Regulating expectation values in time dependent QFTs

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In this note we will show how to regulate expectation values in time-dependent quantum field theories that in principle are UV-divergent. The intuition on how to get to a concrete finite result is taken from the use of `Mathematica`. We also show how this is relevant for the study of quantum quenches. Based on PRL 112 (2014) 171601, arXiv: 1401.0560 [hep-th] and work in progress to appear soon with Sumit R. Das and Robert C. Myers.

I. MOTIVATION

Quantum quenches involve changing a parameter of a closed quantum system and then following its subsequent evolution. Recently, a great deal of attention has been given to the study of such systems, mainly because they have become available in laboratory experiments. Both the Quark and Gluon Plasma (that Heller and Yaffe referred to) and Cold Atoms experiments are examples of those.

A new set of scaling properties in the early time behaviour of quenches were found in the holographic analysis [1, 2] of *smooth* but *fast* quantum quenches where a critical theory was deformed by a relevant operator $\mathcal{O}_\Delta$, with conformal dimension Δ, with a time dependent coupling $\lambda(t)$. Specifically, if $\delta\lambda$ denotes the amplitude of the quench and δt is the time scale of the quench duration, it was found that for fast quenches (*i.e.,* $\delta\lambda(\delta t)^{d-\Delta} \ll 1$), the change in the (renormalized) energy density $\delta\mathcal{E}$ scales as $\delta\lambda^2/\delta t^{2\Delta-d}$ and the peak in the (renormalized) expectation value $\langle\mathcal{O}_\Delta\rangle$ scales as $\delta\lambda/\delta t^{2\Delta-d}$. Note that in the limit $\delta t \to 0$, these scalings yield a physical divergence when the conformal dimension is greater than $d/2$. These divergences noted above for $\delta t \to 0$ suggest that infinitely fast quenches can not be physically realized. One might suspect that this is an holographic effect. However, we show that the same scaling behaviour arises in quenches of simple free field theories.

II. QUENCHING A FREE SCALAR FIELD

We begin here by analyzing mass quenches for a free scalar field ϕ in a general spacetime dimension d. In particular, we consider a time-dependent mass making a smooth transition from m at early times to zero at asymptotically late times, $m^2(t) = \frac{m^2}{2}\left(1 - \tanh t/\delta t\,\right)$. This profile allows an exact solution for arbitrary quench rates. From this perspective, the time-dependent coupling is $\lambda(t) = m^2(t)$ while the operator is $\mathcal{O}_\Delta = \phi^2$ with conformal dimension $\Delta = d-2$.

Using the explicit mode functions, it is straightforward to compute the expectation value

$$\langle\phi^2\rangle \equiv \langle in, 0|\phi^2|in, 0\rangle = \frac{1}{2(2\pi)^d}\int \frac{d^{d-1}k}{\omega_{in}}|{}_2F_1|^2\,, \tag{1}$$

Here ${}_2F_1 = {}_2F_1(1 + i\omega_-\delta t, i\omega_-\delta t; 1 - i\omega_{in}\delta t; \frac{1+\tanh(t/\delta t)}{2})$, $\omega_{in} = \sqrt{\vec{k}^2 + m^2}$ and $\omega_- = (|\vec{k}| - \omega_{in})/2$. Of course, this expectation value (1) contains UV divergences associated with integrating $k = |\vec{k}| \to \infty$.

We write the renormalized expectation value as

$$\langle\phi^2\rangle_{ren} \equiv \frac{\Omega_{d-2}}{2(2\pi)^d}\int dk\left(\frac{k^{d-2}}{\omega_{in}}|{}_2F_1|^2 - f_{ct}(k, m(t))\right)\,. \tag{2}$$

where $f_{ct}(k, m(t))$ designates the counterterm contribution and Ω_{d-2} denotes the volume of a unit (d–2)-sphere.

The problem is how to compute the counterterms. For this we will be using `Mathematica`. A naive approach will be to extract at each time the divergent pieces associated with the instantaneous value of the mass,

$$\langle\phi^2\rangle = \frac{\Omega_{d-2}}{2(2\pi)^d}\int dk\frac{k^{d-2}}{\sqrt{k^2+m^2}} = \frac{\Omega_{d-2}}{2(2\pi)^d}\int dk\left(k^d\left(\frac{1}{k^3} - \frac{m^2}{2k^5} + \frac{3m^4}{8k^7} - \frac{5m^6}{16k^9} + O(k^{-11})\right)\right)\,, \tag{3}$$

While this works up to $d = 5$ we show in the notebook that this is not enough to regulate higher dimensional theories. In fact what we learn from analyzing the expectation value for $d = 7$ is that terms proportional to time derivatives of the mass appear. Doing a similar method as to the one shown in the notebook for $d = 7$, we were able to show that the final result takes the form

$$f_{ct}(k, m(t)) = k^{d-3} - \frac{k^{d-5}}{2} m^2(t) + \frac{k^{d-7}}{8}\left(3m^4(t) + \partial_t^2 m^2(t)\right) - \frac{k^{d-9}}{32}\left(10m^6(t) + \partial_t^4 m^2(t) + 10m(t)^2\,\partial_t^2 m(t)^2 + 5\partial_t m(t)^2\,\partial_t m(t)\right.$$

Only the terms with k^n where $n \geq -1$ should be included. Hence we have all counterterm contributions needed to renormalize the expectation value (2) up to $d = 9$.

And indeed, `Mathematica` also allowed us to prove that is the correct large k behaviour of the integrand analytically and found complete agreement with the coefficients first found numerically. To conclude, we can also say that these expansion shows exactly those terms that appear in an adiabatic expansion of the quench, where the change is taken to be infinitely slow and this is the *physical* understanding of the divergences. However, we wouldn't have been able to figure out the form of these counterterms from scratch without the use of `Mathematica`.

Just to be self-consistent, we show figures with the final results showing that after regulating the expectation values, we found the same scalings as were found in holography, suggesting that these are universal. Moreover, we were able to check this universality by changing the conformal dimension of the operator (quenching the mass of a free Dirac fermion), changing the profile of the quench (using a pulsed profile) and checking the Ward identity and generalizing it for higher spin currents.

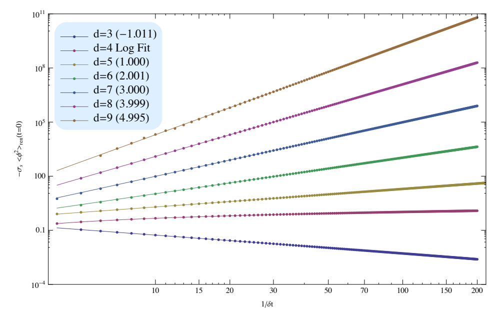

FIG. 1: Expectation value $\langle\phi^2\rangle_{ren}(t = 0)$ as a function of the quench times δt for space-time dimensions from $d = 3$ to $d = 9$. Note that in the plot, the expectation values are multiplied by a numerical factor depending on the dimension: $\sigma_s = 2(2\pi)^d/\Omega_{d-2}$. The slope of the linear fit in each case is shown in the brackets beside the labels. The results support the power law relation $\langle\phi^2\rangle_{ren} \sim \delta t^{-(d-4)}$.

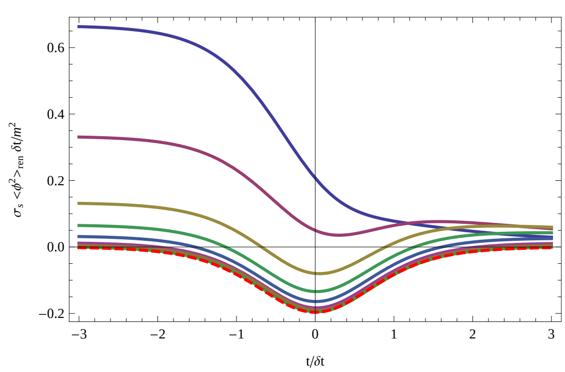

FIG. 2: Scaled expectation value $\delta t/m^2 \times \langle\phi^2\rangle_{ren}$ for $d = 5$. At early times, from top to bottom, the solid curves correspond to $\delta t = 1$, 1/2, 1/5, 1/10, 1/20, 1/50, 1/100, 1/500. As δt gets smaller, the curves approach the leading order solution (shown with the dashed red line) found by expanding for small δt. The last solid curve corresponding to $\delta t = 1/500$ essentially matches this analytic result.

[1] A. Buchel, L. Lehner, R. C. Myers and A. van Niekerk, JHEP **1305**, 067 (2013) [arXiv:1302.2924 [hep-th]].
[2] A. Buchel, R. C. Myers and A. van Niekerk, Phys. Rev. Lett. **111**, 201602 (2013) [arXiv:1307.4740 [hep-th]].