bar{e}_R + J_M . \psi_L \otimes e_R + J_M . \psi_R \otimes e_L}$

•$((D\!\!\!/) \otimes 1_{\mathcal{H}_F}) . \xi \longrightarrow \boxed{(D\!\!\!/) . \chi_L \otimes e_L + (D\!\!\!/) . \chi_R \otimes e_R + (D\!\!\!/) . \psi_L \otimes \bar{e}_R + (D\!\!\!/) . \psi_R \otimes \bar{e}_L}$

•$(\gamma_5 \otimes \mathcal{D}_F) . \xi \longrightarrow \boxed{\gamma_5 . \chi_L \otimes (d[2, 4]\, \bar{e}_L + d^*\, e_R) + \gamma_5 . \chi_R \otimes (d[1, 3]\, \bar{e}_R + d\, e_L) + \gamma_5 . \psi_L \otimes (d^*\, \bar{e}_L) + \gamma_5 . \psi_R \otimes (d\, \bar{e}_R)}$

•$(\gamma^\mu \otimes B_\mu) . \xi \longrightarrow \boxed{\gamma^\mu . \chi_L \otimes (e_L Y_\mu) + \gamma^\mu . \chi_R \otimes (e_R Y_\mu) + \gamma^\mu . \psi_L \otimes (-\bar{e}_R Y_\mu) + \gamma^\mu . \psi_R \otimes (-\bar{e}_L Y_\mu)}$

```
PR["▪From ", $ = $d217[[3]],
 Yield, $ = $ /. $sDA1,
 Yield, $0 =
  $ = $ // tuRepeat[{tuOpDistribute[Dot], tuOpSimplify[Dot], tuOpSimplify[CircleTimes],
      (tt : Tensor[γ, _, _]).(a_ ⊗ b_) → (tt .a ⊗ b), $slashd,
      a_ . 1_ → a, tuOpDistribute[BraKet]}, Simplify],
 NL, "●Evaluate terms ", $0p = $ = tuExtractPositionPattern[BraKet[_, _]][$];
 NL, "•", $ = $0p[[1]]; Framed[$],
 NL, "Define ", $sξt =
  #̃ & /@ $sξ[[1]] //. tuOpDistribute[OverTilde] //.
    tuOpDistribute[OverTilde, CircleTimes] /. ã_ :→ a /; ! FreeQ[a, e],
 Yield, $ = $ /. $sξt /. $sCT //. tuOpDistribute[Dot] //. $sX //. tuOpDistribute[BraKet];
 NL, "e's are orthonormal ",
 $s = {BraKet[a_ ⊗ e1_, b_ ⊗ e2_] :→ If[e1 === e2, BraKet[a, b], 0]},
 Yield, ColumnSumExp[$ = $ //. $s0J /. $s],
 NL, "Symmetry of form ",
 $s = BraKet[J_ . ps_ , d_ . x_] :> BraKet[J . x , d . ps] /; ! FreeQ[x, χ],
 Imply, $ = $ /. $s,
 NL, "Since ", $s = slash[𝒟][ψL] → ψ'R, and, $H[[-2 ;; -1]], " orthogonal, i.e., ",
 Yield, $1 = BraKet[χL + χR, (ψ')L + (ψ')R],
 Yield, $1 = $1 //. tuOpDistribute[BraKet],
 Yield, $1 =
```

```
   $1 /. BraKet[a_, b_] :→ 0 /; FreeQ[a, L] && ! FreeQ[b, L] || FreeQ[a, R] && ! FreeQ[b, R],
  Yield, $1 = $1 /. Reverse[$s] /. Reverse[Swap[{L, R}][$s]],
  NL, "So ", $p1 = $ = $ //. a_R|L → a; $[[2]] = $[[2]] / 2; Framed[$]
 ]
 PR["•", $ = $0p[[2]]; Framed[$], "POFF",
  Yield, $ =
   $ /. $sξt /. $sCT //. tuOpDistribute[Dot] //. $sX //. $s0J //. tuOpDistribute[BraKet],
  Yield, $ = $ //. tuOpSimplify[CircleTimes, {d, Conjugate[d]}] //.
    tuOpSimplify[BraKet, {d, Conjugate[d]}], "PON",
  NL, "e's are orthonormal ",
  $s = {BraKet[a_ ⊗ e1_, b_ ⊗ e2_] :→ If[e1 === e2, BraKet[a, b], 0]},
  Yield, $ = $ /. $s,
  NL, "Move d's back ", $s = d_ BraKet[a_, b_] → BraKet[a, d.b],
  Yield, ColumnSumExp[$ = $ /. $s],
  NL, "Symmetry of form ",
  $s = BraKet[J_ . ps_ , d_ .g_ . x_] :> BraKet[J . x , d .g. ps] /; ! FreeQ[x, χ],
  Imply, $p2 = $ = $ /. $s; ColumnSumExp[$] // Framed
 ]
 PR["•", $ = $0p[[3]]; Framed[$], "POFF",
  Yield, $ =
   $ /. $sξt /. $sCT //. tuOpDistribute[Dot] //. $sX //. $s0J //. tuOpDistribute[BraKet],
  Yield, $ = $ //. tuOpSimplify[CircleTimes,
      {Tensor[Y, _, _], Conjugate[Tensor[Y, _, _]]}] //.
    tuOpSimplify[BraKet, {Tensor[Y, _, _], Conjugate[Tensor[Y, _, _]]}], "PON",
  NL, "e's are orthonormal ",
  $s = {BraKet[a_ ⊗ e1_, b_ ⊗ e2_] :→ If[e1 === e2, BraKet[a, b], 0]},
  Yield, $ = $ /. $s,
  NL, "Move Y's back ", $s = d_ BraKet[a_, b_ . c_] → BraKet[a, b.d.c],
  Yield, ColumnSumExp[$ = $ /. $s],
  NL, "Anti-symmetry of form ",
  $s = BraKet[J_ . ps_ , g_ . d_ . x_] :→ -BraKet[J . x , g. d . ps] /; ! FreeQ[x, χ],
  Imply, $ = $ /. $s // tuRepeat[{tuOpDistribute[Dot], tuOpSimplify[Dot],
      tuOpSimplify[CircleTimes], (tt : Tensor[γ, _, _]).(a_ ⊗ b_) → (tt .a ⊗ b),
      $slashd, a_ . 1_ → a, tuOpDistribute[BraKet], tuOpSimplify[BraKet]}, Simplify];
  ColumnSumExp[$],
  NL, "So ", $p3 = $ //. a_R|L → a; $[[2]] = $[[2]] / 2; Framed[$],
  NL, "● ", $ = tuReplacePart[$0, {$p1, $p2, $p3}]; Framed[$],
  NL, CO["A mass term can be identified by letting ", d → -I m,
   ". Recall 𝒟𝒜 ⇒ d so is the related to the fluctuated Dirac algebra. "]
 ]
```

■From $S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$

$\to S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid (\gamma_5 \otimes \mathcal{D}_F - i\, \gamma^\mu.(i\, 1_{\dim[\mathcal{H}_F]} \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})).\tilde{\xi}\rangle$

$\to S_f \to \frac{1}{2}(\langle J.\tilde{\xi} \mid ((\not{D}) \otimes 1_{\mathcal{H}_F}).\tilde{\xi}\rangle + \langle J.\tilde{\xi} \mid (\gamma_5 \otimes \mathcal{D}_F).\tilde{\xi}\rangle + \langle J.\tilde{\xi} \mid (\gamma^\mu \otimes B_\mu).\tilde{\xi}\rangle)$

●Evaluate terms

• $\boxed{\{2, 2, 1\} \to \langle J.\tilde{\xi} \mid ((\not{D}) \otimes 1_{\mathcal{H}_F}).\tilde{\xi}\rangle}$

Define $\tilde{\xi} \to \tilde{\chi}_L \otimes e_L + \tilde{\chi}_R \otimes e_R + \tilde{\psi}_L \otimes \overline{e_R} + \tilde{\psi}_R \otimes \overline{e_L}$

$\to$

e's are orthonormal $\{\langle a\_ \otimes e1\_ \mid b\_ \otimes e2\_\rangle :\to \mathrm{If}[e1 === e2, \langle a \mid b\rangle, 0]\}$

$$\to \{2, 2, 1\} \to \sum[\begin{matrix} \langle J_M.\tilde{\chi}_L \mid (\not{D}).\tilde{\psi}_R\rangle \\ \langle J_M.\tilde{\chi}_R \mid (\not{D}).\tilde{\psi}_L\rangle \\ \langle J_M.\tilde{\psi}_L \mid (\not{D}).\tilde{\chi}_R\rangle \\ \langle J_M.\tilde{\psi}_R \mid (\not{D}).\tilde{\chi}_L\rangle \end{matrix}]$$

Symmetry of form $\langle (J\_).(ps\_) \mid (d\_).(x\_)\rangle :\to \langle J.x \mid d.ps\rangle$ /; ! FreeQ[x, χ]

$\Rightarrow \{2, 2, 1\} \to 2\langle J_M.\tilde{\chi}_L \mid (\not{D}).\tilde{\psi}_R\rangle + 2\langle J_M.\tilde{\chi}_R \mid (\not{D}).\tilde{\psi}_L\rangle$

Since $(\not{D})[\psi_L] \to \psi'_R$ and $\{\psi_L \in L^2[M, S]^+, \psi_R \in L^2[M, S]^-\}$ orthogonal, i.e.,

$\to \langle \chi_L + \chi_R \mid \psi'_L + \psi'_R\rangle$

$\to \langle \chi_L \mid \psi'_L\rangle + \langle \chi_L \mid \psi'_R\rangle + \langle \chi_R \mid \psi'_L\rangle + \langle \chi_R \mid \psi'_R\rangle$

$\to \langle \chi_L \mid \psi'_L\rangle + \langle \chi_R \mid \psi'_R\rangle$

$\to \langle \chi_L \mid (\not{D})[\psi_R]\rangle + \langle \chi_R \mid (\not{D})[\psi_L]\rangle$

So $\boxed{\{2, 2, 1\} \to 2\langle J_M.\tilde{\chi} \mid (\not{D}).\tilde{\psi}\rangle}$

• $\boxed{\{2, 2, 2\} \to \langle J.\tilde{\xi} \mid (\gamma_5 \otimes \mathcal{D}_F).\tilde{\xi}\rangle}$

e's are orthonormal $\{\langle a\_ \otimes e1\_ \mid b\_ \otimes e2\_\rangle :\to \mathrm{If}[e1 === e2, \langle a \mid b\rangle, 0]\}$

$\to \{2, 2, 2\} \to 0$

Move d's back $\langle a\_ \mid b\_\rangle\, d\_ \to \langle a \mid d.b\rangle$

$\to \{2, 2, 2\} \to 0$

Symmetry of form $\langle (J\_).(ps\_) \mid (d\_).(g\_).(x\_)\rangle :\to \langle J.x \mid d.g.ps\rangle$ /; ! FreeQ[x, χ]

$\Rightarrow \boxed{\{2, 2, 2\} \to 0}$

• $\boxed{\{2, 2, 3\} \to \langle J.\tilde{\xi} \mid (\gamma^\mu \otimes B_\mu).\tilde{\xi}\rangle}$

e's are orthonormal $\{\langle a\_ \otimes e1\_ \mid b\_ \otimes e2\_\rangle :\to \mathrm{If}[e1 === e2, \langle a \mid b\rangle, 0]\}$

$\to \{2, 2, 3\} \to 0$

Move Y's back $\langle a\_ \mid (b\_).(c\_)\rangle\, d\_ \to \langle a \mid b.d.c\rangle$

$\to \{2, 2, 3\} \to 0$

Anti-symmetry of form $\langle (J\_).(ps\_) \mid (g\_).(d\_).(x\_)\rangle :\to -\langle J.x \mid g.d.ps\rangle$ /; ! FreeQ[x, χ]

$\Rightarrow \{2, 2, 3\} \to 0$

So $\boxed{\{2, 2, 3\} \to 0}$

● $\boxed{S_f \to 2\langle J_M.\tilde{\chi} \mid (\not{D}).\tilde{\psi}\rangle}$

A mass term can be identified by letting $d \to -i\, m$

. Recall $\mathcal{D}_{\mathcal{A}} \Rightarrow d$ so is the related to the fluctuated Dirac algebra.

```
PR["●Theorem 4.9. The full Lagrangian is ",
 ℒgrav[T[g, "dd", {μ, ν}]] → 4 ℒM[T[g, "dd", {μ, ν}]] + ℒH[T[g, "dd", {μ, ν}]],
 NL, "E-M Lagrangian ",
 ℒEM[T[g, "dd", {μ, ν}]] →
  -I BraKet[JM.χ̃, (T[γ, "u", {μ}].(("∇"^"S")_μ - I T[Y, "d", {μ}]) - m).ψ̃]_ℒ +
   f[0]/(6 π^2) T[F, "dd", {μ, ν}] T[F, "uu", {μ, ν}],
 NL, "where ", BraKet[ξ, ψ] → IntegralOp[{{x^4, x ∈ M}}, √Abs[det[g]] BraKet[ξ, ψ]_ℒ]
]
```

●Theorem 4.9. The full Lagrangian is $\mathcal{L}_{grav}[g_{\mu\nu}] \to \mathcal{L}_H[g_{\mu\nu}] + 4\,\mathcal{L}_M[g_{\mu\nu}]$

E-M Lagrangian $\mathcal{L}_{EM}[g_{\mu\nu}] \to -i\,\langle J_M.\tilde{\chi} \mid (-m + \gamma^\mu.(\nabla^S{}_\mu - i\,Y_\mu)).\tilde{\psi}\rangle_{\mathcal{L}} + \frac{f[0]\,F_{\mu\nu}\,F^{\mu\nu}}{6\,\pi^2}$

where $\langle \xi \mid \psi\rangle \to \int_{\{x^4, x\in M\}} [\sqrt{\text{Abs}[\det[g]]}\,\langle \xi \mid \psi\rangle_{\mathcal{L}}]$

```
{U[ξ, ζ] → BraKet[J.ξ, 𝒟𝒜.ζ], {ξ, ζ} ∈ ℋ^+}
{B[χ, ψ] → -I BraKet[JM.χ, (T[γ, "u", {μ}].(("∇"^"S")_μ - I T[Y, "d", {μ}]) - m).ψ],
 {χ, ψ} ∈ L^2[M, S]}

{$sξ, χ → χL + χR, ψ → ψL + ψR}
$sDA1
U[ξ, ζ] → 2 B[χ, ψ]
Pf[U] → (IntegralOp[{{D[ξ̃]}}, Exp[1/2 U[ξ̃, ξ̃]]] →
   (IntegralOp[{{D[ξ̃]}, {D[ψ̃]}}, Exp[B[ξ̃, ψ̃]]] →
     Det[B]))
D[η_, θ_] :→ (Table[d[T[η, "d", {i}]].d[T[θ, "d", {i}]], {i, dim[]}])
D[ξ, ψ] /. %
```

$\{U[\xi, \zeta] \to \langle J.\xi \mid \mathcal{D}_{\mathcal{A}}.\zeta\rangle, \{\xi, \zeta\} \in \mathcal{H}^+\}$

$\{B[\chi, \psi] \to -i\,\langle J_M.\chi \mid (-m + \gamma^\mu.(\nabla^S{}_\mu - i\,Y_\mu)).\psi\rangle, \{\chi, \psi\} \in L^2[M, S]\}$

$\{\{\xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \bar{e}_R + \psi_R \otimes \bar{e}_L, \{\chi_L, \psi_L\} \in L^2[M, S]^+, \{\chi_R, \psi_R\} \in L^2[M, S]^-\},$
$\chi \to \chi_L + \chi_R, \psi \to \psi_L + \psi_R\}$

$\mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \mathcal{D}_F - i\,\gamma^\mu.(i\,1_{\dim[\mathcal{H}_F]} \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})$

$U[\xi, \zeta] \to 2\,B[\chi, \psi]$

$Pf[U] \to \int_{\{D[\tilde{\xi}]\}} [e^{\frac{1}{2} U[\tilde{\xi}, \tilde{\xi}]}] \to \int_{\substack{\{D[\tilde{\xi}]\} \\ \{D[\tilde{\psi}]\}}} [e^{B[\tilde{\xi}, \tilde{\psi}]}] \to \text{Det}[B]$

$D[\eta\_, \theta\_] :\to \text{Table}[d[T[\eta, d, \{i\}]].d[T[\theta, d, \{i\}]], \{i, \dim[]\}]$

$\text{Table}[d[T[\xi, d, \{i\}]].d[T[\psi, d, \{i\}]], \{i, \dim[]\}]$

```
tuSaveAllVariables[]
```