Medium/Hard

Summing over spins: one loop free energy of Vasiliev theory in AdS_4

Consider the Vasiliev theory in AdS_{d+1} whose spectrum consists of a scalar field of mass $m^2 = -2(d-2)$ (in units of AdS radius) plus a tower of totally symmetric higher spin gauge fields of spin $s = 1, 2, 3, \ldots$. This is conjecturally dual to the $U(N)$ vector model in the singlet sector. We can also consider the "minimal" Vasiliev theory where the odd spin fields are truncated out: this is dual to the $O(N)$ vector model in the singlet sector. The dual vector model is free if we assign $\Delta = d-2$ boundary condition to the bulk scalar, and interacting if we assign $\Delta = 2$ (this choice is unitary for $d \leq 6$).

The one-loop partition function of the theory in Euclidean AdS (hyperbolic space) is given by the product of determinants

$$Z_{1-\text{loop}} = \frac{1}{\left[\det_{s=0}\left(-\nabla^2 - 2(d-2)\right)\right]^{\frac{1}{2}}} \prod_{s\geq 1} \frac{\left[\det_{s-1}^{STT}\left(-\nabla^2 + (s+d-2)(s-1)\right)\right]^{\frac{1}{2}}}{\left[\det_{s}^{STT}\left(-\nabla^2 + (s+d-2)(s-2) - s\right)\right]^{\frac{1}{2}}} \tag{1}$$

where the numerator factor arises upon gauge fixing the linearized gauge symmetry, and the label "STT" means that the determinant is taken on the space of symmetric traceless transverse tensor fields. The one-loop free energy $F^{(1)}$ is defined in the usual way

$$F^{(1)} = -\log Z_{1-\text{loop}}\,.$$

These determinants can be computed by heat kernel techniques, or equivalently by using spectral ζ-functions (a Mellin transform of the heat kernel). The spectral ζ function for STT fields of any spin with kinetic operator $(-\nabla^2 + \kappa^2)$, with κ^2 being a constant, was computed by Camporesi and Higuchi in the '90's. It takes the following form

$$\begin{aligned}
&\zeta_{(\Delta,s)}(z) = \frac{\text{vol}(\text{AdS}_{d+1})}{\text{vol}(S^d)} \frac{2^{d-1}}{\pi} g_d(s) \int_0^\infty du\, \frac{\mu_s(u)}{\left[u^2+\nu^2\right]^z}\,,\\
&\nu \equiv \Delta - \frac{d}{2}\,, \qquad \Delta(\Delta-d) - s = \kappa^2\,, \qquad g_d(s) = \frac{(2s+d-2)(s+d-3)!}{(d-2)!s!}\,.
\end{aligned} \tag{2}$$

with $\mu_s(u)$ the spin s "spectral density", which takes the form

$$\mu_s(u) = \frac{\pi\left(\left(\frac{d-2}{2}+s\right)^2 + u^2\right)}{\left(2^{d-1}\Gamma\left(\frac{d+1}{2}\right)\right)^2} \left|\frac{\Gamma\left(\frac{d-2}{2}+iu\right)}{\Gamma(iu)}\right|^2\,. \tag{3}$$

Here Δ is of course nothing but the conformal dimension of the dual operator in the CFT_d. The factor $\text{vol}(S^d) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$ is the volume of the unit d-sphere, and $\text{vol}(\text{AdS}_{d+1}) = \pi^{d/2}\Gamma\left(-\frac{d}{2}\right)$ is the dimensionally regularized volume of hyperbolic $d{+}1$-space (if these volume formulae are unfamiliar, try to derive them. The metric of hyperbolic space can be taken to be $ds^2 = \frac{dr^2 + r^2(1-r^2)d\Omega_d}{(1-r^2)^2}$, $0 < r < 1$). Once the spectral ζ-function is known, the functional determinants are obtained as

$$\frac{1}{2}\log \det_{STT_s}(-\nabla^2+\kappa^2) = -\frac{1}{2}\zeta_{(\Delta,s)}(0)\log\Lambda^2 - \frac{1}{2}\zeta'_{(\Delta,s)}(0) \tag{4}$$

where $\zeta'_{(\Delta,s)}(0) = \frac{d}{dz}\zeta_{(\Delta,s)}(z)|_{z=0}$. Here Λ is a UV cutoff, and so each determinant contain a UV logarithmic divergence (this can only appear in even dimensional spacetime, i.e. odd d, namely $\zeta(0)$ vanishes identically in even d).

I. Specializing to AdS_4 $(d = 3)$

In this problem we specialize to the case $d = 3$, i.e. higher spin theory on AdS_4. As a first step, write down in your Mathematica notebook the explicit form of $\zeta(z)$ and specialize to $d = 3$. Show that, given (1 and the definition of κ in (2) above, the conformal dimensions are $\Delta = s + 1$ for the physical STT fields, and $\Delta = s + 2$ for the spin $s - 1$ ghost numerator factor, while for the scalar there are two roots above unitarity: $\Delta = 1$ or $\Delta = 2$. Below we will assume $\Delta = 1$.

II. Cancellation of the UV logarithmic divergence

a) Derive a formula for $\zeta_{(\Delta,s)}(0)$, the coefficient of the logarithmic divergence for a given Δ, s. This should be evaluated by analytic continuation in z. It will be convenient to use that

$$\tanh(\pi u) = 1 - \frac{2}{e^{2\pi u} + 1}$$

and split the u-integral in a term that manifestly converges at $z = 0$, plus a term that can be evaluated at general z and analytically continued to $z = 0$.

b) Given the result in a), derive a formula for the coefficient of the logarithmic divergence in Vasiliev theory in AdS_4

$$F^{(1)}_{\log-\mathrm{div}} = -\frac{1}{2}\zeta_{(\Delta=1,s=0)}(0) - \frac{1}{2}\sum_{s\geq 1}\left(\zeta_{(\Delta=s+1,s)}(0) - \zeta_{(\Delta=s+2,s-1)}(0)\right) \tag{5}$$

Show that, adopting Riemann zeta-function regularization of the sum over spins, this $F^{(1)}_{\log-\mathrm{div}}$ vanishes. Show that this is true both in non-minimal and minimal theories, and regardless of boundary conditions on the scalar field. A convenient way to implement the zeta-function regularization is to insert a regulating factor $e^{-\epsilon s}$ into the sum, and extract the ϵ^0 term in the finite result. This agrees with the analytic continuation of the Riemann zeta-function. (You can use 'SeriesCoefficient' to extract a given term in a series expansion).

III. Evaluating the finite part

The finite part is given by $\zeta'(0)$ and is well defined given that the UV logarithmic term vanishes (in the regularization of the sum over spins described above). It can be shown, starting from the expression of $\zeta_{(\Delta,s)}(z)$ in $d = 3$ (AdS_4) that you derived in part I, that

$$\zeta'_{(\Delta,s)}(0) = \frac{1}{3}(2s+1)\left[\frac{\nu^4}{8} + \frac{\nu^2}{48} + c_1 + \left(s+\frac{1}{2}\right)^2 c_2 + \int_0^\nu dx\left[\left(s+\frac{1}{2}\right)^2 x - x^3\right]\psi(x+\frac{1}{2})\right] \tag{6}$$

where $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ is the digamma function, and c_1, c_2 are constants given by the integrals

$$c_1 = \int_0^\infty du\frac{u^3\log u^2}{e^{2\pi u}+1}\,, \qquad c_2 = \int_0^\infty du\frac{u\log u^2}{e^{2\pi u}+1}\,. \tag{7}$$

You can assume this result and proceed. If you are interested and have time, you can try to derive it. To do so, it is useful to write $\log(u^2+\nu^2) = \log u^2 + \int_0^{\nu^2} dx \frac{1}{u^2+x}$, and use the following integrals

$$\begin{aligned}\int_0^\infty du \frac{u}{(e^{2\pi u}+1)(u^2+x)} &= \frac{1}{2}\psi(\frac{1}{2}+\sqrt{x}) - \frac{1}{2}\log\sqrt{x} \\ \int_0^\infty du \frac{u^3}{(e^{2\pi u}+1)(u^2+x)} &= \frac{1}{48} - \frac{x}{2}\psi(\frac{1}{2}+\sqrt{x}) + \frac{x}{2}\log\sqrt{x}\end{aligned} \tag{8}$$

To be precise, the derivation of the expression (6) is literally correct assuming $\nu > 0$. This is the case for all fields except $s = 0$, for which if we choose $\Delta = 1$, then $\nu = -1/2$. The prescription that gives the correct result for the $\Delta = 1$ scalar, which agrees with other independent methods, is to still use (6) as it is also for $s = 0, \nu = -1/2$.

Let us define for the rest of the problem

$$\mathcal{I}(\nu, s) = \frac{1}{3}(2s+1)\int_0^\nu dx \left[\left(s+\frac{1}{2}\right)^2 x - x^3\right]\psi(x+\frac{1}{2})\,. \tag{9}$$

a) Using Riemann zeta function regularization as in part I above, show that all terms in $\zeta'_{(\Delta,s)}(0)$ except the term including the digamma function cancel when summing over all fields. In other words, the full one-loop free energy is given by

$$F^{(1)} = -\frac{1}{2}\mathcal{I}\left(-\frac{1}{2},0\right) - \frac{1}{2}\sum_{s\geq 1}^{\infty}\left[\mathcal{I}\left(s-\frac{1}{2},s\right) - \mathcal{I}\left(s+\frac{1}{2},s-1\right)\right] \tag{10}$$

Analytically evaluate the $\Delta = 1$ scalar contribution $-\frac{1}{2}\mathcal{I}\left(-\frac{1}{2},0\right)$.

b) Now turn to the sum over spins. To do this, it is convenient to introduce the following representation of the digamma function

$$\psi(y) = \int_0^\infty dt \left(\frac{e^{-t}}{t} - \frac{e^{-yt}}{1-e^{-t}}\right)\,. \tag{11}$$

Insert this into the integral, evaluate the x-integral, and then sum over all spins with the zeta function regulator. This will give you a function of t that we now have to integrate. Expanding the result close to $t = 0$, show that there are only power-like divergences and no $1/t$ term which would give a logarithmic divergence (if you find such term, something went wrong in your calculation).

c) Try first to evaluate the t-integral numerically. You will have to subtract out explicitly the power-like divergent pieces close to $t = 0$ (these can be analytically continued to zero). After summing the result to the scalar contribution, show that this numerical procedure gives results consistent with

$$F^{(1)} = 0 \tag{12}$$

in the theory including all integer spins, and

$$F^{(1)}_{\rm min} = \frac{\log 2}{8} - \frac{3\zeta_3}{16\pi^2} = 0.0638071... \tag{13}$$

in the minimal theory with even spins only.[1] The number $\frac{\log 2}{8} - \frac{3\zeta_3}{16\pi^2}$ appearing here is equal to the S^3

[1] Not directly needed for this problem, but a useful tool to keep in mind is the EZ-face website http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi. You can use this for example to find the coefficients multiplying $\log 2$ and ζ_3/π^2 in a given numerical result if you know it was a combination of those. For instance, if you type lindep[(0.0638071,ln(2),z(3)/Pi^2)] you will find that the coefficients are 1/8 and 3/16 as in our answer above. For fun, you can try the same with the number '0.02594472885413405': assuming you guess that it is equal to $a\log 2 + b\zeta_3/\pi^2$, what are a and b?

free energy of a single 3d conformally coupled free scalar field. This result was interpreted as evidence that in the minimal Vasiliev theory the bulk coupling constant is $G_N \sim \frac{1}{N-1}$ rather than the naive $G_N \sim 1/N$.

d) Now try to evaluate the t-integral analytically (this is a bit harder: optional). To do this, it is convenient to use the integral representation of the Hurwitz-Lerch function

$$\Phi(z, a, v) = \frac{1}{\Gamma(a)} \int_0^\infty dt \frac{t^{a-1} e^{-vt}}{1 - ze^{-t}} , \tag{14}$$

which is implemented in Mathematica as 'HurwitzLerchPhi[z, a, v]'. Each term in the function of t to be integrated can be related to $\Phi(z, s, v)$ and its derivatives in z at $z = 1$. Insert a regulating factor $1/t^\epsilon$ in the t-integral, and represent each term in the t-integral in terms of $\Phi(z, a, v)$, which can be analytically continued to the the relevant values of z, a, v at the end of the calculation (when you send ϵ to zero). The function of t is quite long, so you should implement such substitutions automatically term by term. At the end, you may need to force Mathematica to implement the identity $\Phi(z = 1, a, v) = \zeta(a, v)$, where $\zeta(a, v)$ is the Hurwitz zeta function (implemented in Mathematica as 'HurwitzZeta[a,v]'). Try do this calculation both for non-minimal and minimal theory, and so verify analytically the results of question c). (The procedure described here is just a suggestion: if you have better ideas on how to do it, feel free to explore other ways!).