```
<< Local`QFTToolKit`

$def = {};
ct[a_] := ConjugateTranspose[a];
PR[CO[
  "We use equivalence symbol for isomorphism, and Mod[] symbol for quotient group?"]
]
```

We use equivalence symbol for isomorphism, and Mod[] symbol for quotient group?

```
PR["● M→4-d manifold with canonical triple ", {C"∞"[M], L²[M, S], slash[D]},
 NL, "The connection: ", $connection = "∇"^S[S[]],
 NL, "Dirac operator: ",
 {slash[D][ψ_] → -I T[γ, "u", {μ}].T["∇"^S, "d", {μ}][ψ], ψ ∈ Γ[M, S],
   T["∇"^S, "d", {μ}][f ψ] → f "∇"^S[ψ] + tuPartialD[f, μ] ψ,
   CommutatorM[slash[D], f].ψ → -I T[γ, "u", {μ}]. tuPartialD[f, μ].ψ
  } // Column,
 NL, "Have ℤ2-grading(chirality): ",
 {T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}], T[γ, "d", {5}].T[γ, "d", {5}] → 1,
   ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
   T[γ, "d", {5}][L²[M, S]] -> L²[M, S]⁺ ⊕ L²[M, S]⁻} // Column,
 NL, "Charge conjugation: ", J_M → {J_M.J_M → -1, CommutatorM[J_M, slash[D]] → 0,
    CommutatorM[J_M, T[γ, "d", {5}]] → 0} // ColumnForms
]
```

● M→4-d manifold with canonical triple $\{C^\infty[M], L^2[M, S], \not{D}\}$

The connection: $\nabla^S[S[]]$

Dirac operator:
$$\begin{array}{l} (\not{D})[\psi\_] \to -i\,\gamma^\mu.\nabla^S{}_\mu[\psi] \\ \psi \in \Gamma[M, S] \\ \nabla^S{}_\mu[f\,\psi] \to f\,\nabla^S[\psi] + \psi\,\underset{-\mu}{\partial}[f] \\ [\not{D}, f].\psi \to -i\,\gamma^\mu.\underset{-\mu}{\partial}[f].\psi \end{array}$$

Have $\mathbb{Z}_2$-grading(chirality):
$$\begin{array}{l} \gamma_5 \to \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4 \\ \gamma_5.\gamma_5 \to 1 \\ (\gamma_5)^\dagger \to \gamma_5 \\ \gamma_5[L^2[M, S]] \to L^2[M, S]^+ \oplus L^2[M, S]^- \end{array}$$

Charge conjugation: $J_M \to \begin{array}{l} J_M.J_M \to -1 \\ [J_M, \not{D}] \to 0 \\ [J_M, \gamma_5] \to 0 \end{array}$

```
PR["● F→finite space triple: ", F → {𝒜F, ℋF, 𝒟F},
 " where ", {𝒜F → MN[ℂ], ℋF → "N-dim complex Hilbert space",
   𝒟F -> "hermitian MN[ℂ]", MN[ℂ] → "NxN matrix"} // Column,
 NL, "•ℋF is ℤ2 graded (even) if ∃ a grading operator: ",
 γF ∋ {ConjugateTranspose[γF] → γF, γF γF → 1, γF[ℋF] -> ℋF+ ⊕ ℋF-,
     {γF[ψ ∈ ℋF] → ±ψ},
     CommutatorM[γF, a ∈ AF] → 0,
     CommutatorP[γF, 𝒟F] → 0
    } // ColumnForms
]
```

● F→finite space triple: $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F\}$ where
$\mathcal{A}_F \to M_N[\mathbb{C}]$
$\mathcal{H}_F \to$ N-dim complex Hilbert space
$\mathcal{D}_F \to$ hermitian $M_N[\mathbb{C}]$
$M_N[\mathbb{C}] \to$ NxN matrix

•$\mathcal{H}_F$ is $\mathbb{Z}_2$ graded (even) if ∃ a grading operator: $\gamma_F \ni$
$(\gamma_F)^\dagger \to \gamma_F$
$\gamma_F^2 \to 1$
$\gamma_F[\mathcal{H}_F] \to (\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^-$
$\{\gamma_F[\psi \in \mathcal{H}_F] \to \pm\psi\}$
$[\gamma_F, a \in A_F] \to 0$
$\{\gamma_F, \mathcal{D}_F\} \to 0$

```
εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
     table =
      {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1, }}},
    {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
   ];
PR["Almost-commutative spin manifold: ",
 $ = M × F → {C"∞"[M, 𝒜F], L2[M, S] ⊗ ℋF, 𝒟 -> slash[𝒟] ⊗ 1N + T[γ, "d", {5}] ⊗ 𝒟F};
 ColumnForms[$],
 NL, "with grading: ", γ -> T[γ, "d", {5}] ⊗ γF,
 NL, "•Distance: ", d𝒟[x, y] → sup[‖a[x] - a[y]‖ | a ∈ 𝒜 && ‖CommutatorM[𝒟, a]‖ ≤ 1],
 NL, "●Charge conjugation for F: even space F is real if ∃ ",
 $J = JF[ℋF] ∋ { JF. JF -> ε, JF.𝒟F -> ε'. 𝒟F.JF, JF.γF -> ε''.γF.JF};
 ColumnForms[$J],
 NL, "where the routine εRule[KOdim_] is provided ",
 CR[" What is the meaning of ε's?"],
 NL, "•", $ = ForAll[{a, b}, a | b ∈ 𝒜F,
   {CommutatorM[a, b"0"] → 0, b"0" → JF.ConjugateTranspose[b].ConjugateTranspose[JF]}],
 $def = $def // tuAppendUniq[$];
 NL, "•", $ = ForAll[{a, b}, a | b ∈ 𝒜F, {CommutatorM[CommutatorM[𝒟F, a], b"0"] → 0,
     b"0" → JF.ConjugateTranspose[b].ConjugateTranspose[JF]}],
 $def = $def // tuAppendUniq[$];
]
```

Almost-commutative spin manifold: $M \times F \to$
$C^\infty[M, \mathcal{A}_F]$
$L^2[M, S] \otimes \mathcal{H}_F$
$\mathcal{D} \to (\not{D}) \otimes 1_N + \gamma_5 \otimes \mathcal{D}_F$

with grading: $\gamma \to \gamma_5 \otimes \gamma_F$
•Distance: $d_{\mathcal{D}}[x, y] \to \sup[\|a[x] - a[y]\| \mid a \in \mathcal{A} \,\&\&\, \|[\mathcal{D}, a]\| \le 1]$

●Charge conjugation for F: even space F is real if ∃ $J_F[\mathcal{H}_F] \ni$
$J_F.J_F \to \varepsilon$
$J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F$
$J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F$

where the routine εRule[KOdim_] is provided What is the meaning of ε's?
•$\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$
•$\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[[\mathcal{D}_F, a], b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$Null

```
PR["●Lemma2.7. Definition 2.5: ", $J[[2]],
 NL, "Where γF decomposes ", $h = ℋ → Table[ℋi,j, {i, 2}, {j, 2}];
 MatrixForms[$h],
```

```
  " into ", ℋ → ℋ⁺ ⊕ ℋ⁻ , " i.e. ", $gh = γF.ℋ → {{ℋ⁺, 0}, {0, ℋ⁻}};
  MatrixForms[$gh],
  $gh0 = $gh /. {ℋ⁺ -> ℋ1,1, ℋ⁻ -> ℋ2,2};
  Yield, $gh1 = γF.{{a_, b_}, {c_, d_}} -> DiagonalMatrix[{a, d}];
  MatrixForms[$gh1],
  NL, "Represent ", $j = JF → Table[ji,j, {i, 2}, {j, 2}];
  MatrixForms[$j], " of the same dimensions.",
  NL, "•For: ",
  $JF = {JF → U.cc, U.ConjugateTranspose[U] → 1N, U ∈ U[ℋ"±"], cc → Conjugate},
  NL, "where: ",
  $cc = {ConjugateTranspose[cc] → cc,
    Conjugate[cc] → cc, cc . cc → 1, cc.a_ :→ Conjugate[a].cc},
  Imply, $0 = $ = JF.ConjugateTranspose[JF],
  yield, $ = $0 → ($ /. $JF[[1]] // tuRepeat[$cc, ConjugateCTSimplify1[{cc}]]);
  Framed[$];
  $ = $ /. $JF[[2]]; Framed[$],
  Yield, $ = $ /. ConjugateTranspose → SuperDagger /. Dot → xDot /. $j /.
    SuperDagger[a_] :→ Map[Thread[SuperDagger[#]] &, Transpose[a]] /; MatrixQ[a];
  MatrixForms[$],
  Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
  Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
  Yield, $JJ = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
  FramedColumn[$JJ], CK
 ]
 PR[
  line, "•For ", $s = n → 0; Framed[$s],
  yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
  Column[$1],
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = # . ℋ & /@ $, "POFF",
  Yield, $ = $ /. $gh0;
  Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] → γF.xDot[a];
  Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$];
  Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$];
  Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
  Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
  Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
  NL, "•For ", $ = $1[[1]] /. 1 → 1N; Framed[$],
  Yield, $ = $ /. Dot → xDot /. $j,
  Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
  Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
  Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
  FramedColumn[$JJ1], CK,
  NL, "•Then we have: ", $ = {$JJ1, $JJ, $Jg}; ColumnForms[$],
  Yield, $ = $ /. j1,2 | j2,1 → 0 // ConjugateCTSimplify1[{}]; ColumnForms[$],
  Imply, {ConjugateTranspose[j1,1] -> j1,1, ConjugateTranspose[j2,2] -> j2,2} // FramedColumn
 ]
 PR[
  line, "•For ", $s = n → 2; Framed[$s],
  yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
  Column[$1],
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = # . ℋ & /@ $, "POFF",
  Yield, $ = $ /. $gh0 // tuDotSimplify[],
  Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
    γF.xDot[a], CK,
  Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
```

```
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply,
 {ConjugateTranspose[j1,2] → -j2,1, ConjugateTranspose[j2,1] → -j1,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 4; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
    γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. -1 → -1N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1N → {{1N+, 0}, {0, 1N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,2 | j2,1 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply,
 {ConjugateTranspose[j1,1] → -j1,1, ConjugateTranspose[j2,2] → -j2,2} // FramedColumn
]
PR[
 line, "•For ", $s = n → 6; Framed[$s],
 yield, $1 = $J[[2]] /. εRule[$s[[2]]] // tuDotSimplify[] // Delete[#, 2] &;
 Column[$1],
 Yield, $ = $1[[2]]; Framed[$],
 yield, $ = # . ℋ & /@ $, "POFF",
 Yield, $ = $ /. $gh0 // tuDotSimplify[],
 Yield, $ = $ /. Dot → xDot /. xDot[γF, a__] →
    γF.xDot[a], CK,
 Yield, $ = $ /. $j // MapAt[# /. $h &, #, 2] &; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
```

```
 Yield, $ = $ /. $gh1; MatrixForms[$]; "PONdd",
 Yield, $ = MapThread[Rule, {$[[1]], $[[2]]}, 2]; MatrixForms[$],
 Yield, $Jg = $ // Flatten; FramedColumn[$Jg], CK,
 NL, "•For ", $ = $1[[1]] /. 1 → 1_N; Framed[$],
 Yield, $ = $ /. Dot → xDot /. $j,
 Yield, $ = $ // OrderedxDotMultiplyAll[{}]; MatrixForms[$],
 Yield, $ = $ /. 1_N → {{1_N+, 0}, {0, 1_N-}},
 Yield, $JJ1 = MapThread[Rule, {$[[1]], $[[2]]}, 2] // Flatten;
 FramedColumn[$JJ1], CK,
 NL, "with: ", $sh = {ℋ1,2 | ℋ2,1 → 0},
 NL, "•All conditions: ", $ = {$JJ1, $JJ, $Jg} /. $sh // tuDotSimplify[];
 ColumnForms[$],
 Imply, $s = j1,1 | j2,2 → 0; Framed[$s],
 Yield, $ = $ /. $s // ConjugateCTSimplify1[{}]; ColumnForms[$],
 Imply, {ConjugateTranspose[j1,2] → j2,1, ConjugateTranspose[j2,1] → j1,2} // FramedColumn
]
```

●Lemma2.7. Definition 2.5: $\{J_F.J_F \to \varepsilon,\ J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F,\ J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F\}$

Where γ_F decomposes $\mathcal{H} \to \begin{pmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} \end{pmatrix}$ into $\mathcal{H} \to \mathcal{H}^+ \oplus \mathcal{H}^-$ i.e. $\gamma_F.\mathcal{H} \to \begin{pmatrix} \mathcal{H}^+ & 0 \\ 0 & \mathcal{H}^- \end{pmatrix}$

$\to\ \gamma_F.\begin{pmatrix} a\_ & b\_ \\ c\_ & d\_ \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Represent $J_F \to \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}$ of the same dimensions.

•For: $\{J_F \to U.cc,\ U.U^\dagger \to 1_N,\ U \in U[\mathcal{H}^\pm],\ cc \to \text{Conjugate}\}$
where: $\{cc^\dagger \to cc,\ cc^* \to cc,\ cc.cc \to 1,\ cc.(a\_) \mapsto a^*.cc\}$

$\Rightarrow\ J_F.(J_F)^\dagger \longrightarrow \boxed{J_F.(J_F)^\dagger \to 1_N}$

$\to\ \text{xDot}\left[\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}, \begin{pmatrix} (j_{1,1})^\dagger & (j_{2,1})^\dagger \\ (j_{1,2})^\dagger & (j_{2,2})^\dagger \end{pmatrix}\right] \to 1_N$

$\to\ \begin{pmatrix} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger & j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger & j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \end{pmatrix} \to 1_N$

$\to\ \{\{j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger,\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger\},$
$\{j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger,\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger\}\} \to \{\{1_{N^+}, 0\}, \{0, 1_{N^-}\}\}$

$$\to\ \boxed{\begin{array}{l} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \to 0 \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger \to 0 \\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \to 1_{N^-} \end{array}} \longleftarrow \text{CHECK}$$

•For $\boxed{n \to 0}$ $\longrightarrow$ $\begin{matrix} J_F \cdot J_F \to 1 \\ J_F \cdot \gamma_F \to \gamma_F \cdot J_F \end{matrix}$

$\to$ $\boxed{J_F \cdot \gamma_F \to \gamma_F \cdot J_F}$ $\longrightarrow$ $J_F \cdot \gamma_F \cdot \mathcal{H} \to \gamma_F \cdot J_F \cdot \mathcal{H}$

....

$\to \begin{pmatrix} j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} + j_{1,2} \cdot \mathcal{H}_{2,1} & j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 & j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,1} \cdot \mathcal{H}_{1,2} + j_{2,2} \cdot \mathcal{H}_{2,2} \end{pmatrix}$

$\to \boxed{\begin{matrix} j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} + j_{1,2} \cdot \mathcal{H}_{2,1} \\ j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,1} \cdot \mathcal{H}_{1,2} + j_{2,2} \cdot \mathcal{H}_{2,2} \end{matrix}}$ ⟵CHECK

•For $\boxed{J_F \cdot J_F \to 1_N}$

$\to$ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] $\to 1_N$

$\to \begin{pmatrix} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} & j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} & j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \end{pmatrix} \to 1_N$

$\to$

$\{\{j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1},\ j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2}\},\ \{j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1},\ j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2}\}\} \to \{\{1_{N^+}, 0\}, \{0, 1_{N^-}\}\}$

$\to \boxed{\begin{matrix} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} \to 1_{N^+} \\ j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \to 0 \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} \to 0 \\ j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \to 1_{N^-} \end{matrix}}$ ⟵CHECK

•Then we have:

$$\left\{ \begin{matrix} j_{1,1} \cdot j_{1,1} + j_{1,2} \cdot j_{2,1} \to 1_{N^+} \\ j_{1,1} \cdot j_{1,2} + j_{1,2} \cdot j_{2,2} \to 0 \\ j_{2,1} \cdot j_{1,1} + j_{2,2} \cdot j_{2,1} \to 0 \\ j_{2,1} \cdot j_{1,2} + j_{2,2} \cdot j_{2,2} \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} \cdot (j_{1,1})^\dagger + j_{1,2} \cdot (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} \cdot (j_{2,1})^\dagger + j_{1,2} \cdot (j_{2,2})^\dagger \to 0 \\ j_{2,1} \cdot (j_{1,1})^\dagger + j_{2,2} \cdot (j_{1,2})^\dagger \to 0 \\ j_{2,1} \cdot (j_{2,1})^\dagger + j_{2,2} \cdot (j_{2,2})^\dagger \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} + j_{1,2} \cdot \mathcal{H}_{2,1} \\ j_{1,2} \cdot \mathcal{H}_{2,2} \to 0 \\ j_{2,1} \cdot \mathcal{H}_{1,1} \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,1} \cdot \mathcal{H}_{1,2} + j_{2,2} \cdot \mathcal{H}_{2,2} \end{matrix} \right\}$$

$$\to \left\{ \begin{matrix} j_{1,1} \cdot j_{1,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} \cdot j_{2,2} \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} \cdot (j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} \cdot (j_{2,2})^\dagger \to 1_{N^-} \end{matrix},\ \begin{matrix} j_{1,1} \cdot \mathcal{H}_{1,1} \to j_{1,1} \cdot \mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} \cdot \mathcal{H}_{2,2} \to j_{2,2} \cdot \mathcal{H}_{2,2} \end{matrix} \right\}$$

$\Rightarrow \boxed{\begin{matrix} (j_{1,1})^\dagger \to j_{1,1} \\ (j_{2,2})^\dagger \to j_{2,2} \end{matrix}}$

•For $\boxed{n \to 2}$ ⟶ $J_F . J_F \to -1$
$J_F . \gamma_F \to -\gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to -\gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to (-\gamma_F . J_F) . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} - j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,1} . \mathcal{H}_{1,2} - j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to -1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → -1_N

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to -1_N$

→ $\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1},\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\},\ \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1},\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to$
$\{\{-1_{N^+},\ 0\},\ \{0,\ -1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}}$ ⟵CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions: $\left\{ \begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array},\ \begin{array}{l} j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \end{array},\ \begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to -j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to -j_{2,2} . \mathcal{H}_{2,2} \end{array} \right\}$

⇒ $\boxed{j_{1,1} \mid j_{2,2} \to 0}$

→ $\left\{ \begin{array}{l} j_{1,2} . j_{2,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . j_{1,2} \to -1_{N^-} \end{array},\ \begin{array}{l} j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1} . (j_{2,1})^\dagger \to 1_{N^-} \end{array},\ \begin{array}{l} 0 \to 0 \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{array} \right\}$

⇒ $\boxed{\begin{array}{l} (j_{1,2})^\dagger \to -j_{2,1} \\ (j_{2,1})^\dagger \to -j_{1,2} \end{array}}$

•For $\boxed{n \to 4}$ ⟶ $J_F . J_F \to -1$
$J_F . \gamma_F \to \gamma_F . J_F$

→ $\boxed{J_F . \gamma_F \to \gamma_F . J_F}$ ⟶ $J_F . \gamma_F . \mathcal{H} \to \gamma_F . J_F . \mathcal{H}$

....

→ $\begin{pmatrix} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} & j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 & j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{pmatrix}$

→ $\boxed{\begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} + j_{1,2} . \mathcal{H}_{2,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,1} . \mathcal{H}_{1,2} + j_{2,2} . \mathcal{H}_{2,2} \end{array}}$ ⟵CHECK

•For $\boxed{J_F . J_F \to -1_N}$

→ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] → -1_N

→ $\begin{pmatrix} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} & j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} & j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \end{pmatrix} \to -1_N$

→ $\{\{j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1},\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2}\},\ \{j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1},\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2}\}\} \to$
$\{\{-1_{N^+},\ 0\},\ \{0,\ -1_{N^-}\}\}$

→ $\boxed{\begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array}}$ ⟵CHECK

with: $\{\mathcal{H}_{1,2} \mid \mathcal{H}_{2,1} \to 0\}$

•All conditions: $\left\{ \begin{array}{l} j_{1,1} . j_{1,1} + j_{1,2} . j_{2,1} \to -1_{N^+} \\ j_{1,1} . j_{1,2} + j_{1,2} . j_{2,2} \to 0 \\ j_{2,1} . j_{1,1} + j_{2,2} . j_{2,1} \to 0 \\ j_{2,1} . j_{1,2} + j_{2,2} . j_{2,2} \to -1_{N^-} \end{array},\ \begin{array}{l} j_{1,1} . (j_{1,1})^\dagger + j_{1,2} . (j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1} . (j_{2,1})^\dagger + j_{1,2} . (j_{2,2})^\dagger \to 0 \\ j_{2,1} . (j_{1,1})^\dagger + j_{2,2} . (j_{1,2})^\dagger \to 0 \\ j_{2,1} . (j_{2,1})^\dagger + j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \end{array},\ \begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ j_{1,2} . \mathcal{H}_{2,2} \to 0 \\ j_{2,1} . \mathcal{H}_{1,1} \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{array} \right\}$

⇒ $\boxed{j_{1,2} \mid j_{2,1} \to 0}$

→ $\left\{ \begin{array}{l} j_{1,1} . j_{1,1} \to -1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . j_{2,2} \to -1_{N^-} \end{array},\ \begin{array}{l} j_{1,1} . (j_{1,1})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . (j_{2,2})^\dagger \to 1_{N^-} \end{array},\ \begin{array}{l} j_{1,1} . \mathcal{H}_{1,1} \to j_{1,1} . \mathcal{H}_{1,1} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,2} . \mathcal{H}_{2,2} \to j_{2,2} . \mathcal{H}_{2,2} \end{array} \right\}$

⇒ $\boxed{\begin{array}{l} (j_{1,1})^\dagger \to -j_{1,1} \\ (j_{2,2})^\dagger \to -j_{2,2} \end{array}}$

•For $\boxed{n \to 6}$ $\longrightarrow$ $J_F.J_F \to 1$
$J_F.\gamma_F \to -\gamma_F.J_F$

$\to$ $\boxed{J_F.\gamma_F \to -\gamma_F.J_F}$ $\longrightarrow$ $J_F.\gamma_F.\mathcal{H} \to (-\gamma_F.J_F).\mathcal{H}$

....

$\to$ $\begin{pmatrix} j_{1,1}.\mathcal{H}_{1,1} \to -j_{1,1}.\mathcal{H}_{1,1} - j_{1,2}.\mathcal{H}_{2,1} & j_{1,2}.\mathcal{H}_{2,2} \to 0 \\ j_{2,1}.\mathcal{H}_{1,1} \to 0 & j_{2,2}.\mathcal{H}_{2,2} \to -j_{2,1}.\mathcal{H}_{1,2} - j_{2,2}.\mathcal{H}_{2,2} \end{pmatrix}$

$\to$ $\boxed{\begin{array}{l} j_{1,1}.\mathcal{H}_{1,1} \to -j_{1,1}.\mathcal{H}_{1,1} - j_{1,2}.\mathcal{H}_{2,1} \\ j_{1,2}.\mathcal{H}_{2,2} \to 0 \\ j_{2,1}.\mathcal{H}_{1,1} \to 0 \\ j_{2,2}.\mathcal{H}_{2,2} \to -j_{2,1}.\mathcal{H}_{1,2} - j_{2,2}.\mathcal{H}_{2,2} \end{array}}$ $\longleftarrow$CHECK

•For $\boxed{J_F.J_F \to 1_N}$

$\to$ xDot[{{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}, {{$j_{1,1}$, $j_{1,2}$}, {$j_{2,1}$, $j_{2,2}$}}] $\to 1_N$

$\to$ $\begin{pmatrix} j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1} & j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2} \\ j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1} & j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2} \end{pmatrix} \to 1_N$

$\to$
{{$j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1}$, $j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2}$}, {$j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1}$, $j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2}$}} $\to$ {{1_{N^+}, 0}, {0, 1_{N^-}}}

$\to$ $\boxed{\begin{array}{l} j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1} \to 1_{N^+} \\ j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2} \to 0 \\ j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1} \to 0 \\ j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2} \to 1_{N^-} \end{array}}$ $\longleftarrow$CHECK

with: {$\mathcal{H}_{1,2}$ | $\mathcal{H}_{2,1} \to 0$}

•All conditions: $\left\{ \begin{array}{l} j_{1,1}.j_{1,1} + j_{1,2}.j_{2,1} \to 1_{N^+} \\ j_{1,1}.j_{1,2} + j_{1,2}.j_{2,2} \to 0 \\ j_{2,1}.j_{1,1} + j_{2,2}.j_{2,1} \to 0 \\ j_{2,1}.j_{1,2} + j_{2,2}.j_{2,2} \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,1}.(j_{1,1})^\dagger + j_{1,2}.(j_{1,2})^\dagger \to 1_{N^+} \\ j_{1,1}.(j_{2,1})^\dagger + j_{1,2}.(j_{2,2})^\dagger \to 0 \\ j_{2,1}.(j_{1,1})^\dagger + j_{2,2}.(j_{1,2})^\dagger \to 0 \\ j_{2,1}.(j_{2,1})^\dagger + j_{2,2}.(j_{2,2})^\dagger \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,1}.\mathcal{H}_{1,1} \to -j_{1,1}.\mathcal{H}_{1,1} \\ j_{1,2}.\mathcal{H}_{2,2} \to 0 \\ j_{2,1}.\mathcal{H}_{1,1} \to 0 \\ j_{2,2}.\mathcal{H}_{2,2} \to -j_{2,2}.\mathcal{H}_{2,2} \end{array} \right\}$

$\Rightarrow$ $\boxed{j_{1,1} \mid j_{2,2} \to 0}$

$\to$ $\left\{ \begin{array}{l} j_{1,2}.j_{2,1} \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1}.j_{1,2} \to 1_{N^-} \end{array}, \begin{array}{l} j_{1,2}.(j_{1,2})^\dagger \to 1_{N^+} \\ 0 \to 0 \\ 0 \to 0 \\ j_{2,1}.(j_{2,1})^\dagger \to 1_{N^-} \end{array}, \begin{array}{l} 0 \to 0 \\ j_{1,2}.\mathcal{H}_{2,2} \to 0 \\ j_{2,1}.\mathcal{H}_{1,1} \to 0 \\ 0 \to 0 \end{array} \right\}$

$\Rightarrow$ $\boxed{\begin{array}{l} (j_{1,2})^\dagger \to j_{2,1} \\ (j_{2,1})^\dagger \to j_{1,2} \end{array}}$

Commutative Subalgebras

```
PR["● Define subalgebra of 𝒜: ",
 $sAt = 𝒜̃J → {a ∈ 𝒜, a.J → J.ConjugateTranspose[a], a^"0" → a},
 NL, "•Unitary group: ",
 U[𝒜] → {u ∈ 𝒜, u.ConjugateTranspose[u] | ConjugateTranspose[u].u → 1N},
 Imply, ForAll[x ∈ M,
  u[x].ConjugateTranspose[u[x]] | ConjugateTranspose[u[x]].u[x] → 1N],
 Imply, u ∈ U[𝒜] ⇔ u[x] ∈ U[𝒜F],
 NL, "•Lie algebra: ", u[𝒜] → {X ∈ 𝒜, ConjugateTranspose[X] → -X} → C^"∞"[M, u[𝒜F]],
 NL, "•Special unitary group: ", SU[𝒜F] → {υ ∈ U[𝒜F], Det[u] → 1},
 NL, "•Lie algebra SU[𝒜F]: ", su[𝒜F] → {X ∈ 𝒜F, ConjugateTranspose[X] → -X, Tr[X] → 0},
 line,
 "●Adjoint action. space: ", $F = F → Table[Subscript[i, F], {i, {𝒜, ℋ, 𝒟, γ, J}}],
 NL, "Define: for ", ξ ∈ $F[[2, 2]],
 Yield, $ = {Ad[U[𝒜F]] → Endo[$F[[2, 2]]], ad[u[$F[[2, 1]]]] → Endo[$F[[2, 2]]]};
 Column[$],
 yield, $ = {Ad[u][ξ] → u.ξ.ConjugateTranspose[u] → u.ConjugateTranspose[u]^"0".ξ,
   ad[A][ξ] → A.ξ - ξ.A → (A - A^"0").ξ}; Column[$]
]
```

● Define subalgebra of $\mathcal{A}$: $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A},\ a.J \to J.a^\dagger,\ a^0 \to a\}$
•Unitary group: $U[\mathcal{A}] \to \{u \in \mathcal{A},\ u.u^\dagger \mid u^\dagger.u \to 1_N\}$
$\Rightarrow\ \forall_{x\in M}\,(u[x].u[x]^\dagger \mid u[x]^\dagger.u[x] \to 1_N)$
$\Rightarrow\ u \in U[\mathcal{A}] \Leftrightarrow u[x] \in U[\mathcal{A}_F]$
•Lie algebra: $\mathfrak{u}[\mathcal{A}] \to \{X \in \mathcal{A},\ X^\dagger \to -X\} \to C^\infty[M, \mathfrak{u}[\mathcal{A}_F]]$
•Special unitary group: $SU[\mathcal{A}_F] \to \{\upsilon \in U[\mathcal{A}_F],\ \mathrm{Det}[u] \to 1\}$
•Lie algebra SU[$\mathcal{A}_F$]: $\mathfrak{su}[\mathcal{A}_F] \to \{X \in \mathcal{A}_F,\ X^\dagger \to -X,\ \mathrm{Tr}[X] \to 0\}$

●Adjoint action. space: $F \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F, J_F\}$
Define: for $\xi \in \mathcal{H}_F$

$$\to \begin{array}{l} \mathrm{Ad}[U[\mathcal{A}_F]] \to \mathrm{Endo}[\mathcal{H}_F] \\ \mathrm{ad}[\mathfrak{u}[\mathcal{A}_F]] \to \mathrm{Endo}[\mathcal{H}_F] \end{array} \longrightarrow \begin{array}{l} \mathrm{Ad}[u][\xi] \to u.\xi.u^\dagger \to u.u^{\dagger 0}.\xi \\ \mathrm{ad}[A][\xi] \to A.\xi - \xi.A \to (A - A^0).\xi \end{array}$$

```
PR["●Gauge symmetry. ", {ϕ[M] → M, "diffeomorphism of C^∞[M]"},
 NL, "define automorphism: ", {αϕ[f] → f.inv[ϕ], f ∈ (C^"∞") [M]},
 NL, "define diffeomorphism: ", Diff[M×F] → Aut[(C^"∞") [M, 𝒜F]],
 Imply, {a ∈ (C^"∞") [M, 𝒜F], αϕ[a] → a.inv[ϕ], αϕ[a][x] → a.inv[ϕ][x]} // Column,
 NL, "•Define for ", Inn[a] ->
   {u ∈ (C^"∞") [M, U[𝒜F]], αu[a] → u.a.ConjugateTranspose[u] → Inn[𝒜]} // ColumnForms,
 NL, "•Define outer automorphism: ", Out[𝒜] → Mod[Aut[𝒜], Inn[𝒜]],
 NL, "•Define kernel: ", Ker[ϕ] → {ϕ[U[𝒜]] → Inn[𝒜], ϕ[u → αu],
    u ∈ U[𝒜], ForAll[a ∈ 𝒜, u.a.ConjugateTranspose[u] → a]} // ColumnForms
]
```

●Gauge symmetry. $\{\phi[M] \to M$, diffeomorphism of $C^\infty[M]\}$
define automorphism: $\{\alpha_\phi[f] \to f.\phi^{-1},\ f \in C^\infty[M]\}$
define diffeomorphism: $\mathrm{Diff}[M \times F] \to \mathrm{Aut}[C^\infty[M, \mathcal{A}_F]]$

$$\Rightarrow \begin{array}{l} a \in C^\infty[M, \mathcal{A}_F] \\ \alpha_\phi[a] \to a.\phi^{-1} \\ \alpha_\phi[a][x] \to a.\phi^{-1}[x] \end{array}$$

•Define for $\mathrm{Inn}[a] \to \begin{array}{l} u \in C^\infty[M, U[\mathcal{A}_F]] \\ \alpha_u[a] \to u.a.u^\dagger \to \mathrm{Inn}[\mathcal{A}] \end{array}$

•Define outer automorphism: $\mathrm{Out}[\mathcal{A}] \to \mathrm{Mod}[\mathrm{Aut}[\mathcal{A}], \mathrm{Inn}[\mathcal{A}]]$

•Define kernel: $\mathrm{Ker}[\phi] \to \begin{array}{l} \phi[U[\mathcal{A}]] \to \mathrm{Inn}[\mathcal{A}] \\ \phi[u \to \alpha_u] \\ u \in U[\mathcal{A}] \\ \forall_{a\in\mathcal{A}}\,(u.a.u^\dagger \to a) \end{array}$

```
PR["●Unitary transform. Given a triple: ", {𝒜, ℋ, 𝒟},
  " the representation π of 𝒜 on ℋ: ", π[a][ℋ] ,
  NL, "•Define unitary transform: ",
  $0 = U -> {U[ℋ] → ℋ, {𝒜, ℋ, 𝒟} -> {𝒜, ℋ, U. 𝒟.ConjugateTranspose[U]},
     (a ∈ 𝒜) -> U. π[a].ConjugateTranspose[U],
     γ -> U. γ.ConjugateTranspose[U], J -> U. J.ConjugateTranspose[U]};
  ColumnForms[$0],
  NL, "•EG1. ", {U → π[u], u ∈ U[𝒜]},
  NL, "•EG2. (adjoint action) ", $s = {U → Ad[u] → u.J.u.ConjugateTranspose[J]},
  Yield, $ = U.π[a].ConjugateTranspose[U], "POFF",
  Yield, $ = $ /. ($s[[1, 1]] -> $s[[1, 2, 2]] /. u → π[u]) // ConjugateCTSimplify1[{}],
  Yield, $ = $ /. aa_ . bb_ . π[a] → aa . π[a].bb, (*could be more specific*)
  Yield, $ = $ // tuRepeat[{ConjugateTranspose[J_] . J_ → 1,
      J_ .ConjugateTranspose[J_] → 1}, tuDotSimplify[]],
  Yield, $ = $ /. π[a_].π[b_]. ConjugateTranspose[π[c_]] →
     π[a. b. ConjugateTranspose[c]], "PONdd",
  Yield, $ = $ /. u_ .a_ .ConjugateTranspose[u_] → α_u[a]
 ];
```

●Unitary transform. Given a triple: {𝒜, ℋ, 𝒟} the representation π of 𝒜 on ℋ: π[a][ℋ]

•Define unitary transform: U →
$U[\mathcal{H}] \to \mathcal{H}$
$\{\mathcal{A}, \mathcal{H}, \mathcal{D}\} \to \{\mathcal{A}, \mathcal{H}, U.\mathcal{D}.U^\dagger\}$
$a \in \mathcal{A} \to U.\pi[a].U^\dagger$
$\gamma \to U.\gamma.U^\dagger$
$J \to U.J.U^\dagger$

•EG1. $\{U \to \pi[u], u \in U[\mathcal{A}]\}$
•EG2. (adjoint action) $\{U \to Ad[u] \to u.J.u.J^\dagger\}$
→ $U.\pi[a].U^\dagger$
....
→ $\pi[\alpha_u[a]]$

```
PR["•Define Gauge group: ", 𝒢[M×F] → {u.J.u.ct[J], u ∈ U[𝒜]},
 NL, "Consider: ", {Ad[U[𝒜]] → 𝒢[M×F], Ad[u] → u.ct[u]^"0"} // Column,
 Imply, Ker[Ad] → {u ∈ U[𝒜], (u.J.u.ct[J] → 1) ⇒ (u.J → ct[J].u)},
 NL, "•Define finite gauge group for finite space F: ",
 𝒢[F] → {ℋF → U[(𝒜̃F)JF], hF → u[(𝒜̃F)JF]} // ColumnForms,
 NL, "•Proposition 2.13. ",
 e213 = {𝒢[F] ≃ Mod[SU[𝒜F], SHF], 𝒜F → "complex algebra", SHF → {g ∈ HF, Det[g] → 1}};
 Column[e213],
 NL, "●Proof 2.13: ",
 NL, "•define UH-equivalence: ", $su = u_ ⇔ u_ . h_ -> ForAll[h, h ∈ HF, (u | u.h ∈ U[𝒜F])],
 Yield, $G = {𝒢[F] ≃ Mod[U[𝒜F], HF]} → {u ⇔ u.h},
 Yield, $ = $G /. $su,
 NL, "•define SUSH equivalence: ",
 $su = su_ ⇔ su_ . g_ -> ForAll[g, g ∈ SHF, (su | su.g ∈ SU[𝒜F])],
 Yield, $SU = {Mod[SU[𝒜F], SHF]} → {su ⇔ su.g},
 Yield, $0 = $SU /. $su,
 NL, "(1)•Is SHF a normal subgroup of SU[𝒜F]?: ",
 $ = ForAll[{g, v}, g ∈ SHF && v ∈ SU[𝒜F], (v.g. inv[v]) ∈ SHF],

 NL, "•Evaluate: ", $ = Det[$0 = v.g. inv[v] ∈ HF],
 yield, $ = $ /. a_ ∈ b_ → a,
 yield, $ = Thread[$, Dot] /. Det[inv[a_]] → 1 / Det[a] /. Dot → Times,
 NL, "Since: ", g ∈ SHF,
 imply, $s = Det[g] → 1,
 imply, $0 ∈ SHF,
 imply, "SHF Normal Subgroup of SU[𝒜F]" // Framed
]
```

•Define Gauge group: $\mathcal{G}[M\times F]\to\{u.J.u.J^{\dagger},\ u\in U[\mathcal{A}]\}$

Consider: $\begin{array}{l} Ad[U[\mathcal{A}]]\to\mathcal{G}[M\times F] \\ Ad[u]\to u.u^{\dagger 0}\end{array}$

$\Rightarrow$ $Ker[Ad]\to\{u\in U[\mathcal{A}],\ (u.J.u.J^{\dagger}\to 1)\Rightarrow(u.J\to J^{\dagger}.u)\}$

•Define finite gauge group for finite space F: $\mathcal{G}[F]\to\begin{array}{l}\mathcal{H}_F\to U[\tilde{\mathcal{A}}_{F\,J_F}] \\ h_F\to u[\tilde{\mathcal{A}}_{F\,J_F}]\end{array}$

•Proposition 2.13. $\begin{array}{l}\mathcal{G}[F]\simeq Mod[SU[\mathcal{A}_F],\ SH_F] \\ \mathcal{A}_F\to \text{complex algebra} \\ SH_F\to\{g\in H_F,\ Det[g]\to 1\}\end{array}$

●Proof 2.13:

•define UH-equivalence: $(u\_).(h\_)\Leftrightarrow u\_\to\forall_{h,h\in H_F}\ (u\mid u.h\in U[\mathcal{A}_F])$

$\to$ $\{\mathcal{G}[F]\simeq Mod[U[\mathcal{A}_F],\ H_F]\}\to\{u\Leftrightarrow u.h\}$

$\to$ $\{\mathcal{G}[F]\simeq Mod[U[\mathcal{A}_F],\ H_F]\}\to\{\forall_{h,h\in H_F}\ (u\mid u.h\in U[\mathcal{A}_F])\}$

•define SUSH equivalence: $(su\_).(g\_)\Leftrightarrow su\_\to\forall_{g,g\in SH_F}\ (su\mid su.g\in SU[\mathcal{A}_F])$

$\to$ $\{Mod[SU[\mathcal{A}_F],\ SH_F]\}\to\{su\Leftrightarrow su.g\}$

$\to$ $\{Mod[SU[\mathcal{A}_F],\ SH_F]\}\to\{\forall_{g,g\in SH_F}\ (su\mid su.g\in SU[\mathcal{A}_F])\}$

(1)•Is SH_F a normal subgroup of $SU[\mathcal{A}_F]$?: $\forall_{\{g,v\},g\in SH_F\&\&v\in SU[\mathcal{A}_F]}\ v.g.v^{-1}\in SH_F$

•Evaluate: $Det[v.g.v^{-1}\in H_F]\longrightarrow Det[v.g.v^{-1}]\longrightarrow Det[g]$

Since: $g\in SH_F\Rightarrow Det[g]\to 1\Rightarrow(v.g.v^{-1}\in H_F)\in SH_F\Rightarrow$ $\boxed{SH_F\ \text{Normal Subgroup of}\ SU[\mathcal{A}_F]}$

```
PR["•Property of unitary matrix u: ",
 {Abs[Det[u]] → 1,
   {"Eigenvalues of u", λu ∈ U[1],
    Exists[{u, u'}, u ∈ U[𝒜F] && u' ∈ U[N], u'.u.ct[u'] -> λu 1N]}} // FramedColumn,
 Imply, Exists[λu, λu ∈ U[1] && λu^N → Det[u] && N → dim[ℋF] && U[1] ≤ U[𝒜F]],
 Imply, $ = ($0 = inv[λu].u ∈ SU[𝒜F])⟸{$ = Det[$0[[1]]], $ = Thread[$, Dot],
     $ = $ /. Det[inv[λu]] → λu^(-N), $ = $ /. Det[u] → λu^N, SU[𝒜F]} // ColumnForms,

 NL, "■define group homomorphism from UH->SUSH: ",
 $ph = {φ[$G[[1, 1]]] → Mod[SU[𝒜F], SHF], φ[{u}] → {inv[λu].u}};
 Column[$ph],
 NL, "□Check if φ is independent of representative ", λu,
 NL, "•suppose: ", Implies[Exists[λu', (λu')^N → Det[u]],
  inv[λu].λu' ∈ μN["multiplicative group Nth root of unity"]],
 NL, "•", Implies[Implies[Implies[U[1] ≤ HF, μN ≤ SHF], {inv[λu].u} == {inv[λu'].u}],
  Framed[φ["independent of λu"]]],
 NL, "□Check if φ is independent of representative ", u ∈ U[𝒜F],
 NL, "?: ", $0 = $ = ForAll[u, u ∈ HF, φ[{u}]],
 Yield, $ = $ /. $ph, "POFF",
 NL, "For ", $s = (g -> inv[λh].h) ∈ SHF,
 Yield, $ = $ /. dd : HoldPattern[Dot[a_]] → dd .g,
 Yield, $ = $ /. $s[[1]],
 Yield, $ = $ /. dd : HoldPattern[Dot[_]] :→ tuDotTermLeft[inv[_], {inv[λu]}][dd],
 Yield, $ = $ /. inv[a_]. inv[b_] → inv[b.a],
 Yield, $[[3]] = φ[{u.h}]; $, "PONdd",
 yield, $[[3]] == $0[[3]] // Framed,
 NL, "•Suppose ", $ = ForAll[{u1, u2}, {u1 | u2 ∈ U[𝒜F]}, φ[{u1}] == φ[{u2}]],
 Yield, $ = $ /. φ[{a_}] → {inv[λa].a.ga} /. gu1 → 1 /. gu2 → (g ∈ SHF),
 Yield, $ = $ /. HoldPattern[Dot[a_]] → Dot[λu1, a],
 Yield, $ = $ /. a_ . inv[a_] → 1 /. g ∈ SHF → g // tuDotSimplify[],
 Yield, $3 = $ = $[[3]],
 " for some: ", $ = $[[2]] // First // DeleteCases[#, u2] & // tuDotSimplify[];
 $ ∈ SHF,
 imply, "φ is injective.",
 Imply, $ = $3 /. Thread[Apply[List, $] → 1] // tuDotSimplify[]; Framed[$]
]
```

• Property of unitary matrix u:

$\boxed{\begin{array}{l}\text{Abs[Det[u]]} \to 1 \\ \{\text{Eigenvalues of u, } \lambda_u \in \text{U[1]}, \exists_{\{u,u'\},u\in U[\mathcal{A}_F]\&\&u'\in U[N]} (u'.u.(u')^{\dagger} \to 1_N \lambda_u)\}\end{array}}$

$\Rightarrow \exists_{\lambda_u} (\lambda_u \in \text{U[1]} \;\&\&\; \lambda_u^N \to \text{Det[u]} \;\&\&\; N \to \dim[\mathcal{H}_F] \;\&\&\; \text{U[1]} \le \text{U}[\mathcal{A}_F])$

$$\Rightarrow (\lambda_u^{-1}.u \in SU[\mathcal{A}_F]) \Longleftarrow \begin{array}{l}\text{Det}[\lambda_u^{-1}.u] \\ \text{Det}[\lambda_u^{-1}].\text{Det}[u] \\ \lambda_u^{-N}.\text{Det}[u] \\ \lambda_u^{-N}.\lambda_u^{N} \\ SU[\mathcal{A}_F]\end{array}$$

■define group homomorphism from UH->SUSH: $\begin{array}{l}\varphi[\mathcal{G}[F] \simeq \text{Mod}[U[\mathcal{A}_F], H_F]] \to \text{Mod}[SU[\mathcal{A}_F], SH_F] \\ \varphi[\{u\}] \to \{\lambda_u^{-1}.u\}\end{array}$

□Check if φ is independent of representative λ_u

• suppose: $\exists_{\lambda_u'} ((\lambda_u')^N \to \text{Det[u]}) \Rightarrow \lambda_u^{-1}.\lambda_u' \in \mu_N$[multiplicative group Nth root of unity]

• $((U[1] \le H_F \Rightarrow \mu_N \le SH_F) \Rightarrow \{\lambda_u^{-1}.u\} == \{(\lambda_u')^{-1}.u\}) \Rightarrow \boxed{\varphi[\text{independent of } \lambda_u]}$

□Check if φ is independent of representative $u \in U[\mathcal{A}_F]$

?: $\forall_{u,u\in H_F} \varphi[\{u\}]$

$\to \forall_{u,u\in H_F} \{\lambda_u^{-1}.u\}$

.... $\longrightarrow$ $\boxed{\varphi[\{u.h\}] == \varphi[\{u\}]}$

• Suppose $\forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \varphi[\{u_1\}] == \varphi[\{u_2\}]$

$\to \forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \{\lambda_{u_1}^{-1}.u_1.1\} == \{\lambda_{u_2}^{-1}.u_2.(g \in SH_F)\}$

$\to \forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \{\lambda_{u_1}.\lambda_{u_1}^{-1}.u_1.1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.(g \in SH_F)\}$

$\to \forall_{\{u_1,u_2\},\{u_1|u_2\in U[\mathcal{A}_F]\}} \{u_1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$

$\to \{u_1\} == \{\lambda_{u_1}.\lambda_{u_2}^{-1}.u_2.g\}$ for some: $\lambda_{u_1}.\lambda_{u_2}^{-1}.g \in SH_F \Rightarrow \varphi$ is injective.

$\Rightarrow \boxed{\{u_1\} == \{u_2\}}$

```
PR["●Full symmetry group. ",
 NL, "•Homomorphic action Θ of a group H on group N: ", Θ[H] → Aut[N],
 NL, "•semi-direct product ", $ = N ⊳ H → {{n, h}, n ∈ N && h ∈ H},
 NL, "Properties: ", $sdg = {
   {"product", {n_, h_} · {n1_, h1_} → {n . Θ[h]. n1, h . h1}},
   {"unit", {1, 1}},
   {"inverse", invSDG[{n_, h_}] → {Θ[inv[h]].inv[n], inv[h]}
   }}; FramedColumn[$sdg],
 "POFF",
 NL, "•Check inverse: ",
 NL, "Let: ", $n = {n, h},
 and, "inverse: ", $i = invSDG[$n],
 NL, "For ", $ = $n · $i,
 Yield, $ = $ /. $sdg[[3, 2]] /. $sdg[[1, 2]],
 NL, "If ", $s = {inv[a_] .a_ → 1, a_ .inv[a_] → 1, Θ[a_]. Θ[inv[a_]] → 1,
   Θ[a_] . n1_ . Θ[a_] . n2_ → Θ[a].n1.n2, (*homomorphic property*)
   {Θ[a_], b_} -> {1, b}(* Is Θ[h].1→1? *)
  },
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], OK,
 NL, "For ", $ = $i · $n,
 Yield, $ = $ /. $sdg[[3, 2]] /. $sdg[[1, 2]],
 yield, $ = $ // tuRepeat[$s, tuDotSimplify[]], OK,
 "PONdd",
 NL, "•Invariance under Diff[M]: ", Exists[Θ, Θ → "homomorphism",
  {Θ[Diff[M]] → Aut[𝒢[M×F]] ↦ Θ[ϕ].U → U∘inv[ϕ], ϕ ∈ Diff[M], U ∈ 𝒢[M×F]}],
 Yield, "Full symmetry group: ", 𝒢[M×F] ⊳ Diff[M]
]
```

●Full symmetry group.
•Homomorphic action Θ of a group H on group N: $\Theta[H] \to \mathrm{Aut}[N]$
•semi-direct product $N \triangleright H \to \{\{n, h\}, n \in N \,\&\&\, h \in H\}$

Properties:

{product, {n_, h_} · {n1_, h1_} → {n.Θ[h].n1, h.h1}}
{unit, {1, 1}}
{inverse, invSDG[{n_, h_}] → {Θ[h^{-1}].n^{-1}, h^{-1}}}

....
•Invariance under Diff[M]:
$\exists_{\Theta,\Theta\to\text{homomorphism}} \{\Theta[\mathrm{Diff}[M]] \to \mathrm{Aut}[\mathcal{G}[M\times F]] \mapsto \Theta[\phi].U \to U\circ\phi^{-1},\ \phi \in \mathrm{Diff}[M],\ U \in \mathcal{G}[M\times F]\}$
→ Full symmetry group: $\mathcal{G}[M\times F] \triangleright \mathrm{Diff}[M]$

```
PR["●Principal bundles. ",
 NL, "Let ", $ = {{G → "Lie group", P → "principal G-bundle"} ↦ (π[P] → M),
   Aut[P] -> {f[P] → P, ForAll[{p, g}, p ∈ P && g ∈ G, f[p.g] → f[p].g]},
   Implies[f, Exists[f̄, {(f̄[M] → M) ↦ (f̄[π[p]] → π[f[p]]), f̄ → "diffeomorphism"}]]
  }; Column[$],
 NL, "•Gauge transformation of P: ",
 𝒢[P] → ForAll[g, g ∈ Aut[P], {ḡ == Id_M, π[g[p]] → π[p]}],
 NL, "?Is 𝒢[P] a normal subgroup: ",
 NL, "Since ", $ = f̄[π[p]] → π[f[p]],
 Yield, $ = $ /. f → f∘g∘inv[f],
 NL, "Since: ", $s = {(c__∘a_∘b_)[p_] → (c∘a)[b[p]], (a_ ∘ b_)[p_] → a[b[p]]},
 Yield, $ = MapAt[# //. $s &, $, 2],
 NL, "Using: ", $s = {π[f_[p_]] -> f̄[π[p]], a_[b_[π[p]]] :→ Flatten[a∘b][π[p]]},
 Yield, $ = MapAt[# //. $s &, $, 2]; Framed[Head /@ $],
 NL, "For ", $s = {ḡ → Id_M, f_∘Id_M∘f1_ → f∘f1, f̄_∘inv[f̄_] → Id_M},
 Yield, $ = $ //. $s; $ = Head /@ $,
 imply, $ = $[[1, 1]] ∈ 𝒢[P]; Framed[$ ≤ Aut[P]],
 NL, "Quotient: ", Quotient[Aut[P], 𝒢[P]] ≃ Diff[M]

]
```

●Principal bundles.

Let $\begin{array}{l}\{G\to \text{Lie group},\ P\to \text{principal G-bundle}\}\mapsto (\pi[P]\to M)\\ \text{Aut}[P]\to \{f[P]\to P,\ \forall_{\{p,g\},p\in P\&\&g\in G}\,(f[p.g]\to f[p].g)\}\\ f\Rightarrow \exists_{\bar f}\,\{(\bar f[M]\to M)\mapsto (\bar f[\pi[p]]\to \pi[f[p]]),\ \bar f\to \text{diffeomorphism}\}\end{array}$

•Gauge transformation of P: $\mathcal{G}[P]\to \forall_{g,g\in \text{Aut}[P]}\,\{\bar g == \text{Id}_M,\ \pi[g[p]]\to \pi[p]\}$

?Is $\mathcal{G}[P]$ a normal subgroup:

Since $\bar f[\pi[p]]\to \pi[f[p]]$

$\to\ \overline{f\circ g\circ f^{-1}}[\pi[p]]\to \pi[(f\circ g\circ f^{-1})[p]]$

Since: $\{(c\_\_\circ a\_\circ b\_)[p\_]\to (c\circ a)[b[p]],\ (a\_\circ b\_)[p\_]\to a[b[p]]\}$

$\to\ \overline{f\circ g\circ f^{-1}}[\pi[p]]\to \pi[f[g[f^{-1}[p]]]]$

Using: $\{\pi[f\_[p\_]]\to \bar f[\pi[p]],\ a\_[b\_[\pi[p]]]:\to \text{Flatten}[a\circ b][\pi[p]]\}$

$\to\ \boxed{\overline{f\circ g\circ f^{-1}}\to \bar f\circ \bar g\circ \overline{f^{-1}}}$

For $\{\bar g\to \text{Id}_M,\ f\_\circ \text{Id}_M\circ f1\_\to f\circ f1,\ \bar f\_\circ \overline{f\_}^{-1}\to \text{Id}_M\}$

$\to\ \overline{f\circ g\circ f^{-1}}\to \text{Id}_M\ \Rightarrow\ \boxed{(f\circ g\circ f^{-1}\in \mathcal{G}[P])\le \text{Aut}[P]}$

Quotient: Quotient[Aut[P], $\mathcal{G}$[P]] ≃ Diff[M]

Inner fluctuations

```
PR["●For a Real ACM: ", M×F → {𝒜, ℋ, 𝒟, J},
  NL, "•Define: ", $0 = Ω𝒟^"1" → {xSum[aj. CommutatorM[𝒟, bj], {j}], aj | bj ∈ 𝒜},
  NL, "•inner fluctuations: ",
  𝒜f -> {ForAll[𝒜, 𝒜 ∈ $0[[1]], ConjugateTranspose[𝒜] == 𝒜]},
  NL, "•fluctuated Dirac operator: ", $DA = 𝒟𝒜 → 𝒟 + 𝒜f + ε'.J.𝒜f.ConjugateTranspose[J],
  NL, "■Calculate on inner fluctuations: ",
  NL, $A = $0 = {𝒜 → a.CommutatorM[slash[𝒟], b],
     a | b ∈ C^"∞"[M], slash[𝒟] → -I T[γ, "u", {μ}] tuDs["∇"^S][_, μ]},
  Yield, $ = $0[[1]] /. $0[[-1]] /. CommutatorM → MCommutator //
    tuDotSimplify[{T[γ, "u", {μ}]}],
  yield, $0 = $ = $ /. tuDs["∇"^S][_, μ].b -> tuDs["∇"^S][b, μ] + b.tuDs["∇"^S][_, μ] //
     tuDotSimplify[{T[γ, "u", {μ}]}],
  NL, "Define ", $Am = $ = I T[𝒜, "d", {μ}] -> $[[2]] /. T[γ, "u", {μ}] → I;
  $ = -I # & /@ $;
  Framed[$ ∈ Real[C^"∞"[M]]],
  NL, "Proof:",
  "POFF",
  NL, $0;
  $1 = ConjugateTranspose /@ $0 // ConjugateCTSimplify1[{}, {}, {T[γ, "u", {μ}]}];
  $2 = 𝒜 → ConjugateTranspose[𝒜];
  $ = {$0, $1, $2},
  Yield, $ = tuEliminate[$, {𝒜}],
  yield, $ = Implies[$[[-1]], $[[-1, 2]] ∈ Reals] /. T[γ, "u", {μ}] → 1;
  Framed[$],
  "PONdd",
  NL, "For ", $ = slash[𝒟]𝒜 -> slash[𝒟] + 𝒜 + JM.𝒜.ConjugateTranspose[JM],
  NL, "Since: ", $s = {jj : JM.𝒜 → -Reverse[jj], JM.ConjugateTranspose[JM] → 1},
  imply, $ = slash[𝒟]𝒜 -> slash[𝒟] + 𝒜 + JM.𝒜.ConjugateTranspose[JM]
    // tuRepeat[$s, tuDotSimplify[]]
 ];
```

●For a Real ACM: $M\times F\to\{\mathcal{A},\mathcal{H},\mathcal{D},J\}$

•Define: $\Omega^1_{\mathcal{D}}\to\{\sum_{\{j\}}[a_j.[\mathcal{D},b_j]],\ a_j\mid b_j\in\mathcal{A}\}$

•inner fluctuations: $\mathcal{A}_f\to\{\forall_{\mathcal{A},\mathcal{A}\in\Omega^1_{\mathcal{D}}}\mathcal{A}^\dagger==\mathcal{A}\}$

•fluctuated Dirac operator: $\mathcal{D}_{\mathcal{A}}\to\mathcal{D}+\varepsilon'.J.\mathcal{A}_f.J^\dagger+\mathcal{A}_f$

■Calculate on inner fluctuations:

$\{\mathcal{A}\to a.[\not{D},b],\ a\mid b\in C^\infty[M],\ \not{D}\to-\mathrm{i}\,\gamma^\mu\,\underline{\nabla}^S{}_\mu[\_]\}$

$\to\ \mathcal{A}\to\mathrm{i}\,a.b.\underline{\nabla}^S{}_\mu[\_]\,\gamma^\mu-\mathrm{i}\,a.\underline{\nabla}^S{}_\mu[\_].b\,\gamma^\mu\ \longrightarrow\ \mathcal{A}\to-\mathrm{i}\,a.\underline{\nabla}^S{}_\mu[b]\,\gamma^\mu$

Define $\boxed{(\mathcal{A}_\mu\to-\mathrm{i}\,a.\nabla^S{}_{-\mu}[b])\in\mathrm{Real}[C^\infty[M]]}$

Proof:

....

For $\not{D}_{\mathcal{A}}\to\mathcal{A}+J_M.\mathcal{A}.(J_M)^\dagger+\not{D}$

Since: $\{jj:J_M.\mathcal{A}\to-\mathrm{Reverse}[jj],\ J_M.(J_M)^\dagger\to1\}\ \Rightarrow\ \not{D}_{\mathcal{A}}\to\not{D}$

```
PR["●Inner fluctuations. ",
 NL, "•Dirac operator: ", $d = 𝒟 → slash[𝒟] ⊗ 1_N + T[γ, "d", {5}] ⊗ 𝒟_F,
 NL, "•Examine: ", $ = $A[[1]] /. slash[𝒟] → 𝒟; Framed[$],
 yield, $0 = $ /. $d // simpleCommutator[{}] // tuDotSimplify[],

 NL, "Evaluate[1]: ", $1 = $ = $0[[2, 1]],
 yield, $ = $ /. commutatorDot // tuDotSimplify[],
 NL, "Use: ", $s = {(slash[𝒟] ⊗ 1_N).b -> slash[𝒟] ⊗ b + b. (slash[𝒟] ⊗ 1_N)},
 Yield, $ = $ /. $s // tuDotSimplify[],
 NL, "Use: ", $slashD =
  $s = $sD = {$A[[-1]], a_ .(( c_ tuDs["▽"^S][_, μ]) ⊗ b_) → c ⊗ ( a.tuDs["▽"^S][b, μ]),
     (-I a_) ⊗ b_ → a ⊗ (-I b)},
 Yield, $1 = $1 → ($ //. $s); Framed[$1], CK,

 NL, "Evaluate[2]: ", $2 = $ = $0[[2, 2]],
 NL, "Since: ", CommutatorM[T[γ, "d", {5}], b] → 0,
 NL, "Use: ", $s = {$[[2]] → ($[[2]] /. CommutatorM[a_ ⊗ b_, c_] → a ⊗ CommutatorM[b, c]),
   a_ . ((tt : T[γ, "d", {5}]) ⊗ b_) → tt ⊗ (a.b)},
 Yield, $ = $ /. $s /. $s ; Framed[$2 = $2 -> $],
 yield, "define: ", Framed[$2a = $[[2]] → ϕ],
 NL, "with ", Reverse[$Am],
 Imply, $A1 = $ = $0 /. $2 /. $1 /. $2a /. Reverse[$Am] // Simplify; Framed[$]
]
```

●Inner fluctuations.

•Dirac operator: $\mathcal{D} \to (\not{D}) \otimes 1_N + \gamma_5 \otimes \mathcal{D}_F$

•Examine: $\boxed{\mathcal{A} \to a.[\mathcal{D}, b]}$ $\longrightarrow$ $\mathcal{A} \to a.[(\not{D}) \otimes 1_N, b] + a.[\gamma_5 \otimes \mathcal{D}_F, b]$

Evaluate[1]: $a.[(\not{D}) \otimes 1_N, b]$ $\longrightarrow$ $-a.b.((\not{D}) \otimes 1_N) + a.((\not{D}) \otimes 1_N).b$

Use: $\{((\not{D}) \otimes 1_N).b \to (\not{D}) \otimes b + b.((\not{D}) \otimes 1_N)\}$

$\to$ $a.((\not{D}) \otimes b)$

Use: $\{\not{D} \to -\mathbb{i}\, \gamma^\mu\, \underline{\nabla}^S{}_\mu[\_],\ (a\_).((c\_\ \underline{\nabla}^S{}_\mu[\_]) \otimes b\_) \to c \otimes a.\underline{\nabla}^S{}_\mu[b],\ (-\mathbb{i}\ a\_) \otimes b\_ \to a \otimes (-\mathbb{i}\ b)\}$

$\to$ $\boxed{a.[(\not{D}) \otimes 1_N, b] \to \gamma^\mu \otimes (-\mathbb{i}\ a.\underline{\nabla}^S{}_\mu[b])}$ ⟵CHECK

Evaluate[2]: $a.[\gamma_5 \otimes \mathcal{D}_F, b]$

Since: $[\gamma_5, b] \to 0$

Use: $\{[\gamma_5 \otimes \mathcal{D}_F, b] \to \gamma_5 \otimes [\mathcal{D}_F, b],\ (a\_).((tt : \gamma_5) \otimes b\_) \to tt \otimes a.b\}$

$\to$ $\boxed{a.[\gamma_5 \otimes \mathcal{D}_F, b] \to \gamma_5 \otimes a.[\mathcal{D}_F, b]}$ $\longrightarrow$ define: $\boxed{a.[\mathcal{D}_F, b] \to \phi}$

with $a.\underline{\nabla}^S{}_\mu[b] \to \mathbb{i}\ \mathcal{A}_\mu$

$\Rightarrow$ $\boxed{\mathcal{A} \to \gamma_5 \otimes \phi + \gamma^\mu \otimes \mathcal{A}_\mu}$

```
PR["•Fluctuated Dirac operator: ", $ = $DA,
 Yield, $ = $ /. 𝒜f → 𝒜;
 Yield, $0 = MapAt[# /. $A1 &, $, 2] // tuDotSimplify[],

 NL, "■Examine[𝒜]: ", $ = Select[$0[[2]], ! FreeQ[#, 𝒜] &],
 NL, "J Anticommutes: ",
 $s = e_ .J.(T[γ, "u", {μ}] ⊗ a_).b_ → -T[γ, "u", {μ}] ⊗ (e.J. a.b),
 Yield, $ = $ /. $s,
 yield, $ = $ /. a_ ⊗ b_ + (-a_ ⊗ c_) -> a ⊗ (b - c); Framed[$],
 NL, "Define ", $e216B = e216 = {Bμ -> $[[2]], Bμ ∈ Γ[End["E"]]};
 Framed[$e216B], CG[" (2.16)"],
 NL, "Define twisted connection: ",
 $ = T["∇""E", "d", {μ}] -> T["∇"S, "d", {μ}] ⊗ Id + I Id ⊗ Bμ;
 Framed[$],
 Yield, $ = -I T[γ, "u", {μ}].# & /@ $ // tuDotSimplify[],
 $ = $ /. T[γ, "u", {μ}].(Id ⊗ b_) -> T[γ, "u", {μ}] ⊗ b;
 Yield, $ = $ /. -I a_ . (b_ ⊗ c_) → (-I a b) ⊗ c,
 NL, "Using: ", $s = (I # & /@ Reverse[$A[[-1]]] /. tuDDown[a_][_, m_] → T[a, "d", {m}]),
 Yield, e216a = $ /. $s; Framed[$],

 NL, "■Examine[ϕ]: ",
 NL, "Define ", Φ ∈ Γ[End["E"]] ∋
  ($ = T[γ, "d", {5}] ⊗ Φ -> Select[$0[[2]], ! FreeQ[#, ϕ] &] + T[γ, "d", {5}] ⊗ 𝒟F),
 Imply, e218 = 𝒟A → e216a[[1]] + $[[1]]; Framed[e218]
]
```

•Fluctuated Dirac operator: $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \varepsilon'.J.\mathcal{A}_f.J^\dagger + \mathcal{A}_f$

→

→ $\mathcal{D}_{\mathcal{A}} \to \mathcal{D} + \gamma_5 \otimes \phi + \gamma^\mu \otimes \mathcal{A}_\mu + \varepsilon'.J.(\gamma_5 \otimes \phi).J^\dagger + \varepsilon'.J.(\gamma^\mu \otimes \mathcal{A}_\mu).J^\dagger$

■Examine[𝒜]: $\gamma^\mu \otimes \mathcal{A}_\mu + \varepsilon'.J.(\gamma^\mu \otimes \mathcal{A}_\mu).J^\dagger$

J Anticommutes: $(e\_).J.(\gamma^\mu \otimes a\_).(b\_) \to -\gamma^\mu \otimes e.J.a.b$

→ $-\gamma^\mu \otimes \varepsilon'.J.\mathcal{A}_\mu.J^\dagger + \gamma^\mu \otimes \mathcal{A}_\mu \longrightarrow \boxed{\gamma^\mu \otimes (-\varepsilon'.J.\mathcal{A}_\mu.J^\dagger + \mathcal{A}_\mu)}$

Define $\boxed{\{B_\mu \to -\varepsilon'.J.\mathcal{A}_\mu.J^\dagger + \mathcal{A}_\mu,\ B_\mu \in \Gamma[\mathrm{End}[E]]\}}$ (2.16)

Define twisted connection: $\boxed{\nabla^E{}_\mu \to i\ \mathrm{Id} \otimes B_\mu + \nabla^S{}_\mu \otimes \mathrm{Id}}$

→ $-i\ \gamma^\mu.\nabla^E{}_\mu \to \gamma^\mu.(\mathrm{Id} \otimes B_\mu) - i\ \gamma^\mu.(\nabla^S{}_\mu \otimes \mathrm{Id})$

→ $-i\ \gamma^\mu.\nabla^E{}_\mu \to \gamma^\mu \otimes B_\mu + (-i\ \nabla^S{}_\mu\ \gamma^\mu) \otimes \mathrm{Id}$

Using: $\nabla^S{}_\mu\ \gamma^\mu \to i\ (\not{D})$

→ $\boxed{-i\ \gamma^\mu.\nabla^E{}_\mu \to \gamma^\mu \otimes B_\mu + (-i\ \nabla^S{}_\mu\ \gamma^\mu) \otimes \mathrm{Id}}$

■Examine[ϕ]:

Define $\Phi \in \Gamma[\mathrm{End}[E]] \ni (\gamma_5 \otimes \Phi \to \gamma_5 \otimes \phi + \gamma_5 \otimes \mathcal{D}_F + \varepsilon'.J.(\gamma_5 \otimes \phi).J^\dagger)$

⇒ $\boxed{\mathcal{D}_A \to \gamma_5 \otimes \Phi - i\ \gamma^\mu.\nabla^E{}_\mu}$

```
elementQ[a_, h_List] := tuMemberQ[a, h];
$hermitian = {𝒜μ};
$Iu = {I 𝒜μ};
PR["Since: ",
 $ = Implies[Inactive[elementQ[𝒜μ, $hermitian]], ConjugateTranspose[𝒜μ] == 𝒜μ] ,
 imply, $ = -I # & /@ Activate[$] /. -I ConjugateTranspose[a_] → SuperDagger[I a],
 imply, Framed[I $[[2]] ∈ I u],
 NL, "For ", I g[F] → I Mod[u[F], h[F]],
 imply, e219 = 𝒜μ ∈ C"∞"[M, I g[F]]
]
```

Since: Inactive[elementQ[$\mathcal{A}_\mu$, $hermitian]] ⇒ $(\mathcal{A}_\mu)^\dagger == \mathcal{A}_\mu$ ⇒ $(i\,\mathcal{A}_\mu)^\dagger == -i\,\mathcal{A}_\mu$ ⇒ $\boxed{\mathcal{A}_\mu \in i\,u}$

For $i\,g$[F] → i Mod[u[F], h[F]] ⇒ $\mathcal{A}_\mu \in C^\infty$[M, $i\,g$[F]]

```
PR["Gauge transformation on fluctuating Dirac operator. ",
  Yield, $00 = $0 = 𝒟𝒜 → 𝒟 + 𝒜 + ε'.J.𝒜.ConjugateTranspose[J],
  NL, "Expanding Rules: ",
  $s0 = {U → u.J.u.ConjugateTranspose[J], CommutatorM[a, b"0"] → 0,
    CommutatorM[𝒜, J.u.ConjugateTranspose[J]] → 0,
    CommutatorM[CommutatorM[𝒟, a], b"0"] → 0,
    J.𝒟 → ε'.𝒟.J, b"0" → J.ConjugateTranspose[b].ConjugateTranspose[J],
    J_ .ConjugateTranspose[J_] :> 1 /; MemberQ[{J, u}, J],
    ConjugateTranspose[J_].J_ :> 1 /; MemberQ[{J, u}, J],
    ε^2 → 1};
  Yield, $s0x =
   $s0 /. CommutatorM → MCommutator // tuDotSimplify[{ε'}] // tuRuleEliminate[{b"0"}];
  FramedColumn[$s0x],
  NL, "Evaluate: ",
  $0a = $ = U.#.ConjugateTranspose[U] & /@ $0 // tuDotSimplify[{ε', ε}],

  Yield,
  $1 = $ = $[[2]] // tuRepeat[$s0x, tuDotSimplify[]] // ConjugateCTSimplify1[{ε', ε}];
  $1 = $1 // tuRepeat[$s0x, tuDotSimplify[]], (*Need to repeat for some reason*)
  NL, "From commutation rules: ",
  $s = tuRuleSolve[$s0x[[5]], Dot[𝒟, J]],

  NL, "■Simplify the term: ",
  Yield, $ = $1[[2]]; Framed[$],
  yield, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  yield, $ = $ /. $s0x[[7]] // tuDotSimplify[{ε', ε}],
  NL, "From ", $s = u.CommutatorM[𝒟, ConjugateTranspose[u]] ->
    u.MCommutator[𝒟, ConjugateTranspose[u]],
  $s = $s // tuDotSimplify[];
  yield, $s = $s /. $s0 // tuDotSimplify[],
  yield, $s = tuRuleEliminate[{u.𝒟.ConjugateTranspose[u]}][{$s}];
  Framed[$s],
  Imply, $ = $ /. $s // tuDotSimplify[{ε', ε}],
  Yield, $ = $ //. $s0 // tuDotSimplify[{ε', ε}],
  yield, $1a = $ = $ /. $s; Framed[$], CK
 ];
PR[
  "■Simplify the term: ",
  Yield, $0 = $ = $1[[1]]; Framed[$],
  NL, "Use: ", $s = tuRuleSolve[$s0x /. u → ConjugateTranspose[u], 𝒜._],
  Yield, $ = $ /. $s // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1b = $]
 ];
```

```
$s0x /. xu → ConjugateTranspose[u];
PR[
 "■Simplify the term: ",
 Yield, $0 = $ = $1[[3]]; Framed[$],
 NL, "Append 1→ ", $s = J.ConjugateTranspose[J],
 imply, $ = $. $s // tuDotSimplify[{ε', ε}],
 NL, "Use ",
 $s = tuRuleSolve[$s0x /. u → ConjugateTranspose[u], 𝒜._],
 " with ConjugateTranspose: ", $sa = aa : a | J → ConjugateTranspose[aa],
 Yield, $s = $s /. ConditionalExpression[a_, b_] → a /. $sa //
   tuAddPatternVariable[{a, b}],
 NL, "The Rule applies to: ", $sa = 𝒜 → u.𝒜.ConjugateTranspose[u],
 yield, $s = $s /. $sa,
 Imply, $ = $ /. $s,
 yield, $ = $ // tuRepeat[$s0x, tuDotSimplify[]]; Framed[$1c = $]
]
PR["■Check if equal to (2.20). Our calculation: ",
 $ = $0a[[1]] -> $1a + $1b + $1c; Framed[$],
 NL, "Evaluate (2.20) with ", $ = $00 /. 𝒜 → 𝒜^u, CK,
 Yield, $[[2]] =
  $[[2]] /. 𝒜^u → u.𝒜.ConjugateTranspose[u] + u.CommutatorM[𝒟, ConjugateTranspose[u]] //
   tuDotSimplify[{ε'}];
 Framed[$],
 NL, CR["Almost equal."]

]
```

Gauge transformation on fluctuating Dirac operator.

$\to\ \mathcal{D}_{\mathcal{A}} \to \mathcal{A} + \mathcal{D} + \varepsilon'.J.\mathcal{A}.J^{\dagger}$

Expanding Rules:

$$\to\ \boxed{\begin{array}{l} U \to u.J.u.J^{\dagger} \\ a.J.b^{\dagger}.J^{\dagger} - J.b^{\dagger}.J^{\dagger}.a \to 0 \\ -J.u.J^{\dagger}.\mathcal{A} + \mathcal{A}.J.u.J^{\dagger} \to 0 \\ -a.\mathcal{D}.J.b^{\dagger}.J^{\dagger} + J.b^{\dagger}.J^{\dagger}.a.\mathcal{D} - J.b^{\dagger}.J^{\dagger}.\mathcal{D}.a + \mathcal{D}.a.J.b^{\dagger}.J^{\dagger} \to 0 \\ J.\mathcal{D} \to \mathcal{D}.J\ \varepsilon' \\ (J\_).J\_^{\dagger} :\to 1\ /;\ \text{MemberQ}[\{J, u\}, J] \\ J\_^{\dagger}.(J\_) :\to 1\ /;\ \text{MemberQ}[\{J, u\}, J] \\ \varepsilon^2 \to 1 \end{array}}$$

Evaluate: $U.\mathcal{D}_{\mathcal{A}}.U^{\dagger} \to U.\mathcal{A}.U^{\dagger} + U.\mathcal{D}.U^{\dagger} + U.J.\mathcal{A}.J^{\dagger}.U^{\dagger}\ \varepsilon'$

$\to\ u.J.u.J^{\dagger}.\mathcal{A}.J.u^{\dagger}.J^{\dagger}.u^{\dagger} + u.J.u.J^{\dagger}.\mathcal{D}.J.u^{\dagger}.J^{\dagger}.u^{\dagger} + u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}\ \varepsilon'$

From commutation rules: $\{\mathcal{D}.J \to \dfrac{J.\mathcal{D}}{\varepsilon'}\}$

■Simplify the term:

$$\to\ \boxed{u.J.u.J^{\dagger}.\mathcal{D}.J.u^{\dagger}.J^{\dagger}.u^{\dagger}} \longrightarrow \frac{u.J.u.J^{\dagger}.J.\mathcal{D}.u^{\dagger}.J^{\dagger}.u^{\dagger}}{\varepsilon'} \longrightarrow \frac{u.J.u.\mathcal{D}.u^{\dagger}.J^{\dagger}.u^{\dagger}}{\varepsilon'}$$

From $u.[\mathcal{D}, u^{\dagger}] \to u.(\mathcal{D}.u^{\dagger} - u^{\dagger}.\mathcal{D}) \longrightarrow u.[\mathcal{D}, u^{\dagger}] \to -\mathcal{D} + u.\mathcal{D}.u^{\dagger} \longrightarrow \boxed{\{u.\mathcal{D}.u^{\dagger} \to \mathcal{D} + u.[\mathcal{D}, u^{\dagger}]\}}$

$$\Rightarrow\ \frac{u.J.\mathcal{D}.J^{\dagger}.u^{\dagger}}{\varepsilon'} + \frac{u.J.u.[\mathcal{D}, u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'}$$

$$\to\ u.\mathcal{D}.u^{\dagger} + \frac{u.J.u.[\mathcal{D}, u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'} \longrightarrow \boxed{\mathcal{D} + u.[\mathcal{D}, u^{\dagger}] + \frac{u.J.u.[\mathcal{D}, u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'}} \longleftarrow \text{CHECK}$$

■Simplify the term:

→ $\boxed{u.J.u.J^{\dagger}.\mathcal{A}.J.u^{\dagger}.J^{\dagger}.u^{\dagger}}$

Use: $\{\mathcal{A}.J.u^{\dagger}.J^{\dagger} \to J.u^{\dagger}.J^{\dagger}.\mathcal{A}\}$

→ $\boxed{u.\mathcal{A}.u^{\dagger}}$

■Simplify the term:

→ $\boxed{u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}\ \varepsilon'}$

Append 1→ $J.J^{\dagger}$ ⇒ $u.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}.J.J^{\dagger}\ \varepsilon'$

Use $\{\mathcal{A}.J.u^{\dagger}.J^{\dagger} \to J.u^{\dagger}.J^{\dagger}.\mathcal{A}\}$ with ConjugateTranspose: $aa : a \mid J \to aa^{\dagger}$

→ $\{\mathcal{A}.J^{\dagger}.u^{\dagger}.J \to J^{\dagger}.u^{\dagger}.J.\mathcal{A}\}$

The Rule applies to: $\mathcal{A} \to u.\mathcal{A}.u^{\dagger} \longrightarrow \{u.\mathcal{A}.u^{\dagger}.J^{\dagger}.u^{\dagger}.J \to J^{\dagger}.u^{\dagger}.J.u.\mathcal{A}.u^{\dagger}\}$

⇒ $u.J.J^{\dagger}.u^{\dagger}.J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon' \longrightarrow \boxed{J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$

■Check if equal to (2.20). Our calculation:

$$\boxed{U.\mathcal{D}_{\mathcal{A}}.U^{\dagger} \to \mathcal{D} + u.[\mathcal{D},\ u^{\dagger}] + u.\mathcal{A}.u^{\dagger} + \frac{u.J.u.[\mathcal{D},\ u^{\dagger}].J^{\dagger}.u^{\dagger}}{\varepsilon'} + J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$$

Evaluate (2.20) with $\mathcal{D}_{\mathcal{A}^u} \to \mathcal{A}^u + \mathcal{D} + \varepsilon'.J.\mathcal{A}^u.J^{\dagger}$⟵CHECK

→ $\boxed{\mathcal{D}_{\mathcal{A}^u} \to \mathcal{D} + u.[\mathcal{D},\ u^{\dagger}] + u.\mathcal{A}.u^{\dagger} + J.u.[\mathcal{D},\ u^{\dagger}].J^{\dagger}\ \varepsilon' + J.u.\mathcal{A}.u^{\dagger}.J^{\dagger}\ \varepsilon'}$

Almost equal.

```
PR["●Define bilinear form: ", $0 = $ = 𝔘𝒟[ξ, ξp] → BraKet[J.ξ, 𝒟.ξp](*⟨J.ξ,𝒟.ξp⟩*),
  Yield, $ = $ /. dd : 𝒟.ξp → -J.J.dd //. simpleBraKet[],
  Yield, $ = $ /. BraKet[J.a_, J.b_] → BraKet[b, a] /. J.𝒟 → 𝒟.J,
  Yield, $ = $ /. BraKet[𝒟.a_, b_] → BraKet[a, 𝒟.b](*𝒟 is Hermitian*),
  Yield, $s = Reverse[$0] // tuAddPatternVariable[{ξp, ξ}],
  Yield, $ = $ /. $s; Framed[$]
 ];
```

●Define bilinear form: $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to \langle J.\xi \mid \mathcal{D}.\xi p\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\langle J.\xi \mid J.J.\mathcal{D}.\xi p\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\langle \mathcal{D}.J.\xi p \mid \xi\rangle$

→ $\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\langle J.\xi p \mid \mathcal{D}.\xi\rangle$

→ $\langle J.(\xi\_) \mid \mathcal{D}.(\xi p\_)\rangle \to \mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p]$

→ $\boxed{\mathfrak{U}_{\mathcal{D}}[\xi,\ \xi p] \to -\mathfrak{U}_{\mathcal{D}}[\xi p,\ \xi]}$

```
PR["●Define classical fermions: ", (ℋ⁺)cl → {ξ̃ → Grassmann, ξ ∈ ℋ⁺},
  NL, "●Define action functional: ", $S = S → Sb + Sf → Tr[f[𝒟𝒜 / Λ]] + BraKet[J.ξ̃, 𝒟𝒜.ξ̃] / 2
 ];
```

●Define classical fermions: $\mathcal{H}^{+}{}_{cl} \to \{\tilde{\xi} \to \text{Grassmann},\ \xi \in \mathcal{H}^{+}\}$

●Define action functional: $S \to S_b + S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle + Tr[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

```
PR["●INvariance of action functional under ",
 $s = {𝒟𝒜 → U.𝒟𝒜.ConjugateTranspose[U], xx : ξ̃ → U.xx},
 NL, "■Boson ", $0 = $ = tuExtractPattern[Tr[_]][$S] // First,
 yield, $ = $ /. $s,
 yield, xSum[f[λn / Λ], n], CG[" Invariant"],
 NL, "■Fermion ", $0 = $ = tuExtractPattern[BraKet[_, _]][$S] // First,
 Yield, $ = $ /. $s,
 NL, "Apply ",
 $s = {J.U → U.J, ConjugateTranspose[u_].u_ → 1, BraKet[U . a_, U.b_] -> BraKet[a, b]},
 Yield, $ = $ //. $s // tuDotSimplify[], CG[" Invariant"]

]
```

●INvariance of action functional under $\{\mathcal{D}_{\mathcal{A}} \to U.\mathcal{D}_{\mathcal{A}}.U^{\dagger},\ xx : \tilde{\xi} \to U.xx\}$

■Boson $Tr[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \longrightarrow Tr[f[\frac{U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}}{\Lambda}]] \longrightarrow \sum_{n}[f[\frac{\lambda_n}{\Lambda}]]$ Invariant

■Fermion $\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$

$\to \langle J.U.\tilde{\xi} \mid U.\mathcal{D}_{\mathcal{A}}.U^{\dagger}.U.\tilde{\xi}\rangle$

Apply $\{J.U \to U.J,\ u\_^{\dagger}.(u\_) \to 1,\ \langle U.(a\_) \mid U.(b\_)\rangle \to \langle a \mid b\rangle\}$

$\to \langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$ Invariant

```
PR["●Theorem 2.19. A real even almost-commutative manifold M×F describes
    a gauge theory on M with gauge group 𝒢[M×F]->C∞[M,𝒢[F]]. ",
  NL, "•Sketch of Proof: ",
  $t219 = $ = {{"(2.19)" -> I 𝒜μ[x] ∈ 𝔤[F] → Mod[𝒰[𝒜F], 𝒽F],
       "Total algebra" → 𝒜 → C"∞"[M, 𝒜F] → xSum[section[i, Γ[M×𝒜F]], {i}],
       ω["𝔤[F]-valued 1-form"] → I T[𝒜, "d", {μ}] . DifForm[T[x, "u", {μ}]],
       P["Principal bundle"] → M×𝒢[F],
       "(2.22)" → ω["connection form on P"],
       "group of gauge transform"[P] -> C"∞"[M, 𝒢[F]],
       "(2.12)" ⇒ "group of gauge transform"[P] == 𝒢[M×F],
       "(2.11)" ⇒ 𝒢[M×F] -> {U → u.J.u.ConjugateTranspose[J], u ∈ U[𝒜]},
       rep[𝒜F[ℋF]] → rep[𝒢[ℋF]]
         ⇒ M×ℋF ↔ "vector bundle of principal bundle"[P → M×𝒢[F]]
      }}; Grid[Transpose[$], Frame → All],
  NL, "Note: ", {("E" → M×ℋF) ↔
      (P["Principal bundle"] → M×𝒢[F])⟹"action of gauge group on fermions",
     ℋ["ACM"] → L2[M, S]⊗ℋF → L2[M, S⊗"E"],
     "⇒ particle fields"⟶section[S⊗"E"]} // Column
 ];
```

●Theorem 2.19. A real even almost-commutative manifold M×F describes a gauge theory on M with gauge group $\mathcal{G}$[M×F]->C$^\infty$[M,$\mathcal{G}$[F]].
•Sketch of Proof:

(2.19) → $i\,\mathcal{A}_\mu$[x] ∈ $\mathfrak{g}$[F] → Mod[$\mathcal{U}$[$\mathcal{A}_F$], h_F]
Total algebra → $\mathcal{A}$ → C$^\infty$[M, $\mathcal{A}_F$] → $\sum_{\{i\}}$[section[i, Γ[M×$\mathcal{A}_F$]]]
ω[$\mathfrak{g}$[F]-valued 1-form] → $i\,\mathcal{A}_\mu$.d[x^μ]
P[Principal bundle] → M×$\mathcal{G}$[F]
(2.22) → ω[connection form on P]
group of gauge transform[P] → C$^\infty$[M, $\mathcal{G}$[F]]
(2.12) ⇒ group of gauge transform[P] == $\mathcal{G}$[M×F]
(2.11) ⇒ $\mathcal{G}$[M×F] → {U → u.J.u.$J^\dagger$, u ∈ U[$\mathcal{A}$]}
rep[$\mathcal{A}_F$[$\mathcal{H}_F$]] → rep[$\mathcal{G}$[$\mathcal{H}_F$]] ⇒ M×$\mathcal{H}_F$ ↔ vector bundle of principal bundle[P → M×$\mathcal{G}$[F]]

Note: (E → M×$\mathcal{H}_F$) ↔ (P[Principal bundle] → M×$\mathcal{G}$[F]) ⟹ action of gauge group on fermions
$\mathcal{H}$[ACM] → L^2[M, S]⊗$\mathcal{H}_F$ → L^2[M, S⊗E]
⇒ particle fields ⟶ section[S⊗E]

The Spectral Action

```
PR["●Lichnerowicz formula.",
 NL, "•vector bundle ", "E" → M,
 NL, "•Laplacian ", △"E"["∇""E"["connection on E"]],
 NL, "•generalized Laplacian ", H → {△"E" - F, F ∈ Γ[End["E"]]},
 NL, "•generalized Dirac operator[ℤ2graded vector bundle E]",
 yield, (𝒟[Γ[M, "E""±"]] -> Γ[M, "E""∓"]), imply, 𝒟.𝒟 → H
]
```

●Lichnerowicz formula.
•vector bundle E → M
•Laplacian Δ^E[∇^E[connection on E]]
•generalized Laplacian H → {-F + Δ^E, F ∈ Γ[End[E]]}
•generalized Dirac operator[$\mathbb{Z}_2$graded vector bundle E] ⟶ $\mathcal{D}$[Γ[M, $E^\pm$]] → Γ[M, $E^\mp$] ⇒ $\mathcal{D}$.$\mathcal{D}$ → H

```
PR["■Show ", $ = 𝒟𝒜 -> "generalized Dirac operator" → 𝒟𝒜.𝒟𝒜 → H,
  NL, "•compute ", $[[2, 2, 1]],
  " where ",
  $sDA = $s0 = $s = {𝒟𝒜 -> -I T[γ, "u", {μ}].T["∇"^"E", "d", {μ}] + T[γ, "d", {5}] ⊗ Φ,
      T["∇"^"E", "d", {μ_}] -> T["∇"^"S", "d", {μ}] ⊗ 1ℋF + I 1N ⊗ Bμ,
      T["∇"^"E", "d", {μ}][S ⊗ "E"],
      Φ ∈ Γ[End["E"]] -> "Higg's field"
     }; Column[$s],
  NL, "•Define ",
  $d = {T[𝒟, "d", {μ}][a_] → ad[T["∇"^"E", "d", {μ}]][a], ad[aa_][bb_] → aa.bb - bb.aa},
  "xPOFF",
  Yield, $ = $0 = T[𝒟, "d", {μ}][Φ],
  Yield, $ = $ /. $d,
  Yield, $ = $ /. $d,
  Yield, $ = $ /. $s[[1 ;; 2]],
  Yield, $ = $ // tuDotSimplify[], "PONdd",
  NL, "Using ", $s = {(op_ ⊗ 1ℋF).ph_ → op[ph] ⊗ 1ℋF + ph.(op ⊗ 1ℋF), (1N ⊗ op_).ph_ → 1N ⊗ op.ph,
    ph_ .(1N ⊗ op_) → 1N ⊗ ph.op, ca_ 1N ⊗ a_ + cb_ 1N ⊗ b_ -> 1N ⊗ (ca a + cb b)};
  Column[$s],
  Yield, $ = $0 -> $ //. $s // Simplify,
  Yield, $ = $ /. a_ ⊗ 1ℋF → 1N ⊗ a //
    tuRepeat[{}, (Expand[tuDotSimplify[][#]] //. tuOpDistribute[CircleTimes] //.
        tuOpSimplify[CircleTimes]) &];
  Framed[$D1 = $]
 ];
```

■Show $\mathcal{D}_{\mathcal{A}} \to$ generalized Dirac operator $\to \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to H$

•compute $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}}$ where

$\mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \Phi - i\, \gamma^{\mu}.\nabla^{E}{}_{\mu}$

$\nabla^{E}{}_{\mu\_} \to i\, 1_N \otimes B_{\mu} + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}$

$\nabla^{E}{}_{\mu}[S \otimes E]$

$\Phi \in \Gamma[\mathrm{End}[E]] \to$ Higg's field

•Define $\{\mathcal{D}_{\mu}[a_] \to \mathrm{ad}[\nabla^{E}{}_{\mu}][a],\ \mathrm{ad}[aa_][bb_] \to aa.bb - bb.aa\}$xPOFF

$\to \mathcal{D}_{\mu}[\Phi]$

$\to \mathrm{ad}[\nabla^{E}{}_{\mu}][\Phi]$

$\to -\Phi.\nabla^{E}{}_{\mu} + \nabla^{E}{}_{\mu}.\Phi$

$\to -\Phi.(i\, 1_N \otimes B_{\mu} + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}) + (i\, 1_N \otimes B_{\mu} + \nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).\Phi$

$\to -i\,\Phi.(1_N \otimes B_{\mu}) - \Phi.(\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}) + i\,(1_N \otimes B_{\mu}).\Phi + (\nabla^{S}{}_{\mu} \otimes 1_{\mathcal{H}_F}).\Phi$PONdd

Using

$(op\_ \otimes 1_{\mathcal{H}_F}).(ph\_) \to op[ph] \otimes 1_{\mathcal{H}_F} + ph.(op \otimes 1_{\mathcal{H}_F})$

$(1_N \otimes op\_).(ph\_) \to 1_N \otimes op.ph$

$(ph\_).(1_N \otimes op\_) \to 1_N \otimes ph.op$

$1_N \otimes a\_\ ca\_ + 1_N \otimes b\_\ cb\_ \to 1_N \otimes (a\, ca + b\, cb)$

$\to \mathcal{D}_{\mu}[\Phi] \to 1_N \otimes (-i\,(\Phi.B_{\mu} - B_{\mu}.\Phi)) + \nabla^{S}{}_{\mu}[\Phi] \otimes 1_{\mathcal{H}_F}$

$\to \boxed{\mathcal{D}_{\mu}[\Phi] \to -i\, 1_N \otimes \Phi.B_{\mu} + i\, 1_N \otimes B_{\mu}.\Phi + 1_N \otimes \nabla^{S}{}_{\mu}[\Phi]}$

```
PR["•Define curvature of Bμ: ",
  $F = T[F, "dd", {μ, ν}] → tuDPartial[Bν, μ] - tuDPartial[Bμ, ν] + I CommutatorM[Bμ, Bν],
  NL, "•Define curvature of ", "∇""E", ": ",
  $O = {Ω"E"[X, Y] -> T["∇""E", "d", {X}].T["∇""E", "d", {Y}] - T["∇""E", "d", {Y}].T["∇""E",
        "d", {X}] - T["∇""E", "d", {CommutatorM[X, Y]}], {X, Y} → "vector fields"},
  NL, CO["▪For local coordinates: "], CommutatorM[tuDPartial[_, μ],
    tuDPartial[_, ν]] → 0,
  NL, "define ", {tuDPartial[_, μ] → X, tuDPartial[_, ν] → Y},
  Yield, $s = {CommutatorM[X, Y] → 0, X -> μ, Y → ν, T["∇""E", "d", {0}] → 0},
  Imply, e33 = $ = $O[[1]] //. $s,
  Yield, $ = $ /. $sDA[[1 ;; 2]],
  Yield, $ = $ // tuDotSimplify[],
  NL, "Using: ", $scc = $s = {
     (a_ ⊗ b_).(c_ ⊗ d_) :→ a . c ⊗ b . d +
       If[! FreeQ[a, "∇"] && ! FreeQ[d, B | Φ], c ⊗ a[d] , 0] +
       If[! FreeQ[b, "∇"] && ! FreeQ[d, B | Φ], a ⊗ b[d], 0],
     1N_ . a_ → a, a_ . 1N_ → a , (a_ ⊗ 1ℋF) - (b_ ⊗ 1ℋF) → (a - b) ⊗ 1ℋF,
     (1N_ ⊗ a_) - (1N_ ⊗ b_) → 1N ⊗ (a - b)};
  ColumnSumExp[$s],
  Yield, $ = $ //. $s // Simplify // Expand; $ // ColumnSumExp // Framed,
  NL, "Use ", $s = {I 1N ⊗ a_ - I 1N ⊗ b_ -> 1N ⊗ ( I a - I b),
    1N ⊗ a_ + 1N ⊗ b_ -> 1N ⊗ ( a + b), T["∇""S", "d", {a_}][b_] → tuDPartial[b, a]
   }; Column[$s],
  Yield, $ = $ //. $s,
  NL, "Apply (3.2) ",
  $s = tuRuleSolve[$F, CommutatorM[_, _]] /. CommutatorM → MCommutator // First //
    Map[-# &, #] &,
  NL, "Define ", $s1 = $O[[1]] //. {"E" → S, CommutatorM[X, Y] → 0,
     X -> μ, Y → ν, T["∇""S", "d", {0}] → 0}, CK,
  Yield, $s34 = e34 = $ = $ /. $s /. Reverse[$s1] // Simplify;
  Framed[$], CG[" (3.4)"]
 ];
```

•Define curvature of B_μ: $F_{\mu\nu} \to \mathrm{i}\,[B_\mu, B_\nu] - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

•Define curvature of ∇^E: $\{\Omega^E[X, Y] \to \nabla^E{}_X.\nabla^E{}_Y - \nabla^E{}_Y.\nabla^E{}_X - \nabla^E{}_{[X,Y]}, \{X, Y\} \to \text{vector fields}\}$

▪For local coordinates: $[\underline{\partial}_\mu[\_], \underline{\partial}_\nu[\_]] \to 0$

define $\{\underline{\partial}_\mu[\_] \to X, \underline{\partial}_\nu[\_] \to Y\}$

$\to \{[X, Y] \to 0, X \to \mu, Y \to \nu, \nabla^E{}_0 \to 0\}$

$\Rightarrow \Omega^E[\mu, \nu] \to \nabla^E{}_\mu.\nabla^E{}_\nu - \nabla^E{}_\nu.\nabla^E{}_\mu$

$\to \Omega^E[\mu, \nu] \to (\mathrm{i}\,1_N \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(\mathrm{i}\,1_N \otimes B_\nu + \nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) - (\mathrm{i}\,1_N \otimes B_\nu + \nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}).(\mathrm{i}\,1_N \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})$

$\to \Omega^E[\mu, \nu] \to -(1_N \otimes B_\mu).(1_N \otimes B_\nu) + \mathrm{i}\,(1_N \otimes B_\mu).(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) + (1_N \otimes B_\nu).(1_N \otimes B_\mu) - \mathrm{i}\,(1_N \otimes B_\nu).(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}) + \mathrm{i}\,(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(1_N \otimes B_\nu) + (\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) - \mathrm{i}\,(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}).(1_N \otimes B_\mu) - (\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})$

Using: $\{(a\_ \otimes b\_).(c\_ \otimes d\_) :\to \sum[\begin{matrix} a.c \otimes b.d \\ \text{If[! FreeQ[a, } \nabla] \text{ \&\& ! FreeQ[d, B} \mid \Phi], c \otimes a[d], 0] \\ \text{If[! FreeQ[b, } \nabla] \text{ \&\& ! FreeQ[d, B} \mid \Phi], a \otimes b[d], 0] \end{matrix}],$

$1_N\_.(a\_) \to a, (a\_).1_N\_ \to a, \sum[\begin{matrix} a\_ \otimes 1_{\mathcal{H}_F} \\ -(b\_ \otimes 1_{\mathcal{H}_F}) \end{matrix}] \to \sum[\begin{matrix} a \\ -b \end{matrix}] \otimes 1_{\mathcal{H}_F}, \sum[\begin{matrix} 1_N\_ \otimes a\_ \\ -(1_N\_ \otimes b\_) \end{matrix}] \to 1_N \otimes \sum[\begin{matrix} a \\ -b \end{matrix}]\}$

$$\to \boxed{\Omega^E[\mu, \nu] \to \sum[\begin{matrix} (\nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu) \otimes 1_{\mathcal{H}_F} \\ 1_N \otimes (-B_\mu.B_\nu + B_\nu.B_\mu) \\ \mathrm{i}\,1_N \otimes \nabla^S{}_\mu[B_\nu] \\ -\mathrm{i}\,1_N \otimes \nabla^S{}_\nu[B_\mu] \end{matrix}]}$$

Use $\begin{matrix} \mathrm{i}\,1_N \otimes a\_ - \mathrm{i}\,1_N \otimes b\_ \to 1_N \otimes (\mathrm{i}\,a - \mathrm{i}\,b) \\ 1_N \otimes a\_ + 1_N \otimes b\_ \to 1_N \otimes (a + b) \\ \nabla^S{}_{a\_}[b\_] \to \underline{\partial}_a[b] \end{matrix}$

$\to \Omega^E[\mu, \nu] \to (\nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu) \otimes 1_{\mathcal{H}_F} + 1_N \otimes (-B_\mu.B_\nu + B_\nu.B_\mu - \mathrm{i}\,\underline{\partial}_\nu[B_\mu] + \mathrm{i}\,\underline{\partial}_\mu[B_\nu])$

Apply (3.2) $-B_\mu.B_\nu + B_\nu.B_\mu \to \mathrm{i}\,(F_{\mu\nu} + \underline{\partial}_\nu[B_\mu] - \underline{\partial}_\mu[B_\nu])$

Define $\Omega^S[\mu, \nu] \to \nabla^S{}_\mu.\nabla^S{}_\nu - \nabla^S{}_\nu.\nabla^S{}_\mu \longleftarrow$ CHECK

$$\to \boxed{\Omega^E[\mu, \nu] \to 1_N \otimes (\mathrm{i}\,F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F}} \quad (3.4)$$

```
$d;
PR["•Calculate ", $0 = $ = CommutatorM[T[𝒟, "d", {μ}], T[𝒟, "d", {ν}]].Φ,
  NL, "From the definition: ", $d,
  Yield, $ = $ /. CommutatorM → MCommutator // tuDotSimplify[],
  yield, $ = $ //. a_ . b_ → a[b],
  Yield, $ = $ //. $d,
  Yield, $ = $ // tuDotSimplify[],
  NL, "Use ", $s =
   {a_ .Φ - b_ .Φ → (a - b).Φ, Φ.a_ - Φ. b_ → Φ.(a - b), a_ .b_ - b_ .a_ → CommutatorM[a, b],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}]},
  Yield, $ = $ // tuRepeat[$s, tuDotSimplify[]]; Framed[$0 → $],
  NL, "From ", $s1 = e33,
  yield, $s1 = $s1 /. $s // Reverse // tuAddPatternVariable[{μ, ν}],
  Imply, $ = $ /. $s1; Framed[$0 → $],
  yield, $ = $ /. CommutatorM → MCommutator /.
    ($d[[2]] // RemovePatterns // Reverse // tuAddPatternVariable[{aa, bb}]);
  Framed[$0 → $]
 ];
```

•Calculate $[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi$

From the definition: $\{\mathcal{D}_\mu[a\_] \to \mathrm{ad}[\nabla^E{}_\mu][a], \mathrm{ad}[aa\_][bb\_] \to aa.bb - bb.aa\}$

$\to \mathcal{D}_\mu.\mathcal{D}_\nu.\Phi - \mathcal{D}_\nu.\mathcal{D}_\mu.\Phi \longrightarrow \mathcal{D}_\mu[\mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[\mathcal{D}_\mu[\Phi]]$

$\to \nabla^E{}_\mu.\mathcal{D}_\nu[\Phi] - \mathcal{D}_\nu[\Phi].\nabla^E{}_\mu - \mathcal{D}_\nu[-\Phi.\nabla^E{}_\mu + \nabla^E{}_\mu.\Phi]$

$\to \nabla^E{}_\mu.\mathcal{D}_\nu[\Phi] - \mathcal{D}_\nu[\Phi].\nabla^E{}_\mu - \mathcal{D}_\nu[-\Phi.\nabla^E{}_\mu + \nabla^E{}_\mu.\Phi]$

Use $\{(a\_).\Phi - (b\_).\Phi \to (a-b).\Phi, \Phi.(a\_) - \Phi.(b\_) \to \Phi.(a-b),$
$(a\_).(b\_) - (b\_).(a\_) \to [a, b], [a\_, b\_] :\to -[b, a] /; \mathrm{OrderedQ}[\{b, a\}]\}$

$\to \boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi \to [\nabla^E{}_\mu, \mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[-[\Phi, \nabla^E{}_\mu]]}$

From $\Omega^E[\mu, \nu] \to \nabla^E{}_\mu.\nabla^E{}_\nu - \nabla^E{}_\nu.\nabla^E{}_\mu \longrightarrow [\nabla^E{}_{\mu\_}, \nabla^E{}_{\nu\_}] \to \Omega^E[\mu, \nu]$

$\Rightarrow \boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi \to [\nabla^E{}_\mu, \mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[-[\Phi, \nabla^E{}_\mu]]} \longrightarrow \boxed{[\mathcal{D}_\mu, \mathcal{D}_\nu].\Phi \to \mathrm{ad}[\nabla^E{}_\mu][\mathcal{D}_\nu[\Phi]] - \mathcal{D}_\nu[-\Phi.\nabla^E{}_\mu + \nabla^E{}_\mu.\Phi]}$

```
$scc;
PR["Local Laplacian: ",
  $0 = $ = △"E" → -T[g, "uu", {μ, ν}].(T["∇""E", "d", {μ}].T["∇""E", "d", {ν}] -
        T[Γ, "udd", {ρ, μ, ν}].T["∇""E", "d", {ρ}]),
  NL, "Use definition ", $s = $sDA[[2]],
  Yield, $ = $ /. $s // tuDotSimplify[],
  Yield, $ = $ //. $scc /. {a_ . (b_ ⊗ c : 1_) → (a.b) ⊗ c};
  ColumnSumExp[$] // Framed,
  NL, "Define ", $s = $0 /. "E" → S,
  yield, $s = Map[# ⊗ 1_{H_F} &, $s] // tuDotSimplify[];
  $s = $s /. (a_ + b_) ⊗ c_ → a ⊗ c + b ⊗ c /. tuOpSimplify[CircleTimes] // Reverse,
  Imply, $ =
   $ /. $s /. a_ (tt : T[g, "uu", {μ, ν}]) .b_ → tt.(a b) //. (tt : T[g, "uu", {μ, ν}]).(a_) +
       (tt : T[g, "uu", {μ, ν}]) .b_ → tt.(a + b) // ExpandAll;
  ColumnSumExp[$],
  NL, "Use ", $s = {a_ . (1_N ⊗ c_) → a ⊗ c, a_ ⊗ B_μ :> (a /. ν → μ) ⊗ B_ν},
  $ = $ /. $s; Framed[e35 = $], CG[" (3.5)"]
 ];
```

Local Laplacian: $\triangle^E \to -g^{\mu\nu}.(\nabla^E{}_\mu.\nabla^E{}_\nu - \Gamma^\rho{}_{\mu\nu}.\nabla^E{}_\rho)$

Use definition $\nabla^E{}_{\mu\_} \to i\, 1_N \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}$

$\to$ $\triangle^E \to g^{\mu\nu}.(1_N \otimes B_\mu).(1_N \otimes B_\nu) - i\, g^{\mu\nu}.(1_N \otimes B_\mu).(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) - i\, g^{\mu\nu}.(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(1_N \otimes B_\nu) - g^{\mu\nu}.(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).(\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) + i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N \otimes B_\rho) + g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(\nabla^S{}_\rho \otimes 1_{\mathcal{H}_F})$

$$\to \boxed{\triangle^E \to \sum\left[\begin{array}{l} -(g^{\mu\nu}.\nabla^S{}_\mu.\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) \\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.\nabla^S{}_\rho \otimes 1_{\mathcal{H}_F} \\ g^{\mu\nu}.(1_N \otimes B_\mu.B_\nu) \\ -i\, g^{\mu\nu}.(1_N \otimes \nabla^S{}_\mu[B_\nu] + \nabla^S{}_\mu \otimes B_\nu) \\ -i\, g^{\mu\nu}.(\nabla^S{}_\nu \otimes B_\mu) \\ i\, g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.(1_N \otimes B_\rho) \end{array}\right]}$$

Define $\triangle^S \to -g^{\mu\nu}.(\nabla^S{}_\mu.\nabla^S{}_\nu - \Gamma^\rho{}_{\mu\nu}.\nabla^S{}_\rho)$ $\longrightarrow$ $-(g^{\mu\nu}.\nabla^S{}_\mu.\nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}) + g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu}.\nabla^S{}_\rho \otimes 1_{\mathcal{H}_F} \to \triangle^S \otimes 1_{\mathcal{H}_F}$

$$\Rightarrow \triangle^E \to \sum\left[\begin{array}{l} \triangle^S \otimes 1_{\mathcal{H}_F} \\ g^{\mu\nu}.(1_N \otimes B_\mu.B_\nu - i\, 1_N \otimes \nabla^S{}_\mu[B_\nu] - i\, \nabla^S{}_\mu \otimes B_\nu - i\, \nabla^S{}_\nu \otimes B_\mu + i\, \Gamma^\rho{}_{\mu\nu}.(1_N \otimes B_\rho)) \end{array}\right]$$

Use $\{(a\_).(1_N \otimes c\_) \to a \otimes c,\ a\_ \otimes B_\mu :\to (a /. \nu \to \mu) \otimes B_\nu\}$

$$\boxed{\triangle^E \to \triangle^S \otimes 1_{\mathcal{H}_F} + g^{\mu\nu}.(1_N \otimes B_\mu.B_\nu - i\, 1_N \otimes \nabla^S{}_\mu[B_\nu] - 2\, i\, \nabla^S{}_\mu \otimes B_\nu + i\, \Gamma^\rho{}_{\mu\nu} \otimes B_\rho)} \quad (3.5)$$

```
$sDA;
$sD;
$scc;
PR["•Given the Lichnerowicz formula: ",
  $L = slash[𝒟].slash[𝒟] → △^S + s / 4,
  NL, "Show(prop.3.1) ", $31 = $0 = $ =
    {𝒟𝒜.𝒟𝒜 → △^"E" - Q, Q → -( s ⊗ 1_ℋF ) / 4 - 1_N ⊗ (Φ . Φ) + I / 2 (T[γ, "u", {μ}].T[γ, "u", {ν}]) ⊗
        T[F, "dd", {μ, ν}] - I T[γ, "u", {μ}]. T[γ, "d", {5}] ⊗ T[𝒟, "d", {μ}].Φ},
  Yield, $ = $0[[1, 1]], CK,
  Yield, xtmp = $ = $ //. $sDA[[1 ;; 2]] /. a_ . b_ :→ a. (b /. μ → ν), CK, (***)
  NL, "Use ",
  $s = $ss = {
    T[γ, "d", {5}].T[γ, "d", {5}] → 1_N,
    (tt : T[γ, "u", {μ_}]).(1_N ⊗ b_) :→ (tt ⊗ b) /; ! FreeQ[b, μ],
    (a_ ⊗ b_).(c_ ⊗ d_) :→ a . c ⊗ b . d +
      If[ ! FreeQ [a, "∇"] && ! FreeQ [d, B], c ⊗ a[d] , 0] +
      If[ ! FreeQ [b, "∇"] && ! FreeQ [d, B], a ⊗ b[d], 0],
    (tt : T[γ, "u", {μ_}]).(a_ ⊗ b_) :→ (tt.a ⊗ b) /; ! FreeQ[a, "∇"],
    (tt : T[γ, "u", {μ_}]). Shortest[a_] .b_ :→
     I slash[𝒟].b /; ! FreeQ[a, "∇" && ! (FreeQ[a, μ])],
    (tt : T[γ, "u", {μ_}]) . a_ .b_ :→ I slash[𝒟] /;
      ! FreeQ[a, "∇" && ! (FreeQ[a, μ] && FreeQ[b, "∇"])],
    b__ .(tt : T[γ, "u", {μ_}]). a_ :→ I b.slash[𝒟] /; ! FreeQ[a, "∇" && ! (FreeQ[a, μ])],
    1_N_ . a_ → a, a_ . 1_N_ → a ,
    (a_ ⊗ 1_N_) - (b_ ⊗ 1_N_) → (a - b) ⊗ 1_N, (1_N_ ⊗ a_) - (1_N_ ⊗ b_) → 1_N ⊗ (a - b)};
  Column[$s],
  $ = $ // tuRepeat[$s, tuDotSimplify[]];
  $pass = $ = $ /. tuOpSimplify[CircleTimes]; ColumnSumExp[$] // Framed
 ];
```

•Given the Lichnerowicz formula: $(\not{D}).(\not{D}) \to \frac{s}{4} + \triangle^S$

Show(prop.3.1) $\{\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to -Q + \triangle^E,\ Q \to -\frac{1}{4} s \otimes 1_{\mathcal{H}_F} - i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi\}$

$\to\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}}$ ⟵CHECK

$\to\ (\gamma_5 \otimes \Phi - i\,\gamma^\mu.(i\,1_N \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})).(\gamma_5 \otimes \Phi - i\,\gamma^\nu.(i\,1_N \otimes B_\nu + \nabla^S{}_\nu \otimes 1_{\mathcal{H}_F}))$ ⟵CHECK

Use

```
γ5.γ5 → 1N
(tt : γ^μ_).(1N⊗b_) ⧴ tt⊗b /; ! FreeQ[b, μ]
(a_⊗b_).(c_⊗d_) ⧴ a.c⊗b.d +
   If[! FreeQ[a, ∇] && ! FreeQ[d, B], c⊗a[d], 0] + If[! FreeQ[b, ∇] && ! FreeQ[d, B], a⊗b[d], 0]
(tt : γ^μ_).(a_⊗b_) ⧴ tt.a⊗b /; ! FreeQ[a, ∇]
(tt : γ^μ_).Shortest[a_].(b_) ⧴ i (D̸).b /; ! FreeQ[a, ∇ && ! FreeQ[a, μ]]
(tt : γ^μ_).(a_).(b_) ⧴ i (D̸) /; ! FreeQ[a, ∇ && ! (FreeQ[a, μ] && FreeQ[b, ∇])]
(b__).(tt : γ^μ_).(a_) ⧴ i b.(D̸) /; ! FreeQ[a, ∇ && ! FreeQ[a, μ]]
1N_.(a_) → a
(a_).1N_ → a
a_⊗1N_ - b_⊗1N_ → (a - b)⊗1N
1N_⊗a_ - 1N_⊗b_ → 1N⊗(a - b)
```

$$\boxed{\sum\left[\begin{array}{l} (\not{D}).(\not{D}) \otimes 1_{\mathcal{H}_F} \\ (\not{D}).\gamma_5 \otimes \Phi \\ (\not{D}).\gamma^\nu \otimes B_\nu \\ \gamma_5.(\not{D}) \otimes \Phi \\ \gamma_5.\gamma^\nu \otimes \Phi.B_\nu \\ \gamma^\mu.(\not{D}) \otimes B_\mu \\ \gamma^\mu.\gamma_5 \otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\nu \otimes B_\mu.B_\nu \\ 1_N \otimes \Phi.\Phi \\ -i\,\gamma^\mu.(\gamma^\nu \otimes \nabla^S{}_\mu[B_\nu]) \end{array}\right]}$$

```
PR["•Examine different terms of: ", $0 = $pass; ColumnSumExp[$0],
  NL, "•1: ", $ = $0[[1]] → "Lichnerowicz formula" → Framed[$p[1] = $L[[2]] ⊗ 1HF],
  NL, "•2,4: ", $ = $0[[{2, 4}]],
  NL, "Use ", CommutatorM[T[γ, "d", {5}], slash[D]] → 0,
  imply, $ → Framed[0],
  CO[back, "Liebnitz like rule accounted for by[[10]] ", $p[6] = $0[[10]]],

  NL, "•3,6,10: ", $p[2] = $ = $0[[{3, 6, 10}]]; Framed[$], CK,
  NL, "•5,7: ", $ = $0[[{5, 7}]],
  NL, "Use ", $s = CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  yield, $s = $s /. CommutatorP → ACommutator,
  yield, $s = -$s[[1, 2]] + # & /@ $s // tuAddPatternVariable[{μ}],
  Imply, $ = $ /. $s /. tuOpSimplify[CircleTimes] /. ν → μ,
  yield, $ = $ /. (a_ ⊗ b_ ) - (a_ ⊗ c_) → a ⊗ (b - c); Framed[$p[3] = $],
  NL, "•8: ", $ = $0[[8]],
  NL, "Use symmetic and antisymmetric form: ",
  $s = $[[2]] → 1 / 2 (Apply[CommutatorP, $[[2]]] + Apply[CommutatorM, $[[2]]]),
  Yield, $ = $ /. $s /. tuOpSimplify[CircleTimes],
  Yield, $ = $ /. a_ ⊗ (b_ + c_) -> a ⊗ (b) + a ⊗ (c); Framed[$p[4] = $],
  NL, "•9: ", $ = $0[[9]]; Framed[$p[5] = $],
  NL, "●All terms: ", $pass1 = Sum[$p[i], {i, 6}]; ColumnSumExp[$pass1]
 ];
```

•Examine different terms of: $\sum\left[\begin{array}{l} (\not{D}).(\not{D})\otimes 1_{\mathcal{H}_F} \\ (\not{D}).\gamma_5\otimes\Phi \\ (\not{D}).\gamma^\nu\otimes B_\nu \\ \gamma_5.(\not{D})\otimes\Phi \\ \gamma_5.\gamma^\nu\otimes\Phi.B_\nu \\ \gamma^\mu.(\not{D})\otimes B_\mu \\ \gamma^\mu.\gamma_5\otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu \\ 1_N\otimes\Phi.\Phi \\ -i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu]) \end{array}\right]$

•1: $(\not{D}).(\not{D})\otimes 1_{\mathcal{H}_F}\to$ Lichnerowicz formula $\to \boxed{\left(\frac{s}{4}+\triangle^S\right)\otimes 1_{\mathcal{H}_F}}$

•2,4: $(\not{D}).\gamma_5\otimes\Phi+\gamma_5.(\not{D})\otimes\Phi$

Use $[\gamma_5,\ \not{D}]\to 0\ \Rightarrow\ (\not{D}).\gamma_5\otimes\Phi+\gamma_5.(\not{D})\otimes\Phi\to\boxed{0}\ \longleftarrow$

Liebnitz like rule accounted for by[[10]] $-i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu])$

•3,6,10: $\boxed{(\not{D}).\gamma^\nu\otimes B_\nu+\gamma^\mu.(\not{D})\otimes B_\mu-i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu])}$ ⟵CHECK

•5,7: $\gamma_5.\gamma^\nu\otimes\Phi.B_\nu+\gamma^\mu.\gamma_5\otimes B_\mu.\Phi$

Use $\{\gamma_5,\ \gamma^\mu\}\to 0\ \longrightarrow\ \gamma_5.\gamma^\mu+\gamma^\mu.\gamma_5\to 0\ \longrightarrow\ \gamma_5.\gamma^\mu-\to-\gamma^\mu.\gamma_5$

$\Rightarrow\ -(\gamma^\mu.\gamma_5\otimes\Phi.B_\mu)+\gamma^\mu.\gamma_5\otimes B_\mu.\Phi\ \longrightarrow\ \boxed{\gamma^\mu.\gamma_5\otimes(-\Phi.B_\mu+B_\mu.\Phi)}$

•8: $\gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu$

Use symmetic and antisymmetric form: $B_\mu.B_\nu\to\frac{1}{2}([B_\mu,\ B_\nu]+\{B_\mu,\ B_\nu\})$

$\to\ \frac{1}{2}\gamma^\mu.\gamma^\nu\otimes([B_\mu,\ B_\nu]+\{B_\mu,\ B_\nu\})$

$\to\ \boxed{\frac{1}{2}(\gamma^\mu.\gamma^\nu\otimes[B_\mu,\ B_\nu]+\gamma^\mu.\gamma^\nu\otimes\{B_\mu,\ B_\nu\})}$

•9: $\boxed{1_N\otimes\Phi.\Phi}$

●All terms: $\sum\left[\begin{array}{l} \left(\frac{s}{4}+\triangle^S\right)\otimes 1_{\mathcal{H}_F} \\ (\not{D}).\gamma^\nu\otimes B_\nu \\ \gamma^\mu.(\not{D})\otimes B_\mu \\ \gamma^\mu.\gamma_5\otimes(-\Phi.B_\mu+B_\mu.\Phi) \\ \frac{1}{2}(\gamma^\mu.\gamma^\nu\otimes[B_\mu,\ B_\nu]+\gamma^\mu.\gamma^\nu\otimes\{B_\mu,\ B_\nu\}) \\ 1_N\otimes\Phi.\Phi \\ -2\,i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu]) \end{array}\right]$

```
PR["●Manipulate (3.5) to apply to this form: ", $35 = $ = e35,
  NL, "Use ",
  $s = $s2 = {T[g, "uu", {μ, ν}] →
      1 / 2 (T[γ, "u", {μ}].T[γ, "u", {ν}] + T[γ, "u", {ν}].T[γ, "u", {μ}]),
     a_ .(1N ⊗ c_) → (a) ⊗ c, T["∇"S, "d", {a_}][b_] → tuDPartial[b, a]},
  Imply, $ = $ // tuRepeat[Join[$s, $ss], tuDotSimplify[]];
  ColumnSumExp[$];
  Yield, $ = $ //. $s /. tuOpSimplify[CircleTimes]; ColumnSumExp[$],
  NL, "■Evaluate parts of RHS: ", $1 = $[[2]];

  NL, CB["■", $i = {3, 6}, ": "], $ = $1[[$i]],
  Yield, $ = MapAt[Swap[{μ, ν}][#] &, $, 2] /. a_ (b_ ⊗ c_) + a_ (b_ ⊗ c1_) → a (b ⊗ (c + c1)),
  NL, "From definition: ", $F,
  yield, $s = Map[# - $F[[2, {1, 2}]] &, $F] // Reverse,
  Yield, $p[1] = {#} & /@ $i -> $ /. $s //. tuOpDistribute[CircleTimes] /.
     tuOpSimplify[CircleTimes] // Expand;
  Framed[$p[1]], "POFF",
  $i1 = $i;
  NL, "■{}: ", Delete[$1, ({#} & /@ $i1)] // ColumnSumExp, CK, "PON",

  NL, CB["■", $i = {2, 5}, ": "], $ = $1[[$i]],
  yield, $ = MapAt[Swap[{μ, ν}][#] &, $, 2] /. a_ (b_ ⊗ c_) + a_ (b_ ⊗ c1_) → a (b ⊗ (c + c1)),
  NL, "Use ", $s = ACommutator[a_, b_] -> CommutatorP[a, b],
  Yield, $p[2] = {#} & /@ $i -> $ /. $s //. tuOpDistribute[CircleTimes] /.
     tuOpSimplify[CircleTimes] // Expand;
  Framed[$p[2]], "POFF",
  $i1 = Join[$i1, $i];
  NL, "■{}: ", $3 = Delete[$1, ({#} & /@ $i1)]; ColumnSumExp[$3], "PON",
  Yield, $s = {$p[1], $p[2]},
  Imply, $35 = $35[[1]] → ($3 + Apply[Plus, #[[2]] & /@ $s]);
  ColumnSumExp[$35]
 ];
```

●Manipulate (3.5) to apply to this form:
$\triangle^E \to \triangle^S\otimes 1_{\mathcal{H}_F} + g^{\mu\nu}.(1_N\otimes B_\mu.B_\nu - \mathbb{i}\,1_N\otimes\nabla^S{}_\mu[B_\nu] - 2\,\mathbb{i}\,\nabla^S{}_\mu\otimes B_\nu + \mathbb{i}\,\Gamma^\rho{}_{\mu\nu}\otimes B_\rho)$

Use $\{g^{\mu\nu}\to\frac{1}{2}(\gamma^\mu.\gamma^\nu+\gamma^\nu.\gamma^\mu),\ (a\_).(1_N\otimes c\_)\to a\otimes c,\ \nabla^S{}_{a\_}[b\_]\to\underline{\partial}_a[b]\}$

⇒

$$\to\ \triangle^E\to\sum\left[\begin{array}{l}\triangle^S\otimes 1_{\mathcal{H}_F}\\ \frac{1}{2}\gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu\\ -\frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu]\\ \gamma^\nu.(\not{D})\otimes B_\nu\\ \frac{1}{2}\gamma^\nu.\gamma^\mu\otimes B_\mu.B_\nu\\ -\frac{1}{2}\mathbb{i}\,\gamma^\nu.\gamma^\mu\otimes\underline{\partial}_\mu[B_\nu]\\ -\mathbb{i}\,\gamma^\mu.\gamma^\nu.\nabla^S{}_\mu\otimes B_\nu\\ \frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho)\\ \frac{1}{2}\mathbb{i}\,\gamma^\nu.\gamma^\mu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho)\end{array}\right]$$

■Evaluate parts of RHS:

■{3, 6}: $-\frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu]-\frac{1}{2}\mathbb{i}\,\gamma^\nu.\gamma^\mu\otimes\underline{\partial}_\mu[B_\nu]$

$\to\ -\frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes(\underline{\partial}_\nu[B_\mu]+\underline{\partial}_\mu[B_\nu])$

From definition: $F_{\mu\nu}\to\mathbb{i}\,[B_\mu,B_\nu]-\underline{\partial}_\nu[B_\mu]+\underline{\partial}_\mu[B_\nu]\ \longrightarrow\ \underline{\partial}_\mu[B_\nu]\to-\mathbb{i}\,[B_\mu,B_\nu]+F_{\mu\nu}+\underline{\partial}_\nu[B_\mu]$

$\to\ \boxed{\{\{3\},\{6\}\}\to-\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes[B_\mu,B_\nu]-\frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu}-\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu]}$

■{2, 5}: $\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes B_\mu.B_\nu+\frac{1}{2}\gamma^\nu.\gamma^\mu\otimes B_\mu.B_\nu\ \longrightarrow\ \frac{1}{2}\gamma^\mu.\gamma^\nu\otimes(B_\mu.B_\nu+B_\nu.B_\mu)$

Use $(a\_).(b\_)+(b\_).(a\_)\to\{a,b\}$

$\to\ \boxed{\{\{2\},\{5\}\}\to\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes\{B_\mu,B_\nu\}}$

→

$\{\{\{3\},\{6\}\}\to-\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes[B_\mu,B_\nu]-\frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu}-\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu],\ \{\{2\},\{5\}\}\to\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes\{B_\mu,B_\nu\}\}$

$$\Rightarrow\ \triangle^E\to\sum\left[\begin{array}{l}\triangle^S\otimes 1_{\mathcal{H}_F}\\ -\frac{1}{2}\gamma^\mu.\gamma^\nu\otimes[B_\mu,B_\nu]\\ \frac{1}{2}\gamma^\mu.\gamma^\nu\otimes\{B_\mu,B_\nu\}\\ -\frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu}\\ -\mathbb{i}\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu]\\ \gamma^\nu.(\not{D})\otimes B_\nu\\ -\mathbb{i}\,\gamma^\mu.\gamma^\nu.\nabla^S{}_\mu\otimes B_\nu\\ \frac{1}{2}\mathbb{i}\,\gamma^\mu.\gamma^\nu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho)\\ \frac{1}{2}\mathbb{i}\,\gamma^\nu.\gamma^\mu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho)\end{array}\right]$$

```
PR["Simplifying ", $ =
   $31[[1, 1]] -> $pass1 /. tuOpDistribute[CircleTimes] /. tuOpSimplify[CircleTimes],
  $ = $ /. tuRuleSolve[$35, △S ⊗ 1ℋF] // Simplify; ColumnSumExp[$],
  Yield, $ = $ //. {a_ . (b_ ⊗ c_) → (a. b) ⊗ c, a_ (b_ ⊗ c_) + a_ (b1_ ⊗ c_) → a ( b + b1) ⊗ c,
        a_ .b_ + a1_ . b_ → (a + a1).b, aa : a_ ⊗ Bν :→ (aa /. ν → μ) /; FreeQ[aa, μ],
        T["∇"S, "d", {a_}][b_] → tuDPartial[b, a],
        T[γ, "u", {μ}]. a_ . T["∇"S, "d", {μ}] → I slash[𝒟].a,
         b_ ⊗ c_ - b_ ⊗ d_ → b ⊗ (c - d),
        b_ ⊗ c_ - I b_ ⊗ d_ → b ⊗ (c - I d),
        b_ ⊗ c_ - a1_ b_ ⊗ d_ → b ⊗ (c - a1 d),
        Reverse[2 T[g, "uu", {μ, ν}] →
          (T[γ, "u", {μ}].T[γ, "u", {ν}] + T[γ, "u", {ν}].T[γ, "u", {μ}])]
      } /. tuOpSimplify[CircleTimes] /. CommutatorM → MCommutator // tuDotSimplify[];
  ColumnSumExp[$],
  NL, "Using ",
  $s = {-I # & /@ $D1 /. tuOpSimplify[CircleTimes] //.
        { 1_ ⊗ a_ → a, a_ ⊗ 1_ → a, T["∇"S, "d", {a_}][b_] → tuDPartial[b, a]
        } // Simplify // Reverse // tuAddPatternVariable[Φ],
    T[g, "uu", {μ, ν}].T[Γ, "udd", {ρ, μ, ν}] → 0, I b_ ⊗ c_ - I b_ ⊗ d_ → I b ⊗ (c - d),
    0 ⊗ _ → 0
   },
  Yield, $ = $ //. $s /. tuOpSimplify[CircleTimes] //. $s;
  Yield, $ = $ /. a_ . b_ - b_ . a_ → CommutatorM[a, b] /.
     tuRuleSolve[$F, CommutatorM[_, _]] // ExpandAll,
  Yield, $ = $ //. tuOpDistribute[CircleTimes] //. tuOpSimplify[CircleTimes],
  ColumnSumExp[$] // Framed,
  CG[" QED"]
 ];
```

Simplifying $\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \frac{s\otimes 1_{\mathcal{H}_F}}{4} + \Delta^S\otimes 1_{\mathcal{H}_F} + (\not{D}).\gamma^\nu\otimes B_\nu + \gamma^\mu.(\not{D})\otimes B_\mu - \gamma^\mu.\gamma_5\otimes\Phi.B_\mu + \gamma^\mu.\gamma_5\otimes B_\mu.\Phi +$
$\frac{1}{2}(\gamma^\mu.\gamma^\nu\otimes[B_\mu, B_\nu] + \gamma^\mu.\gamma^\nu\otimes\{B_\mu, B_\nu\}) + 1_N\otimes\Phi.\Phi - 2\,i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu])\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum[\begin{matrix}\Delta^E \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ (\not{D}).\gamma^\nu\otimes B_\nu \\ \gamma^\mu.(\not{D})\otimes B_\mu \\ -(\gamma^\mu.\gamma_5\otimes\Phi.B_\mu) \\ \gamma^\mu.\gamma_5\otimes B_\mu.\Phi \\ \gamma^\mu.\gamma^\nu\otimes[B_\mu, B_\nu] \\ \frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] \\ -(\gamma^\nu.(\not{D})\otimes B_\nu) \\ i\,\gamma^\mu.\gamma^\nu.\nabla^S{}_\mu\otimes B_\nu \\ 1_N\otimes\Phi.\Phi \\ -2\,i\,\gamma^\mu.(\gamma^\nu\otimes\nabla^S{}_\mu[B_\nu]) \\ -\frac{1}{2}i\,\gamma^\mu.\gamma^\nu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho) \\ -\frac{1}{2}i\,\gamma^\nu.\gamma^\mu.(\Gamma^\rho{}_{\mu\nu}\otimes B_\rho)\end{matrix}]$

$\to\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum[\begin{matrix}\Delta^E \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ -\frac{1}{2}i\,(2\,g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu})\otimes B_\rho \\ \gamma^\mu.\gamma_5\otimes(-\Phi.B_\mu + B_\mu.\Phi) \\ \gamma^\mu.\gamma^\nu\otimes(B_\mu.B_\nu - B_\nu.B_\mu) \\ \frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] \\ -2\,i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] \\ 1_N\otimes\Phi.\Phi\end{matrix}]$

Using
$\{-(\Phi\_).B_\mu + B_\mu.(\Phi\_) - i\,\underline{\partial}_\mu[\Phi\_] \to -i\,\mathcal{D}_\mu[\Phi],\ g^{\mu\nu}.\Gamma^\rho{}_{\mu\nu} \to 0,\ i\,b\_\otimes c\_ - i\,b\_\otimes d\_ \to i\,b\otimes(c-d),\ 0\otimes\_ \to 0\}$
$\to$
$\to\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \Delta^E + \frac{s\otimes 1_{\mathcal{H}_F}}{4} + \gamma^\mu.\gamma_5\otimes[B_\mu, \Phi] + \frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} +$
$i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] + \gamma^\mu.\gamma^\nu\otimes(-i\,F_{\mu\nu} - i\,\underline{\partial}_\nu[B_\mu] + i\,\underline{\partial}_\mu[B_\nu]) - 2\,i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] + 1_N\otimes\Phi.\Phi$
$\to\ \mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \Delta^E + \frac{s\otimes 1_{\mathcal{H}_F}}{4} + \gamma^\mu.\gamma_5\otimes[B_\mu, \Phi] - \frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} - i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] + 1_N\otimes\Phi.\Phi$

$\boxed{\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{A}} \to \sum[\begin{matrix}\Delta^E \\ \frac{s\otimes 1_{\mathcal{H}_F}}{4} \\ \gamma^\mu.\gamma_5\otimes[B_\mu, \Phi] \\ -\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ -i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] \\ 1_N\otimes\Phi.\Phi\end{matrix}]}$ QED

Alternative calculation UNFINISHED

```
$s = Join[$scc, {(tt : T[γ, "u", {μ_}]).(a_ ⊗ b_) -> (tt.a ⊗ b),
    T[γ, "d", {5}] . T[γ, "d", {5}] → 1N,
    gg : T[γ, "u", {ν_}].T[γ, "d", {5}] :> -Reverse[gg]
   }]; Column[$s];
Bi_ := T[B, "d", {i}];
xtmp // ColumnSumExp;
$ = xtmp //
    tuRepeat[$s, (Simplify[tuDotSimplify[][#]] //. tuOpSimplify[CircleTimes]) &];

$s1 = {aa : a_ ⊗ b_ :→ (aa /. ν :> μ) /; FreeQ[aa, μ], (*
    (aa_⊗a_)-(aa_⊗b_)→aa⊗(a-b),*)(*
    a_⊗((gg:Tensor[γ,_,_]). b_)[c_]:>(a.gg.b)⊗c/;¬FreeQ[b,"∇"],*)
    a_ ⊗ ((gg : Tensor[γ, _, _]). b_)[c_] :> (a.gg) ⊗ b[c] /; ¬ FreeQ[b, "∇"],
    Bμ.Bν → (CommutatorM[Bμ, Bν] + CommutatorP[Bμ, Bν]) / 2,
    (T[γ, "u", {μ_}].T[γ, "u", {ν_}]) ⊗ CommutatorP[a_, b_] →
     2 T[g, "uu", {μ, ν}] 1N ⊗ a.b
   };
FramedColumn[$s1]
$ = $ //. $s1;
ColumnSumExp[$];
$ = $ // tuRepeat[{}, (Expand[tuDotSimplify[][#]] //. tuOpDistribute[CircleTimes] //.
        tuOpSimplify[CircleTimes]) &];
ColumnSumExp[$];
$ = $ /. Join[{(T[γ, "u", {μ_}] . T[γ, "u", {ν_}]) ⊗ CommutatorP[a_, b_] →
        2 × 1N ⊗ (a.b T[g, "uu", {μ, ν}])},
      tuRuleSolve[$F, CommutatorM[_, _]]] // ContractUpDn[g];
$ = $ /. {a_ .(tt : T["∇"^S, "d", {μ}]).b ⊗ Φ → a.b ⊗ tt[Φ]};
ColumnSumExp[$]


$d1 = Map[T[γ, "d", {5}] . T[γ, "u", {μ}].# &, $D1] // tuRepeat[{a_ .(1N ⊗ b_) → a ⊗ b},
    (Expand[tuDotSimplify[][#]] //. tuOpSimplify[CircleTimes]) &]
$d1 = tuRuleSolve[$d1, $d1[[2, -1]]] // Expand // First
$ = $ /. $d1 // Expand;


$ = $ /. tt : T["∇"^S, "d", {μ}].T[γ, "d", {5}] :→ Reverse[tt];
$ = $ /. tuRuleSolve[$x = tuIndicesLower[5][ps371], $x[[1, 2]]] //
    tuRepeat[{}, (Expand[tuDotSimplify[][#]] //. tuOpDistribute[CircleTimes] //.
        tuOpSimplify[CircleTimes]) &];
$ = $ /. a_ . T["∇"^S, "d", {μ_}] . b___ ⊗ c_ :> a . b ⊗ tuDDown["∂"][c, μ] /; FreeQ[c, 1] /.
   T["∇"^S, "d", {μ_}][a_] → tuDDown["∂"][a, μ]
ColumnSumExp[$]


$ = Select[$, ! FreeQ[#, B] &]
$s = {tt : a_ ⊗ tuDDown["∂"][T[B, "d", {μ}], ν] :→ tuIndexSwap[{μ, ν}][tt],
   XX a1_ (aa_ ⊗ a_) + b1_ (aa_ ⊗ b_) :> aa ⊗ (a1 a + b1 b) /; ! FreeQ[a, B] && ! FreeQ[b, B]};
Column[$s]
$ = $ //. $s ;
ColumnSumExp[$]
```

```
aa : a_⊗b_ :→ (aa /. ν :→ μ) /; FreeQ[aa, μ]
a_⊗((gg : Tensor[γ, _, _]).(b_))[c_] :→ a.gg⊗b[c] /; ! FreeQ[b, ∇]
B_μ.B_ν → 1/2 ([B_μ, B_ν] + {B_μ, B_ν})
γ^μ_.γ^ν_⊗{a_, b_} → 2 × 1_N⊗a.b g^μν
```

$$\sum\left[\begin{array}{l} \gamma_5.\gamma^\mu\otimes\Phi.B_\mu \\ -(\gamma_5.\gamma^\mu\otimes B_\mu.\Phi) \\ i\,\gamma_5.\gamma^\mu\otimes\nabla^S{}_\mu[\Phi] \\ \frac{1}{2}\gamma^\mu.\gamma^\nu\otimes(-i\,(F_{\mu\nu}+\underline{\partial}_\nu[B_\mu]-\underline{\partial}_\mu[B_\nu])) \\ -i\,\gamma^\nu.\gamma^\mu\otimes\nabla^S{}_\mu[B_\nu] \\ -i\,\gamma_5.\gamma^\mu.\nabla^S{}_\mu\otimes\Phi \\ -i\,\gamma^\mu.\nabla^S{}_\mu.\gamma_5\otimes\Phi \\ -i\,\gamma^\mu.\nabla^S{}_\mu.\gamma^\nu\otimes B_\nu \\ -i\,\gamma^\mu.\gamma^\nu.\nabla^S{}_\nu\otimes B_\mu \\ -(\gamma^\mu.\nabla^S{}_\mu.\gamma^\nu.\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) \\ 1_N\otimes\Phi.\Phi \\ 1_N\otimes B^\nu.B_\nu \end{array}\right]$$

$$\gamma_5.\gamma^\mu.\mathcal{D}_\mu[\Phi]\to -i\,\gamma_5.\gamma^\mu\otimes\Phi.B_\mu+i\,\gamma_5.\gamma^\mu\otimes B_\mu.\Phi+\gamma_5.\gamma^\mu\otimes\nabla^S{}_\mu[\Phi]$$

$$\gamma_5.\gamma^\mu\otimes\nabla^S{}_\mu[\Phi]\to i\,\gamma_5.\gamma^\mu\otimes\Phi.B_\mu-i\,\gamma_5.\gamma^\mu\otimes B_\mu.\Phi+\gamma_5.\gamma^\mu.\mathcal{D}_\mu[\Phi]$$

$$-\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu}-\frac{3}{2}i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu]-\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu]-$$
$$i\,\gamma^\nu.\gamma^\mu\otimes\underline{\partial}_\mu[B_\nu]-\gamma^\mu.\nabla^S{}_\mu.\gamma^\nu.\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}+1_N\otimes\Phi.\Phi+1_N\otimes B^\nu.B_\nu+i\,\gamma_5.\gamma^\mu.\mathcal{D}_\mu[\Phi]$$

$$\sum\left[\begin{array}{l} -\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes F_{\mu\nu} \\ -\frac{3}{2}i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu] \\ -\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] \\ -i\,\gamma^\nu.\gamma^\mu\otimes\underline{\partial}_\mu[B_\nu] \\ -(\gamma^\mu.\nabla^S{}_\mu.\gamma^\nu.\nabla^S{}_\nu\otimes 1_{\mathcal{H}_F}) \\ 1_N\otimes\Phi.\Phi \\ 1_N\otimes B^\nu.B_\nu \\ i\,\gamma_5.\gamma^\mu.\mathcal{D}_\mu[\Phi] \end{array}\right]$$

$$-\frac{3}{2}i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\nu[B_\mu]-\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu]-i\,\gamma^\nu.\gamma^\mu\otimes\underline{\partial}_\mu[B_\nu]+1_N\otimes B^\nu.B_\nu$$

```
tt : a_⊗∂_ν[B_μ] :→ tuIndexSwap[{μ, ν}][tt]
XX aa_⊗a_ a1_ + aa_⊗b_ b1_ :→ aa⊗(a1 a + b1 b) /; ! FreeQ[a, B] && ! FreeQ[b, B]
```

$$\sum\left[\begin{array}{l} -\frac{1}{2}i\,\gamma^\mu.\gamma^\nu\otimes\underline{\partial}_\mu[B_\nu] \\ -\frac{5}{2}i\,\gamma^\nu.\gamma^\mu\otimes\underline{\partial}_\mu[B_\nu] \\ 1_N\otimes B^\nu.B_\nu \end{array}\right]$$

Heat expansion

```
PR["●Theorem 3.2. ",
  $t32 = {Tr[Exp[-t H]] ~ xSum[t^((k - n) / 2) ak[H], {k ≥ 0}],
    H → "Laplacian"["E"],
    n → dim[M],
    ak[H] → IntegralOp[{{M}}, ak[x, H] √Det[g] ]
   }; Column[$t32]
 ];
```

●Theorem 3.2. $\mathrm{Tr}[e^{-H t}] \sim \sum_{\{k\geq 0\}} [t^{\frac{k-n}{2}} a_k[H]]$

$H \to \mathrm{Laplacian}[E]$

$n \to \dim[M]$

$a_k[H] \to \int_{\{M\}} [\sqrt{\mathrm{Det}[g]}\, a_k[x, H]]$

```
PR["●Theorem 3.3. ",
  $t33 = {a0[x, H] → (4 π)^(-n / 2) Tr"E"x[1N],
    a2[x, H] -> (4 π)^(-n / 2) Tr"E"x[s / 6 1N + F],
    a4[x, H] -> (4 π)^(-n / 2) (1 / 360)
      Tr"E"x[(-12 △[s] + 5 s.s - 2 T[R, "dd", {μ, ν}].T[R, "uu", {μ, ν}] +
         2 T[R, "dddd", {μ, ν, ρ, σ}]. T[R, "uuuu", {μ, ν, ρ, σ}] + 60 s. F +
         180 F . F - 60 △[F] + 30 T[Ω"E", "dd", {μ, ν}]. T[Ω"E", "uu", {μ, ν}])],
    s → "scalar curvature of ∇",
    △ → "scalar Laplacian",
    T[Ω"E", "dd", {μ, ν}] → "curvature of connnection ∇E"
   }; Column[$t33]
 ];
```

●Theorem 3.3.

$a_0[x, H] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[1_N]$

$a_2[x, H] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[F + \frac{s 1_N}{6}]$

$a_4[x, H] \to$

$\frac{1}{45} 2^{-3-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[180\, F.F + 60\, s.F + 5\, s.s - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[F] - 12\, \triangle[s]]$

$s \to$ scalar curvature of ∇

$\triangle \to$ scalar Laplacian

$\Omega^E{}_{\mu\nu} \to$ curvature of connnection ∇^E

```
PR["●Proposition 3.4. ",
  $t34 =
   {Tr[f[𝒟𝒜 / Λ]] ~ a4[𝒟𝒜^2] f[0] + 2 xSum[f4-k Λ^(4-k) ak[𝒟𝒜^2] / Γ[(4 - k) / 2], {k, 0, 4, even}],
    fi → IntegralOp[{{v}}, v^(j-1) f[v]]},
  Yield, $t34 = $t34 /. {k, 0, 4, even} → {k, {0, 2}} /. xSum → Sum
 ];
```

●Proposition 3.4. $\{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \sim 2 \sum_{\{k,0,4,\mathrm{even}\}} [\frac{\Lambda^{4-k} f_{4-k} a_k[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[\frac{4-k}{2}]}] + f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\ f_i \to \int_{\{v\}} [v^{-1+j} f[v]]\}$

$\to \{\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \sim 2 (\frac{\Lambda^4 f_4 a_0[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2 a_2[\mathcal{D}_{\mathcal{A}}^2]}{\Gamma[1]}) + f[0]\, a_4[\mathcal{D}_{\mathcal{A}}^2],\ f_i \to \int_{\{v\}} [v^{-1+j} f[v]]\}$

```
PR["●Proposition 3.5. For canonical triple ", {C"∞"[M], L²[M, S], slash[𝒟]},
  Yield,
  $p35 = $ = {Tr[f[slash[𝒟] / Λ]] ~ IntegralOp[{{x⁴}}, ℒM[T[g, "dd", {μ, ν}]]],
     ℒM[T[g, "dd", {μ, ν}]] → f₄ Λ⁴ / (2 π²) - f₂ Λ²
         / (24 π²) + f[0] / (16 π²) (△[s] / 30 -
          T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] / 20 + 11 / 360 R*.R*)};
  Column[$],
  NL, CO["Sketch proof: with ",
   $s0 = {m → dim[M], dim[M] → 4, Tr"E"x[1N] → dim[S], dim[S] → 2^(m/2)}],
  NL, "■Evaluate terms in T.3.4. ", $t34s = $t34 /. 𝒟𝒜 → slash[𝒟],
  NL, "■ ", $0 = $ = tuExtractPattern[a₀[_]][$t34s[[1, 2]]] // First,
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {M} → {x, x ∈ M} /. g → g[x],
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[1]]],
  Yield, $a0 = $0 -> $ /. $t32[[3 ;; -1]] //. $s0 // tuSimpleIntegralOp;
  Framed[$a0],

  NL, "■ ", $0 = $ = tuExtractPattern[a₂[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = F → -s / 4 1N,
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {{M} → {x, x ∈ M}, g → g[x]},
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[2]]] /. $sF,
  Yield, $ = $ //. tuOpSimplify[Tr"E"x, {s}] /. s → s[x],
  Yield, $a2 = $0 -> $ /. $t32[[3 ;; -1]] //. $s0 // tuSimpleIntegralOp;
  Framed[$a2],

  NL, "■ ", $0 = $ = tuExtractPattern[a₄[_]][$t34s[[1, 2]]] // First,
  " using ", $sF = {s → s . 1N, F → -s / 4 1N, Ω"E" → Ω^S},
  Yield,
  $ = $ /. tuAddPatternVariable[{H, k}][$t32[[-1]]] /. {{M} → {x, x ∈ M}, g → g[x]},
  Yield, $ = $ /. tuAddPatternVariable[{H, x}][$t33[[3]]] /. $sF, "POFF",
  Yield, $ = $ // tuDotSimplify[{s}],
  Yield, $ = $ //. tuOpSimplify[△, {1N}] /. 1N_ . 1N_ → 1N,
  Yield, $ = $ //. tuOpSimplify[Tr"E"x, {s}] /. s → s[x],
  Yield, $ = $0 -> $ /. $t32[[3 ;; -1]] //. $s0 // tuSimpleIntegralOp, "PONdd",
  Yield, $ = $ //. tuOpDistribute[ Tr"E"x ],
  Yield, $ = $ //. tuOpSimplify[Tr"E"x, {s[x], △[_]}] //. $s0 // Simplify;
  Framed[$a4b = $]
 ];
```

●Proposition 3.5. For canonical triple $\{C^{\infty}[M], L^2[M, S], D\!\!\!/\}$

$$\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim \int_{\{x^4\}}[\mathcal{L}_M[g_{\mu\nu}]]$$

$$\to \mathcal{L}_M[g_{\mu\nu}] \to -\frac{\Lambda^2 f_2}{24\pi^2} + \frac{\Lambda^4 f_4}{2\pi^2} + \frac{f[0]\,(\frac{11 R^*.R^*}{360} - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{\Delta[s]}{30})}{16\pi^2}$$

Sketch proof: with $\{m \to \dim[M], \dim[M] \to 4, \mathrm{Tr}_{E_x}[1_N] \to \dim[S], \dim[S] \to 2^{m/2}\}$

■Evaluate terms in T.3.4.

$$\{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim 2\,(\frac{\Lambda^4 f_4 a_0[(D\!\!\!/)^2]}{\Gamma[2]} + \frac{\Lambda^2 f_2 a_2[(D\!\!\!/)^2]}{\Gamma[1]}) + f[0]\, a_4[(D\!\!\!/)^2],\ f_i \to \int_{\{v\}}[v^{-1+j} f[v]]\}$$

■ $a_0[(D\!\!\!/)^2]$

$$\to \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ a_0[x, (D\!\!\!/)^2]]$$

$$\to \int_{\{x,x\in M\}}[2^{-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[1_N]]$$

$$\to \boxed{a_0[(D\!\!\!/)^2] \to \frac{\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\,]}{4\pi^2}}$$

■ $a_2[(D\!\!\!/)^2]$ using $F \to -\frac{s\,1_N}{4}$

$$\to \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ a_2[x, (D\!\!\!/)^2]]$$

$$\to \int_{\{x,x\in M\}}[2^{-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[-\frac{s\,1_N}{12}]]$$

$$\to \int_{\{x,x\in M\}}[-\frac{1}{3}2^{-2-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[s[x]\,1_N]]$$

$$\to \boxed{a_2[(D\!\!\!/)^2] \to -\frac{\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[s[x]\,1_N]]}{192\pi^2}}$$

■ $a_4[(D\!\!\!/)^2]$ using $\{s \to s.1_N,\ F \to -\frac{s\,1_N}{4},\ \Omega^E \to \Omega^S\}$

$$\to \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ a_4[x, (D\!\!\!/)^2]]$$

$$\to \int_{\{x,x\in M\}}[\frac{1}{45}2^{-3-n}\pi^{-n/2}\sqrt{\mathrm{Det}[g[x]]}\ \mathrm{Tr}_{E_x}[180\,(-\frac{s\,1_N}{4}).(-\frac{s\,1_N}{4}) - 2R_{\mu\nu}.R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu} + 60\,s.1_N.(-\frac{s\,1_N}{4}) + 5\,s.1_N.s.1_N - 12\,\Delta[s.1_N] - 60\,\Delta[-\frac{s\,1_N}{4}]]]$$

....

$$\to a_4[(D\!\!\!/)^2] \to \frac{1}{5760\pi^2}\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ (\mathrm{Tr}_{E_x}[-2R_{\mu\nu}.R^{\mu\nu}] + \mathrm{Tr}_{E_x}[2R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + \mathrm{Tr}_{E_x}[30\,\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] + \mathrm{Tr}_{E_x}[\frac{5}{4}s[x]^2\,1_N] + \mathrm{Tr}_{E_x}[3\,\Delta[s[x]\,1_N]]))]$$

$$\to \boxed{a_4[(D\!\!\!/)^2] \to \frac{1}{5760\pi^2}\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ (-2\,\mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + 30\,\mathrm{Tr}_{E_x}[\Omega^S{}_{\mu\nu}.\Omega^{S\mu\nu}] + \frac{5}{4}\mathrm{Tr}_{E_x}[s[x]^2\,1_N] + 3\,\mathrm{Tr}_{E_x}[\Delta[s[x]\,1_N]]))]}$$

```
PR["From (3.14): ", $s = e314 =
    T[Ω^S, "dd", {μ, ν}] → 1 / 4 T[R, "dddd", {μ, ν, ρ, σ}] T[γ, "u", {ρ}].T[γ, "u", {σ}],
  yield, $s314 = {e314, e314 /. ρ → ρ1 /. σ → σ1 // tuIndicesRaise[{μ, ν}]} //
    tuAddPatternVariable[{μ, ν}],
  NL, "Evaluate ", $ = $a4b // tuExtractPattern[T[Ω^S, "dd", {μ, ν}].T[Ω^S, "uu", {μ, ν}]] //
    First;
  $TO = $ = Tr[$],
  Yield, $ = $ /. $s314 // tuDotSimplify[{Tensor[R, __]}],
  Yield, $ = $ //. tuOpSimplify[Tr, {Tensor[R, _, _]}] /. subTraceGamma0,
  Yield, $ = $ // Expand // ContractUpDn[g],
  NL, "Use ", $s = {T[R, "ddud", {μ_, ν_, ρ_, ρ_}] → 0,
    T[R, "dduu", {μ, ν, ρ1, σ1}] → -T[R, "dduu", {μ, ν, σ1, ρ1}]},
  Yield, $TO = $TO -> $ /. $s /. Tr -> Tr"E"x; Framed[$TO],
  Imply, $ = $a4b /. $TO; Framed[$],
  NL, "Remaining Dot[] are scalars: ",
  Yield, $ = $ /. dd : HoldPattern[Dot[_]] → 1N dd /.
     tuOpSimplify[Tr"E"x, {HoldPattern[Dot[_]]}] //. $s0,
  Yield, $ = UpDownIndexSwap[{ρ1, σ1}][$] /. ρ1 → ρ /. σ1 → σ /.
      tt : T[R, "dddd", {_, _, _, _}] :→ tuTensorAntiSymmetricOrdered[tt, {3, 4}] /. Dot →
      Times // Simplify;
  Framed[$a4c = $]
 ];
PR["■Transform using: ",
 NL, "•Weyl tensor: ", T[C, "dddd", {μ, ν, ρ, σ}],
 Yield, $ = T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] ->
   T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}] -
    2 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] + s[x]^2 / 3,
 NL, "•Pontryagin class ",
 $1 = R*.R* → s[x]^2 - 4 T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}] +
    T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
 NL, "In integrand ",
 $2 = $a4c // tuExtractIntegrand;
 $2a = $2[[1, 2, 2]];
 $2a = test → $2a,
 $ = {$, $1, $2a};
 Imply,
 $ = tuEliminate[$, {T[R, "dddd", {μ, ν, ρ, σ}] T[R, "uuuu", {μ, ν, ρ, σ}],
     T[R, "dd", {μ, ν}] T[R, "uu", {μ, ν}]}];
 $ = tuRuleSolve[$, test],
 $2[[1, 2, 2]] = $[[1, 2]]; $2,
 Yield, $a4d = tuReplacePart[$a4c, $2]; Framed[$], CG[" QED"]
]
```

From (3.14): $\Omega^{S}{}_{\mu\nu} \to \frac{1}{4}\gamma^{\rho}.\gamma^{\sigma} R_{\mu\nu\rho\sigma} \longrightarrow \{\Omega^{S}{}_{\mu\_\nu\_} \to \frac{1}{4}\gamma^{\rho}.\gamma^{\sigma} R_{\mu\nu\rho\sigma},\ \Omega^{S\mu\_\nu\_} \to \frac{1}{4}\gamma^{\rho1}.\gamma^{\sigma1} R^{\mu\nu}{}_{\rho1\,\sigma1}\}$

Evaluate $\mathrm{Tr}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}]$

$\to \mathrm{Tr}[\frac{1}{16}\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\,\sigma1}]$

$\to \frac{1}{16}\mathrm{Tr}[\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\,\sigma1}]$

$\to \frac{1}{16}\mathrm{Tr}[\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\,\sigma1}]$

Use $\{R_{\mu\_\nu\_}{}^{\rho\_}{}_{\rho\_} \to 0,\ R_{\mu\nu}{}^{\rho1\,\sigma1} \to -R_{\mu\nu}{}^{\sigma1\,\rho1}\}$

$\to \boxed{\mathrm{Tr}_{E_x}[\Omega^{S}{}_{\mu\nu}.\Omega^{S\mu\nu}] \to \frac{1}{16}\mathrm{Tr}_{E_x}[\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\,\sigma1}]}$

$$\Rightarrow \boxed{\begin{aligned} &a_4[(\not{D})^2] \to \frac{1}{5760\,\pi^2}\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ (-2\,\mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}] + 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}] + \\ &\quad \frac{5}{4}\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] + \frac{15}{8}\mathrm{Tr}_{E_x}[\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\,\sigma1}] + 3\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]])]\end{aligned}}$$

Remaining Dot[] are scalars:

$\to$

$$a_4[(\not{D})^2] \to \frac{1}{5760\,\pi^2}\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ (-2\,\mathrm{Tr}_{E_x}[R_{\mu\nu}.R^{\mu\nu}\, 1_N] + 2\,\mathrm{Tr}_{E_x}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}\, 1_N] + \frac{5}{4}\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] + \frac{15}{8}\mathrm{Tr}_{E_x}[\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\rho1}.\gamma^{\sigma1}\, 1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\rho1\,\sigma1}] + 3\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]])]$$

$$\to \boxed{\begin{aligned} &a_4[(\not{D})^2] \to \frac{1}{5760\,\pi^2}\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ (\frac{5}{4}\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] - 2\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu} R^{\mu\nu}] + \\ &\quad 2\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] + \frac{15}{8}\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\,\gamma_{\rho}\,\gamma_{\sigma}\,\gamma^{\rho}\,\gamma^{\sigma}] + 3\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]])]\end{aligned}}$$

▪Transform using:

•Weyl tensor: $C_{\mu\nu\rho\sigma}$

$\to C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \to \frac{s[x]^2}{3} - 2\,R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

•Pontryagin class $R^{*}.R^{*} \to s[x]^2 - 4\,R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

In integrand test $\to \frac{5}{4}\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] - 2\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu} R^{\mu\nu}] +$

$2\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] + \frac{15}{8}\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\,\gamma_{\rho}\,\gamma_{\sigma}\,\gamma^{\rho}\,\gamma^{\sigma}] + 3\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]]$

$\Rightarrow \{\mathrm{test} \to \frac{1}{8}(10\,\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] - 16\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu} R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] +$
$15\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\,\gamma_{\rho}\,\gamma_{\sigma}\,\gamma^{\rho}\,\gamma^{\sigma}] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]])\}$

$\{\{2, 3, 2\} \to \frac{1}{8}\sqrt{\mathrm{Det}[g[x]]}\ (10\,\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] - 16\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu} R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] +$
$15\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\,\gamma_{\rho}\,\gamma_{\sigma}\,\gamma^{\rho}\,\gamma^{\sigma}] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]])\}$

$$\to \boxed{\begin{aligned} &\{\mathrm{test} \to \frac{1}{8}(10\,\mathrm{Tr}_{E_x}[s[x]^2\, 1_N] - 16\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu} R^{\mu\nu}] + \\ &\quad 16\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] + 15\,\mathrm{Tr}_{E_x}[1_N\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\,\gamma_{\rho}\,\gamma_{\sigma}\,\gamma^{\rho}\,\gamma^{\sigma}] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_N]])\}\end{aligned}}$$ QED

```
PR["•NOTE: In 4-dim compact orientable manifold M without boundary ",
  Yield,
  {IntegralOp[{{M}}, R*.R* νg] → 8 π^2 χ[M], χ[M] → "Euler Characteristic"} // Column,
  imply, "Topological term",
  yield, "Constant",
  yield, "Ignore",
  NL, "With no boundaries the ", △[s[x]], " term does not contribute."
 ];
```

•NOTE: In 4-dim compact orientable manifold M without boundary

→ $\int_{\{M\}}[R^{*}.R^{*}\,\nu_g] \to 8\,\pi^2\,\chi[M]$, $\chi[M] \to$ Euler Characteristic ⇒ Topological term ⟶ Constant ⟶ Ignore

With no boundaries the △[s[x]] term does not contribute.

```
PR[imply, "Proposition 3.5 ",
 $ = $t34s /. {$a0, $a2, $a4d} /. {R*.R* → 0, △[s[x]] → 0} /. tt : Tensor[C, _, _] → tt[x],
 Yield, $t34s1 = $ // gatherIntegralOp // Simplify,
 NL, "•Compare with (3.19). The integrand: ", $ = $t34s1[[1, 2]] // tuExtractIntegrand,
 Yield, $ = $ /. Γ → Gamma /. √_ → 1 // Expand,
 Yield, $LM = ℒM[T[g, "dd", {μ, ν}]] -> $[[1, 2]], CG[" Agrees."]
]
```

⇒ Proposition 3.5

$$\{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim \frac{1}{5760\,\pi^2} f[0] \int_{\{x,x\in M\}}[\frac{1}{8}\sqrt{\mathrm{Det}[g[x]]}\,(10\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N] - 16\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + 15\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]])] + 2\,(-\frac{\Lambda^2\,f_2\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\,\mathrm{Tr}_{E_x}[s[x]\,1_N]]}{192\,\pi^2\,\Gamma[1]} + \frac{\Lambda^4\,f_4\int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\,]}{4\,\pi^2\,\Gamma[2]}),\ f_i \to \int_{\{v\}}[v^{-1+j}\,f[v]]\}$$

$$\to \{\mathrm{Tr}[f[\frac{D\!\!\!/}{\Lambda}]] \sim \int_{\{x,x\in M\}}[\frac{1}{46\,080\,\pi^2\,\Gamma[1]\,\Gamma[2]}\sqrt{\mathrm{Det}[g[x]]}\,(23\,040\,\Lambda^4\,f_4\,\Gamma[1] + \Gamma[2]\,(-480\,\Lambda^2\,f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N] + f[0]\,\Gamma[1]\,(10\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N] - 16\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + 15\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]])))],\ f_i \to \int_{\{v\}}[v^{-1+j}\,f[v]]\}$$

•Compare with (3.19). The integrand: $\{\{2\} \to \frac{1}{46\,080\,\pi^2\,\Gamma[1]\,\Gamma[2]}\sqrt{\mathrm{Det}[g[x]]}\,(23\,040\,\Lambda^4\,f_4\,\Gamma[1] + \Gamma[2]\,(-480\,\Lambda^2\,f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N] + f[0]\,\Gamma[1]\,(10\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N] - 16\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}] + 16\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}] + 15\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma] + 24\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]])))\}$

$$\to \{\{2\} \to \frac{\Lambda^4\,f_4}{2\,\pi^2} - \frac{\Lambda^2\,f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N]}{96\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N]}{4608\,\pi^2} - \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma]}{3072\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]]}{1920\,\pi^2}\}$$

$$\to \mathcal{L}_M[g_{\mu\nu}] \to \frac{\Lambda^4\,f_4}{2\,\pi^2} - \frac{\Lambda^2\,f_2\,\mathrm{Tr}_{E_x}[s[x]\,1_N]}{96\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N]}{4608\,\pi^2} - \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu}\,R^{\mu\nu}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}]}{2880\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,\gamma_\rho\,\gamma_\sigma\,\gamma^\rho\,\gamma^\sigma]}{3072\,\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[\triangle[s[x]\,1_N]]}{1920\,\pi^2}$$ Agrees.

```
PR[CO["p.35"],
  "●Proposition 3.7. The spectral action of the fluctuated Dirac operator is ",
  $p37 = $ = {Tr[f[𝒟𝒜 / Λ]] ~ IntegralOp[{{x, x ∈ M}},
        √Det[g[x]] ℒ[T[g, "dd", {μ, ν}], Bμ, Φ]],
      ℒ[T[g, "dd", {μ, ν}], Bμ, Φ] → N ℒM[T[g, "dd", {μ, ν}]] +
        ℒB[Bμ] + ℒH[T[g, "dd", {μ, ν}], Bμ, Φ],
      $LM,
      N → dim[ℋF],
      ℒB[Bμ] → f[0] / (24 π^2) Tr[T[F, "dd", {μ, ν}] T[F, "uu", {μ, ν}]],
      ℒB[Bμ] → "Kinetic term gauge fields",
      ℒH[T[g, "dd", {μ, ν}], Bμ, Φ] →
       -2 f2 Λ^2 / (4 π^2) Tr[Φ.Φ] + f[0] / (8 π^2) Tr[Φ.Φ.Φ.Φ] + f[0] / (24 π^2) Δ[Tr[Φ.Φ]] +
        f[0] / (48 π^2) s[x] Tr[Φ.Φ] + f[0] / (8 π^2) Tr[T[𝒟, "d", {μ}][Φ].T[𝒟, "u", {μ}][Φ]],
      ℒH[T[g, "dd", {μ, ν}], Bμ, Φ] → "Higgs lagrangian",
      N → Tr[1ℋF]
     }; FramedColumn[$]
 ];
```

p.35●Proposition 3.7. The spectral action of the fluctuated Dirac operator is

$$\mathrm{Tr}[f[\tfrac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]] \sim \int_{\{x,x\in M\}}[\sqrt{\mathrm{Det}[g[x]]}\ \mathcal{L}[g_{\mu\nu}, B_\mu, \Phi]]$$

$$\mathcal{L}[g_{\mu\nu}, B_\mu, \Phi] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] + N\,\mathcal{L}_M[g_{\mu\nu}]$$

$$\mathcal{L}_M[g_{\mu\nu}] \to \frac{\Lambda^4 f_4}{2\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}_{E_x}[s[x]\,1_N]}{96\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[s[x]^2\,1_N]}{4608\pi^2} - \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu} R^{\mu\nu}]}{2880\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}]}{2880\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[1_N R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\gamma^\sigma]}{3072\pi^2} + \frac{f[0]\,\mathrm{Tr}_{E_x}[\Delta[s[x]\,1_N]]}{1920\pi^2}$$

$N \to \mathrm{dim}[\mathcal{H}_F]$

$$\mathcal{L}_B[B_\mu] \to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$$

$\mathcal{L}_B[B_\mu] \to$ Kinetic term gauge fields

$$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to \frac{f[0]\,s[x]\,\mathrm{Tr}[\Phi.\Phi]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[\Phi.\Phi]}{2\pi^2} + \frac{f[0]\,\mathrm{Tr}[\mathcal{D}_\mu[\Phi].\mathcal{D}^\mu[\Phi]]}{8\pi^2} + \frac{f[0]\,\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\pi^2} + \frac{f[0]\,\Delta[\mathrm{Tr}[\Phi.\Phi]]}{24\pi^2}$$

$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \Phi] \to$ Higgs lagrangian

$N \to \mathrm{Tr}[1_{\mathcal{H}_F}]$

```
PR["●For the formulas from Theorem 3.3 ", $ = $t33[[1 ;; 3]],
  NL, "let ",
  $s = {F → Q, H → 𝒟𝒜},
  " ", "explicit tensor notation. ", H → S × ℋF,
  Yield,
  $t33a = {{$ /. $s, $31[[-1]]} /. (tt : Tr_)[1N] :→ tt[1N ⊗ 1ℋF] /. s 1N → s /. s ⊗ 1ℋF → s /.
       s → (s 1N ⊗ 1ℋF) /. 1Nx → 1N ⊗ 1ℋ
        1N → "Identity for spinor field"} // Flatten; FramedColumn[$t33a]
 ];
```

●For the formulas from Theorem 3.3

$$\{a_0[x, H] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[1_N],\ a_2[x, H] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[F + \frac{s\,1_N}{6}],\ a_4[x, H] \to \frac{1}{45}2^{-3-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[180\,F.F + 60\,s.F + 5\,s.s - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\Delta[F] - 12\,\Delta[s]]\}$$

let $\{F \to Q, H \to \mathcal{D}_{\mathcal{A}}\}$ explicit tensor notation. $H \to S \times \mathcal{H}_F$

$$a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_2[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[Q + \tfrac{1}{6}s\,1_N \otimes 1_{\mathcal{H}_F}]$$

$$\to\ a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45}2^{-3-n}\pi^{-n/2}\mathrm{Tr}_{E_x}[180\,Q.Q + 60\,(s\,1_N\otimes 1_{\mathcal{H}_F}).Q + 5\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\Delta[Q] - 12\,\Delta[s\,1_N\otimes 1_{\mathcal{H}_F}]]$$

$$Q \to -i\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \tfrac{1}{2}i\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \tfrac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}$$

```
PR["●Compute the a[] terms of ", $t34[[1, 1]], (*
  " relative to ",$p35[[1,1]],*)
  NL, "for ", $s04 = Join[$s0, {Tr[1_N] → dim[S], n → dim[M]}],
  Yield, $t33a // FramedColumn
 ];
```

●Compute the a[] terms of $\mathrm{Tr}[f[\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda}]]$

for $\{m \to \dim[M], \dim[M] \to 4, \mathrm{Tr}_{E_x}[1_N] \to \dim[S], \dim[S] \to 2^{m/2}, \mathrm{Tr}[1_N] \to \dim[S], n \to \dim[M]\}$

→

$$a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_2[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[Q + \frac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]$$

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{45} 2^{-3-n} \pi^{-n/2} \mathrm{Tr}_{E_x}[180\, Q.Q + 60\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).Q + 5\, (s\, 1_N \otimes 1_{\mathcal{H}_F}).(s\, 1_N \otimes 1_{\mathcal{H}_F}) - 2\, R_{\mu\nu}.R^{\mu\nu} + 2\, R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\, \Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\, \triangle[Q] - 12\, \triangle[s\, 1_N \otimes 1_{\mathcal{H}_F}]]$$

$$Q \to -\mathbb{i}\, \gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2} \mathbb{i}\, \gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4} s\, 1_N \otimes 1_{\mathcal{H}_F}$$

```
PR["■For ", $ = $t33a[[1]],
  NL, "■For : ", $ = $t33a[[1]] /. Tr_ → Tr /. $t32[[3]] /. $s0,
  Yield,
  $ = $ /. tuOpDistribute[Tr, CircleTimes] //. tuOpSimplify[CircleTimes, {Tr[_]}],
  " ", "Recall ", $s = $t33[[1]] //. Join[{H → slash[𝒟], Tr_ → Tr}, $s04[[{2, -1}]]],
  Imply, $a0a = tuRuleEliminate[{Tr[1N]}][{$s, $}] // First; Framed[$a0a],

  NL, "■For : ", $ = $t33a[[2]] /. Tr_ → Tr /. $t32[[3]] /. $s0,
  Yield, $ = $ /. $t33a[[4]] //. tuOpDistribute[Tr] //. tuOpSimplify[Tr, {s}] /.
     tuOpDistribute[Tr, CircleTimes] //. tuOpSimplify[CircleTimes, {Tr[_]}],
  NL, "• ", T[F, "dd", {μ, ν}], " is anti-symmetric: ",
  $s = Tr[T[γ, "u", {μ}].T[γ, "u", {ν}]] Tr[T[F, "dd", {μ, ν}]] → 0,
  and,
  $sg = T[γ, "d", {5}] -> T[γ, "u", {5}],
  Yield, $ = $ /. $s /. $sg /. simpleGamma,
  NL, "Recall ",
  $s = $t33[[2]] //. Join[{H → slash[𝒟], Tr_ → Tr, $sF[[2]]}, $s04[[{2, -1}]]] //.
    tuOpSimplify[Tr, {s}],
  Imply, $a2a = $ /. tuRuleSolve[$s, {s Tr[1N]}] // Expand; Framed[$a2a]
 ];
```

■For $a_0[x, \mathcal{D}_{\mathcal{A}}] \to 2^{-n}\,\pi^{-n/2}\,\mathrm{Tr}_{E_x}[1_N \otimes 1_{\mathcal{H}_F}]$

■For : $a_0[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{\mathrm{Tr}[1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

→ $a_0[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{\mathrm{Tr}[1_N] \otimes \mathrm{Tr}[1_{\mathcal{H}_F}]}{16\,\pi^2}$ Recall $a_0[x, \not{D}] \to \dfrac{\mathrm{Tr}[1_N]}{16\,\pi^2}$

⇒ $\boxed{\left\{a_0[x, \not{D}] \to \dfrac{\mathrm{Tr}[1_N]}{16\,\pi^2},\; a_0[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{\mathrm{Tr}[1_N] \otimes \mathrm{Tr}[1_{\mathcal{H}_F}]}{16\,\pi^2}\right\}}$

■For : $a_2[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{\mathrm{Tr}[Q + \frac{1}{6} s\, 1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

→ $a_2[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{-i\,\mathrm{Tr}[\gamma^{\mu}.\gamma_5] \otimes \mathrm{Tr}[\mathcal{D}_{\mu}.\Phi] + \frac{1}{2} i\,\mathrm{Tr}[\gamma^{\mu}.\gamma^{\nu}] \otimes \mathrm{Tr}[F_{\mu\nu}] - \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[\Phi.\Phi] - \frac{1}{12}\mathrm{Tr}[s\, 1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

• $F_{\mu\nu}$ is anti-symmetric: $\mathrm{Tr}[\gamma^{\mu}.\gamma^{\nu}]\,\mathrm{Tr}[F_{\mu\nu}] \to 0$ and $\gamma_5 \to \gamma^5$

→ $a_2[x, \mathcal{D}_{\mathcal{A}}] \to \dfrac{-i\, 0 \otimes \mathrm{Tr}[\mathcal{D}_{\mu}.\Phi] + \frac{1}{2} i\,(4\, g^{\mu\nu}) \otimes \mathrm{Tr}[F_{\mu\nu}] - \mathrm{Tr}[1_N] \otimes \mathrm{Tr}[\Phi.\Phi] - \frac{1}{12}\mathrm{Tr}[s\, 1_N \otimes 1_{\mathcal{H}_F}]}{16\,\pi^2}$

Recall $a_2[x, \not{D}] \to -\dfrac{\mathrm{Tr}[s\, 1_N]}{192\,\pi^2}$

⇒ $\boxed{a_2[x, \mathcal{D}_{\mathcal{A}}] \to -\dfrac{i\, 0 \otimes \mathrm{Tr}[\mathcal{D}_{\mu}.\Phi]}{16\,\pi^2} + \dfrac{i\,(4\, g^{\mu\nu}) \otimes \mathrm{Tr}[F_{\mu\nu}]}{32\,\pi^2} - \dfrac{\mathrm{Tr}[1_N] \otimes \mathrm{Tr}[\Phi.\Phi]}{16\,\pi^2} - \dfrac{\mathrm{Tr}[s\, 1_N \otimes 1_{\mathcal{H}_F}]}{192\,\pi^2}}$

```
PR["■For: ", $ = $t33a[[3]] /. Tr_ → Tr /. s → s ⊗ 1ℋF /. $t32[[3]] /. $s0;
  Framed[$],
  NL, "Let: ", $sQ = {Map[#.(# /. {μ → μ1, ν → ν1}) &, $t33a[[4]]], $t33a[[4]]};
  $sQ, CK
 ];

PR["■For: ", $ = $t33a[[3]] /. Tr_ → Tr /. $t32[[3]] /. $s0 /.
    {tt : Tensor[R, _, _].Tensor[R, _, _] → tt 1N ⊗ 1ℋF},
  NL, "Scalars: ", $scal = {s, △[s], Tensor[R, _, _]},
  NL, "Use: ", $s = Join[($sQ /. s → s 1N), {$s34}, $s314];
  FramedColumn[$s],
```

```
Yield, $ = $ //. $s; ColumnSumExp[$];
Yield, $ = $ // tuDotSimplify[]; ColumnSumExp[$];
Yield,
$ = $ //. tuOpDistribute[△] //. tuOpSimplify[△] //. {△[ a_ ⊗ b_ ] → △[a] ⊗ b + a ⊗ △[b],
   △[ s_  a_ ⊗ b_ ] → △[s a] ⊗ b + s a ⊗ △[b], △[a_  b_ ] → △[a] b + a △[b]};
$sT = {tuOpDistribute[Tr], tuOpSimplify[Tr, $scal], tuOpDistribute[CircleTimes],
   tuOpSimplify[CircleTimes, $scal], tuOpSimplify[Dot, $scal]} // Flatten;
Yield, $ = $ // tuRepeat[$sT]; ColumnSumExp[$];
NL, "Use: ", $sX = { (a_ ⊗ b_ ) .( c_ ⊗ d_ ) → a.c ⊗ b.d,
  1n_ . a_ → a, a_ . 1n_ → a,
  ((SS : s | s^_ ) a_) ⊗ b_ → SS (a ⊗ b)},
Yield, $ = $ /. $sX // tuRepeat[$sT]; ColumnSumExp[$];
NL, "Use: ", $s = {△[1_] → 0, △[Tensor[γ, a_, b_].Tensor[γ, c_, d_]] → 0,
   1_.a_ → a, a_ . 1_ → a, {T[γ, "d", {5}] -> T[γ, "u", {5}],
    T[γ, "u", {5}].T[γ, "u", {a_}].T[γ, "u", {5}] :→ -T[γ, "u", {a}]}
  } // Flatten,
Yield, $ = $ //. tuOpDistribute[Tr, CircleTimes] //
    tuRepeat[Flatten[Join[$s, simpleGamma, $sT]]] //
   (# //. tuOpSimplify[Dot, {Tensor[R, _, _]}] &) // Expand;
ColumnSumExp[$];

$s = {a_  ⊗ b_ :→ 0 /; ($$ = ExtractPattern[T[g, "uu", {μ_, ν_}]][a] // First;
      $$ = ($$ /. g → F) // UpDownIndexSwap[1, 1] // UpDownIndexSwap[2, 2];
      ! FreeQ[b, $$]),
   aa : a_  ⊗ b_ :→ (aa //. μ1 → ν) /; FreeQ[aa, ν],
   aa : a_  ⊗ b_ :→ (aa //. μ1 → μ) /; FreeQ[aa, μ],
   aa : a_  ⊗ b_ :→ (aa //. ν1 → μ) /; FreeQ[aa, μ],
   (g_ gg : T[g, "uu", {μ_, ν_}]) ⊗ Tr[a_] → g ⊗ Tr[ gg a],
   ( gg : T[g, "uu", {μ_, ν_}]) ⊗ Tr[a_] → Tr[1N] ⊗ Tr[ gg a]
  } // Flatten;
Yield, $ = $ //. $s // tuMetricContractAll[g] // OrderTensorDummyIndices;

NL, "Manipulate indices: ",
$s = {aa : (a_  ⊗ b_) | (a_  b_) :→ (aa //. ν1 → μ) /; FreeQ[aa, μ],
  aa : (a_  ⊗ b_) | (a_  b_) :→ (aa //. μ1 → ν) /; FreeQ[aa, ν],
  aa : (a_  ⊗ b_) | (a_  b_) :→ (aa //. ρ1 → ρ) /; FreeQ[aa, ρ],
  aa : (a_  ⊗ b_) | (a_  b_) :→ (aa //. σ1 → σ) /; FreeQ[aa, σ],
  aa : (a_  ⊗ b_) | (a_  b_) :→ (aa //. σ1 → ρ) /; FreeQ[aa, ρ]
 },
and, $sR = {
  T[R, "dddu", {μ_, ν_, ρ_, ρ_}] → 0,
  T[R, "dudu", {μ_, ν_, ρ_, ρ_}] → 0,
  aa : (a_  ⊗ b_) | (a_  b_) :→ (aa //. σ1 → ρ) /; FreeQ[aa, ρ]
 },
and, $sR1 = {tt : Tensor[R, a_, b_] Tensor[R, c_, d_] :→ UpDownIndexSwap[μ][tt],
  T[R, "dddu", {μ_, ν_, ρ_, ρ_}] → 0,
  T[R, "uuuu", {μ, ν, σ, ρ}] → -T[R, "uuuu", {μ, ν, ρ, σ}]},
and, $sR2 = {T[R, "uuuu", {μ, ν, σ, ρ}] → -T[R, "uuuu", {μ, ν, ρ, σ}],
  T[F, "dd", {ν, μ}] → -T[F, "dd", {μ, ν}], T[F, "uu", {ν, μ}] → -T[F, "uu", {μ, ν}]},
Yield, $ = $ //. $s //. $sR /. $sR1 /. $sR2 //. tuOpSimplify[Dot] //. tuOpSimplify[Tr] //.
  tuOpSimplify[CircleTimes];
NL, "Apply factor to compare with p.37: ",
$ = (4 π)^2 360 # & /@ $ /. a_ ⊗ b_ → a b /. Tr[1N] → 4 // Expand;
ColumnSumExp[$],
CR["The coefficients 1320 and 2880 do not match."]
];
```

▪For: $\boxed{a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\mathrm{Tr}[180\,Q.Q + 60\,(s \otimes 1_{\mathcal{H}_F}\ 1_N \otimes 1_{\mathcal{H}_F}).Q + 5\,(s \otimes 1_{\mathcal{H}_F}\ 1_N \otimes 1_{\mathcal{H}_F}).(s \otimes 1_{\mathcal{H}_F}\ 1_N \otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s \otimes 1_{\mathcal{H}_F}\ 1_N \otimes 1_{\mathcal{H}_F}]]}$

Let: $\{Q.Q \to (-\mathrm{i}\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}).$
$(-\mathrm{i}\,\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}),$
$Q \to -\mathrm{i}\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}\}$ ⟵CHECK

▪For: $a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\mathrm{Tr}[180\,Q.Q + 60\,(s\,1_N \otimes 1_{\mathcal{H}_F}).Q + 5\,(s\,1_N \otimes 1_{\mathcal{H}_F}).(s\,1_N \otimes 1_{\mathcal{H}_F}) -$
$2 \times 1_N \otimes 1_{\mathcal{H}_F}\,R_{\mu\nu}.R^{\mu\nu} + 2 \times 1_N \otimes 1_{\mathcal{H}_F}\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s\,1_N \otimes 1_{\mathcal{H}_F}]]$
Scalars: {s, △[s], Tensor[R, _, _]}

Use:
$$\boxed{\begin{array}{l}
Q.Q \to (-\mathrm{i}\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}\ 1_N). \\
\quad (-\mathrm{i}\,\gamma^{\mu 1}.\gamma_5 \otimes \mathcal{D}_{\mu 1}.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^{\mu 1}.\gamma^{\nu 1} \otimes F_{\mu 1\,\nu 1} - 1_N \otimes \Phi.\Phi - \frac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}\ 1_N) \\
Q \to -\mathrm{i}\,\gamma^\mu.\gamma_5 \otimes \mathcal{D}_\mu.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^\mu.\gamma^\nu \otimes F_{\mu\nu} - 1_N \otimes \Phi.\Phi - \frac{1}{4}s\,1_N \otimes 1_{\mathcal{H}_F}\ 1_N \\
\Omega^E[\mu, \nu] \to 1_N \otimes (\mathrm{i}\,F_{\mu\nu}) + \Omega^S[\mu, \nu] \otimes 1_{\mathcal{H}_F} \\
\Omega S_{\mu_\,\nu_} \to \frac{1}{4}\gamma^\rho.\gamma^\sigma\,R_{\mu\nu\rho\sigma} \\
\Omega S^{\mu_\,\nu_} \to \frac{1}{4}\gamma^{\rho 1}.\gamma^{\sigma 1}\,R^{\mu\nu}{}_{\rho 1\,\sigma 1}
\end{array}}$$

→
→
→
→
Use: $\{(a\_ \otimes b\_).(c\_ \otimes d\_) \to a.c \otimes b.d,\ 1_{n\_}.(a\_) \to a,\ (a\_).1_{n\_} \to a,\ (a\_\,(SS : s \mid s-)) \otimes b\_ \to SS\ a \otimes b\}$
→
Use: $\{\triangle[1\_] \to 0,\ \triangle[\mathrm{Tensor}[\gamma, a\_, b\_].\mathrm{Tensor}[\gamma, c\_, d\_]] \to 0,$
$1\_.(a\_) \to a,\ (a\_).1\_ \to a,\ \gamma_5 \to \gamma^5,\ \gamma^5.\gamma^{a}{}_{-}.\gamma^5 \mapsto -T[\gamma, u, \{a\}]\}$
→
→
Manipulate indices: $\{aa : a\_ \otimes b\_ \mid a\_\,b\_ \mapsto (aa\ //.\ \nu 1 \to \mu)\ /;\ \mathrm{FreeQ}[aa, \mu],$
$aa : a\_ \otimes b\_ \mid a\_\,b\_ \mapsto (aa\ //.\ \mu 1 \to \nu)\ /;\ \mathrm{FreeQ}[aa, \nu],\ aa : a\_ \otimes b\_ \mid a\_\,b\_ \mapsto (aa\ //.\ \rho 1 \to \rho)\ /;\ \mathrm{FreeQ}[aa, \rho],$
$aa : a\_ \otimes b\_ \mid a\_\,b\_ \mapsto (aa\ //.\ \sigma 1 \to \sigma)\ /;\ \mathrm{FreeQ}[aa, \sigma],\ aa : a\_ \otimes b\_ \mid a\_\,b\_ \mapsto (aa\ //.\ \sigma 1 \to \rho)\ /;\ \mathrm{FreeQ}[aa, \rho]\}$
and $\{R_{\mu\_\,\nu\_\,\rho\_}{}^{\rho}{}_{-} \to 0,\ R_{\mu\_}{}^{\nu}{}_{-\,\rho\_}{}^{\rho}{}_{-} \to 0,\ aa : a\_ \otimes b\_ \mid a\_\,b\_ \mapsto (aa\ //.\ \sigma 1 \to \rho)\ /;\ \mathrm{FreeQ}[aa, \rho]\}$ and
$\{tt : \mathrm{Tensor}[R, a\_, b\_]\ \mathrm{Tensor}[R, c\_, d\_] \mapsto \mathrm{UpDownIndexSwap}[\mu][tt],\ R_{\mu\_\,\nu\_\,\rho\_}{}^{\rho}{}_{-} \to 0,\ R^{\mu\nu\sigma\rho} \to -R^{\mu\nu\rho\sigma}\}$
and $\{R^{\mu\nu\sigma\rho} \to -R^{\mu\nu\rho\sigma},\ F_{\nu\mu} \to -F_{\mu\nu},\ F^{\nu\mu} \to -F^{\mu\nu}\}$
→
Apply factor to compare with p.37:

$$5760\,\pi^2\, a_4[x, \mathcal{D}_{\mathcal{A}}] \to \sum\Big[\begin{array}{l}
45\, i\, \mathrm{Tr}[(\mathcal{D}^{\mu}.\Phi\, \gamma_{\mu}.\gamma_5).(s\, 1_N^2\, 1_{\mathcal{H}_F})] \\
45\, \mathrm{Tr}[(\Phi.\Phi\, 1_N).(s\, 1_N^2\, 1_{\mathcal{H}_F})] \\
-60\, i\, \mathrm{Tr}[(s\, 1_N\, 1_{\mathcal{H}_F}).(\mathcal{D}^{\mu}.\Phi\, \gamma_{\mu}.\gamma_5)] \\
-60\, \mathrm{Tr}[(s\, 1_N\, 1_{\mathcal{H}_F}).(\Phi.\Phi\, 1_N)] \\
5\, \mathrm{Tr}[(s\, 1_N\, 1_{\mathcal{H}_F}).(s\, 1_N\, 1_{\mathcal{H}_F})] \\
-15\, \mathrm{Tr}[(s\, 1_N\, 1_{\mathcal{H}_F}).(s\, 1_N^2\, 1_{\mathcal{H}_F})] \\
30\, i\, \mathrm{Tr}[(s\, 1_N\, 1_{\mathcal{H}_F}).(\gamma_{\mu}.\gamma_{\nu}\, F^{\mu\nu})] \\
45\, i\, \mathrm{Tr}[(s\, 1_N^2\, 1_{\mathcal{H}_F}).(\mathcal{D}^{\nu}.\Phi\, \gamma_{\nu}.\gamma_5)] \\
45\, \mathrm{Tr}[(s\, 1_N^2\, 1_{\mathcal{H}_F}).(\Phi.\Phi\, 1_N)] \\
\frac{45}{4}\, \mathrm{Tr}[(s\, 1_N^2\, 1_{\mathcal{H}_F}).(s\, 1_N^2\, 1_{\mathcal{H}_F})] \\
-\frac{45}{2}\, i\, \mathrm{Tr}[(s\, 1_N^2\, 1_{\mathcal{H}_F}).(-\gamma_{\nu}.\gamma_{\mu}\, F^{\mu\nu})] \\
-\frac{45}{2}\, i\, \mathrm{Tr}[(\gamma_{\mu}.\gamma_{\nu}\, F^{\mu\nu}).(s\, 1_N^2\, 1_{\mathcal{H}_F})] \\
30\, \mathrm{Tr}[\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu}] \\
90\, i\, \mathrm{Tr}[\Phi.\Phi.F^{\mu\nu}]\, \mathrm{Tr}[1_N.\gamma_{\nu}.\gamma_{\mu}] \\
-90\, i\, \mathrm{Tr}[F^{\mu\nu}.\Phi.\Phi]\, \mathrm{Tr}[\gamma_{\mu}.\gamma_{\nu}.1_N] \\
180\, \mathrm{Tr}[1_N.1_N]\, \mathrm{Tr}[\Phi.\Phi.\Phi.\Phi] \\
180\, i\, \mathrm{Tr}[1_N.\gamma_{\nu}.\gamma_5]\, \mathrm{Tr}[\Phi.\Phi.\mathcal{D}^{\nu}.\Phi] \\
180\, i\, \mathrm{Tr}[\gamma_{\mu}.\gamma_5.1_N]\, \mathrm{Tr}[\mathcal{D}^{\mu}.\Phi.\Phi.\Phi] \\
-180\, \mathrm{Tr}[\mathcal{D}^{\mu}.\Phi.\mathcal{D}^{\nu}.\Phi]\, \mathrm{Tr}[\gamma_{\mu}.\gamma_5.\gamma_{\nu}.\gamma_5] \\
90\, \mathrm{Tr}[\mathcal{D}^{\mu}.\Phi.F^{\nu\nu 1}]\, \mathrm{Tr}[\gamma_{\mu}.\gamma_5.\gamma_{\nu}.\gamma_{\nu 1}] \\
90\, \mathrm{Tr}[F^{\mu\nu}.\mathcal{D}^{\mu 1}.\Phi]\, \mathrm{Tr}[\gamma_{\mu}.\gamma_{\nu}.\gamma_{\mu 1}.\gamma_5] \\
-45\, \mathrm{Tr}[F^{\mu\nu}.F^{\mu 1\nu 1}]\, \mathrm{Tr}[\gamma_{\mu}.\gamma_{\nu}.\gamma_{\mu 1}.\gamma_{\nu 1}] \\
-2\, \mathrm{Tr}[R_{\mu\nu}.R^{\mu\nu}\, 1_N\, 1_{\mathcal{H}_F}] \\
2\, \mathrm{Tr}[R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma}\, 1_N\, 1_{\mathcal{H}_F}] \\
240\, \mathrm{Tr}[\triangle[\Phi.\Phi]] \\
60\, i\, \mathrm{Tr}[\gamma_{\mu}.\gamma_5]\, \mathrm{Tr}[\triangle[\mathcal{D}^{\mu}.\Phi]] \\
60\, i\, \mathrm{Tr}[\mathcal{D}^{\mu}.\Phi]\, \mathrm{Tr}[\triangle[\gamma_{\mu}.\gamma_5]] \\
-30\, i\, \mathrm{Tr}[F^{\mu\nu}]\, \mathrm{Tr}[\triangle[\gamma_{\mu}.\gamma_{\nu}]] \\
60\, \mathrm{Tr}[\Phi.\Phi]\, \mathrm{Tr}[\triangle[1_N]] \\
-12\, \mathrm{Tr}[1_{\mathcal{H}_F}]\, \mathrm{Tr}[1_N\, \triangle[s] + s\, \triangle[1_N]] \\
15\, \mathrm{Tr}[1_{\mathcal{H}_F}]\, \mathrm{Tr}[1_N^2\, \triangle[s] + s\, \triangle[1_N^2]] \\
-12\, \mathrm{Tr}[s\, 1_N\, \triangle[1_{\mathcal{H}_F}]] \\
15\, \mathrm{Tr}[s\, 1_N^2\, \triangle[1_{\mathcal{H}_F}]] \\
-30\, i\, \mathrm{Tr}[\gamma_{\mu}.\gamma_{\nu}]\, \mathrm{Tr}[\triangle[F^{\mu\nu}]]
\end{array}\Big]$$

The coefficients 1320 and 2880 do not match.

```
PR[aside,
  NL, "•Evaluate: ", $ = $sQ[[1]] // tuDotSimplify[],
  NL, CO["Is there a Logical order to the operatoins? "],
  Yield, $ = $ /. s → s 1N,
  $sX = { (a_ ⊗ b_) .( c_ ⊗ d_) → a.c ⊗ b.d,
    1n_ . a_ → a, a_ . 1n_ → a,
    ((SS : s | s^_) a_) ⊗ b_ → SS (a ⊗ b)};
  $ = $ // tuRepeat[$sX, tuDotSimplify[{s}]] ;
  $ = Tr[#] & /@ $ //. tuTrSimplify[{s}];
  $[[2]] = $[[2]] // tuDistributeOp[Tr[_], CircleTimes];

  $ = $ //. {T[γ, "d", {5}] -> T[γ, "u", {5}],
     T[γ, "u", {5}].T[γ, "u", {a_}].T[γ, "u", {5}] ⧴ -T[γ, "u", {a}]};
  $ = $ //. simpleGamma /. 0 ⊗ a_ → 0 //. tuOpSimplify[CircleTimes] //.
    tuOpDistribute[CircleTimes ];
  $ = $ //. ( g_ T[g, "uu", {a_, b_}]) ⊗ Tr[ c_ ] ⧴ 0 /; ! FreeQ[c, T[F, "dd", {a, b}]] /.
      g_ T[g, "uu", {a_, b_}] ⊗ Tr[ c_ ] ⧴ 0 /; ! FreeQ[c, T[F, "dd", {a, b}]] /.
     tuTrSimplify[] /. tuOpSimplify[CircleTimes] /. simpleGamma;
  $ = $ /. (gg : Tensor[g, _, _] g_ ) ⊗ Tr[a_] → 1N ⊗ Tr[gg a] // ContractUpDn[g];
  $ = $ /. T[F, "uu", {a_, b_}] ⧴ -T[F, "uu", {b, a}] /; OrderedQ[{b, a}] /.
     tuOpSimplify[CircleTimes] // tuDotSimplify[];
  $ = $ /. tuTrSimplify[] /. tuOpSimplify[CircleTimes];
  ColumnSumExp[$sQQ = $] // Framed, OK
 ];
```

←←←←←←←←Side Note

•Evaluate:

$$Q.Q \to -(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) + \frac{1}{2}(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + \frac{1}{2}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{4}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) - \frac{1}{2}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(1_N\otimes\Phi.\Phi) - \frac{1}{8}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(s\,1_N\otimes 1_{\mathcal{H}_F}) + i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{2}i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + (1_N\otimes\Phi.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}(1_N\otimes\Phi.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + \frac{1}{4}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{8}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + \frac{1}{4}(s\,1_N\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi) + \frac{1}{16}(s\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F})$$

Is there a Logical order to the operatoins?

$$\to\ Q.Q \to -(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) + \frac{1}{2}(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}i\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N) + \frac{1}{2}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{4}(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) - \frac{1}{2}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(1_N\otimes\Phi.\Phi) - \frac{1}{8}i\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\,\nu}).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N) + i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{2}i\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + (1_N\otimes\Phi.\Phi).(1_N\otimes\Phi.\Phi) + \frac{1}{4}(1_N\otimes\Phi.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N) + \frac{1}{4}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{1}{8}i\,(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + \frac{1}{4}(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(1_N\otimes\Phi.\Phi) + \frac{1}{16}(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N).(s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N)$$

$$\boxed{Tr[Q.Q] \to \sum\left[\begin{array}{l} 2\times 1_N\otimes Tr[F^{\mu 1\,\nu 1}.F_{\mu 1\,\nu 1}] \\ 4\times 1_N\otimes Tr[\mathcal{D}^{\mu 1}.\Phi.\mathcal{D}_{\mu 1}.\Phi] \\ \frac{1}{2}s\,Tr[1_N]\otimes Tr[\Phi.\Phi] \\ Tr[1_N]\otimes Tr[\Phi.\Phi.\Phi.\Phi] \\ \frac{1}{16}s^2\,Tr[1_N]\otimes Tr[1_{\mathcal{H}_F}] \end{array}\right]}$$ OK

```
PR["■For: ", $ = $t33a[[3]] /. Tr_ → Tr /. s → s⊗1_ℋF /. $t32[[3]] /. $s0;
  Framed[$],
  NL, "Let: ", $sQ = {Map[#.(# /. {μ → μ1, ν → ν1}) &, $t33a[[4]]], $t33a[[4]]};
  $sQ, CK,

  Yield, $ = $ /. $sQ // tuDotSimplify[]; Framed[$],
  NL, "Apply (3.4): ", $s34,
  Yield, $ = $ /. $s34; Framed[$], CK,
  "POFF",
  Yield, $ = $ /. (a_⊗b_).(c_⊗d_) → (a.c)⊗(b.d) /. s → s 1N /. 1_N_ . 1_N_ → 1N //
    tuDotSimplify[{s}], CK, "POFF",
  Yield, $ = $ /. 1_N_ . 1_N_ → 1N //. tuOpSimplify[CircleTimes, {s}] // tuDotSimplify[{s}],
  ColumnSumExp[$],
  NL, "Simplify indices: ",
  $ = $ /. {aa : a_⊗b_ :→ (aa /. μ1 → μ /. ν1 → ν) /; FreeQ[aa, μ | ν]};
  ColumnSumExp[$];
  NL, "Simpliy 1_ with γ's Φ's: ",
  Yield, $ = $ /. HoldPattern[Dot[a__]] :→ Apply[Dot, Select[{a}, # =!= 1N &]] /;
       ¬ FreeQ[{a}, 1N] && ¬ FreeQ[{a}, γ] /. HoldPattern[Dot[a__]] :→
     Apply[Dot, Select[{a}, # =!= 1ℋF &]] /; ¬ FreeQ[{a}, 1ℋF] && ¬ FreeQ[{a}, Φ | F];
  ColumnSumExp[$];
  Yield, $ = $ /. tt : △[_] :> Distribute[tt] //. tuOpSimplify[△, {}] // Simplify;
  Yield, $ = $ /. tt : Tr[_] :→ Distribute[tt] //. tuOpSimplify[Tr, {s}] // Simplify;
  ColumnSumExp[$],
  Yield, $ = $ /. tt : Tr[a__] :> Distribute[tt, CircleTimes] /; Head[a] === CircleTimes //
    simpleTrGamma1[{}];
  Yield, $ = $ //. tuOpDistribute[CircleTimes] /. tuOpSimplify[CircleTimes] /.
     0⊗a_ → 0 //. tuOpSimplify[CircleTimes, {s}], "PON",

  NL, "• ", T[F, "dd", {μ, ν}], " is anti-symmetric ",
  Yield, $s = {T[g, "uu", {μ, ν}]⊗Tr[b___ . T[F, "dd", {μ, ν}]. a___] → 0,
    T[g, "uu", {μ, ν}]⊗Tr[ T[F, "dd", {μ, ν}]] → 0,

    (a_ T[g, "uu", {μ, ν}])⊗Tr[ T[F, "dd", {μ, ν}] . b_] → 0,
    (g_ gg : T[g, "uu", {μ_, ν_}])⊗Tr[a_] → g ⊗Tr[ gg a],
    ( gg : T[g, "uu", {μ_, ν_}])⊗Tr[a_] → Tr[1N]⊗Tr[ gg a],
    CircleTimes[a__] :→ 0 /; ¬ FreeQ[{a}, 0]
   }; Column[$s],
  Yield, $ = $ //. $s // tuMetricContractAll[g] // OrderTensorDummyIndices;
  Yield, (*simplify F.F*)
  $pass2 = $ = $ /. tt : Tensor[F, _, _] .Tensor[F, _, _] :→ tt /. ν → μ1 /.
        T[F, "dd", {ν1, μ1}] → -T[F, "dd", {μ1, ν1}] /. tuOpSimplify[Dot] /.
      tuOpSimplify[Tr] /. tuOpSimplify[CircleTimes];
  ColumnSumExp[$pass3 = $]
 ];
```

■For: $\boxed{a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\mathrm{Tr}[180\,Q.Q + 60\,(s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}).Q + 5\,(s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}).(s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^E{}_{\mu\nu}.\Omega^{E\mu\nu} - 60\,\triangle[Q] - 12\,\triangle[s\otimes 1_{\mathcal{H}_F}\ 1_N\otimes 1_{\mathcal{H}_F}]]}$

Let: $\{Q.Q \to (-i\,\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi + \frac{1}{2}i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \frac{1}{4}s\,1_N\otimes 1_{\mathcal{H}_F}).$
$(-i\,\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi + \frac{1}{2}i\,\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1} - 1_N\otimes\Phi.\Phi - \frac{1}{4}s\,1_N\otimes 1_{\mathcal{H}_F}),$
$Q \to -i\,\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi + \frac{1}{2}i\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \frac{1}{4}s\,1_N\otimes 1_{\mathcal{H}_F}\}$ ⟵CHECK

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\mathrm{Tr}[-180\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) + 90\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + 180\,\mathrm{i}\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(1_N\otimes\Phi.\Phi) + 45\,\mathrm{i}\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 90\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - 45\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) - 90\,\mathrm{i}\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(1_N\otimes\Phi.\Phi) - \frac{45}{2}\mathrm{i}\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 180\,\mathrm{i}\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - 90\,\mathrm{i}\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + 180\,(1_N\otimes\Phi.\Phi).(1_N\otimes\Phi.\Phi) + 45\,(1_N\otimes\Phi.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 45\,\mathrm{i}\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{45}{2}\mathrm{i}\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + 45\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi) + \frac{45}{4}(s\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F}) - 60\,\mathrm{i}\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi) + 30\,\mathrm{i}\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}) - 60\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi) - 15\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 5\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu} - 12\,\triangle[s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}] - 60\,\triangle[-\mathrm{i}\,\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \frac{1}{4}s\,1_N\otimes 1_{\mathcal{H}_F}]]$$

Apply (3.4): $\Omega^{E}[\mu,\nu] \to 1_N\otimes(\mathrm{i}\,F_{\mu\nu}) + \Omega^{S}[\mu,\nu]\otimes 1_{\mathcal{H}_F}$

$$a_4[x, \mathcal{D}_{\mathcal{A}}] \to \frac{1}{5760\,\pi^2}\mathrm{Tr}[-180\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) + 90\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + 180\,\mathrm{i}\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(1_N\otimes\Phi.\Phi) + 45\,\mathrm{i}\,(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 90\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - 45\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) - 90\,\mathrm{i}\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(1_N\otimes\Phi.\Phi) - \frac{45}{2}\mathrm{i}\,(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 180\,\mathrm{i}\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - 90\,\mathrm{i}\,(1_N\otimes\Phi.\Phi).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + 180\,(1_N\otimes\Phi.\Phi).(1_N\otimes\Phi.\Phi) + 45\,(1_N\otimes\Phi.\Phi).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 45\,\mathrm{i}\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma_5\otimes\mathcal{D}_{\mu 1}.\Phi) - \frac{45}{2}\mathrm{i}\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu 1}.\gamma^{\nu 1}\otimes F_{\mu 1\,\nu 1}) + 45\,(s\,1_N\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi) + \frac{45}{4}(s\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F}) - 60\,\mathrm{i}\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi) + 30\,\mathrm{i}\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu}) - 60\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi) - 15\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(s\,1_N\otimes 1_{\mathcal{H}_F}) + 5\,(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}).(s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}) - 2\,R_{\mu\nu}.R^{\mu\nu} + 2\,R_{\mu\nu\rho\sigma}.R^{\mu\nu\rho\sigma} + 30\,\Omega^{E}{}_{\mu\nu}.\Omega^{E\mu\nu} - 12\,\triangle[s\otimes 1_{\mathcal{H}_F}\,1_N\otimes 1_{\mathcal{H}_F}] - 60\,\triangle[-\mathrm{i}\,\gamma^{\mu}.\gamma_5\otimes\mathcal{D}_{\mu}.\Phi + \frac{1}{2}\mathrm{i}\,\gamma^{\mu}.\gamma^{\nu}\otimes F_{\mu\nu} - 1_N\otimes\Phi.\Phi - \frac{1}{4}s\,1_N\otimes 1_{\mathcal{H}_F}]]$$

⟵CHECK

- $F_{\mu\nu}$ is anti-symmetric

$g^{\mu\nu}\otimes\mathrm{Tr}[(b\_\_\_).F_{\mu\nu}.(a\_\_\_)] \to 0$
$g^{\mu\nu}\otimes\mathrm{Tr}[F_{\mu\nu}] \to 0$
$(a\_\,g^{\mu\nu})\otimes\mathrm{Tr}[F_{\mu\nu}.(b\_)] \to 0$
$(g\_\,(gg : g^{\mu\_\,\nu\_}))\otimes\mathrm{Tr}[a\_] \to g\otimes\mathrm{Tr}[a\,gg]$
$(gg : g^{\mu\_\,\nu\_})\otimes\mathrm{Tr}[a\_] \to \mathrm{Tr}[1_N]\otimes\mathrm{Tr}[a\,gg]$
$\otimes a\_\_ :\to 0\ /;\ !\,\mathrm{FreeQ}[\{a\}, 0]$

$$
\rightarrow\ a_4[x,\mathcal{D}_{\mathcal{A}}] \to \sum\left[\begin{array}{l}
\frac{\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[F_{\mu1\,\nu1}.F^{\mu1\,\nu1}]}{16\,\pi^2} \\
\frac{\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]}{32\,\pi^2} \\
\frac{\mathrm{Tr}[1_N]\otimes\mathrm{Tr}[\mathcal{D}_{\mu1}.\Phi.\mathcal{D}^{\mu1}.\Phi]}{8\,\pi^2} \\
-\frac{i\,\mathrm{Tr}[s\,(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi).(1_N\otimes 1_{\mathcal{H}_F})]}{128\,\pi^2} \\
-\frac{i\,\mathrm{Tr}[s\,(\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1}).(1_N\otimes 1_{\mathcal{H}_F})]}{256\,\pi^2} \\
-\frac{i\,\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}).(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi)]}{128\,\pi^2} \\
-\frac{i\,\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}).(\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1})]}{256\,\pi^2} \\
\frac{i\,\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi)]}{96\,\pi^2} \\
\frac{i\,\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1})]}{192\,\pi^2} \\
-\frac{\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(1_N\otimes\Phi.\Phi)]}{96\,\pi^2} \\
\frac{\mathrm{Tr}[(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F})]}{1152\,\pi^2} \\
-\frac{\mathrm{Tr}[R_{\mu\,\mu1}.R^{\mu\,\mu1}]}{2880\,\pi^2} \\
\frac{\mathrm{Tr}[R_{\mu\,\mu1\,\rho\,\sigma}.R^{\mu\,\mu1\,\rho\,\sigma}]}{2880\,\pi^2} \\
\frac{\mathrm{Tr}[\Omega^E{}_{\mu\,\mu1}.\Omega^{E\,\mu\,\mu1}]}{192\,\pi^2} \\
\frac{\mathrm{Tr}[s\,(1_N\otimes\Phi.\Phi).(1_N\otimes 1_{\mathcal{H}_F}).1_N]}{128\,\pi^2} \\
\frac{\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}).1_N.(1_N\otimes\Phi.\Phi)]}{128\,\pi^2} \\
-\frac{\mathrm{Tr}[s\,(1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}).(1_N\otimes 1_{\mathcal{H}_F}).1_N]}{384\,\pi^2} \\
\frac{\mathrm{Tr}[s^2\,(1_N\otimes 1_{\mathcal{H}_F}).1_N.(1_N\otimes 1_{\mathcal{H}_F}).1_N]}{512\,\pi^2} \\
\frac{i\,\mathrm{Tr}[\triangle[-(\gamma_5.\gamma_\mu\otimes\mathcal{D}^\mu.\Phi)]]}{96\,\pi^2} \\
-\frac{i\,\mathrm{Tr}[\triangle[\gamma_\mu.\gamma_{\mu1}\otimes F^{\mu\,\mu1}]]}{192\,\pi^2} \\
\frac{\mathrm{Tr}[\triangle[1_N\otimes\Phi.\Phi]]}{96\,\pi^2} \\
-\frac{\mathrm{Tr}[\triangle[1_N\otimes 1_{\mathcal{H}_F}\,(s\,1_N)\otimes 1_{\mathcal{H}_F}]]}{480\,\pi^2} \\
\frac{\mathrm{Tr}[\triangle[s\,1_N\otimes 1_{\mathcal{H}_F}\,1_N]]}{384\,\pi^2}
\end{array}\right]
$$

```
PR["•Compare with p37: ", "POFF",
 $ = $pass3,
 Yield, $ = (4 π)^2 360 # & /@ $ // Expand; ColumnSumExp[$],
 Yield, $ = $ //. {(a_ ⊗ b_).(c_ ⊗ d_) :→ a.c ⊗ b.d, 1n_ . a_ | a_ . 1n_ → a},
 Yield, $ = $ /. Tr[a_] :→ Tr[a /. μ1 :> μ /; FreeQ[a, μ]] /.
     Tr[a_] :→ Tr[a /. ν1 :→ ν /; FreeQ[a, ν]] /.
   Tr[a_] :→ Tr[a /. μ1 :→ ν /; FreeQ[a, ν]],
 Yield, $ = $ /. aa : a_ ⊗ T[F, "dd", {μ, ν}] :→ UpDownIndexSwap[{μ, ν}][aa],
 NL, "Let ", $s = {△[a_ ⊗ b_] → △[a] ⊗ b + a ⊗ △[b], △[a_ . b_] → △[a]. b + a.△[b],
   △[a_ b_] → △[a] b + a △[b], a_ ⊗ b_ :→ 0 /; ! FreeQ[{a, b}, 0], △[] → 0,
   △[a_] :→ 0 /; MatchQ[a, 1n_]},
 Yield,
 $ = $ // tuRepeat[$s, (# //. tuOpSimplify[△, {Tensor[γ, _, _]}] & // tuDotSimplify[])],
 Yield, $ = $ /. tuTrExpand /. Tr[a_ (b_ ⊗ c_)] → a Tr[b] ⊗ Tr[c] /.
     Tr[(b_ ⊗ c_)] → Tr[b] ⊗ Tr[c] /. simpleGamma //
   tuRepeat[$s, (# //. tuOpSimplify[CircleTimes] & // tuDotSimplify[])],
 Yield, $ = $ /. T[g, "dd", {μ, ν}] ⊗ a_ :→ 0 /; ! FreeQ[a, F],
 "PON",
 ColumnSumExp[$]
]
```

```
•Compare with p37: 5760 π² a₄[x, 𝒟𝒜] → ∑[
  -60 ⅈ Tr[γ₅.△[γμ] + △[γ₅].γμ]⊗Tr[𝒟^μ.Φ]
  -30 ⅈ Tr[γμ.△[γν] + △[γμ].γν]⊗Tr[F^μν]
  45 s Tr[1_N]⊗Tr[Φ.Φ]
  360 Tr[1_N]⊗Tr[Fμν.F^μν]
  60 Tr[1_N]⊗Tr[Φ.△[Φ] + △[Φ].Φ]
  180 Tr[1_N]⊗Tr[Φ.Φ.Φ.Φ]
  720 Tr[1_N]⊗Tr[𝒟μ.Φ.𝒟^μ.Φ]
  -12 Tr[1_N]⊗Tr[1_ℋF] (1_N △[s])⊗1_ℋF
  60 ⅈ Tr[(1_N⊗1_ℋF (s 1_N)⊗1_ℋF).(γ₅.γμ⊗𝒟^μ.Φ)]
  30 ⅈ Tr[(1_N⊗1_ℋF (s 1_N)⊗1_ℋF).(γμ.γν⊗F^μν)]
  -60 Tr[(1_N⊗1_ℋF (s 1_N)⊗1_ℋF).(1_N⊗Φ.Φ)]
  -15 Tr[s (1_N⊗1_ℋF (s 1_N)⊗1_ℋF).(1_N⊗1_ℋF)]
  5 Tr[(1_N⊗1_ℋF (s 1_N)⊗1_ℋF).(1_N⊗1_ℋF (s 1_N)⊗1_ℋF)]
  -2 Tr[Rμν.R^μν]
  2 Tr[Rμνρσ.R^μνρσ]
  30 Tr[Ω^E_μν.Ω^Eμν]
  45 Tr[s (1_N⊗1_ℋF).1_N.(1_N⊗Φ.Φ)]
  45/4 Tr[s² (1_N⊗1_ℋF).1_N.(1_N⊗1_ℋF)]
  15 Tr[1_N]⊗Tr[1_ℋF] 1_N △[s]
]
```

4. Electrodynamics p.38

```
PR["●EG: Two point space.", {X -> {x, y}, C[X] → ℂ², C → "complex functions"},
  NL, "•Construct even finite space ",
  {F_X → {C[X], ℋF, 𝒟F, γF}, dim[ℋF] ≥ 2, γF → "ℤ²grading" },
  NL, "Let ", ℋF → ℂ²,
  Yield, γF ⇒ ℋF → {ℋF⁺ ⊕ ℋF⁻ → ℂ ⊕ ℂ, ℋF^"±" → {ψ ∈ ℋF | γF.ψ → ±ψ}},
  imply, $ = γF → {{1, 0}, {0, -1}}; MatrixForms[$],
  NL, "Since ", $sD0 = {CommutatorM[γF, a] → 0,
    CommutatorP[𝒟F, γF] → 0, 𝒟F → "offDiagonal", 𝒟F → {{0, du}, {dl, 0}}},
  Imply, {a.ψ → Inactive[Dot][{{a₊, 0}, {0, a₋}}, {{ψ₊}, {ψ₋}}], a ∈ 𝒜F, ψ ∈ ℋF} //
   MatrixForms,
  Imply, F_X → {{𝒜F, ℋF, 𝒟F, γF} → {ℂ², ℂ², {{0, t}, {t̄, 0}}, {{1, 0}, {0, -1}}}, t ∈ ℂ} //
   MatrixForms,
  NL, "■Prop.4.1. A real structure ", $ = JF ⇒ {𝒟F → 0},
  NL, "Determine 𝒟F for even KO dimensions by requiring: ",
  $c = $ = Join[$J[[2]], $def]; Column[$],

  NL, "▪KOdim->0: ", $sj = {JF → {{j₊, 0}, {0, j₋}}.cc, j"±" ∈ U[1]},
  NL, "for ", $sa = ab : a | b -> {{ab₊, 0}, {0, ab₋}}; MatrixForms[$sa],
  NL, "•Compute ", $0 = $ = tuExtractPattern[b"0" → _][$c] // First,
  yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];
  yield, $[[2]] = $[[2]] // tuRepeat[$cc, ConjugateCTSimplify1[{}]];
  MatrixForms[$],
  yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
  MatrixForms[$sb = $] // Framed,
  NL, "Diagonal", imply, $c[[4]] // Framed,

  Imply, $c[[5]],
  NL, "•Evaluate: ", $ = $c[[5, -1, 1]],
  $sa = ab : a | xb -> {{ab₊, 0}, {0, ab₋}};
  Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
  yield, $ = $ //. CommutatorM → MCommutator //
    tuRepeat[Join[tuOpSimplify[Dot], {tuOpDistribute[Dot]}]];
  yield, $ = $ /. $sD0[[-1]] // Simplify;
  MatrixForms[$],
  $x = tuExtractPattern[du _][$][[1]] / du;
```

```
  yield, $ = $x.(# / $x) & /@ $ /. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
  imply, Framed[𝒟F → 0]
 ];
PR[
  NL, "■KOdim->0: ", $sj = {JF → {{0, j}, {-j, 0}}.cc, j ∈ U[1]},
  NL, "for ", $sa = ab : a | b -> {{ab+, 0}, {0, ab-}}; MatrixForms[$sa],
  NL, "Compute ", $0 = $ = tuExtractPattern[b"0" → _][$c] // First,
  yield, $[[2]] = $[[2]] /. $sa /. $sj[[1]];

  yield, $[[2]] = $[[2]] // tuRepeat[$cc, ConjugateCTSimplify1[{}]];
  MatrixForms[$sb = $],
  yield, $ = $ /. x_ Conjugate[x_] :> 1 /; ! FreeQ[x, j];
  MatrixForms[$sb = $],
  NL, "Diagonal", imply, $c[[4]] // Framed,

  NL, "•Evaluate: ", $ = $c[[5, -1, 1]],
  $sa = ab : a | xb -> {{ab+, 0}, {0, ab-}};
  Yield, $ = $ /. $sb /. $sa; MatrixForms[$],
  yield, $ = $ //. CommutatorM → MCommutator //
    tuRepeat[Join[tuOpSimplify[Dot], {tuOpDistribute[Dot]}]];
  yield, $ = $ /. $sD0[[-1]] // Simplify;
  MatrixForms[$],
  $x = tuExtractPattern[du _][$][[1]] / du;
  yield, $ = $x.(# / $x) & /@ $ //. tuOpSimplify[Dot] /. Reverse[$sD0[[-1]]],
  imply, Framed[𝒟F → 0]
 ];
```

●EG: Two point space. $\{X \to \{x, y\}, C[X] \to \mathbb{C}^2, C \to \text{complex functions}\}$

•Construct even finite space $\{F_X \to \{C[X], \mathcal{H}_F, \mathcal{D}_F, \gamma_F\}, \dim[\mathcal{H}_F] \ge 2, \gamma_F \to \mathbb{Z}^2\text{grading}\}$

Let $\mathcal{H}_F \to \mathbb{C}^2$

$\to\ \gamma_F \Rightarrow \mathcal{H}_F \to \{(\mathcal{H}_F)^+ \oplus (\mathcal{H}_F)^- \to \mathbb{C}\oplus\mathbb{C},\ \mathcal{H}_F^\pm \to \{\psi \in \mathcal{H}_F \mid \gamma_F.\psi \to \pm\psi\}\} \Rightarrow \gamma_F \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Since $\{[\gamma_F, a] \to 0, \{\mathcal{D}_F, \gamma_F\} \to 0, \mathcal{D}_F \to \text{offDiagonal}, \mathcal{D}_F \to \{\{0, du\}, \{dl, 0\}\}\}$

$\Rightarrow\ \{a.\psi \to \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \circ \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, a \in \mathcal{A}_F, \psi \in \mathcal{H}_F\}$

$\Rightarrow\ F_X \to \{\{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F\} \to \{\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}, t \in \mathbb{C}\}$

■Prop.4.1. A real structure $J_F \Rightarrow \{\mathcal{D}_F \to 0\}$

Determine $\mathcal{D}_F$ for even KO dimensions by requiring:

$J_F.J_F \to \varepsilon$
$J_F.\mathcal{D}_F \to \varepsilon'.\mathcal{D}_F.J_F$
$J_F.\gamma_F \to \varepsilon''.\gamma_F.J_F$
$\forall_{\{a,b\},a|b\in\mathcal{A}_F}\ \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$
$\forall_{\{a,b\},a|b\in\mathcal{A}_F}\ \{[[\mathcal{D}_F, a], b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$

■KOdim->0: $\{J_F \to \{\{j_+, 0\}, \{0, j_-\}\}.cc, j_\pm \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

•Compute $b^0 \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow \quad \longrightarrow b^0 \to \begin{pmatrix} (j_+)^* b_+ j_+ & 0 \\ 0 & (j_-)^* b_- j_- \end{pmatrix} \longrightarrow \boxed{b^0 \to \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}}$

Diagonal $\Rightarrow \boxed{\forall_{\{a,b\},a|b\in\mathcal{A}_F}\ \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}}$

$\Rightarrow\ \forall_{\{a,b\},a|b\in\mathcal{A}_F}\ \{[[\mathcal{D}_F, a], b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$

•Evaluate: $[[\mathcal{D}_F, a], b^0] \to 0$

$\to\ [[\mathcal{D}_F, \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}], \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}] \to 0 \longrightarrow \quad \longrightarrow$

$\begin{pmatrix} 0 & du\,(a_- - a_+)\,(b_- - b_+) \\ dl\,(a_- - a_+)\,(b_- - b_+) & 0 \end{pmatrix} \to 0 \longrightarrow ((a_- - a_+)\,(b_- - b_+)).\mathcal{D}_F \to 0 \Rightarrow \boxed{\mathcal{D}_F \to 0}$

■KOdim->0: $\{J_F \to \{\{0, j\}, \{-j, 0\}\}.cc, j \in U[1]\}$

for $ab : a \mid b \to \begin{pmatrix} ab_+ & 0 \\ 0 & ab_- \end{pmatrix}$

Compute $b^0 \to J_F.b^\dagger.(J_F)^\dagger \longrightarrow \longrightarrow b^0 \to \begin{pmatrix} j\,j^*\,b_- & 0 \\ 0 & j\,j^*\,b_+ \end{pmatrix} \longrightarrow b^0 \to \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}$

Diagonal $\Rightarrow$ $\boxed{\forall_{\{a,b\},a|b\in\mathcal{A}_F} \{[a, b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\}}$

•Evaluate: $[[\mathcal{D}_F, a], b^0] \to 0$

$\to [[\mathcal{D}_F, \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}], \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}] \to 0 \longrightarrow \longrightarrow$

$\begin{pmatrix} 0 & -du\,(a_- - a_+)\,(b_- - b_+) \\ -dl\,(a_- - a_+)\,(b_- - b_+) & 0 \end{pmatrix} \to 0 \longrightarrow -((a_- - a_+)\,(b_- - b_+)).\mathcal{D}_F \to 0 \Rightarrow \boxed{\mathcal{D}_F \to 0}$

```
PR["From M, 4-dim Riemann spin manifold and Fx two-point space, form ",
 M×Fx → {C"∞"[M, ℂ²], L²[M, S]⊗ℂ², slash[𝒟]⊗𝟙, γ5⊗γF, JM⊗JF}
]
```

From M, 4-dim Riemann spin manifold and F_X two-point space, form

$M\times F_X \to \{C^\infty[M, \mathbb{C}^2], L^2[M, S]\otimes\mathbb{C}^2, (\not{D})\otimes\mathbb{1}, \gamma_5\otimes\gamma_F, J_M\otimes J_F\}$

```
PR["●U[1] gauge theory ",
 NL, "gauge group ", 𝒢[𝒜] → Mod[U[𝒜], U[$sAt[[1]]]],
 NL, "where ", {$t219[[1, -2]], U[𝒜] ≠ U[$sAt[[1]]], $sAt} // Column,
 Imply, "KOdim[JF]" → {2, 6},
 ", i.e., off diagonal. only KOdim→6 for Standard Model used in this case. ",
 Imply, "Can use Def.2.17 for action functional ",
 $d217 = {S → Sb + Sf, Sb → Tr[f[𝒟𝒜 / Λ]], Sf → 1 / 2 BraKet[J.ξ̃, 𝒟𝒜.ξ̃],
   ξ̃ ∈ ℋcl⁺, ℋcl⁺ → {ξ̃ | ξ ∈ ℋ⁺}, ξ̃ → "GrassmannVariable"};
 Column[$d217],
 NL, "•Consider ", $Fx = Fx → {ℂ², ℂ², 0, γF → {{1, 0}, {0, -1}}, JF → {{0, C}, {C, 0}}};
 MatrixForms[$Fx]
]
```

●U[1] gauge theory

gauge group $\mathcal{G}[\mathcal{A}] \to Mod[U[\mathcal{A}], U[\tilde{\mathcal{A}}_J]]$

$(2.11) \Rightarrow \mathcal{G}[M\times F] \to \{U \to u.J.u.J^\dagger, u \in U[\mathcal{A}]\}$

where $U[\mathcal{A}] \neq U[\tilde{\mathcal{A}}_J]$

$\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A}, a.J \to J.a^\dagger, a^0 \to a\}$

$\Rightarrow$ $KOdim[J_F] \to \{2, 6\}$

, i.e., off diagonal. only KOdim→6 for Standard Model used in this case.

$\Rightarrow$ Can use Def.2.17 for action functional

$S \to S_b + S_f$

$S_b \to Tr[f[\frac{\mathcal{D}_\mathcal{A}}{\Lambda}]]$

$S_f \to \frac{1}{2}\left\langle J.\tilde{\xi} \mid \mathcal{D}_\mathcal{A}.\tilde{\xi}\right\rangle$

$\tilde{\xi} \in (\mathcal{H}_{cl})^+$

$(\mathcal{H}_{cl})^+ \to \{\tilde{\xi} \mid \xi \in \mathcal{H}^+\}$

$\tilde{\xi} \to GrassmannVariable$

•Consider $F_X \to \{\mathbb{C}^2, \mathbb{C}^2, 0, \gamma_F \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J_F \to \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}\}$

```
PR["Prop.4.2. The gauge group ", 𝒢[𝒜F] → U[1],
  NL, "Note: ", U[𝒜F] → U[1] × U[1],
  NL, "From ", $sAt,
  yield, $ = ForAll[a,
    a ∈ ℂ² && a ∈ ($sAtj = ($sAt[[1]] /. J → F)JF), (JF.ConjugateTranspose[a].JF → a)],
  NL, "Compute ", $ = tuExtractPattern[Rule[__]][$][[1]],
  yield, $ = $ /. $Fx[[2, -2 ;; -1]]; MatrixForms[$],
  NL, "Let ", $sCC = $s = {a -> DiagonalMatrix[{a1, a2}],
     C.a_ :> Conjugate[a].C /; FreeQ[a, C], Conjugate[C] → C, C.C → 1},
  Yield, $ = $ /. Dot → xDot /. $s // OrderedxDotMultiplyAll[];
  MatrixForms[$],
  yield, $ = $ // tuRepeat[$s, ConjugateCTSimplify1[{}]];
  MatrixForms[$] // Framed,
  Imply, a1 → a2, imply, a → "diagonal",
  imply, $pass4 = $ = $sAtj ≃ ℂ,
  imply, (U[$[[1]]] → U[1]) ⊂ U[𝒜F]
 ];
```

Prop.4.2. The gauge group $\mathcal{G}[\mathcal{A}_F]\to U[1]$

Note: $U[\mathcal{A}_F]\to U[1]\times U[1]$

From $\tilde{\mathcal{A}}_J\to\{a\in\mathcal{A},\ a.J\to J.a^\dagger,\ a^0\to a\}\longrightarrow\forall_{a,a\in\mathbb{C}^2\&\&a\in\tilde{\mathcal{A}}_{F\,J_F}}(J_F.a^\dagger.J_F\to a)$

Compute $J_F.a^\dagger.J_F\to a\longrightarrow\begin{pmatrix}0&C\\C&0\end{pmatrix}.a^\dagger.\begin{pmatrix}0&C\\C&0\end{pmatrix}\to a$

Let $\{a\to\{\{a1,0\},\{0,a2\}\},\ C.(a\_)\mapsto a^*.C\ /;\ FreeQ[a,C],\ C^*\to C,\ C.C\to 1\}$

$\to\begin{pmatrix}C.a2^*.C&0\\0&C.a1^*.C\end{pmatrix}\to\begin{pmatrix}a1&0\\0&a2\end{pmatrix}\longrightarrow\boxed{\begin{pmatrix}a2&0\\0&a1\end{pmatrix}\to\begin{pmatrix}a1&0\\0&a2\end{pmatrix}}$

$\Rightarrow a1\to a2\ \Rightarrow\ a\to diagonal\ \Rightarrow\ \tilde{\mathcal{A}}_{F\,J_F}\simeq\mathbb{C}\ \Rightarrow\ (U[\tilde{\mathcal{A}}_{F\,J_F}]\to U[1])\subset U[\mathcal{A}_F]$

```
PR["■Determine Bμ. Since ", $pass4,
  yield, (hF → u[$sAtj]) ≃ I ℝ,
  NL, "Gauge field: ",
  Aμ[x] ∈ (I gF -> I Mod[u[($a = $sAt[[1]] /. J → F)], I ℝ]) → (I su[$a] ≃ ℝ),
  NL, "Arbitrary hermitian field ",
  $sA = {Aμ → -I a tuDPartial[b, μ], Aμ → {{T[X"1", "d", {μ}], 0}, {0, T[X², "d", {μ}]}},
    {T[X"1", "d", {μ}], T[X², "d", {μ}]} ∈ C"∞"[M, ℝ], C.tt : T[X"1"|², "d", {μ}] → tt.C},
  NL, "Since ", Aμ , " is always in form ", $ = Bμ -> Aμ - JF.Aμ.inv[JF],
  Yield, $ = $ /. $Fx[[2, -1]] /. inv[cc : 0 | C] → cc /. Dot → xDot /.
       dd : xDot[__] :> (dd /. $sA[[2]] //. $sA[[-1]]) /.
      Plus → xPlus /. $sA[[2]] // OrderedxDotMultiplyAll[];
  Yield, $ = $ /. xPlus → Plus /. $sA[[-1]] /. $sCC /. tuOpSimplify[Dot];
  MatrixForms[$B = $],
  " define ", $ = $ -> {{T[Y, "d", {μ}], 0}, {0, -T[Y, "d", {μ}]}};
  $ = Flatten /@ ($[[1, 2]] -> $[[-1]]);
  $sb = Thread[$] // DeleteCases[#, 0 → 0] & // First,
  imply, $B = $B /. {$sb, -1 # & /@ $sb};
  MatrixForms[$B -> T[Y, "d", {μ}] ⊗ γF] // Framed
 ];
```

■Determine B_μ. Since $\tilde{\mathcal{A}}_{F\,J_F}\simeq\mathbb{C}\longrightarrow(h_F\to u[\tilde{\mathcal{A}}_{F\,J_F}])\simeq i\,\mathbb{R}$

Gauge field: $A_\mu[x]\in(i\,g_F\to i\,Mod[u[\tilde{\mathcal{A}}_F],\ i\,\mathbb{R}])\to I\,su[\tilde{\mathcal{A}}_F]\simeq\mathbb{R}$

Arbitrary hermitian field

$\{A_\mu\to -i\,a\,\partial_\mu[b],\ A_\mu\to\{\{X^1{}_\mu,0\},\{0,X^2{}_\mu\}\},\ \{X^1{}_\mu,X^2{}_\mu\}\in C^\infty[M,\mathbb{R}],\ C.(tt:X^{1|2}{}_\mu)\to tt.C\}$

Since A_μ is always in form $B_\mu\to -J_F.A_\mu.J_F^{-1}+A_\mu$

$\to$

$\to B_\mu\to\begin{pmatrix}-X^2{}_\mu+X^1{}_\mu&0\\0&X^2{}_\mu-X^1{}_\mu\end{pmatrix}$ define $-X^2{}_\mu+X^1{}_\mu\to Y_\mu\ \Rightarrow\ \boxed{\left(B_\mu\to\begin{pmatrix}Y_\mu&0\\0&-Y_\mu\end{pmatrix}\right)\to Y_\mu\otimes\gamma_F}$

```
PR["●Prop.4.3. Inner fluctuations for
   ACM M×F_X are parameterized by a U[1]-gauge field Y_μ ",
 Yield, 𝒟 ↦ 𝒟' → 𝒟 + T[γ, "u", {μ}].T[Y, "d", {μ}] ⊗ γ_F,
 NL, "The action of gauge group ", 𝒢[𝒜] ≃ C"∞"[M, U[1]][𝒟'],
 Yield,
 {T[Y, "d", {μ}] ↦ T[Y, "d", {μ}] - I u.tuDPartial[ConjugateTranspose[u], μ], u ∈ 𝒢[𝒜]}
]
```

●Prop.4.3. Inner fluctuations
   for ACM M×F_X are parameterized by a U[1]-gauge field Y_μ

→ $\mathcal{D}\mapsto\mathcal{D}'\to\mathcal{D}+\gamma^\mu.Y_\mu\otimes\gamma_F$

The action of gauge group $\mathcal{G}[\mathcal{A}]\simeq C^\infty[M, U[1]][\mathcal{D}']$

→ $\{Y_\mu\mapsto -i\, u.\partial_\mu[u^\dagger]+Y_\mu,\ u\in\mathcal{G}[\mathcal{A}]\}$

```
PR["■Two modifications needed for E-M: ", {𝒟_F → ! 0, S_f → "2 independent spinors"},
 NL, "•Let ", {{e, ē} → "basis of ℋ_F",
   e → "basis of ℋ_F^+",
   ē → "basis of ℋ_F^-",
   J_F.e -> ē,
   J_F.ē → e,
   γ_F.e → e,
   γ_F.ē → -ē
  } // Column,
 imply,
 $H = {ℋ → L^2[M, S] ⊗ ℋ_F, L^2[M, S] -> L^2[M, S]^+ ⊕ L^2[M, S]^-,
   ℋ^+ → "positiveEigenSpace of γ→γ5⊗γ_F",
   ℋ^+ -> L^2[M, S]^+ ⊗ ℋ_F^+ ⊕ L^2[M, S]^- ⊗ ℋ_F^-,
   ξ ∈ ℋ^+,
   ξ → ψ_L ⊗ e + ψ_R ⊗ ē,
   ψ_L ∈ L^2[M, S]^+,
   ψ_R ∈ L^2[M, S]^-
  }; Column[$H],
 NL, "•Doubling space ", C"∞"[M, ℂ^2] ↔ (N → M × X ≃ M ⊔ M),
 NL, "Let ",
 $se = {{e_R, e_L, ē_R, ē_L} → basis [ℋ_F → ℂ^4], γ_F.e_L → e_L, γ_F.e_R → -e_R, J_F.e_R → -ē_L, J_F.e_L → -ē_R,
   KOdim → 6, J_F.J_F → 𝕀, J_F.γ_F → -γ_F.J_F}; Column[$se],
 NL, "Chirality ", {J_F.γ_F.e_L → -γ_F.J_F.e_L, J_F.γ_F.e_R → -γ_F.J_F.e_R} //
   tuRepeat[Join[$se, tuOpSimplify[Dot]]] // Column,
 Imply, $sgj = {γ_F → DiagonalMatrix[{-1, 1, 1, -1}],
    J_F → SparseArray[{Band[{1, 3}] → C, Band[{3, 1}] → C}, {4, 4}]} // Normal;
 MatrixForms[$sgj],
 NL, "•The elements ",
 $sa = {a ∈ (𝒜_F → ℂ^2), a[{e_R, e_L, ē_R, ē_L}] → DiagonalMatrix[{a1, a1, a2, a2}]};
 MatrixForms[$sa]
]
PR["■Prop.4.5. ", F_ED → {ℂ^2, ℂ^4, 0, γ_F, J_F}, " is a real even finite space of KOdim→6."
]
```

■Two modifications needed for E-M: $\{\mathcal{D}_F \to !\,0,\ S_f \to 2 \text{ independent spinors}\}$

•Let
$$\begin{array}{l} \{e, \overline{e}\} \to \text{basis of } \mathcal{H}_F \\ e \to \text{basis of } \mathcal{H}_F{}^+ \\ \overline{e} \to \text{basis of } \mathcal{H}_F{}^- \\ J_F.e \to \overline{e} \\ J_F.\overline{e} \to e \\ \gamma_F.e \to e \\ \gamma_F.\overline{e} \to -\overline{e} \end{array} \Rightarrow \begin{array}{l} \mathcal{H} \to L^2[M, S] \otimes \mathcal{H}_F \\ L^2[M, S] \to L^2[M, S]^+ \oplus L^2[M, S]^- \\ \mathcal{H}^+ \to \text{positiveEigenSpace of } \gamma \to \gamma_5 \otimes \gamma_F \\ \mathcal{H}^+ \to L^2[M, S]^+ \otimes (\mathcal{H}_F)^+ \oplus L^2[M, S]^- \otimes (\mathcal{H}_F)^- \\ \xi \in \mathcal{H}^+ \\ \xi \to \psi_L \otimes e + \psi_R \otimes \overline{e} \\ \psi_L \in L^2[M, S]^+ \\ \psi_R \in L^2[M, S]^- \end{array}$$

•Doubling space $C^\infty[M, \mathbb{C}^2] \leftrightarrow (N \to M \times X \simeq M \vee M)$

Let
$$\begin{array}{l} \{e_R, e_L, \overline{e_R}, \overline{e_L}\} \to \text{basis}[\mathcal{H}_F \to \mathbb{C}^4] \\ \gamma_F.e_L \to e_L \\ \gamma_F.e_R \to -e_R \\ J_F.e_R \to -\overline{e_L} \\ J_F.e_L \to -\overline{e_R} \\ \text{KOdim} \to 6 \\ J_F.J_F \to \mathbb{I} \\ J_F.\gamma_F \to -\gamma_F.J_F \end{array}$$

Chirality
$$\begin{array}{l} -\overline{e_R} \to \gamma_F.\overline{e_R} \\ \overline{e_L} \to \gamma_F.\overline{e_L} \end{array}$$

$$\Rightarrow \{\gamma_F \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ J_F \to \begin{pmatrix} 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \\ C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \end{pmatrix}\}$$

•The elements $\{a \in (\mathcal{A}_F \to \mathbb{C}^2),\ a[\{e_R, e_L, \overline{e_R}, \overline{e_L}\}] \to \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}\}$

■Prop.4.5. $F_{ED} \to \{\mathbb{C}^2, \mathbb{C}^4, 0, \gamma_F, J_F\}$ is a real even finite space of KOdim→6.

```
PR["■Add non-trivial Dirac operator.
Since ",
  $ = 𝒟F.γF → -γF.𝒟F, "POFF",
  NL, "𝒟F Hermitian condition: ",
  $d = Table[d[i, j], {i, 4}, {j, 4}]; MatrixForms[$d],
  $ct = ct[$d]; MatrixForms[$ct],
  $ct = $d → $ct //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates,
  $ct = Select[$ct, OrderedQ[{#[[1, 2]], #[[1, 1]]}] &],
  $d = 𝒟F → $d;
  Yield, $ = $ /. $d /. $sgj; MatrixForms[$],
  Yield, $ = $ //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates,
  Yield, $s = tuRuleSolve[$, Flatten[$d[[2]]]],
  "PON",
  imply, $d0 = $d = $d /. $s /. $ct; MatrixForms[$d],

  NL, "Since ", $ = 𝒟F.JF → JF.𝒟F, "POFF",
  Yield, $ = $ /. Dot → xDot /. $d /. $sgj // OrderedxDotMultiplyAll[];
  MatrixForms[$],
  Yield, $ = $ /. C.d_ → Conjugate[d].C; MatrixForms[$],
  Yield, $ = $ //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates;
  Yield, $ = $ /. a_ .C → a; "PON",
  Imply, $d = $d /. $; MatrixForms[$d],
  NL, "Comparing these: ",
  $ = $d[[2]] → $d0[[2]] //. rr : Rule[_, _] :→ Thread[rr] // Flatten // DeleteDuplicates;

  Yield, $ = $ /. List → And /. Rule → Equal,
  Yield, $ = Reduce[$, {d[1, 2]}, Complexes] /. And → List /. Equal → Rule,
  Imply, $d = $d /. $; MatrixForms[$d] // Framed,

  NL, "•Order one condition: ",
  $Da = $ = CommutatorM[𝒟F, a]; Framed[$],
  Yield, $ = $ /. $d /. a → $sa[[-1, -1]] /. CommutatorM → MCommutator // Simplify;
  MatrixForms[$],
  NL, "Simplifying ",
  yield, $s = Flatten[$] /. List → Plus // Simplify;
  $s = Apply[List, $s, {0}];
  yield, $1 = $Da -> $s[[2]]. ($ / $s[[2]]) // Simplify;
  MatrixForms[$1] // Framed
 ];
```

■Add non-trivial Dirac operator.

Since $\mathcal{D}_F.\gamma_F \to -\gamma_F.\mathcal{D}_F$ ⇒ $\mathcal{D}_F \to \begin{pmatrix} 0 & d[1,2] & d[1,3] & 0 \\ d[1,2]^* & 0 & 0 & d[2,4] \\ d[1,3]^* & 0 & 0 & d[3,4] \\ 0 & d[2,4]^* & d[3,4]^* & 0 \end{pmatrix}$

Since $\mathcal{D}_F.J_F \to J_F.\mathcal{D}_F$

⇒ $\mathcal{D}_F \to \begin{pmatrix} 0 & d[3,4]^* & d[1,3] & 0 \\ d[3,4] & 0 & 0 & d[2,4] \\ d[1,3]^* & 0 & 0 & d[1,2]^* \\ 0 & d[2,4]^* & d[1,2] & 0 \end{pmatrix}$

Comparing these:

→ $d[3,4]^* == d[1,2]$ && $d[3,4] == d[1,2]^*$ && $d[1,2]^* == d[3,4]$ && $d[1,2] == d[3,4]^*$

→ $d[1,2] \to d[3,4]^*$

⇒ $\boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d[3,4]^* & d[1,3] & 0 \\ d[3,4] & 0 & 0 & d[2,4] \\ d[1,3]^* & 0 & 0 & d[3,4] \\ 0 & d[2,4]^* & d[3,4]^* & 0 \end{pmatrix}}$

•Order one condition: $\boxed{[\mathcal{D}_F, a]}$

→ $\begin{pmatrix} 0 & 0 & d[1,3](-a_1+a_2) & 0 \\ 0 & 0 & 0 & d[2,4](-a_1+a_2) \\ d[1,3]^*(a_1-a_2) & 0 & 0 & 0 \\ 0 & d[2,4]^*(a_1-a_2) & 0 & 0 \end{pmatrix}$

Simplifying ⟶ ⟶ $\boxed{[\mathcal{D}_F, a] \to (a_1-a_2).\begin{pmatrix} 0 & 0 & -d[1,3] & 0 \\ 0 & 0 & 0 & -d[2,4] \\ d[1,3]^* & 0 & 0 & 0 \\ 0 & d[2,4]^* & 0 & 0 \end{pmatrix}}$

```
PR["The condition ", $ = $c[[5, -1]],
 Yield, $ = $[[1]] /. $1 /. $[[2]] /. ($saa = (a₁ - a₂) → a1m2);
 Yield, $ = $ /. b -> DiagonalMatrix[{b₁, b₁, b₂, b₂}] /. Dot → xDot /. $sgj //
   OrderedxDotMultiplyAll[];
 Yield, $ = $ /. C.d_ → Conjugate[d].C // ConjugateCTSimplify1[{}];
 Yield, $ = $ /. C.C → 1 /. tuOpSimplify[Dot] /. CommutatorM → MCommutator /.
    tuOpSimplify[Dot, {a1m2}] // Simplify;
 MatrixForms[$],

 NL, "Move common factors outside ",
 Yield, $s = Flatten[$[[1]]],
 Yield, $s = $s /. List → Plus // Simplify,
 Yield, $s = Apply[List, $s, {0}],
 Yield, $2 = ($s[[1]] $s[[3]]).($[[1]] / ($s[[1]] $s[[3]])) // Simplify;
 MatrixForms[($2 → 0) /. Reverse[$saa]] // Framed,
 NL, "Since a's and b's arbitrary ",
 imply, $ = (tuExtractPattern[List[__]][$2] // Flatten // DeleteDuplicates) → 0,
 Yield, $s = Thread[$]; FramedColumn[$s],
 imply, $d = $d /. $s; MatrixForms[$d] // Framed,
 " relabel ", $Dd = $d /. d[3, 4] → Conjugate[d];
 MatrixForms[$Dd] // Framed
]
```

The condition $\{[[\mathcal{D}_F, a], b^0] \to 0,\ b^0 \to J_F.b^\dagger.(J_F)^\dagger\}$

→

→

→

$$\to \text{a1m2}.\begin{pmatrix} 0 & 0 & -d[1,3]\,b_1 & 0 \\ 0 & 0 & 0 & -d[2,4]\,b_1 \\ d[1,3]^*\,b_2 & 0 & 0 & 0 \\ 0 & d[2,4]^*\,b_2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1 \end{pmatrix}.\text{a1m2}.\begin{pmatrix} 0 & 0 & -d[1,3] & 0 \\ 0 & 0 & 0 & -d[2,4] \\ d[1,3]^* & 0 & 0 & 0 \\ 0 & d[2,4]^* & 0 & 0 \end{pmatrix} \to 0$$

Move common factors outside

→

a1m2.{{0, 0, -d[1, 3] b_1, 0}, {0, 0, 0, -d[2, 4] b_1}, {d[1, 3]* b_2, 0, 0, 0}, {0, d[2, 4]* b_2, 0, 0}} - {{b_2, 0, 0, 0}, {0, b_2, 0, 0}, {0, 0, b_1, 0}, {0, 0, 0, b_1}}.a1m2.{{0, 0, -d[1, 3], 0}, {0, 0, 0, -d[2, 4]}, {d[1, 3]*, 0, 0, 0}, {0, d[2, 4]*, 0, 0}}

→ a1m2.(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2) - (2 (b_1 + b_2)).a1m2.(d[1, 3]* + d[2, 4]* - d[1, 3] - d[2, 4])

→ {a1m2.(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2), -(2 (b_1 + b_2)).a1m2.(d[1, 3]* + d[2, 4]* - d[1, 3] - d[2, 4])}

→

((a₁ − a₂).(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2)
{(a_1 - a_2).(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2),
-(2 (b_1 + b_2)).(a_1 - a_2).(d[1, 3]* + d[2, 4]* - d[1, 3] - d[2, 4])}⟦3⟧).

$$(((a_1 - a_2).\begin{pmatrix} 0 & 0 & -d[1,3]\,b_1 & 0 \\ 0 & 0 & 0 & -d[2,4]\,b_1 \\ d[1,3]^*\,b_2 & 0 & 0 & 0 \\ 0 & d[2,4]^*\,b_2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1 \end{pmatrix}.(a_1 - a_2).\begin{pmatrix} 0 & 0 & -d[1,3] & 0 \\ 0 & 0 & 0 & -d[2,4] \\ d[1,3]^* & 0 & 0 & 0 \\ 0 & d[2,4]^* & 0 & 0 \end{pmatrix})\,/$$

((a_1 - a_2).(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2)
{(a_1 - a_2).(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2),
-(2 (b_1 + b_2)).(a_1 - a_2).(d[1, 3]* + d[2, 4]* - d[1, 3] - d[2, 4])}⟦3⟧)) → 0

Since a's and b's arbitrary ⇒ {a1m2.(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2), -(2 (b_1 + b_2)).a1m2.(d[1, 3]* + d[2, 4]* - d[1, 3] - d[2, 4]), 0, -d[1, 3] b_1, -d[2, 4] b_1, d[1, 3]* b_2, d[2, 4]* b_2, b_2, b_1, -d[1, 3], -d[2, 4], d[1, 3]*, d[2, 4]*} → 0

→

a1m2.(-(d[1, 3] + d[2, 4]) b_1 + (d[1, 3]* + d[2, 4]*) b_2) → 0
-(2 (b_1 + b_2)).a1m2.(d[1, 3]* + d[2, 4]* - d[1, 3] - d[2, 4]) → 0
0 → 0
-d[1, 3] b_1 → 0
-d[2, 4] b_1 → 0
d[1, 3]* b_2 → 0
d[2, 4]* b_2 → 0
b_2 → 0
b_1 → 0
-d[1, 3] → 0
-d[2, 4] → 0
d[1, 3]* → 0
d[2, 4]* → 0

⇒

$$\boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d[3,4]^* & d[1,3] & 0 \\ d[3,4] & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d[3,4] \\ 0 & 0 & d[3,4]^* & 0 \end{pmatrix}} \quad \text{relabel} \quad \boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}}$$

```
PR["●Then ", $MF = M × FX → {C"∞"[M, ℂ²], L²[M, S] ⊗ ℂ², slash[𝒟] ⊗ 𝟙, γ5 ⊗ γF, JM ⊗ JF};
 ColumnForms[$MF],
 " becomes ",
 M × FED → {C"∞"[M, ℂ²], L²[M, S] ⊗ ℂ⁴, slash[𝒟] ⊗ 𝟙 + T[γ, "d", {5}] ⊗ 𝒟F, γ5 ⊗ γF, JM ⊗ JF} //
  ColumnForms,
 NL, "Decompose ", {𝒜 ← C"∞"[M, ℂ²] -> C"∞"[M, ℂ] ⊕ C"∞"[M, ℂ],
    (ℋ ← L²[M, S] ⊗ ℂ⁴) -> L²[M, S] ⊗ ℋe ⊕ L²[M, S] ⊗ ℋe,
    a ∈ 𝒜 → $sa[[2]]
   } // Column // MatrixForms,
 NL, "Gauge group ", 𝒢[𝒜F] ≃ U[1],
 Yield, $B = {T[B, "d", {μ}] →
    DiagonalMatrix[{T[Y, "d", {μ}], T[Y, "d", {μ}], -T[Y, "d", {μ}], -T[Y, "d", {μ}]}],
   T[Y, "d", {μ}][x] ∈ ℝ}; MatrixForms[$B]
]
```

●Then $M\times F_X\to \begin{array}{l} C^\infty[M,\mathbb{C}^2] \\ L^2[M,S]\otimes\mathbb{C}^2 \\ (\not{D})\otimes\mathbb{1} \\ \gamma_5\otimes\gamma_F \\ J_M\otimes J_F \end{array}$ becomes $M\times F_{ED}\to \begin{array}{l} C^\infty[M,\mathbb{C}^2] \\ L^2[M,S]\otimes\mathbb{C}^4 \\ (\not{D})\otimes\mathbb{1}+\gamma_5\otimes\mathcal{D}_F \\ \gamma_5\otimes\gamma_F \\ J_M\otimes J_F \end{array}$

Decompose $\begin{array}{l} \mathcal{A}\leftarrow C^\infty[M,\mathbb{C}^2]\to C^\infty[M,\mathbb{C}]\oplus C^\infty[M,\mathbb{C}] \\ \mathcal{H}\leftarrow L^2[M,S]\otimes\mathbb{C}^4\to L^2[M,S]\otimes\mathcal{H}_e\oplus L^2[M,S]\otimes\mathcal{H}_{\bar{e}} \\ a\in\mathcal{A}\to a[\{e_R,e_L,\overline{e_R},\overline{e_L}\}]\to\left(\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{array}\right) \end{array}$

Gauge group $\mathcal{G}[\mathcal{A}_F]\simeq U[1]$

$\to \left\{B_\mu\to\left(\begin{array}{cccc} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & -Y_\mu & 0 \\ 0 & 0 & 0 & -Y_\mu \end{array}\right), Y_\mu[x]\in\mathbb{R}\right\}$

■ 4.2.4 Lagrangian

● Spectral Action

```
PR["Insert ", $sΦ = $s = {Φ → 𝒟F, N → dim[ℋF], dim[ℋF] → 4, Tr[1ℋF] → N},
 and, $ = { $B[[1]], $Dd}; MatrixForms[$],
 NL, "into Prop.3.7 Lagrangian ", $ = $p37[[{2, 3, 5, 7}]] /. $s;
 Column[$0 = $],
 NL, "•Evaluate term ", $ = $0[[3]],
 " where ", $s =
  {$$ = T[F, "dd", {μ, ν}] → tuDPartial[T[Y, "d", {ν}], μ] - tuDPartial[T[Y, "d", {μ}], ν],
   tuIndicesRaise[{μ, ν}][$$]},
 Imply, $ = $ /. $s; Framed[$],
 NL, "•Evaluate term ", $ = $0[[4]],
 Yield, $[[2]] = $[[2]] /. $Dd; MatrixForms[$],
 NL, "Evaluate Tr[]'s ",
 $1 = $ // tuExtractPositionPattern[Tr[_]];
 Yield, $1 = $1 /. tt : (T[𝒟, "d", {μ}] | T[𝒟, "u", {μ}])[a_] :→ Thread[tt] /.
    tt : (T[𝒟, "d", {μ}] | T[𝒟, "u", {μ}])[a_] :→ Thread[tt] /.
   (T[𝒟, "d", {μ}] | T[𝒟, "u", {μ}])[0] → 0,
 Yield, $ = tuReplacePart[$, $1]; Framed[$]
]
```

Insert $\{\Phi \to \mathcal{D}_F,\ N \to \dim[\mathcal{H}_F],\ \dim[\mathcal{H}_F] \to 4,\ \mathrm{Tr}[1_{\mathcal{H}_F}] \to N\}$

and $\left\{B_\mu \to \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & -Y_\mu & 0 \\ 0 & 0 & 0 & -Y_\mu \end{pmatrix},\ \mathcal{D}_F \to \begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}\right\}$

into Prop.3.7 Lagrangian

$$\mathcal{L}[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \mathcal{L}_B[B_\mu] + \mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] + \dim[\mathcal{H}_F]\, \mathcal{L}_M[g_{\mu\nu}]$$

$$\mathcal{L}_M[g_{\mu\nu}] \to \frac{\Lambda^4 f_4}{2\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}_{E_x}[s[x]\, 1_{\dim[\mathcal{H}_F]}]}{96\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[s[x]^2\, 1_{\dim[\mathcal{H}_F]}]}{4608\pi^2} - \frac{f[0]\, \mathrm{Tr}_{E_x}[1_{\dim[\mathcal{H}_F]}\, R_{\mu\nu} R^{\mu\nu}]}{2880\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[1_{\dim[\mathcal{H}_F]}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}]}{2880\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[1_{\dim[\mathcal{H}_F]}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \gamma_\rho \gamma_\sigma \gamma^\rho \gamma^\sigma]}{3072\pi^2} + \frac{f[0]\, \mathrm{Tr}_{E_x}[\triangle[s[x]\, 1_{\dim[\mathcal{H}_F]}]]}{1920\pi^2}$$

$$\mathcal{L}_B[B_\mu] \to \frac{f[0]\, \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$$

$$\mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]}{2\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_\mu[\mathcal{D}_F].\mathcal{D}^\mu[\mathcal{D}_F]]}{8\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F.\mathcal{D}_F.\mathcal{D}_F]}{8\pi^2} + \frac{f[0]\, \triangle[\mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]]}{24\pi^2}$$

•Evaluate term $\mathcal{L}_B[B_\mu] \to \frac{f[0]\, \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}{24\pi^2}$ where $\{F_{\mu\nu} \to -\underline{\partial}_\nu[Y_\mu] + \underline{\partial}_\mu[Y_\nu],\ F^{\mu\nu} \to -\underline{\partial}^\nu[Y^\mu] + \underline{\partial}^\mu[Y^\nu]\}$

$$\Rightarrow \boxed{\mathcal{L}_B[B_\mu] \to \frac{f[0]\, \mathrm{Tr}[(-\underline{\partial}_\nu[Y_\mu] + \underline{\partial}_\mu[Y_\nu])\,(-\underline{\partial}^\nu[Y^\mu] + \underline{\partial}^\mu[Y^\nu])]}{24\pi^2}}$$

•Evaluate term $\mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{f[0]\, s[x]\, \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]}{48\pi^2} - \frac{\Lambda^2 f_2 \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]}{2\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_\mu[\mathcal{D}_F].\mathcal{D}^\mu[\mathcal{D}_F]]}{8\pi^2} + \frac{f[0]\, \mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F.\mathcal{D}_F.\mathcal{D}_F]}{8\pi^2} + \frac{f[0]\, \triangle[\mathrm{Tr}[\mathcal{D}_F.\mathcal{D}_F]]}{24\pi^2}$

$$\to \mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{d^2 d^{*2} f[0]}{2\pi^2} + \frac{d\, d^*\, f[0]\, s[x]}{12\pi^2} - \frac{2\, d\, \Lambda^2\, d^*\, f_2}{\pi^2} + \frac{f[0]\, \mathrm{Tr}\left[\mathcal{D}_\mu\left[\begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}\right].\mathcal{D}^\mu\left[\begin{pmatrix} 0 & d & d[1,3] & 0 \\ d^* & 0 & 0 & d[2,4] \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix}\right]\right]}{8\pi^2} + \frac{f[0]\, \triangle[4\, d\, d^*]}{24\pi^2}$$

Evaluate Tr[]'s

$$\to \{\{2, 4, 4\} \to 2\, \mathcal{D}_\mu[d]\, \mathcal{D}^\mu[d] + 2\, \mathcal{D}_\mu[d^*]\, \mathcal{D}^\mu[d^*] + \mathcal{D}_\mu[d[1,3]]\, \mathcal{D}^\mu[d[1,3]] + \mathcal{D}_\mu[d[2,4]]\, \mathcal{D}^\mu[d[2,4]]\}$$

$$\to \boxed{\mathcal{L}_H[g_{\mu\nu}, B_\mu, \mathcal{D}_F] \to \frac{d^2 d^{*2} f[0]}{2\pi^2} + \frac{d\, d^*\, f[0]\, s[x]}{12\pi^2} - \frac{2\, d\, \Lambda^2\, d^*\, f_2}{\pi^2} + \frac{f[0]\, \triangle[4\, d\, d^*]}{24\pi^2} + \frac{f[0]\, (2\, \mathcal{D}_\mu[d]\, \mathcal{D}^\mu[d] + 2\, \mathcal{D}_\mu[d^*]\, \mathcal{D}^\mu[d^*] + \mathcal{D}_\mu[d[1,3]]\, \mathcal{D}^\mu[d[1,3]] + \mathcal{D}_\mu[d[2,4]]\, \mathcal{D}^\mu[d[2,4]])}{8\pi^2}}$$

4.2.5 Fermionic action

```
PR[$H // Column,
 NL, "Basis ", $sa[[2, 1, 1]],
 Yield, $H[[4]],
 NL, "Spanning basis ", {ℋF^+[{eL, ēR}], ℋF^-[{eR, ēL}]},
 NL, "Arbitrary vector ",
 $sξ = {ξ -> χR ⊗ eR + χL ⊗ eL + ψL ⊗ ēR + ψR ⊗ ēL, {χL, ψL} ∈ L^2[M, S]^+, {χR, ψR} ∈ L^2[M, S]^-};

 Column[$sξ],
 NL, "Then fermionic action for ", $MF,
 Yield,
 $Sf = $ = Sf → -I BraKet[JM.χ̃, T[γ, "u", {μ}].(T["∇"^S, "d", {μ}] - I T[Y, "d", {μ}]).ψ̃] +
     BraKet[JM.χ̃L, ct[d].ψ̃L] - BraKet[JM.χ̃R, d.ψ̃R];
 Framed[$], CO["Prop.4.7"],
 NL, "where the ~ means ", $sAt, CK,
 NL, "■Proof: ",
 NL, "The fluctuated Dirac operator ",
 Yield, $sDA1 = $ = $sDA[[1]] /. $sDA[[2]] /. $sΦ, "POFF",
 Yield, $ = $ /. tuOpDistribute[Dot] //. tuOpSimplify[Dot] // Expand,
 Yield, $ = $ //. {a_ . (b_ ⊗ c_) → (a.b) ⊗ c, a_ . 1_ → a}, "PON",
 NL, "Since ", $s = $slashD[[1]] /. a_ tuDDown[tt : _][_, i_] ⧴ a .
      T[tt, "d", {i}] //. tuOpSimplify[Dot],
 $slashd = $s = tuRuleSolve[$s, Dot[_, _]];
 yield, $ = $ /. Reverse[$s] //. tuOpSimplify[CircleTimes] //. tuOpSimplify[Dot];
 Framed[$sDA0 = $], CO["p.48"],
 NL, "▪Using ", $sCT = {J -> $MF[[2, -1]]}, and,
 $s = Map[$Dd[[1]].# &, $sa[[2, 1, 1]]];
 $s = $s → ($Dd[[2]].Transpose[{$sa[[2, 1, 1]]}] // Transpose // Flatten) // Thread;
 $s1 = $B[[1, 2]].Transpose[{$sa[[2, 1, 1]]}] // Flatten;
 $s1 = Map[$B[[1, 1]].# &, $sa[[2, 1, 1]]] → $s1 // Thread;
 Yield,
 $s0J = {JF.ei_ -> ēi, JF.ēi_ → ei, γF.ei_ → ei, γF.ēi_ → -ēi, $s, $s1
   } // Flatten,

 NL, "Compute ",
 NL, "•", $ = J.ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$],
 NL, "•", $ = $sDA0[[2, 1]].ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$],
 NL, "•", $ = $sDA0[[2, 2]].ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$],
 NL, "•", $ = $sDA0[[2, 3]].ξ,
 yield, $ = $ /. $sξ[[1]] /. $sCT //. tuOpDistribute[Dot] //. $sX /. $s0J;
 Framed[$]

]
```

$\mathcal{H} \to L^2[M, S] \otimes \mathcal{H}_F$
$L^2[M, S] \to L^2[M, S]^+ \oplus L^2[M, S]^-$
$\mathcal{H}^+ \to$ positiveEigenSpace of $\gamma \to \gamma_5 \otimes \gamma_F$
$\mathcal{H}^+ \to L^2[M, S]^+ \otimes (\mathcal{H}_F)^+ \oplus L^2[M, S]^- \otimes (\mathcal{H}_F)^-$
$\xi \in \mathcal{H}^+$
$\xi \to \psi_L \otimes e + \psi_R \otimes \overline{e}$
$\psi_L \in L^2[M, S]^+$
$\psi_R \in L^2[M, S]^-$

Basis $\{e_R, e_L, \overline{e_R}, \overline{e_L}\}$
$\to \mathcal{H}^+ \to L^2[M, S]^+ \otimes (\mathcal{H}_F)^+ \oplus L^2[M, S]^- \otimes (\mathcal{H}_F)^-$
Spanning basis $\{(\mathcal{H}_F)^+[\{e_L, \overline{e_R}\}], (\mathcal{H}_F)^-[\{e_R, \overline{e_L}\}]\}$

Arbitrary vector $\xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \overline{e_R} + \psi_R \otimes \overline{e_L}$
$\{\chi_L, \psi_L\} \in L^2[M, S]^+$
$\{\chi_R, \psi_R\} \in L^2[M, S]^-$

Then fermionic action for $M \times F_X \to \{C^\infty[M, \mathbb{C}^2], L^2[M, S] \otimes \mathbb{C}^2, (\not{D}) \otimes \mathbb{1}, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\}$

$\to \boxed{S_f \to -i \left\langle J_M.\tilde{\chi} \mid \gamma^\mu.(\nabla^S{}_\mu - i Y_\mu).\tilde{\psi} \right\rangle + \left\langle J_M.\tilde{\chi_L} \mid d^\dagger.\tilde{\psi_L} \right\rangle - \left\langle J_M.\tilde{\chi_R} \mid d.\tilde{\psi_R} \right\rangle}$ Prop.4.7

where the ~ means $\tilde{\mathcal{A}}_J \to \{a \in \mathcal{A}, a.J \to J.a^\dagger, a^0 \to a\} \longleftarrow$ CHECK
■Proof:
The fluctuated Dirac operator
$\to \mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \mathcal{D}_F - i \gamma^\mu.(i\, 1_{\dim[\mathcal{H}_F]} \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})$

Since $\not{D} \to -i \gamma^\mu.\nabla^S{}_\mu \longrightarrow \boxed{\mathcal{D}_{\mathcal{A}} \to (\not{D}) \otimes 1_{\mathcal{H}_F} + \gamma_5 \otimes \mathcal{D}_F + \gamma^\mu \otimes B_\mu}$ p.48

■Using $\{J \to J_M \otimes J_F\}$ and
$\to \{J_F.e_{i\_} \to \overline{e_i}, J_F.\overline{e_{i\_}} \to e_i, \gamma_F.e_{i\_} \to e_i, \gamma_F.\overline{e_{i\_}} \to -\overline{e_i}, \mathcal{D}_F.e_R \to d[1, 3]\, \overline{e_R} + d\, e_L, \mathcal{D}_F.e_L \to d[2, 4]\, \overline{e_L} + d^*\, e_R,$
$\mathcal{D}_F.\overline{e_R} \to d^*\, \overline{e_L}, \mathcal{D}_F.\overline{e_L} \to d\, \overline{e_R}, B_\mu.e_R \to e_R\, Y_\mu, B_\mu.e_L \to e_L\, Y_\mu, B_\mu.\overline{e_R} \to -\overline{e_R}\, Y_\mu, B_\mu.\overline{e_L} \to -\overline{e_L}\, Y_\mu\}$
Compute

- $J.\xi \longrightarrow \boxed{J_M.\chi_L \otimes \overline{e_L} + J_M.\chi_R \otimes \overline{e_R} + J_M.\psi_L \otimes e_R + J_M.\psi_R \otimes e_L}$

- $((\not{D}) \otimes 1_{\mathcal{H}_F}).\xi \longrightarrow \boxed{(\not{D}).\chi_L \otimes e_L + (\not{D}).\chi_R \otimes e_R + (\not{D}).\psi_L \otimes \overline{e_R} + (\not{D}).\psi_R \otimes \overline{e_L}}$

- $(\gamma_5 \otimes \mathcal{D}_F).\xi \longrightarrow \boxed{\gamma_5.\chi_L \otimes (d[2, 4]\, \overline{e_L} + d^*\, e_R) + \gamma_5.\chi_R \otimes (d[1, 3]\, \overline{e_R} + d\, e_L) + \gamma_5.\psi_L \otimes (d^*\, \overline{e_L}) + \gamma_5.\psi_R \otimes (d\, \overline{e_R})}$

- $(\gamma^\mu \otimes B_\mu).\xi \longrightarrow \boxed{\gamma^\mu.\chi_L \otimes (e_L\, Y_\mu) + \gamma^\mu.\chi_R \otimes (e_R\, Y_\mu) + \gamma^\mu.\psi_L \otimes (-\overline{e_R}\, Y_\mu) + \gamma^\mu.\psi_R \otimes (-\overline{e_L}\, Y_\mu)}$

```
PR["■From ", $ = $d217[[3]],
 Yield, $ = $ /. $sDA1,
 Yield, $0 =
  $ = $ // tuRepeat[{tuOpDistribute[Dot], tuOpSimplify[Dot], tuOpSimplify[CircleTimes],
     (tt : Tensor[γ, _, _]).(a_ ⊗ b_) → (tt .a ⊗ b), $slashd,
     a_ . 1_ → a, tuOpDistribute[BraKet]}, Simplify],
 NL, "●Evaluate terms ", $0p = $ = tuExtractPositionPattern[BraKet[_, _]][$];
 NL, "•", $ = $0p[[1]]; Framed[$],
 NL, "Define ", $sξt =
  #̃ & /@ $sξ[[1]] //. tuOpDistribute[OverTilde] //.
    tuOpDistribute[OverTilde, CircleTimes] /. ã_ :→ a /; ! FreeQ[a, e],
 Yield, $ = $ /. $sξt /. $sCT //. tuOpDistribute[Dot] //. $sX //. tuOpDistribute[BraKet];
 NL, "e's are orthonormal ",
 $s = {BraKet[a_ ⊗ e1_, b_ ⊗ e2_] :→ If[e1 === e2, BraKet[a, b], 0]},
 Yield, ColumnSumExp[$ = $ //. $s0J /. $s],
 NL, "Symmetry of form ",
 $s = BraKet[J_ . ps_ , d_ . x_] :> BraKet[J . x , d . ps] /; ! FreeQ[x, χ],
 Imply, $ = $ /. $s,
 NL, "Since ", $s = slash[𝒟][ψL] → ψ'R, and, $H[[-2 ;; -1]], " orthogonal, i.e., ",
 Yield, $1 = BraKet[χL + χR, (ψ')L + (ψ')R],
 Yield, $1 = $1 //. tuOpDistribute[BraKet],
 Yield, $1 =
```

```
   $1 /. BraKet[a_, b_] :→ 0 /; FreeQ[a, L] && ! FreeQ[b, L] || FreeQ[a, R] && ! FreeQ[b, R],
  Yield, $1 = $1 /. Reverse[$s] /. Reverse[Swap[{L, R}][$s]],
  NL, "So ", $p1 = $ = $ //. a_R|L → a; $[[2]] = $[[2]] / 2; Framed[$]
 ]
 PR["•", $ = $0p[[2]]; Framed[$], "POFF",
  Yield, $ =
   $ /. $sξt /. $sCT //. tuOpDistribute[Dot] //. $sX //. $s0J //. tuOpDistribute[BraKet],
  Yield, $ = $ //. tuOpSimplify[CircleTimes, {d, Conjugate[d]}] //.
    tuOpSimplify[BraKet, {d, Conjugate[d]}], "PON",
  NL, "e's are orthonormal ",
  $s = {BraKet[a_ ⊗ e1_, b_ ⊗ e2_] :→ If[e1 === e2, BraKet[a, b], 0]},
  Yield, $ = $ /. $s,
  NL, "Move d's back ", $s = d_ BraKet[a_, b_] → BraKet[a, d.b],
  Yield, ColumnSumExp[$ = $ /. $s],
  NL, "Symmetry of form ",
  $s = BraKet[J_ . ps_ , d_ .g_ . x_] :> BraKet[J . x , d .g. ps] /; ! FreeQ[x, χ],
  Imply, $p2 = $ = $ /. $s; ColumnSumExp[$] // Framed
 ]
 PR["•", $ = $0p[[3]]; Framed[$], "POFF",
  Yield, $ =
   $ /. $sξt /. $sCT //. tuOpDistribute[Dot] //. $sX //. $s0J //. tuOpDistribute[BraKet],
  Yield, $ = $ //. tuOpSimplify[CircleTimes,
      {Tensor[Y, _, _], Conjugate[Tensor[Y, _, _]]}] //.
    tuOpSimplify[BraKet, {Tensor[Y, _, _], Conjugate[Tensor[Y, _, _]]}], "PON",
  NL, "e's are orthonormal ",
  $s = {BraKet[a_ ⊗ e1_, b_ ⊗ e2_] :→ If[e1 === e2, BraKet[a, b], 0]},
  Yield, $ = $ /. $s,
  NL, "Move Y's back ", $s = d_ BraKet[a_, b_ . c_] → BraKet[a, b.d.c],
  Yield, ColumnSumExp[$ = $ /. $s],
  NL, "Anti-symmetry of form ",
  $s = BraKet[J_ . ps_ , g_ . d_ . x_] :→ -BraKet[J . x , g. d . ps] /; ! FreeQ[x, χ],
  Imply, $ = $ /. $s // tuRepeat[{tuOpDistribute[Dot], tuOpSimplify[Dot],
      tuOpSimplify[CircleTimes], (tt : Tensor[γ, _, _]).(a_ ⊗ b_) → (tt .a ⊗ b),
      $slashd, a_ . 1_ → a, tuOpDistribute[BraKet], tuOpSimplify[BraKet]}, Simplify];
  ColumnSumExp[$],
  NL, "So ", $p3 = $ //. a_R|L → a; $[[2]] = $[[2]] / 2; Framed[$],
  NL, "● ", $ = tuReplacePart[$0, {$p1, $p2, $p3}]; Framed[$],
  NL, CO["A mass term can be identified by letting ", d → -I m,
   ". Recall 𝒟𝒜 ⇒ d so is the related to the fluctuated Dirac algebra. "]
 ]
```

■From $S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_{\mathcal{A}}.\tilde{\xi}\rangle$

$\to S_f \to \frac{1}{2}\langle J.\tilde{\xi} \mid (\gamma_5 \otimes \mathcal{D}_F - i\,\gamma^\mu.(i\, 1_{\dim[\mathcal{H}_F]} \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})).\tilde{\xi}\rangle$

$\to S_f \to \frac{1}{2}(\langle J.\tilde{\xi} \mid (\gamma_5 \otimes \mathcal{D}_F).\tilde{\xi}\rangle + \langle J.\tilde{\xi} \mid \gamma^\mu.(1_{\dim[\mathcal{H}_F]} \otimes B_\mu).\tilde{\xi}\rangle + \langle J.\tilde{\xi} \mid -i\,\gamma^\mu.(\nabla^S{}_\mu \otimes 1_{\mathcal{H}_F}).\tilde{\xi}\rangle)$

●Evaluate terms

• $\boxed{\{2,2,1\} \to \langle J.\tilde{\xi} \mid (\gamma_5 \otimes \mathcal{D}_F).\tilde{\xi}\rangle}$

Define $\tilde{\xi} \to \tilde{\chi}_L \otimes e_L + \tilde{\chi}_R \otimes e_R + \tilde{\psi}_L \otimes \overline{e_R} + \tilde{\psi}_R \otimes \overline{e_L}$

→

e's are orthonormal $\{\langle a\_ \otimes e1\_ \mid b\_ \otimes e2\_\rangle \mapsto \text{If}[e1 === e2, \langle a \mid b\rangle, 0]\}$

$\to \{2,2,1\} \to 0$

Symmetry of form $\langle (J\_).(ps\_) \mid (d\_).(x\_)\rangle \mapsto \langle J.x \mid d.ps\rangle$ /; ! FreeQ[x, χ]

$\Rightarrow \{2,2,1\} \to 0$

Since $(\not{D})[\psi_L] \to \psi'_R$ and $\{\psi_L \in L^2[M,S]^+, \psi_R \in L^2[M,S]^-\}$ orthogonal, i.e.,

$\to \langle \chi_L + \chi_R \mid \psi'_L + \psi'_R\rangle$

$\to \langle \chi_L \mid \psi'_L\rangle + \langle \chi_L \mid \psi'_R\rangle + \langle \chi_R \mid \psi'_L\rangle + \langle \chi_R \mid \psi'_R\rangle$

$\to \langle \chi_L \mid \psi'_L\rangle + \langle \chi_R \mid \psi'_R\rangle$

$\to \langle \chi_L \mid (\not{D})[\psi_R]\rangle + \langle \chi_R \mid (\not{D})[\psi_L]\rangle$

So $\boxed{\{2,2,1\} \to 0}$

• $\boxed{\{2,2,2\} \to \langle J.\tilde{\xi} \mid \gamma^\mu.(1_{\dim[\mathcal{H}_F]} \otimes B_\mu).\tilde{\xi}\rangle}$

e's are orthonormal $\{\langle a\_ \otimes e1\_ \mid b\_ \otimes e2\_\rangle \mapsto \text{If}[e1 === e2, \langle a \mid b\rangle, 0]\}$

$\to \{2,2,2\} \to \langle J_M.\tilde{\chi}_L \otimes \overline{e_L} \mid \gamma^\mu.(\tilde{\chi}_L \otimes (e_L\, Y_\mu))\rangle + \langle J_M.\tilde{\chi}_L \otimes \overline{e_L} \mid \gamma^\mu.(\tilde{\chi}_R \otimes (e_R\, Y_\mu))\rangle +$
$\langle J_M.\tilde{\chi}_L \otimes \overline{e_L} \mid \gamma^\mu.(-(\tilde{\psi}_L \otimes (\overline{e_R}\, Y_\mu)))\rangle + \langle J_M.\tilde{\chi}_L \otimes \overline{e_L} \mid \gamma^\mu.(-(\tilde{\psi}_R \otimes (\overline{e_L}\, Y_\mu)))\rangle +$
$\langle J_M.\tilde{\chi}_R \otimes \overline{e_R} \mid \gamma^\mu.(\tilde{\chi}_L \otimes (e_L\, Y_\mu))\rangle + \langle J_M.\tilde{\chi}_R \otimes \overline{e_R} \mid \gamma^\mu.(\tilde{\chi}_R \otimes (e_R\, Y_\mu))\rangle + \langle J_M.\tilde{\chi}_R \otimes \overline{e_R} \mid \gamma^\mu.(-(\tilde{\psi}_L \otimes (\overline{e_R}\, Y_\mu)))\rangle +$
$\langle J_M.\tilde{\chi}_R \otimes \overline{e_R} \mid \gamma^\mu.(-(\tilde{\psi}_R \otimes (\overline{e_L}\, Y_\mu)))\rangle + \langle J_M.\tilde{\psi}_L \otimes e_R \mid \gamma^\mu.(\tilde{\chi}_L \otimes (e_L\, Y_\mu))\rangle + \langle J_M.\tilde{\psi}_L \otimes e_R \mid \gamma^\mu.(\tilde{\chi}_R \otimes (e_R\, Y_\mu))\rangle +$
$\langle J_M.\tilde{\psi}_L \otimes e_R \mid \gamma^\mu.(-(\tilde{\psi}_L \otimes (\overline{e_R}\, Y_\mu)))\rangle + \langle J_M.\tilde{\psi}_L \otimes e_R \mid \gamma^\mu.(-(\tilde{\psi}_R \otimes (\overline{e_L}\, Y_\mu)))\rangle + \langle J_M.\tilde{\psi}_R \otimes e_L \mid \gamma^\mu.(\tilde{\chi}_L \otimes (e_L\, Y_\mu))\rangle +$
$\langle J_M.\tilde{\psi}_R \otimes e_L \mid \gamma^\mu.(\tilde{\chi}_R \otimes (e_R\, Y_\mu))\rangle + \langle J_M.\tilde{\psi}_R \otimes e_L \mid \gamma^\mu.(-(\tilde{\psi}_L \otimes (\overline{e_R}\, Y_\mu)))\rangle + \langle J_M.\tilde{\psi}_R \otimes e_L \mid \gamma^\mu.(-(\tilde{\psi}_R \otimes (\overline{e_L}\, Y_\mu)))\rangle$

Move d's back $\langle a\_ \mid b\_\rangle\, d\_ \to \langle a \mid d.b\rangle$

$$
\begin{array}{l}
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle \\
\to \{2,2,2\}\to\sum[\ \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle\ ] \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\to \{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle
\end{array}
$$

Symmetry of form ⟨(J_).(ps_) | (d_).(g_).(x_)⟩ :> ⟨J.x | d.g.ps⟩ /; ! FreeQ[x, χ]

$$
\Rightarrow\ \boxed{
\begin{array}{l}
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle \\
\{2,2,2\}\to\sum[\ \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle\ ] \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(\tilde{\chi}_L\otimes(e_L\,Y_\mu))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(\tilde{\chi}_R\otimes(e_R\,Y_\mu))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(-(\tilde{\psi}_L\otimes(\overline{e_R}\,Y_\mu)))\rangle \\
\phantom{\{2,2,2\}\to\sum[\ } \langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.(-(\tilde{\psi}_R\otimes(\overline{e_L}\,Y_\mu)))\rangle
\end{array}}
$$

- $\boxed{\{2,2,3\}\to\langle J.\tilde{\xi} \mid -i\,\gamma^\mu.({\nabla^S}_\mu\otimes 1_{\mathcal{H}_F}).\tilde{\xi}\rangle}$

e's are orthonormal {⟨a_⊗e1_ | b_⊗e2_⟩ :> If[e1 === e2, ⟨a | b⟩, 0]}

$$
\begin{aligned}
\to \{2,2,3\}\to -i\,&\langle J_M.\tilde{\chi}_L\otimes\overline{e_L} \mid \gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_L\otimes e_L)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_R\otimes e_R)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_L\otimes\overline{e_R})+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_R\otimes\overline{e_L})\rangle - \\
i\,&\langle J_M.\tilde{\chi}_R\otimes\overline{e_R} \mid \gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_L\otimes e_L)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_R\otimes e_R)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_L\otimes\overline{e_R})+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_R\otimes\overline{e_L})\rangle - \\
i\,&\langle J_M.\tilde{\psi}_L\otimes e_R \mid \gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_L\otimes e_L)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_R\otimes e_R)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_L\otimes\overline{e_R})+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_R\otimes\overline{e_L})\rangle - \\
i\,&\langle J_M.\tilde{\psi}_R\otimes e_L \mid \gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_L\otimes e_L)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\chi}_R\otimes e_R)+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_L\otimes\overline{e_R})+\gamma^\mu.({\nabla^S}_\mu.\tilde{\psi}_R\otimes\overline{e_L})\rangle
\end{aligned}
$$

Move Y's back ⟨a_ | (b_).(c_)⟩ d_ → ⟨a | b.d.c⟩

$$\rightarrow \{2,2,3\}\rightarrow\textstyle\sum\Big[\begin{array}{l}
-\mathrm{i}\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle\\
-\mathrm{i}\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle\\
-\mathrm{i}\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle\\
-\mathrm{i}\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})+\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle
\end{array}\Big]$$

Anti-symmetry of form $\langle (J\_).(ps\_) \mid (g\_).(d\_).(x\_)\rangle :\rightarrow -\langle J.x \mid g.d.ps\rangle$ /; ! FreeQ[x, χ]

$$\Rightarrow \{2,2,3\}\rightarrow -\mathrm{i}\textstyle\sum\Big[\begin{array}{l}
\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle\\
\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle\\
\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle\\
\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle\\
\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle\\
\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle\\
\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle\\
\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle\\
\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle\\
\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle\\
\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle\\
\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle\\
\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle\\
\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle\\
\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle\\
\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle
\end{array}\Big]$$

So

$$\boxed{\begin{array}{l}
\{2,2,3\}\rightarrow\\
-\frac{1}{2}\mathrm{i}\,\Big(\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle+\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle+\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle+\\
\quad\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle+\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle+\\
\quad\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle+\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle+\\
\quad\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle+\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle+\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle+\\
\quad\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle+\left\langle J_M.\tilde\psi_L\otimes e_R\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle+\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_L\otimes e_L)\right\rangle+\\
\quad\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\chi_R\otimes e_R)\right\rangle+\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_L\otimes\overline{e_R})\right\rangle+\left\langle J_M.\tilde\psi_R\otimes e_L\mid \gamma^\mu.(\nabla^S{}_\mu.\tilde\psi_R\otimes\overline{e_L})\right\rangle\Big)
\end{array}}$$

•

$$\boxed{\begin{array}{l}
S_f\rightarrow\frac{1}{2}\Big(-\mathrm{i}\,\Big(4\left\langle J_M.\tilde\chi\otimes\overline{e}\mid\gamma^\mu.(\nabla^S{}_\mu.\tilde\chi\otimes e)\right\rangle+4\left\langle J_M.\tilde\chi\otimes\overline{e}\mid\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi\otimes\overline{e})\right\rangle+\\
\qquad 4\left\langle J_M.\tilde\psi\otimes e\mid\gamma^\mu.(\nabla^S{}_\mu.\tilde\chi\otimes e)\right\rangle+4\left\langle J_M.\tilde\psi\otimes e\mid\gamma^\mu.(\nabla^S{}_\mu.\tilde\psi\otimes\overline{e})\right\rangle\Big)+\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid\gamma^\mu.(\tilde\chi_L\otimes(e_L\,Y_\mu))\right\rangle+\\
\quad\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid\gamma^\mu.(\tilde\chi_R\otimes(e_R\,Y_\mu))\right\rangle+\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid\gamma^\mu.(-(\tilde\psi_L\otimes(\overline{e_R}\,Y_\mu)))\right\rangle+\left\langle J_M.\tilde\chi_L\otimes\overline{e_L}\mid\gamma^\mu.(-(\tilde\psi_R\otimes(\overline{e_L}\,Y_\mu)))\right\rangle+\\
\quad\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid\gamma^\mu.(\tilde\chi_L\otimes(e_L\,Y_\mu))\right\rangle+\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid\gamma^\mu.(\tilde\chi_R\otimes(e_R\,Y_\mu))\right\rangle+\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid\gamma^\mu.(-(\tilde\psi_L\otimes(\overline{e_R}\,Y_\mu)))\right\rangle+\\
\quad\left\langle J_M.\tilde\chi_R\otimes\overline{e_R}\mid\gamma^\mu.(-(\tilde\psi_R\otimes(\overline{e_L}\,Y_\mu)))\right\rangle+\left\langle J_M.\tilde\psi_L\otimes e_R\mid\gamma^\mu.(\tilde\chi_L\otimes(e_L\,Y_\mu))\right\rangle+\left\langle J_M.\tilde\psi_L\otimes e_R\mid\gamma^\mu.(\tilde\chi_R\otimes(e_R\,Y_\mu))\right\rangle+\\
\quad\left\langle J_M.\tilde\psi_L\otimes e_R\mid\gamma^\mu.(-(\tilde\psi_L\otimes(\overline{e_R}\,Y_\mu)))\right\rangle+\left\langle J_M.\tilde\psi_L\otimes e_R\mid\gamma^\mu.(-(\tilde\psi_R\otimes(\overline{e_L}\,Y_\mu)))\right\rangle+\left\langle J_M.\tilde\psi_R\otimes e_L\mid\gamma^\mu.(\tilde\chi_L\otimes(e_L\,Y_\mu))\right\rangle+\\
\quad\left\langle J_M.\tilde\psi_R\otimes e_L\mid\gamma^\mu.(\tilde\chi_R\otimes(e_R\,Y_\mu))\right\rangle+\left\langle J_M.\tilde\psi_R\otimes e_L\mid\gamma^\mu.(-(\tilde\psi_L\otimes(\overline{e_R}\,Y_\mu)))\right\rangle+\left\langle J_M.\tilde\psi_R\otimes e_L\mid\gamma^\mu.(-(\tilde\psi_R\otimes(\overline{e_L}\,Y_\mu)))\right\rangle\Big)
\end{array}}$$

A mass term can be identified by letting d → -ⅈ m
. Recall $\mathcal{D}_{\mathcal{A}}$ ⇒ d so is the related to the fluctuated Dirac algebra.

```
PR["●Theorem 4.9. The full Lagrangian is ",
 ℒgrav[T[g, "dd", {μ, ν}]] → 4 ℒM[T[g, "dd", {μ, ν}]] + ℒH[T[g, "dd", {μ, ν}]],
 NL, "E-M Lagrangian ",
 ℒEM[T[g, "dd", {μ, ν}]] →
  -I BraKet[JM.χ̃, (T[γ, "u", {μ}].(("∇"^S)_μ - I T[Y, "d", {μ}] ) - m).ψ̃]_ℒ +
    f[0]/(6 π^2) T[F, "dd", {μ, ν}] T[F, "uu", {μ, ν}],
 NL, "where ", BraKet[ξ, ψ] → IntegralOp[{{x^4, x ∈ M}}, Sqrt[Abs[det[g]]] BraKet[ξ, ψ]_ℒ]
]
```

●Theorem 4.9. The full Lagrangian is $\mathcal{L}_{grav}[g_{\mu\nu}] \to \mathcal{L}_H[g_{\mu\nu}] + 4\,\mathcal{L}_M[g_{\mu\nu}]$

E-M Lagrangian $\mathcal{L}_{EM}[g_{\mu\nu}] \to -i \left\langle J_M.\tilde{\chi} \mid (-m + \gamma^\mu.(\nabla^S{}_\mu - i\, Y_\mu)).\tilde{\psi} \right\rangle_{\mathcal{L}} + \frac{f[0]\, F_{\mu\nu} F^{\mu\nu}}{6\,\pi^2}$

where $\langle \xi \mid \psi \rangle \to \int_{\{x^4, x\in M\}} [\sqrt{\mathrm{Abs}[\det[g]]}\ \langle \xi \mid \psi \rangle_{\mathcal{L}}]$

```
{𝕌[ξ, ζ] → BraKet[J.ξ, 𝒟𝒜.ζ], {ξ, ζ} ∈ ℋ^+}
{𝔹[χ, ψ] → -I BraKet[JM.χ, (T[γ, "u", {μ}].(("∇"^S)_μ - I T[Y, "d", {μ}] ) - m).ψ],
 {χ, ψ} ∈ L^2[M, S]}

{$sξ, χ → χL + χR, ψ → ψL + ψR}
$sDA1
𝕌[ξ, ζ] → 2 𝔹[χ, ψ]
Pf[𝕌] → (IntegralOp[{{𝔻[ξ̃]}}, Exp[1 / 2 𝕌[ξ̃, ξ̃]]] →
   (IntegralOp[{{𝔻[ξ̃]}, {𝔻[ψ̃]}}, Exp[𝔹[ξ̃, ψ̃]]] →
     Det[𝔹]))
𝔻[η_, θ_] :→ (Table[d[T[η, "d", {i}]].d[T[θ, "d", {i}]], {i, dim[]}])
𝔻[ξ, ψ] /. %
```

$\{\mathbb{U}[\xi, \zeta] \to \langle J.\xi \mid \mathcal{D}_{\mathcal{A}}.\zeta \rangle, \{\xi, \zeta\} \in \mathcal{H}^+\}$

$\{\mathbb{B}[\chi, \psi] \to -i \langle J_M.\chi \mid (-m + \gamma^\mu.(\nabla^S{}_\mu - i\, Y_\mu)).\psi \rangle, \{\chi, \psi\} \in L^2[M, S]\}$

$\{\{\xi \to \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_L \otimes \overline{e_R} + \psi_R \otimes \overline{e_L}, \{\chi_L, \psi_L\} \in L^2[M, S]^+, \{\chi_R, \psi_R\} \in L^2[M, S]^-\},$
$\chi \to \chi_L + \chi_R, \psi \to \psi_L + \psi_R\}$

$\mathcal{D}_{\mathcal{A}} \to \gamma_5 \otimes \mathcal{D}_F - i\,\gamma^\mu.(i\, 1_{\dim[\mathcal{H}_F]} \otimes B_\mu + \nabla^S{}_\mu \otimes 1_{\mathcal{H}_F})$

$\mathbb{U}[\xi, \zeta] \to 2\,\mathbb{B}[\chi, \psi]$

$\mathrm{Pf}[\mathbb{U}] \to \int_{\{\mathbb{D}[\tilde{\xi}]\}} [e^{\frac{1}{2}\mathbb{U}[\tilde{\xi},\tilde{\xi}]}] \to \int_{\{\mathbb{D}[\tilde{\xi}]\} \{\mathbb{D}[\tilde{\psi}]\}} [e^{\mathbb{B}[\tilde{\xi},\tilde{\psi}]}] \to \mathrm{Det}[\mathbb{B}]$

$\mathbb{D}[\eta\_, \theta\_] :\to \mathrm{Table}[d[T[\eta, d, \{i\}]].d[T[\theta, d, \{i\}]], \{i, \dim[]\}]$

$\mathrm{Table}[d[T[\xi, d, \{i\}]].d[T[\psi, d, \{i\}]], \{i, \dim[]\}]$

5. Glashow-Weinberg-Salam

■ Construction of finite space F_{GWS}

```
PR["Basis of space: ",
  $basis = {($ = {eR, eL, ēR, ēL}), ($ /. e → ν)} // Flatten,
  NL, "Lepton basis ", $lep = {$ = (Select[$basis, Head[#] =!= OverBar &] // Sort[#] & //
        Permute[#, Cycles[{{1, 4}}]] &) ∈ (ℋ1 → ℂ^4)},
  NL, "AntiLepton basis ", $antilep = {$ = (Select[$basis, Head[#] == OverBar &] //
         Sort[#] & // Permute[#, Cycles[{{1, 4}}]] &) ∈ (ℋ1̄ → ℂ^4)},
  NL, "Decompose ", ℋ -> ℋ1 ⊕ ℋ1̄,
  NL, "Expand E-M algebra ℂ⊕ℂ to accomodate weak interactions ",
  {𝒜F → ℂ ⊕ ℍ, ℍ → quarterions},

  NL, "where ", $q = $ = {q ∈ ℍ, q → α + β j, {α, β} ∈ ℂ,
     q → {{α, β}, {-Conjugate[β], Conjugate[α]}}, qλ → {{λ, 0}, {0, Conjugate[λ]}}};
  MatrixForms[$],
  NL, "Algebra: ", $ = a → ({λ, q} ∈ 𝒜F),
  yield, $ = {{qλ, 0}, {0, q}},
  yield, $ = $ /. $q[[-2 ;; -1]] // ArrayFlatten; MatrixForms[$],

  NL, "For ", ℋ → ℋ1R ⊕ ℋ1L ⊕ ℋ1̄R ⊕ ℋ1̄L,
  Yield, $alg = {a.1 → $.1, 1 ∈ ℋ1, a.1̄ → λ 1̄, 1̄ ∈ ℋ1̄}; MatrixForms[$alg], CK,

  NL, "Derived rules ", $sr = {JF.1_ -> 1̄, JF.1̄_ → 1,
     γF → DiagonalMatrix[{-1, 1, 1, -1}],
     JF → SparseArray[{Band[{1, 3}] → C, Band[{3, 1}] → C}, {4, 4}] } // Normal;
  MatrixForms[$sr] // Column // Framed, CK
 ];
```

Basis of space: $\{e_R, e_L, \overline{e_R}, \overline{e_L}, \nu_R, \nu_L, \overline{\nu_R}, \overline{\nu_L}\}$

Lepton basis $\{\{\nu_R, e_R, \nu_L, e_L\} \in (\mathcal{H}_1 \to \mathbb{C}^4)\}$

AntiLepton basis $\{\{\overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\} \in (\mathcal{H}_{\overline{1}} \to \mathbb{C}^4)\}$

Decompose $\mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_{\overline{1}}$

Expand E-M algebra $\mathbb{C}\oplus\mathbb{C}$ to accomodate weak interactions $\{\mathcal{A}_F \to \mathbb{C}\oplus\mathbb{H},\ \mathbb{H} \to \text{quarterions}\}$

where $\{q \in \mathbb{H},\ q \to \alpha + j\,\beta,\ \{\alpha, \beta\} \in \mathbb{C},\ q \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},\ q_\lambda \to \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}\}$

Algebra: $a \to \{\lambda, q\} \in \mathcal{A}_F \longrightarrow \{\{q_\lambda, 0\}, \{0, q\}\} \longrightarrow \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$

For $\mathcal{H} \to \mathcal{H}_{1_R} \oplus \mathcal{H}_{1_L} \oplus \mathcal{H}_{\overline{1}_R} \oplus \mathcal{H}_{\overline{1}_L}$

$\to \{a.l \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}.l,\ l \in \mathcal{H}_1,\ a.\overline{l} \to \lambda\,\overline{l},\ \overline{l} \in \mathcal{H}_{\overline{1}}\} \longleftarrow$ CHECK

Derived rules
$$\boxed{\begin{array}{l} J_F.(l\_) \to \overline{l} \\ J_F.\overline{l\_} \to l \\ \gamma_F \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ J_F \to \begin{pmatrix} 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \\ C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \end{pmatrix} \end{array}}$$
$\longleftarrow$ CHECK

5.1.1 Finite Dirac Operator

```
PR["•Decompose Hermitian ", $D = $ = {𝒟F → {{S , ct[T]}, {T, S'}}, {S, S'} → Hermitian };
  MatrixForms[$],
  NL, "•Constrain 𝒟F require ", $ = CommutatorM[𝒟F, JF] → 0,
  NL, "Use ", $s = JF → {{0, C}, {C, 0}},
  Yield, $ =
   $ /. CommutatorM → MCommutator /. Dot → xDot /. $D /. $s // OrderedxDotMultiplyAll[];
  yield, $ = $ /. {a_ .C ⧴ C. Conjugate[a]} // ConjugateCTSimplify1[{}] //
    collectDotLeft;
  MatrixForms[$],
  Imply, $ = $ /. C.a_ → a; $c1 = $ = Thread[Flatten[$[[1]]] → 0]; Framed[$],
  Imply, $D[[1]] = $D[[1]] /. tuRuleSolve[$c1, {ct[T], S'}];
  MatrixForms[$D[[1]]],
  NL, "In 4x4 form, Let ",
  $s = {S -> Table[si,j, {i, 2}, {j, 2}], T -> Table[ti,j, {i, 2}, {j, 2}]}, "POFF",
  Yield, $0 = $ = $D[[1, 2]] /. $s // ArrayFlatten; MatrixForm[$],
  Yield, $ht = ct[$]; MatrixForm[$ht],
  Yield, $ = $ → $ht //. rr : Rule[__] ⧴ Thread[rr]; MatrixForms[$], "PON",
  Yield, $s1 = tuRuleSolve[Flatten[$], {s2,1, t2,1}],
  Yield, $D44 = $ = $D[[1, 1]] -> $0 /. $s1 /. Conjugate[si_,i_] → si,i;
  MatrixForms[$],
  line,
  NL, "•Also require: ", $ = CommutatorP[𝒟F, γF] → 0, "POFF",
  Yield, $ = $ /. $sr /. $D44; MatrixForms[$],
  Yield, $ = $ /. CommutatorP → ACommutator; MatrixForms[$], "PON",
  yield, $ = $ //. rr : Rule[__] ⧴ Thread[rr] // Flatten // DeleteDuplicates //
    tuRuleSolve[#, {s1,1, s2,2, t1,2}] &,
  Yield, $ = $D44 /. $; MatrixForms[$], "PON",
  NL, "Switching notation ", $s = {s1,2 → Conjugate[Y0], t1,1 → TR, t2,2 → TL},
  Imply, $D44 = $ = $ /. $s; MatrixForms[$] // Framed,
  NL, "In the ", {$lep[[1, 1]], $antilep[[1, 1]]} // Flatten, " basis ",
  {Y0, TR, TL}, " are symmetric 2x2 matrices."
 ];
```

•Decompose Hermitian $\{\mathcal{D}_F \to \begin{pmatrix} S & T^\dagger \\ T & S' \end{pmatrix}, \{S, S'\} \to \text{Hermitian}\}$

•Constrain $\mathcal{D}_F$ require $[\mathcal{D}_F, J_F] \to 0$
Use $J_F \to \{\{0, C\}, \{C, 0\}\}$

$\to \;\longrightarrow \begin{pmatrix} C.(-T + T^T) & C.(S^* - S') \\ C.(-S + (S')^*) & C.(T^* - T^\dagger) \end{pmatrix} \to 0$

$\Rightarrow \boxed{\{-T + T^T \to 0,\ S^* - S' \to 0,\ -S + (S')^* \to 0,\ T^* - T^\dagger \to 0\}}$

$\Rightarrow \mathcal{D}_F \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix}$

In 4x4 form, Let $\{S \to \{\{s_{1,1}, s_{1,2}\}, \{s_{2,1}, s_{2,2}\}\}, T \to \{\{t_{1,1}, t_{1,2}\}, \{t_{2,1}, t_{2,2}\}\}\}$

$\to \{s_{2,1} \to (s_{1,2})^*, t_{2,1} \to t_{1,2}\}$

$\to \mathcal{D}_F \to \begin{pmatrix} s_{1,1} & s_{1,2} & (t_{1,1})^* & (t_{1,2})^* \\ (s_{1,2})^* & s_{2,2} & (t_{1,2})^* & (t_{2,2})^* \\ t_{1,1} & t_{1,2} & s_{1,1} & (s_{1,2})^* \\ t_{1,2} & t_{2,2} & s_{1,2} & s_{2,2} \end{pmatrix}$

•Also require: $\{\mathcal{D}_F, \gamma_F\} \to 0 \;\longrightarrow\; \{s_{1,1} \to 0, s_{2,2} \to 0, t_{1,2} \to 0\}$

$\to \mathcal{D}_F \to \begin{pmatrix} 0 & s_{1,2} & (t_{1,1})^* & 0 \\ (s_{1,2})^* & 0 & 0 & (t_{2,2})^* \\ t_{1,1} & 0 & 0 & (s_{1,2})^* \\ 0 & t_{2,2} & s_{1,2} & 0 \end{pmatrix}$

Switching notation $\{s_{1,2} \to (Y_0)^*, t_{1,1} \to T_R, t_{2,2} \to T_L\}$

$\Rightarrow \boxed{\mathcal{D}_F \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix}}$

In the $\{\nu_R, e_R, \nu_L, e_L, \overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\}$ basis $\{Y_0, T_R, T_L\}$ are symmetric 2x2 matrices.

```
$basis8 = {$lep[[1, 1]], $antilep[[1, 1]]} // Flatten
$basis

PR["■How does the restriction: ",
  {T.$basis[[5]] → YR.$basis[[7]], T.l ⧴ 0 /; FreeQ[l, $basis[[7]]]},
  " constrain T? ",
  NL, $t = T → DiagonalMatrix[{TR, TL}],
  Yield, $t = $t /. tt : TR_ -> Table[t[R]i,j, {i, 2}, {j, 2}];
  $t[[2]] = $t[[2]] // ArrayFlatten;
  Yield, MatrixForms[$t],
  NL, "•Hermiticity of 𝒟F", imply, $st = {t[L]2,1 -> t[L]1,2, t[R]2,1 -> t[R]1,2},
  Yield, $t44 = $t = $t /. $st; MatrixForms[$t], CK,
  NL, "In the 8x8 context: ",
  Yield, $ = {{0, Conjugate[T]}, {T, 0}};
  Yield, $t = T → ($ /. $t // ArrayFlatten);
  Yield, $ = T . Transpose[{$basis8}]; $ = $ → $;
  Yield, $[[2]] = $[[2]] /. $t;
  MatrixForms[$],
  NL, "The only non-zero element of T: ", t[R]1,1 // Framed,
  NL, "also ", y2,1 → y1,2,
  NL, "Require ℋF be mass eigenstates ", $Y = Y0 → DiagonalMatrix[{Yν, Ye}],
  line,
  NL, "Rules for making 8x8 GWS ", $D44[[1]],
  Yield, $sDAgws = {$D44, tt : TR_ → Table[t[R]i,j, {i, 2}, {j, 2}],
    t[RL_]i_,j_ ⧴ 0 /; (i ≠ 1 || j ≠ 1 || RL =!= R), $Y}
 ];
```

$\{\nu_R, e_R, \nu_L, e_L, \overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\}$

$\{e_R, e_L, \overline{e_R}, \overline{e_L}, \nu_R, \nu_L, \overline{\nu_R}, \overline{\nu_L}\}$

■How does the restriction: $\{T.\nu_R \to Y_R.\overline{\nu_R},\ T.l :\to 0 \,/;\, \text{FreeQ}[l, \$basis[[7]]]\}$ constrain T?

$T \to \{\{T_R, 0\}, \{0, T_L\}\}$

→

$$\to\ T \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} & 0 & 0 \\ t[R]_{2,1} & t[R]_{2,2} & 0 & 0 \\ 0 & 0 & t[L]_{1,1} & t[L]_{1,2} \\ 0 & 0 & t[L]_{2,1} & t[L]_{2,2} \end{pmatrix}$$

•Hermiticity of $\mathcal{D}_F \Rightarrow \{t[L]_{2,1} \to t[L]_{1,2},\ t[R]_{2,1} \to t[R]_{1,2}\}$

$$\to\ T \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} & 0 & 0 \\ t[R]_{1,2} & t[R]_{2,2} & 0 & 0 \\ 0 & 0 & t[L]_{1,1} & t[L]_{1,2} \\ 0 & 0 & t[L]_{1,2} & t[L]_{2,2} \end{pmatrix} \longleftarrow \text{CHECK}$$

In the 8x8 context:

→

→

→

$$\to\ T.\begin{pmatrix} \nu_R \\ e_R \\ \nu_L \\ e_L \\ \overline{\nu_R} \\ \overline{e_R} \\ \overline{\nu_L} \\ \overline{e_L} \end{pmatrix} \to \begin{pmatrix} (t[R]_{1,2})^* \overline{e_R} + (t[R]_{1,1})^* \overline{\nu_R} \\ (t[R]_{2,2})^* \overline{e_R} + (t[R]_{1,2})^* \overline{\nu_R} \\ (t[L]_{1,2})^* \overline{e_L} + (t[L]_{1,1})^* \overline{\nu_L} \\ (t[L]_{2,2})^* \overline{e_L} + (t[L]_{1,2})^* \overline{\nu_L} \\ \nu_R\, t[R]_{1,1} + e_R\, t[R]_{1,2} \\ \nu_R\, t[R]_{1,2} + e_R\, t[R]_{2,2} \\ \nu_L\, t[L]_{1,1} + e_L\, t[L]_{1,2} \\ \nu_L\, t[L]_{1,2} + e_L\, t[L]_{2,2} \end{pmatrix}$$

The only non-zero element of T: $\boxed{t[R]_{1,1}}$

also $y_{2,1} \to y_{1,2}$

Require $\mathcal{H}_F$ be mass eigenstates $Y_0 \to \{\{Y_\nu, 0\}, \{0, Y_e\}\}$

Rules for making 8x8 GWS $\mathcal{D}_F$

$\to\ \{\mathcal{D}_F \to \{\{0, (Y_0)^*, (T_R)^*, 0\}, \{Y_0, 0, 0, (T_L)^*\}, \{T_R, 0, 0, Y_0\}, \{0, T_L, (Y_0)^*, 0\}\},$
$\quad tt : T_{R\_} \to \{\{t[R]_{1,1}, t[R]_{1,2}\}, \{t[R]_{2,1}, t[R]_{2,2}\}\},$
$\quad t[RL\_]_{i\_,j\_} :\to 0 \,/;\, i \neq 1 \,||\, j \neq 1 \,||\, RL =!= R,\ Y_0 \to \{\{Y_\nu, 0\}, \{0, Y_e\}\}\}$

```
PR["Prop.5.1. ", FGWS → Map[# /. a_ → aF &, {𝒜, ℋ, 𝒟, γ, J}],
  " define a real even KOdim→6 space.",
  NL, "Show that ", $ = $def[[2]],
  yield, $ = $[[3, 1]],
  NL, "•Within each subspace: ", $ = $D[[1]] /. T → 0,
  " where ", $h = ℋ -> ℋ1 ⊕ ℋI,
  Yield, {CommutatorM[Conjugate[S], a] → 0, a ∈ $h[[2, 2]], a.l → λl},
  NL, "For ", $ = {a ∈ $h[[2, 1]], $alg[[1 ;; 2]]},
  NL, "Expand to 8x8 representation ",
  $aa = $alg[[1]] /. a_ . _ → a;
  $aa = {{$aa[[2]], 0}, {0, DiagonalMatrix[{λ, λ, λ, λ}]}} // Normal // ArrayFlatten;
  MatrixForms[$a = a -> $aa],
  " ",
  $jj = J -> ($sr[[-1, 2]] /. C → DiagonalMatrix[{C, C}] // ArrayFlatten);
  MatrixForms[$jj],
  NL, "Compute ",
  $ = a"0" -> Dot[J , ct[a], ct[J]],
  Yield, $[[2]] = $[[2]] /. Dot → xDot /. $jj /. $a // OrderedxDotMultiplyAll[] //
    tuRepeat[{Conjugate[C] → C, Conjugate[C].C → 1, C . a_ → Conjugate[a] . C},
     ConjugateCTSimplify1[{}]];
  MatrixForms[$],
  NL, "i.e., the action of ", $[[1]], " on ", $h[[2, 1]],
  " is equal to multiplication by a diagonal matrix; hence, condition satisfied."
 ];
PR["The action of a on basis: ",
  $ = a.l -> a . l;
  $[[2]] = $[[2]] /. $a;
  $ = $ /. l :> Transpose[{$basis8}] // MatrixForms
 ];
```

Prop.5.1. $F_{GWS} \to \{\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F, J_F\}$ define a real even KOdim→6 space.

Show that $\forall_{\{a,b\},a|b\in\mathcal{A}_F}\{[[\mathcal{D}_F, a], b^0] \to 0, b^0 \to J_F.b^\dagger.(J_F)^\dagger\} \longrightarrow [[\mathcal{D}_F, a], b^0] \to 0$

•Within each subspace: $\mathcal{D}_F \to \{\{S, 0\}, \{0, S^*\}\}$ where $\mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_I$

$\to \{[S^*, a] \to 0, a \in \mathcal{H}_I, a.l \to \lambda l\}$

For $\{a \in \mathcal{H}_1, \{a.l \to \{\{\lambda, 0, 0, 0\}, \{0, \lambda^*, 0, 0\}, \{0, 0, \alpha, \beta\}, \{0, 0, -\beta^*, \alpha^*\}\}.l, l \in \mathcal{H}_1\}\}$

Expand to 8x8 representation $a \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^* & \alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$ $J \to \begin{pmatrix} 0 & 0 & 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C \\ C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 & 0 & 0 \end{pmatrix}$

Compute $a^0 \to J.a^\dagger.J^\dagger$

$\to a^0 \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & -\beta^* \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & \alpha^* \end{pmatrix}$

i.e., the action of a^0 on $\mathcal{H}_1$ is equal to multiplication by a diagonal matrix; hence, condition satisfied.

The action of a on basis: $a.\begin{pmatrix} \nu_R \\ e_R \\ \nu_L \\ e_L \\ \overline{\nu_R} \\ \overline{e_R} \\ \overline{\nu_L} \\ \overline{e_L} \end{pmatrix} \to \begin{pmatrix} \lambda\, \nu_R \\ \lambda^*\, e_R \\ \beta\, e_L + \alpha\, \nu_L \\ \alpha^*\, e_L - \beta^*\, \nu_L \\ \lambda\, \overline{\nu_R} \\ \lambda\, \overline{e_R} \\ \lambda\, \overline{\nu_L} \\ \lambda\, \overline{e_L} \end{pmatrix}$

5.2 Gauge Theory

```
PR["Local gauge group from ", F_GWS,
  NL, "Examine subalgebra ", $AFJ = {Ã_{F J_F}, A_F → ℂ⊕ℍ, a.J_F → J_F.ct[a],
    a ∈ Ã_{F J_F}, a → {λ, q}, {λ, Conjugate[λ], α, Conjugate[α]} → λ, β → 0},
  NL, "Recall algebra ", $alg[[1]] /. a_ . _ → a // MatrixForms,
  imply, $e54 = $AFJ[[1]] → λ 1_{H_F} ≃ ℝ, CG[" (5.4)"]
 ];
```

Local gauge group from F_{GWS}

Examine subalgebra $\{\tilde{\mathcal{A}}_{F J_F}, \mathcal{A}_F \to \mathbb{C}\oplus\mathbb{H}, a.J_F \to J_F.a^\dagger, a \in \tilde{\mathcal{A}}_{F J_F}, a \to \{\lambda, q\}, \{\lambda, \lambda^*, \alpha, \alpha^*\} \to \lambda, \beta \to 0\}$

Recall algebra $a \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix} \Rightarrow \tilde{\mathcal{A}}_{F J_F} \to \lambda\, 1_{\mathcal{H}_F} \simeq \mathbb{R}$ (5.4)

```
PR["Lie algebra ", h_F → u[$AFJ[[1]]],
 Yield,
 {u ∈ u[𝒜_F], u → {λ, q}, λ ∈ I ℝ, I q → xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 0, 3}]},
 Imply, Conjugate[λ] → -λ,
 imply, h_F → u[$AFJ[[1]]],
 imply, {λ, Conjugate[λ], α, Conjugate[α]} → 0,
 imply, h_F → {0},
 NL, "●Prop.5.2. local gauge group of F_GWS is ",
 $G = 𝒢[F_GWS] ≃ Mod[U[1] × SU[2], {1, -1}_SU2],
 NL, "■ ", U[𝒜_F] ≃ U[1] × U[ℍ],
 NL, "where ", {q ∈ ℍ, q -> xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 0, 3}]},
 NL, "Unitarity of q ",
 imply, Abs[q]^2 → Det[q] → 1,
 imply, U[ℍ] ≃ SU[2],

 NL, "Since ", $e54,
 imply, ℋ_F → U[$AFJ[[1]]] → {1, -1},
 imply, $G,
 NL, "The gauge field ", T[𝒜, "d", {μ}],
 CR[" takes values "], " in the Lie subalgebra ",
 g_F → Mod[u[𝒜_F], h_F] -> u[𝒜_F] → u[1] ⊕ su[2],
 NL, "■For Gauge field ", {T[𝒜, "d", {μ}], ϕ},
 NL, "Let ", {a → {λ, q}, b → {λ', q'}, {a, b} ∈ (𝒜 → C^"∞"[M, ℂ ⊕ ℍ])},
 NL, "▪The inner fluctuation ",
 $A = T[𝒜, "d", {μ}] → -I a.tuDPartial[b, μ],
 NL, "Let ", $s = $alg[[1]] /. a_ . _ → a; $sb = $s /. {a → b, λ → λ', β → β', α → α'};
 $sab = {$s, $sb} // Flatten,
 Imply, $A = $A /. $sab //. tt : tuDDown["∂"][__] :→ Thread[tt] /. tuDDown["∂"][0, _] → 0;
 MatrixForms[$A],
 NL, "Hermiticity",

 imply, ($A[[2, 1, 1]] → -$A[[2, 2, 2]]) ∈ ℝ,
 NL, "Represent ",
 $A3 = {T[𝒜, "d", {μ}] → DiagonalMatrix[{T[Λ, "d", {μ}], -T[Λ, "d", {μ}], T[Q, "d", {μ}]}],
   T[Q, "d", {μ}] → I xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 0, 3}], T[q, "d", {i}] ∈ ℝ}
]
```

```
Lie algebra hF → u[𝒜̃F JF]
→ {u ∈ u[𝒜F], u → {λ, q}, λ ∈ i ℝ, i q → Σ_{i,0,3} [qi σ^i]}
⇒ λ* → -λ ⇒ hF → u[𝒜̃F JF] ⇒ {λ, λ*, α, α*} → 0 ⇒ hF → {0}
●Prop.5.2. local gauge group of FGWS is 𝒢[FGWS] ≃ Mod[U[1] × SU[2], {1, -1}SU2]
■ U[𝒜F] ≃ U[1] × U[ℍ]
where {q ∈ ℍ, q → Σ_{i,0,3} [qi σ^i]}
Unitarity of q ⇒ Abs[q]^2 → Det[q] → 1 ⇒ U[ℍ] ≃ SU[2]
Since 𝒜̃F JF → λ 1ℋF ≃ ℝ ⇒ ℋF → U[𝒜̃F JF] → {1, -1} ⇒ 𝒢[FGWS] ≃ Mod[U[1] × SU[2], {1, -1}SU2]
The gauge field 𝒜μ takes values
  in the Lie subalgebra 𝔤F → Mod[u[𝒜F], hF] → u[𝒜F] → u[1] ⊕ su[2]
■For Gauge field {𝒜μ, ϕ}
Let {a → {λ, q}, b → {λ', q'}, {a, b} ∈ (𝒜 → C^∞[M, ℂ⊕ℍ])}
▪The inner fluctuation 𝒜μ → -i a.∂μ[b]
Let {a → {{λ, 0, 0, 0}, {0, λ*, 0, 0}, {0, 0, α, β}, {0, 0, -β*, α*}},
  b → {{λ', 0, 0, 0}, {0, (λ')*, 0, 0}, {0, 0, α', β'}, {0, 0, -(β')*, (α')*}}}
```

$$\Rightarrow \mathcal{A}_\mu \to \begin{pmatrix} -i\,\lambda\,\partial_\mu[\lambda'] & 0 & 0 & 0 \\ 0 & -i\,\lambda^*\,\partial_\mu[(\lambda')^*] & 0 & 0 \\ 0 & 0 & -i\,(\beta\,\partial_\mu[-(\beta')^*] + \alpha\,\partial_\mu[\alpha']) & -i\,(\beta\,\partial_\mu[(\alpha')^*] + \alpha\,\partial_\mu[\beta']) \\ 0 & 0 & -i\,(\alpha^*\,\partial_\mu[-(\beta')^*] - \beta^*\,\partial_\mu[\alpha']) & -i\,(\alpha^*\,\partial_\mu[(\alpha')^*] - \beta^*\,\partial_\mu[\beta']) \end{pmatrix}$$

```
Hermiticity ⇒ (-i λ ∂μ[λ'] → i λ* ∂μ[(λ')*]) ∈ ℝ
Represent {𝒜μ → {{Λμ, 0, 0}, {0, -Λμ, 0}, {0, 0, Qμ}}, Qμ → i Σ_{i,0,3} [qi σ^i], qi ∈ ℝ}

PR["▪From the definition ", ϕ → a CommutatorM[𝒟F, b],
  NL, "For this case ", $ = {$D44, $Y}; MatrixForms[$],
  NL, "Previous calculation show that only the
    upper left quadrant (S) does not commute with the algebra. ",
  Imply, $sD = S → ($D44[[2, 1 ;; 2, 1 ;; 2]] /. $Y // ArrayFlatten);
  MatrixForms[$sD],
  NL, "• ", $ = ϕ -> a . CommutatorM[S, b]; $,
  yield, $ = $ /. $sD /. $sab; MatrixForms[$],
  Yield, $ph = $ = $ /. CommutatorM → MCommutator // Simplify;
  MatrixForms[$],
  NL, "Requiring: ", $ = ϕ → ct[ϕ],
  Yield, $ = $ /. $ph //. tt : Rule[__] :→ Thread[tt] // Flatten // DeleteDuplicates //
    DeleteCases[#, 0 → 0] &;
  Column[$];
  Yield, $ = $ //. (a_ b_ → a_ c_) → (b → c) // Simplify; Column[$];
  Yield, $ = $[[1 ;; 4]]; Column[$ // Expand],
  NL, "There only 2 independent relationships: ",
  FramedColumn[$ph12 = Thread[{ϕ1, ϕ2} -> $[[1 ;; 2]]]],
  NL, "Put ϕ[ϕ1,ϕ2]: ",
  NL, "Reverse relationships for ϕ1,ϕ2: ",
  $s = Map[Apply[List, Thread[#, Rule]] &, $ph12] // Flatten // Map[Reverse[#] &, #] &;
  Column[$s],
  NL, "Add Conjugate relationships: ",
  $sc = Thread[Conjugate[#], Rule] & /@ $s // ConjugateCTSimplify1[{}] // Simplify;
  Column[$sc],
  Yield, $ = $ph //. $s //. $sc; MatrixForms[$];
  $ϕ = $ = $ //. $s //. $sc /. Simplify[Thread[Times[-1 #], Rule] & /@ $s];
  MatrixForms[$] // Framed, CG[" (5.6)"]

];
```

■From the definition $\phi \to a[\mathcal{D}_F, b]$

For this case $\{\mathcal{D}_F \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix}, Y_0 \to \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}\}$

Previous calculation show that only the
upper left quadrant (S) does not commute with the algebra.

$$\Rightarrow S \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}$$

• $\phi \to a.[S, b] \longrightarrow \phi \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}.[\begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & (\lambda')^* & 0 & 0 \\ 0 & 0 & \alpha' & \beta' \\ 0 & 0 & -(\beta')^* & (\alpha')^* \end{pmatrix}]$

$$\to \phi \to \begin{pmatrix} 0 & 0 & \lambda (Y_\nu)^* (\alpha' - \lambda') & \lambda (Y_\nu)^* \beta' \\ 0 & 0 & -\lambda^* (Y_e)^* (\beta')^* & \lambda^* (Y_e)^* ((\alpha')^* - (\lambda')^*) \\ Y_\nu (\beta (\beta')^* + \alpha (-\alpha' + \lambda')) & -Y_e (\beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta') & 0 & 0 \\ Y_\nu (\alpha^* (\beta')^* + \beta^* (\alpha' - \lambda')) & -Y_e (\alpha^* ((\alpha')^* - (\lambda')^*) - \beta^* \beta') & 0 & 0 \end{pmatrix}$$

Requiring: $\phi \to \phi^\dagger$

→

→

$$\to \begin{array}{l} \lambda \alpha' - \lambda \lambda' \to -\alpha^* (\alpha')^* + \alpha^* (\lambda')^* + \beta^* \beta' \\ \lambda \beta' \to \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta' \\ \lambda^* (\beta')^* \to (\alpha \beta')^* + \beta^* \alpha' - \beta^* \lambda' \\ \lambda^* (\alpha')^* - \lambda^* (\lambda')^* \to \beta (\beta')^* - \alpha \alpha' + \alpha \lambda' \end{array}$$

There only 2 independent relationships: $\boxed{\begin{array}{l} \phi_1 \to \lambda (\alpha' - \lambda') \to \alpha^* (-(\alpha')^* + (\lambda')^*) + \beta^* \beta' \\ \phi_2 \to \lambda \beta' \to \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta' \end{array}}$

Put $\phi[\phi_1, \phi_2]$:

Reverse relationships for ϕ_1, ϕ_2: $\begin{array}{l} \lambda (\alpha' - \lambda') \to \phi_1 \\ \alpha^* (-(\alpha')^* + (\lambda')^*) + \beta^* \beta' \to \phi_1 \\ \lambda \beta' \to \phi_2 \\ \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta' \to \phi_2 \end{array}$

Add Conjugate relationships: $\begin{array}{l} \lambda^* ((\alpha')^* - (\lambda')^*) \to (\phi_1)^* \\ \beta (\beta')^* + \alpha (-\alpha' + \lambda') \to (\phi_1)^* \\ \lambda^* (\beta')^* \to (\phi_2)^* \\ \alpha^* (\beta')^* + \beta^* (\alpha' - \lambda') \to (\phi_2)^* \end{array}$

$$\to \boxed{\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 \end{pmatrix}} \quad (5.6)$$

```
PR["●Note: ϕ's is generally a sum of like terms: ",
 $ = Map[# /. tt : λ' | α' | β' | λ | α | β :→ T[tt, "d", {j}] &, $ph12];
 Yield, $ph12p = $ =
   Map[#[[1]] -> xSum[#[[2]], {j}] &, $] /. xSum[a_ -> b_, c_] → xSum[a, c] -> xSum[b, c];
 Column[$]
]
```

●Note: ϕ's is generally a sum of like terms:

$$\to \begin{array}{l} \phi_1 \to \sum_{\{j\}} [\lambda_j (\alpha'_j - \lambda'_j)] \to \sum_{\{j\}} [(\alpha_j)^* (-(\alpha'_j)^* + (\lambda'_j)^*) + (\beta_j)^* \beta'_j] \\ \phi_2 \to \sum_{\{j\}} [\lambda_j \beta'_j] \to \sum_{\{j\}} [(\alpha'_j)^* \beta_j - (\lambda'_j)^* \beta_j + \alpha_j \beta'_j] \end{array}$$

```
PR["Summary: ",
  NL, $e57 = $ = {$A3, T[Λ, "d", {μ}] ∈ ℝ,
      ϕ → {{0, Conjugate[Y]}, {Y, 0}},
      $ϕ,
      $ph12,
      T[B, "d", {μ}]_ℋℓ →
       {{0, 0, 0}, {0, -2 T[Λ, "d", {μ}], 0}, {0, 0, T[Q, "d", {μ}] - T[Λ, "d", {μ}] 1₂}},
      T[B, "d", {μ}]_ℋr̄ → {{0, 0, 0}, {0, 2 T[Λ, "d", {μ}], 0},
        {0, 0, -T[Λ, "d", {μ}] 1₂ - Conjugate[T[Q, "d", {μ}]]}}
     } // Flatten;
  MatrixForms[$],
  NL, "Calculate ", $ = e216[[1]] /. εRule[2] //. tuOpSimplify[Dot],
  NL, "Expand to 8x8 ", "POFF",
  $q = T[Q, "d", {μ}] → Table[T[q, "d", {μ}]_i,j, {i, 2}, {j, 2}];
  MatrixForms[$q],
  yield, $s = $e57[[1]] /. $q;
  $s[[2]] = ArrayFlatten[$s[[2]]];
  MatrixForms[$s];
  $s[[2]] =
   {{$s[[2]], 0}, {0, DiagonalMatrix[Table[T[Λ, "d", {μ}], {4}]]}} // ArrayFlatten;
  MatrixForms[$s],
  Yield, $ = $ /. Dot → xDot /. Plus → Inactive[Plus] /. $s /. $jj;
  MatrixForms[$],
  Yield, $ = $ // OrderedxDotMultiplyAll[] // ConjugateSimplify[{C}];
  MatrixForms[$];
  Yield, "PON", $e58 =
   $ = $ // tuRepeat[{Conjugate[C] → C, C.C → 1, Conjugate[C].C → 1, C . a_ :> Conjugate[a].
           C /; a =!= C}, ConjugateSimplify[{C, T[Λ, "d", {μ}]}]] // Activate;
  MatrixForms[$], CG[" (5.8)"]
 ];
```

Summary:

$\{\mathcal{A}_\mu \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix}, Q_\mu \to i \sum_{\{i,0,3\}} [q_i\, \sigma^i], q_i \in \mathbb{R}, \Lambda_\mu \in \mathbb{R}, \phi \to \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix},$

$\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 \\ 0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^*\,(\phi_1)^* \\ (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 \\ (\phi_2)^*\,Y_\nu & Y_e\,\phi_1 & 0 & 0 \end{pmatrix}, \phi_1 \to \lambda\,(\alpha' - \lambda') \to \alpha^*\,(-(\alpha')^* + (\lambda')^*) + \beta^*\,\beta',$

$\phi_2 \to \lambda\,\beta' \to \beta\,(\alpha')^* - \beta\,(\lambda')^* + \alpha\,\beta', B_{\mu\mathcal{H}_\ell} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu - 1_2\,\Lambda_\mu \end{pmatrix}, B_{\mu\mathcal{H}_{\bar r}} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\,\Lambda_\mu & 0 \\ 0 & 0 & -(Q_\mu)^* - 1_2\,\Lambda_\mu \end{pmatrix}\}$

Calculate $B_\mu \to -J.\mathcal{A}_\mu.J^\dagger + \mathcal{A}_\mu$

Expand to 8x8 $B_\mu \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 1,1} - \Lambda_\mu & q_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 2,1} & q_{\mu 2,2} - \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 1,1})^* + \Lambda_\mu & -(q_{\mu 1,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 2,1})^* & -(q_{\mu 2,2})^* + \Lambda_\mu \end{pmatrix}$ (5.8)

```
PR["●Higgs field ", $ = Φ → 𝒟F + tt[{{ϕ, 0}, {0, 0}}] + JF.tt[{{ϕ, 0}, {0, 0}}].ct[JF] /.
    Plus → Inactive[Plus] /. tt[a_] → a;
 MatrixForms[$],
 NL, "Expand to 8x8: ",
 $sϕ = {{ϕ, 0}, {0, 0}} → ArrayFlatten[DiagonalMatrix[{ϕ, 0, 0, 0, 0}] /. $ϕ];
 MatrixForms[$sϕ], "POFF",
 $ = $ /. JF → J;
 $ = $ /. Dot → xDot /. $jj /. $sϕ; MatrixForms[$],
 Yield, $ = $ // OrderedxDotMultiplyAll[] //
   tuRepeat[{Conjugate[C] → C, C.C → 1, Conjugate[C].C → 1,
     C . a_ :> Conjugate[a].C /; a =!= C}, ConjugateSimplify[{C, T[Λ, "d", {μ}]}]];
 "PON",
 MatrixForms[$],
 NL, "From ", $ = $D[[1]]; MatrixForms[$],
 yield, $ = Φ -> $[[2]] + {{ϕ, 0}, {0, Conjugate[ϕ]}};
 MatrixForms[$e59 = $] // Framed, CG[" (5.9)"]
]
```

●Higgs field $\Phi \to J_F . \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} . (J_F)^{\dagger} + \mathcal{D}_F + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}$

Expand to 8x8: $\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$\Phi \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_\nu . (\phi_1)^* & Y_\nu . (\phi_2)^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y_e . \phi_2 & Y_e . \phi_1 \\ 0 & 0 & 0 & 0 & \phi_1 . (Y_\nu)^* & -(Y_e)^* . (\phi_2)^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_2 . (Y_\nu)^* & (Y_e)^* . (\phi_1)^* & 0 & 0 \end{pmatrix} +$$

$$\mathcal{D}_F + \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From $\mathcal{D}_F \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix} \longrightarrow \boxed{\Phi \to \begin{pmatrix} S + \phi & T^* \\ T & S^* + \phi^* \end{pmatrix}}$ (5.9)

Prop.5.3.

```
PR["●Prop.5.3. The action on gauge group ",
 𝒢[M × F_GWS][𝒟_𝒜 → slash[𝒟] ⊗ 𝕀 + T[γ, "u", {μ}] ⊗ T[B, "d", {μ}] + T[γ, "d", {5}] ⊗ Φ],
 NL, "is given by: ",
 $ = {T[Λ, "d", {μ}] -> T[Λ, "d", {μ}] - I λ.tuDPartial[Conjugate[λ], μ],
   T[Q, "d", {μ}] → q.T[Q, "d", {μ}].ct[q] - I q.tuDPartial[Conjugate[q], μ],
   {{ϕ1}, {ϕ2}} → Conjugate[λ].q.{{ϕ1}, {ϕ2}} + (Conjugate[λ].q - 1).{{1}, {0}},
   λ ∈ C"∞"[M, U[1]], q ∈ C"∞"[M, SU[2]]};
 MatrixForms[$e221a = $] // Column,
 NL, "For the fields (5.7) compute the transformations (2.21).",
 Yield, $e221 = {T[A, "d", {μ}] → u.T[A, "d", {μ}].ct[u] - I u.tuDPartial[ct[u], μ],
   ϕ -> u.ϕ.ct[u] + u.CommutatorM[𝒟_F, ct[u]],
   {u → {λ, q}} ∈ C"∞"[M, U[1] × SU[2]]
  }; Column[$e221],
 NL, "In 8x8 form ", $ = $e57[[1, 2]]; $ = ArrayPad[$, {0, 1}];
 $[[4, 4]] = DiagonalMatrix[Table[T[Λ, "d", {μ}], {i, 4}]];
 $ = $e57[[1, 1]] → ($ /. $q // ArrayFlatten);
 MatrixForms[$a88 = $],
 NL, "Check if ", u. $a88[[1]] . ct[u],
 imply, T[Q, "d", {μ}] → u.T[Q, "d", {μ}].ct[u], "POFF",
 $u = $a88[[2]] /. T[Λ, "d", {μ}] :→ λ /. q → qu; MatrixForms[$];
 $ = $u .$a88[[2]].ct[$u]; MatrixForms[$], "PON", OK,
 NL, "Check statements on ", $ = I u.tuDPartial[ct[u], μ], "POFF",
 $s = u → $u;
 Yield, $ = $ /. $s //. tt : tuDDown["∂"][_, _] :→ Thread[tt] /. tuDDown["∂"][0, _] → 0;
 MatrixForms[$], "PON", OK,
 Imply, $e221a[[2]], OK
]
```

●Prop.5.3. The action on gauge group $\mathcal{G}[M\times F_{GWS}][\mathcal{D}_{\mathcal{A}}\to(\not{\mathcal{D}})\otimes\mathbb{I}+\gamma_5\otimes\Phi+\gamma^{\mu}\otimes B_{\mu}]$

is given by:
$$\begin{array}{l}\Lambda_{\mu}\to -i\,\lambda.\underline{\partial}_{\mu}[\lambda^*]+\Lambda_{\mu}\\ Q_{\mu}\to -i\,q.\underline{\partial}_{\mu}[q^*]+q.Q_{\mu}.q^{\dagger}\\ \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\to(-1+\lambda^*.q).\begin{pmatrix}1\\ 0\end{pmatrix}+\lambda^*.q.\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\\ \lambda\in C^{\infty}[M,U[1]]\\ q\in C^{\infty}[M,SU[2]]\end{array}$$

For the fields (5.7) compute the transformations (2.21).

$$\to\ \begin{array}{l}A_{\mu}\to -i\,u.\underline{\partial}_{\mu}[u^{\dagger}]+u.A_{\mu}.u^{\dagger}\\ \phi\to u.[\mathcal{D}_F,u^{\dagger}]+u.\phi.u^{\dagger}\\ \{u\to\{\lambda,q\}\}\in C^{\infty}[M,U[1]\times SU[2]]\end{array}$$

In 8x8 form
$$\mathcal{A}_{\mu}\to\begin{pmatrix}\Lambda_{\mu}&0&0&0&0&0&0&0\\ 0&-\Lambda_{\mu}&0&0&0&0&0&0\\ 0&0&q_{\mu 1,1}&q_{\mu 1,2}&0&0&0&0\\ 0&0&q_{\mu 2,1}&q_{\mu 2,2}&0&0&0&0\\ 0&0&0&0&\Lambda_{\mu}&0&0&0\\ 0&0&0&0&0&\Lambda_{\mu}&0&0\\ 0&0&0&0&0&0&\Lambda_{\mu}&0\\ 0&0&0&0&0&0&0&\Lambda_{\mu}\end{pmatrix}$$

Check if $u.\mathcal{A}_{\mu}.u^{\dagger}$ $\Rightarrow$ $Q_{\mu}\to u.Q_{\mu}.u^{\dagger}$ OK

Check statements on $i\,u.\underline{\partial}_{\mu}[u^{\dagger}]$ OK

$\Rightarrow$ $Q_{\mu}\to -i\,q.\underline{\partial}_{\mu}[q^*]+q.Q_{\mu}.q^{\dagger}$ OK

```
PR["Check transformation ", $ = $e221[[2]],
 NL, "Collect relevant pieces ", "POFF",
 Yield,
 $s08 = {ϕ -> $sϕ[[2]], u -> $aa, $D44[[1]] → ($D44[[1]] //. $sDAgws // ArrayFlatten)};
 MatrixForms[$s08],
 line,
 NL, "Calculate RHS:",
 Yield, $[[2]] = $[[2]] /. Plus → Inactive[Plus] /. $s08;
 MatrixForms[$0 = $], "PON",
 line,
 NL, "The commutator term: ", $ = $0 // tuExtractPositionPattern[CommutatorM[__]];
 Yield, $ = $ /. CommutatorM → MCommutator;
 MatrixForms[$],
 NL, "Recombine ",
 Yield, $pht = $ = tuReplacePart[$0, $] // Activate // Simplify;
 MatrixForms[$]
]
```

Check transformation $\phi \to u.[\mathcal{D}_F, u^\dagger] + u.\phi.u^\dagger$
Collect relevant pieces

The commutator term:

$$\to \{\{2,1,2\} \to \begin{pmatrix} 0 & 0 & \alpha^* (Y_\nu)^* - \lambda^* (Y_\nu)^* & -\beta (Y_\nu)^* & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^* (Y_e)^* & \alpha (Y_e)^* - \lambda (Y_e)^* & 0 & 0 & 0 & 0 \\ -\alpha^* Y_\nu + \lambda^* Y_\nu & \beta Y_e & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta^* Y_\nu & -\alpha Y_e + \lambda Y_e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\}$$

Recombine

$$\to \phi \to \begin{pmatrix} 0 & 0 & \lambda (Y_\nu)^* (-\lambda^* + \alpha^* (1+ \\ 0 & 0 & \lambda^* (Y_e)^* (\beta^* (1 + (\phi_1)^* \\ (-\alpha\,\alpha^* - \beta\,\beta^* + \lambda^* (\alpha + \alpha (\phi_1)^* + \beta (\phi_2)^*)) Y_\nu & \lambda Y_e (\beta + \beta\,\phi_1 - \alpha\,\phi_2) & 0 \\ -\lambda^* (\beta^* (1 + (\phi_1)^*) - \alpha^* (\phi_2)^*) Y_\nu & Y_e (\alpha^* (-\alpha + \lambda + \lambda\,\phi_1) + \beta^* (-\beta + \lambda\,\phi_2)) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
PR["So the correspondence between φ and the transformed φ: ", "POFF",
 Imply, $ = $s08[[1]] → ($pht /. φ → φt), "PON",
 Yield, $ =
  $[[1, 2]] - $[[2, 2]] // Flatten // DeleteDuplicates // Simplify // DeleteCases[#, 0] &;
 $ = (# → 0) & /@ $;
 Column[$],
 NL, "Y's are multiplicative factors that can be removed: ",
 Yield, $1 = $ = $ /. Y_ → 1; Column[$],
 NL,
 "We get several diffent transformation that will work since there are 8 equations
   and only 2 complex unknowns; However, using the following substitutions
   reduce the number of possible solution to one: ",
 NL, "The ",
 $sq = $aa[[3 ;; 4, 3 ;; 4]], " is ∈SU[2] ⇒ Det[]→1 and λ∈U[1]: ",
 $sq = Det[$sq] → 1; $sq = {$sq, -1 # & /@ $sq};
 $sq = {$sq, Conjugate[$sq] // ConjugateSimplify[{}],
   λ Conjugate[λ] → 1, β Conjugate[β] → 1 - α Conjugate[α]},

 NL, "Generate equation selections: ",
 $p = Permutations[Table[i, {i, 8}], {4}] // Sort /@ # & // DeleteDuplicates;
 Yield, $ = Map[($ = $1 /. Rule → Equal; $ = $[[#]];
     $ = tuRepeat[$sq, Simplify][Solve[$, {φt₁, φt₂}, Complexes]];
     $ = tuRepeat[{α Conjugate[α] → 1 - β Conjugate[β]}, Expand][$] // Simplify;
     {#, $}) &, $p];
 Column[$];
 NL, "Possible transformation: ",
 $ = Map[Flatten[#[[2]]] &, Select[$, Length[#[[2]]] > 0 &]] // DeleteDuplicates;
 Framed[$[[1]] /. λ Conjugate[λ] → 1],
 CR["Using ", Conjugate[β] → -β, " makes this consistent with text."]


]
```

So the correspondence between ϕ and the transformed ϕ:

$$\to \begin{array}{l} (Y_\nu)^* (\lambda \lambda^* + \phi_1 - \lambda \alpha^* (1 + \phi t_1) - \lambda \beta^* \phi t_2) \to 0 \\ (Y_\nu)^* (\phi_2 + \lambda (\beta + \beta \phi t_1 - \alpha \phi t_2)) \to 0 \\ -(Y_e)^* ((\phi_2)^* + \beta^* \lambda^* (1 + (\phi t_1)^*) - \alpha^* \lambda^* (\phi t_2)^*) \to 0 \\ (Y_e)^* ((\phi_1)^* - \lambda^* (\alpha - \lambda + \alpha (\phi t_1)^* + \beta (\phi t_2)^*)) \to 0 \\ (\alpha \alpha^* + \beta \beta^* + (\phi_1)^* - \lambda^* (\alpha + \alpha (\phi t_1)^* + \beta (\phi t_2)^*)) Y_\nu \to 0 \\ -Y_e (\phi_2 + \lambda (\beta + \beta \phi t_1 - \alpha \phi t_2)) \to 0 \\ ((\phi_2)^* + \beta^* \lambda^* (1 + (\phi t_1)^*) - \alpha^* \lambda^* (\phi t_2)^*) Y_\nu \to 0 \\ Y_e (\phi_1 + \alpha^* (\alpha - \lambda - \lambda \phi t_1) + \beta^* (\beta - \lambda \phi t_2)) \to 0 \end{array}$$

Y's are multiplicative factors that can be removed:

$$\to \begin{array}{l} \lambda \lambda^* + \phi_1 - \lambda \alpha^* (1 + \phi t_1) - \lambda \beta^* \phi t_2 \to 0 \\ \phi_2 + \lambda (\beta + \beta \phi t_1 - \alpha \phi t_2) \to 0 \\ -(\phi_2)^* - \beta^* \lambda^* (1 + (\phi t_1)^*) + \alpha^* \lambda^* (\phi t_2)^* \to 0 \\ (\phi_1)^* - \lambda^* (\alpha - \lambda + \alpha (\phi t_1)^* + \beta (\phi t_2)^*) \to 0 \\ \alpha \alpha^* + \beta \beta^* + (\phi_1)^* - \lambda^* (\alpha + \alpha (\phi t_1)^* + \beta (\phi t_2)^*) \to 0 \\ -\phi_2 - \lambda (\beta + \beta \phi t_1 - \alpha \phi t_2) \to 0 \\ (\phi_2)^* + \beta^* \lambda^* (1 + (\phi t_1)^*) - \alpha^* \lambda^* (\phi t_2)^* \to 0 \\ \phi_1 + \alpha^* (\alpha - \lambda - \lambda \phi t_1) + \beta^* (\beta - \lambda \phi t_2) \to 0 \end{array}$$

We get several diffent transformation that will work since there are 8 equations and only 2 complex unknowns; However, using the following substitutions reduce the number of possible solution to one:

The $\{\{\alpha, \beta\}, \{-\beta^*, \alpha^*\}\}$ is $\in SU[2] \Rightarrow$ Det[]→1 and $\lambda \in U[1]$:

$\{\{\alpha \alpha^* + \beta \beta^* \to 1, -\alpha \alpha^* - \beta \beta^* \to -1\}, \{\alpha \alpha^* + \beta \beta^* \to 1, -\alpha \alpha^* - \beta \beta^* \to -1\}, \lambda \lambda^* \to 1, \beta \beta^* \to 1 - \alpha \alpha^*\}$

Generate equation selections:

→

Possible transformation: $\boxed{\left\{\phi t_1 \to \frac{\alpha - \lambda + \alpha \phi_1 - \beta^* \phi_2}{\lambda}, \phi t_2 \to \frac{\beta + \beta \phi_1 + \alpha^* \phi_2}{\lambda}\right\}}$

Using $\beta^* \to -\beta$ makes this consistent with text.

5.3 Spectral Action

```
$p37;
$e57;
$e58;
$F;
PR["●Lemma 5.4: ",
  NL, $0 = $ = {Tr[F_{μ ν} F^{μ ν}] → 12 T[Λ, "dd", {μ, ν}] T[Λ, "uu", {μ, ν}] +
       2 Tr[T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]],
     T[Λ, "dd", {μ, ν}] → tuDPartial[T[Λ, "d", {ν}], μ] - tuDPartial[T[Λ, "d", {μ}], ν],
     T[Q, "dd", {μ, ν}] → tuDPartial[T[Q, "d", {ν}], μ] -
       tuDPartial[T[Q, "d", {μ}], ν] + I CommutatorM[T[Q, "d", {μ}], T[Q, "d", {ν}]],
     "q's" → "hermitian",
     (qq : T[q, "d", {μ_}])_{i_,j_} :> Conjugate[qq_{j,i}] /; j < i,
     Conjugate[qq : q_{i_,i_}] → q_{i,i}
    }; FramedColumn[$]
 ];
PR["Start with (5.7): ", $ = $e57; $ = $[[-2 ;; -1]];
 $ = T[B, "d", {μ_}] → ({{$[[1, 2]], 0}, {0, $[[2, 2]]}} // ArrayFlatten);
 MatrixForms[$b = $],
 NL, "Compute: ",
 $ = ($F /. CommutatorM → MCommutator /. Dot → xDot) /. $b /. Plus → Inactive[Plus] //.
      tt : tuDDown["∂"][__] :→ Thread[tt] /. tuDDown["∂"][0, _] → 0 //
    tuRepeat[{tuOpDistribute[tuDDown["∂"], Inactive[Plus]],
      tuOpSimplify[tuDDown["∂"]]}, Simplify];
 MatrixForms[$];
```

```
$ = $ /. CommutatorM → MCommutator // OrderedxDotMultiplyAll[] // Activate;
$ = $ //. tuOpSimplify[Dot]; MatrixForms[$];
$u = tuIndicesRaise[{μ, ν}][$];
$s = {$, $u} // Flatten;
$ = $0[[1, 1, 1]] /. Times → Dot,
$ = $ /. $s;
$ = $ /. Dot → xDot // OrderedxDotMultiplyAll[{Tensor[Λ, _, _]}];
Yield,
$ = $ //. tuDExpand[tuDDown["∂"], {1_}] //. tuDExpand[tuDUp["∂"], {1_}] // Simplify;
MatrixForms[$xFF = $],

NL, "▪Compare: ",
$qq0 = $ = $0[[1, 2, 2, 2, 1]],
yield, $sqq = {$0[[3]], tuIndicesRaise[{μ, ν}][$0[[3]]]};
yield, $qq = $ = $ /. $sqq /. CommutatorM → MCommutator,

NL, "•with the {3,3} non-Λ terms: ",
$2 = Select[Expand[$xFF[[3, 3]]], Length[tuExtractPattern[Λ][#]] == 0 &] // Simplify,
Yield, $qq == $2 // Simplify, imply, "equal", $part[1] = $qq0;

NL, "•with the Conjugate of the {6,6} non-Λ terms: ",
Yield,
$2 = Select[Expand[$xFF[[6, 6]]], Length[tuExtractPattern[Λ][#]] == 0 &] // Simplify,
Yield, Conjugate[$2] == $ // ConjugateCTSimplify1[{}], imply, "equal",
$part[2] = Conjugate[$qq0];

line,
NL, "Since ", $qq0[[1]], " is Hermitian ", $ = Conjugate[$qq0] → Transpose[$qq0],
NL, "However for the Tr[]: ", $ = Tr /@ $ → Tr[$qq0];
Framed[$], $trQQ = {$[[1, 1]] → $[[2]], Tr[Conjugate[$qq0[[2]]]] → Tr[$qq0[[2]]]};

NL, "•The {3,3} terms linear in Λ: ",
$2 = Select[Expand[$xFF[[3, 3]]], Length[tuExtractPattern[Λ][#]] == 1 &] // Simplify;
Yield, $2 = $2 /. tt : a_ . 1_ ⧴ Reverse[tt] /. 1_ . QQ_ ⧴ QQ /; ! FreeQ[QQ, Q] // Simplify //
   (# /. 1_ → 1) &,
Yield, $2 = $2 // Collect[#, tuDPartial[T[Λ, "d", {μ}, ν]]] &,
Yield, $2 =
 Collect[Expand[$2], {tuDPartial[T[Λ, "d", {μ}], ν], tuDPartialu[T[Λ, "u", {μ}], ν],
      tuDPartial[T[Λ, "d", {ν}], μ], tuDPartialu[T[Λ, "u", {ν}], μ]}] /.
    tt : tuDPartialu[T[Λ, "u", {μ}], ν] a_ ⧴ tuIndexSwap[{μ, ν}][tt] /.
   tt : tuDPartial[T[Λ, "d", {μ}], ν] a_ ⧴ tuIndexSwap[{μ, ν}][tt] /.
  tt : tuDPartialu[T[Λ, "u", {ν}], μ] a_ ⧴ UpDownIndexSwap[{μ, ν}][tt];
Framed[$qΛ = $2],
NL, "which can be written in terms of: ",
$ = {$sqq, test -> $qΛ} /. CommutatorM → MCommutator // Flatten;
$ = tuEliminate[$, tuDPartialu[T[Q, "u", {ν}], μ]];
$ = Apply[List, $] /. Equal → Rule;
$ = tuRuleSolve[$, test]; Framed[$part[3] = $qΛ = $[[1, 2]]],

NL, "•The {6,6} terms linear in Λ: ",
$2 = Select[Expand[$xFF[[6, 6]]], Length[tuExtractPattern[Λ][#]] == 1 &] // Simplify;
Yield, $2 = $2 /. tt : a_ . 1_ ⧴ Reverse[tt] /. 1_ . QQ_ ⧴ QQ /; ! FreeQ[QQ, Q] // Simplify //
   (# /. 1_ → 1) &,
Yield, $2 = $2 // Collect[#, tuDPartial[T[Λ, "d", {μ}, ν]]] &,
Yield, $2 =
 Collect[Expand[$2], {tuDPartial[T[Λ, "d", {μ}], ν], tuDPartialu[T[Λ, "u", {μ}], ν],
      tuDPartial[T[Λ, "d", {ν}], μ], tuDPartialu[T[Λ, "u", {ν}], μ]}] /.
```

```
      tt : tuDPartialu[T[Λ, "u", {μ}], ν] a_ ⧴ tuIndexSwap[{μ, ν}][tt] /.
     tt : tuDPartial[T[Λ, "d", {μ}], ν] a_ ⧴ tuIndexSwap[{μ, ν}][tt] /.
    tt : tuDPartialu[T[Λ, "u", {ν}], μ] a_ ⧴ UpDownIndexSwap[{μ, ν}][tt];
  (*TO DO*)
  Framed[$qΛ = $2],

  $sqqc = Conjugate /@ $sqq;
  $ = {$sqqc, test -> $qΛ} /. CommutatorM → MCommutator //
     ConjugateCTSimplify1[{}] // Flatten;
  $ = tuEliminate[$, Conjugate[tuDPartialu[T[Q, "u", {ν}], μ]]];
  $ = Apply[List, $] /. Equal → Rule;
  NL, "which can be written in terms of: ",
  $ = tuRuleSolve[$, test]; Framed[$part[4] = $qΛ = $[[1, 2]]],
  line,

  NL, "•The {3,3} terms quadratic in Λ: ",
  $2 = Select[Expand[$xFF[[3, 3]]], Length[tuExtractPattern[Λ][#]] == 2 &] // Simplify;
  yield, $part[5] = $2 = $2 /. tt : a_ . 1_ ⧴ Reverse[tt] // Simplify,
  NL, "•The {6,6} terms quadratic in Λ: ",
  $2 = Select[Expand[$xFF[[6, 6]]], Length[tuExtractPattern[Λ][#]] == 2 &] // Simplify;
  yield, $part[6] = $2 = $2 /. tt : a_ . 1_ ⧴ Reverse[tt] // Simplify

 ]
 PR["Take Tr[]: ", Tr[$0[[1, 1, 1]]],
  Yield,
  Yield, $ = Sum[Tr[$part[$i]], {$i, 6}],
  Yield, $ = $ //. tuTrSimplify[{tuDPartial[T[Λ, "d", {ν}], μ]}],
  Yield, $ = $ /. $trQQ /. Tr[1₂^2 a_] → a Tr[1₂] /. Tr[1₂] → 2,
  NL, "Add the {2,2} and {5.5} term: ",
  Yield, $ = $ + $xFF[[2, 2]] + $xFF[[5, 5]],
  Yield, ColumnSumExp[$] // Framed,
  yield, $ /. Reverse[-1 # & /@ $0[[2]]] /.
    Reverse[-1 # & /@ tuIndicesRaise[{μ, ν}][$0[[2]]]] // Framed, OK
 ]
```

●Lemma 5.4:

$$\boxed{\begin{array}{l} \mathrm{Tr}[F_{\mu\nu}\, F^{\mu\nu}] \to 12\, \Lambda_{\mu\nu}\, \Lambda^{\mu\nu} + 2\, \mathrm{Tr}[Q_{\mu\nu}\, Q^{\mu\nu}] \\ \Lambda_{\mu\nu} \to -\underline{\partial}_{\nu}[\Lambda_{\mu}] + \underline{\partial}_{\mu}[\Lambda_{\nu}] \\ Q_{\mu\nu} \to i\, [Q_{\mu},\, Q_{\nu}] - \underline{\partial}_{\nu}[Q_{\mu}] + \underline{\partial}_{\mu}[Q_{\nu}] \\ \text{q's} \to \text{hermitian} \\ qq : q_{\mu\_i\_,j\_} \mapsto (qq_{j,i})^{*} \,/;\, j < i \\ (qq : q_{\_i\_,i\_})^{*} \to q_{i,i} \end{array}}$$

Start with (5.7): $B_{\mu\_} \to \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_\mu - 1_2\,\Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\,\Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -(Q_\mu)^* - 1_2\,\Lambda_\mu \end{array}\right)$

Compute: $F_{\mu\nu}.F^{\mu\nu}$

$$\to \left(\begin{array}{ccc} 0 & 0 & \\ 0 & 4\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu])\,(\partial^\nu[\Lambda^\mu] - \partial^\mu[\Lambda^\nu]) & \\ 0 & 0 & (i\,(Q_\mu.Q_\nu - Q_\nu.Q_\mu - 1_2.Q_\nu\,\Lambda_\mu + Q_\nu.1_2\,\Lambda_\mu + 1_2.Q_\mu\,\Lambda_\nu - Q_\mu.1_2\,\Lambda_\nu) - \partial_\nu[Q_\mu \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \end{array}\right.$$

▪Compare: $Q_{\mu\nu}\,Q^{\mu\nu} \longrightarrow \longrightarrow$

$(i\,(Q_\mu.Q_\nu - Q_\nu.Q_\mu) - \partial_\nu[Q_\mu] + \partial_\mu[Q_\nu])\,(i\,(Q^\mu.Q^\nu - Q^\nu.Q^\mu) - \partial^\nu[Q^\mu] + \partial^\mu[Q^\nu])$

•with the {3,3} non-Λ terms:

$-(Q_\mu.Q_\nu - Q_\nu.Q_\mu + i\,(\partial_\nu[Q_\mu] - \partial_\mu[Q_\nu]))\,(Q^\mu.Q^\nu - Q^\nu.Q^\mu + i\,(\partial^\nu[Q^\mu] - \partial^\mu[Q^\nu]))$

→ True ⇒ equal

•with the Conjugate of the {6,6} non-Λ terms:

$\to (\partial_\nu[Q_\mu]^* - \partial_\mu[Q_\nu]^* + i\,((Q_\mu)^*.(Q_\nu)^* - (Q_\nu)^*.(Q_\mu)^*))\,(\partial^\nu[Q^\mu]^* - \partial^\mu[Q^\nu]^* + i\,((Q^\mu)^*.(Q^\nu)^* - (Q^\nu)^*.(Q^\mu)^*))$

→ True ⇒ equal

Since $Q_{\mu\nu}$ is Hermitian $(Q_{\mu\nu}\,Q^{\mu\nu})^* \to Q_{\mu\nu}\,Q^{\mu\nu T}$

However for the Tr[]: $\boxed{(\mathrm{Tr}[(Q_{\mu\nu}\,Q^{\mu\nu})^*] \to \mathrm{Tr}[Q_{\mu\nu}\,Q^{\mu\nu T}]) \to \mathrm{Tr}[Q_{\mu\nu}\,Q^{\mu\nu}]}$

•The {3,3} terms linear in Λ:

$\to i\,Q^\mu.Q^\nu\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) - i\,Q^\nu.Q^\mu\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) - \partial_\nu[\Lambda_\mu]\,\partial^\nu[Q^\mu] +$
$\partial_\mu[\Lambda_\nu]\,\partial^\nu[Q^\mu] + \partial_\nu[\Lambda_\mu]\,\partial^\mu[Q^\nu] - \partial_\mu[\Lambda_\nu]\,\partial^\mu[Q^\nu] + i\,Q_\mu.Q_\nu\,\partial^\nu[\Lambda^\mu] - i\,Q_\nu.Q_\mu\,\partial^\nu[\Lambda^\mu] -$
$\partial_\nu[Q_\mu]\,\partial^\nu[\Lambda^\mu] + \partial_\mu[Q_\nu]\,\partial^\nu[\Lambda^\mu] - i\,Q_\mu.Q_\nu\,\partial^\mu[\Lambda^\nu] + i\,Q_\nu.Q_\mu\,\partial^\mu[\Lambda^\nu] + \partial_\nu[Q_\mu]\,\partial^\mu[\Lambda^\nu] - \partial_\mu[Q_\nu]\,\partial^\mu[\Lambda^\nu]$

$\to i\,Q^\mu.Q^\nu\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) - i\,Q^\nu.Q^\mu\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) - \partial_\nu[\Lambda_\mu]\,\partial^\nu[Q^\mu] + \partial_\mu[\Lambda_\nu]\,\partial^\nu[Q^\mu] +$
$\partial_\nu[\Lambda_\mu]\,\partial^\mu[Q^\nu] - \partial_\mu[\Lambda_\nu]\,\partial^\mu[Q^\nu] + i\,Q_\mu.Q_\nu\,\partial^\nu[\Lambda^\mu] - i\,Q_\nu.Q_\mu\,\partial^\nu[\Lambda^\mu] - \partial_\nu[Q_\mu]\,\partial^\nu[\Lambda^\mu] +$
$\partial_\mu[Q_\nu]\,\partial^\nu[\Lambda^\mu] - i\,Q_\mu.Q_\nu\,\partial^\mu[\Lambda^\nu] + i\,Q_\nu.Q_\mu\,\partial^\mu[\Lambda^\nu] + \partial_\nu[Q_\mu]\,\partial^\mu[\Lambda^\nu] - \partial_\mu[Q_\nu]\,\partial^\mu[\Lambda^\nu]$

$\to \boxed{4\,\partial_\mu[\Lambda_\nu]\,(-i\,Q^\mu.Q^\nu + i\,Q^\nu.Q^\mu + \partial^\nu[Q^\mu] - \partial^\mu[Q^\nu])}$

which can be written in terms of: $\boxed{-4\,Q^{\mu\nu}\,\partial_\mu[\Lambda_\nu]}$

•The {6,6} terms linear in Λ:

$\to \partial^\nu[Q^\mu]^*\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) + \partial^\mu[Q^\nu]^*\,(-\partial_\nu[\Lambda_\mu] + \partial_\mu[\Lambda_\nu]) +$
$i\,((Q^\mu)^*.(Q^\nu)^*\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) + (Q^\nu)^*.(Q^\mu)^*\,(-\partial_\nu[\Lambda_\mu] + \partial_\mu[\Lambda_\nu]) +$
$(-i\,\partial_\nu[Q_\mu]^* + i\,\partial_\mu[Q_\nu]^* + (Q_\mu)^*.(Q_\nu)^* - (Q_\nu)^*.(Q_\mu)^*)\,(\partial^\nu[\Lambda^\mu] - \partial^\mu[\Lambda^\nu]))$

$\to \partial^\nu[Q^\mu]^*\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) + \partial^\mu[Q^\nu]^*\,(-\partial_\nu[\Lambda_\mu] + \partial_\mu[\Lambda_\nu]) +$
$i\,((Q^\mu)^*.(Q^\nu)^*\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu]) + (Q^\nu)^*.(Q^\mu)^*\,(-\partial_\nu[\Lambda_\mu] + \partial_\mu[\Lambda_\nu]) +$
$(-i\,\partial_\nu[Q_\mu]^* + i\,\partial_\mu[Q_\nu]^* + (Q_\mu)^*.(Q_\nu)^* - (Q_\nu)^*.(Q_\mu)^*)\,(\partial^\nu[\Lambda^\mu] - \partial^\mu[\Lambda^\nu]))$

$\to \boxed{4\,(-\partial^\nu[Q^\mu]^* + \partial^\mu[Q^\nu]^* - i\,(Q^\mu)^*.(Q^\nu)^* + i\,(Q^\nu)^*.(Q^\mu)^*)\,\partial_\mu[\Lambda_\nu]}$

which can be written in terms of: $\boxed{4\,(Q^{\mu\nu})^*\,\partial_\mu[\Lambda_\nu]}$

•The {3,3} terms quadratic in Λ: $\longrightarrow 1_2^2\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu])\,(\partial^\nu[\Lambda^\mu] - \partial^\mu[\Lambda^\nu])$

•The {6,6} terms quadratic in Λ: $\longrightarrow 1_2^2\,(\partial_\nu[\Lambda_\mu] - \partial_\mu[\Lambda_\nu])\,(\partial^\nu[\Lambda^\mu] - \partial^\mu[\Lambda^\nu])$

Take Tr[]: $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]$

→

→ $\mathrm{Tr}[(Q_{\mu\nu} Q^{\mu\nu})^*] + \mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}] + \mathrm{Tr}[4 (Q^{\mu\nu})^* \underline{\partial}_\mu[\Lambda_\nu]] +$
$\mathrm{Tr}[-4 Q^{\mu\nu} \underline{\partial}_\mu[\Lambda_\nu]] + 2\,\mathrm{Tr}[1_2^2 (\underline{\partial}_\nu[\Lambda_\mu] - \underline{\partial}_\mu[\Lambda_\nu]) (\underline{\partial}^\nu[\Lambda^\mu] - \underline{\partial}^\mu[\Lambda^\nu])]$

→ $\mathrm{Tr}[(Q_{\mu\nu} Q^{\mu\nu})^*] + \mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}] + 2\,\mathrm{Tr}[1_2^2 (\underline{\partial}_\nu[\Lambda_\mu] - \underline{\partial}_\mu[\Lambda_\nu]) (\underline{\partial}^\nu[\Lambda^\mu] - \underline{\partial}^\mu[\Lambda^\nu])] +$
$4\,\mathrm{Tr}[(Q^{\mu\nu})^*]\, \underline{\partial}_\mu[\Lambda_\nu] - 4\,\mathrm{Tr}[Q^{\mu\nu}]\, \underline{\partial}_\mu[\Lambda_\nu]$

→ $2\,\mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}] + 4 (\underline{\partial}_\nu[\Lambda_\mu] - \underline{\partial}_\mu[\Lambda_\nu]) (\underline{\partial}^\nu[\Lambda^\mu] - \underline{\partial}^\mu[\Lambda^\nu])$

Add the {2,2} and {5.5} term:

→ $2\,\mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}] + 12 (\underline{\partial}_\nu[\Lambda_\mu] - \underline{\partial}_\mu[\Lambda_\nu]) (\underline{\partial}^\nu[\Lambda^\mu] - \underline{\partial}^\mu[\Lambda^\nu])$

→ $\boxed{\sum\left[\begin{array}{l} 2\,\mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}] \\ 12 (\underline{\partial}_\nu[\Lambda_\mu] - \underline{\partial}_\mu[\Lambda_\nu]) (\underline{\partial}^\nu[\Lambda^\mu] - \underline{\partial}^\mu[\Lambda^\nu]) \end{array}\right]} \longrightarrow \boxed{12\, \Lambda_{\mu\nu} \Lambda^{\mu\nu} + 2\,\mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}]}$ OK

```
PR["●Lemma 5.5: ",
  $155 = $ = {Tr[Φ^2] → 4 a Abs[H']^2 + 2 c,
     Tr[Φ^4] → 4 b Abs[H']^4 + 8 e Abs[H']^2 + 2 d,
     H' → {ϕ1 + 1, ϕ2},
     a → Abs[Yν]^2 + Abs[Ye]^2,
     b → Abs[Yν]^4 + Abs[Ye]^4,
     c -> Abs[YR]^2, d -> Abs[YR]^4, e -> Abs[YR]^2 Abs[Yν]^2
    }; Column[$],
  $155a = Association[$155];
  Imply, $155x = {
     Abs[H']^2 → (H' . Conjugate[H'] /. $155) // FullSimplify // Reverse,
     t[_]i_,j_ :→ 0 /; i ≠ 1 || j ≠ 1,
     t[_]i,j → "GWS basis"
    } /. Re[x_] → (x + Conjugate[x]) / 2,
  NL, "",
  $ = ($ = Conjugate[T].T) → ($ /. $t44 /. $155x // Simplify);
  MatrixForms[$],
  Yield, $ = $[[2]] -> Abs[YR]^2,
  AppendTo[$155x, $];
  Yield, $ = Tr /@ $ // FullSimplify,
  AppendTo[$155x, $];
  line,
  NL, "Proof: ",
  NL, "Use the 8x8 representation of: ",
  $ = $e59 // Inactivate[#, Plus] &,
  Yield, $ = $ /. $sD /. $ϕ /. $t44 // Activate;
  Yield, $sΦ1 = $ = MapAt[ArrayFlatten[#] &, $, 2]; MatrixForms[$]
 ];
```

●Lemma 5.5:
$\mathrm{Tr}[\Phi^2] \to 2\,c + 4\,a\,\mathrm{Abs}[H']^2$
$\mathrm{Tr}[\Phi^4] \to 2\,d + 8\,e\,\mathrm{Abs}[H']^2 + 4\,b\,\mathrm{Abs}[H']^4$
$H' \to \{1 + \phi_1, \phi_2\}$
$a \to \mathrm{Abs}[Y_e]^2 + \mathrm{Abs}[Y_\nu]^2$
$b \to \mathrm{Abs}[Y_e]^4 + \mathrm{Abs}[Y_\nu]^4$
$c \to \mathrm{Abs}[Y_R]^2$
$d \to \mathrm{Abs}[Y_R]^4$
$e \to \mathrm{Abs}[Y_R]^2\,\mathrm{Abs}[Y_\nu]^2$

⇒ $\{1 + \mathrm{Abs}[\phi_1]^2 + \mathrm{Abs}[\phi_2]^2 + (\phi_1)^* + \phi_1 \to \mathrm{Abs}[H']^2,\ t[\_]_{i\_,j\_} :\to 0 /;\ i \neq 1 \,||\, j \neq 1,\ t[\_]_{i,j} \to \text{GWS basis}\}$

$$T^*.T \to \begin{pmatrix} (t[R]_{1,1})^*\, t[R]_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (t[L]_{1,1})^*\, t[L]_{1,1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

→ $\{\{(t[R]_{1,1})^*\, t[R]_{1,1}, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, (t[L]_{1,1})^*\, t[L]_{1,1}, 0\}, \{0, 0, 0, 0\}\} \to \mathrm{Abs}[Y_R]^2$
→ $\mathrm{Abs}[t[L]_{1,1}]^2 + \mathrm{Abs}[t[R]_{1,1}]^2 \to \mathrm{Tr}[\mathrm{Abs}[Y_R]^2]$

Proof:
Use the 8x8 representation of: $\Phi \to \{\{S + \phi, T^*\}, \{T, S^* + \phi^*\}\}$
→

$$\to\ \Phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 & (t[R]_{1,1})^* & (t[R]_{1,2})^* & 0 \\ 0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^* + (Y_e)^*\,(\phi_1)^* & (t[R]_{1,2})^* & (t[R]_{2,2})^* & 0 \\ Y_\nu + (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 & 0 & 0 & (t[L]_{1,} \\ (\phi_2)^*\,Y_\nu & Y_e + Y_e\,\phi_1 & 0 & 0 & 0 & 0 & (t[L]_{1,} \\ t[R]_{1,1} & t[R]_{1,2} & 0 & 0 & 0 & 0 & Y_\nu + (\phi_1 \\ t[R]_{1,2} & t[R]_{2,2} & 0 & 0 & 0 & 0 & -Y_e\,\phi \\ 0 & 0 & t[L]_{1,1} & t[L]_{1,2} & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & -(Y_e\,\phi_2)^* & 0 \\ 0 & 0 & t[L]_{1,2} & t[L]_{2,2} & (Y_\nu)^*\,\phi_2 & (Y_e)^* + (Y_e\,\phi_1)^* & 0 \end{pmatrix}$$

```
PR["•Compute: ", $01 = $ = Inactive[Tr][Φ.Φ],
  Yield, $ = $ /. $sΦ1; MatrixForms[$]; "POFF",
  Yield, $ = $ // Activate // FullSimplify;
  Yield,
  $ = $01 -> $ //. $155x /. Re[x_] → (x + Conjugate[x]) / 2 //. tuTrSimplify[{Abs[_]}];
  $ = {$, $155[[-5 ;; -1]]} // Flatten;
  $ = tuEliminate[$, {Abs[Ye]^2, Abs[YR]^2}];
  $ = Apply[List, $] /. Equal → Rule /. tuRuleSolve[$155x[[1]], $155x[[1, 1, 2]]];
  "PON",
  $ = tuRuleSolve[$, Tr[Φ.Φ]] // First; Framed[$], OK
 ];
```

•Compute: Tr[Φ.Φ]

→ $\boxed{\mathrm{Tr}[\Phi.\Phi] \to 2\,(c + 2\,a\,\mathrm{Abs}[H']^2)}$ OK

```
$sexp = {Conjugate[a_ b_] → Conjugate[a] Conjugate[b], Abs[a_ b_] → Abs[a] Abs[b],
   a_ Conjugate[a_] → Abs[a]^2, a_^2 Conjugate[a_]^2 → Abs[a]^4}
PR["•In the same way Compute: ", $01 = $ = Inactive[Tr][Φ.Φ.Φ.Φ],
 Yield, $ = $ /. $sΦ1; MatrixForms[$];
 Yield, $ = $ // Activate;
 Yield, $ = Expand[$] //. $155x //. $sexp;
 Yield, $ = $01 -> $ //. $155x //. tuTrSimplify[{Abs[_]}] // Simplify;
 Yield, $ = $ /.
     turuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] /.
    tuRuleSolve[$155x[[1]], $155x[[1, 1, 2]]] // Simplify;
 Yield, $ = $ /. tuRuleSolve[$155x[[1]], $155x[[1, 1, 4]]] /.
    tuRuleSolve[$155x[[1]], $155x[[1, 1, 3]]] // Simplify;
 Yield, $ = $ /. (#^2 & /@ tuRuleSolve[$155x[[1]], $155x[[1, 1, 3]]][[1]] // Expand) /.
    $sexp // Simplify;
 Yield, $ = $ /. $sexp /. tuRuleSolve[$155x[[1]], $155x[[1, 1, 2]]];
 Yield, $ =
  $ /. tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] // Collect[#, Abs[H'], Simplify] &;
 ColumnSumExp[$];
 Yield, $ = $ /. tuRuleSolve[Expand[#^2 & /@ $155x[[-1]]], Abs[t[L]1,1^4]] // Collect[#,
     {Abs[H'], Tr[Abs[YR]^2], Conjugate[Yν], Yν, t[L]1,1, Abs[t[R]1,1]}, Simplify] &;
 ColumnSumExp[$] // Framed,
 NL, CR["There are extra terms: ",
  $[[2]] /. {Abs[H'] → 0, Tr[Abs[YR]^2] → 0} // ColumnSumExp,
  " unable to show that this is 0. Alternative calculation
    does not eliminate these terms. There are notational issues
    in the mixing of Tr[] and matrix notation in the text."]
]
```

{(a_b_)* → a* b*, Abs[a_b_] → Abs[a] Abs[b], a_* a_ → Abs[a]^2, a_^{*2} a_^{2} → Abs[a]^4}

•In the same way Compute: Tr[Φ.Φ.Φ.Φ]

→
→
→
→
→
→
→
→
→

→ Tr[Φ.Φ.Φ.Φ] → ∑[
-4 (2 Abs[Y_e]^2 Abs[ϕ_2]^2 – 2 Abs[Y_ν]^2 Abs[ϕ_2]^2 + Abs[t[L]$_{1,1}$]^2) Abs[t[R]$_{1,1}$]^2
4 (Abs[Y_e]^4 + Abs[Y_ν]^4) Abs[H′]^4
4 (1 + (ϕ_1)*)^2 (t[R]$_{1,1}$)* Y_ν^2 t[L]$_{1,1}$
4 (Y_ν)^{*2} (t[L]$_{1,1}$)* (1 + ϕ_1)^2 t[R]$_{1,1}$
8 (Abs[Y_e]^2 – Abs[Y_ν]^2) Abs[ϕ_2]^2 Tr[Abs[Y_R]^2]
8 Abs[Y_ν]^2 Abs[H′]^2 Tr[Abs[Y_R]^2]
2 Tr[Abs[Y_R]^2]^2
]

There are extra terms:

∑[
-4 (2 Abs[Y_e]^2 Abs[ϕ_2]^2 – 2 Abs[Y_ν]^2 Abs[ϕ_2]^2 + Abs[t[L]$_{1,1}$]^2) Abs[t[R]$_{1,1}$]^2
4 (1 + (ϕ_1)*)^2 (t[R]$_{1,1}$)* Y_ν^2 t[L]$_{1,1}$
4 (Y_ν)^{*2} (t[L]$_{1,1}$)* (1 + ϕ_1)^2 t[R]$_{1,1}$
]

unable to show that this is 0. Alternative calculation
does not eliminate these terms. There are notational issues
in the mixing of Tr[] and matrix notation in the text.

```
$s = {tuRuleSolve[$155x[[-1]], Abs[t[R]1,1]^2][[1]],
    #^2 & /@ tuRuleSolve[$155x[[-1]], Abs[t[R]1,1]^2][[1]]} // Expand;
$0 = $;
$1 = Select[$, ! FreeQ[#, t] &]; ColumnSumExp[$1];
$1 = Collect[$1, Abs[Yν]]; ColumnSumExp[$1];
$1 = $1 /. $s //. $sexp // Expand;
$1 = Collect[$1, {Conjugate[t[R]1,1], Conjugate[t[L]1,1], Tr[Abs[YR]^2]}];
ColumnSumExp[$1];
$0 = $1 + Select[$, FreeQ[#, t] &] // Expand;
ColumnSumExp[$0];

$1 = Select[$0, ! FreeQ[#, ϕ] &];
$1 = Collect[$1, {Abs[Yν], Abs[Ye]}, Simplify];
$1 =
  $1 /. Conjugate[a_] + a_ → 2 Re[a] /. tuRuleSolve[$155x[[1]], Re[ϕ1]][[1]] // Expand;
$1 = Collect[$1, {Abs[Yν], Abs[Ye]}, Simplify];
$1 = $1 /. tuRuleSolve[b -> $155a[b], $155a[b][[1]]] // Simplify;
$1 = Collect[$1, {Abs[Yν], b}, Simplify];
ColumnSumExp[$1];
$0 = $1 = $1 + Select[$0, FreeQ[#, ϕ] &] // Expand;

$1 = $1 /. Map[Expand[#^2] &, tuRuleSolve[$155x[[1]], Abs[ϕ2]^2][[1]]] /. tuRuleSolve[
      b -> $155a[b], $155a[b][[1]]] /. tuRuleSolve[a -> $155a[a], $155a[a][[1]]] /.
    tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] // Expand;
$s = Map[Expand[#^2] &, $155x[[-1]]];
$1 = $1 /. tuRuleSolve[$s, $s[[1, 1]]];
$1 = Collect[$1, {b, t[R]1,1, Conjugate[t[R]1,1], t[L]1,1,
    Conjugate[t[L]1,1], Tr[Abs[YR]^2], Yν^2, Abs[Yν^2], Abs[t[R]1,1], a}, Simplify];
ColumnSumExp[$1];

$1 = Select[$0, ! FreeQ[#, ϕ] &];
$1 = Collect[$1, {Abs[Yν], Abs[Ye], Abs[ϕ1], Abs[ϕ2]}];
$1 = $1 /. tuRuleSolve[b -> $155a[b], $155a[b][[1]]];
$1 = Collect[$1, {b, Abs[Yν], Abs[Ye], Abs[ϕ1], Abs[ϕ2]}];
$0 = $1 = $1 + Select[$0, FreeQ[#, ϕ] &] // Expand;
(*b coef*)
$1 = Select[$0, ! FreeQ[#, b] &] // Simplify
$1 = $1 /. Map[#^2 &, tuRuleSolve[$155x[[1]], Abs[ϕ2]^2][[1]]] /.
        Re[a_] → (a + Conjugate[a]) / 2 //. Conjugate[a_]^2 a_ → Abs[a]^2 Conjugate[a] /.
      Conjugate[a_] a_^2 → Abs[a]^2 a /. tuRuleSolve[$155x[[1]], Abs[ϕ2]^2] /.
    Conjugate[a : ϕ1] → 2 Re[a] - a // Simplify;
$0 = $1 = $1 + Select[$0, FreeQ[#, b] &] // Expand;
ColumnSumExp[$1]
```

$$4\,(b\,\mathrm{Abs}[\phi_1]^4 + b\,\mathrm{Abs}[\phi_2]^4 + 2\,b\,(\phi_1)^* + b\,(\phi_1)^{*2} + 2\,b\,\phi_1 + 2\,b\,(\phi_1)^{*2}\,\phi_1 + b\,\phi_1^2 + 2\,b\,(\phi_1)^*\,\phi_1^2 + 2\,(\phi_1)^*\,(t[R]_{1,1})^*\,Y_\nu^2\,t[L]_{1,1} + (\phi_1)^{*2}\,(t[R]_{1,1})^*\,Y_\nu^2\,t[L]_{1,1} + 2\,(Y_\nu)^{*2}\,(t[L]_{1,1})^*\,\phi_1\,t[R]_{1,1} + (Y_\nu)^{*2}\,(t[L]_{1,1})^*\,\phi_1^2\,t[R]_{1,1} + 4\,\mathrm{Abs}[Y_\nu]^2\,\mathrm{Re}[\phi_1]\,\mathrm{Tr}[\mathrm{Abs}[Y_R]^2] + 2\,\mathrm{Abs}[\phi_1]^2\,(2\,b + b\,\mathrm{Abs}[\phi_2]^2 + \mathrm{Abs}[Y_\nu]^2\,\mathrm{Tr}[\mathrm{Abs}[Y_R]^2]) + 2\,\mathrm{Abs}[\phi_2]^2\,(b + \mathrm{Abs}[Y_e]^2\,\mathrm{Abs}[t[L]_{1,1}]^2 + 2\,b\,\mathrm{Re}[\phi_1] + \mathrm{Abs}[Y_\nu]^2\,(-\mathrm{Abs}[t[L]_{1,1}]^2 + \mathrm{Tr}[\mathrm{Abs}[Y_R]^2])))\ /.\ \{\}[[1]]$$

```
    4 Abs[Ye]^4
    4 Abs[Yν]^4
    4 Abs[t[L]1,1]^4
    4 (b Abs[ϕ1]^4 + 4 b Re[ϕ1] + b (1 + Abs[ϕ1]^2 - Abs[H′]^2 + 2 Re[ϕ1])^2 + 2 b Abs[ϕ1]^2 (2 Re[ϕ1] - ϕ1) +
        2 b Abs[ϕ1]^2 ϕ1 + b ϕ1^2 + b (-2 Re[ϕ1] + ϕ1)^2 + 2 (t[R]1,1)^* Yν^2 (2 Re[ϕ1] - ϕ1) t[L]1,1 +
        (t[R]1,1)^* Yν^2 (-2 Re[ϕ1] + ϕ1)^2 t[L]1,1 + 2 (Yν)^*2 (t[L]1,1)^* ϕ1 t[R]1,1 +
        (Yν)^*2 (t[L]1,1)^* ϕ1^2 t[R]1,1 + 4 Abs[Yν]^2 Re[ϕ1] Tr[Abs[YR]^2] +
∑[      2 Abs[ϕ1]^2 (b - b Abs[ϕ1]^2 + b Abs[H′]^2 - 2 b Re[ϕ1] + Abs[Yν]^2 Tr[Abs[YR]^2]) +
        2 (-1 - Abs[ϕ1]^2 + Abs[H′]^2 - 2 Re[ϕ1]) (b + Abs[Ye]^2 Abs[t[L]1,1]^2 +
           2 b Re[ϕ1] + Abs[Yν]^2 (-Abs[t[L]1,1]^2 + Tr[Abs[YR]^2]))) /. {}〚1〛
    4 (t[R]1,1)^* Yν^2 t[L]1,1
    4 (Yν)^*2 (t[L]1,1)^* t[R]1,1
    8 Abs[Yν]^2 Tr[Abs[YR]^2]
    -4 Abs[t[L]1,1]^2 Tr[Abs[YR]^2]
    2 Tr[Abs[YR]^2]^2
```

```
"Some useful relationships"
$155a
tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]]
#^2 & /@ $155x[[-1]] // Expand
tuRuleSolve[Expand[#^2 & /@ $155x[[-1]]], a_^4 + b_^4]
$ = H' → $155a[H']
$h2 = Abs[H']^2 → Last[Thread[$ . Conjugate /@ $, Rule] // Expand]
$sh2 = tuRuleSolve[$h2, Conjugate[ϕ2] ϕ2][[1]]
$sh2 = {$sh2, Abs[ϕ2]^2 → $sh2[[1]]}
```

```
Some useful relationships

⟨|Tr[Φ^2] → 2 c + 4 a Abs[H′]^2, Tr[Φ^4] → 2 d + 8 e Abs[H′]^2 + 4 b Abs[H′]^4,
  H′ → {1 + ϕ1, ϕ2}, a → Abs[Ye]^2 + Abs[Yν]^2, b → Abs[Ye]^4 + Abs[Yν]^4,
  c → Abs[YR]^2, d → Abs[YR]^4, e → Abs[YR]^2 Abs[Yν]^2 |⟩

{Abs[t[L]1,1]^2 → -Abs[t[R]1,1]^2 + Tr[Abs[YR]^2]}

Abs[t[L]1,1]^4 + 2 Abs[t[L]1,1]^2 Abs[t[R]1,1]^2 + Abs[t[R]1,1]^4 → Tr[Abs[YR]^2]^2

{Abs[t[L]1,1]^4 + Abs[t[R]1,1]^4 → -2 Abs[t[L]1,1]^2 Abs[t[R]1,1]^2 + Tr[Abs[YR]^2]^2}

H′ → {1 + ϕ1, ϕ2}

Abs[H′]^2 → 1 + (ϕ1)^* + ϕ1 + (ϕ1)^* ϕ1 + (ϕ2)^* ϕ2

(ϕ2)^* ϕ2 → -1 + Abs[H′]^2 - (ϕ1)^* - ϕ1 - (ϕ1)^* ϕ1

{(ϕ2)^* ϕ2 → -1 + Abs[H′]^2 - (ϕ1)^* - ϕ1 - (ϕ1)^* ϕ1, Abs[ϕ2]^2 → (ϕ2)^* ϕ2}
```

Alternative computation

```
$sexp = {Conjugate[a_ b_] → Conjugate[a] Conjugate[b], Abs[a_ b_] → Abs[a] Abs[b],
    a_ Conjugate[a_] → Abs[a]^2, a_^2 Conjugate[a_]^2 → Abs[a]^4};
PR["•Compute: ", $01 = $ = Inactive[Tr][Φ.Φ.Φ.Φ],
   " Defining ", $s = {$$ = S + ϕ → Sϕ, ConjugateSimplify[Conjugate[$$], {}]},
   NL, "(5.9)", yield, $s = $e59 /. $s,
   Yield,
   $ = Φ.Φ.Φ.Φ /. Dot → xDot /. $s;
   $ = $ // OrderedxDotMultiplyAll[] // Simplify;
   MatrixForms[$ss = $],
   NL, "•Note that the [[1,1]] and [[2,2]] components are Conjugate[]s: ",
   $ss[[1, 1]] == ConjugateSimplify[Conjugate[$ss[[2, 2]]], {}] // Simplify,
   NL, "•Take Tr[]: ",
   $ = Tr[$] //. tuTrSimplify[],
   NL, "•The elements Sϕ,T are 4x4 matrices so each term needs to be Tr[]d: ",
   $ = Tr[$] //. tuTrSimplify[],
   NL, "•Use cyclic permutation equivalance
     of Tr[], T is a symmetric matrix, and Sϕ is hermitian: ",

   $sTr = {Tr[a_] :→ Tr[Transpose[a]] /; FreeQ[Transpose[a], Transpose],
     T.Conjugate[T] -> Conjugate[T].T,
     Conjugate[Sϕ] → Transpose[Sϕ], Tr[Transpose[a_]] → Tr[a],
     Tr[a_] :→ Tr[Transpose[a]] /; Count[tuExtractPattern[Transpose][a], Transpose] ≥ 2,
     Transpose[T] → T, Transpose[Conjugate[T]] → Conjugate[T]},
   Yield, $tr4 = $ // tuRepeat[{tt : Tr[a_] :> tuTrCanonicalOrder[tt], $sTr}],
   line
  ];
```

```
•Compute: Tr[Φ.Φ.Φ.Φ] Defining {S + ϕ → Sϕ, S* + ϕ* → Sϕ*}
(5.9) ⟶ Φ → {{Sϕ, T*}, {T, Sϕ*}}
→ ( Sϕ.Sϕ.Sϕ.Sϕ + Sϕ.Sϕ.T*.T + Sϕ.T*.T.Sϕ + Sϕ.T*.Sϕ*.T + T*.T.Sϕ.Sϕ + T*.T.T*.T + T*.Sϕ*.T.Sϕ + T*.Sϕ*.Sϕ*
    T.Sϕ.Sϕ.Sϕ + T.Sϕ.T*.T + T.T*.T.Sϕ + T.T*.Sϕ*.T + Sϕ*.T.Sϕ.Sϕ + Sϕ*.T.T*.T + Sϕ*.Sϕ*.T.Sϕ + Sϕ*.Sϕ*.Sϕ*
•Note that the [[1,1]] and [[2,2]] components are Conjugate[]s: True
•Take Tr[]: Sϕ.Sϕ.Sϕ.Sϕ + Sϕ.Sϕ.T*.T + Sϕ.T*.T.Sϕ + Sϕ.T*.Sϕ*.T +
  T.Sϕ.Sϕ.T* + T.Sϕ.T*.Sϕ* + T.T*.T.T* + T.T*.Sϕ*.Sϕ* + Sϕ*.T.Sϕ.T* + Sϕ*.T.T*.Sϕ* +
  Sϕ*.Sϕ*.T.T* + Sϕ*.Sϕ*.Sϕ*.Sϕ* + T*.T.Sϕ.Sϕ + T*.T.T*.T + T*.Sϕ*.T.Sϕ + T*.Sϕ*.Sϕ*.T
•The elements Sϕ,T are 4x4 matrices so each term needs to be Tr[]d:
 Tr[Sϕ.Sϕ.Sϕ.Sϕ] + Tr[Sϕ.Sϕ.T*.T] + Tr[Sϕ.T*.T.Sϕ] + Tr[Sϕ.T*.Sϕ*.T] +
  Tr[T.Sϕ.Sϕ.T*] + Tr[T.Sϕ.T*.Sϕ*] + Tr[T.T*.T.T*] + Tr[T.T*.Sϕ*.Sϕ*] +
  Tr[Sϕ*.T.Sϕ.T*] + Tr[Sϕ*.T.T*.Sϕ*] + Tr[Sϕ*.Sϕ*.T.T*] + Tr[Sϕ*.Sϕ*.Sϕ*.Sϕ*] +
  Tr[T*.T.Sϕ.Sϕ] + Tr[T*.T.T*.T] + Tr[T*.Sϕ*.T.Sϕ] + Tr[T*.Sϕ*.Sϕ*.T]
•Use cyclic permutation equivalance of Tr[], T is
   a symmetric matrix, and Sϕ is hermitian:
 {Tr[a_] :→ Tr[a^T] /; FreeQ[a^T, Transpose], T.T* → T*.T, Sϕ* → Sϕ^T, Tr[a_^T] → Tr[a],
  Tr[a_] :→ Tr[a^T] /; Count[tuExtractPattern[Transpose][a], Transpose] ≥ 2, T^T → T, T*^T → T*}
→ 2 Tr[Sϕ.Sϕ.Sϕ.Sϕ] + 8 Tr[Sϕ.Sϕ.T*.T] + 4 Tr[Sϕ.T*.Sϕ^T.T] + 2 Tr[T*.T.T.T*]
```

```
PR["●Useful relationships: ",
 $ = $sΦ1; MatrixForms[$];
]
ConjugateTranspose[$] == $ // ConjugateCTSimplify1[{}];
$S = Sϕ -> $sΦ1[[2, 1 ;; 4, 1 ;; 4]]; MatrixForms[$S]
$T = ($T0 = T -> $sΦ1[[2, 5 ;; 8, 1 ;; 4]]) /. t[R | L]i_,j_ :→ 0 /; i ≠ 1 || j ≠ 1;
MatrixForms[$T]

PR[
 NL, "Evaluate each term: ",
 NL, "■ ", $ = $tr4[[1]],
 yield, $ = $ /. $S // Simplify,
 yield, $ = $ /. (Conjugate[a_] a_)^2 → Abs[a]^4 /. $155a[b] → b;
 Yield, $ = $ /. $sh2 // Simplify;
 Framed[$], OK,

 NL, "■ ", $ = $tr4[[2]],
 yield, $ = $ /. {$S, $T} // Simplify,
 Yield, $ = $ //. $sexp // Simplify,
 Yield, $ = $ /. tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] // Simplify;
 Framed[$],

 NL, "■ ", $ = $tr4[[3]],
 yield, $ = $ /. {$S, $T} // Simplify,
 Yield, $ = $ //. $sexp // Simplify, (*
 Yield,$=$/.tuRuleSolve[$155x[[-1]],$155x[[-1,1,1]]]//Simplify;*)Framed[$],

 NL, "■ ", $ = $tr4[[4]],
 yield, $ = $ /. {$S, $T} // Simplify,
 Yield, xtmp = $ = $ //. $sexp //. $sh2 // Simplify; Framed[$],

 NL, "■ ", $ = $tr4[[4]],
 yield, $ = $ /. {$S, $T} // Simplify;
 Yield, $ = $ //. $sexp // Simplify,
 Yield, $ = $ /. tuRuleSolve[Expand[#^2 & /@ $155x[[-1]]], a_^4 + b_^4] // Simplify;
 Framed[$]
]
```

●Useful relationships: Null

$$S\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 \\ 0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^* + (Y_e)^*\,(\phi_1)^* \\ Y_\nu + (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 \\ (\phi_2)^*\,Y_\nu & Y_e + Y_e\,\phi_1 & 0 & 0 \end{pmatrix}$$

$$T \to \begin{pmatrix} t[R]_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t[L]_{1,1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Evaluate each term:

- $2\,\mathrm{Tr}[S\phi.S\phi.S\phi.S\phi] \longrightarrow 4\,((Y_e)^{*2}\,Y_e^2 + (Y_\nu)^{*2}\,Y_\nu^2)\,(1+\phi_1+(\phi_1)^*\,(1+\phi_1)+(\phi_2)^*\,\phi_2)^2 \longrightarrow$

$\rightarrow \boxed{4\,b\,\mathrm{Abs}[H']^4}$ OK

- $8\,\mathrm{Tr}[S\phi.S\phi.T^*.T] \longrightarrow 8\,((t[L]_{1,1})^*\,((Y_\nu)^*\,(1+(\phi_1)^*)\,Y_\nu\,(1+\phi_1)+(Y_e)^*\,(\phi_2)^*\,Y_e\,\phi_2)\,t[L]_{1,1} + (Y_\nu)^*\,(t[R]_{1,1})^*\,Y_\nu\,(1+\phi_1+(\phi_1)^*\,(1+\phi_1)+(\phi_2)^*\,\phi_2)\,t[R]_{1,1})$

$\rightarrow 8\,(\mathrm{Abs}[Y_\nu]^2\,\mathrm{Abs}[t[R]_{1,1}]^2\,(\mathrm{Abs}[\phi_2]^2+(1+(\phi_1)^*)\,(1+\phi_1)) + \mathrm{Abs}[t[L]_{1,1}]^2\,(\mathrm{Abs}[Y_e]^2\,\mathrm{Abs}[\phi_2]^2+\mathrm{Abs}[Y_\nu]^2\,(1+(\phi_1)^*)\,(1+\phi_1)))$

$\rightarrow \boxed{8\,(\mathrm{Abs}[Y_e]^2\,\mathrm{Abs}[\phi_2]^2\,(-\mathrm{Abs}[t[R]_{1,1}]^2+\mathrm{Tr}[\mathrm{Abs}[Y_R]^2]) + \mathrm{Abs}[Y_\nu]^2\,(\mathrm{Abs}[\phi_2]^2\,\mathrm{Abs}[t[R]_{1,1}]^2+(1+(\phi_1)^*)\,(1+\phi_1)\,\mathrm{Tr}[\mathrm{Abs}[Y_R]^2]))}$

- $4\,\mathrm{Tr}[S\phi.T^*.S\phi^T.T] \longrightarrow 4\,((1+(\phi_1)^*)^2\,(t[R]_{1,1})^*\,Y_\nu^2\,t[L]_{1,1}+(Y_\nu)^{*2}\,(t[L]_{1,1})^*\,(1+\phi_1)^2\,t[R]_{1,1})$

$\rightarrow 4\,((1+(\phi_1)^*)^2\,(t[R]_{1,1})^*\,Y_\nu^2\,t[L]_{1,1}+(Y_\nu)^{*2}\,(t[L]_{1,1})^*\,(1+\phi_1)^2\,t[R]_{1,1})$

$\boxed{4\,((1+(\phi_1)^*)^2\,(t[R]_{1,1})^*\,Y_\nu^2\,t[L]_{1,1}+(Y_\nu)^{*2}\,(t[L]_{1,1})^*\,(1+\phi_1)^2\,t[R]_{1,1})}$

- $2\,\mathrm{Tr}[T^*.T.T.T^*] \longrightarrow 2\,((t[L]_{1,1})^{*2}\,t[L]_{1,1}^2+(t[R]_{1,1})^{*2}\,t[R]_{1,1}^2)$

$\rightarrow \boxed{2\,(\mathrm{Abs}[t[L]_{1,1}]^4+\mathrm{Abs}[t[R]_{1,1}]^4)}$

- $2\,\mathrm{Tr}[T^*.T.T.T^*] \longrightarrow$

$\rightarrow 2\,(\mathrm{Abs}[t[L]_{1,1}]^4+\mathrm{Abs}[t[R]_{1,1}]^4)$

$\rightarrow \boxed{2\,(-2\,\mathrm{Abs}[t[L]_{1,1}]^2\,\mathrm{Abs}[t[R]_{1,1}]^2+\mathrm{Tr}[\mathrm{Abs}[Y_R]^2]^2)}$

Lemma 5.6

```
PR["●Lemma 5.6. ",
 $156 = $ = {Tr[tuDDown[𝒟][Φ, μ] tuDUp[𝒟][Φ, μ]] → 4 a Abs[tuDDown[𝒟̃][H', μ]]^2,
    tuDDown[𝒟̃][H', μ] →
     tuDDown["∂"][H', μ] + I T[Q, "ud", {a, μ}] T[σ, "u", {a}] H' - I T[Λ, "d", {μ}] H',
    $e31 = tuDDown[𝒟][Φ, μ] → tuDPartial[Φ, μ] + I CommutatorM[T[B, "d", {μ}], Φ],
    $e59
   }; Column[$],
 NL, back, "From ", $D1,
 NL, "•Calculate ", $ = $156[[3, 2, 1]], "POFF",
 NL, "Use: ", $s = {$sΦ1, $e58}; MatrixForms[$s],
 Yield, $part[1] = $ = $ /. $s /. CommutatorM → MCommutator // Simplify;
 MatrixForms[$],
 "PON",
 NL, "•Calculate ", $ = $156[[3, 2, 2]],
 Yield, $ = $ /. $s //. tt : tuDDown["∂"][a_, b_] :→ Thread[tt] /. tuDDown["∂"][0, _] → 0;
 Yield, $part[2] = $ = $ /. tuOpDistribute[tuDDown["∂"]] /.
     tuOpSimplify[tuDDown["∂"]] // tuDerivativeExpand[{}];
 NL, "Summing: ", $ = $part[1] + $part[2] // Simplify; MatrixForms[$]
]
```

●Lemma 5.6.
$\mathrm{Tr}[\underline{\mathcal{D}}_{\mu}[\Phi]\,\underline{\mathcal{D}}^{\mu}[\Phi]] \to 4\,a\,\mathrm{Abs}[\underline{\tilde{\mathcal{D}}}_{\mu}[H']]^2$
$\underline{\tilde{\mathcal{D}}}_{\mu}[H'] \to -i\,\Lambda_{\mu}\,H' + i\,Q^{a}{}_{\mu}\,\sigma^{a}\,H' + \underline{\partial}_{\mu}[H']$
$\underline{\mathcal{D}}_{\mu}[\Phi] \to i\,[B_{\mu}, \Phi] + \underline{\partial}_{\mu}[\Phi]$
$\Phi \to \{\{S+\phi, T^*\}, \{T, S^*+\phi^*\}\}$

⟵From $\mathcal{D}_{\mu}[\Phi] \to -i\,1_N\otimes\Phi.B_{\mu} + i\,1_N\otimes B_{\mu}.\Phi + 1_N\otimes\nabla^{S}{}_{\mu}[\Phi]$
•Calculate $i\,[B_{\mu}, \Phi]$
•Calculate $\underline{\partial}_{\mu}[\Phi]$
→
→

Summing: (
0
0
$\underline{\partial}_{\mu}[\phi_1]^*\,Y_{\nu} + i\,Y_{\nu}\,((\phi_2)^*\,q_{\mu 1,2} + (1+(\phi_1)^*)\,(q_{\mu 1,1} - \Lambda_{\mu})) + \underline{\partial}_{\mu}[Y_{\nu}] + (\phi_1)^*\,\underline{\partial}_{\mu}[Y_{\nu}]$ $\quad i\,Y_e\,((1+\phi_1)\,q$
$\underline{\partial}_{\mu}[\phi_2]^*\,Y_{\nu} + i\,Y_{\nu}\,((1+(\phi_1)^*)\,q_{\mu 2,1} + (\phi_2)^*\,(q_{\mu 2,2} - \Lambda_{\mu})) + (\phi_2)^*\,\underline{\partial}_{\mu}[Y_{\nu}]$ $\quad -i\,Y_e\,(\phi_2\,q_{\mu 2,1} - (1$
$\underline{\partial}_{\mu}[t[R]_{1,1}]$
$2\,i\,t[R]_{1,2}\,\Lambda_{\mu} + \underline{\partial}_{\mu}[t[R]_{1,2}]$
0
0

```
T[B, "d", {μ}]
```

B_{μ}

```
$D1
$sΦ1
$l55a
$e57
$e58
$e59
```

$\mathcal{D}_\mu[\Phi] \to -\mathbb{i}\, 1_N \otimes \Phi . B_\mu + \mathbb{i}\, 1_N \otimes B_\mu . \Phi + 1_N \otimes \nabla^S{}_\mu[\Phi]$

$\Phi \to \{\{0, 0, (Y_\nu)^* + (Y_\nu)^* \phi_1, (Y_\nu)^* \phi_2, (t[R]_{1,1})^*, (t[R]_{1,2})^*, 0, 0\},$
$\{0, 0, -(Y_e)^* (\phi_2)^*, (Y_e)^* + (Y_e)^* (\phi_1)^*, (t[R]_{1,2})^*, (t[R]_{2,2})^*, 0, 0\},$
$\{Y_\nu + (\phi_1)^* Y_\nu, -Y_e \phi_2, 0, 0, 0, 0, (t[L]_{1,1})^*, (t[L]_{1,2})^*\},$
$\{(\phi_2)^* Y_\nu, Y_e + Y_e \phi_1, 0, 0, 0, 0, (t[L]_{1,2})^*, (t[L]_{2,2})^*\},$
$\{t[R]_{1,1}, t[R]_{1,2}, 0, 0, 0, 0, Y_\nu + (\phi_1)^* Y_\nu, (\phi_2)^* Y_\nu\},$
$\{t[R]_{1,2}, t[R]_{2,2}, 0, 0, 0, 0, -Y_e \phi_2, Y_e + Y_e \phi_1\},$
$\{0, 0, t[L]_{1,1}, t[L]_{1,2}, (Y_\nu)^* + (Y_\nu)^* \phi_1, -(Y_e \phi_2)^*, 0, 0\},$
$\{0, 0, t[L]_{1,2}, t[L]_{2,2}, (Y_\nu)^* \phi_2, (Y_e)^* + (Y_e \phi_1)^*, 0, 0\}\}$

$\langle| \mathrm{Tr}[\Phi^2] \to 2 c + 4 a \,\mathrm{Abs}[H']^2, \mathrm{Tr}[\Phi^4] \to 2 d + 8 e \,\mathrm{Abs}[H']^2 + 4 b \,\mathrm{Abs}[H']^4,$
$H' \to \{1 + \phi_1, \phi_2\}, a \to \mathrm{Abs}[Y_e]^2 + \mathrm{Abs}[Y_\nu]^2, b \to \mathrm{Abs}[Y_e]^4 + \mathrm{Abs}[Y_\nu]^4,$
$c \to \mathrm{Abs}[Y_R]^2, d \to \mathrm{Abs}[Y_R]^4, e \to \mathrm{Abs}[Y_R]^2 \mathrm{Abs}[Y_\nu]^2 |\rangle$

$\{\mathcal{A}_\mu \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\}, Q_\mu \to \mathbb{i} \sum_{\{i,0,3\}} [q_i \sigma^i],$
$q_i \in \mathbb{R}, \Lambda_\mu \in \mathbb{R}, \phi \to \{\{0, Y^*\}, \{Y, 0\}\}, \phi \to \{\{0, 0, (Y_\nu)^* \phi_1, (Y_\nu)^* \phi_2\},$
$\{0, 0, -(Y_e)^* (\phi_2)^*, (Y_e)^* (\phi_1)^*\}, \{(\phi_1)^* Y_\nu, -Y_e \phi_2, 0, 0\}, \{(\phi_2)^* Y_\nu, Y_e \phi_1, 0, 0\}\},$
$\phi_1 \to \lambda (\alpha' - \lambda') \to \alpha^* (-(\alpha')^* + (\lambda')^*) + \beta^* \beta', \phi_2 \to \lambda \beta' \to \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta',$
$B_{\mu\mathcal{H}_\ell} \to \{\{0, 0, 0\}, \{0, -2 \Lambda_\mu, 0\}, \{0, 0, Q_\mu - 1_2 \Lambda_\mu\}\},$
$B_{\mu\mathcal{H}_{\bar{\ell}}} \to \{\{0, 0, 0\}, \{0, 2 \Lambda_\mu, 0\}, \{0, 0, -(Q_\mu)^* - 1_2 \Lambda_\mu\}\}\}$

$B_\mu \to \{\{0, 0, 0, 0, 0, 0, 0, 0\}, \{0, -2 \Lambda_\mu, 0, 0, 0, 0, 0, 0\}, \{0, 0, q_{\mu_{1,1}} - \Lambda_\mu, q_{\mu_{1,2}}, 0, 0, 0, 0\},$
$\{0, 0, q_{\mu_{2,1}}, q_{\mu_{2,2}} - \Lambda_\mu, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 2 \Lambda_\mu, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, -(q_{\mu_{1,1}})^* + \Lambda_\mu, -(q_{\mu_{1,2}})^*\}, \{0, 0, 0, 0, 0, 0, -(q_{\mu_{2,1}})^*, -(q_{\mu_{2,2}})^* + \Lambda_\mu\}\}$

$\Phi \to \{\{S + \phi, T^*\}, \{T, S^* + \phi^*\}\}$