Perimeter Institute, August 24-29 2015
Mathematica Summer School

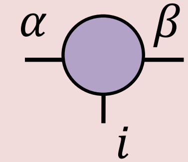
Lectures on Tensor Networks, Guifre Vidal (Perimeter Institute)

1- Tensor networks and many-body entanglement
   Matrix product state (MPS)
2- Multi-scale entanglement renormalization ansatz (MERA)
3- Tensor network renormalization (TNR)

Slides used during the lectures
(Tuesday 25th - Thursday 27th 2015)

LECTURE 2

Summary of matrix product state (MPS)



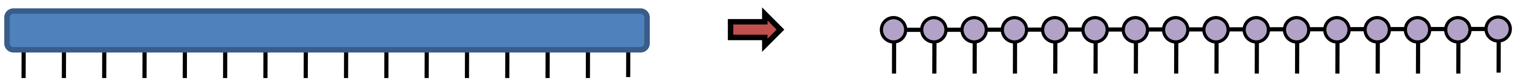
$|\alpha| = |\beta| = \chi$

$|i| = d$

$O(d\chi^2)$ parameters

$$|\Psi\rangle = \sum_{i_1 i_2 \cdots i_N} \Psi_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$

$\Psi_{i_1 i_2 \cdots i_N}$

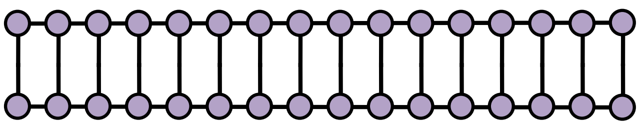


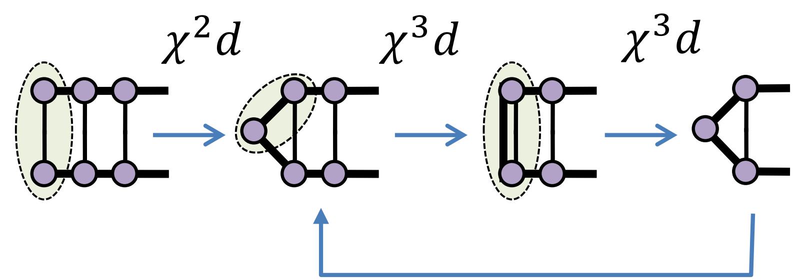
2^N

$O(Nd\chi^2)$

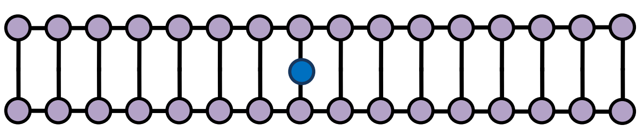
Efficient representation!

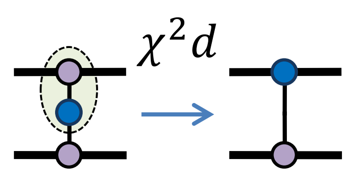
Efficient computation?

$\langle\Psi|\Psi\rangle =$ 



$O(Nd\chi^3)$!!!

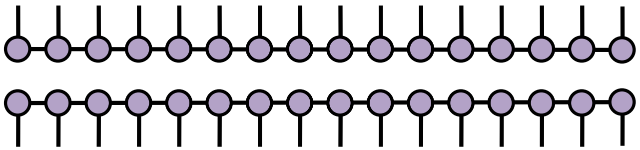
$\langle\Psi|\hat{o}|\Psi\rangle =$ 



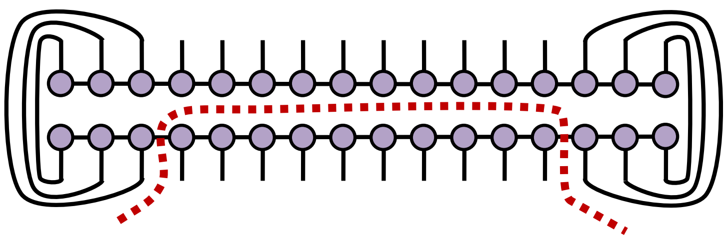
2.B Physics? Structural properties:

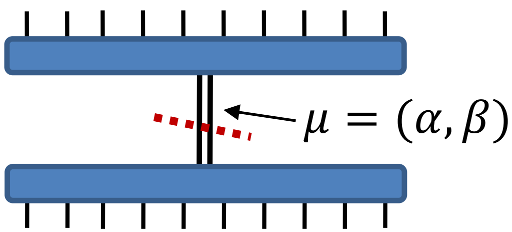
(a) entanglement entropy
(b) correlations

- entanglement entropy

$|\Psi\rangle\langle\Psi| =$ 

$\rho_A = tr_B\, |\Psi\rangle\langle\Psi|$

$\rho_A =$ 

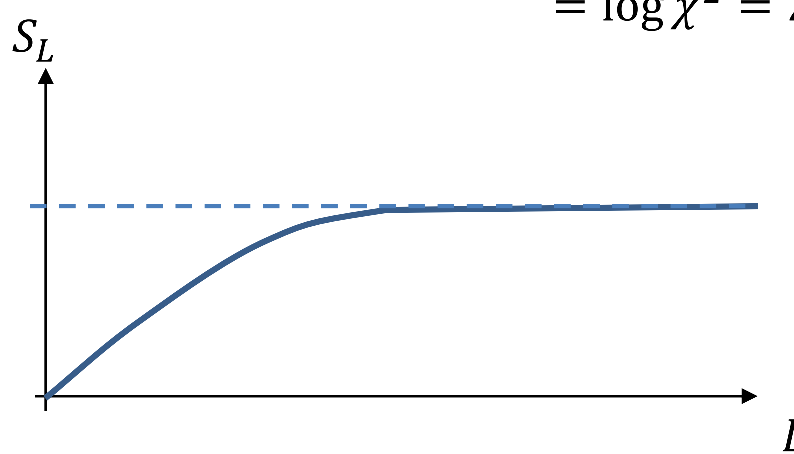
$=$ 

$$\rho_A = \sum_{\mu=1}^{\chi^2} |\phi_\mu\rangle\langle\tilde{\phi}_\mu|$$

$$S(\rho_A) = S\left(p_1, p_2, \cdots, p_{\chi^2}\right)$$

$$\leq S\left(\frac{1}{\chi^2}, \frac{1}{\chi^2}, \cdots, \frac{1}{\chi^2}\right)$$

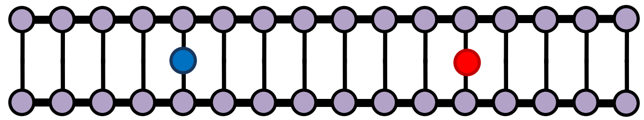
$$= \log \chi^2 = 2 \log \chi$$



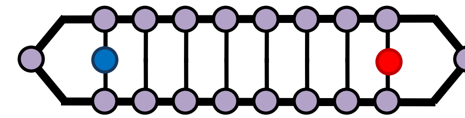
$S_L \leq const$

- correlations

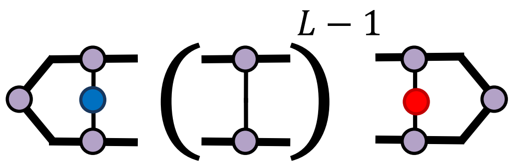
$$\langle\Psi|\hat{o}(0)\,\hat{o}(L)|\Psi\rangle =$$



$$=$$



$$=$$



$$\approx a\,\lambda^{L} = a e^{-L/\xi}$$

$$\xi \equiv -\frac{1}{\log\lambda}$$

Structural properties of MPS

correlations $C(L) \approx e^{-L/\xi}$

entanglement $S_L \leq 2\log\chi$

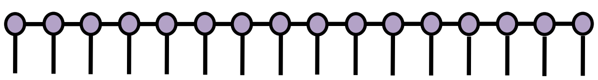
match with
ground states of 1D
gapped Hamiltonians

MERA: definition

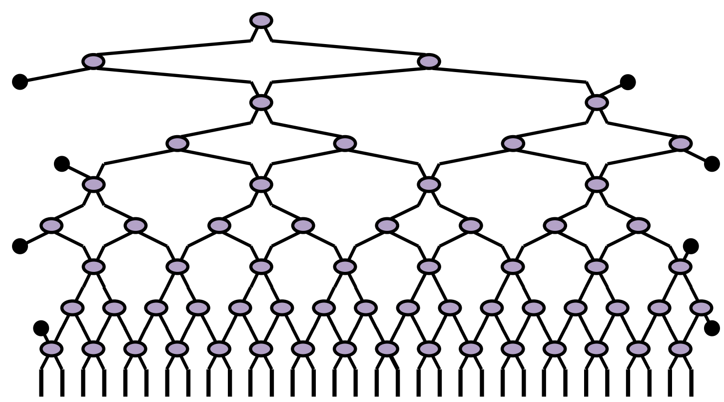
$$|\Psi\rangle \in (\mathbb{C}^d)^{\otimes N}$$

d^N complex numbers

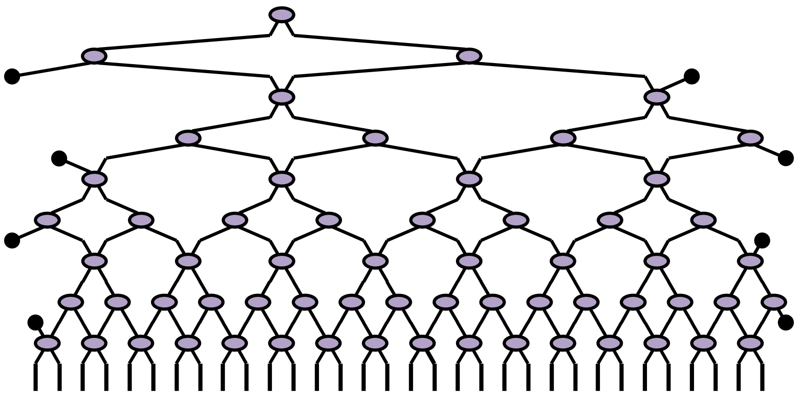
Matrix product state
(MPS)



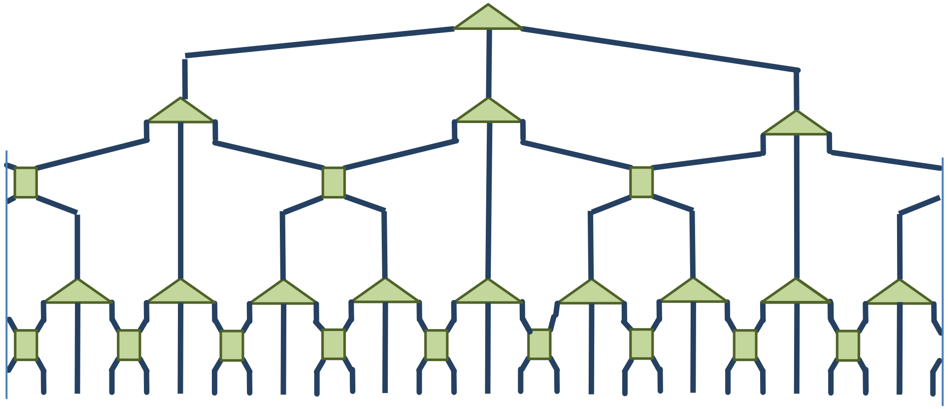
Multi-scale entanglement
renormalization ansatz
(MERA)



MERA



also MERA !



Efficiency

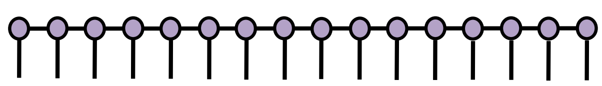
$$|\Psi\rangle \in (\mathbb{C}^d)^{\otimes N}$$

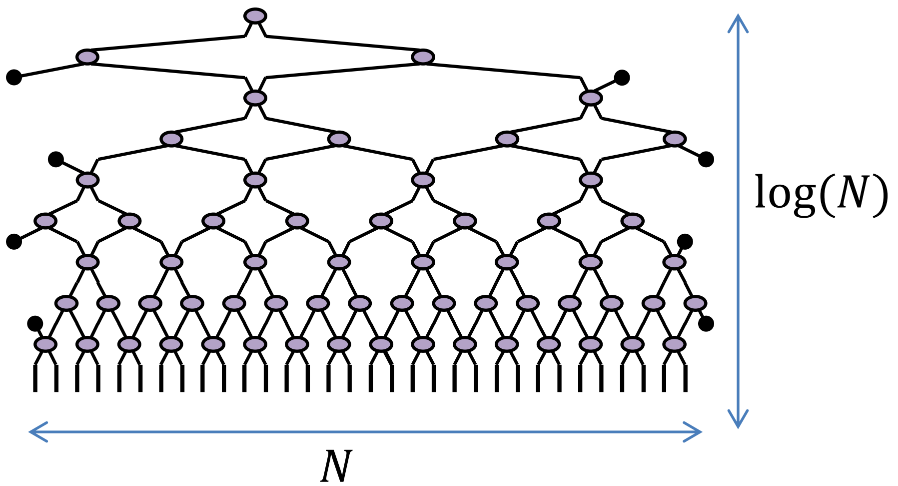
d^N complex numbers

$$\boxed{N + \frac{N}{2} + \frac{N}{4} + \ldots = N\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \leq 2N}$$

Multi-scale entanglement renormalization ansatz (MERA)

Matrix product state (MPS)





N spins $\Rightarrow$ N tensors
$\Rightarrow O(N)$ parameters

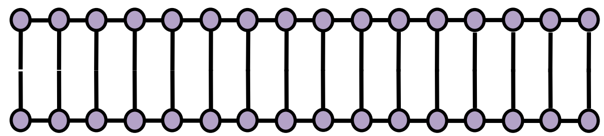
N spins $\Rightarrow N\log(N)$ tensors ?

$2N$ tensors $\Rightarrow O(N)$ parameters

efficiency

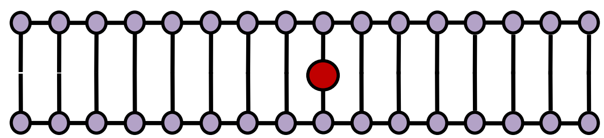
Matrix product state
(MPS)

$$\langle \Psi | \Psi \rangle$$



cost $O(N)$

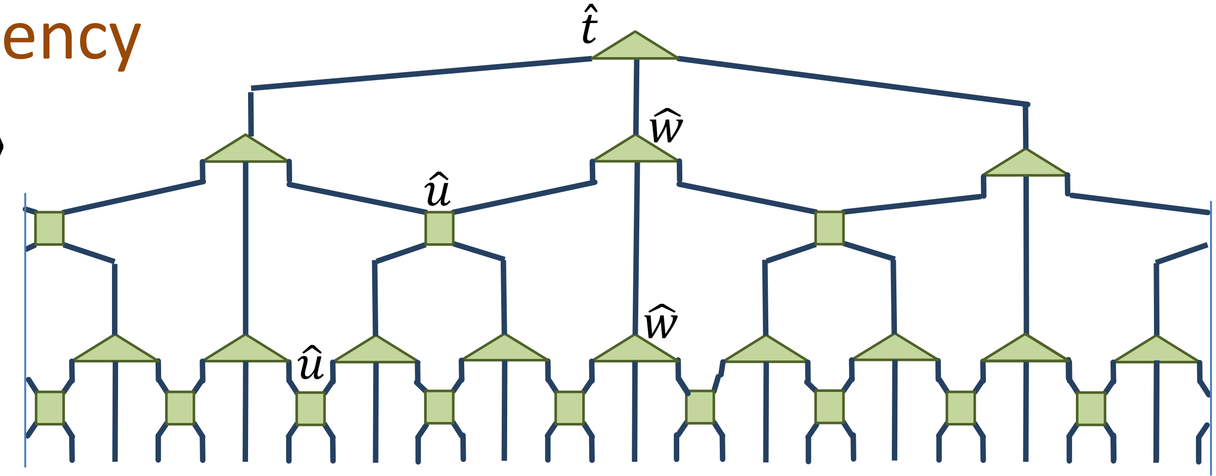
$$\langle \Psi | \hat{o} | \Psi \rangle$$



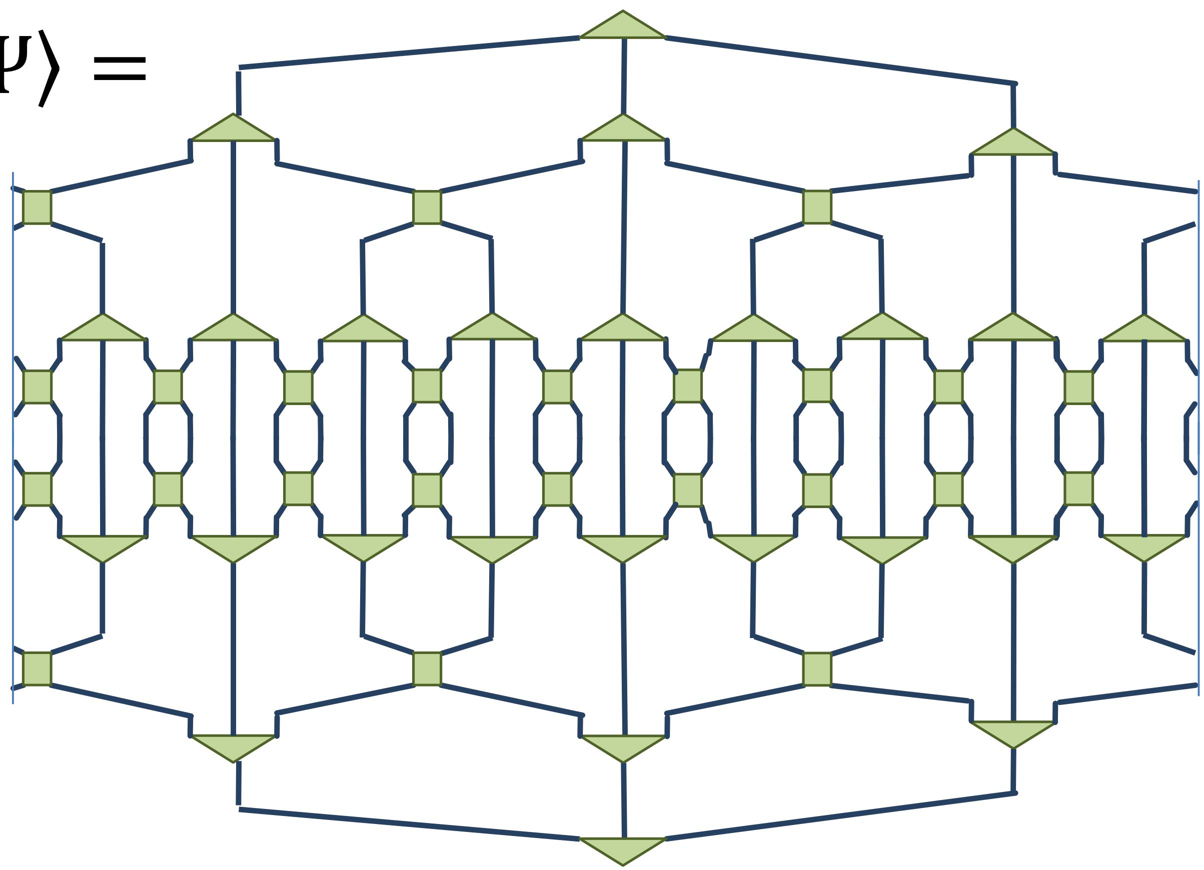
cost $O(N)$

efficiency

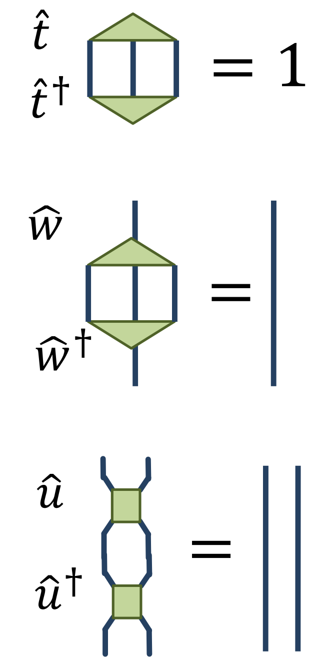
$|\Psi\rangle$



$\langle\Psi|\Psi\rangle =$



isometric tensors!

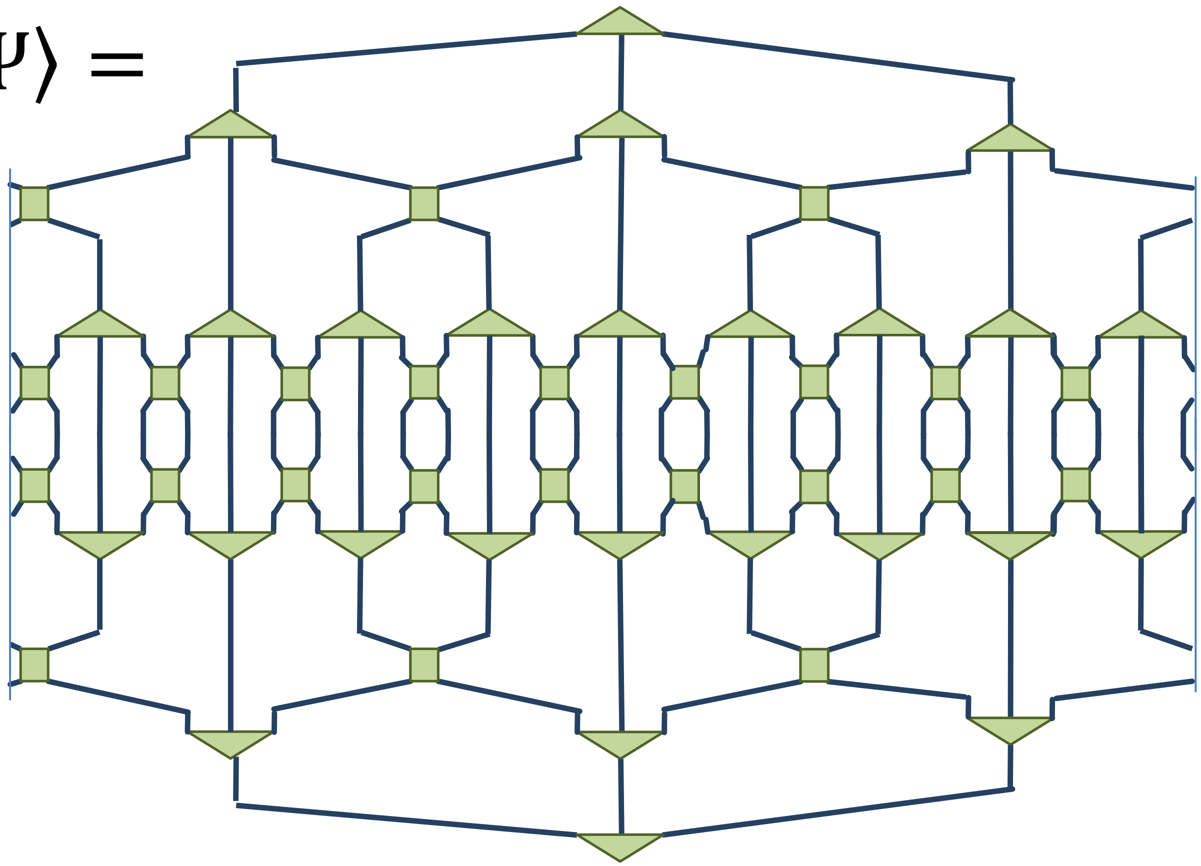


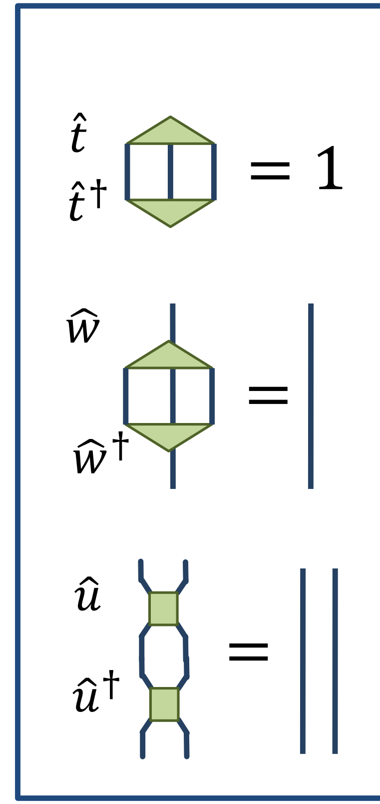
cost $= 0$!!!

efficiency

isometric tensors!

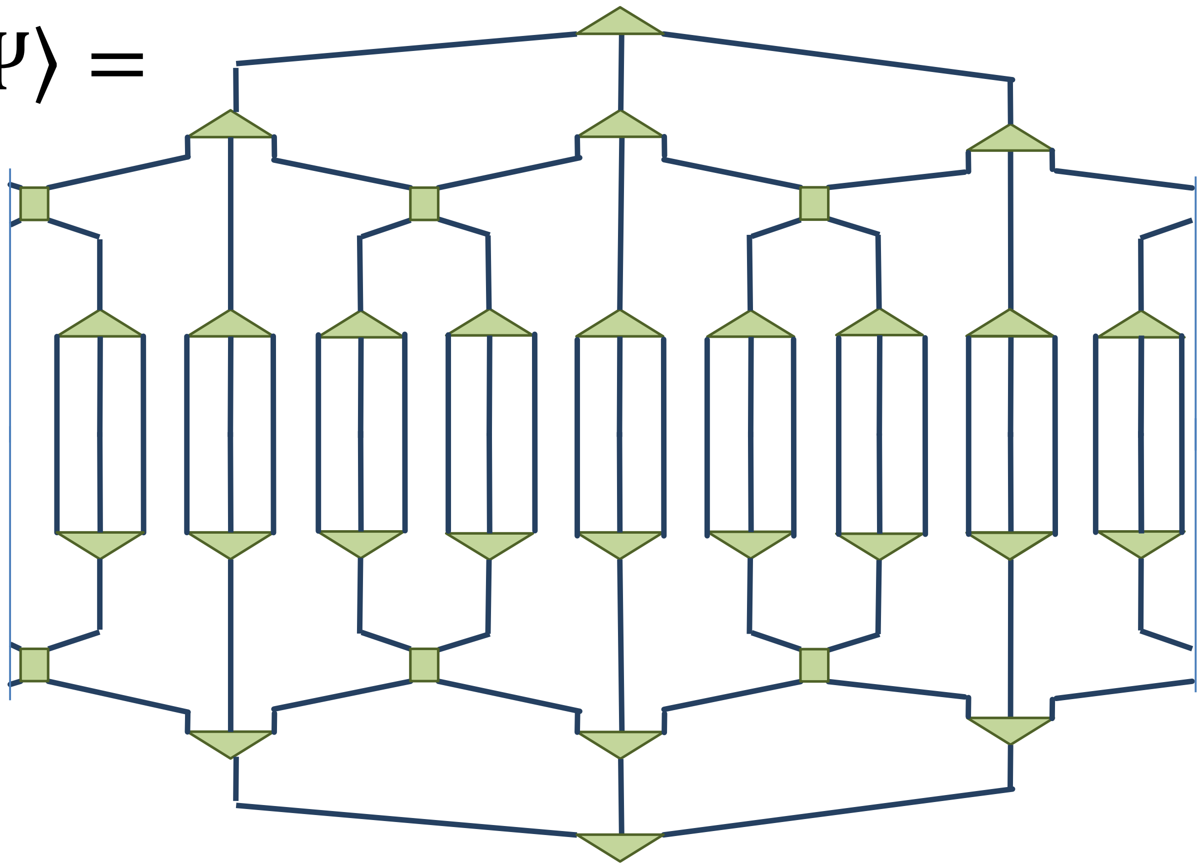
$$\langle \Psi | \Psi \rangle =$$



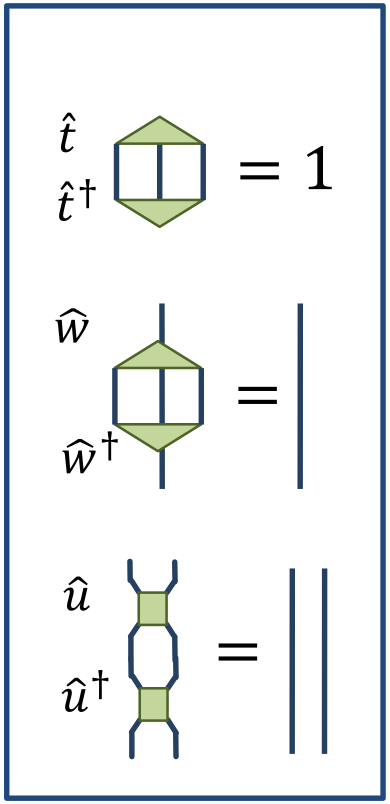


efficiency

$$\langle\Psi|\Psi\rangle =$$

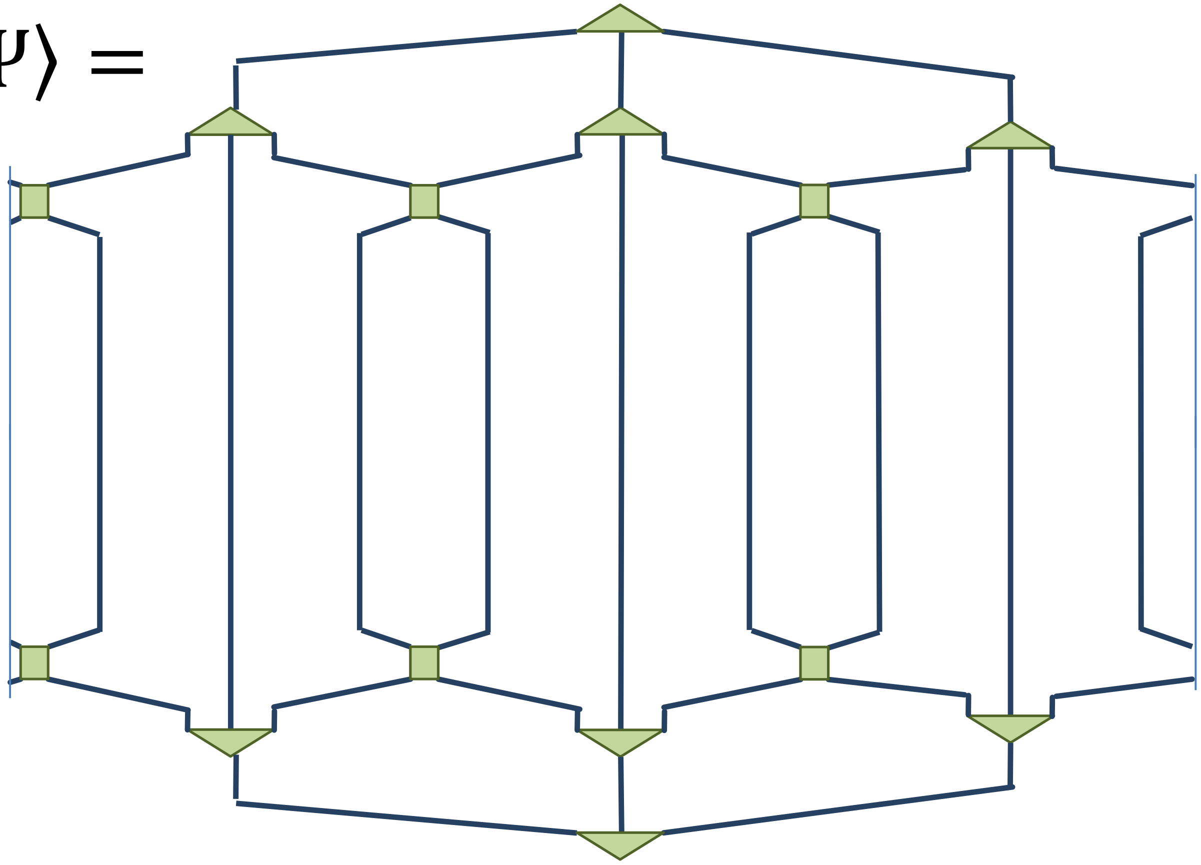


isometric tensors!

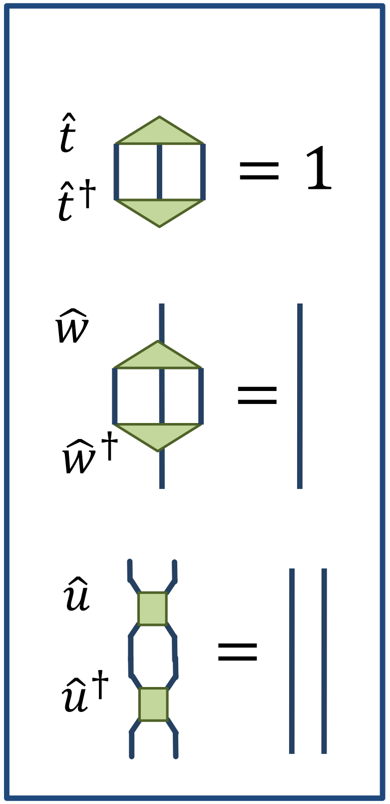


efficiency

$\langle\Psi|\Psi\rangle =$

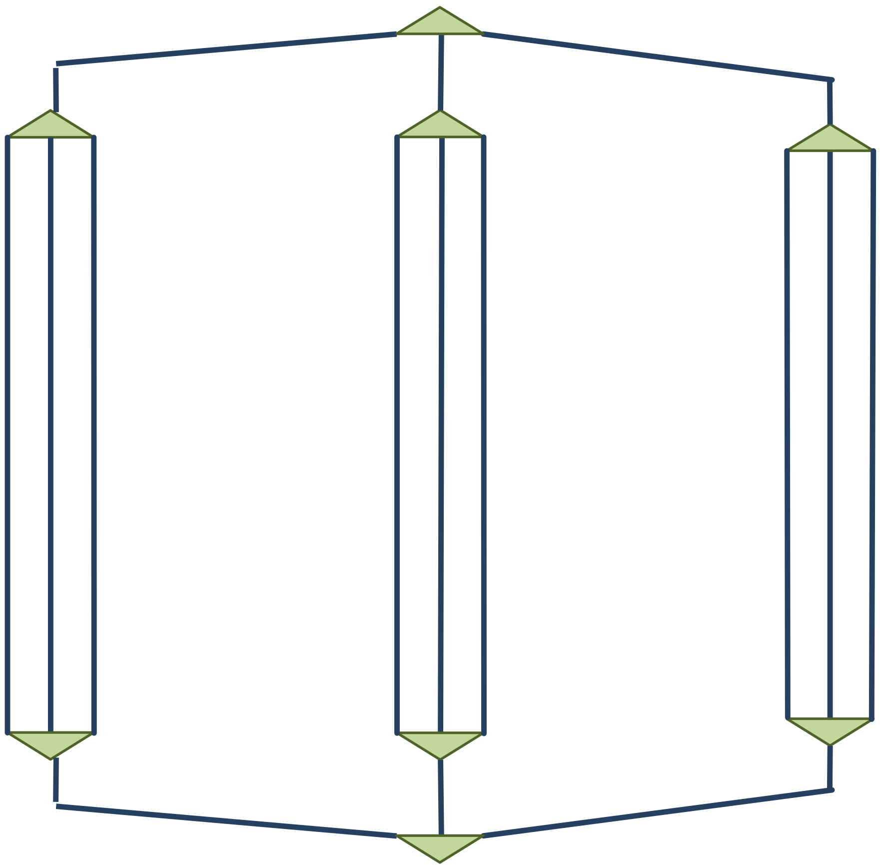


isometric tensors!

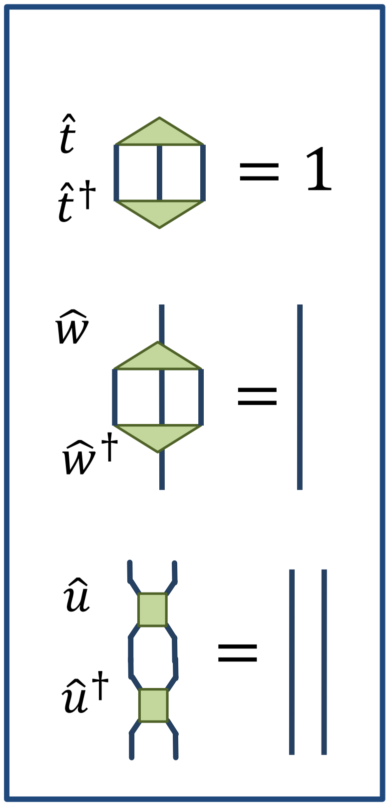


efficiency

$\langle\Psi|\Psi\rangle =$

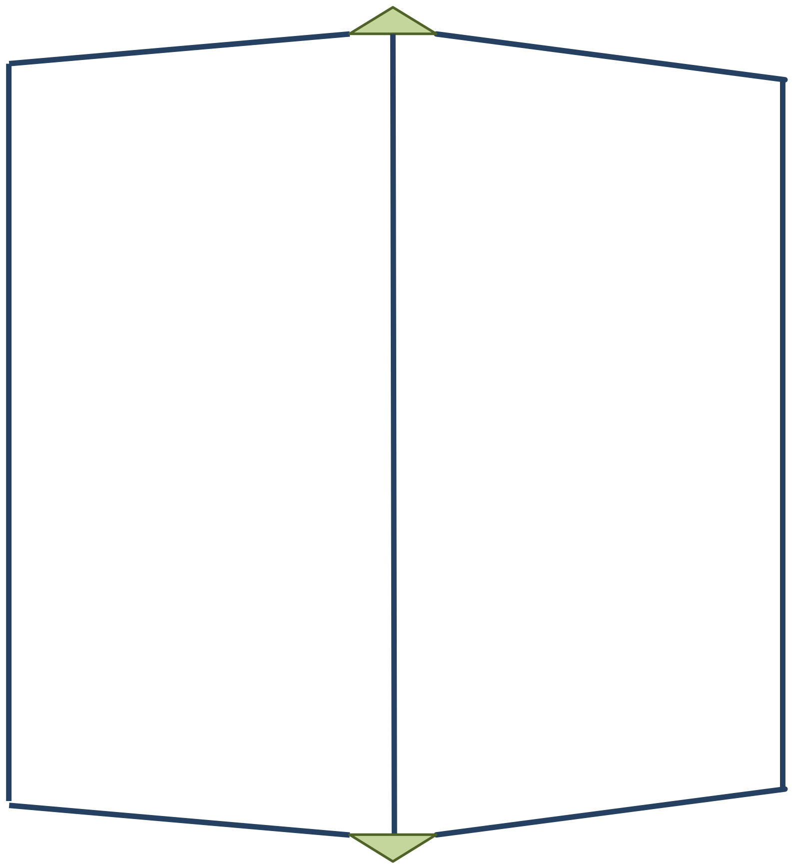


isometric tensors!

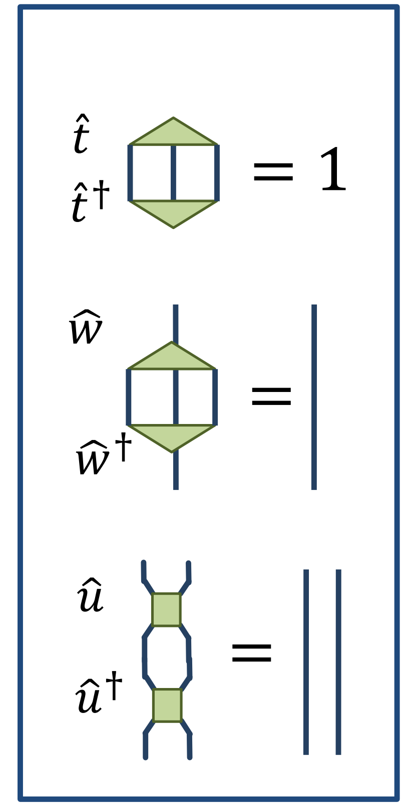


efficiency

$$\langle \Psi | \Psi \rangle =$$



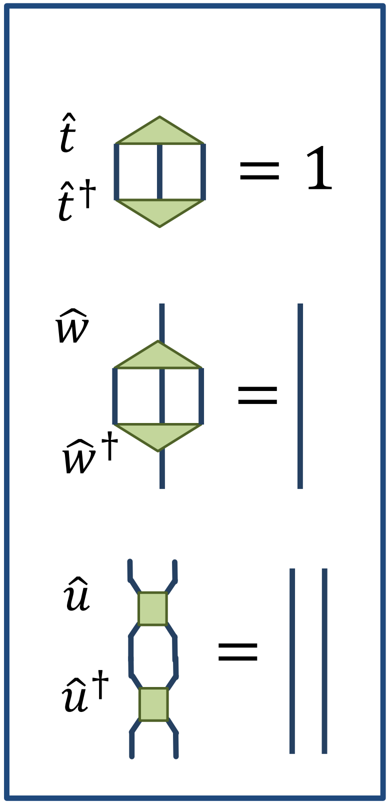
isometric tensors!



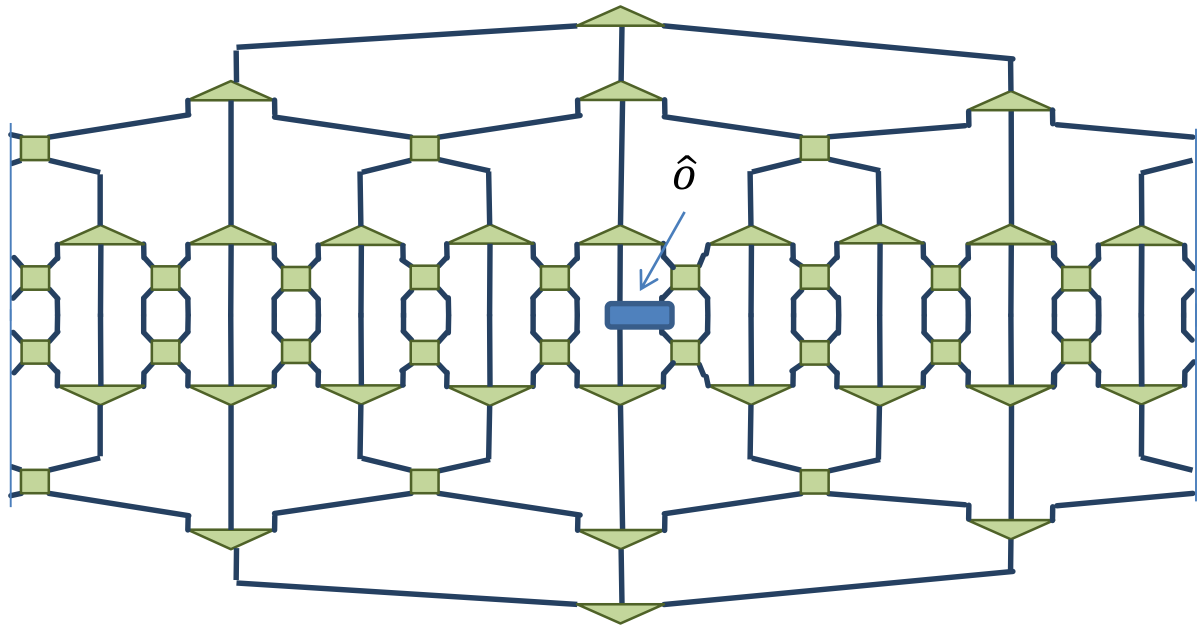
efficiency

$$\langle \Psi | \Psi \rangle = 1$$

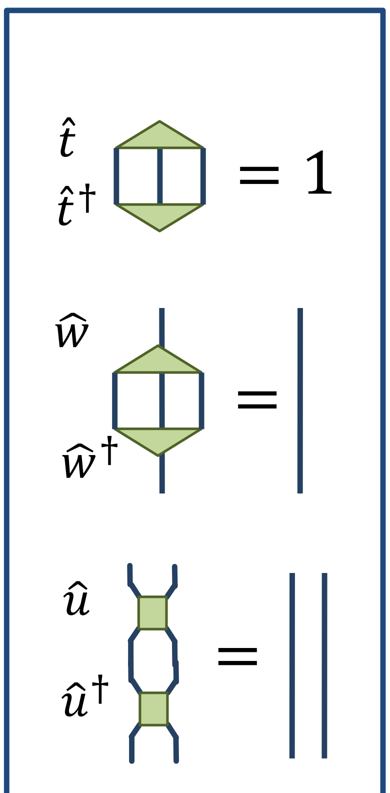
isometric tensors!



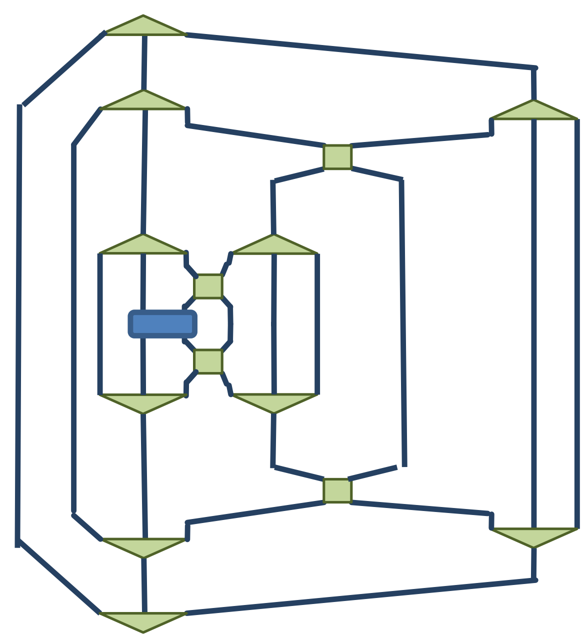
$$\langle \Psi | \hat{o} | \Psi \rangle =$$

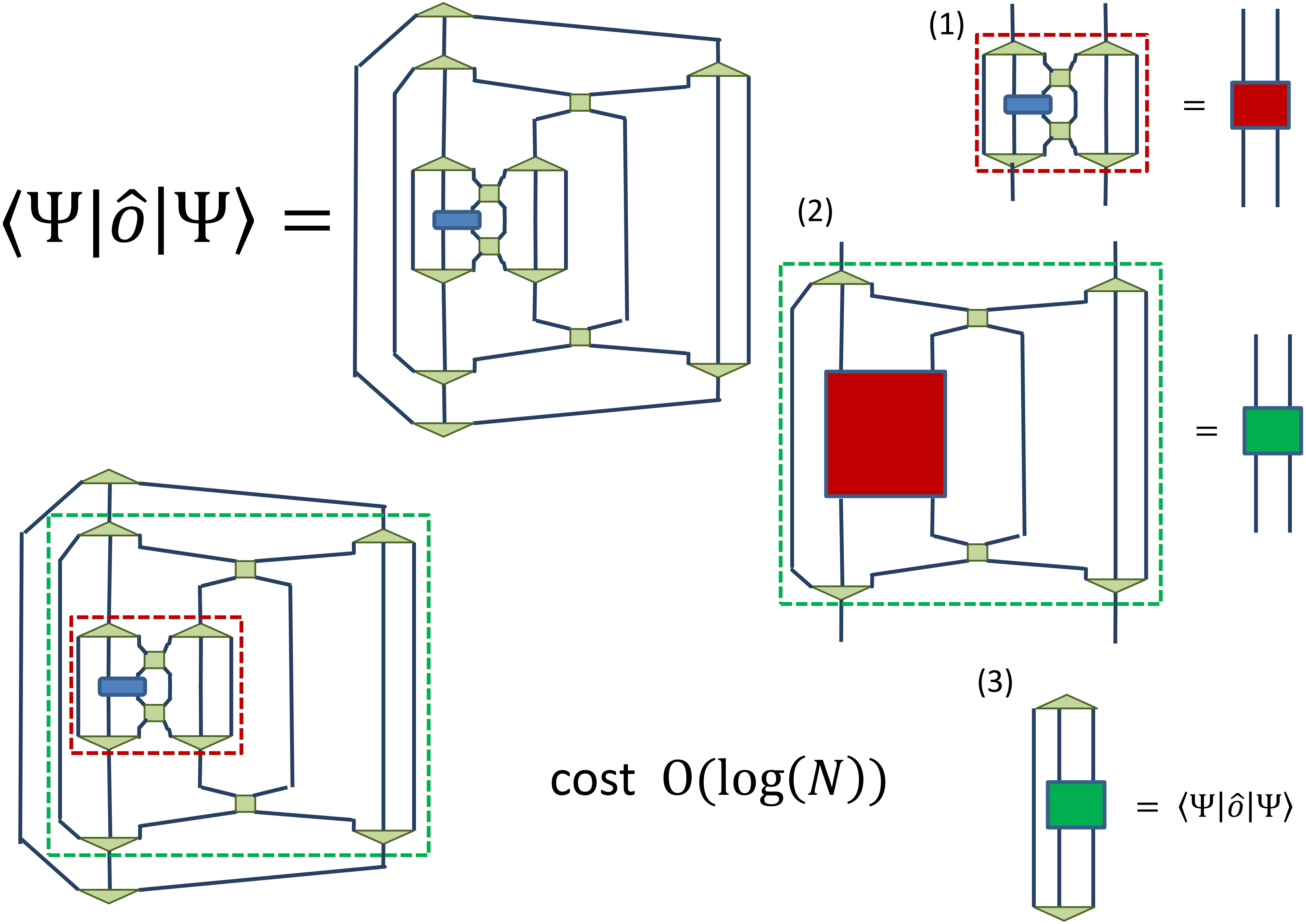


isometric tensors!



$=$



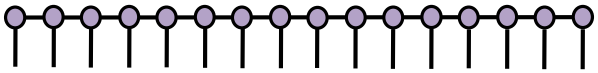


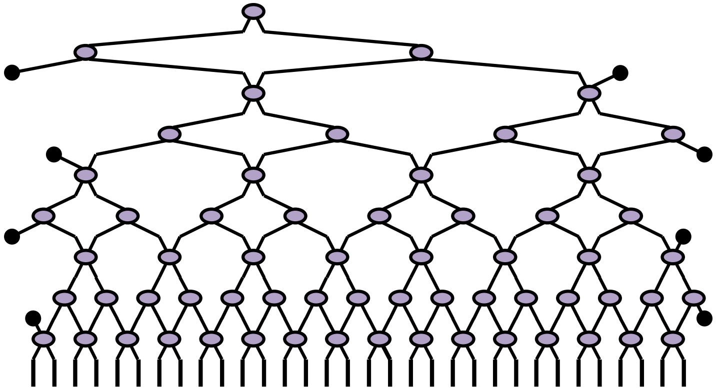
$\langle\Psi|\hat{o}|\Psi\rangle =$
(1)
=
(2)
=
(3)
cost $O(\log(N))$
$= \langle\Psi|\hat{o}|\Psi\rangle$

Structural properties

$$|\Psi\rangle \in (\mathbb{C}^d)^{\otimes N}$$

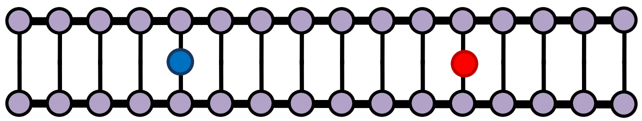
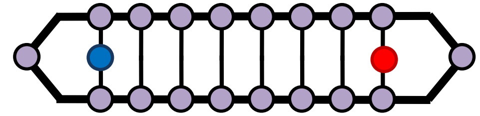
d^N complex numbers

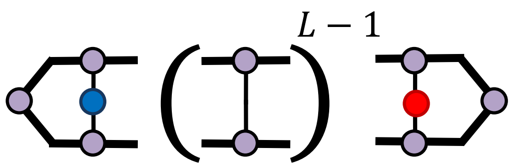




- Decay of correlations
- Scaling of entanglement

$\langle\Psi|\hat{o}(0)\ \hat{o}(L)|\Psi\rangle$ MPS

$=$  $=$ 

$=$ 

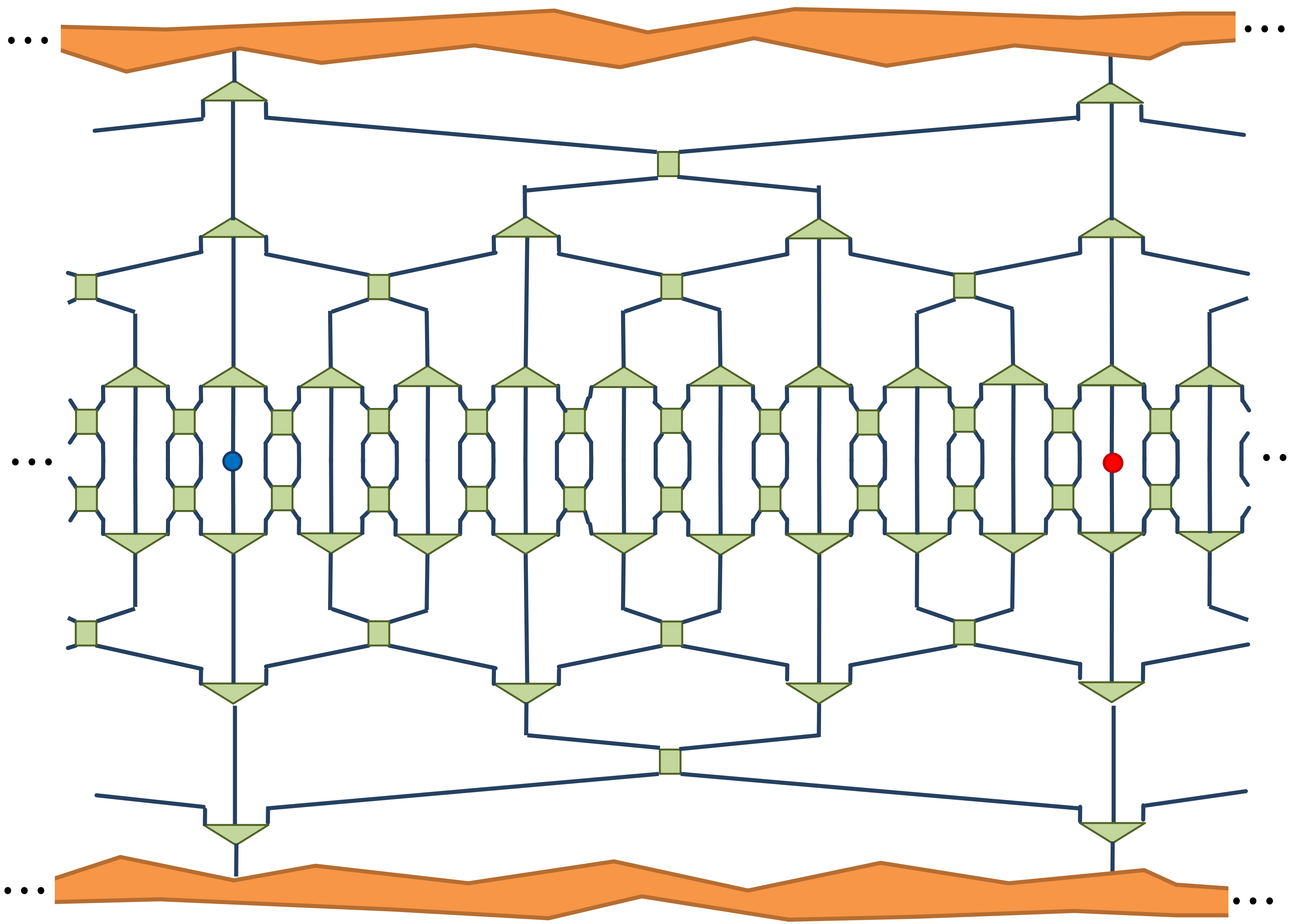
$\approx\ a\lambda^{L} = a e^{-L/\xi}$

$$\xi \equiv -\frac{1}{\log\lambda}$$

$\Rightarrow$ exponential decay of correlations

MERA

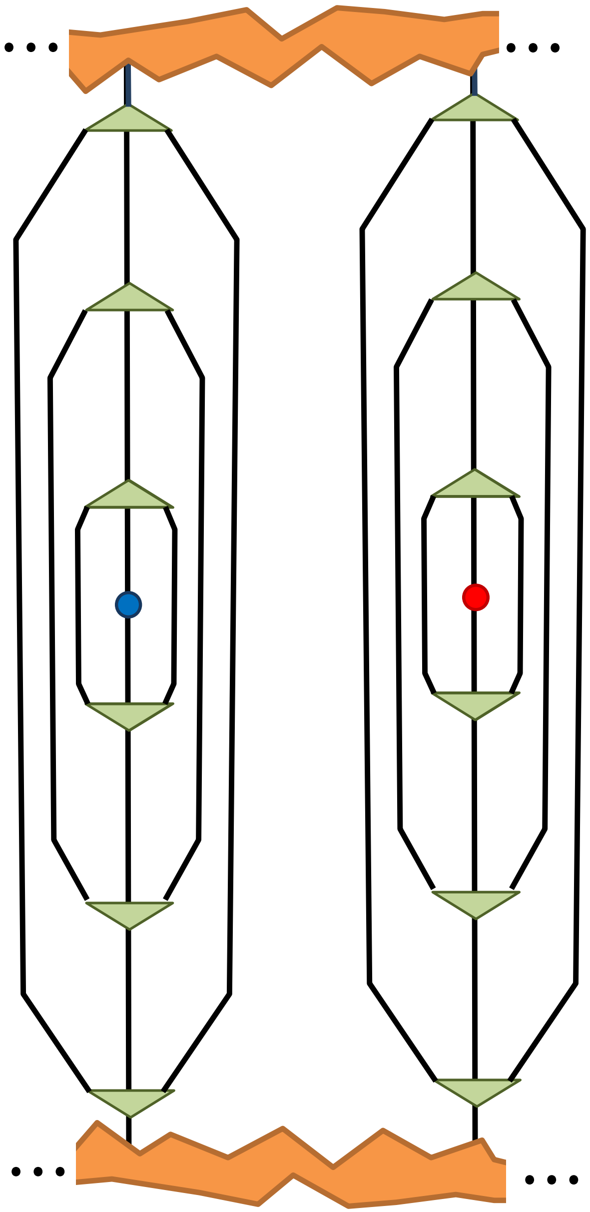
$$\langle \Psi | \hat{o}(0)\, \hat{o}(L) | \Psi \rangle \; =$$



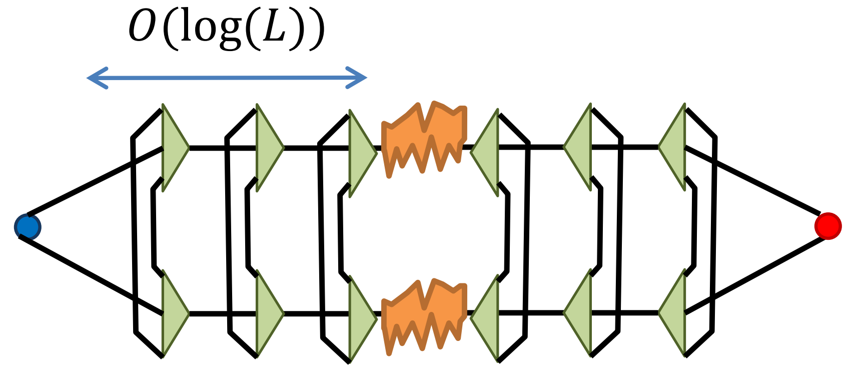
$\langle\Psi|\hat{o}(0)\ \hat{o}(L)|\Psi\rangle$

MERA

$=$



$=$



$$\approx (\lambda)^{\log_3(L)}(\lambda)^{\log_3(L)}$$

$$= \lambda^{2\log_3(L)} \quad = L^{2\log_3(\lambda)} = L^{-p}$$

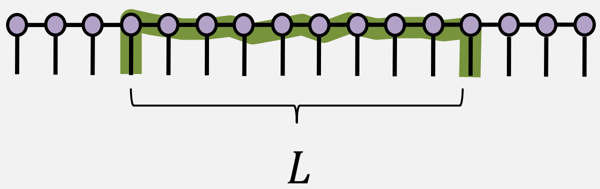
$$x^{\log_3(y)} = y^{\log_3(x)} \qquad p \equiv -2\log_3(\lambda)$$

$\Rightarrow$ polynomial decay of correlations

Correlations: summary and interpretation

matrix product state

(MPS)



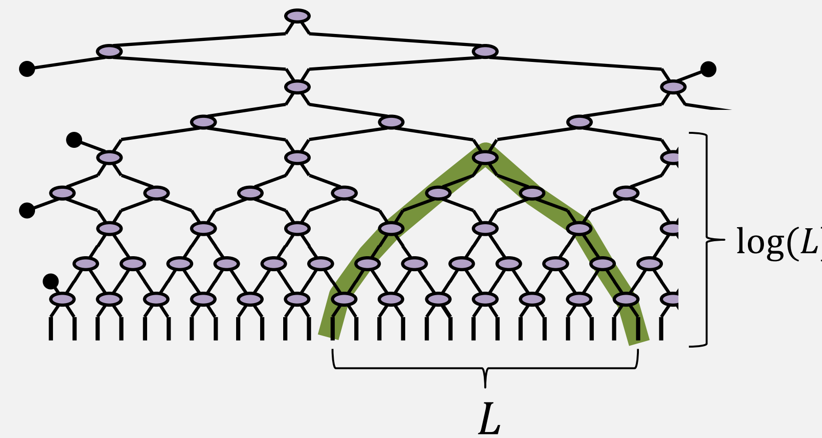
structure of geodesics:

$$\langle \hat{o}(0)\hat{o}(L)\rangle \approx e^{-L/\xi}$$

exponential

multi-scale entanglement renormalization ansatz

(MERA)



structure of geodesics:

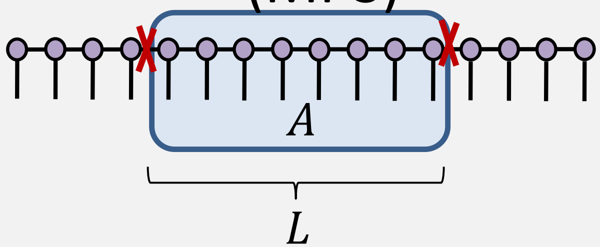
$$\langle \hat{o}(0)\hat{o}(L)\rangle \approx L^{-p}$$

power-law

Entanglement entropy

matrix product state

(MPS)



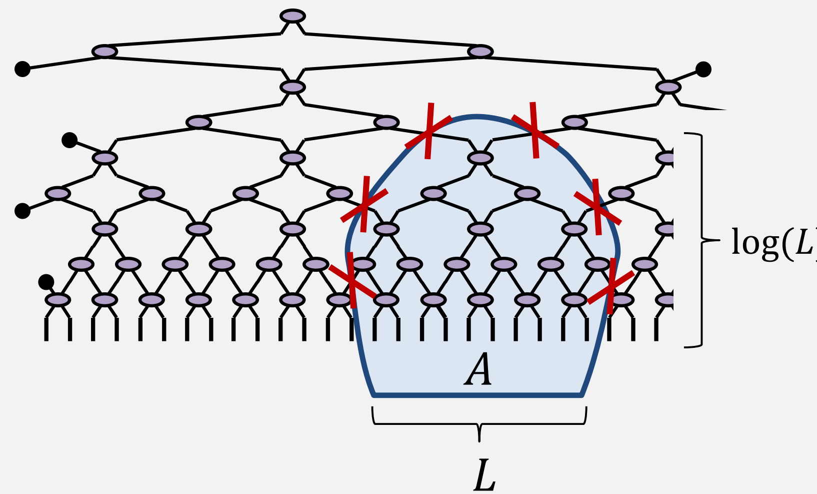
connectivity:

$S(A) \leq const$

boundary law!

multi-scale entanglement renormalization ansatz

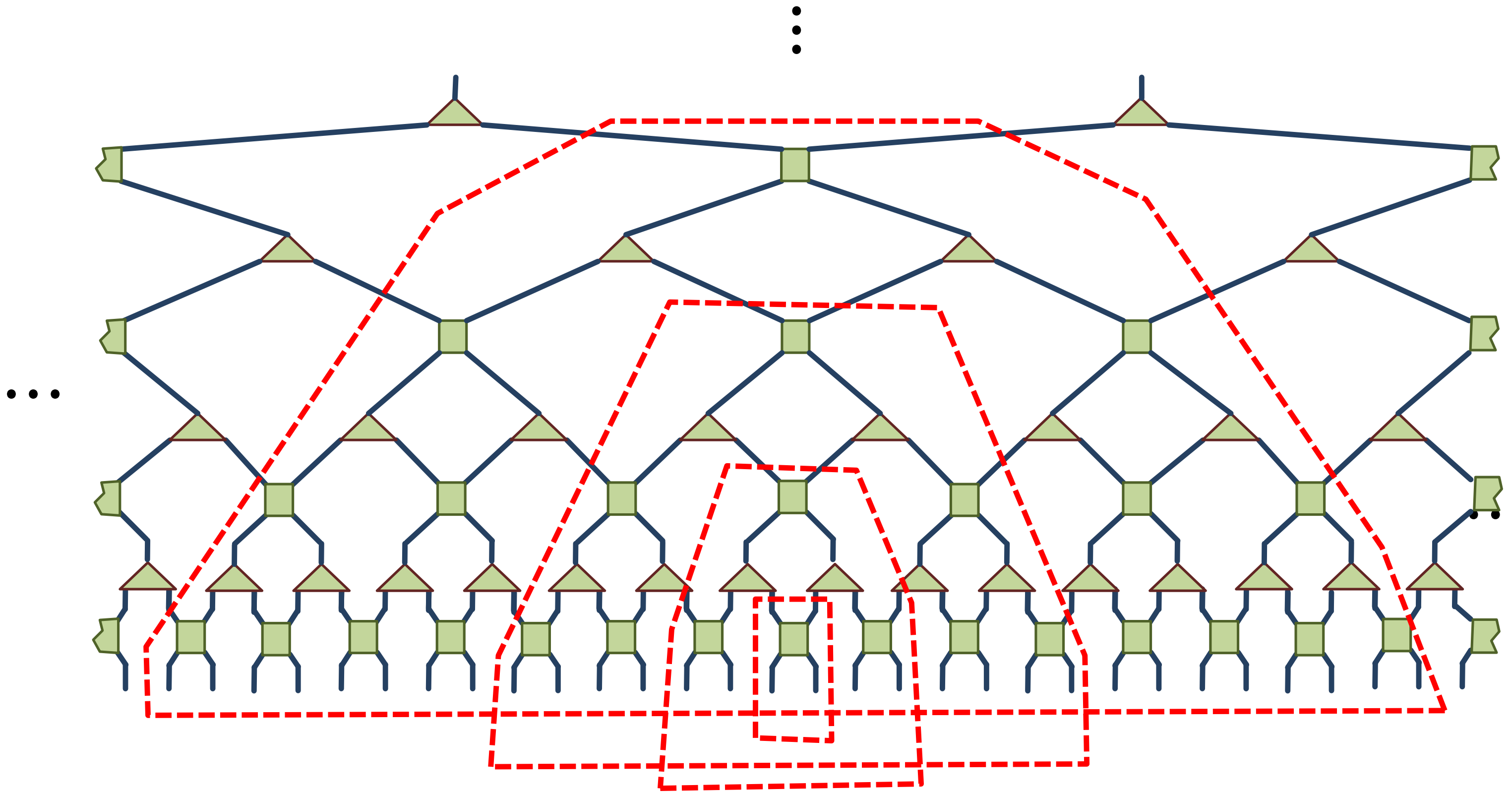
(MERA)



connectivity:

$S(A) \leq \log L$

logarithmic correction!



$$n(A) \approx \log L$$

$L = 2, \quad n(A) = 2$

$L = 6, \quad n(A) = 4$

$L = 14, \quad n(A) = 6$

$L = 30, \quad n(A) = 8$

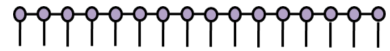
Conclusions

- What is a tensor network state?

- Important aspects of a TN?

    - efficient representation and computation
    - structural properties (correlations and entanglement)

MPS

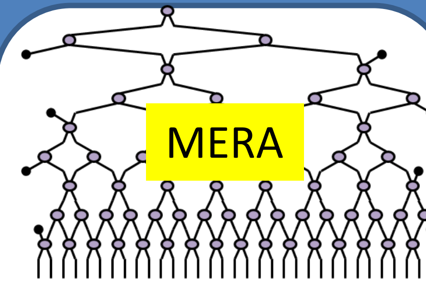


$$\langle \hat{o}(0)\hat{o}(L)\rangle \approx e^{-L/\xi}$$

$$S(A) \leq const$$

ground state
gapped Hamiltonian

MERA



$$\langle \hat{o}(0)\hat{o}(L)\rangle \approx \frac{1}{L^p}$$

$$S(A) \approx \log L$$

ground state
gapless Hamiltonian

Aspects not covered

e.g. DMRG !!!!

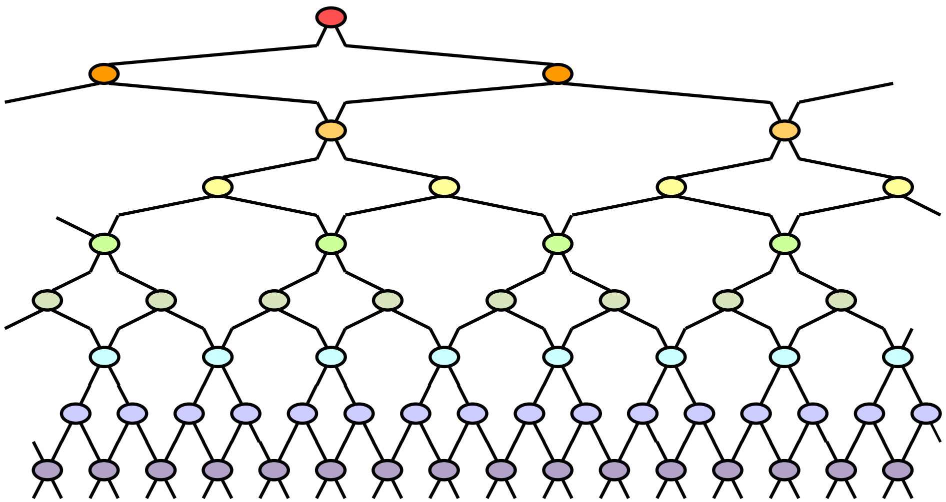
- How to optimize variational parameters (energy minimization; imaginary time evolution)

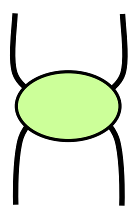
- Simulation of time evolution (MPS)

- continuous MPS, continuous MERA for quantum field theories

- D>1 spatial dimensions (PEPS, MERA, branching MERA)

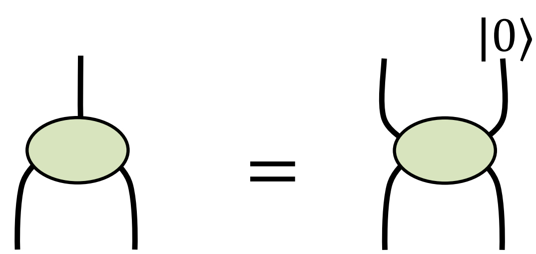
MERA as a quantum circuit





disentangler

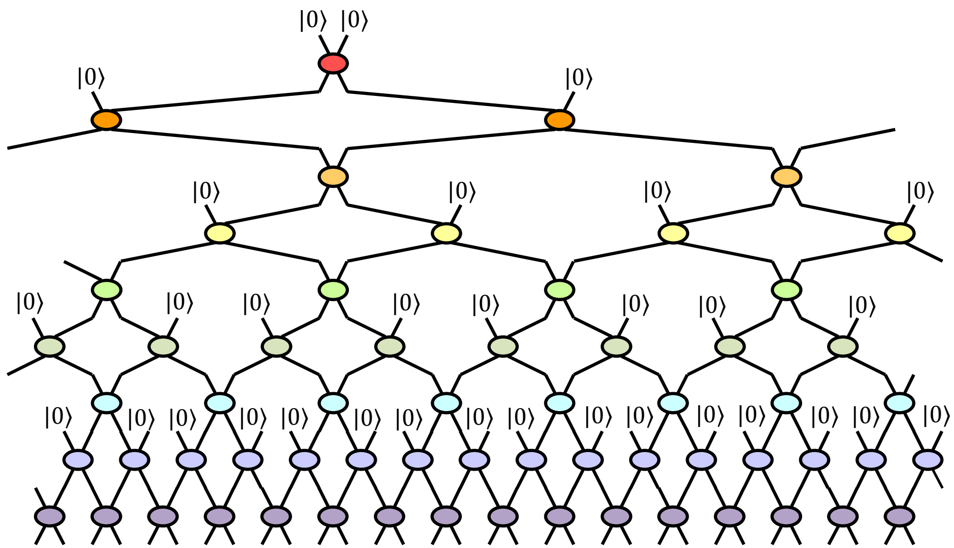
two-body unitary gate

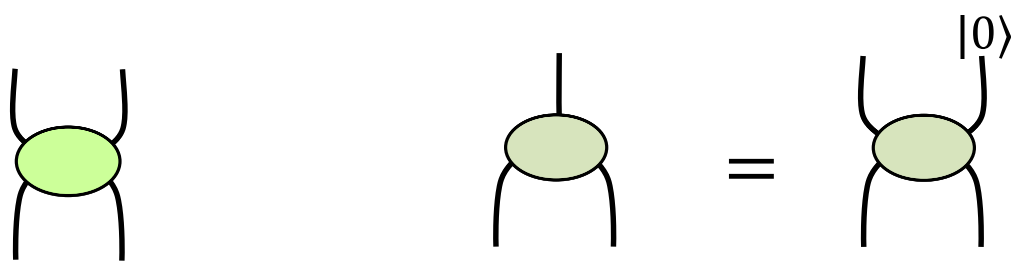


isometry

also a
two-body unitary gate

MERA as a quantum circuit





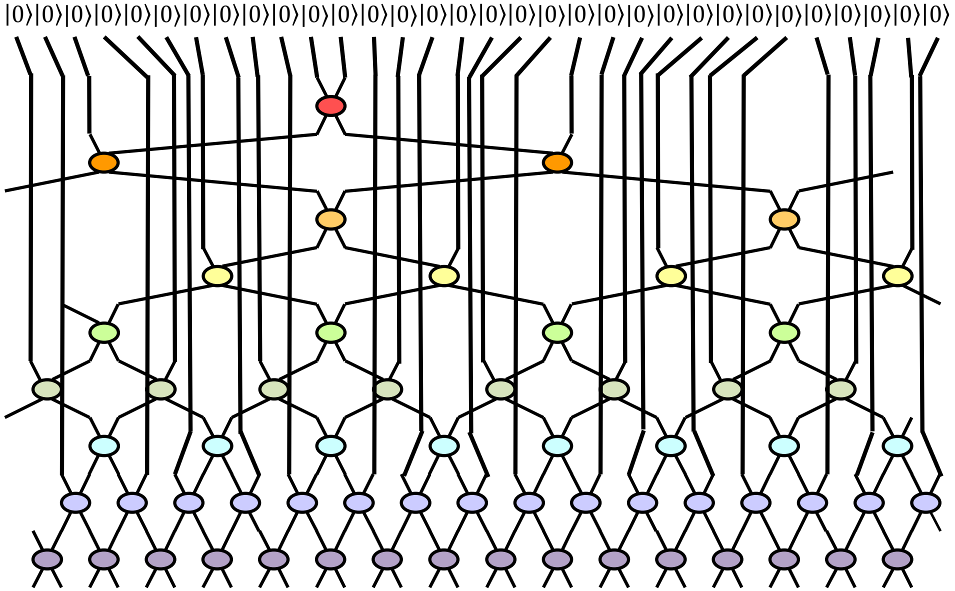
disentangler

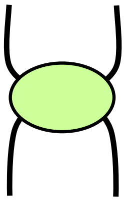
two-body unitary gate

isometry

also a
two-body unitary gate

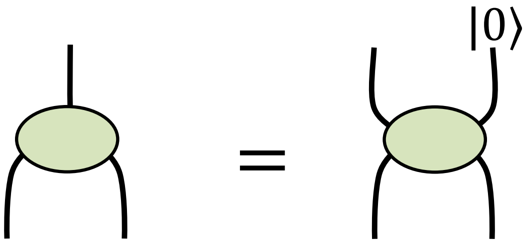
MERA as a quantum circuit





disentangler

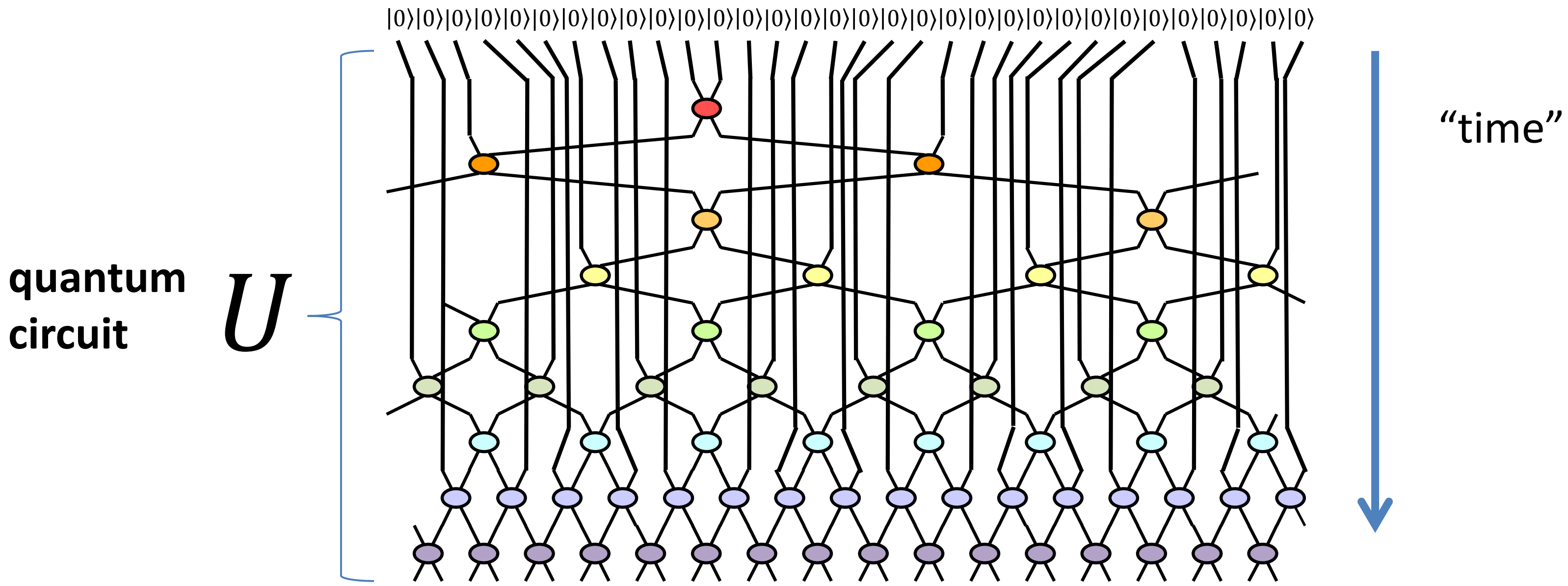
two-body unitary gate



isometry

also a
two-body unitary gate

MERA as a quantum circuit



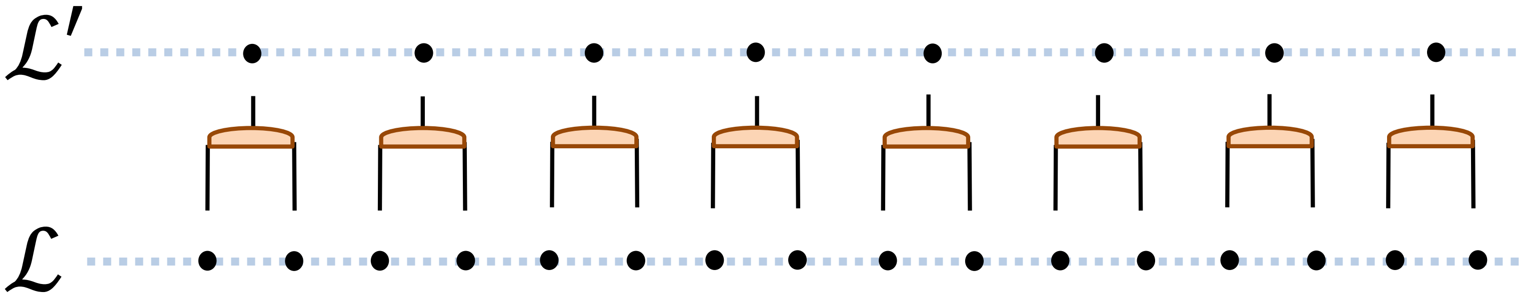
ground state ansatz $|\Psi\rangle = U\ |0\rangle^{\otimes N}$

Entanglement introduced by gates at different "times" (= length scales)

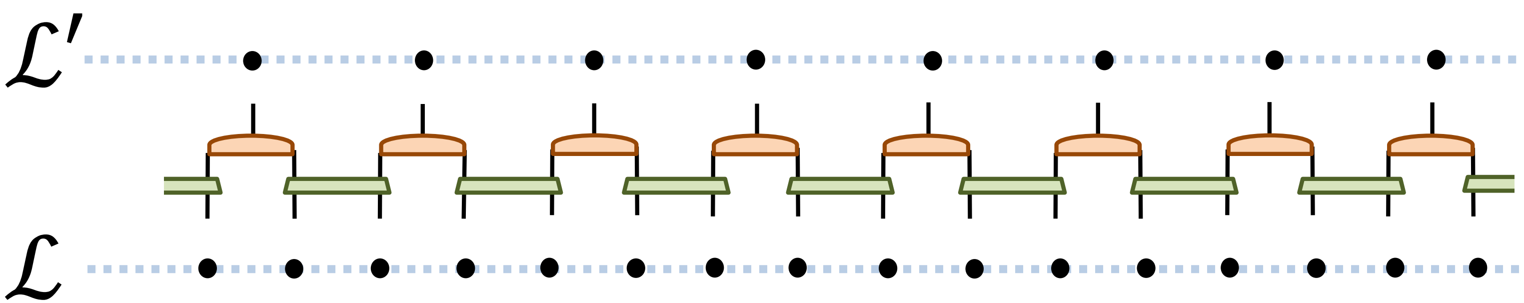
MERA as a (real space) Renormalizatin Group transformation

Kadanoff (1966) + White (1992)

blocking variational optimization



Entanglement renormalization (2005)

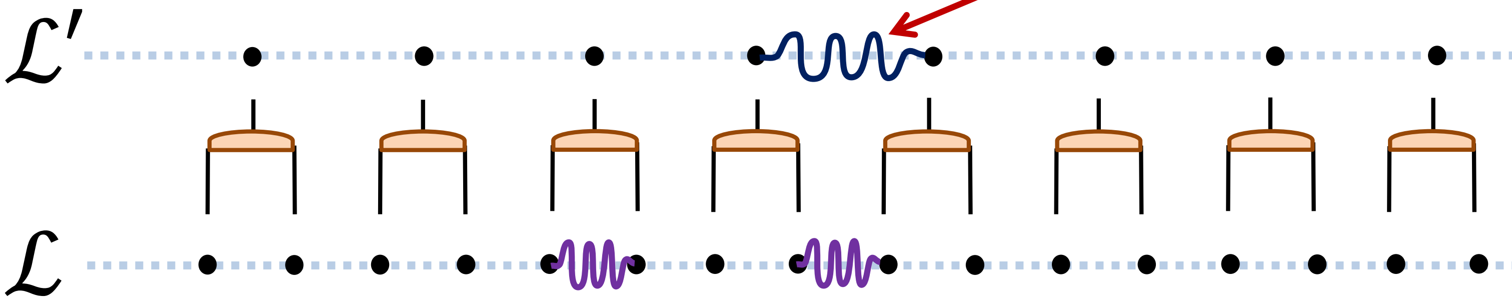


MERA as a (real space) Renormalizatin Group transformation

Kadanoff (1966) + White (1992)

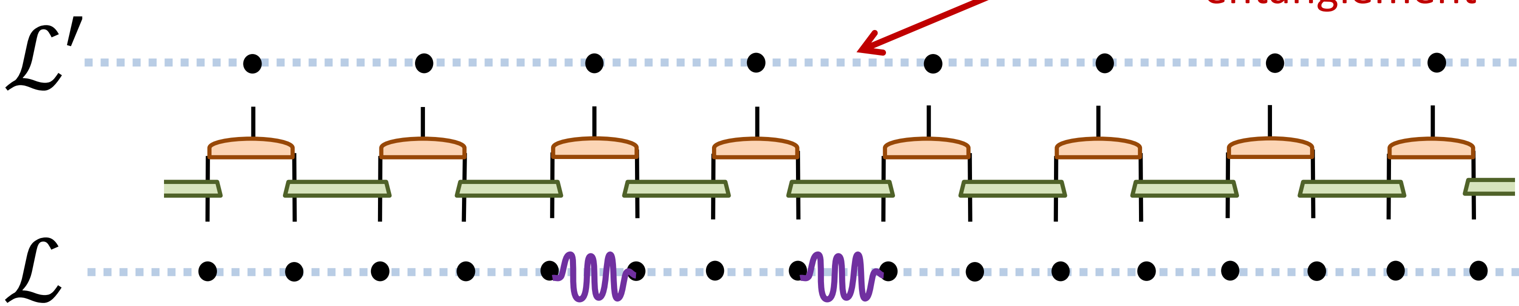
blocking variational optimization

failure to remove ***some*** short-range entanglement !



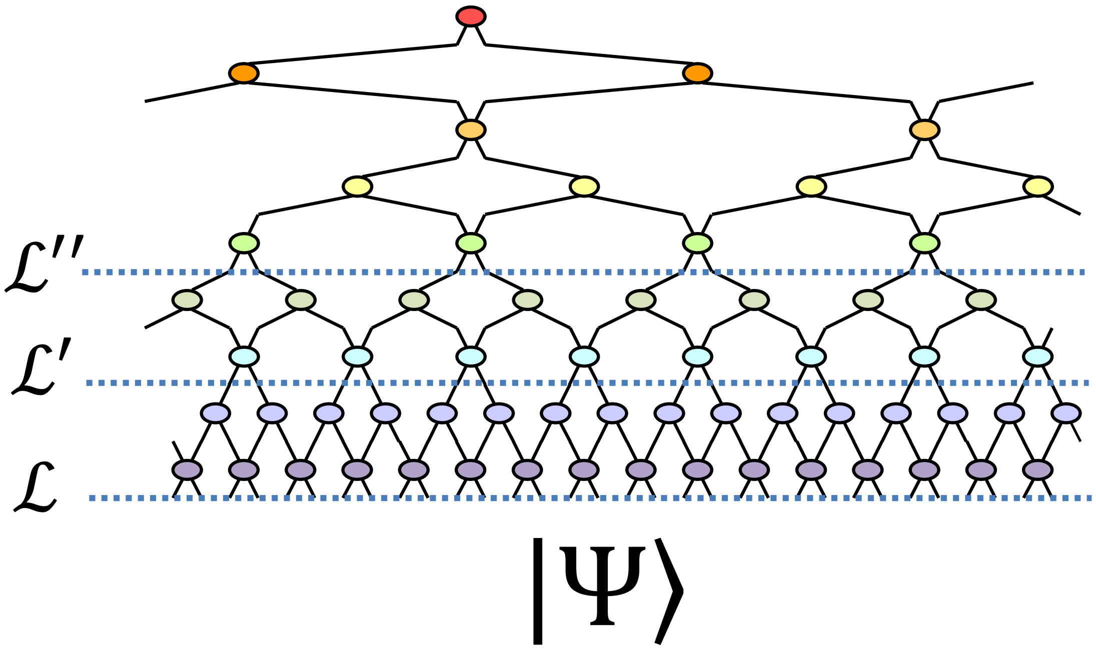
Entanglement renormalization (2005)

removal of ***all*** short-range entanglement



MERA as a (real space) Renormalizatin Group transformation

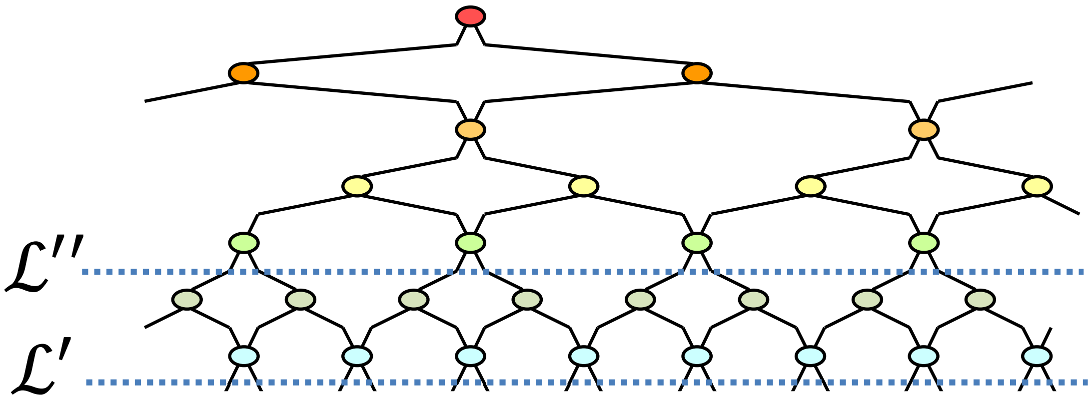
sequence of ground state wave-functions



$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$

MERA as a (real space) Renormalizatin Group transformation

sequence of ground state wave-functions

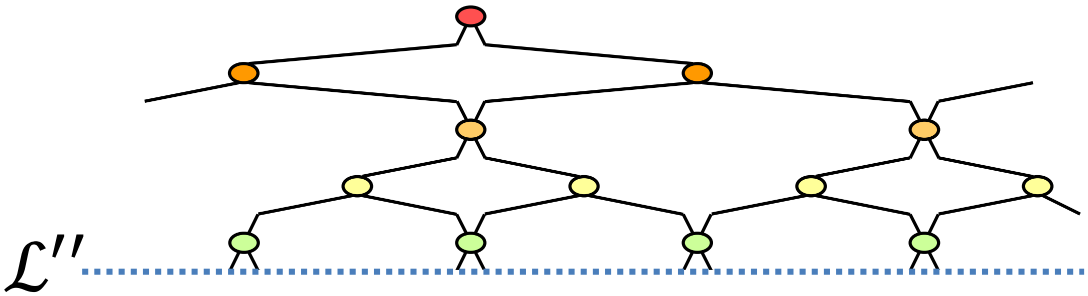


$$|\Psi'\rangle$$

$$|\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow \cdots$$

MERA as a (real space) Renormalizatin Group transformation

sequence of ground state wave-functions

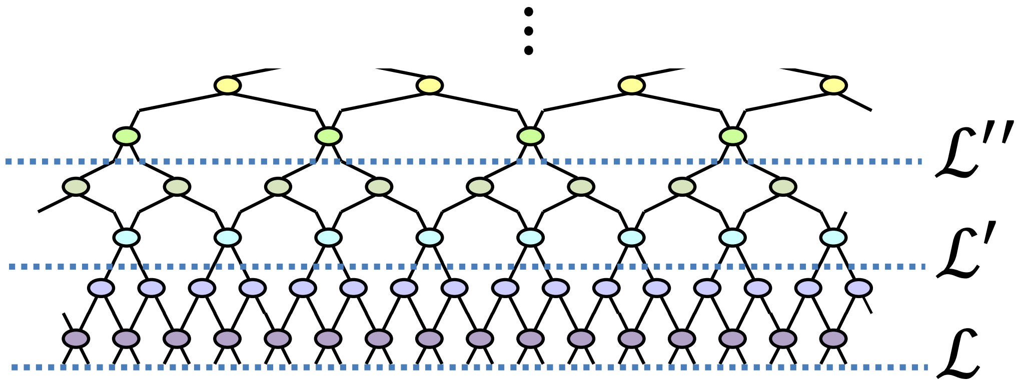


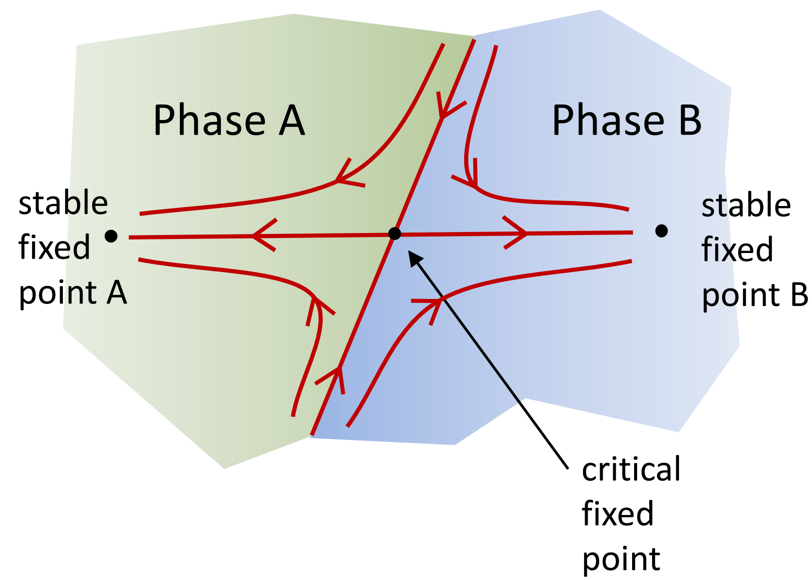
$$|\Psi''\rangle$$

$$|\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow \cdots$$

MERA defines an RG flow in the space of wave-functions

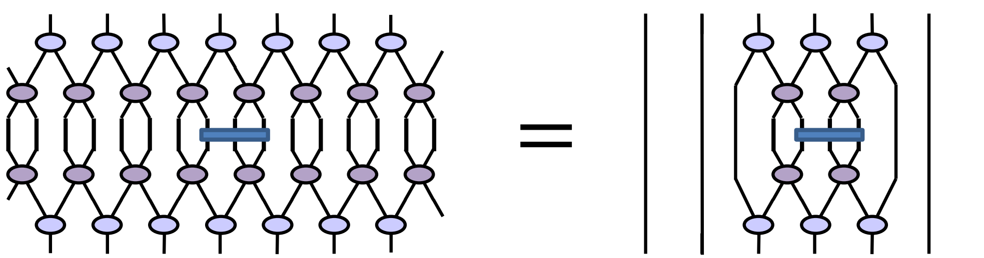
$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$





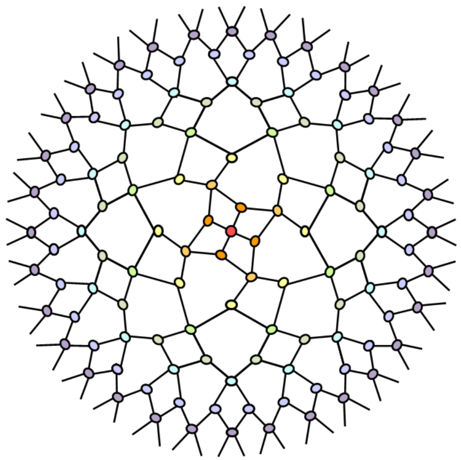
… and in the space of Hamiltonians

$$H \to H' \to H'' \to \cdots$$



local operators are mapped into local operators !

MERA and CFT



MERA

input

1D quantum Hamiltonian

- **on the lattice**
- **at a critical point**

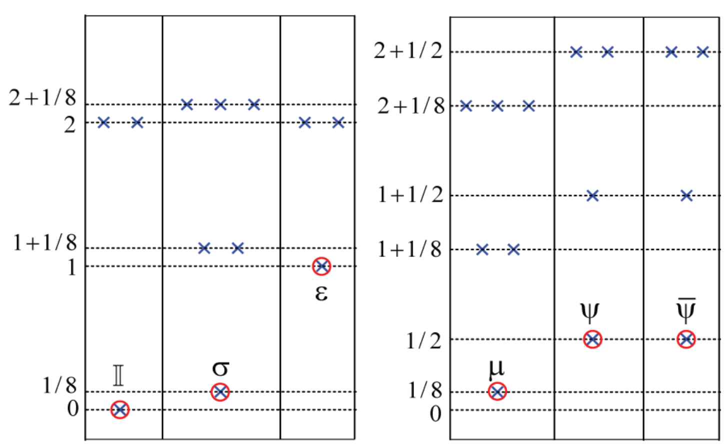
output

Numerical determination of conformal data:

- **central charge** c
- **scaling dimensions** $\Delta_\alpha \equiv h_\alpha + \bar{h}_\alpha$
  and conformal spins $s_\alpha \equiv h_\alpha - \bar{h}_\alpha$
- **OPE coefficients** $C_{\alpha\beta\gamma}$

e.g. critical Ising model (approx. an hour on your laptop)

Pfeifer, Evenbly, Vidal 08



$(\Delta_{\mathbb{I}} = 0)$

$\Delta_\sigma \approx 0.124997$

$\Delta_\varepsilon \approx 0.99993$

$\Delta_\mu \approx 0.125002$

$\Delta_\psi \approx 0.500001$

$\Delta_{\bar{\psi}} \approx 0.500001$

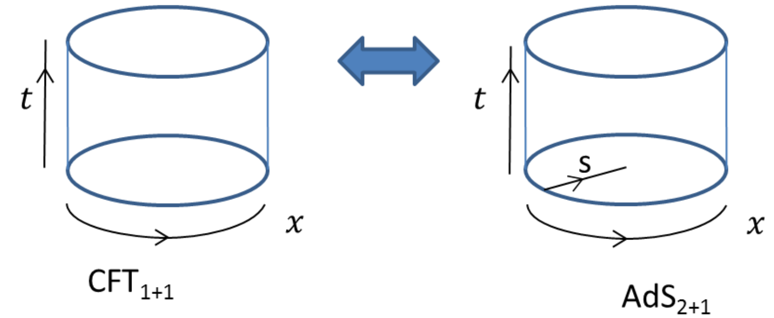
$C_{\epsilon\sigma\sigma} = \frac{1}{2}$ $\quad C_{\epsilon\mu\mu} = -\frac{1}{2}$

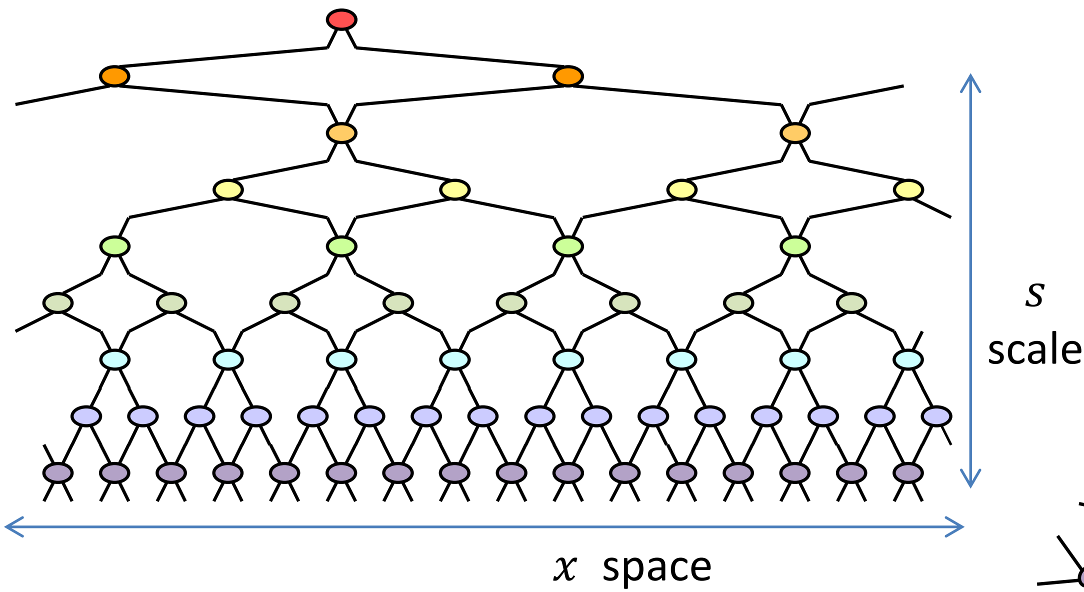
$C_{\epsilon\psi\bar{\psi}} = i$ $\quad C_{\epsilon\bar{\psi}\psi} = -i$

$C_{\psi\mu\sigma} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}$ $\quad C_{\bar{\psi}\mu\sigma} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}$

$(\pm 6 \times 10^{-4})$

MERA and holography?





- entanglement entropy

$$S_L \approx \log(L)$$

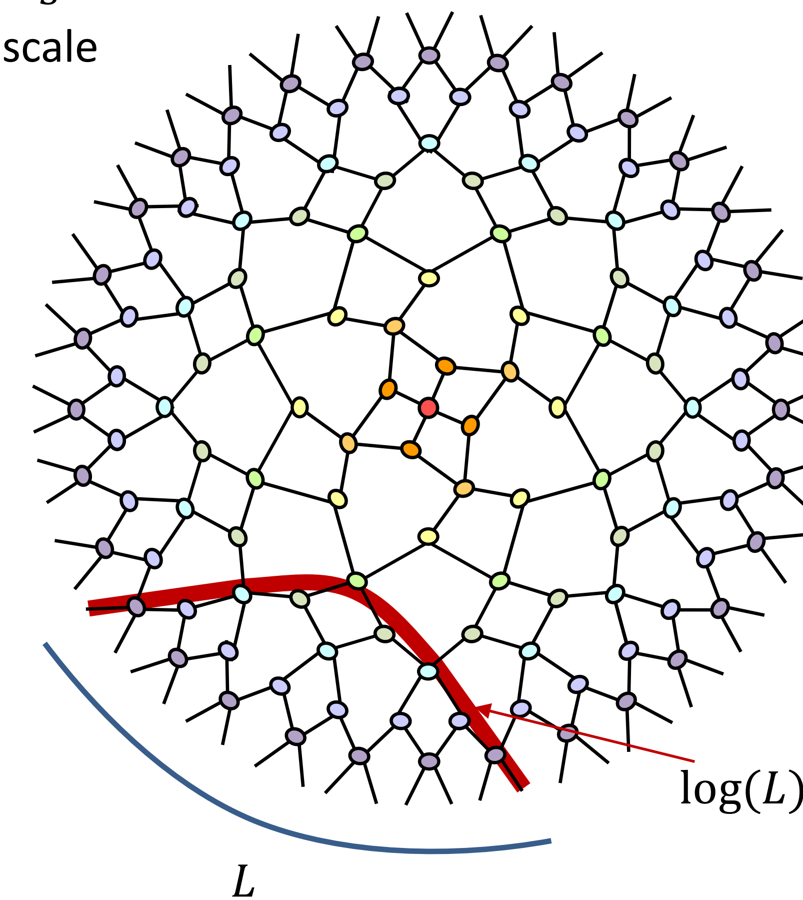
parallel to area of minimal surface in Ryu-Takayanagi

- two-point correlations

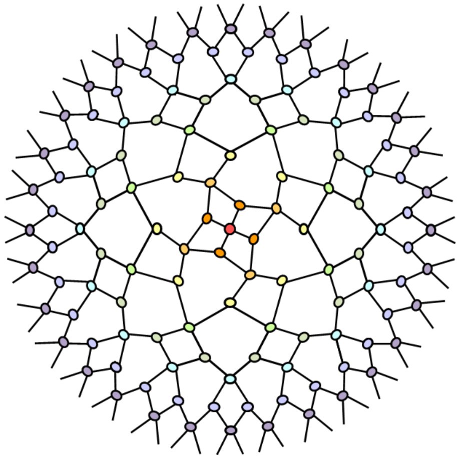
$$C(L) \approx L^{-2\Delta}$$

geodesic distance $D \approx \log(L)$ as in hyperbolic space

$$C(L) \approx e^{-D} = e^{-2\Delta \log(L)} = L^{-2\Delta}$$



MERA and holography?



MERA
(2005)

MERA ↔ AdS/CFT

Swingle, 2009

"Entanglement renormalization for quantum fields"
Haegeman, Osborne, Verschelde, Verstraete, 2011

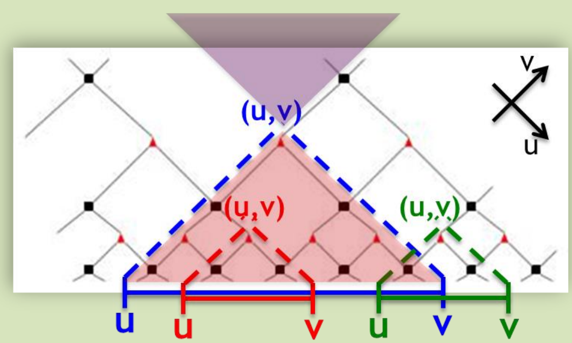
"Holographic Geometry of Entanglement Renormalization in Quantum Field Theories"
Nozaki, Ryu, Takayanagi, 2012

"Time Evolution of Entanglement Entropy from Black Hole Interiors"
Hartman, Maldacena, 2013

"Exact holographic mapping and emergent space-time geometry"
Xiaoliang Qi, 2013

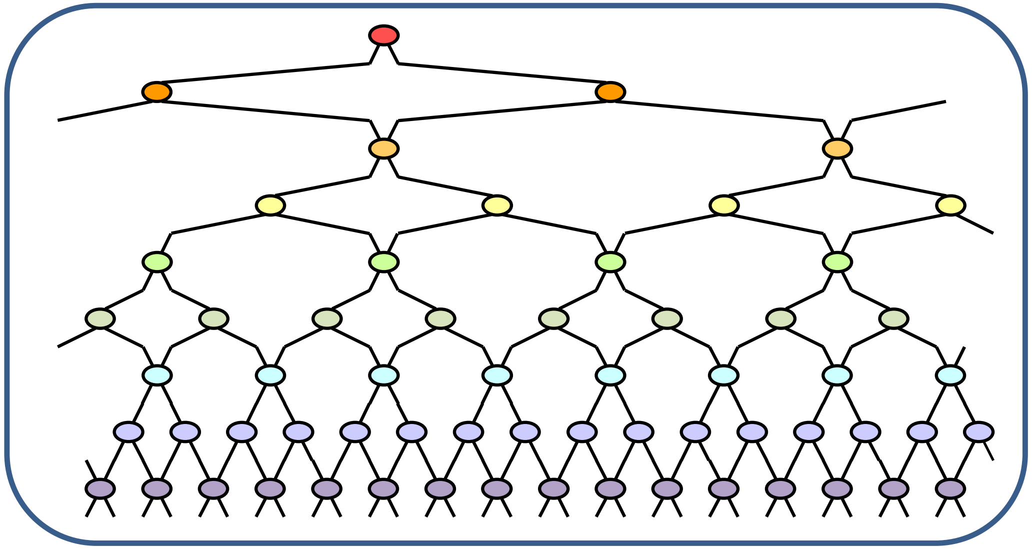
"Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence"
Pastawki, Yoshida, Harlow, Preskill, 2015

"Integral Geometry and Holography"
Czech, Lamprou, McCandlish, Sully, 2015



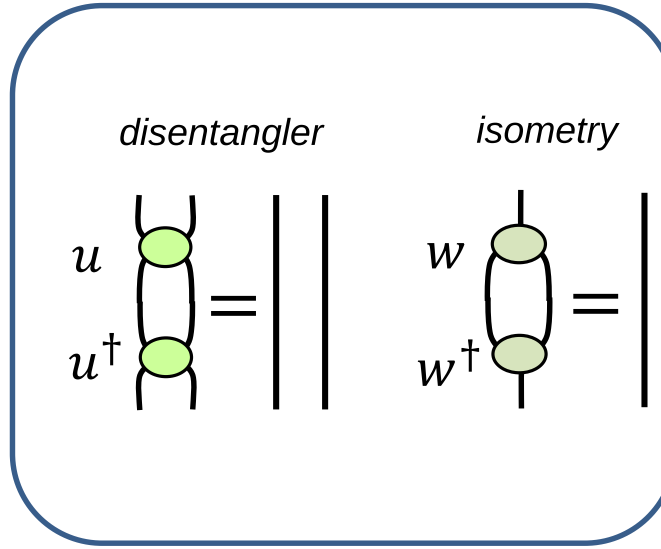
tion
code
Coarse -graining
(Dis)entangler

MERA = tensor network + isometric/unitary constraints



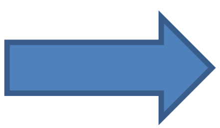
~ hyperbolic plane?

(Swingle 2009)



~ de Sitter space?

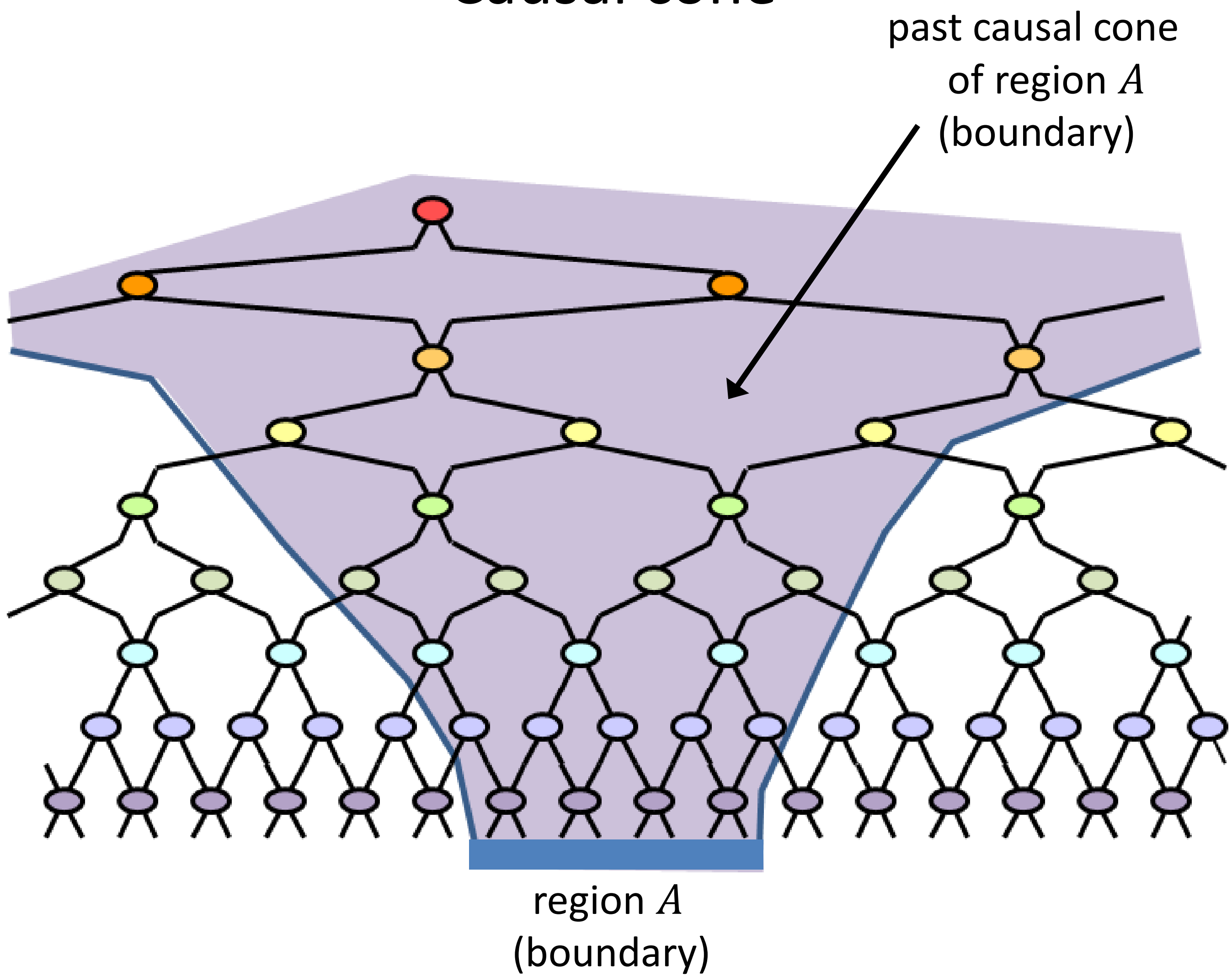
(Beny 2011, Czech 2015)



Causal structure

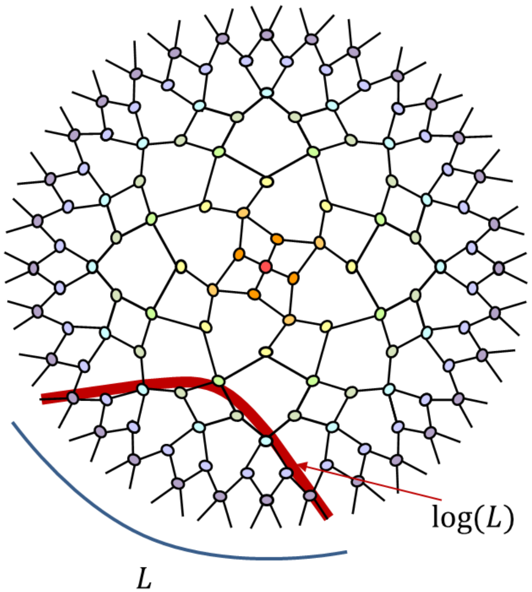
essential for many MERA properties
and computational efficiency

Causal cone



MERA = RG

Tensor network for ground state/Hilbert space of CFT, organized in extra dimension corresponding to scale



generic CFT (no large N, strong interactions)

e.g. for Ising model

dictionary

boundary	bulk
state of CFT (or Hilbert space of CFT)	tensor network (?) (or unitary map, EHM)
scaling dimension Δ	mass $\sim\Delta$
entanglement entropy	"minimal connecting surface"
global on-site symmetry (e.g. Z_2)	local/gauge symmetry (e.g. Z_2)

MERA operates at scale of AdS radius
For smaller scale? → cMERA

Useful test bed

Generalized notion of *holographic* description?