```
<< Local`QFTToolKit2`
tuItalics
```

{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z}

```
expandDC[sub_ : {}, scalar_ : {}, func_ : {}] :=
 tuRepeat[{sub, tuOpDistribute[dotOps], tuOpSimplify[dotOps, scalar]}, {func}]
```

```
Get[NotebookDirectory[] <> "AZeeGroup.V.3.out"];
$gellmann = e[1];
$gellmannR = $gellmann // tuRule
normalize[EXP_List] := Module[{}, EXP / Sqrt[Tr[EXP.EXP]]]
```

$\{\lambda_1 \to \{\{0, 1, 0\}, \{1, 0, 0\}, \{0, 0, 0\}\}, \lambda_2 \to \{\{0, -i, 0\}, \{i, 0, 0\}, \{0, 0, 0\}\},$
$\lambda_3 \to \{\{1, 0, 0\}, \{0, -1, 0\}, \{0, 0, 0\}\}, \lambda_4 \to \{\{0, 0, 1\}, \{0, 0, 0\}, \{1, 0, 0\}\},$
$\lambda_5 \to \{\{0, 0, -i\}, \{0, 0, 0\}, \{i, 0, 0\}\}, \lambda_6 \to \{\{0, 0, 0\}, \{0, 0, 1\}, \{0, 1, 0\}\},$
$\lambda_7 \to \{\{0, 0, 0\}, \{0, 0, -i\}, \{0, i, 0\}\}, \lambda_8 \to \{\{\frac{1}{\sqrt{3}}, 0, 0\}, \{0, \frac{1}{\sqrt{3}}, 0\}, \{0, 0, -\frac{2}{\sqrt{3}}\}\},$
$\mathrm{Tr}[\lambda_a \cdot \lambda_b] \to 2\,\delta_{a\,b}$, FontFamily → Helvetica}

■ VI.2 Roots and Weights for Orthogonal, Unitary, and Symplectic Algebras

Setting the stage with SU[2] and SU[3]

Rank and the maximal number of mutually commuting generators

Onward to the orthogonals: Our friend the square appears

```
PR[$ = {{T[iH, "u", {i}], {i, il[CG["number of matrices"]]}}[
     CG["mathematician label for Lie algebra generators"]],
    {{CG["Find weights ", T[iH, "u", {i}]]},
      CG["Find maximal set of mutually commuting generators and diagonalize them"],
      CG["Find l-root vectors connecting weights"]}[
     CO["a method for determing root vectors"]]
  }; (ColumnForms[#1, 3] &)[$]
]
```

H^i
{i, l[number of matrices]} [mathematician label for Lie algebra generators]
{Find weights , H^i}
Find maximal set of mutually commuting generators and diagonalize them [a method for determing root vectors]
Find l-root vectors connecting weights

Review of SU[3]

```
PR[$ = {λ3, λ8};
 e[1] = $ = MapAt[normalize, #, 2] & /@ Thread[($ /. λ → iH) → ($ /. $gellmannR)];
 {$ // MatrixForms}[CG["diagonal generators"]],
 NL, "Weights: ", e[2] = Table[T[w, "u", {i}] →
     ((DeleteCases[#[[2, i]], 0] /. {} → {0}) & /@ e[1] // Flatten ), {i, 3}],
 CG["note correspondence to up,down,strange quark states",
  {iI"+", iU"+", iV"+"} Sqrt[2]],
 NL, "Root vectors ", $ = Table[T[α, "u", {i}] ->
     (T[w, "u", {Mod[i, 3] + 1}] - T[w, "u", {Mod[i + 1, 3] + 1}]) Sign[3.5 - i], {i, 6}],
 Yield, e[3] = $ = $ /. e[2] // Simplify; (ColumnForms[#1, 2] &)[$]
]
```

$\{\{H_3 \to \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_8 \to \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix}\}\}$[diagonal generators]

Weights: $\{w^1 \to \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\}, w^2 \to \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\}, w^3 \to \{0, -\sqrt{\frac{2}{3}}\}\}$

note correspondence to up,down,strange quark states$\{\sqrt{2}\, I_+, \sqrt{2}\, U_+, \sqrt{2}\, V_+\}$

Root vectors $\{\alpha^1 \to w^2 - w^3, \alpha^2 \to -w^1 + w^3, \alpha^3 \to w^1 - w^2, \alpha^4 \to -w^2 + w^3, \alpha^5 \to w^1 - w^3, \alpha^6 \to -w^1 + w^2\}$

$\to$

$\alpha^1 \to \{-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\}$

$\alpha^2 \to \{-\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}\}$

$\alpha^3 \to \{\sqrt{2}, 0\}$

$\alpha^4 \to \{\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}\}$

$\alpha^5 \to \{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\}$

$\alpha^6 \to \{-\sqrt{2}, 0\}$

Positive and simple roots

Onward to the orthogonals: Our friend the square appears

```
PR["Pedagogically do SO[4] first. ",
 NL, $ = {{N (N - 1) / 2}[CG["generators"]],
   {T[J, "dd", {1, 2}], T[J, "dd", {2, 3}]}[
    CR["maximal subset of mutually commuting generators"]],
   $ = e[4] = {T[H, "u", {1}] → DiagonalMatrix[{1, -1, 0, 0}],
      T[H, "u", {2}] → DiagonalMatrix[{0, 0, 1, -1}]};
   ($ // MatrixForms)[CG["diagonalize"]],
   $ = e[5] = Table[T[w, "u", {i}] →
       ((DeleteCases[#[[2, i]], 0] /. {} → {0}) & /@ e[4] // Flatten ), {i, 4}];
   {$}[CG["Weights e[5]"]],

   CG["● Root vectors "],
   CG["Choose shortest distances pairs"],
   $s = Sort /@ Permutations[{1, 2, 3, 4}, {2}] // DeleteDuplicates;
   $ = (# -> T[w, "u", {#[[1]]}] - T[w, "u", {#[[2]]}]) & /@ $s /. e[5];
   $ = #[[1]] -> #[[2]].#[[2]] & /@ $;
   $smallest = #[[2]] & /@ $ // Sort // First;
   $ = Select[$, #[[2]] === $smallest &];

   CG["Root vectors "],
   $ = T[w, "u", {#[[1, 1]]}] - T[w, "u", {#[[1, 2]]}] & /@ $;
   $ = Table[T[α, "u", {i}] → $[[i]], {i, 4}];
   e[6] = $ = $ /. e[5]; (ColumnForms[#1, 2] &)[$][CG["e[6]"]],
   {T[e, "u", {1}] → {1, 0}, T[e, "u", {2}] → {0, 1}
    }[CG["Cartisian basis which αs are composed"]]

  }; (ColumnForms[#1, 2] &)[$]
]
```

Pedagogically do SO[4] first.

$\{\frac{1}{2}(-1+N)N\}$[generators]

$\{J_{12}, J_{23}\}$[maximal subset of mutually commuting generators]

$\{H^1 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, H^2 \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\}$[diagonalize]

$\{\{w^1 \to \{1, 0\}, w^2 \to \{-1, 0\}, w^3 \to \{0, 1\}, w^4 \to \{0, -1\}\}\}$[Weights e[5]]

● Root vectors

Choose shortest distances pairs

Root vectors

$\alpha^1 \to \{1, -1\}$
$\alpha^2 \to \{1, 1\}$
$\alpha^3 \to \{-1, -1\}$ [e[6]]
$\alpha^4 \to \{-1, 1\}$

$\{e^1 \to \{1, 0\}, e^2 \to \{0, 1\}\}$[Cartisian basis which αs are composed]

SO[5] and a vew feature about roots

```
PR["SO[5]",
 NL, $ = {$ = e[9] = {T[H, "u", {1}] → DiagonalMatrix[{1, -1, 0, 0, 0}],
      T[H, "u", {2}] → DiagonalMatrix[{0, 0, 1, -1, 0}]};
   {$ // MatrixForms}[CG["maximal subset of mutually commuting generators,
       there are no others"]],

   e[9.1] = $ = Table[T[w, "u", {i}] →
       ((DeleteCases[#[[2, i]], 0] /. {} → {0}) & /@ e[9] // Flatten ), {i, 5}];
   {$}[CG["Weights e[9.1]"]],
   CG["● Root vectors "],
   CG["Choose shortest distances pairs"],
   $s = Sort /@ Permutations[{1, 2, 3, 4, 5}, {2}] // DeleteDuplicates;
   $ = (# -> T[w, "u", {#[[1]]}] - T[w, "u", {#[[2]]}]) & /@ $s /. e[9.1];
   $ = #[[1]] -> #[[2]].#[[2]] & /@ $;
   $smallest = #[[2]] & /@ $ // Sort // First;
   $ = Select[$, #[[2]] === $smallest &];

   CG["Additional Root vectors to e[6] (short roots) "],
   $ = T[w, "u", {#[[1, 1]]}] - T[w, "u", {#[[1, 2]]}] & /@ $;
   $ = Table[T[α, "u", {i + 4}] → $[[i]], {i, Length[$]}];
   e[10] = $ = $ /. e[9.1]; (ColumnForms[#1, 2] &)[$][CG["e[10]"]],
   CG["Same Cartesian basis as SO[4]",
    α ⊂ {pm[T[e, "u", {1}]] + pm[T[e, "u", {2}]], pm[T[e, "u", {1}]], pm[T[e, "u", {2}]]}
   ]
  }; (ColumnForms[#1, 2] &)[$]
]
```

SO[5]

$$\left\{\left\{H^1 \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ H^2 \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right\}\right\}[$$

maximal subset of mutually commuting generators, there are no others]

$\{\{w^1 \to \{1, 0\},\ w^2 \to \{-1, 0\},\ w^3 \to \{0, 1\},\ w^4 \to \{0, -1\},\ w^5 \to \{0, 0\}\}\}$[Weights e[9.1]]

● Root vectors

Choose shortest distances pairs

Additional Root vectors to e[6] (short roots)

$\alpha^5 \to \{1, 0\}$
$\alpha^6 \to \{-1, 0\}$
$\alpha^7 \to \{0, 1\}$
$\alpha^8 \to \{0, -1\}$
[e[10]]

Same Cartesian basis as SO[4]

$\alpha \subset \{\pm[e^1] + \pm[e^2], \pm[e^1], \pm[e^2]\}$

SO[6]

```
PR["SO[6]",
 NL, $ = {$ = e[14] = {T[H, "u", {1}] → DiagonalMatrix[{1, -1, 0, 0, 0, 0}],
      T[H, "u", {2}] → DiagonalMatrix[{0, 0, 1, -1, 0, 0}],
      T[H, "u", {3}] → DiagonalMatrix[{0, 0, 0, 0, 1, -1}]};
   {$ // MatrixForms}[
    CR["maximal subset of mutually commuting generators, there are no others"]],

   e[15] = $ = Table[T[w, "u", {i}] →
       ((DeleteCases[#[[2, i]], 0] /. {} → {0}) & /@ e[14] // Flatten ), {i, 6}];
   {$}[CG["Weights e[15]"]],
   CG["● Root vectors "],
   CG["Choose shortest distances pairs"],
   $s = Sort /@ Permutations[{1, 2, 3, 4, 5, 6}, {2}] // DeleteDuplicates;
   $ = (# -> T[w, "u", {#[[1]]}] - T[w, "u", {#[[2]]}]) & /@ $s /. e[15];
   $ = #[[1]] -> #[[2]].#[[2]] & /@ $;
   $smallest = #[[2]] & /@ $ // Sort // First;
   $ = Select[$, #[[2]] === $smallest &];

   CG["Root vectors grouped by position of 0"],
   $ = T[w, "u", {#[[1, 1]]}] - T[w, "u", {#[[1, 2]]}] & /@ $;
   $ = Table[T[α, "u", {i}] → $[[i]], {i, Length[$]}];
   e[15.1] = $ = $ /. e[15]; (ColumnForms[#1, 2] &)[$][CG["e[15.1]"]];
   {Select[$, #[[2, 1]] == 0 &], Select[$, #[[2, 2]] == 0 &], Select[$, #[[2, 3]] == 0 &]},
   {pm[T[e, "u", {1}]] + pm[T[e, "u", {2}]], pm[T[e, "u", {2}]] + pm[T[e, "u", {3}]],
     pm[T[e, "u", {2}]] + pm[T[e, "u", {3}]]}[CG["Cartesian roots "]],
   {{1, -1, 0}, {0, 1, -1}, {0, 1, 1}}[
    CG["simple roots, can get all roots from linear combination of these"]]

  }; (ColumnForms[#1, 2] &)[$]
]
```

SO[6]

$$\left\{\left\{H^1\to\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\ H^2\to\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\ H^3\to\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\end{pmatrix}\right\}\right\}[$$

maximal subset of mutually commuting generators, there are no others]

$\{\{w^1\to\{1,0,0\},\ w^2\to\{-1,0,0\},\ w^3\to\{0,1,0\},\ w^4\to\{0,-1,0\},\ w^5\to\{0,0,1\},\ w^6\to\{0,0,-1\}\}\}[$
Weights e[15]]

● Root vectors

Choose shortest distances pairs

Root vectors grouped by position of 0

$\{\alpha^9\to\{0,1,-1\},\ \alpha^{10}\to\{0,1,1\},\ \alpha^{11}\to\{0,-1,-1\},\ \alpha^{12}\to\{0,-1,1\}\}$

$\{\alpha^3\to\{1,0,-1\},\ \alpha^4\to\{1,0,1\},\ \alpha^7\to\{-1,0,-1\},\ \alpha^8\to\{-1,0,1\}\}$

$\{\alpha^1\to\{1,-1,0\},\ \alpha^2\to\{1,1,0\},\ \alpha^5\to\{-1,-1,0\},\ \alpha^6\to\{-1,1,0\}\}$

$\{\pm[e^1]+\pm[e^2],\ \pm[e^2]+\pm[e^3],\ \pm[e^2]+\pm[e^3]\}$[Cartesian roots]

{{1, -1, 0}, {0, 1, -1}, {0, 1, 1}}[simple roots, can get all roots from linear combination of these]

SO[2*l*] versus SO[2*l*+1]

```
PR["For SO[2l]", soN = 2 il;
 NL, $ = {e[19] = {T[H, "u", {i_?# ≤ soN / 2 &}] :>
      SparseArray[{{2 i - 1} → 1, {2 i} → -1, {soN / 2} → 0}], i ≤ soN / 2},
    {e[20] = {T[w, "u", {i_}] :> SparseArray[{i} → -1^(i-1)], i <= soN,
       T[w, "u", {i_}] ∈ dim[soN / 2]}}[CG["weights"]],
    {pm[T[e, "u", {i}]] + pm[T[e, "u", {j}]], i < j}[CG["Roots"]],
    {T[e, "u", {i - 1}] - T[e, "u", {i}],
      T[e, "u", {il - 1}] + T[e, "u", {il}]}[CG["simple roots"]]

  }; (ColumnForms[#1, 2] &)[$]
]
```

```
For SO[2l]
H^(i_?#1≤ soN/2 &) :→ SparseArray[{{2 i - 1} → 1, {2 i} → -1, {soN/2} → 0}]
i ≤ l
{{w^(i_) :→ SparseArray[{i} → -1^(i-1)], i ≤ 2 l, w^(i_) ∈ dim[l]}}[weights]
{±[e^i] + ±[e^j], i < j}[Roots]
{e^(-1+i) - e^i, e^(-1+l) + e^l}[simple roots]
```

```
PR["For SO[2l+1]", soN1 = 2 il + 1;
 NL, $ = {{T[H, "u", {i_?# ≤ (soN + 1) / 2 &}] :>
     SparseArray[{{2 i - 1} → 1, {2 i} → -1, {(soN1 + 1) / 2} → 0}], i ≤ (soN1 - 1) / 2},
    {{T[w, "u", {i_}] :→ If[i <= soN1 - 1, SparseArray[{i} → -1^(i-1)], 0],
      T[w, "u", {i_}] ∈ dim[(soN1 + 1) / 2]}}[CG["weights"]],
    {pm[T[e, "u", {i}]] + pm[T[e, "u", {j}]], i < j}[CG["Roots"]],
    {T[e, "u", {i - 1}] - T[e, "u", {i}],
      T[e, "u", {il - 1}] + T[e, "u", {il}]}[CG["simple roots"]]

  }; (ColumnForms[#1, 2] &)[$]
]
```

```
For SO[2l+1]
H^(i_?#1≤ (soN+1)/2 &) :→ SparseArray[{{2 i - 1} → 1, {2 i} → -1, {(soN1+1)/2} → 0}]
i ≤ l
{{w^(i_) :→ If[i ≤ soN1 - 1, SparseArray[{i} → -1^(i-1)], 0], w^(i_) ∈ dim[1/2 (2 + 2 l)]}}[weights]
{±[e^i] + ±[e^j], i < j}[Roots]
{e^(-1+i) - e^i, e^(-1+l) + e^l}[simple roots]
```

The roots of SU[N]

```
PR[
 "SU[N] ",
 NL, $ = {
   {e[25] = T[H, "u", {i_}] :→ DiagonalMatrix[($ = If[i < suN - 2, Flatten[
                {Table[1, {ii, i}], -i, Table[0, {ii, i + 2, suN}]}] / Sqrt[i (i + 1)],
              If[i === suN - 2, Flatten[{Table[1, {ii, i}], -i, {0}}] / Sqrt[i (i - 1)],
               If[i === suN - 1,
                Flatten[{Table[1, {ii, i}], -i}] / Sqrt[i (i + 1)]]
              ]
             ]) / √($.$) ]}[CG["N-1 traceless commuting N×N matrices"]],
   {CG["Example suN=", suN = 6],
    $ = T[H, "u", {5}],
    yield, $ /. e[25]},
   CG["N-weights w^i are columns of H."],
   {T[w, "u", {m}] - T[w, "u", {n}], {m, n} ∈ {1, …, N}, T[H, "u", {i}]}[CG["roots"]]

  }; (ColumnForms[#1, 2] &)[$]
]
```

SU[N]

$\{H^{i\_} :\to \text{DiagonalMatrix}[\frac{1}{\sqrt{\$.\$}}(\$ = \text{If}[i < \text{suN} - 2, \frac{\text{Flatten[\{Table[1,\{ii,i\}],-i,Table[0,\{ii,i+2,suN\}]\}]}}{\sqrt{i\,(i+1)}}, \text{If}[i === \text{suN} - 2, \frac{\text{Flatten[\{Table[1,\{ii,i\}],-i,\{0\}\}]}}{\sqrt{i\,(i-1)}}, \text{If}[i === \text{suN} - 1, \frac{\text{Flatten[\{Table[1,\{ii,i\}],-i\}]}}{\sqrt{i\,(i+1)}}]]])]\}$[N-1 traceless commuting N×N matrices]

Example suN=
6
H^5
$\longrightarrow$
$\{\{\frac{1}{\sqrt{30}}, 0, 0, 0, 0, 0\}, \{0, \frac{1}{\sqrt{30}}, 0, 0, 0, 0\}, \{0, 0, \frac{1}{\sqrt{30}}, 0, 0, 0\}, \{0, 0, 0, \frac{1}{\sqrt{30}}, 0, 0\}, \{0, 0, 0, 0, \frac{1}{\sqrt{30}}, 0\}, \{0, 0, 0, 0, 0, -\sqrt{\frac{5}{6}}\}\}$

N-weights w^i are columns of H.

$\{w^m - w^n, \{m, n\} \in \{1, \ldots, N\}, H^i\}$[roots]

From line segment to equilateral triangle to tetrahedron and so on

```
PR["Example for SU[4] ", suN = 4;
 NL, $ = Table[T[H, "u", {i}], {i, 3}];
 $ = Thread[$ → ($h = $ /. e[25])]; $ // MatrixForms,
 NL, "Weights",
 e[28] = $ = Table[T[w, "u", {j}], {j, 4}] -> Table[$h[[i, j, j]], {j, 4}, {i, 3}] // Thread;
 (ColumnForms[#1, 2] &)[$],
 NL, "Roots",

 $ = Table[{T[w, "u", {i}] - T[w, "u", {j}]}, {j, 2, 4}, {i, j - 1}] //
    Flatten[#, 1] & // Sort;

 e[29] = $ = $ → ($ /. e[28] // Simplify // Flatten[#, 1] &) // Thread;
 (ColumnForms[#1, 2] &)[$],
 NL, CO["Is there an obvious choice of simple roots? Examine Dot[]'s "],
 $ = Permutations[e[29], {2}];
 $ = tuRuleTimes /@ $ // DeleteDuplicates;
 $ = MapAt[Apply[Plus, #] &, #, 2] & /@ $;
 $ = Sort[$, #1[[2]] < #2[[2]] &]; $ // Column



]
```

Example for SU[4]

$$\left\{H^1\to\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ H^2\to\begin{pmatrix}\frac{1}{\sqrt{6}} & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{6}} & 0 & 0\\ 0 & 0 & -\sqrt{\frac{2}{3}} & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ H^3\to\begin{pmatrix}\frac{1}{2\sqrt{3}} & 0 & 0 & 0\\ 0 & \frac{1}{2\sqrt{3}} & 0 & 0\\ 0 & 0 & \frac{1}{2\sqrt{3}} & 0\\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2}\end{pmatrix}\right\}$$

Weights

$$w^1\to\left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}},\frac{1}{2\sqrt{3}}\right\}$$
$$w^2\to\left\{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}},\frac{1}{2\sqrt{3}}\right\}$$
$$w^3\to\left\{0,-\sqrt{\frac{2}{3}},\frac{1}{2\sqrt{3}}\right\}$$
$$w^4\to\left\{0,0,-\frac{\sqrt{3}}{2}\right\}$$

Roots

$$\{w^1-w^2\}\to\{\sqrt{2},0,0\}$$
$$\{w^1-w^3\}\to\left\{\frac{1}{\sqrt{2}},\sqrt{\frac{3}{2}},0\right\}$$
$$\{w^2-w^3\}\to\left\{-\frac{1}{\sqrt{2}},\sqrt{\frac{3}{2}},0\right\}$$
$$\{w^1-w^4\}\to\left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{3}}\right\}$$
$$\{w^2-w^4\}\to\left\{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{3}}\right\}$$
$$\{w^3-w^4\}\to\left\{0,-\sqrt{\frac{2}{3}},\frac{2}{\sqrt{3}}\right\}$$

Is there an obvious choice of simple roots? Examine Dot[]'s

$\{(w^2-w^3)\ (w^3-w^4)\}\to -1$
$\{(w^1-w^3)\ (w^3-w^4)\}\to -1$
$\{(w^1-w^2)\ (w^2-w^4)\}\to -1$
$\{(w^1-w^2)\ (w^2-w^3)\}\to -1$
$\{(w^2-w^3)\ (w^1-w^4)\}\to 0$
$\{(w^1-w^3)\ (w^2-w^4)\}\to 0$
$\{(w^1-w^2)\ (w^3-w^4)\}\to 0$
$\{(w^2-w^4)\ (w^3-w^4)\}\to 1$
$\{(w^1-w^4)\ (w^3-w^4)\}\to 1$
$\{(w^1-w^4)\ (w^2-w^4)\}\to 1$
$\{(w^2-w^3)\ (w^2-w^4)\}\to 1$
$\{(w^1-w^3)\ (w^1-w^4)\}\to 1$
$\{(w^1-w^3)\ (w^2-w^3)\}\to 1$
$\{(w^1-w^2)\ (w^1-w^4)\}\to 1$
$\{(w^1-w^2)\ (w^1-w^3)\}\to 1$