Physics 234A: String Theory

Prof. Mina Aganagic
Homework 1.
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1 Problem

Consider the following action for a point particle of mass m, moving on a $d+1$ dimensional manifold $\mathcal{M}_{d+11}$ with coordinates X^μ and the metric

$$ds^2_{\mathcal{M}} = -g_{\mu,\nu}(X) dX^\mu dX^\nu, \qquad \mu, \nu = 0, \dots d,$$

$$S(X,h) = \frac{1}{2}\int_\Sigma d\tau \sqrt{-h}\big(h^{\tau\tau}\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu,\nu} - m^2\big) \tag{1.1}$$

Here

$$ds^2_\Sigma = -h_{\tau\tau}(d\tau)^2$$

is the world-line metric, Σ is a $1d$ interval, (and $h = \det(-h) = -h_{\tau\tau}$).

1.1

Show that extrema of the action S with respect to X^μ give the geodesic equation on $\mathcal{M}_{d+1}$.

1.2

What is the momentum conjugate to X^μ?

1.3

Show that, giving the particle an electric charge e and coupling it to a background electro-magnetic field A_μ the action S changes by the addition of a term

$$\Delta S = e \int A_\mu dX^\mu = e \int d\tau A_\mu \partial_\tau X^\mu$$

2 Problem

2.1

Show that the action of the previous section is invariant under diffeomerphisms that change the coordinate on the world-line τ,

$$\tau \to \tau'(\tau).$$

How do the world-line metric $h_{\tau\tau}$ and the length of the interval $\int_{\tau_1}^{\tau_2} ds_\Sigma$ change under diffeomorphisms?

2.2

Find a specific diffeomorphism that sets $h_{\tau\tau} = e(\tau)^2$ to 1 point wise on Σ.

2.3

What is the interpretation of extrema of S with respect to the world line metric $h_{\tau,\tau}$?

3 Problem

In quantum fiend theory, the momentum-space propagator for a particle of mass m in $D = d + 1$ dimensional Minkowski space equals

$$(2\pi)^D \delta(p - p') \frac{1}{p^2 + m^2},$$

Here, $p^2 = p^\mu \partial^\nu \eta_{\mu\nu}$, $p$and p' are the momenta of the incoming and outgoing particle and δ is a D-dimensional delta function. In this problem, we will derive this propagator from the path integral

$$\int \mathcal{D}h\mathcal{D}Xe^{iS(X,h)}$$

where S is the action in (1.1), and the integral over h should be understood as integral over all metrics $h_{\tau\tau}$ on the interval, modulo diffeomorphisms.

It is somewhat simpler to consider first the amplitude for a particle to propagate from X to X':

$$\langle X|X'\rangle = \int_{X(0)=X}^{X(1)=X'} \mathcal{D}h\mathcal{D}Xe^{\frac{-i}{2}\int_\Sigma d\tau\sqrt{-h_{\tau\tau}}\left(h^{\tau\tau}\partial_\tau X^\mu\partial_\tau X^\nu\eta_{\mu,\nu}-m^2\right)}$$

Here we have arbitrarily set the propagation from $\tau = 0$ to $\tau = 1$; due to reparametrization invariance, the answer only depends on

$$L = \int_{\tau=0}^{\tau=1} d\tau\sqrt{-h} = \int_{\tau=0}^{\tau=1} d\tau e(\tau)$$

where $h_{\tau\tau} = -e(\tau)^2$. Without changing the endpoints, we can use reparametrization invariance to set $e(\tau)$ to a constant, in this case, $e(\tau) = L$.

3.1

Show how to rewrite the path integral as an integral over L, and the path integral X. Go to euclidian space, by rotating $\tau \to i\tau$, $X^0 \to iX^0$.

3.2

The path integral over X is a Gaussian integral. Evaluate it, and express the answer as an integral over L with integrand that depends on X and X', positions of the endpoints. (You will need to know how to compute Gaussian path integrals of this kind, in terms of zeta function regularization).

3.3

Transform the answer of the previous part to momentum space, using

$$|p\rangle = \int d^Dxe^{ip_\mu X^\mu}|x\rangle,$$

and evaluate the integral over L and X to show that the standard momentum space propagator of QFT emerges – after undoing the euclidian rotation.