```
<< Local`QFTToolKit`
Put[SaveFile = NBname["stub"] <> ".out"]
```

TensorFormRoutines must be run first by opening TensorialForms.m

```
DefineTensor[α, μ, ν, T, y, x, V, u, U]

PR1["Pullback definition and notation: ",
  ea1 = ϕ*[f] -> f∘ϕ, " explicitly",
  yield, ϕ*[f][p ∈ N] -> f∘ϕ[p ∈ N],
  Yield, ea9 = T[ϕ*[T], "d"][T[μ, "d"]["{i,imax}"]] -> Product[
      xPartialD[y@u[α@d[i]], x@u[μ@d[i]]], {i, imax}] T[T, "d"][α@d["{i,imax}"]],
  NL, "Pushforward (of vectors) definition and notation: ",
  Yield,
  ea2 = ϕ*[V][f] -> V[ea1[[1]]],
  Yield, tmp = ea2 /. ea1,
  Yield, tmp = tmp /. V[a_] -> V@u[μ] xPartialD[a, μ],
  Yield, tmp = tmp /. xPartialD[f ∘ a_, μ] -> xPartialD[e@u[α].a, μ] . xPartialD[f, α],
  Yield, tmp = tmp /. ϕ*[V][f] -> T[ϕ*[V], "u"][α] xPartialD[f, α],
  Yield, tmp = tmp /. xPartialD[f, α] -> 1 /. simpleDot2[{}],
  Yield, tmp = tmp /. T[ϕ*[V], "u"][α] -> T[ϕ*, "ud"][α, μ] V@u[μ],
  Yield, tmp = tmp /. V@u[μ] -> 1,
  " ⟵ matrix definition of pushforward (A.4). ",
  Yield, ea10 = T[ϕ*[S], "u"][T[α, "d"]["{i,imax}"]] -> Product[
      xPartialD[y@u[α@d[i]], x@u[μ@d[i]]], {i, imax}] T[S, "u"][μ@d["{i,imax}"]]
 ];
```

Pullback definition and notation: $\phi^*[f]\to f\circ\phi$ explicitly $\longrightarrow$ $\phi^*[f][p\in N]\to f\circ\phi[p\in N]$

$\to\ \phi^*[T]_{\mu_{\{i,imax\}}}\to\left(\prod_i^{imax}\partial_{x^{\mu_i}}\left[y^{\alpha_i}\right]\right)T_{\alpha_{\{i,imax\}}}$

Pushforward (of vectors) definition and notation:

$\to\ \phi_*[V][f]\to V[\phi^*[f]]$

$\to\ \phi_*[V][f]\to V[f\circ\phi]$

$\to\ \phi_*[V][f]\to V^\mu\,\partial_\mu[f\circ\phi]$

$\to\ \phi_*[V][f]\to\partial_\mu[e^\alpha.\phi].\partial_\alpha[f]\,V^\mu$

$\to\ \phi_*[V]^\alpha\,\partial_\alpha[f]\to\partial_\mu[e^\alpha.\phi].\partial_\alpha[f]\,V^\mu$

$\to\ \phi_*[V]^\alpha\to V^\mu\,\partial_\mu[e^\alpha.\phi]$

$\to\ V^\mu\,\phi_*{}^\alpha{}_\mu\to V^\mu\,\partial_\mu[e^\alpha.\phi]$

$\to\ \phi_*{}^\alpha{}_\mu\to\partial_\mu[e^\alpha.\phi]$ ⟵ matrix definition of pushforward (A.4).

$\to\ \phi_*[S]^{\alpha_{\{i,imax\}}}\to\left(\prod_i^{imax}\partial_{x^{\mu_i}}\left[y^{\alpha_i}\right]\right)S^{\mu_{\{i,imax\}}}$

```
PR1["Manifold transformation: ",
  tmpϕ = ϕ[M] -> N,
  " where the coordinates are: ",
  sub = {
    M -> {θ, ϕ},
    N -> {x -> Sin[θ] Cos[ϕ], y -> Sin[θ] Sin[ϕ], z -> Cos[θ]}
   },
  Imply, tmpϕ /. sub,
  NL, "The correspondence to Tensor notation: ",
  sub1 = {Thread[Table[y@u[i], {i, 3}] -> {x, y, z}],
     Thread[Table[x@u[i], {i, 2}] -> {θ, ϕ}]
    } // Flatten,
  NL, "and the partial derivatives: ", (tmpdydx0 = xPartialD[y@u[i3_], x@u[i2_]] ->
      Table[xPartialD[y@u[i3], x@u[i2]], {i2, 2}, {i3, 3}]) // MatrixForms,
  yield, (subdydx = MapAt[# //. sub1 /. sub[[2, 2]] /. xPartialD -> D &, tmpdydx0, 2]) //
    MatrixForms,
  NL, "Pulling back ℝ^3 metric, g, gives S^2 metric: ",
  Yield, (tmp = T[ϕ*[g], "dd"][i2, j2] -> xPartialD[y@u[i3], x@u[i2]].
       g@dd[i3, j3] .Transpose[ xPartialD[y@u[j3], x@u[j2]] ]) // MatrixForms,
  Yield, (tmp = tmp /. subdydx) // MatrixForms,
  Yield, (sub = g@dd[a_, b_] -> IdentityMatrix[3] ) // MatrixForms ,
  Yield, (tmp = tmp /. sub // Simplify) // MatrixForms, "  (A.13)"
 ];
```

Manifold transformation: $\phi[M] \to N$ where the coordinates are:

$\{M \to \{\theta, \phi\}, N \to \{x \to \text{Cos}[\phi]\,\text{Sin}[\theta], y \to \text{Sin}[\theta]\,\text{Sin}[\phi], z \to \text{Cos}[\theta]\}\}$

$\Rightarrow \phi[\{\theta, \phi\}] \to \{x \to \text{Cos}[\phi]\,\text{Sin}[\theta], y \to \text{Sin}[\theta]\,\text{Sin}[\phi], z \to \text{Cos}[\theta]\}$

The correspondence to Tensor notation: $\{y^1 \to x, y^2 \to y, y^3 \to z, x^1 \to \theta, x^2 \to \phi\}$

and the partial derivatives: $\partial_{x^{i2\_}}\left[y^{i3\_}\right] \to \begin{pmatrix} \partial_{x^1}[y^1] & \partial_{x^1}[y^2] & \partial_{x^1}[y^3] \\ \partial_{x^2}[y^1] & \partial_{x^2}[y^2] & \partial_{x^2}[y^3] \end{pmatrix}$

$\longrightarrow \partial_{x^{i2\_}}\left[y^{i3\_}\right] \to \begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\phi] & \text{Cos}[\theta]\,\text{Sin}[\phi] & -\text{Sin}[\theta] \\ -\text{Sin}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\phi]\,\text{Sin}[\theta] & 0 \end{pmatrix}$

Pulling back ℝ^3 metric, g, gives S^2 metric:

$\to \phi^*[g]_{i2\,j2} \to \partial_{x^{i2}}[y^{i3}] \cdot g_{i3\,j3} \cdot \partial_{x^{j2}}[y^{j3}]^T$

$\to \phi^*[g]_{i2\,j2} \to \begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\phi] & \text{Cos}[\theta]\,\text{Sin}[\phi] & -\text{Sin}[\theta] \\ -\text{Sin}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\phi]\,\text{Sin}[\theta] & 0 \end{pmatrix} \cdot g_{i3\,j3} \cdot \begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\phi] & -\text{Sin}[\theta]\,\text{Sin}[\phi] \\ \text{Cos}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\phi]\,\text{Sin}[\theta] \\ -\text{Sin}[\theta] & 0 \end{pmatrix}$

$\to g_{a\_\,b\_} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\to \phi^*[g]_{i2\,j2} \to \begin{pmatrix} 1 & 0 \\ 0 & \text{Sin}[\theta]^2 \end{pmatrix}$ (A.13)

```
PR1["Definition and notational
    check: the value of tensor at φ[p] pulled back to p: ",
  b0pb = T[φ*[T[φ[p]][p]], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]] ->
    T[T[φ[p]]∘φ[p], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]],
  NL, "Or: ",
  b0pb1 = T[φ*[T[φ[p]]], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]][p] ->
    T[T[φ[p]], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]]∘φ[p]
 ];
```

Definition and notational check: the value of tensor at $\phi[p]$ pulled back to p:

$$\phi^*[T[\phi[p]][p]]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}} \to T[\phi[p]]\circ\phi[p]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}$$

Or: $\phi^*[T[\phi[p]]]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}[p] \to T[\phi[p]]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}\circ\phi[p]$

```
PR1["Integral curves: ",
  tmpx = (x@u[μ])[t], " of ", tmpV = (V@u[μ])[t],
  " are solutions of: ",
  tmpIC = xPartialD[tmpx, t] == tmpV
 ];
```

Integral curves: $x^\mu[t]$ of $V^\mu[t]$ are solutions of: $\partial_t[x^\mu[t]] == V^\mu[t]$

```
PR1["Lie Derivative: ",
  tmpL = xLieD[Tensor[f], V] -> V[f] -> V@u[μ] xPartialD[f, μ],
  NL, "Difference between tensor and its pullback: ",
  b0pb[[1]] - T[T[p][p], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]],
  NL, "Or: ",
  b0pb1[[1]] - T[T[p], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]][p],
  NL, "For a one parameter (t) transformation φt: ",
  Yield, tmp = Δt[T[T[p], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]]] ->
    (b0pb1[[1]] /. φ -> φt) -
     T[T[p], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]],
  NL, "Define Lie Derivative: ",
  xLieD[T[T[p], "ud"][T[μ, "d"]["{j,jmax}"], T[ν, "d"]["{i,imax}"]], V] ->
   xLimit[tmp[[1]] / t, t -> 0],
  " where ", V -> V@u[μ] -> xPartialD[φ[p], t]
 ];
```

Lie Derivative: $\mathcal{L}_V[f] \to V[f] \to V^\mu\,\partial_\mu[f]$

Difference between tensor and its pullback:

$$\phi^*[T[\phi[p]][p]]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}} - T[p][p]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}$$

Or: $-T[p]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}[p] + \phi^*[T[\phi[p]]]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}[p]$

For a one parameter (t) transformation ϕ_t:

$$\to \Delta_t\left[T[p]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}\right] \to -T[p]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}} + (\phi_t)^*[T[\phi_t[p]]]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}[p]$$

Define Lie Derivative:

$$\mathcal{L}_V\left[T[p]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}\right] \to \text{xLimit}\left[\frac{\Delta_t\left[T[p]^{\mu_{\{j,jmax\}}}{}_{\nu_{\{i,imax\}}}\right]}{t},\ t \to 0\right] \text{ where } V \to V^\mu \to \partial_t[\phi[p]]$$

```
PR1["Lie derivative on one-form:
From the two definitions of Lie Derivatives: ",
  {subL = xLieD[T[U_, "u"][μ_], V_] -> CommutatorM[V, U][μ],
   subL1 = xLieD[f_, V_] -> V@u[μ9] xPartialD[f, μ9],
   subV[V_] := V[a_] -> V@u[$i = μ9] xPartialD[a, $i];
   subCom = CommutatorM[V_, U_][μ_] -> V[U@u[μ]] - U[V@u[μ]]
  },
  NL, "From: ",
  tmp = xLieD[ω@d[μ] U@u[μ], V],
  Yield, tmp = tmp // DerivativeExpand[{}],
  Yield, tmp = tmp /. subL,
  Yield, tmp = tmp /. subCom,
  Yield, tmp0 = tmp /. {subV[V], subV[U]} // Expand,
  NL, "From: ",
  tmp = xLieD[ω@d[μ] U@u[μ], V],
  Yield, tmp = tmp /. subL1,
  Yield, tmp = tmp // DerivativeExpand[{}] // Expand,
  NL, "Equating the two: ", tmp = tmp0 == tmp,
  yield, tmp = tmp // Simplify // Expand;
  {tmp[[1]] = tmp[[1]] /. μ9 -> ν, tmp[[2]] = tmp[[2]] /. {μ -> ν, μ9 -> μ}};
  Imply, tmp = Map[# / U@u[μ] &, tmp] // Expand,
  yield, Framed[tmp = Solve[tmp, tmp[[1, 1]]][[1, 1]]]
 ];
```

Lie derivative on one-form:
From the two definitions of Lie Derivatives:
$\{\underline{\mathcal{L}}_{V\_}[U\_^{\mu\_}] \to [V, U][\mu],\ \underline{\mathcal{L}}_{V\_}[f\_] \to V^{\mu 9}\, \underline{\partial}_{\mu 9}[f],\ [V\_, U\_][\mu\_] \to -U[V^{\mu}] + V[U^{\mu}]\}$

From: $\underline{\mathcal{L}}_V[U^{\mu}\,\omega_{\mu}]$

$\to\ \omega_{\mu}\,\underline{\mathcal{L}}_V[U^{\mu}] + U^{\mu}\,\underline{\mathcal{L}}_V[\omega_{\mu}]$

$\to\ U^{\mu}\,\underline{\mathcal{L}}_V[\omega_{\mu}] + \omega_{\mu}\,[V, U][\mu]$

$\to\ \omega_{\mu}\,(-U[V^{\mu}] + V[U^{\mu}]) + U^{\mu}\,\underline{\mathcal{L}}_V[\omega_{\mu}]$

$\to\ U^{\mu}\,\underline{\mathcal{L}}_V[\omega_{\mu}] + V^{\mu 9}\,\omega_{\mu}\,\underline{\partial}_{\mu 9}[U^{\mu}] - U^{\mu 9}\,\omega_{\mu}\,\underline{\partial}_{\mu 9}[V^{\mu}]$

From: $\underline{\mathcal{L}}_V[U^{\mu}\,\omega_{\mu}]$

$\to\ V^{\mu 9}\,\underline{\partial}_{\mu 9}[U^{\mu}\,\omega_{\mu}]$

$\to\ V^{\mu 9}\,\omega_{\mu}\,\underline{\partial}_{\mu 9}[U^{\mu}] + U^{\mu}\,V^{\mu 9}\,\underline{\partial}_{\mu 9}[\omega_{\mu}]$

Equating the two: $U^{\mu}\,\underline{\mathcal{L}}_V[\omega_{\mu}] + V^{\mu 9}\,\omega_{\mu}\,\underline{\partial}_{\mu 9}[U^{\mu}] - U^{\mu 9}\,\omega_{\mu}\,\underline{\partial}_{\mu 9}[V^{\mu}] == V^{\mu 9}\,\omega_{\mu}\,\underline{\partial}_{\mu 9}[U^{\mu}] + U^{\mu}\,V^{\mu 9}\,\underline{\partial}_{\mu 9}[\omega_{\mu}] \longrightarrow$

$\Rightarrow\ \underline{\mathcal{L}}_V[\omega_{\mu}] - V^{\nu}\,\underline{\partial}_{\nu}[\omega_{\mu}] == \omega_{\nu}\,\underline{\partial}_{\mu}[V^{\nu}] \longrightarrow \boxed{\underline{\mathcal{L}}_V[\omega_{\mu}] \to \omega_{\nu}\,\underline{\partial}_{\mu}[V^{\nu}] + V^{\nu}\,\underline{\partial}_{\nu}[\omega_{\mu}]}$

```
PR1["• Diffeomorphism ϕ is a symmetry of tensor T if ", ϕ*[T] == T,
  New, "If family of symmetries generates a vector field: ",
  ϕt ⟶ V@u[μ] ⟺ xLieD[T, V] == 0,
  New, "Killing vector field, ", K@u[μ], imply, xLieD[g@dd[μ, ν], K] == 0, imply,
  Symmetrize2[{μ, ν}][xCovariantD[K@d[ν], μ]] == 0
 ];
```

- Diffeomorphism ϕ is a symmetry of tensor T if $\phi^*[T] == T$
- If family of symmetries generates a vector field: $(\phi_t \longrightarrow V^{\mu}) \Longleftrightarrow \underline{\mathcal{L}}_V[T] == 0$
- Killing vector field, $K^{\mu}\ \Rightarrow\ \underline{\mathcal{L}}_K[g_{\mu\nu}] == 0\ \Rightarrow\ \frac{1}{2}\left(\underline{\mathcal{D}}_{\nu}[K_{\mu}] + \underline{\mathcal{D}}_{\mu}[K_{\nu}]\right) == 0$

```
PR1["B.1.1: In Euclidean three-space,
    find and draw th integral curves of the vector fields ",
  subAB = {A[a_] -> (y - x)/r xPartialD[a, x] - (x + y)/r xPartialD[a, y],
    B[a_] -> x y xPartialD[a, x] - y y xPartialD[a, y]},
  NL, "Calculate ", tmpC = C -> xLieD[B, A], " and draw the integral curves of C.",
  NL, "The integral curves are solutions of : ", tmpIC,
  Imply, "For A: ",
  (tmp0 = tmp = xPartialD[{{x}, {y}}, t] -> {{(y - x)/r}, {-(x + y)/r}}) // MatrixForms,
  NL, "A streamline plot of the components show general behavior near origin:",
  tmp1 = tmp0[[2]] /. r -> 10 // Flatten
 ];
StreamPlot[tmp1, {x, -.1, .1}, {y, -.1, .1}]
PR1["The integral curves are defined by: ",
  tmp = tmp /. {x -> x[t], y -> y[t], Rule -> Equal};
  tmp = tmp //. {xPartialD[a_List, b_] :> Map[D[#, b] &, a]};
  tmp = Map[Thread[#] &, Thread[tmp]] // Flatten;
  Yield, tmp = DSolve[tmp, {x[t], y[t]}, t][[1]],
  NL, "For B: ",
  (tmp0 = tmp = xPartialD[{{x}, {y}}, t] -> {{y x}, {-y^2}}) // MatrixForms,
  NL, "A streamline plot of the components show general behavior near origin:",
  tmp1 = tmp0[[2]] // Flatten
 ];
StreamPlot[tmp1, {x, -.1, .1}, {y, -.1, .1}]
PR1["The integral curves are defined by: ",
  tmp = tmp /. {x -> x[t], y -> y[t], Rule -> Equal};
  tmp = tmp //. {xPartialD[a_List, b_] :> Map[D[#, b] &, a]};
  tmp = Map[Thread[#] &, Thread[tmp]] // Flatten;
  Yield, tmp = DSolve[tmp, {x[t], y[t]}, t][[1]]
 ];
```

B.1.1: In Euclidean three-space,
find and draw th integral curves of the vector fields

$$\left\{A[a\_] \to \frac{(-x+y)\,\partial_x[a]}{r} - \frac{(x+y)\,\partial_y[a]}{r},\ B[a\_] \to x\,y\,\partial_x[a] - y^2\,\partial_y[a]\right\}$$

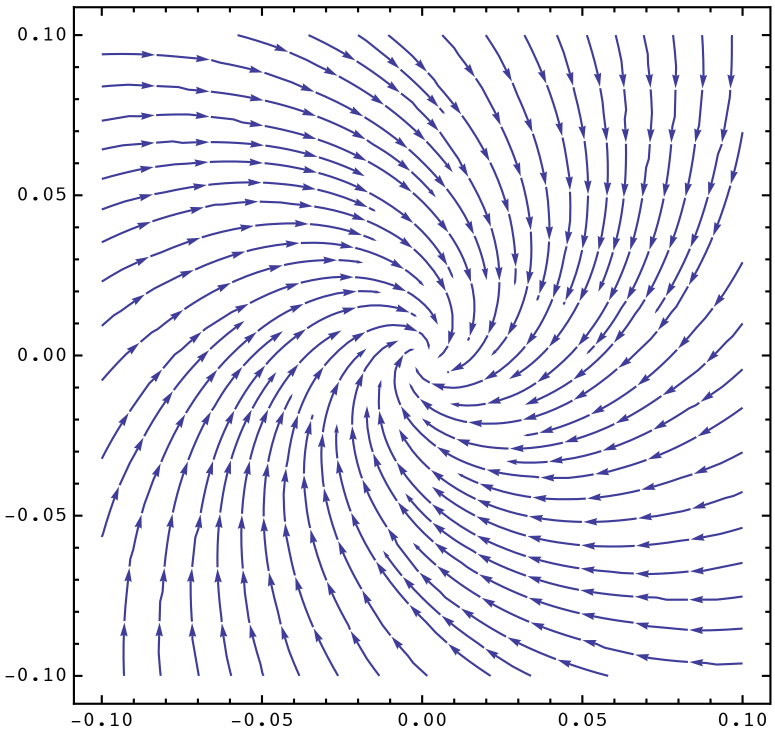
Calculate $C \to \mathcal{L}_A[B]$ and draw the integral curves of C.

The integral curves are solutions of : $\partial_t\left[x^\mu[t]\right] == V^\mu[t]$

$\Rightarrow$ For A: $\partial_t\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] \to \begin{pmatrix} \frac{-x+y}{r} \\ -\frac{x+y}{r} \end{pmatrix}$

A streamline plot of the components show general behavior near origin:

$$\left\{\frac{1}{10}(-x+y),\ \frac{1}{10}(-x-y)\right\}$$

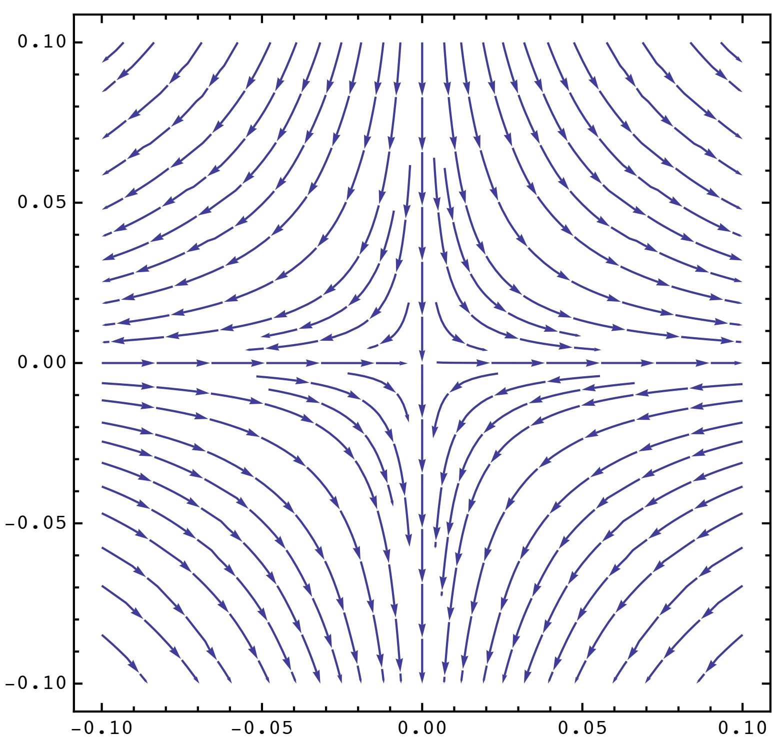


The integral curves are defined by:

$\rightarrow \left\{x[t] \rightarrow e^{-\frac{t}{r}} C[1] \operatorname{Cos}\left[\frac{t}{r}\right] + e^{-\frac{t}{r}} C[2] \operatorname{Sin}\left[\frac{t}{r}\right],\ y[t] \rightarrow e^{-\frac{t}{r}} C[2] \operatorname{Cos}\left[\frac{t}{r}\right] - e^{-\frac{t}{r}} C[1] \operatorname{Sin}\left[\frac{t}{r}\right]\right\}$

For B: $\partial_t\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] \rightarrow \begin{pmatrix} x\,y \\ -y^2 \end{pmatrix}$

A streamline plot of the components show general behavior near origin: $\{x\,y,\ -y^2\}$



The integral curves are defined by:

$$\rightarrow \left\{y[t] \to \frac{1}{t - C[1]},\ x[t] \to (t - C[1])\, C[2]\right\}$$

```
subL1 = xLieD[f_, V_] -> V[f]
PR1[tmp = tmpC,
 Yield, tmp = tmp /. subL1,
 Yield, tmp = tmp /. subAB /. B -> B[_] /. subAB[[2]],
 Yield, tmp = tmp // DerivativeExpand[{x, y}] // Expand,
 Yield, tmp = tmp /. xPartialD[xPartialD[_, _], _] -> 0 // Simplify,
 Yield, tmp1 = Coefficient[tmp[[2]], {xPartialD[_, x], xPartialD[_, y]}]
]
```

$$\mathcal{L}_{V\_}[f\_] \to V[f]$$

$$C \to \mathcal{L}_{A}[B]$$

$$\rightarrow C \to A[B]$$

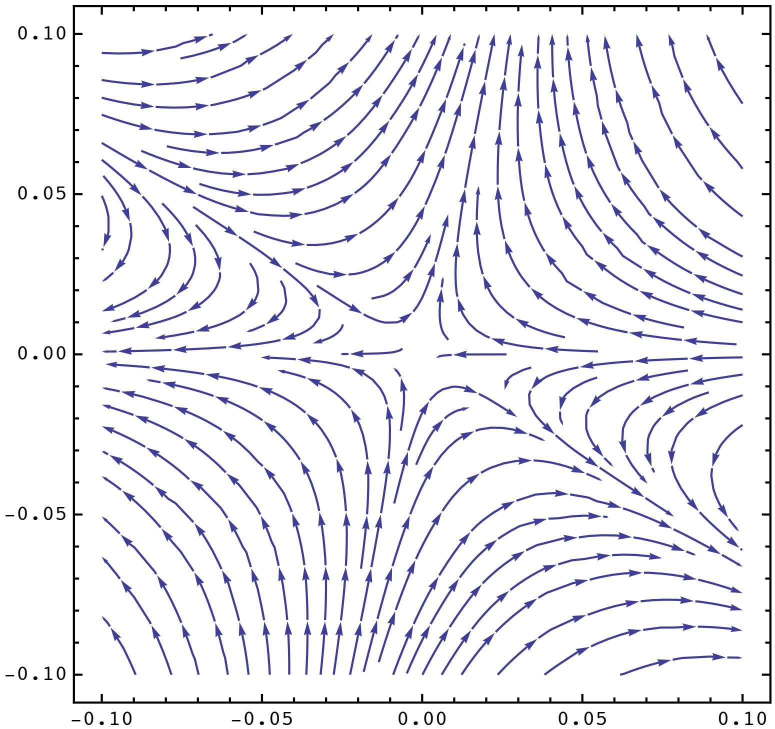
$$\rightarrow C \to \frac{(-x + y)\, \partial_x\left[x\, y\, \partial_x[\_] - y^2\, \partial_y[\_]\right]}{r} - \frac{(x + y)\, \partial_y\left[x\, y\, \partial_x[\_] - y^2\, \partial_y[\_]\right]}{r}$$

$$\rightarrow C \to -\frac{x^2\, \partial_x[\_]}{r} - \frac{2\, x\, y\, \partial_x[\_]}{r} + \frac{y^2\, \partial_x[\_]}{r} + \frac{2\, x\, y\, \partial_y[\_]}{r} + \frac{2\, y^2\, \partial_y[\_]}{r} - \frac{x^2\, y\, \partial_x[\partial_x[\_]]}{r} + \frac{x\, y^2\, \partial_x[\partial_x[\_]]}{r} - \frac{x^2\, y\, \partial_y[\partial_x[\_]]}{r} - \frac{x\, y^2\, \partial_y[\partial_x[\_]]}{r} + \frac{x\, y^2\, \partial_x\left[\partial_y[\_]\right]}{r} - \frac{y^3\, \partial_x\left[\partial_y[\_]\right]}{r} + \frac{x\, y^2\, \partial_y\left[\partial_y[\_]\right]}{r} + \frac{y^3\, \partial_y\left[\partial_y[\_]\right]}{r}$$

$$\rightarrow C \to \frac{\left(-x^2 - 2\, x\, y + y^2\right)\, \partial_x[\_] + 2\, y\, (x + y)\, \partial_y[\_]}{r}$$

$$\rightarrow \left\{\frac{-x^2 - 2\, x\, y + y^2}{r},\ \frac{2\, y\, (x + y)}{r}\right\}$$

```
tmp1 = tmp1 /. r -> 1;
StreamPlot[tmp1, {x, -.1, .1}, {y, -.1, .1}]
PR1["The integral curves are defined by: ",
  tmp = Thread[{xPartialD[x, t], xPartialD[y, t]} -> tmp1],
  Yield, tmp = tmp /. {x -> x[t], y -> y[t], Rule -> Equal},
  Yield, tmp = tmp //. {xPartialD[a_, b_] :> D[a, b]},
  Yield, xtmp = xDSolve[tmp, {x[t], y[t]}, t]
 ];
```



The integral curves are defined by: $\{\underline{\partial}_t[x] \to -x^2 - 2\,x\,y + y^2,\ \underline{\partial}_t[y] \to 2\,y\,(x+y)\}$

$\to\ \{\underline{\partial}_t[x[t]] == -x[t]^2 - 2\,x[t]\,y[t] + y[t]^2,\ \underline{\partial}_t[y[t]] == 2\,y[t]\,(x[t]+y[t])\}$

$\to\ \{x'[t] == -x[t]^2 - 2\,x[t]\,y[t] + y[t]^2,\ y'[t] == 2\,y[t]\,(x[t]+y[t])\}$

$\to$ xDSolve$[\{x'[t] == -x[t]^2 - 2\,x[t]\,y[t] + y[t]^2,\ y'[t] == 2\,y[t]\,(x[t]+y[t])\},\ \{x[t], y[t]\},\ t]$

```
DSolve[tmp, {x[t], y[t]}, t]
```

DSolve$[\{x'[t] == -x[t]^2 - 2\,x[t]\,y[t] + y[t]^2,\ y'[t] == 2\,y[t]\,(x[t]+y[t])\},\ \{x[t], y[t]\},\ t]$

Stoke's Theorem

```
PR1["Exterior derivative of p-form(2.76): ",
  e276 = T[d[A_], "d"][μ[{i, p + 1}]] -> (p + 1)
     xAntiSymmetric[{μ[p + 1], μ[{i, p}]}][xPartialD[T[A, "d"][μ[{i, p}]], μ[p + 1]]]
 ];
```

Exterior derivative of p-form(2.76):

$d[A\_]_{\mu[\{i,1+p\}]} \to (1+p)$ xAntiSymmetric$[\{\mu[1+p],\ \mu[\{i,p\}]\}][\underline{\partial}_{\mu[1+p]}[A_{\mu[\{i,p\}]}]]$

```
PR1["Stoke's Theorem: ",
  IntegralOp[{M}, d[ω]] == IntegralOp[{xPartialD[M, boundary]}, ω],
  " where ", {d[ω] == "n-form", ω == "(n-1)-form"}, and,
  ω -> HodgeStar[V],
  NL, "HodgeStar: ", subHS = T[HodgeStar[A_], "d"][μ[{i, n - p}]] ->
     1/p! T[ϵ, "ud"][ν[{j, p}], μ[{i, n - p}]] A@d[ν[{j, p}]],
  Imply, tmp = ω@d[μ[{i, n - 1}]] -> T[HodgeStar[V], "d"][μ[{i, n - 1}]],
  Yield, tmp = tmp[[1]] -> ϵ@ud[ν, μ[{i, n - 1}]] V@d[ν],
  Yield, Framed[tmp = tmp[[1]] -> ϵ@dd[ν, μ[{i, n - 1}]] V@u[ν]],
  NL, "Also, ",
  tmp = V -> (-1)^(s + n - 1) HodgeStar[HodgeStar[V]],
  yield, tmp = tmp /. HodgeStar[V] -> ω, " where ", {s[Lorentz] -> -1, s[Euclid] -> 1},
  NL, "Exterior derivative of ", tmp = ω -> HodgeStar[V],
  Imply, tmp = Map[T[d[#], "dd"][λ, μ[{i, n - 1}]] &, tmp],
  Yield, tmp = tmp /. T[d[B_], "dd"][a_, b_] -> T[d[T[B, "d"][b]], "d"][a],
  NL, "From the HodgeStar definition: ", sub0 = subHS //. {p -> 1, ν[{j_, 1}] → ν1},
  Imply, tmp = tmp /. sub0,
  NL, "Convenient index notation change where μ is associated with ϵ: ",
  sub0 = T[d[T[A_, "d"][ν_] T[ϵ, "ud"][ν_, μ_]], "d"][λ_] →
    T[d[T[A, "d"][ν] T[ϵ, "u"][ν]], "dd"][λ, μ],
  " and (2.76)",
  sub = e276 /. {p → n - 1};
  sub =
   sub /. T[d[A_], "d"][μ_[{i_, n_}]] → T[d[A], "dd"][μ[n], μ[{i, n - 1}]] /. μ[n] → λ;
  sub = sub /. μ[n] → λ,
  Yield, tmp = tmp /. sub0,
  Yield, Framed[tmp = tmp /. sub], " (E.5) "
 ];
```

Stoke's Theorem: $\int_M [d[\omega]] == \int_{\underline{\partial}_{\text{boundary}}[M]} [\omega]$ where $\{d[\omega] == \text{n-form}, \omega == (n-1)\text{-form}\}$ and $\omega \to \underline{\star}[V]$

HodgeStar: $\underline{\star}[A\_]_{\mu[\{i,n-p\}]} \to \dfrac{A_{\nu[\{j,p\}]}\, \epsilon^{\nu[\{j,p\}]}{}_{\mu[\{i,n-p\}]}}{p!}$

$\Rightarrow\ \omega_{\mu[\{i,-1+n\}]} \to \underline{\star}[V]_{\mu[\{i,-1+n\}]}$

$\to\ \omega_{\mu[\{i,-1+n\}]} \to V_\nu\, \epsilon^{\nu}{}_{\mu[\{i,-1+n\}]}$

$\to\ \boxed{\omega_{\mu[\{i,-1+n\}]} \to V^{\nu}\, \epsilon_{\nu\,\mu[\{i,-1+n\}]}}$

Also, $V \to (-1)^{-1+n+s}\, \underline{\star}[\underline{\star}[V]] \longrightarrow V \to (-1)^{-1+n+s}\, \underline{\star}[\omega]$ where $\{s[\text{Lorentz}] \to -1, s[\text{Euclid}] \to 1\}$

Exterior derivative of $\omega \to \underline{\star}[V]$

$\Rightarrow\ d[\omega]_{\lambda\,\mu[\{i,-1+n\}]} \to d[\underline{\star}[V]]_{\lambda\,\mu[\{i,-1+n\}]}$

$\to\ d\left[\omega_{\mu[\{i,-1+n\}]}\right]_\lambda \to d\left[\underline{\star}[V]_{\mu[\{i,-1+n\}]}\right]_\lambda$

From the HodgeStar definition: $\underline{\star}[A\_]_{\mu[\{i,-1+n\}]} \to A_{\nu 1}\, \epsilon^{\nu 1}{}_{\mu[\{i,-1+n\}]}$

$\Rightarrow\ d\left[\omega_{\mu[\{i,-1+n\}]}\right]_\lambda \to d\left[V_{\nu 1}\, \epsilon^{\nu 1}{}_{\mu[\{i,-1+n\}]}\right]_\lambda$

Convenient index notation change where μ is associated with ϵ:

$d\left[\epsilon^{\nu\_}{}_{\mu\_}\, A\_{}_{\nu\_}\right]_{\lambda\_} \to d[A_\nu\, \epsilon^{\nu}]_{\lambda\,\mu}$ and (2.76)

$d[A\_]_{\lambda\,\mu[\{i,-1+n\}]} \to n\ \text{xAntiSymmetric}[\{\lambda, \mu[\{i, -1+n\}]\}]\left[\underline{\partial}_\lambda\left[A_{\mu[\{i,-1+n\}]}\right]\right]$

$\to\ d\left[\omega_{\mu[\{i,-1+n\}]}\right]_\lambda \to d\left[V_{\nu 1}\, \epsilon^{\nu 1}\right]_{\lambda\,\mu[\{i,-1+n\}]}$

$\to\ \boxed{d\left[\omega_{\mu[\{i,-1+n\}]}\right]_\lambda \to n\ \text{xAntiSymmetric}[\{\lambda, \mu[\{i, -1+n\}]\}]\left[\underline{\partial}_\lambda\left[V_{\nu 1}\, \epsilon^{\nu 1}{}_{\mu[\{i,-1+n\}]}\right]\right]}$ (E.5)

Raychaudhuri equation

```
PR1["Covariant derivative along path: ",
  sub = xDDeltaD[A_, τ_] -> T[U, "u"][σ1] xDeltaD[A, σ1],
  " of ", tmpB = B@ud[μ, ν] -> xDeltaD[T[ U, "u"][μ] , ν], " in ",
  (tmp = xDDeltaD[ T[V, "u"][μ], τ]) -> (tmp /. sub),
  NL, "Projection operator: ",
  eF4 = T[P, "ud"][μ_, ν_] -> T[δ, "ud"][μ, ν] + T[U, "u"][μ] T[U, "d"][ν],
  NL, "Then we can write: ",
  eF6 = T[B, "dd"][μ_, ν_] -> θ T[P, "dd"][μ, ν] / 3 + T[σ, "dd"][μ, ν] + T[ω, "dd"][μ, ν],
  " where the scalar, symmetric, and antisymmetric components are: ",
  {θ, T[σ, "dd"][μ, ν], T[ω, "dd"][μ, ν]},
  NL, "Then: ",
  eF10 = xDDeltaD[T[B, "dd"][μ_, ν_], τ] -> T[B, "ud"][σ1, ν] T[B, "dd"][μ, σ1] -
     T[R, "dddd"][λ, μ, ν, σ] T[U, "u"][σ] T[U, "u"][λ],
  NL, "The Trace yields the Raychaudhuri equation: ",
  eF11 = xDDeltaD[θ, τ] -> -θ θ / 3 - T[σ, "dd"][μ, ν] T[σ, "uu"][μ, ν] +
     T[ω, "dd"][μ, ν] T[ω, "uu"][μ, ν] - T[R, "dd"][μ, ν] T[U, "u"][μ] T[U, "u"][ν]

 ];
```

Covariant derivative along path: $\underline{D}_{\tau\_}[A\_] \to U^{\sigma 1}\, \underline{\nabla}_{\sigma 1}[A]$ of $B^{\mu}{}_{\nu} \to \underline{\nabla}_{\nu}[U^{\mu}]$ in $\underline{D}_{\tau}[V^{\mu}] \to U^{\sigma 1}\, \underline{\nabla}_{\sigma 1}[V^{\mu}]$

Projection operator: $P^{\mu\_}{}_{\nu\_} \to U_{\nu}\, U^{\mu} + \delta^{\mu}{}_{\nu}$

Then we can write: $B_{\mu\_\,\nu\_} \to \frac{1}{3}\,\theta\, P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$

where the scalar, symmetric, and antisymmetric components are: $\{\theta,\ \sigma_{\mu\nu},\ \omega_{\mu\nu}\}$

Then: $\underline{D}_{\tau}[B_{\mu\_\,\nu\_}] \to B_{\mu\,\sigma 1}\, B^{\sigma 1}{}_{\nu} - R_{\lambda\mu\nu\sigma}\, U^{\lambda}\, U^{\sigma}$

The Trace yields the Raychaudhuri equation: $\underline{D}_{\tau}[\theta] \to -\frac{\theta^2}{3} - R_{\mu\nu}\, U^{\mu}\, U^{\nu} - \sigma_{\mu\nu}\, \sigma^{\mu\nu} + \omega_{\mu\nu}\, \omega^{\mu\nu}$

Conformal Transformations

```
PR1["Conformal transformation is a local change of scale. Via metric: ",
 T[g̃, "dd"][μ_, ν_] -> ω[x]^2 T[g, "dd"][μ, ν]
]
```

Conformal transformation is a local change of scale. Via metric: $\tilde{g}_{\mu\_\,\nu\_} \to g_{\mu\nu}\, \omega[x]^2$

Exercise G.1

```
PR1["G.1.1: Show that the conformal transformations leave
    null geodesics invariant, that is, that the null geodesics of ",
  ω^2 T[g, "dd"][μ, ν], " are the same as those of ", T[g̃, "dd"][μ, ν],
  ". (We already know that they leave null curves invariant; you have to show that
    the transformed curves still are geodesics.)  What is the relationship
    between the affine parameters in the orginal and conformal metrics?"
 ];
```

G.1.1: Show that the conformal transformations leave null geodesics invariant, that is, that the null geodesics of $\omega^2\, g_{\mu\nu}$ are the same as those of $\tilde{g}_{\mu\nu}$

. (We already know that they leave null curves invariant; you have to show that the transformed curves still are geodesics.) What is the relationship between the affine parameters in the orginal and conformal metrics?

```
PR1["The geodesic equation ( λ->affine parameter ): ",
  e344 = xDDeltaD[xDDeltaD[T[x, "u"][μ], λ], λ] + T[Γ, "udd"][μ, ρ1, σ1]
      xDDeltaD[T[x, "u"][ρ1], λ] xDDeltaD[T[x, "u"][σ1], λ] -> 0, " (3.44)",
  NL, "with ",
  e327 = T[Γ, "udd"][σ_, μ_, ν_] -> 1/2 g@uu[σ, ρ2] (xPartialD[g@dd[ν, ρ2], μ] +
       xPartialD[g@dd[ρ2, μ], ν] - xPartialD[g@dd[μ, ν], ρ2]), " (3.27)",
  Imply, e355 = e344 /. e327, " (3.55)",
  NL,
  "Null geodesics are path the light rays follow and the geodesics satisfy (3.55).
Applying the conformal transformation: ",
  subConf = T[g̃, "dd"][μ_, ν_] -> ω^2 T[g, "dd"][μ, ν];
  yield, subConf = {subConf, subConf /. ω -> 1 / ω // RaiseIndexTU1[{μ, ν}, {μ, ν}]},
  Imply, tmp = e355 /. g -> g̃,
  Yield, tmp = tmp /. subConf,
  Yield, tmp = tmp // DerivativeExpand[{}] // Expand;
  Yield, (xtmp = tmp = tmp[[1]]) // ColumnSum,
  NL, "Examine ω terms: ",
  Yield, tmpω = tmp = Apply[Plus, ExtractPattern[tmp, a__ / ω]],
  NL, "Remove ω factor: ",
  Yield, tmp = tmpω ω // Simplify // Expand,
  Yield, tmp = tmp // Simplify,
  Yield, tmp = tmp /. xDDeltaD[_, _] -> 1 // Expand // MetricSimplify[g], " (G.6)"
 ];
```

The geodesic equation (λ->affine parameter): $\Gamma^{\mu}{}_{\rho 1\,\sigma 1}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}] + \underline{D}_{\lambda}[\underline{D}_{\lambda}[x^{\mu}]] \to 0$ (3.44)

with $\Gamma^{\sigma\_}{}_{\mu\_\,\nu\_} \to \frac{1}{2} g^{\sigma\,\rho 2}\,(-\underline{\partial}_{\rho 2}[g_{\mu\,\nu}] + \underline{\partial}_{\mu}[g_{\nu\,\rho 2}] + \underline{\partial}_{\nu}[g_{\rho 2\,\mu}])$ (3.27)

$\Rightarrow \underline{D}_{\lambda}[\underline{D}_{\lambda}[x^{\mu}]] + \frac{1}{2} g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,(-\underline{\partial}_{\rho 2}[g_{\rho 1\,\sigma 1}] + \underline{\partial}_{\sigma 1}[g_{\rho 2\,\rho 1}] + \underline{\partial}_{\rho 1}[g_{\sigma 1\,\rho 2}]) \to 0$ (3.55)

Null geodesics are path the light rays follow and the geodesics satisfy (3.55).

Applying the conformal transformation: $\longrightarrow \left\{\tilde{g}_{\mu\_\,\nu\_} \to \omega^2 g_{\mu\,\nu},\ \tilde{g}^{\mu\_\,\nu\_} \to \frac{g^{\mu\,\nu}}{\omega^2}\right\}$

$\Rightarrow \underline{D}_{\lambda}[\underline{D}_{\lambda}[x^{\mu}]] + \frac{1}{2} \tilde{g}^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,(-\underline{\partial}_{\rho 2}[\tilde{g}_{\rho 1\,\sigma 1}] + \underline{\partial}_{\sigma 1}[\tilde{g}_{\rho 2\,\rho 1}] + \underline{\partial}_{\rho 1}[\tilde{g}_{\sigma 1\,\rho 2}]) \to 0$

$\to \underline{D}_{\lambda}[\underline{D}_{\lambda}[x^{\mu}]] + \frac{g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,(-\underline{\partial}_{\rho 2}[\omega^2 g_{\rho 1\,\sigma 1}] + \underline{\partial}_{\sigma 1}[\omega^2 g_{\rho 2\,\rho 1}] + \underline{\partial}_{\rho 1}[\omega^2 g_{\sigma 1\,\rho 2}])}{2\,\omega^2} \to 0$

$\to$

$$\to \begin{array}{l}
\underline{D}_{\lambda}[\underline{D}_{\lambda}[x^{\mu}]] \\
\frac{g_{\sigma 1\,\rho 2}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 1}[\omega]}{\omega} \\
-\frac{g_{\rho 1\,\sigma 1}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 2}[\omega]}{\omega} \\
\frac{g_{\rho 2\,\rho 1}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\sigma 1}[\omega]}{\omega} \\
-\frac{1}{2} g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 2}[g_{\rho 1\,\sigma 1}] \\
\frac{1}{2} g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\sigma 1}[g_{\rho 2\,\rho 1}] \\
\frac{1}{2} g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 1}[g_{\sigma 1\,\rho 2}]
\end{array}$$

Examine ω terms:

$\to \frac{g_{\sigma 1\,\rho 2}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 1}[\omega]}{\omega} - \frac{g_{\rho 1\,\sigma 1}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 2}[\omega]}{\omega} + \frac{g_{\rho 2\,\rho 1}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\sigma 1}[\omega]}{\omega}$

Remove ω factor:

$\to g_{\sigma 1\,\rho 2}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 1}[\omega] - g_{\rho 1\,\sigma 1}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\rho 2}[\omega] + g_{\rho 2\,\rho 1}\,g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,\underline{\partial}_{\sigma 1}[\omega]$

$\to g^{\mu\,\rho 2}\,\underline{D}_{\lambda}[x^{\rho 1}]\,\underline{D}_{\lambda}[x^{\sigma 1}]\,(g_{\sigma 1\,\rho 2}\,\underline{\partial}_{\rho 1}[\omega] - g_{\rho 1\,\sigma 1}\,\underline{\partial}_{\rho 2}[\omega] + g_{\rho 2\,\rho 1}\,\underline{\partial}_{\sigma 1}[\omega])$

$\to g^{\mu}{}_{\sigma 1}\,\underline{\partial}_{\rho 1}[\omega] - g_{\rho 1\,\sigma 1}\,g^{\mu\,\rho 2}\,\underline{\partial}_{\rho 2}[\omega] + g^{\mu}{}_{\rho 1}\,\underline{\partial}_{\sigma 1}[\omega]$ (G.6)

```
PR1["Affine parameter is defined ( τ is the proper time ): ",
  tAffine = λ -> C[1] τ + C[2]
 ];
subConf
```

Affine parameter is defined (τ is the proper time): $\lambda \to \tau\,C[1] + C[2]$

$$\left\{\tilde{g}_{\mu\_\,\nu\_} \to \omega^2 g_{\mu\,\nu},\ \tilde{g}^{\mu\_\,\nu\_} \to \frac{g^{\mu\,\nu}}{\omega^2}\right\}$$

```
PR1["Wald.3.1.7: ",
  eW317 = xD["∇", T[ω, "d"][b], a] ->
    xD["∇̃", T[ω, "d"][b], a] - T[C, "udd"][c, a, b] T[ω, "d"][c],
  NL, "Wald.D.1: ",
  tmpD1 = T[C, "udd"][c, a, b] -> 1/2 T[g̃, "uu"][c, d] (xDeltaD[T[g̃, "dd"][b, d], a] +
       xDeltaD[T[g̃, "dd"][a, d], b] - xDeltaD[T[g̃, "dd"][a, b], d]),
  NL, "Wald.D.2: ", tmp = xDeltaD[T[g̃, "dd"][b, c], a] ->
    xDeltaD[Ω^2 T[g, "dd"][b, c], a],
  imply, sub = MapAt[# //. DifExpand[xDeltaD, {T[g, "dd"][_, _]}] &, tmp, 2] //
    RuleVarPattern[{a, b, c}],
  and, sub1 = T[g̃, "uu"][a_, b_] -> Ω^(-2) T[g, "uu"][a, b],
  Imply, subC = tmpD1 /. sub /. sub1 // Expand,
  Imply, tmp = eW317 /. subC,
  NL, "Or (3.1.13): ",
  eW3113 = xD["∇", T[t_, "u"][b_], a_] ->
    xD["∇̃", T[t, "u"][b], a] + T[C, "udd"][b, a, c] T[t, "u"][c],
  Imply, tmp = xD["∇̃", T[v, "u"][b], a],
  yield, tmp = tmp -> (tmp /. RuleX1[eW3113, xD["∇̃", T[t, "u"][b], a], {t, a, b}]),
  Imply, tmp = Map[T[v, "u"][a] # &, tmp],
  Imply, tmp = tmp /. RuleX2PatternVar[subC, {a, b, c}] // Expand,
  NL, "For null geodesics: ", sub = T[g, "dd"][a_, b_] T[v, "u"][a_] T[v, "u"][b_] -> 0,
  Imply, tmp = tmp /. sub,
  Yield, tmp = MapAt[MetricSimplify[g][#] &, tmp, 2] /. c -> a,
  NL, "which is always transformable to a conformally invariant equation(3.2.2)."
 ];
```

Wald.3.1.7: $\underline{\nabla}_a[\omega_b] \to -C^c{}_{a\,b}\,\omega_c + \tilde{\underline{\nabla}}_a[\omega_b]$

Wald.D.1: $C^c{}_{a\,b} \to \frac{1}{2}\,\tilde{g}^{c\,d}\left(-\underline{\nabla}_d[\tilde{g}_{a\,b}] + \underline{\nabla}_b[\tilde{g}_{a\,d}] + \underline{\nabla}_a[\tilde{g}_{b\,d}]\right)$

Wald.D.2: $\underline{\nabla}_a[\tilde{g}_{b\,c}] \to \underline{\nabla}_a[\Omega^2\, g_{b\,c}] \Rightarrow \underline{\nabla}_{a\_}[\tilde{g}_{b\_\,c\_}] \to 2\,\Omega\, g_{b\,c}\,\underline{\nabla}_a[\Omega]$ and $\tilde{g}^{a\_\,b\_} \to \frac{g^{a\,b}}{\Omega^2}$

$\Rightarrow C^c{}_{a\,b} \to \frac{g^{c\,d}\, g_{b\,d}\,\underline{\nabla}_a[\Omega]}{\Omega} + \frac{g^{c\,d}\, g_{a\,d}\,\underline{\nabla}_b[\Omega]}{\Omega} - \frac{g^{c\,d}\, g_{a\,b}\,\underline{\nabla}_d[\Omega]}{\Omega}$

$\Rightarrow \underline{\nabla}_a[\omega_b] \to \tilde{\underline{\nabla}}_a[\omega_b] - \omega_c\left(\frac{g^{c\,d}\, g_{b\,d}\,\underline{\nabla}_a[\Omega]}{\Omega} + \frac{g^{c\,d}\, g_{a\,d}\,\underline{\nabla}_b[\Omega]}{\Omega} - \frac{g^{c\,d}\, g_{a\,b}\,\underline{\nabla}_d[\Omega]}{\Omega}\right)$

Or (3.1.13): $\underline{\nabla}_{a\_}[t\_^{b\_}] \to C^b{}_{a\,c}\, t^c + \tilde{\underline{\nabla}}_a[t^b]$

$\Rightarrow \tilde{\underline{\nabla}}_a[v^b] \longrightarrow \tilde{\underline{\nabla}}_a[v^b] \to -C^b{}_{a\,c}\, v^c + \underline{\nabla}_a[v^b]$

$\Rightarrow v^a\,\tilde{\underline{\nabla}}_a[v^b] \to v^a\left(-C^b{}_{a\,c}\, v^c + \underline{\nabla}_a[v^b]\right)$

$\Rightarrow v^a\,\tilde{\underline{\nabla}}_a[v^b] \to v^a\,\underline{\nabla}_a[v^b] - \frac{g^{b\,d}\, g_{c\,d}\, v^a\, v^c\,\underline{\nabla}_a[\Omega]}{\Omega} - \frac{g^{b\,d}\, g_{a\,d}\, v^a\, v^c\,\underline{\nabla}_c[\Omega]}{\Omega} + \frac{g^{b\,d}\, g_{a\,c}\, v^a\, v^c\,\underline{\nabla}_d[\Omega]}{\Omega}$

For null geodesics: $g_{a\_\,b\_}\, v^{a\_}\, v^{b\_} \to 0$

$\Rightarrow v^a\,\tilde{\underline{\nabla}}_a[v^b] \to v^a\,\underline{\nabla}_a[v^b] - \frac{g^{b\,d}\, g_{c\,d}\, v^a\, v^c\,\underline{\nabla}_a[\Omega]}{\Omega} - \frac{g^{b\,d}\, g_{a\,d}\, v^a\, v^c\,\underline{\nabla}_c[\Omega]}{\Omega}$

$\to v^a\,\tilde{\underline{\nabla}}_a[v^b] \to v^a\,\underline{\nabla}_a[v^b] - \frac{2\, v^a\, v^b\,\underline{\nabla}_a[\Omega]}{\Omega}$

which is always transformable to a conformally invariant equation(3.2.2).

Conformal Diagrams

```
PR1["Conformal Diagrams: light cones at 45° and ",
  x@u[ν], " are time-like.  Minkowski metric: ",
  eH1 = {ds^2 -> -dt^2 + dr^2 + r^2 dΩ^2, dΩ^2 -> dθ^2 + Sin[θ]^2 dϕ^2},
  NL, "Choose transformation: ", eH5 = {u -> t - r, v -> t + r},
  NL, "with range: ", eH6 = {-∞ < u < ∞, -∞ < v < ∞, u ≤ v},
  Yield, tmp = subuv = Solve[eH5 /. Rule -> Equal, {t, r}][[1]],
  Yield, subd = Map[Map[Dt[#] &, #] &, tmp],
  Yield, tmp = eH1 /. {dt -> Dt[t], dr -> Dt[r], du -> Dt[u], dv -> Dt[v]},
  Yield, tmp = tmp /. subd /. subuv // Expand // Simplify,
  NL, "use: ", subUV = eH8 = {U -> ArcTan[u], V -> ArcTan[v]}, " with range ",
  {-π / 2 < U < π / 2, -π / 2 < V < π / 2, U ≤ V},
  Imply, subUVi = Map[Map[Tan[#] &, Reverse[#]] &, subUV],
  NL, "The metric: ", tmp = tmp /. subUVi,
  Yield, tmp = tmp // TrigReduce // Simplify,
  NL, "Transform to: ", subTR = eH13 = {T -> V + U, R -> V - U}, " with range ",
  {0 <= R < π, Abs[T] + R < π},
  Imply, subTRi = Solve[subTR /. Rule -> Equal, {U, V}][[1]],
  Imply, tmp = tmp /. subTRi // TrigReduce // Simplify
 ];
```

Conformal Diagrams: light cones at 45° and x^{ν} are time-like. Minkowski metric: $\{ds^2 \to dr^2 - dt^2 + d\Omega^2\, r^2,\ d\Omega^2 \to d\theta^2 + d\phi^2\, \mathrm{Sin}[\theta]^2\}$

Choose transformation: $\{u \to -r + t,\ v \to r + t\}$

with range: $\{-\infty < u < \infty,\ -\infty < v < \infty,\ u \le v\}$

$\to \left\{t \to \frac{u+v}{2},\ r \to -\frac{u}{2} + \frac{v}{2}\right\}$

$\to \left\{dt \to \frac{1}{2}(du + dv),\ dr \to -\frac{du}{2} + \frac{dv}{2}\right\}$

$\to \{ds^2 \to d\Omega^2\, r^2 + (dr)^2 - (dt)^2,\ d\Omega^2 \to d\theta^2 + d\phi^2\, \mathrm{Sin}[\theta]^2\}$

$\to \left\{ds^2 \to \frac{1}{4}\left(d\Omega^2\,(u - v)^2 - 4\, du\, dv\right),\ d\Omega^2 \to d\theta^2 + d\phi^2\, \mathrm{Sin}[\theta]^2\right\}$

use: $\{U \to \mathrm{ArcTan}[u],\ V \to \mathrm{ArcTan}[v]\}$ with range $\left\{-\frac{\pi}{2} < U < \frac{\pi}{2},\ -\frac{\pi}{2} < V < \frac{\pi}{2},\ U \le V\right\}$

$\Rightarrow \{u \to \mathrm{Tan}[U],\ v \to \mathrm{Tan}[V]\}$

The metric: $\left\{ds^2 \to \frac{1}{4}\left(-4\, dU\, dV\, \mathrm{Sec}[U]^2\, \mathrm{Sec}[V]^2 + d\Omega^2\,(\mathrm{Tan}[U] - \mathrm{Tan}[V])^2\right),\ d\Omega^2 \to d\theta^2 + d\phi^2\, \mathrm{Sin}[\theta]^2\right\}$

$\to \left\{ds^2 \to -\frac{1}{4}\mathrm{Sec}[U]^2\, \mathrm{Sec}[V]^2\left(4\, dU\, dV - d\Omega^2\, \mathrm{Sin}[U - V]^2\right),\ d\Omega^2 \to \frac{1}{2}\left(2\, d\theta^2 + d\phi^2 - d\phi^2\, \mathrm{Cos}[2\,\theta]\right)\right\}$

Transform to: $\{T \to U + V,\ R \to -U + V\}$ with range $\{0 \le R < \pi,\ R + \mathrm{Abs}[T] < \pi\}$

$\Rightarrow \left\{U \to \frac{1}{2}(-R + T),\ V \to \frac{R}{2} + \frac{T}{2}\right\}$

$\Rightarrow \left\{ds^2 \to \frac{(dR)^2 - (dT)^2 + d\Omega^2\, \mathrm{Sin}[R]^2}{(\mathrm{Cos}[R] + \mathrm{Cos}[T])^2},\ d\Omega^2 \to \frac{1}{2}\left(2\, d\theta^2 + d\phi^2 - d\phi^2\, \mathrm{Cos}[2\,\theta]\right)\right\}$

```
PR1["Transformation summary: ", {subuv, subUVi, subTRi} // Column,
  NL, "Or: ",
  (subRTrt = {eH5, eH8, eH13} // Flatten) // Column,
  NL, "Then ", tmp = {T, R},
  yield, subTRtr = Thread[tmp -> (tmp //. subRTrt)],
  NL, "For example, ", sub = {t -> ∞, r -> 0},
  imply, subTRtr /. sub
 ];
```

Transformation summary:
$\left\{t \to \frac{u+v}{2},\ r \to -\frac{u}{2} + \frac{v}{2}\right\}$
$\{u \to \mathrm{Tan}[U],\ v \to \mathrm{Tan}[V]\}$
$\left\{U \to \frac{1}{2}(-R + T),\ V \to \frac{R}{2} + \frac{T}{2}\right\}$

Or:
$u \to -r + t$
$v \to r + t$
$U \to \mathrm{ArcTan}[u]$
$V \to \mathrm{ArcTan}[v]$
$T \to U + V$
$R \to -U + V$

Then $\{T, R\} \longrightarrow \{T \to -\mathrm{ArcTan}[r - t] + \mathrm{ArcTan}[r + t],\ R \to \mathrm{ArcTan}[r - t] + \mathrm{ArcTan}[r + t]\}$
For example, $\{t \to \infty,\ r \to 0\} \Rightarrow \{T \to \pi,\ R \to 0\}$

Noncoordinate Bases

```
Clear[spinorcoordinate];
spinorcoordinate[a_] := MemberQ[CharacterRange["a", "z"], ToString[a]];
PR1["Natural basis for tangent and cotangent spaces at p: ",
  eJ0 = {T[ê, "d"][μ_] -> T["∂", "d"][μ], T[θ̂, "u"][μ_] -> T["dx", "u"][μ]},
  NL, "Metric tensor g[] and Minkowski metric η[]: ",
  eJ1 = g[T[ê, "d"][a], T[ê, "d"][b]] -> η@dd[a, b],
  NL, "For ", T[e, "du"][μ, a],
  ", n x n invertible matrix (vielbein) where {μ,a} are in different basis: ",
  NL,
  eJ2 = {T[ê, "d"][μ_] :> T[e, "du"][μ, a1] T[ê, "d"][a1] /; ! spinorcoordinate[μ],
    T[ê, "d"][a_] :> T[e, "ud"][μ1, a] T[ê, "d"][μ1] /; spinorcoordinate[a]
   }, CR[" ← Note left-right position."],
  NL, "Inverse relationships: ",
  eJ3 = {T[e, "ud"][μ_, a_] T[e, "du"][ν_, a_] -> δ@ud[μ, ν],
    T[e, "du"][μ_, a_] T[e, "ud"][μ_, b_] -> δ@ud[a, b]},
  NL, "Spin connection, a connection for noncoordinate basis: ", T[ω, "dud"][μ, a, b],
  Yield, e[J17] = xD["∇", T[X, "ud"][a, b], μ] -> xPartialD[T[X, "ud"][a, b], μ] +
     T[ω, "dud"][μ, a, c] T[X, "ud"][c, b] - T[ω, "dud"][μ, c, b] T[X, "ud"][a, c]
 ];
```

Natural basis for tangent and cotangent spaces at p: $\{\hat{e}_{\mu\_} \to \partial_\mu, \hat{\theta}^{\mu\_} \to dx^\mu\}$

Metric tensor g[] and Minkowski metric η[]: $g[\hat{e}_a, \hat{e}_b] \to \eta_{ab}$

For $e_\mu{}^a$, n x n invertible matrix (vielbein) where {μ,a} are in different basis:

{ê_μ_ :→ T[e, du][μ, a1] T[ê, d][a1] /; ! spinorcoordinate[μ],
ê_a_ :→ T[e, ud][μ1, a] T[ê, d][μ1] /; spinorcoordinate[a]} ← Note left-right position.

Inverse relationships: $\{e_{\nu\_}{}^{a\_}\, e^{\mu\_}{}_{a\_} \to \delta^\mu{}_\nu,\ e_{\mu\_}{}^{a\_}\, e^{\mu\_}{}_{b\_} \to \delta^a{}_b\}$

Spin connection, a connection for noncoordinate basis: $\omega_\mu{}^a{}_b$

$\to \underline{\nabla}_\mu[X^a{}_b] \to X^c{}_b\,\omega_\mu{}^a{}_c - X^a{}_c\,\omega_\mu{}^c{}_b + \underline{\partial}_\mu[X^a{}_b]$

```
PR1["Relationship of spinor connection to coordinate connection: ",
  tmp = ∇X -> (xD["∇", T[X, "u"][ν], μ].(T[dx, "u"][μ] ⊗ T["∂", "d"][ν])) ↔
     xD["∇", T[X, "u"][a], μ].(T[dx, "u"][μ] ⊗ T[ê, "d"][a]),
  NL, "Using definition of covariant derivative and expanding: ",
  sub = {xD["∇", T[X_, "u"][ν_], μ_] :> xPartialD[T[X, "u"][ν], μ] +
        T[Γ, "udd"][ν, μ, λ1] T[X, "u"][λ1] /; ! spinorcoordinate[ν],
     xD["∇", T[X_, "u"][a_], μ_] :> xPartialD[T[X, "u"][a], μ] +
        T[ω, "dud"][μ, a, b] T[X, "u"][b] /; spinorcoordinate[a]
    },
  Yield, tmp = tmp[[2]] /. sub,
  NL,
  sub = {T[X, "u"][a_] :> T[e, "du"][μ2, a] T[X, "u"][μ2] /; spinorcoordinate[a], eJ2} //
     Flatten,
  Yield, tmp = tmp /. sub,
  Yield, tmp = tmp /. simpleNC[CircleTimes, {T[e, "du"][μ_, a_], T[e, "ud"][μ_, a_]}] //.
       simpleDot2[{T[e, "du"][μ_, a_]}] /. eJ0 // DerivativeExpand[{}],
  Yield, tmp = tmp /. Dot -> Times,
  Yield, tmp = tmp /. CircleTimes[__] -> 1 // ExpandAll,
  Yield, tmp = tmp /. eJ3 // KroneckerAbsorb[δ],
  Yield, tmp = tmp /. μ1 -> ν,
  NL, "Both sides should equal: ",
  tmp = tmp /. LeftRightArrow -> Equal // Simplify,
  Imply, Framed[eJ20 = tmp /. μ2 -> λ1 /. T[X, "u"][λ1] -> 1], "  (J.20)"
 ];
```

Relationship of spinor connection to coordinate connection:

$\nabla X \to \underline{\nabla}_{\mu}[X^{\nu}].(dx^{\mu}\otimes\partial_{\nu}) \leftrightarrow \underline{\nabla}_{\mu}[X^{a}].(dx^{\mu}\otimes\hat{e}_{a})$

Using definition of covariant derivative and expanding:

$\{\underline{\nabla}_{\mu\_}[X\_^{\nu\_}] :\to \underline{\partial}_{\mu}[X^{\nu}] + T[\Gamma, udd][\nu, \mu, \lambda1]\ T[X, u][\lambda1]$ /; ! spinorcoordinate[ν],

$\underline{\nabla}_{\mu\_}[X\_^{a\_}] :\to \underline{\partial}_{\mu}[X^{a}] + T[\omega, dud][\mu, a, b]\ T[X, u][b]$ /; spinorcoordinate[a]}

$\to\ (X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1} + \underline{\partial}_{\mu}[X^{\nu}]).(dx^{\mu}\otimes\partial_{\nu}) \leftrightarrow (X^{b}\,\omega_{\mu}{}^{a}{}_{b} + \underline{\partial}_{\mu}[X^{a}]).(dx^{\mu}\otimes\hat{e}_{a})$

$\{X^{a\_} :\to T[e, du][\mu2, a]\ T[X, u][\mu2]$ /; spinorcoordinate[a],

$\hat{e}_{\mu\_} :\to T[e, du][\mu, a1]\ T[\hat{e}, d][a1]$ /; ! spinorcoordinate[μ],

$\hat{e}_{a\_} :\to T[e, ud][\mu1, a]\ T[\hat{e}, d][\mu1]$ /; spinorcoordinate[a]}

$\to\ (X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1} + \underline{\partial}_{\mu}[X^{\nu}]).(dx^{\mu}\otimes\partial_{\nu}) \leftrightarrow (e_{\mu2}{}^{b}\,X^{\mu2}\,\omega_{\mu}{}^{a}{}_{b} + \underline{\partial}_{\mu}[e_{\mu2}{}^{a}\,X^{\mu2}]).(dx^{\mu}\otimes(e^{\mu1}{}_{a}\,\hat{e}_{\mu1}))$

$\to\ (X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1}).(dx^{\mu}\otimes\partial_{\nu}) + \underline{\partial}_{\mu}[X^{\nu}].(dx^{\mu}\otimes\partial_{\nu}) \leftrightarrow$

$(X^{\mu2}\,\underline{\partial}_{\mu}[e_{\mu2}{}^{a}] + e_{\mu2}{}^{a}\,\underline{\partial}_{\mu}[X^{\mu2}]).(dx^{\mu}\otimes\partial_{\mu1}\,e^{\mu1}{}_{a}) + X^{\mu2}.\omega_{\mu}{}^{a}{}_{b}.(dx^{\mu}\otimes\partial_{\mu1}\,e^{\mu1}{}_{a})\,e_{\mu2}{}^{b}$

$\to\ dx^{\mu}\otimes\partial_{\nu}\,X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1} + dx^{\mu}\otimes\partial_{\nu}\,\underline{\partial}_{\mu}[X^{\nu}] \leftrightarrow dx^{\mu}\otimes\partial_{\mu1}\,e_{\mu2}{}^{b}\,e^{\mu1}{}_{a}\,X^{\mu2}\,\omega_{\mu}{}^{a}{}_{b} + dx^{\mu}\otimes\partial_{\mu1}\,e^{\mu1}{}_{a}\,(X^{\mu2}\,\underline{\partial}_{\mu}[e_{\mu2}{}^{a}] + e_{\mu2}{}^{a}\,\underline{\partial}_{\mu}[X^{\mu2}])$

$\to\ X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1} + \underline{\partial}_{\mu}[X^{\nu}] \leftrightarrow e_{\mu2}{}^{b}\,e^{\mu1}{}_{a}\,X^{\mu2}\,\omega_{\mu}{}^{a}{}_{b} + e^{\mu1}{}_{a}\,X^{\mu2}\,\underline{\partial}_{\mu}[e_{\mu2}{}^{a}] + e_{\mu2}{}^{a}\,e^{\mu1}{}_{a}\,\underline{\partial}_{\mu}[X^{\mu2}]$

$\to\ X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1} + \underline{\partial}_{\mu}[X^{\nu}] \leftrightarrow e_{\mu2}{}^{b}\,e^{\mu1}{}_{a}\,X^{\mu2}\,\omega_{\mu}{}^{a}{}_{b} + e^{\mu1}{}_{a}\,X^{\mu2}\,\underline{\partial}_{\mu}[e_{\mu2}{}^{a}] + \underline{\partial}_{\mu}[X^{\mu1}]$

$\to\ X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1} + \underline{\partial}_{\mu}[X^{\nu}] \leftrightarrow e_{\mu2}{}^{b}\,e^{\nu}{}_{a}\,X^{\mu2}\,\omega_{\mu}{}^{a}{}_{b} + e^{\nu}{}_{a}\,X^{\mu2}\,\underline{\partial}_{\mu}[e_{\mu2}{}^{a}] + \underline{\partial}_{\mu}[X^{\nu}]$

Both sides should equal: $X^{\lambda1}\,\Gamma^{\nu}{}_{\mu\,\lambda1} == e^{\nu}{}_{a}\,X^{\mu2}\,(e_{\mu2}{}^{b}\,\omega_{\mu}{}^{a}{}_{b} + \underline{\partial}_{\mu}[e_{\mu2}{}^{a}])$

$\Rightarrow\ \boxed{\Gamma^{\nu}{}_{\mu\,\lambda1} == e^{\nu}{}_{a}\,(e_{\lambda1}{}^{b}\,\omega_{\mu}{}^{a}{}_{b} + \underline{\partial}_{\mu}[e_{\lambda1}{}^{a}])}$ (J.20)

```
PR1["Derive (J.23): Starting with (J.21) ",
  eJ21 = T[ω, "dud"][μ, a, b] -> T[e, "du"][ν, a] T[e, "ud"][λ, b] T[Γ, "udd"][ν, μ, λ] -
     T[e, "ud"][λ, b] xPartialD[T[e, "du"][λ, a], μ],
  NL, "Apply transformation like (J16): ",
  eJ16 = {T[e, "du"][ν, ap] -> T[Λ, "ud"][ap, a1].T[e, "du"][ν, a1],
    T[e, "ud"][ν, ap] -> T[Λ, "ud"][a1, ap].T[e, "ud"][ν, a1]},
  NL, "(J.21) in the p-coordinates: ",
  Yield, tmp = eJ21 /. {a -> ap, b -> bp},
  NL, "Apply (J.16): ",
  sub = RuleX2PatternVar[{a1, ap, ν}][eJ16],
  Yield, tmp = tmp /. sub,
  Yield, tmp = MapAt[# /. a1 -> a2 &, tmp, {{2, 1, 2}, {2, 2, 2}}],
  Yield, tmp = tmp /. Dot -> Times,
  NL, "Apply (J.21): ", subJ21 = Rule4Pattern[eJ21, a__ T[Γ, "udd"][_, _, _]][[1]] //
    RuleX2PatternVar[{a, b, μ, ν, λ}],
  Imply, tmp = tmp /. subJ21 // DerivativeExpand[{}] // Expand,
  Yield, tmp = MapAt[Swap[{a1, a2}][#] &, tmp, {2, 2}],
  NL, "In primed notation: ",
  Framed[eJ23 = tmp /. ap -> a' /. bp -> b' /. eJ3 // KroneckerAbsorb[δ]], " (J.23)"
 ];
```

Derive (J.23): Starting with (J.21) $\omega_\mu{}^a{}_b \to e_\nu{}^a e^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - e^\lambda{}_b \partial_\mu[e_\lambda{}^a]$

Apply transformation like (J16): $\{e_\nu{}^{ap} \to \Lambda^{ap}{}_{a1} . e_\nu{}^{a1},\ e^\nu{}_{ap} \to \Lambda^{a1}{}_{ap} . e^\nu{}_{a1}\}$

(J.21) in the p-coordinates:

$\to\ \omega_\mu{}^{ap}{}_{bp} \to e_\nu{}^{ap} e^\lambda{}_{bp} \Gamma^\nu{}_{\mu\lambda} - e^\lambda{}_{bp} \partial_\mu[e_\lambda{}^{ap}]$

Apply (J.16): $\{e_{\nu\_}{}^{ap\_} \to \Lambda^{ap}{}_{a1} . e_\nu{}^{a1},\ e^{\nu\_}{}_{ap\_} \to \Lambda^{a1}{}_{ap} . e^\nu{}_{a1}\}$

$\to\ \omega_\mu{}^{ap}{}_{bp} \to \Lambda^{a1}{}_{bp} . e^\lambda{}_{a1} \Lambda^{ap}{}_{a1} . e_\nu{}^{a1} \Gamma^\nu{}_{\mu\lambda} - \Lambda^{a1}{}_{bp} . e^\lambda{}_{a1} \partial_\mu[\Lambda^{ap}{}_{a1} . e_\lambda{}^{a1}]$

$\to\ \omega_\mu{}^{ap}{}_{bp} \to \Lambda^{a1}{}_{bp} . e^\lambda{}_{a1} \Lambda^{ap}{}_{a2} . e_\nu{}^{a2} \Gamma^\nu{}_{\mu\lambda} - \Lambda^{a2}{}_{bp} . e^\lambda{}_{a2} \partial_\mu[\Lambda^{ap}{}_{a1} . e_\lambda{}^{a1}]$

$\to\ \omega_\mu{}^{ap}{}_{bp} \to e_\nu{}^{a2} e^\lambda{}_{a1} \Gamma^\nu{}_{\mu\lambda} \Lambda^{a1}{}_{bp} \Lambda^{ap}{}_{a2} - e^\lambda{}_{a2} \Lambda^{a2}{}_{bp} \partial_\mu[e_\lambda{}^{a1} \Lambda^{ap}{}_{a1}]$

Apply (J.21): $\left\{e_{\nu\_}{}^{a\_} e^{\lambda\_}{}_{b\_} \Gamma^{\nu\_}{}_{\mu\_\lambda\_} \to \omega_\mu{}^a{}_b + e^\lambda{}_b \partial_\mu[e_\lambda{}^a]\right\}$

$\Rightarrow\ \omega_\mu{}^{ap}{}_{bp} \to \Lambda^{a1}{}_{bp} \Lambda^{ap}{}_{a2} \omega_\mu{}^{a2}{}_{a1} - e^\lambda{}_{a2} \Lambda^{a2}{}_{bp} \Lambda^{ap}{}_{a1} \partial_\mu[e_\lambda{}^{a1}] + e^\lambda{}_{a1} \Lambda^{a1}{}_{bp} \Lambda^{ap}{}_{a2} \partial_\mu[e_\lambda{}^{a2}] - e_\lambda{}^{a1} e^\lambda{}_{a2} \Lambda^{a2}{}_{bp} \partial_\mu[\Lambda^{ap}{}_{a1}]$

$\to\ \omega_\mu{}^{ap}{}_{bp} \to \Lambda^{a1}{}_{bp} \Lambda^{ap}{}_{a2} \omega_\mu{}^{a2}{}_{a1} - e_\lambda{}^{a1} e^\lambda{}_{a2} \Lambda^{a2}{}_{bp} \partial_\mu[\Lambda^{ap}{}_{a1}]$

In primed notation: $\boxed{\omega_\mu{}^{a'}{}_{b'} \to \Lambda^{a1}{}_{b'} \Lambda^{a'}{}_{a2} \omega_\mu{}^{a2}{}_{a1} - \Lambda^{a2}{}_{b'} \partial_\mu[\Lambda^{a'}{}_{a2}]}$ (J.23)

```
PR1["Switch to TensorForm notation: ",
  eJ26 = {T[ê, "u"][a] -> e@u[a], e@u[a] -> e@du[μ, a] ExteriorD[T[x, "u"][μ]],
    T[ω, "ud"][a, b] -> T[ω, "dud"][μ, a, b] T[dx, "u"][μ]},
  NL, "Define torsion and curvature: ",
  eJ28 = {T[T, "u"][a] -> ExteriorD[T[e, "u"][a]] + T[ω, "ud"][a, b] ∧ T[e, "u"][b],
    T[R, "ud"][a, b] -> ExteriorD[T[ω, "ud"][a, b]] + T[ω, "ud"][a, c] ∧ T[ω, "ud"][c, b]},
  " (μ,ν indices suppressed)."
 ];
```

Switch to TensorForm notation: $\{\hat{e}^a \to e^a,\ e^a \to dx^\mu e_\mu{}^a,\ \omega^a{}_b \to dx^\mu \omega_\mu{}^a{}_b\}$

Define torsion and curvature:

$\{T^a \to de^a + \omega^a{}_b \wedge e^b,\ R^a{}_b \to d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\}$ (μ,ν indices suppressed).

```
PR1[CR["Review of definition of the differentials: "],
  NL, "Exterior derivative(J.24) of ", tmp = T[X, "du"][ν, a], " over μ",
  yield,
  ExteriorD[T[X, "ddu"][μ, ν, a]] -> (tmp1 = xPartialD[tmp, μ]) - Swap[{μ, ν}][tmp1]
 ];
```

Review of definition of the differentials:
Exterior derivative(J.24) of $X_{\nu}{}^{a}$ over μ $\longrightarrow$ $dX_{\mu\nu}{}^{a} \to -\underline{\partial}_{\nu}[X_{\mu}{}^{a}] + \underline{\partial}_{\mu}[X_{\nu}{}^{a}]$

```
PR1["Check standard result for curvature from: ",
  tmp = eJ28[[2]],
  yield,
  eJ28b2 = tmp = tmp //. {T[A_, "ud"][a_, b_] :> AddDnIndex[1, ν][T[A, "ud"][a, b]] /;
        MemberQ[{R, dω}, A], T[A_, "dud"][d_, a_, b_] :>
       AddDnIndex[1, μ][T[A, "dud"][d, a, b]] /; MemberQ[{R, dω}, A],
     T[ω, "ud"][a1_, b1_] :> AddDnIndex[1, ν][T[ω, "ud"][a1, b1]] /; MemberQ[{b}, b1],
     T[ω, "ud"][a1_, b1_] :> AddDnIndex[1, μ][T[ω, "ud"][a1, b1]] /;
        MemberQ[{a}, a1]},
  NL, "which is: ", T[R, "uddd"][ρ, σ, μ, ν] -> 2 AntiSymmetric[{μ, ν}][
     xPartialD[Γ@udd[ρ, ν, σ], μ] + Γ@udd[ρ, μ, λ] Γ@udd[λ, ν, σ]] // Expand,
  yield,
  NL, "Apply vielbein for coordinate indices: ",
  tmp = Map[T[e, "du"][σ, b] T[e, "ud"][λ, a] # &, tmp],
  Yield, tmp = tmp // Expand,
  Yield, tmp[[1]] = tmp[[1]] // KroneckerAbsorb[e];
  yield, tmpR = tmp = tmp /. a1_ ∧ b1_ :> a1 b1 - (b1 a1 // Swap[{μ, ν}])
 ];
```

Check standard result for curvature from: $R^{a}{}_{b} \to d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}$ $\longrightarrow$ $R_{\mu\nu}{}^{a}{}_{b} \to d\omega_{\nu}{}^{a}{}_{b} + \omega_{\mu}{}^{a}{}_{c} \wedge \omega_{\nu}{}^{c}{}_{b}$
which is: $R^{\rho}{}_{\sigma\mu\nu} \to \Gamma^{\lambda}{}_{\nu\sigma}\Gamma^{\rho}{}_{\mu\lambda} - \Gamma^{\lambda}{}_{\mu\sigma}\Gamma^{\rho}{}_{\nu\lambda} - \underline{\partial}_{\nu}[\Gamma^{\rho}{}_{\mu\sigma}] + \underline{\partial}_{\mu}[\Gamma^{\rho}{}_{\nu\sigma}]$ $\longrightarrow$
Apply vielbein for coordinate indices: $e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, R_{\mu\nu}{}^{a}{}_{b} \to e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, (d\omega_{\nu}{}^{a}{}_{b} + \omega_{\mu}{}^{a}{}_{c} \wedge \omega_{\nu}{}^{c}{}_{b})$
$\to$ $e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, R_{\mu\nu}{}^{a}{}_{b} \to d\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b}\, e^{\lambda}{}_{a} + e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, \omega_{\mu}{}^{a}{}_{c} \wedge \omega_{\nu}{}^{c}{}_{b}$
$\to$ $\longrightarrow$ $R_{\mu\nu}{}^{\lambda}{}_{\sigma} \to d\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b}\, e^{\lambda}{}_{a} + e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, (-\omega_{\nu}{}^{a}{}_{c}\, \omega_{\mu}{}^{c}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{b})$

```
(**)
PR1["Evaluate the term: ",
  tmp = tmpR[[2, 2, 3]],
  NL, "From (J.21): ", subJ21,
  Yield, tmpDot = tmp = tmp /. subJ21 /. simpleDot2[{}] // Expand,
  NL, "Unique dummy indices: ",
  tmp = tmp /. Dot[a_, b__] :→ (Dot[a, b] //. {ν1 -> Unique["ν"], λ1 -> Unique["λ"]}),
  NL, "Remove Dot: ", tmpR1 = tmp = tmp /. Dot → Times,
  NL, "CheckIndices: ",
  CheckIndices[tmpR1]
 ];
```

Evaluate the term: $-\omega_{\nu}{}^{a}{}_{c}\, \omega_{\mu}{}^{c}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{b}$
From (J.21): $\{e_{\nu\_}{}^{a\_}\, e^{\lambda\_}{}_{b\_}\, \Gamma^{\nu\_}{}_{\mu\_\lambda\_} \to \omega_{\mu}{}^{a}{}_{b} + e^{\lambda}{}_{b}\, \underline{\partial}_{\mu}[e_{\lambda}{}^{a}]\}$
$\to$ $-\omega_{\nu}{}^{a}{}_{c}\, \omega_{\mu}{}^{c}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{b}$
Unique dummy indices: $-\omega_{\nu}{}^{a}{}_{c}\, \omega_{\mu}{}^{c}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{b}$
Remove Dot: $-\omega_{\nu}{}^{a}{}_{c}\, \omega_{\mu}{}^{c}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{b}$
CheckIndices: {{{c}, {{a}, {ν, μ, b}}, {}}, {{c}, {{a}, {μ, ν, b}}, {}}}

```
eJ22 = xPartialD[T[e, "du"][ν_, a_], μ_] :>
    T[Γ, "udd"][(tmpλ = Unique["λ"]), μ, ν] T[e, "du"][tmpλ, a] -
     T[ω, "dud"][μ, a, tmpb = Unique["b"]] T[e, "du"][ν, tmpb];
PR1["Determine term: ",
  tmpR[[2, 1, 1]], " from ",
  tmp = eJ21 /. {λ -> λ1, ν -> ν1},
  tmp = Map[d[#] &, tmp];
  tmp = tmp /. d[T[ω, "dud"][μ, a, b]] → T[dω, "ddud"][ν, μ, a, b];
  Imply, tmp = tmp /. d[a_] :→ (tmpx = xPartialD[a, ν]) - Swap[{ν, μ}][tmpx],
  tmp = tmp // DerivativeExpand[{}];
  Yield, (tmp = MapAt[Swap[{μ, ν}][#] &, tmp, {2, -1}] // Expand) // ColumnSumExp,
  NL, "Apply J.22: ",
  Imply, (tmpR2 = tmp //. eJ22 // Expand) // ColumnSumExp,
  NL, "CheckIndices: ",
  CheckIndices[tmpR2[[-1]]] // Column,
  Yield, tmpR2 = tmpR2 // RuleX2PatternVar[{ν, μ, a, b}]
 ];
```

Determine term: $d\omega_{\nu}{}^{a}{}_{b}$ from $\omega_{\mu}{}^{a}{}_{b} \to e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\mu\lambda1} - e^{\lambda1}{}_{b} \underline{\partial}_{\mu}[e_{\lambda1}{}^{a}]$

$\Rightarrow d\omega_{\nu\mu}{}^{a}{}_{b} \to \underline{\partial}_{\nu}[e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\mu\lambda1} - e^{\lambda1}{}_{b} \underline{\partial}_{\mu}[e_{\lambda1}{}^{a}]] - \underline{\partial}_{\mu}[e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\nu\lambda1} - e^{\lambda1}{}_{b} \underline{\partial}_{\nu}[e_{\lambda1}{}^{a}]]$

$$\rightarrow d\omega_{\nu\mu}{}^{a}{}_{b} \to \sum\left[\begin{array}{l} -e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\nu\lambda1} \underline{\partial}_{\mu}[e_{\nu1}{}^{a}] \\ e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\mu\lambda1} \underline{\partial}_{\nu}[e_{\nu1}{}^{a}] \\ -e_{\nu1}{}^{a} \Gamma^{\nu1}{}_{\nu\lambda1} \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] \\ \underline{\partial}_{\nu}[e_{\lambda1}{}^{a}]\, \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] \\ e_{\nu1}{}^{a} \Gamma^{\nu1}{}_{\mu\lambda1} \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] \\ -\underline{\partial}_{\mu}[e_{\lambda1}{}^{a}]\, \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] \\ e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \underline{\partial}_{\nu}[\Gamma^{\nu1}{}_{\mu\lambda1}] \\ -e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \underline{\partial}_{\mu}[\Gamma^{\nu1}{}_{\nu\lambda1}] \end{array}\right]$$

Apply J.22:

$$\Rightarrow d\omega_{\nu\mu}{}^{a}{}_{b} \to \sum\left[\begin{array}{l} e_{\lambda7}{}^{a} e^{\lambda1}{}_{b} \Gamma^{\lambda7}{}_{\nu\nu1} \Gamma^{\nu1}{}_{\mu\lambda1} \\ -e_{\lambda3}{}^{a} e^{\lambda1}{}_{b} \Gamma^{\lambda3}{}_{\mu\nu1} \Gamma^{\nu1}{}_{\nu\lambda1} \\ e_{\nu1}{}^{b6} e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\nu\lambda1} \omega_{\mu}{}^{a}{}_{b6} \\ -e_{\nu1}{}^{b8} e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\mu\lambda1} \omega_{\nu}{}^{a}{}_{b8} \\ e_{\lambda9}{}^{a} \Gamma^{\lambda9}{}_{\nu\lambda1} \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] \\ -e_{\nu1}{}^{a} \Gamma^{\nu1}{}_{\nu\lambda1} \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] \\ -e_{\lambda1}{}^{b10} \omega_{\nu}{}^{a}{}_{b10} \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] \\ -e_{\lambda11}{}^{a} \Gamma^{\lambda11}{}_{\mu\lambda1} \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] \\ e_{\nu1}{}^{a} \Gamma^{\nu1}{}_{\mu\lambda1} \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] \\ e_{\lambda1}{}^{b12} \omega_{\mu}{}^{a}{}_{b12} \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] \\ e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \underline{\partial}_{\nu}[\Gamma^{\nu1}{}_{\mu\lambda1}] \\ -e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \underline{\partial}_{\mu}[\Gamma^{\nu1}{}_{\nu\lambda1}] \end{array}\right]$$

CheckIndices:

```
{{λ7, λ1, ν1}, {{a}, {b, ν, μ}}, {}}
{{λ3, λ1, ν1}, {{a}, {b, μ, ν}}, {}}
{{ν1, b6, λ1}, {{a}, {b, ν, μ}}, {}}
{{ν1, b8, λ1}, {{a}, {b, μ, ν}}, {}}
{{λ9, λ1}, {{a}, {b, ν, μ}}, {}}
{{ν1, λ1}, {{a}, {b, ν, μ}}, {}}
{{λ1, b10}, {{a}, {b, ν, μ}}, {}}
{{λ11, λ1}, {{a}, {b, μ, ν}}, {}}
{{ν1, λ1}, {{a}, {b, μ, ν}}, {}}
{{λ1, b12}, {{a}, {b, μ, ν}}, {}}
{{ν1, λ1}, {{a}, {b, μ, ν}}, {}}
{{ν1, λ1}, {{a}, {b, ν, μ}}, {}}
```

$\rightarrow$

$$d\omega_{\nu\_\mu\_}{}^{a\_}{}_{b\_} \to e_{\lambda7}{}^{a} e^{\lambda1}{}_{b} \Gamma^{\lambda7}{}_{\nu\nu1} \Gamma^{\nu1}{}_{\mu\lambda1} - e_{\lambda3}{}^{a} e^{\lambda1}{}_{b} \Gamma^{\lambda3}{}_{\mu\nu1} \Gamma^{\nu1}{}_{\nu\lambda1} + e_{\nu1}{}^{b6} e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\nu\lambda1} \omega_{\mu}{}^{a}{}_{b6} - e_{\nu1}{}^{b8} e^{\lambda1}{}_{b} \Gamma^{\nu1}{}_{\mu\lambda1} \omega_{\nu}{}^{a}{}_{b8} + e_{\lambda9}{}^{a} \Gamma^{\lambda9}{}_{\nu\lambda1} \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] - e_{\nu1}{}^{a} \Gamma^{\nu1}{}_{\nu\lambda1} \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] - e_{\lambda1}{}^{b10} \omega_{\nu}{}^{a}{}_{b10} \underline{\partial}_{\mu}[e^{\lambda1}{}_{b}] - e_{\lambda11}{}^{a} \Gamma^{\lambda11}{}_{\mu\lambda1} \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] + e_{\nu1}{}^{a} \Gamma^{\nu1}{}_{\mu\lambda1} \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] + e_{\lambda1}{}^{b12} \omega_{\mu}{}^{a}{}_{b12} \underline{\partial}_{\nu}[e^{\lambda1}{}_{b}] + e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \underline{\partial}_{\nu}[\Gamma^{\nu1}{}_{\mu\lambda1}] - e_{\nu1}{}^{a} e^{\lambda1}{}_{b} \underline{\partial}_{\mu}[\Gamma^{\nu1}{}_{\nu\lambda1}]$$

```
PR1["Continuing: ",
  tmpR[[2, 2, 3]] = tmpR1;
  tmp = tmpR /. tmpR2 // Expand;
  tmp = tmp //. eJ3 // KroneckerAbsorb[δ] // SymmetrizeSlots[];
  tmp /. checkeindex;
  CheckIndices[tmp[[2]]] // Column;
  tmp11 = tmp
 ];
```

Continuing: $R_{\mu\nu}{}^{\lambda}{}_{\sigma} \to \mathbb{d}\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b}\, e^{\lambda}{}_{a} - e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, \omega_{\nu}{}^{a}{}_{c}\, \omega_{\mu}{}^{c}{}_{b} + e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, \omega_{\mu}{}^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{b}$

```
letters = Join[CharacterRange["A", "Z"], CharacterRange["a", "z"]];
firstsymbol[var_] := Module[{chars = Characters[ToString[var]]},
    chars[[1]]
  ];
firstSymbolLength[var_] := Module[{chars = Characters[ToString[var]]},
    xPrint[chars, ":", Length[chars]];
    {chars[[1]], Length[chars]}
  ];
checkeindex := {T[e, "du"][λ_, a_] :> T[Style[e, Red], "du"][λ, a] /;
     ! MemberQ[letters, firstsymbol[a] ], T[e, "ud"][λ_, a_] :>
    T[Style[e, Red], "ud"][λ, a] /; ! MemberQ[letters, firstsymbol[a] ]
 }
(**)
PR1["Continuing: ",
  tmp = tmp11 /.
    {νq_ :> ν1 /; firstSymbolLength[νq][[1]] === "ν" && firstSymbolLength[νq][[2]] > 1,
     λq_ :> λ1 /; firstSymbolLength[λq][[1]] === "λ" && firstSymbolLength[λq][[2]] > 1,
     bq_ :> b1 /; firstSymbolLength[bq][[1]] === "b" && firstSymbolLength[bq][[2]] > 1};
  tmp /. checkeindex;
  CheckIndices[tmp] // Column;
  tmp =
   tmp /. (subJ21 /. λ1 -> λ2 /. ν1 -> ν2) /. simpleDot2[{}] /. Dot -> Times // Expand;
  tmp = tmp //. eJ3 // KroneckerAbsorb[δ];
  CheckIndices[tmp[[2]]] // Column;
  tmp[[2]] // ColumnSum;
  tmp12 = tmp,
  NL, CR["The cancellation of the e-terms has been difficult to show."]
 ];
```

Continuing: $R_{\mu\nu}{}^{\lambda}{}_{\sigma} \to \mathbb{d}\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b}\, e^{\lambda}{}_{a} - e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, \omega_{\nu}{}^{a}{}_{c}\, \omega_{\mu}{}^{c}{}_{b} + e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\, \omega_{\mu}{}^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{b}$

The cancellation of the e-terms has been difficult to show.

```
(tmp = ExtractPattern[tmp12, a__T[e, "du"][_, _]]) // Column;
Apply[Plus, tmp] // Simplify
% / e@ud[λ, a];
% /. λ1 -> ν1 // Expand
Collect[%, {e@du[ν1, b1], e@du[σ, b]}]
```

$e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\left(d\omega_{\nu}{}^{a}{}_{b} - \omega_{\nu}{}^{a}{}_{c}\,\omega_{\mu}{}^{c}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\,\omega_{\nu}{}^{c}{}_{b}\right)$

$d\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b} - e_{\sigma}{}^{b}\,\omega_{\nu}{}^{a}{}_{c}\,\omega_{\mu}{}^{c}{}_{b} + e_{\sigma}{}^{b}\,\omega_{\mu}{}^{a}{}_{c}\,\omega_{\nu}{}^{c}{}_{b}$

$e_{\sigma}{}^{b}\left(d\omega_{\nu}{}^{a}{}_{b} - \omega_{\nu}{}^{a}{}_{c}\,\omega_{\mu}{}^{c}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\,\omega_{\nu}{}^{c}{}_{b}\right)$

```
BlankDummyIndices[exp_] := Module[{tmp = Apply[List, exp], indices},
    indices = Map[ParseTermIndices[#] &, tmp]; xPrint[indices // Column];
    MapIndexed[tmp[[#2]] /. Thread[#1[[1]] -> _] &, indices]
   ];
tmp = tmp12
tmp = tmp /. eJ22
ColumnSum[tmp[[2]]]
tmp = BlankDummyIndices[tmp[[2]]];
Apply[Plus, tmp][[1]] // Simplify;
ColumnSum[%];
```

$R_{\mu\nu}{}^{\lambda}{}_{\sigma} \to d\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b}\, e^{\lambda}{}_{a} - e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\,\omega_{\nu}{}^{a}{}_{c}\,\omega_{\mu}{}^{c}{}_{b} + e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\,\omega_{\mu}{}^{a}{}_{c}\,\omega_{\nu}{}^{c}{}_{b}$

$R_{\mu\nu}{}^{\lambda}{}_{\sigma} \to d\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b}\, e^{\lambda}{}_{a} - e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\,\omega_{\nu}{}^{a}{}_{c}\,\omega_{\mu}{}^{c}{}_{b} + e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\,\omega_{\mu}{}^{a}{}_{c}\,\omega_{\nu}{}^{c}{}_{b}$

$d\omega_{\nu}{}^{a}{}_{b}\, e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}$
$-e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\,\omega_{\nu}{}^{a}{}_{c}\,\omega_{\mu}{}^{c}{}_{b}$
$e_{\sigma}{}^{b}\, e^{\lambda}{}_{a}\,\omega_{\mu}{}^{a}{}_{c}\,\omega_{\nu}{}^{c}{}_{b}$

Exercise J.1.2

```
PR1["2. Calculate the connection one-forms(",
  eJ27 = ω@ud[a, b] -> ω@dud[μ, a, b] ExteriorD[x@u[μ]], "), curvature two-forms(",
  eJ29 = R@ud[a, b] -> ExteriorD[ω@ud[a, b]] + ω@ud[a, c] ∧ ω@ud[c, b],
  "), and hence the components of the Riemann tensor for
    the Mixmaster universe.  The metric is given by: ", tmpds =
   ds^2 -> -dt ⊗ dt + α^2 σ@u[1] ⊗ σ@u[1] + β^2 σ@u[2] ⊗ σ@u[2] + γ^2 σ@u[3] ⊗ σ@u[3],
  " where α, β, γ are functions of t only and the one-forms ",
  (tmpσ = {σ@u[1] -> Cos[ψ] dθ + Sin[ψ] Sin[θ] dϕ,
      σ@u[2] -> Sin[ψ] dθ - Cos[ψ] Sin[θ] dϕ,
      σ@u[3] -> dψ + Cos[θ] dϕ}) // Column
 ];
```

2. Calculate the connection one-forms($\omega^{a}{}_{b} \to \mathbb{d}x^{\mu}\, \omega_{\mu}{}^{a}{}_{b}$), curvature two-forms($R^{a}{}_{b} \to \mathbb{d}\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}$), and hence the components of the Riemann tensor for the Mixmaster universe. The metric is given by: $ds^2 \to -dt \otimes dt + \alpha^2\, \sigma^1 \otimes \sigma^1 + \beta^2\, \sigma^2 \otimes \sigma^2 + \gamma^2\, \sigma^3 \otimes \sigma^3$ where α, β, γ are functions of t only and the one-forms

$\sigma^1 \to d\theta\, \mathrm{Cos}[\psi] + d\phi\, \mathrm{Sin}[\theta]\, \mathrm{Sin}[\psi]$

$\sigma^2 \to -d\phi\, \mathrm{Cos}[\psi]\, \mathrm{Sin}[\theta] + d\theta\, \mathrm{Sin}[\psi]$

$\sigma^3 \to d\psi + d\phi\, \mathrm{Cos}[\theta]$

```
(*****)
PR1["Translating into TensorForm notation
Follow similar steps from J.35: ",
  NL, "The σ-metric: ",
  tmpds /. T[σ, "u"][a_] -> d[T[σ, "u"][a]] /. dt -> d[t] /. d[a_] -> ExteriorD[a],
  NL, "by taking ", coef = {α, β, γ};
  tmpe = {e@u[a_] :> If[a > 0, c[a] T[σ, "u"][a], d[t]],
     c[a_] :> If[a > 0, coef[[a]], c0]} /. d[a_] -> ExteriorD[a],
  NL, "the metric: ", tmp = eJ37 = ds^2 -> η@dd[a, b] e@u[a] ⊗ e@u[b],
  yield, tmp = tmp /. tmpe,
  NL, "Using the conditions: ",
  eJ39 = {ω@ud[0, 0] -> 0, ω@ud[0, j_] -> ω@ud[j, 0],
    ω@ud[i_, j_] :> -ω@ud[j, i] /; i > 0 && j > 0},
  NL, "Compute the spin connection from this and (J.35): ",
  eJ35 = ω@ud[a, b1] ∧ e@u[b1] == -ExteriorD[e@u[a]],
  NL, "Computing RHS of J.35: ",
  tmpd = -ExteriorD[e@u[i]],
  yield, tmp2 = Table[tmpd, {i, 0, 3}],
  imply, (tmpr = Table[tmpd -> (
         ExpandExteriorD0[coef, {} ][ tmp2[[i + 1]] //. tmpe ]
         ), {i, 0, 3}]
   ) // Column,
  NL, "σ are 1-forms: ", sub = ExteriorD[σ@u[_]] → 0,
  imply, (tmpr = tmpr /. sub) // Column,
  NL, coef, " are only functions of t: ",
  sub = ExteriorD[a_] :→ xPartialD[a, t] ExteriorD[t]
     /; MemberQ[coef, a],
  imply, (tmpr = tmpr /. sub // WedgeSimplify[{}]) // Column
 ];
```

Translating into TensorForm notation
Follow similar steps from J.35:
The σ-metric: $ds^2 \to -dt \otimes dt + \alpha^2 \, d\sigma^1 \otimes d\sigma^1 + \beta^2 \, d\sigma^2 \otimes d\sigma^2 + \gamma^2 \, d\sigma^3 \otimes d\sigma^3$
by taking $\{e^{a\_} :\to \mathrm{If}[a > 0, c[a]\, T[\sigma, u][a], dt], c[a\_] :\to \mathrm{If}[a > 0, \mathrm{coef}[\![a]\!], c0]\}$
the metric: $ds^2 \to e^a \otimes e^b \, \eta_{ab} \longrightarrow$
$ds^2 \to \mathrm{If}[a > 0, c[a]\, T[\sigma, u][a], dt] \otimes \mathrm{If}[b > 0, c[b]\, T[\sigma, u][b], dt]\, \eta_{ab}$
Using the conditions: $\{\omega^0{}_0 \to 0, \omega^0{}_{j\_} \to \omega^j{}_0, \omega^{i\_}{}_{j\_} :\to -\omega[\mathrm{ud}[j, i]] \,/;\, i > 0 \,\&\&\, j > 0\}$
Compute the spin connection from this and (J.35): $\omega^a{}_{b1} \wedge e^{b1} == -d e^a$

Computing RHS of J.35: $-d e^i \longrightarrow \{-d e^0, -d e^1, -d e^2, -d e^3\} \Rightarrow$

$$\begin{array}{l} -d e^0 \to 0 \\ -d e^1 \to -\alpha \,\mathrm{Wedge}[d\sigma^1] - d\alpha \wedge \sigma^1 \\ -d e^2 \to -\beta \,\mathrm{Wedge}[d\sigma^2] - d\beta \wedge \sigma^2 \\ -d e^3 \to -\gamma \,\mathrm{Wedge}[d\sigma^3] - d\gamma \wedge \sigma^3 \end{array}$$

σ are 1-forms: $d\sigma\_ \to 0 \Rightarrow$

$$\begin{array}{l} -d e^0 \to 0 \\ -d e^1 \to -(d\alpha \wedge \sigma^1) \\ -d e^2 \to -(d\beta \wedge \sigma^2) \\ -d e^3 \to -(d\gamma \wedge \sigma^3) \end{array}$$

$\{\alpha, \beta, \gamma\}$ are only functions of t: $d(a\_) :\to \underline{\partial}_t[a]\, dt \,/;\, \mathrm{MemberQ}[\mathrm{coef}, a] \Rightarrow$

$$\begin{array}{l} -d e^0 \to 0 \\ -d e^1 \to \sigma^1 \wedge (dt\, \underline{\partial}_t[\alpha]) \\ -d e^2 \to \sigma^2 \wedge (dt\, \underline{\partial}_t[\beta]) \\ -d e^3 \to \sigma^3 \wedge (dt\, \underline{\partial}_t[\gamma]) \end{array}$$

```
PR1["The LHS: ",
  tmpl = eJ35[[1]],
  imply, (tmpl = Table[tmpl, {a, 0, 3}]) // Column,
  tmpr1 = Map[#[[2]] &, tmpr];
  imply, (xtmp = tmp = Thread[tmpl -> tmpr1]) // Column,
  Yield, tmpa = Map[MapAt[Sum[( # /. b1 -> ii), {ii, 0, 3}] &, #, {1}] &, tmp]
    // Column,
  NL, "Using the conditions J.39: ", eJ39,
  Yield, subs = {eJ39[[1]]},
  NL, "From the first equation: ", tmp = tmpa[[1, 1]],
  yield, tmp = tmp //. tmpe /. subs // WedgeSimplify[coef],
  NL, "Is satisfied by: ", sub = {ω@ud[0, i_] :> c1[i] σ@u[i] /; i > 0},
  NL, "Imply the conditions: ", subs = Join[sub, eJ39],
  NL, "We are left with: ",
  tmpa = tmpa //. tmpe /. sub /. subs /. sub // WedgeSimplify[Append[coef, c1[_]]]
 ];
```

The LHS: $\omega^{a}{}_{b1}\wedge e^{b1} \Rightarrow \begin{matrix} \omega^{0}{}_{b1}\wedge e^{b1} \\ \omega^{1}{}_{b1}\wedge e^{b1} \\ \omega^{2}{}_{b1}\wedge e^{b1} \\ \omega^{3}{}_{b1}\wedge e^{b1} \end{matrix} \Rightarrow \begin{matrix} \omega^{0}{}_{b1}\wedge e^{b1}\to 0 \\ \omega^{1}{}_{b1}\wedge e^{b1}\to \sigma^{1}\wedge (dt\,\partial_t[\alpha]) \\ \omega^{2}{}_{b1}\wedge e^{b1}\to \sigma^{2}\wedge (dt\,\partial_t[\beta]) \\ \omega^{3}{}_{b1}\wedge e^{b1}\to \sigma^{3}\wedge (dt\,\partial_t[\gamma]) \end{matrix}$

$$\to \begin{matrix} \omega^{0}{}_{0}\wedge e^{0}+\omega^{0}{}_{1}\wedge e^{1}+\omega^{0}{}_{2}\wedge e^{2}+\omega^{0}{}_{3}\wedge e^{3}\to 0 \\ \omega^{1}{}_{0}\wedge e^{0}+\omega^{1}{}_{1}\wedge e^{1}+\omega^{1}{}_{2}\wedge e^{2}+\omega^{1}{}_{3}\wedge e^{3}\to \sigma^{1}\wedge (dt\,\partial_t[\alpha]) \\ \omega^{2}{}_{0}\wedge e^{0}+\omega^{2}{}_{1}\wedge e^{1}+\omega^{2}{}_{2}\wedge e^{2}+\omega^{2}{}_{3}\wedge e^{3}\to \sigma^{2}\wedge (dt\,\partial_t[\beta]) \\ \omega^{3}{}_{0}\wedge e^{0}+\omega^{3}{}_{1}\wedge e^{1}+\omega^{3}{}_{2}\wedge e^{2}+\omega^{3}{}_{3}\wedge e^{3}\to \sigma^{3}\wedge (dt\,\partial_t[\gamma]) \end{matrix}$$

Using the conditions J.39: $\{\omega^{0}{}_{0}\to 0,\ \omega^{0}{}_{j\_}\to \omega^{j}{}_{0},\ \omega^{i\_}{}_{j\_} :\to -\omega[ud[j, i]] /; i>0\ \&\&\ j>0\}$

$\to \{\omega^{0}{}_{0}\to 0\}$

From the first equation:

$\omega^{0}{}_{0}\wedge e^{0}+\omega^{0}{}_{1}\wedge e^{1}+\omega^{0}{}_{2}\wedge e^{2}+\omega^{0}{}_{3}\wedge e^{3}\to 0 \longrightarrow -\alpha\,\sigma^{1}\wedge\omega^{0}{}_{1}-\beta\,\sigma^{2}\wedge\omega^{0}{}_{2}-\gamma\,\sigma^{3}\wedge\omega^{0}{}_{3}\to 0$

Is satisfied by: $\{\omega^{0}{}_{i\_} :\to c1[i]\,\sigma[u[i]] /; i>0\}$

Imply the conditions:

$\{\omega^{0}{}_{i\_} :\to c1[i]\,\sigma[u[i]] /; i>0,\ \omega^{0}{}_{0}\to 0,\ \omega^{0}{}_{j\_}\to \omega^{j}{}_{0},\ \omega^{i\_}{}_{j\_} :\to -\omega[ud[j, i]] /; i>0\ \&\&\ j>0\}$

We are left with: $\begin{matrix} 0\to 0 \\ -(dt\wedge\omega^{1}{}_{0})+\alpha\,\sigma^{1}\wedge\omega^{1}{}_{1}+\beta\,\sigma^{2}\wedge\omega^{2}{}_{1}+\gamma\,\sigma^{3}\wedge\omega^{3}{}_{1}\to \sigma^{1}\wedge (dt\,\partial_t[\alpha]) \\ -(dt\wedge\omega^{2}{}_{0})+\alpha\,\sigma^{1}\wedge\omega^{1}{}_{2}+\beta\,\sigma^{2}\wedge\omega^{2}{}_{2}+\gamma\,\sigma^{3}\wedge\omega^{3}{}_{2}\to \sigma^{2}\wedge (dt\,\partial_t[\beta]) \\ -(dt\wedge\omega^{3}{}_{0})+\alpha\,\sigma^{1}\wedge\omega^{1}{}_{3}+\beta\,\sigma^{2}\wedge\omega^{2}{}_{3}+\gamma\,\sigma^{3}\wedge\omega^{3}{}_{3}\to \sigma^{3}\wedge (dt\,\partial_t[\gamma]) \end{matrix}$

```
tmpa
PR1["Using the condition: ", sub = {ω@ud[i_, j_] :>
      If[(i + j == j > 0 || i + j == i) && i + j > 0, σ@u[i + j] xPartialD[coef[[i + j]], t], 0]},
  Yield, tmp = tmpa /. sub // WedgeSimplify[Join[1 / coef, {xPartialD[_, _]}]],
  Yield, tmp /. Rule -> Equal // Simplify,
  NL, "We find solutions for c1[i].",

  NL, "We have a solution for the curvature 2-forms: ",
  subs = Join[sub, eJ39]
 ];
```

$0 \to 0$

$-\left(\mathrm{d}t \wedge \omega^1{}_0\right) + \alpha\,\sigma^1 \wedge \omega^1{}_1 + \beta\,\sigma^2 \wedge \omega^2{}_1 + \gamma\,\sigma^3 \wedge \omega^3{}_1 \to \sigma^1 \wedge (\mathrm{d}t\,\underline{\partial}_t[\alpha])$

$-\left(\mathrm{d}t \wedge \omega^2{}_0\right) + \alpha\,\sigma^1 \wedge \omega^1{}_2 + \beta\,\sigma^2 \wedge \omega^2{}_2 + \gamma\,\sigma^3 \wedge \omega^3{}_2 \to \sigma^2 \wedge (\mathrm{d}t\,\underline{\partial}_t[\beta])$

$-\left(\mathrm{d}t \wedge \omega^3{}_0\right) + \alpha\,\sigma^1 \wedge \omega^1{}_3 + \beta\,\sigma^2 \wedge \omega^2{}_3 + \gamma\,\sigma^3 \wedge \omega^3{}_3 \to \sigma^3 \wedge (\mathrm{d}t\,\underline{\partial}_t[\gamma])$

Using the condition:

$\left\{\omega^{i\_}{}_{j\_} \mapsto \mathrm{If}[(i + j == j > 0 \;||\; i + j == i)\ \&\&\ i + j > 0,\ \sigma[u[i + j]]\ \underline{\partial}_t[\{\alpha, \beta, \gamma\}[\![i + j]\!]],\ 0]\right\}$

$\to$
$0 \to 0$
$-\left(\mathrm{d}t \wedge \sigma^1\right) \underline{\partial}_t[\alpha] \to -\left(\mathrm{d}t \wedge \sigma^1\right) \underline{\partial}_t[\alpha]$
$-\left(\mathrm{d}t \wedge \sigma^2\right) \underline{\partial}_t[\beta] \to -\left(\mathrm{d}t \wedge \sigma^2\right) \underline{\partial}_t[\beta]$
$-\left(\mathrm{d}t \wedge \sigma^3\right) \underline{\partial}_t[\gamma] \to -\left(\mathrm{d}t \wedge \sigma^3\right) \underline{\partial}_t[\gamma]$

$\to$
True
True
True
True

We find solutions for c1[i].
We have a solution for the curvature 2-forms:

$\left\{\omega^{i\_}{}_{j\_} \mapsto \mathrm{If}[(i + j == j > 0 \;||\; i + j == i)\ \&\&\ i + j > 0,\ \sigma[u[i + j]]\ \underline{\partial}_t[\{\alpha, \beta, \gamma\}[\![i + j]\!]],\ 0],\right.$
$\left.\omega^0{}_0 \to 0,\ \omega^0{}_{j\_} \to \omega^j{}_0,\ \omega^{i\_}{}_{j\_} \mapsto -\omega[ud[j, i]]\ /;\ i > 0\ \&\&\ j > 0\right\}$

```
PR1["Check if σ's are 1-forms.
Take exterior derivative of σ's in {ϕ,ψ,θ} coordinates: ",
  tmp = tmpσ[[3]] /. {dθ -> ExteriorD[θ], dϕ -> ExteriorD[ϕ], dψ -> ExteriorD[ψ]},
  Yield, tmp = Map[ExteriorD[#] &, tmp],
  yield, tmp = tmp[[2]] // ExpandExteriorD0[{Cos[_], Sin[_]}, {}],
  yield, tmp // ExpandExteriorD[labels, {i, j}] // ExteriorDContract //
   ExpandExteriorD0[{Cos[_], Sin[_]}, {}],
  CR[" They don't seem to be."]
 ];
```

Check if σ's are 1-forms.
Take exterior derivative of σ's in {ϕ,ψ,θ} coordinates: $\sigma^3 \to \mathrm{Cos}[\theta]\,\mathrm{d}\phi + \mathrm{d}\psi$
$\to \mathrm{d}\sigma^3 \to \mathrm{d}(\mathrm{Cos}[\theta]\,\mathrm{d}\phi) \longrightarrow -(\mathrm{d}\phi \wedge \mathrm{d}(\mathrm{Cos}[\theta])) \longrightarrow -\mathrm{Sin}[\theta]\,\mathrm{d}\theta \wedge \mathrm{d}\phi$ They don't seem to be.

```
e3113 = R@uddd[ρ, σ, μ, ν] -> xPartialD[Γ@udd[ρ, ν, σ], μ] - xPartialD[Γ@udd[ρ, μ, σ], ν] +
    Γ@udd[ρ, μ, λ] Γ@udd[λ, ν, σ] - Γ@udd[ρ, ν, λ] Γ@udd[λ, μ, σ];
PR1["The curvature 2-form using the spin connection is given by J.45: ",
  eJ45 = eJ28[[2]],
  CR[" Recall the two Greek indices are suppressed."],
  NL, "Substitute definitions: ", tmp0 = eJ45, " for ",
  NL,
  (tmp = Table[
        tmp = tmp0;
        tmp = MapAt[Sum[#, {c, 0, 3}] &, tmp, {2, 2}] /. subs, {a, 0, 3}, {b, 0, 3}]
       // WedgeSimplify[{xPartialD[_, _]}]
      // ExpandExteriorD0[{xPartialD[_, _]}, {}];
    tmp = tmp /. ExteriorD[σ@u[_]] -> 0)
   // MatrixForms,
  NL, "Substitute: ", sub = {ExteriorD[xPartialD[a_, b_]] -> xPartialD[ExteriorD[a], b],
    ExteriorD[a : (α | β | γ)] -> ExteriorD[t] xPartialD[a, t]},
  Imply, (tmp = tmp //. sub) // MatrixForms,
  Yield,
  (tmpb = tmp //. xPartialDExpand[{ExteriorD[t]}] // WedgeSimplify[{xPartialD[_, _]}])
   // MatrixForms
 ];
```

The curvature 2-form using the spin connection is given by J.45:

$R^a{}_b \to d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ Recall the two Greek indices are suppressed.

Substitute definitions: $R^a{}_b \to d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ for

$$\begin{pmatrix} R^0{}_0 \to 0 & R^0{}_1 \to -\left(d(\underline{\partial}_t[\alpha]) \wedge \sigma^1\right) & R^0{}_2 \to -\left(d(\underline{\partial}_t[\beta]) \wedge \sigma^2\right) & R^0{}_3 \to -\left(d(\underline{\partial}_t[\gamma]) \wedge \sigma^3\right) \\ R^1{}_0 \to -\left(d(\underline{\partial}_t[\alpha]) \wedge \sigma^1\right) & R^1{}_1 \to 0 & R^1{}_2 \to \sigma^1 \wedge \sigma^2\, \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\beta] & R^1{}_3 \to \sigma^1 \wedge \sigma^3\, \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\gamma] \\ R^2{}_0 \to -\left(d(\underline{\partial}_t[\beta]) \wedge \sigma^2\right) & R^2{}_1 \to -\left(\sigma^1 \wedge \sigma^2\right) \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\beta] & R^2{}_2 \to 0 & R^2{}_3 \to \sigma^2 \wedge \sigma^3\, \underline{\partial}_t[\beta]\, \underline{\partial}_t[\gamma] \\ R^3{}_0 \to -\left(d(\underline{\partial}_t[\gamma]) \wedge \sigma^3\right) & R^3{}_1 \to -\left(\sigma^1 \wedge \sigma^3\right) \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\gamma] & R^3{}_2 \to -\left(\sigma^2 \wedge \sigma^3\right) \underline{\partial}_t[\beta]\, \underline{\partial}_t[\gamma] & R^3{}_3 \to 0 \end{pmatrix}$$

Substitute: $\left\{d\left(\underline{\partial}_{b\_}[a\_]\right) \to \underline{\partial}_b[d a],\ d(a : \alpha \mid \beta \mid \gamma) \to d t\, \underline{\partial}_t[a]\right\}$

$$\Rightarrow \begin{pmatrix} R^0{}_0 \to 0 & R^0{}_1 \to -\left(\underline{\partial}_t[d t\, \underline{\partial}_t[\alpha]] \wedge \sigma^1\right) & R^0{}_2 \to -\left(\underline{\partial}_t[d t\, \underline{\partial}_t[\beta]] \wedge \sigma^2\right) & R^0{}_3 \to -\left(\underline{\partial}_t[d t\, \underline{\partial}_t[\gamma]] \wedge \right. \\ R^1{}_0 \to -\left(\underline{\partial}_t[d t\, \underline{\partial}_t[\alpha]] \wedge \sigma^1\right) & R^1{}_1 \to 0 & R^1{}_2 \to \sigma^1 \wedge \sigma^2\, \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\beta] & R^1{}_3 \to \sigma^1 \wedge \sigma^3\, \underline{\partial}_t[\alpha]\, \underline{\partial}_t \\ R^2{}_0 \to -\left(\underline{\partial}_t[d t\, \underline{\partial}_t[\beta]] \wedge \sigma^2\right) & R^2{}_1 \to -\left(\sigma^1 \wedge \sigma^2\right) \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\beta] & R^2{}_2 \to 0 & R^2{}_3 \to \sigma^2 \wedge \sigma^3\, \underline{\partial}_t[\beta]\, \underline{\partial}_t \\ R^3{}_0 \to -\left(\underline{\partial}_t[d t\, \underline{\partial}_t[\gamma]] \wedge \sigma^3\right) & R^3{}_1 \to -\left(\sigma^1 \wedge \sigma^3\right) \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\gamma] & R^3{}_2 \to -\left(\sigma^2 \wedge \sigma^3\right) \underline{\partial}_t[\beta]\, \underline{\partial}_t[\gamma] & R^3{}_3 \to 0 \end{pmatrix}$$

$$\to \begin{pmatrix} R^0{}_0 \to 0 & R^0{}_1 \to -\left(d t \wedge \sigma^1\right) \underline{\partial}_t[\underline{\partial}_t[\alpha]] & R^0{}_2 \to -\left(d t \wedge \sigma^2\right) \underline{\partial}_t[\underline{\partial}_t[\beta]] & R^0{}_3 \to -\left(d t \wedge \sigma^3\right) \underline{\partial}_t[\underline{\partial}_t \\ R^1{}_0 \to -\left(d t \wedge \sigma^1\right) \underline{\partial}_t[\underline{\partial}_t[\alpha]] & R^1{}_1 \to 0 & R^1{}_2 \to \sigma^1 \wedge \sigma^2\, \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\beta] & R^1{}_3 \to \sigma^1 \wedge \sigma^3\, \underline{\partial}_t[\alpha]\, \underline{\partial}_t \\ R^2{}_0 \to -\left(d t \wedge \sigma^2\right) \underline{\partial}_t[\underline{\partial}_t[\beta]] & R^2{}_1 \to -\left(\sigma^1 \wedge \sigma^2\right) \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\beta] & R^2{}_2 \to 0 & R^2{}_3 \to \sigma^2 \wedge \sigma^3\, \underline{\partial}_t[\beta]\, \underline{\partial}_t \\ R^3{}_0 \to -\left(d t \wedge \sigma^3\right) \underline{\partial}_t[\underline{\partial}_t[\gamma]] & R^3{}_1 \to -\left(\sigma^1 \wedge \sigma^3\right) \underline{\partial}_t[\alpha]\, \underline{\partial}_t[\gamma] & R^3{}_2 \to -\left(\sigma^2 \wedge \sigma^3\right) \underline{\partial}_t[\beta]\, \underline{\partial}_t[\gamma] & R^3{}_3 \to 0 \end{pmatrix}$$

```
PR1["How do you translate ", tmp = eJ28[[2]], " to all indices?  Recall J.27: ",
  sJ27 = eJ27 /. μ -> μ1 // RulesVarPattern[{a, b}],
  imply, xtmp = tmp = tmp /. sJ27,
  yield, tmp = tmp // ExpandExteriorD0[{}, {}],
  yield, tmp = tmp // UniqueDummyIndices[{μ1}];
  Yield, xtmp = tmp = tmp // ExpandExteriorD0[{T[ω, "dud"][_, _, _]}, {}],
  NL, "With J.24: ", sub =
   ExteriorD[ω@dud[μ1_, a_, b_]] -> -ExteriorD[x@u[ν1]] xPartialD[ω@dud[μ1, a, b], ν1] +
     ExteriorD[x@u[μ1]] xPartialD[ω@dud[ν1, a, b], μ1],
  Imply, tmp = tmp /. sub,
  Yield, tmp = tmp // WedgeSimplify[{xPartialD[a_, b_]}],
  NL, "Removing ", sub = ExteriorD[x@u[μ1_]] ∧ ExteriorD[x@u[μ2_]], Yield,
  sub = ExteriorD[x@u[μ1_]] ∧ ExteriorD[x@u[μ2_]] a__ :→
    (($t = Times[a] //. {μ1 → μ, μ2 → ν}) - Swap[{μ, ν}][$t]),
  tmp[[1]] = tmp[[1]] // AddDnIndex[-1, μ] // AddDnIndex[-1, ν];
  tmpJ25 = tmp /. sub,
  NL, "which is J.25(J.29 with J.27 inserted) for ", ω@dud[μ, a, b],
  NL, "Writing out (J.49): ",
  eJ49 = R@uddd[ρ, σ, μ, ν] -> e@ud[ρ, a] e@du[σ, b] R@uddd[a, b, μ, ν],
  CO["The greek-latin index notation is confused here, since up to this
     point the latin indices were on the left of the greek indices. The
     confusing point is does ", R@uddd[a, b, μ, ν] == R@ddud[μ, ν, a, b]],
  NL, "From ", tmpe,
  imply, "Correspondence between local and coordinate basis ",
  tmp0 = Table[e@u[i], {i, 0, 3}],
  yield, tmp = tmp0 //. tmpe,
  NL, "and local coordinate forms: ",
  dtmpg = {ExteriorD[t], Table[σ@u[i], {i, 3}]} // Flatten,
  Imply, tmpa = Thread[tmp0 → (tmp0 //. tmpe)],
  NL, "The  local 1-form basis correspondence: ",
  tmpdg = Thread[Table[ExteriorD[x@u[i]], {i, 0, 3}] ->
     Flatten[{ExteriorD[t], Table[σ@u[i], {i, 3}]}]],
  Imply, "From J.26: ", tmp = eJ26[[2]],
  imply, sub = RuleX[tmp, e@du[μ, a]][[1]],
  Yield, "The non-zero vielbein values ",
  tmpv = sub[[1]] -> Table[sub /. {a → i, μ → i}, {i, 0, 3}] /. tmpa /. tmpdg,
  NL, tmpv1 = tmpv /. Tensor[a__] :> UpDownIndexSwap[1, 2][Tensor[a]],
  yield, tmpv1 = tmpv1[[1]] -> Map[#[[1]] -> 1 / #[[2]] &, tmpv1[[2]]]
 ];
```

How do you translate $R^a{}_b \to d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ to all indices? Recall J.27: $\omega^{a\_}{}_{b\_} \to dx^{\mu 1}\,\omega_{\mu 1}{}^a{}_b$ ⇒

$R^a{}_b \to d(dx^{\mu 1}\,\omega_{\mu 1}{}^a{}_b) + (dx^{\mu 1}\,\omega_{\mu 1}{}^a{}_c) \wedge (dx^{\mu 1}\,\omega_{\mu 1}{}^c{}_b) \longrightarrow R^a{}_b \to dx^{\mu 1} \wedge d\omega_{\mu 1}{}^a{}_b + (dx^{\mu 1}\,\omega_{\mu 1}{}^a{}_c) \wedge (dx^{\mu 1}\,\omega_{\mu 1}{}^c{}_b) \longrightarrow$

→ $R^a{}_b \to dx^{\mu 1\$51442} \wedge d\omega_{\mu 1\$51442}{}^a{}_b + \omega_{\mu 1\$51503}{}^a{}_{c\$51484}\,\omega_{\mu 1\$51526}{}^{c\$51484}{}_b\,dx^{\mu 1\$51503} \wedge dx^{\mu 1\$51526}$

With J.24: $d\omega_{\mu 1\_}{}^{a\_}{}_{b\_} \to -dx^{\nu 1}\,\partial_{\nu 1}[\omega_{\mu 1}{}^a{}_b] + dx^{\mu 1}\,\partial_{\mu 1}[\omega_{\nu 1}{}^a{}_b]$

⇒ $R^a{}_b \to -(dx^{\mu 1\$51442} \wedge (dx^{\nu 1}\,\partial_{\nu 1}[\omega_{\mu 1\$51442}{}^a{}_b])) +$
$dx^{\mu 1\$51442} \wedge (dx^{\mu 1\$51442}\,\partial_{\mu 1\$51442}[\omega_{\nu 1}{}^a{}_b]) + \omega_{\mu 1\$51503}{}^a{}_{c\$51484}\,\omega_{\mu 1\$51526}{}^{c\$51484}{}_b\,dx^{\mu 1\$51503} \wedge dx^{\mu 1\$51526}$

→ $R^a{}_b \to \omega_{\mu 1\$51503}{}^a{}_{c\$51484}\,\omega_{\mu 1\$51526}{}^{c\$51484}{}_b\,dx^{\mu 1\$51503} \wedge dx^{\mu 1\$51526} - dx^{\mu 1\$51442} \wedge dx^{\nu 1}\,\partial_{\nu 1}[\omega_{\mu 1\$51442}{}^a{}_b]$

Removing $dx^{\mu 1\_} \wedge dx^{\mu 2\_}$

→ a__ $dx^{\mu 1\_} \wedge dx^{\mu 2\_}$:→ ($t = Times[a] //. {μ1 → μ, μ2 → ν}) − Swap[{μ, ν}][$t]

$R^a{}_{b\mu\nu} \to -\omega_\nu{}^a{}_{c\$51484}\,\omega_\mu{}^{c\$51484}{}_b + \omega_\mu{}^a{}_{c\$51484}\,\omega_\nu{}^{c\$51484}{}_b - \partial_\nu[\omega_\mu{}^a{}_b] + \partial_\mu[\omega_\nu{}^a{}_b]$

which is J.25 (J.29 with J.27 inserted) for $\omega_\mu{}^a{}_b$

Writing out (J.49): $R^\rho{}_{\sigma\mu\nu} \to e_\sigma{}^b\,e^\rho{}_a\,R^a{}_{b\mu\nu}$

The greek-latin index notation is confused here,
since up to this point the latin indices were on the left of
the greek indices. The confusing point is does $R^a{}_{b\mu\nu} == R_{\mu\nu}{}^a{}_b$

From {$e^{a\_}$:→ If[a > 0, c[a] T[σ, u][a], dt], c[a_] :→ If[a > 0, coef⟦a⟧, c0]} ⇒
Correspondence between local and coordinate basis
$\{e^0, e^1, e^2, e^3\} \longrightarrow \{dt, \alpha\,\sigma^1, \beta\,\sigma^2, \gamma\,\sigma^3\}$

and local coordinate forms: $\{dt, \sigma^1, \sigma^2, \sigma^3\}$

⇒ $\{e^0 \to dt, e^1 \to \alpha\,\sigma^1, e^2 \to \beta\,\sigma^2, e^3 \to \gamma\,\sigma^3\}$

The local 1-form basis correspondence: $\{dx^0 \to dt, dx^1 \to \sigma^1, dx^2 \to \sigma^2, dx^3 \to \sigma^3\}$

⇒ From J.26: $e^a \to dx^\mu\,e_\mu{}^a$ ⇒ $e_\mu{}^a \to \dfrac{e^a}{dx^\mu}$

→ The non-zero vielbein values $e_\mu{}^a \to \{e_0{}^0 \to 1, e_1{}^1 \to \alpha, e_2{}^2 \to \beta, e_3{}^3 \to \gamma\}$

$e^\mu{}_a \to \{e^0{}_0 \to 1, e^1{}_1 \to \alpha, e^2{}_2 \to \beta, e^3{}_3 \to \gamma\} \longrightarrow e^\mu{}_a \to \{e^0{}_0 \to 1, e^1{}_1 \to \frac{1}{\alpha}, e^2{}_2 \to \frac{1}{\beta}, e^3{}_3 \to \frac{1}{\gamma}\}$

```
PR1[
  "The individual terms ", tmp = tmpR0 = R@ud[a, b],
  " expand to: ",
  tmpRx = tmp -> tmpJ25[[1]] (tmpbase0 = ExteriorD[x@u[μ]] ∧ ExteriorD[x@u[ν]]),
  NL, "R is anti-symmetric in last 2 indices: ",
  TensorSymmetry[R, 4] = AntiSymmetric[3, 4],
  Yield, tmpRx = tmpRx // ExpandIndex[{μ, 0, 3}] // ExpandIndex[{ν, 0, 3}] //
    SymmetrizeSlots[],
  Yield, tmpRx = tmpRx /. (sub = Table[ExteriorD[x@u[μ]] -> dtmpg[[μ + 1]], {μ, 0, 3}]) //
    WedgeSimplify[{}],
  NL, "Compare for different a,b: ", tmpb,
  NL, "with ",
  (tmpR4 = Table[tmpRx /. {a -> i, b -> j} /. Flatten[tmpb], {i, 0, 3}, {j, 0, 3}]) //
   MatrixForms, check,
  NL, "Match coefficients of exterior products for each a,b to determine: ",
  tmpJ25[[1]],
  " (show only non-zero)",
  (tmp1 = Table[
      tmp = tmpR4[[i, j]];
      tmp = tmp[[2]] - tmp[[1]] // FullSimplify;
      tmp = ExtractPattern[tmp, a__ Wedge[__]] /. Wedge[__] → 1, {i, 4}, {j, 4}]) //
   MatrixForms;
  xtmp = tmp = tmp1 // Flatten;
  tmp = Map[Solve4Pattern[# == 0, T[R, "uddd"][_, _, _, _]][[1, 1]] &, tmp];
  tmpR5 = DeleteCases[tmp, a__ → 0]
 ];
```

The individual terms $R^a{}_b$ expand to: $R^a{}_b \to R^a{}_{b\mu\nu}\, dx^\mu \wedge dx^\nu$

R is anti-symmetric in last 2 indices: AntiSymmetric[3, 4]

$\to$

$$R^a{}_b \to R^a{}_{b01}\, dx^0 \wedge dx^1 + R^a{}_{b02}\, dx^0 \wedge dx^2 + R^a{}_{b03}\, dx^0 \wedge dx^3 - R^a{}_{b01}\, dx^1 \wedge dx^0 + R^a{}_{b12}\, dx^1 \wedge dx^2 + R^a{}_{b13}\, dx^1 \wedge dx^3 - R^a{}_{b02}\, dx^2 \wedge dx^0 - R^a{}_{b12}\, dx^2 \wedge dx^1 + R^a{}_{b23}\, dx^2 \wedge dx^3 - R^a{}_{b03}\, dx^3 \wedge dx^0 - R^a{}_{b13}\, dx^3 \wedge dx^1 - R^a{}_{b23}\, dx^3 \wedge dx^2$$

$$\to R^a{}_b \to 2 R^a{}_{b01}\, dt \wedge \sigma^1 + 2 R^a{}_{b02}\, dt \wedge \sigma^2 + 2 R^a{}_{b03}\, dt \wedge \sigma^3 + 2 R^a{}_{b12}\, \sigma^1 \wedge \sigma^2 + 2 R^a{}_{b13}\, \sigma^1 \wedge \sigma^3 + 2 R^a{}_{b23}\, \sigma^2 \wedge \sigma^3$$

Compare for different a,b:

$$\{\{R^0{}_0 \to 0,\ R^0{}_1 \to -(dt \wedge \sigma^1)\, \partial_t[\partial_t[\alpha]],\ R^0{}_2 \to -(dt \wedge \sigma^2)\, \partial_t[\partial_t[\beta]],\ R^0{}_3 \to -(dt \wedge \sigma^3)\, \partial_t[\partial_t[\gamma]]\},$$
$$\{R^1{}_0 \to -(dt \wedge \sigma^1)\, \partial_t[\partial_t[\alpha]],\ R^1{}_1 \to 0,\ R^1{}_2 \to \sigma^1 \wedge \sigma^2\, \partial_t[\alpha]\, \partial_t[\beta],\ R^1{}_3 \to \sigma^1 \wedge \sigma^3\, \partial_t[\alpha]\, \partial_t[\gamma]\},$$
$$\{R^2{}_0 \to -(dt \wedge \sigma^2)\, \partial_t[\partial_t[\beta]],\ R^2{}_1 \to -(\sigma^1 \wedge \sigma^2)\, \partial_t[\alpha]\, \partial_t[\beta],\ R^2{}_2 \to 0,\ R^2{}_3 \to \sigma^2 \wedge \sigma^3\, \partial_t[\beta]\, \partial_t[\gamma]\},$$
$$\{R^3{}_0 \to -(dt \wedge \sigma^3)\, \partial_t[\partial_t[\gamma]],\ R^3{}_1 \to -(\sigma^1 \wedge \sigma^3)\, \partial_t[\alpha]\, \partial_t[\gamma],\ R^3{}_2 \to -(\sigma^2 \wedge \sigma^3)\, \partial_t[\beta]\, \partial_t[\gamma],\ R^3{}_3 \to 0\}\}$$

with

$$\left(\begin{array}{l}
0 \to 2 R^0{}_{001}\, dt \wedge \sigma^1 + 2 R^0{}_{002}\, dt \wedge \sigma^2 + 2 R^0{}_{003}\, dt \wedge \sigma^3 + 2 R^0{}_{012}\, \sigma^1 \wedge \sigma^2 + 2 R^0{}_{013}\, \sigma^1 \wedge \sigma^3 + 2 R^0{}_{023}\, \sigma \\
-(dt \wedge \sigma^1)\, \partial_t[\partial_t[\alpha]] \to 2 R^1{}_{001}\, dt \wedge \sigma^1 + 2 R^1{}_{002}\, dt \wedge \sigma^2 + 2 R^1{}_{003}\, dt \wedge \sigma^3 + 2 R^1{}_{012}\, \sigma^1 \wedge \sigma^2 + 2 R^1{}_{013}\, \sigma^1 \wedge \sigma^3 + \\
-(dt \wedge \sigma^2)\, \partial_t[\partial_t[\beta]] \to 2 R^2{}_{001}\, dt \wedge \sigma^1 + 2 R^2{}_{002}\, dt \wedge \sigma^2 + 2 R^2{}_{003}\, dt \wedge \sigma^3 + 2 R^2{}_{012}\, \sigma^1 \wedge \sigma^2 + 2 R^2{}_{013}\, \sigma^1 \wedge \sigma^3 + \\
-(dt \wedge \sigma^3)\, \partial_t[\partial_t[\gamma]] \to 2 R^3{}_{001}\, dt \wedge \sigma^1 + 2 R^3{}_{002}\, dt \wedge \sigma^2 + 2 R^3{}_{003}\, dt \wedge \sigma^3 + 2 R^3{}_{012}\, \sigma^1 \wedge \sigma^2 + 2 R^3{}_{013}\, \sigma^1 \wedge \sigma^3 +
\end{array}\right.$$

Match coefficients of exterior products for each a,b to determine:

$R^a{}_{b\mu\nu}$ (show only non-zero)

$$\{R^0{}_{101} \to -\frac{1}{2}\partial_t[\partial_t[\alpha]],\ R^0{}_{202} \to -\frac{1}{2}\partial_t[\partial_t[\beta]],\ R^0{}_{303} \to -\frac{1}{2}\partial_t[\partial_t[\gamma]],\ R^1{}_{001} \to -\frac{1}{2}\partial_t[\partial_t[\alpha]],$$
$$R^1{}_{212} \to \frac{1}{2}\partial_t[\alpha]\,\partial_t[\beta],\ R^1{}_{313} \to \frac{1}{2}\partial_t[\alpha]\,\partial_t[\gamma],\ R^2{}_{002} \to -\frac{1}{2}\partial_t[\partial_t[\beta]],\ R^2{}_{112} \to -\frac{1}{2}\partial_t[\alpha]\,\partial_t[\beta],$$
$$R^2{}_{323} \to \frac{1}{2}\partial_t[\beta]\,\partial_t[\gamma],\ R^3{}_{003} \to -\frac{1}{2}\partial_t[\partial_t[\gamma]],\ R^3{}_{113} \to -\frac{1}{2}\partial_t[\alpha]\,\partial_t[\gamma],\ R^3{}_{223} \to -\frac{1}{2}\partial_t[\beta]\,\partial_t[\gamma]\}$$

```
PR1["Compute ",
  tmp = eJ49,
  Yield, tmp = Table[MapAt[EinsteinSum[][#] &, tmp, {2}], {ρ, 0, 3}, {σ, 0, 3}];
  NL, "Apply vielbein: ",
  Yield, sub = {tmpv[[2]], tmpv1[[2]]} // Flatten,
  NL,
  sub = Join[sub, {T[e, "ud"][i_, j_] :> 0 /; i =!= j, T[e, "du"][i_, j_] :> 0 /; i =!= j}],
  Yield, (tmp = tmp /. sub) // MatrixForms,
  NL, "Expand Greek indices and apply values for R: ",
  tmp = Table[tmp, {μ, 0, 3}, {ν, 0, 3}] // Flatten;
  Yield, tmp =
   Map[#[[1]] -> If[! FreeQ[$tmp = (#[[2]] /. tmpR5), Tensor[R, _, _]], 0, $tmp] &, tmp];
  DeleteCases[tmp, a__ → 0],
  NL, CR["The factor of 2 difference due definition of anti-symmetric."]
 ];
```

Compute $R^{\rho}{}_{\sigma\mu\nu} \to e_{\sigma}{}^{b}\, e^{\rho}{}_{a}\, R^{a}{}_{b\mu\nu}$

→

Apply vielbein:

$\to \left\{e_0{}^0 \to 1,\ e_1{}^1 \to \alpha,\ e_2{}^2 \to \beta,\ e_3{}^3 \to \gamma,\ e^0{}_0 \to 1,\ e^1{}_1 \to \frac{1}{\alpha},\ e^2{}_2 \to \frac{1}{\beta},\ e^3{}_3 \to \frac{1}{\gamma}\right\}$

$\left\{e_0{}^0 \to 1,\ e_1{}^1 \to \alpha,\ e_2{}^2 \to \beta,\ e_3{}^3 \to \gamma,\ e^0{}_0 \to 1,\ e^1{}_1 \to \frac{1}{\alpha},\ e^2{}_2 \to \frac{1}{\beta},\ e^3{}_3 \to \frac{1}{\gamma},\ e^{i\_}{}_{j\_} :\to 0 \,/;\, i =!= j,\ e_{i\_}{}^{j\_} :\to 0 \,/;\, i =!= j\right\}$

$$\to \begin{pmatrix} R^0{}_{0\mu\nu} \to R^0{}_{0\mu\nu} & R^0{}_{1\mu\nu} \to \alpha R^0{}_{1\mu\nu} & R^0{}_{2\mu\nu} \to \beta R^0{}_{2\mu\nu} & R^0{}_{3\mu\nu} \to \gamma R^0{}_{3\mu\nu} \\ R^1{}_{0\mu\nu} \to \frac{R^1{}_{0\mu\nu}}{\alpha} & R^1{}_{1\mu\nu} \to R^1{}_{1\mu\nu} & R^1{}_{2\mu\nu} \to \frac{\beta R^1{}_{2\mu\nu}}{\alpha} & R^1{}_{3\mu\nu} \to \frac{\gamma R^1{}_{3\mu\nu}}{\alpha} \\ R^2{}_{0\mu\nu} \to \frac{R^2{}_{0\mu\nu}}{\beta} & R^2{}_{1\mu\nu} \to \frac{\alpha R^2{}_{1\mu\nu}}{\beta} & R^2{}_{2\mu\nu} \to R^2{}_{2\mu\nu} & R^2{}_{3\mu\nu} \to \frac{\gamma R^2{}_{3\mu\nu}}{\beta} \\ R^3{}_{0\mu\nu} \to \frac{R^3{}_{0\mu\nu}}{\gamma} & R^3{}_{1\mu\nu} \to \frac{\alpha R^3{}_{1\mu\nu}}{\gamma} & R^3{}_{2\mu\nu} \to \frac{\beta R^3{}_{2\mu\nu}}{\gamma} & R^3{}_{3\mu\nu} \to R^3{}_{3\mu\nu} \end{pmatrix}$$

Expand Greek indices and apply values for R:

$$\to \Big\{R^0{}_{101} \to -\frac{1}{2}\alpha\, \partial_t[\partial_t[\alpha]],\ R^1{}_{001} \to -\frac{\partial_t[\partial_t[\alpha]]}{2\alpha},\ R^0{}_{202} \to -\frac{1}{2}\beta\, \partial_t[\partial_t[\beta]],\ R^2{}_{002} \to -\frac{\partial_t[\partial_t[\beta]]}{2\beta},$$
$$R^0{}_{303} \to -\frac{1}{2}\gamma\, \partial_t[\partial_t[\gamma]],\ R^3{}_{003} \to -\frac{\partial_t[\partial_t[\gamma]]}{2\gamma},\ R^1{}_{212} \to \frac{\beta\, \partial_t[\alpha]\, \partial_t[\beta]}{2\alpha},\ R^2{}_{112} \to -\frac{\alpha\, \partial_t[\alpha]\, \partial_t[\beta]}{2\beta},$$
$$R^1{}_{313} \to \frac{\gamma\, \partial_t[\alpha]\, \partial_t[\gamma]}{2\alpha},\ R^3{}_{113} \to -\frac{\alpha\, \partial_t[\alpha]\, \partial_t[\gamma]}{2\gamma},\ R^2{}_{323} \to \frac{\gamma\, \partial_t[\beta]\, \partial_t[\gamma]}{2\beta},\ R^3{}_{223} \to -\frac{\beta\, \partial_t[\beta]\, \partial_t[\gamma]}{2\gamma}\Big\}$$

The factor of 2 difference due definition of anti-symmetric.