PHYSICS 234A String Theory — # 3

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Problem 1.
(1.1). Let $A : \mathbb{Z} \to \mathbb{R}$ be given, let V be a real vector space with basis $\{L_m : m \in \mathbb{Z}\}$ and equip V with a Lie bracket[1] by defining

$$[L_m, L_n] := (m-n)L_{m+n} + A(m)\delta_{m+n,0}. \tag{1}$$

(i). **Theorem.** *There exist a basis $\{K_m : m \in \mathbb{Z}\}$ of V such that*

$$[K_m, K_n] = (m-n)K_{m+n} + B(m)\delta_{m+n,0},$$

where $B : \mathbb{Z} \to \mathbb{R}$ is such that $B(1) = 0$.
Proof. From the definition of the Lie bracket on V, in particular, we have that

$$[L_1, L_{-1}] = 2L_0 + A(1) = 2\left(L_0 + \frac{1}{2}A(1)\right).$$

This motivates us to define $K_0 := L_0 + \frac{1}{2}A(1)$ and $K_m := L_m$ for $m \neq 0$. Another way of writing this is

$$K_m := L_m + \frac{1}{2}A(1)\delta_{m,0}.$$

Then,

$$\begin{aligned}
[K_m, K_n] &= \left[L_m + \frac{1}{2}A(1)\delta_{m,0}, L_n + \frac{1}{2}A(1)\delta_{n,0}\right] = [L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0} \\
&= (m-n)K_{m+n} - \frac{m-n}{2}A(1)\delta_{m+n,0} + A(m)\delta_{m+n,0} = (m-n)K_{m+n} + (A(m) - mA(1))\,\delta_{m+n,0} \\
&= (m-n) + B(m)\delta_{m+n,0},
\end{aligned}$$

where we have defined

$$B(m) := A(m) - mA(1).$$

Trivially, we see that $B(1) = 0$. $\square$

(ii). **Theorem.** *Let $\{K_m : m \in \mathbb{Z}\}$ be a basis for V such that*

$$[K_m, K_n] = (m-n)K_{m+n} + B(m)\delta_{m+n,0}, \tag{2}$$

where $B : \mathbb{Z} \to \mathbb{R}$ is such that $B(1) = 0$. Then, $\{K_{-1}, K_0, K_1\}$ generate a subalgebra of V.
Proof. Let S be the subspace of V spanned by $\{K_{-1}, K_0, K_1\}$. However, simply applying (2), we find that

$$[K_0, K_{-1}] = K_{-1}; [K_1, K_0] = K_1; [K_1, K_{-1}] = 2K_0 + B(1) = 2K_0.$$

Thus, S is closed under the Lie bracket, and hence yields a subalgebra of V. $\square$

[1] Not just any old function A will make this definition actually yield a Lie bracket. For the moment, we assume that A is chosen so that this definition actually does in fact give us a Lie bracket.

(1.2). **Theorem.** *If* (1) *defines a Lie bracket and* $A(1) = 0$,[2] *then*

$$A(m) = \frac{A(2)}{6}(m^3 - m). \tag{3}$$

Proof. Suppose that (1) defines a Lie bracket and that $A(1) = 0$. Antisymmetry of the bracket implies that

$$-(m-n)L_{m+n} + A(-m)\delta_{m+n,0} = [L_n, L_m] = -[L_m, L_n] = -\left((m-n)L_{m+n} + A(m)\delta_{m+n,0}\right),$$

from which it follows that

$$A(-m) = -A(m).$$

In particular, $A(0) = 0$, and we need only prove (3) for positive m. However, we have assumed that $A(1) = 0$, so in fact we need only prove (3) for $m \geq 2$, which we prove by induction (after first deriving a recursive formula).

We see that the Jacobi identity implies

$$\begin{aligned}
0 &= [[L_k, L_l], L_m] + [[L_m, L_k], L_l] + [[L_l, L_m], L_k] \\
&= [(k-l)L_{k+l} + A(k)\delta_{k+l,0}, L_m] + [(m-k)L_{m+k} + A(m)\delta_{m+k,0}, L_l] + [(l-m)L_{l+m} + A(l)\delta_{l+m,0}, L_k] \\
&= (k-l)[L_{k+l}, L_m] + (m-k)[L_{m+k}, L_l] + (l-m)[L_{l+m}, L_k] \\
&= (k-l)\left((k+l-m)L_{k+l+m} + A(k+l)\delta_{k+l+m,0}\right) + (m-k)\left((m+k-l)L_{m+k+l} + A(m+k)\delta_{m+k+l,0}\right) \\
&\quad + (l-m)\left((l+m-k)L_{l+m+k} + A(l+m)\delta_{l+m+k,0}\right) \\
&= \left((k-l)A(k+l) + (m-k)A(m+k) + (l-m)A(l+m)\right)\delta_{k+l+m,0}
\end{aligned}$$

Replacing in the coefficient of $\delta_{k+l+m,0}$ m with $-(k+l)$, we find that this becomes

$$0 = (k-l)A(k+l) - (2k+l)A(-l) + (k+2l)A(-k) = (k-l)A(k+l) + (2k+l)A(l) - (k+2l)A(k).$$

Plugging in $l = 1$ and using the assumption that $A(1) = 0$, this simplifies to

$$0 = (k-1)A(k+1) - (k+2)A(k).$$

For $k \geq 2$, this gives us that

$$A(k+1) = \frac{k+2}{k-1}A(k).$$

From here, we proceed by induction. The result (3) is trivially true for $m = 2$, so suppose it is true for m. Then,

$$A(m+1) = \frac{m+2}{m-1}A(m) = \frac{m+2}{m-1}\frac{A(2)}{6}(m^3 - m) = \frac{A(2)}{6}\left((m+1)^3 - (m+1)\right),$$

which completes the proof. □

Problem 2. Let $P := \{z \in \mathbb{C} : z \neq 0\}$ denote the punctured complex plane, let $X : P \to \mathbb{R}$ be a field, and define an action[3]

$$S(X) := \mu \int \mathrm{d}^2 z\, \partial X \bar{\partial} X, \tag{4}$$

where

$$\mathrm{d}^2 z := \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

and

$$\partial := \partial_z ; \bar{\partial} := \partial_{\bar{z}}.$$

[2] The point of the previous part was to show that we can always assume this (i.e. that $A(1) = 0$) without loss of generality.
[3] μ *is for mass density.*

234A HW 3 Problem 2 Ken McElvain

Free Boson CFT

Consider a theory of a single free boson $X(z,\bar{z})$ on a plane,

$$S=\frac{1}{2\pi\alpha'}\int d^2z\,\partial X\bar{\partial}X \quad \text{where} \quad \partial=\partial_z,\ \bar{\partial}=\partial_{\bar{z}}$$

2.1 Normal Ordering

Work out the relationship between

$$X(z_1,\bar{z}_1)X(z_2,\bar{z}_2) \ \ and \ \ :X(z_1,\bar{z}_1)X(z_2,\bar{z}_2):$$

Then compute

$$\langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)\rangle \ \text{Implicitly in vacuum.}$$

Recall also that, in writing the product of operators, time ordering is implicit: we write operators at earlier times to the right of those at later times. In going from the cylinder to the plane, the slices of constant time are slices of constant |z|.

Solution

The mode expansion for X on the plane comes from a variable substitution into the result from problem 2 last time (after Wick rotation τ→-iτ)

$$X_{\pm}(\tau\mp\theta)=\frac{x}{2}+\alpha'\frac{p}{2}(\tau\mp\theta)+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\alpha_n e^{-in(\tau\mp\theta)} \qquad x,p \textit{ are operators too}$$

$$\rightarrow\frac{x}{2}-\frac{i\alpha'}{2}p(\tau\mp i\theta)+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\alpha_n e^{-n(\tau\mp i\theta)} \qquad \textit{Wick rotation}$$

We take $\quad z=e^{\tau-i\theta},\ \ \bar{z}=e^{\tau+i\theta}$

$$X_R(z)=\frac{1}{2}x-\frac{i}{2}\alpha' p\ln z+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\alpha_n z^{-n}$$

$$X_L(\bar{z})=\frac{1}{2}x-\frac{i}{2}\alpha' p\ln\bar{z}+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\tilde{\alpha}_n \bar{z}^{-n}$$

$$X(z,\bar{z})=x-\frac{i}{2}\alpha' p\ln z\bar{z}+i\sqrt{\frac{\alpha'}{2}}\sum_{n>0}\frac{1}{n}\left(\alpha_n z^{-n}-\alpha_{-n}z^{n}+\tilde{\alpha}_n\bar{z}^{-n}-\tilde{\alpha}_{-n}\bar{z}^{n}\right)$$

This leaves the α unchanged, along with their commutation relations.

$$[x,p]=i \qquad \textit{Normal ordered is p to right of x}$$
$$[\alpha_n,\alpha_{-m}]=[\alpha_n,\alpha_m^\dagger]=n\delta_{n,m}$$
$$\alpha_n\alpha_n^\dagger=\alpha_n^\dagger\alpha_n-n$$

$$X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)=:X(z_1,\bar{z}_1)X(z_2,\bar{z}_2):$$
$$-\frac{i}{2}\alpha'[p,x]\log(z\bar{z})-\frac{\alpha'}{2}\sum_{n>0}\frac{1}{n^2}\left([\alpha_n,\alpha_{-n}]\left(\frac{z_2}{z_1}\right)^n+[\tilde{\alpha}_n,\tilde{\alpha}_{-n}]\left(\frac{\bar{z}_2}{\bar{z}_1}\right)^n\right)$$
$$X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)=:X(z_1,\bar{z}_1)X(z_2,\bar{z}_2):$$
$$-\frac{i}{2}\alpha'[p,x]\log(z\bar{z})+\frac{\alpha'}{2}\sum_{n>0}\frac{1}{n}\left(\left(\frac{z_2}{z_1}\right)^n+\left(\frac{\bar{z}_2}{\bar{z}_1}\right)^n\right)$$

I recognize the Taylor series for log $-\log(1-a)=\sum a^n/n$

$$X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)=:X(z_1,\bar{z}_1)X(z_2,\bar{z}_2):-\frac{1}{2}\alpha'\left[\log(z\bar{z})+\left(\log\left(1-\frac{z_2}{z_1}\right)+\log\left(1-\frac{\bar{z}_2}{\bar{z}_1}\right)\right)\right]$$

Simplifying the log expression

$$X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)=:X(z_1,\bar{z}_1)X(z_2,\bar{z}_2):-\frac{1}{2}\alpha'\log\left((z_1-z_2)(\bar{z}_1-\bar{z}_2)\right)$$

Evaluating the propagator using the above normal ordering result

$$\langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)\rangle=-\frac{1}{2}\alpha'\log\left((z_1-z_2)(\bar{z}_1-\bar{z}_2)\right)$$

I'll do it the long way too just to be sure ...

I remember this technique from David Tong's notes. We know that the expectation value of a single insertion is 0.

$$0=\langle X(z_2,\bar{z}_2)\rangle=\int \mathcal{D}X\, e^S X(z_2,\bar{z}_2) \quad \textit{with } S=\frac{1}{2\pi\alpha'}\int d^2w\,\partial X\bar{\partial}X$$

I form the two point function by taking a functional derivative

$$0=\frac{\delta}{\delta X(z_1,\bar{z}_1)}\int \mathcal{D}X\, e^S X(z_2,\bar{z}_2)$$

$$0=\frac{\delta}{\delta X(z_1,\bar{z}_1)}\int \mathcal{D}X\left(\frac{\delta}{\delta X(z_1,\bar{z}_1)}e^S\right)X(z_2,\bar{z}_2)+e^S\frac{\delta}{\delta X(z_1,\bar{z}_1)}X(z_2,\bar{z}_2)$$

$$0=\int \mathcal{D}X\, e^S\left[\frac{2}{2\pi\alpha'}\left(\partial\bar{\partial}X(z_1,\bar{z}_1)\right)X(z_2,\bar{z}_2)+\delta(z_2-z_1)\delta(\bar{z}_2-\bar{z}_1)\right]$$

$$\left\langle \partial_{z_1}\bar{\partial}_{\bar{z}_1}X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)\right\rangle=-\pi\alpha'\delta(z_2-z_1)\delta(\bar{z}_2-\bar{z}_1)$$

$$\partial_{z_1}\bar{\partial}_{\bar{z}_1}\left\langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)\right\rangle=-\pi\alpha'\delta(z_2-z_1)\delta(\bar{z}_2-\bar{z}_1)$$

It is probably best to retreat to ordinary coordinates for the plane now.

$$\partial_{z_1}\bar{\partial}_{\bar{z}_1}=\frac{1}{2}\left(\partial_{\sigma_1}-i\partial_{\sigma_2}\right)\frac{1}{2}\left(\partial_{\sigma_1}+i\partial_{\sigma_2}\right)=\frac{1}{4}\left(\partial^2_{\sigma_1}+\partial^2_{\sigma_2}\right)$$

$$\delta(z)\delta(\bar{z})=\frac{1}{2}\delta(\sigma_1)\delta(\sigma_2)\quad \textit{because}\quad dzd\bar{z}=2d\sigma_1 d\sigma_2\ \textit{ and normalization}$$

$$\left(\partial^2_{\sigma_1}+\partial^2_{\sigma_2}\right)\left\langle X(\sigma_1,\sigma_2)X(\omega_1,\omega_2)\right\rangle=-2\pi\alpha'\delta(\sigma_1-\omega_1)\delta(\sigma_2-\omega_2)$$

In polar coordinates about (ω_1,ω_2) this is

$$\partial^2_r f(r)=-2\pi\alpha'\delta(r)\quad \text{where}\quad f(r)=\left\langle X(\sigma_1,\sigma_2)X(\omega_1,\omega_2)\right\rangle$$

The standard result we need is $4\pi\delta(r)=\partial^2\log(r^2)$. Away from r=0 we have $\partial^2_r\log(r^2)=0$. Mathematica will do this quickly for you, but will miss the delta function spike at r=0. To see it, we have to use Stokes theorem.

$$\int dA\,\partial^2_r\ln(r^2)=-2\int dA\frac{1}{r^2}=2\oint d\ell\,\frac{1}{r}=2\int_0^{2\pi} r\,d\theta\,\frac{1}{r}=4\pi$$

Using this result, we have

$$\left\langle X(\sigma_1,\sigma_2)X(\omega_1,\omega_2)\right\rangle=-\alpha'\log(r^2)=-\alpha'\log\left((\sigma_1-\omega_1)^2+(\sigma_2+\omega_2)^2\right)$$

$$=-\frac{\alpha'}{2}\log\left((z_1-z_2)(\bar{z}_1-\bar{z}_2)\right)$$

Which agrees with the normal ordering result

2.2 Compute

$$\langle :\exp\left(ikX\left(z_1,\overline{z}_1\right)\right): \; :\exp\left(ik'X\left(z_2,\overline{z}_2\right)\right): \rangle = \left\langle \; 0 \; \middle| \; :\exp\left(ikX\left(z_1,\overline{z}_1\right)\right): \; :\exp\left(ik'X\left(z_2,\overline{z}_2\right)\right): \; \middle| \; 0 \; \right\rangle$$

The field X is a linear combination of operators that fail to commute in pairs and for which the commutation relations in those pairs result in numbers, not operators.

Let a and $a^\dagger$ be such a pair, then using Baker-Campbell-Hausdorff

$$e^{\alpha a^\dagger + \beta a} = e^{\alpha a^\dagger} e^{\beta a} e^{-(1/2)\alpha\beta\left[a^\dagger, a\right]} = e^{\beta a} e^{\alpha a^\dagger} e^{+(1/2)\alpha\beta\left[a^\dagger, a\right]}$$

$$e^{\beta a} e^{\alpha a^\dagger} = e^{\alpha a^\dagger} e^{\beta a} e^{-\alpha\beta\left[a^\dagger, a\right]}$$

The normal order of $\exp\left[\alpha a^\dagger + \beta a\right]$ can be computed by expanding. (This points out that normal ordering is tricky. Rewriting the expression as above, evaluating the commutator and then applying normal ordering yields a different result.)

$$:e^{\alpha a^\dagger + \beta a}: = \sum_{n=0}^{\infty} \frac{1}{n!} :\left(\alpha a^\dagger + \beta a\right)^n: = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \left(\alpha a^\dagger\right)^m (\beta a)^{n-m}$$

$$= \sum_{m \le n} \frac{1}{m!(n-m!)} \left(\alpha a^\dagger\right)^m (\beta a)^{n-m} = e^{\alpha a^\dagger} e^{\beta a}$$

The above sum can be recognized by expanding and merging the product $e^A e^B$.

I will also need: $\langle X_1 X_2 \rangle = \left\langle \; 0 \; \middle| \; \left(\alpha_1 a^\dagger + \beta_1 a\right)\left(\alpha_2 a^\dagger + \beta_2 a\right) \; \middle| \; 0 \; \right\rangle = -\beta_1 \alpha_2 \left[a^\dagger, a\right]$

Suppose I have two such linear combinations of the creation and annihilation operators $X_1 = \alpha_1 a^\dagger + \beta_1 a, \; and \; X_2 = \alpha_2 a^\dagger + \beta_2 a$. Then I can show that

$$:e^{X_1 + X_2}: e^{\langle X_1 X_2 \rangle} = :e^{X_1}::e^{X_2}:$$

$$:e^{(\alpha_1+\alpha_2)a^\dagger + (\beta_1+\beta_2)a}: e^{-\beta_1\alpha_2\left[a^\dagger, a\right]} = e^{\alpha_1 a^\dagger} e^{\beta_1 a} e^{\alpha_2 a^\dagger} e^{\beta_2 a}$$

$$e^{(\alpha_1+\alpha_2)a^\dagger} e^{(\beta_1+\beta_2)a} \cancel{e^{-\beta_1\alpha_2\left[a^\dagger, a\right]}} = e^{\alpha_1 a^\dagger} \left(e^{\alpha_2 a^\dagger} e^{\beta_1 a} \cancel{e^{-\beta_1\alpha_2\left[a^\dagger, a\right]}} \right) e^{\beta_2 a}$$

And we have equality. If I expand X_1 and X_2 to now include many such pairs of operators, which commute pairwise, then by linearity, the same relationship holds.

$$:e^{ikX(z_1)}::e^{ik'X(z_2)}: = \; :e^{ikX(z_1) + ik'X(z_2)}: \; e^{-kk'\langle X(z_1) X(z_2) \rangle}$$

In problem 2.1 we computed the expectation value

$$\langle X(z_1)X(z_2)\rangle = -\frac{\alpha'}{2}\log\left((z_1-z_2)(\overline{z}_1-\overline{z}_2)\right)$$

The final exponential becomes $e^{kk'\frac{\alpha'}{2}\log\left((z_1-z_2)(\overline{z}_1-\overline{z}_2)\right)} = \left((z_1-z_2)(\overline{z}_1-\overline{z}_2)\right)^{\frac{\alpha'}{2}kk'} = |z_1-z_2|^{\alpha' kk'}$

Now I need the expectation value for $\left\langle\ 0\ \middle|\ :e^{ikX(z_1)+ik'X(z_2)}:\ \middle|\ 0\ \right\rangle$

In terms of the mode expansion, the normal ordered exponential has the form

$$X(z,\overline{z}) = x - \frac{i}{2}\alpha' p \ln z\overline{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n>0}\frac{1}{n}\left(\alpha_n z^{-n} - \alpha_{-n}z^n + \tilde{\alpha}_n\overline{z}^{-n} - \tilde{\alpha}_{-n}\overline{z}^n\right)$$

$$\begin{aligned} ikX(z_1)+ik'X(z_2) &= i(k+k')x - \frac{i}{2}\alpha' p\left(ik\ln z_1\overline{z}_1 + ik'\ln z_2\overline{z}_2\right) \\ &\quad -\sqrt{\frac{\alpha'}{2}}\sum_{n>0}\frac{1}{n}\begin{pmatrix}\alpha_n\left(kz_1^{-n}+k'z_2^{-n}\right)-\alpha_{-n}\left(kz_1^{n}+k'z_2^{n}\right) \\ +\tilde{\alpha}_n\left(k\overline{z}_1^{-n}+k'\overline{z}_2^{-n}\right)-\tilde{\alpha}_{-n}\left(k\overline{z}_1^{n}+k'\overline{z}_2^{n}\right)\end{pmatrix}\end{aligned}$$

We want

$$\left\langle \begin{aligned} &:\left(\sum_{j=0}^{\infty}\frac{1}{j!}:\left(i(k+k')x+\frac{1}{2}\alpha' p\left(k\ln z_1\overline{z}_1 + k'\ln z_2\overline{z}_2\right)\right)^j:\right) \\ &\quad\times\prod_{n>0}\left(\sum_{j=0}^{\infty}\frac{1}{j!}:\left(i\sqrt{\frac{\alpha'}{2}}\left(\alpha_n\left(kz_1^{-n}+k'z_2^{-n}\right)-\alpha_{-n}\left(kz_1^{n}+k'z_2^{n}\right)\right)\right)^j:\right) \\ &\quad\times\prod_{n>0}\left(\sum_{j=0}^{\infty}\frac{1}{j!}:\left(i\sqrt{\frac{\alpha'}{2}}\left(\alpha_n\left(k\overline{z}_1^{-n}+k'\overline{z}_2^{-n}\right)-\alpha_{-n}\left(k\overline{z}_1^{n}+k'\overline{z}_2^{n}\right)\right)\right)^j:\right): \end{aligned}\right\rangle$$

Where I have separated the operators into non-commuting pairs. I added an outer normal ordering to finish the job of moving annihilation operators all the way to the right. This has no real effect since they commute with everything to the right anyway.

In the second and third lines, the generated operators will annihilate against the vacuum leaving the only contribution from j=0, which is just 1.

Now we are left with only the x-p piece. All terms with operator p will annihilate leaving the simple sum.

$$\sum_{j=0}^{\infty}\frac{1}{j!}\left(i(k+k')x\right)^j = e^{i(k+k)x}$$

In problem 2.3 we show that this yields a state of momentum k+k', so we know that

$$\left\langle\ 0\ \middle|\ e^{i(k+k')x}\ \middle|\ 0\ \right\rangle = \delta(k+k')$$

The net result is: $\left\langle :e^{ikX(z_1)}: :e^{ikX(z_2)}: \right\rangle = |z_1 - z_2|^{\alpha' kk'}\,\delta(k+k')$

What constraint do k and k' have to satisfy for this to be non vanishing?

These operations create momentum states so $\quad k + k' = 0$

2.3 Operator-State Correspondence

Show that operator-state correspondence relates $:\exp\left(ikX(z,\bar{z})\right):$ to the eigenstate $|k\rangle$ of space-time momentum p.

We want the limit as z→0

$$\lim_{z\to 0} :e^{ikX(z,\bar{z})}:|0\rangle = \lim_{z\to 0} :\exp\begin{pmatrix} ikx + \frac{1}{2}kp\alpha' \log(z\bar{z}) \\ -k\sqrt{\frac{\alpha'}{2}}\sum_{n>0}\frac{1}{n}\left(\alpha_n z^{-n} - \alpha_{-n}z^n + \tilde{\alpha}_n z^{-n} - \tilde{\alpha}_{-n}z^n\right)\end{pmatrix}:|0\rangle$$

The creation operators α_{-n} are suppressed by z→0, and the annihilation operators a_n (and p) cancel out against the vacuum. This leaves

$$\lim_{z\to 0} :e^{ikX(z,\bar{z})}:|0\rangle = exp(ikx)|0\rangle$$

Now, whatever this state is, if I hit it with the momentum operator $-i\partial_x$, then I get

$-i\partial_x e^{ikx}|0\rangle = -i(ik)e^{ikx}|0\rangle = ke^{ikx}|0\rangle$, so $e^{ikx}|0\rangle$ is an eigenfunction of the momentum operator with eigenvalue k.

3.~~0~~1

Derive $T(z) X^\mu(w,\bar w) \sim \frac{1}{z-w} \partial X^\mu(w,\bar w) + \dots$

OPE of some field and stress-energy tensor may be derived ~~from the~~ by requiring the proper transformation law for this field under conformal transformation. It becomes easy if the field is primary. If we know the conformal weights of primary field we can determine OPE of this field and set

$$T(z)\Phi(w,\bar w) = \frac{h}{(z-w)^2}\Phi(w,\bar w) + \frac{1}{z-w}\partial\Phi(w,\bar w) + \dots$$

Here, $\Phi(z) = X(z)$ (we deal with right-movers) and $T(z) = -2 :\partial X\cdot\partial X:$

the Green function for X is:

$$\langle X_\mu(z,\bar z) X_\nu(w,\bar w)\rangle = -\frac{1}{4}\eta_{\mu\nu}\left(\ln(z-w) + \ln(\bar z - \bar w)\right)$$

$$\Rightarrow \langle \partial X_\mu(z)\, X_\nu(w)\rangle = \frac{1}{4}\eta_{\mu\nu}\frac{1}{z-w}$$

Using Wick's theorem, we can now calculate

$$T(z) X^\mu(w) = -2 :\partial X\cdot\partial X X^\mu: - 4\partial X^\nu(z)\langle \partial X_\nu(z) X^\mu(w)\rangle$$

$$= \frac{\partial X^\mu(z)}{z-w} + \dots$$

$$\frac{\partial X^\mu(z)}{z-w} = \frac{\partial X^\mu(w) + (z-w)\partial^2 X^\mu}{z-w}$$

we can conclude that $h=0$ is a conformal dimension of primary field X^μ.

3.2

(i). Considering the $w, \bar{w}$-differentiation of

$$T(z)X^\mu(w) = \frac{\partial X^\mu(w)}{z-w} + \cdots$$

Consequently, we get:

$$T(z)\partial X^\mu(w) = \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w} + \cdots$$

$$T(z)\bar{\partial} X^\mu(w) = \frac{\partial\bar{\partial} X^\mu(w,\bar{w})}{z-w} + \cdots$$

$$T(z)\partial^2 X^\mu(w) = \frac{2\partial X^\mu(w)}{(z-w)^3} + \frac{2\partial^2 X^\mu(w)}{(z-w)^2} + \frac{\partial^3 X^\mu(w)}{z-w} + \cdots$$

Considering

$$X^\mu(w,\bar{w}) = X^\mu_L(\bar{w}) + X^\mu_R(w),$$

we get

$$\tilde{T}(\bar{z})X^\mu(w,\bar{w}) = \frac{\bar{\partial} X^\mu(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial^2 X^\mu(\bar{w})}{\bar{z}-\bar{w}} + \cdots$$

4. CORRELATION FUNCTION OF PRIMARY OPERATORS IN A CFT

4.1 [BBS] 3.7

Laurent expansion of $T(z)$ reads

$$T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}} \Rightarrow L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z)$$

While a conformal field $\Phi(z)$ of $(h, 0)$ expands as

$$\Phi(z) = \sum_{n=-\infty}^{+\infty} \frac{\Phi_n}{z^{n+h}} \Rightarrow \Phi_n = \frac{1}{2\pi i}\oint dz\, z^{n+h-1}\Phi(z)$$

Such that $\Phi^\dagger(z) = \sum_{n=-\infty}^{+\infty} \frac{\Phi_n^\dagger}{\bar{z}^{n+h}} = \frac{1}{\bar{z}^{2h}}\Phi\left(\frac{1}{\bar{z}}\right)$

provided $\Phi_n^\dagger = \Phi_{-n}$.

The above construction is required to make the operator-state correspondence compatible.

$$|\Phi_{in}\rangle = \lim_{z\to 0}\Phi(z)|0\rangle$$

$$\langle\Phi_{out}| = \lim_{z\to 0}\langle 0|\Phi^\dagger(z) = |\Phi_{in}\rangle^\dagger$$

$$\Rightarrow \langle\Phi_{out}|\Phi_{in}\rangle = \lim_{z_1,z_2\to 0}\langle 0|\Phi^\dagger(z_1)\Phi(z_2)|0\rangle$$

$$= \lim_{z_1,z_2\to 0}\langle 0|\bar{z}_1^{-2h}\Phi\left(\frac{1}{\bar{z}_1}\right)\Phi(z_2)|0\rangle$$

$$= \lim_{\xi\to\infty}\xi^{2h}\langle 0|\Phi(\xi)\Phi(0)|0\rangle$$

$$= \lim_{\xi\to\infty}\xi^{2h}\frac{C}{\xi^{2h}} = C$$

which is then well defined

So that

$$[L_m, \Phi_n] = \left[\oint_z \frac{dz}{2\pi i} z^{m+1}T(z), \oint_w \frac{dw}{2\pi i} w^{n+h-1}\Phi(w)\right]$$

$$= \oint\frac{dw}{2\pi i} w^{n+h-1}\oint_w \frac{dz}{2\pi i} z^{m+1}\mathcal{R}\left[T(z)\Phi(w)\right]$$

$$= \oint\frac{dw}{2\pi i} w^{n+h-1}\oint_w \frac{dz}{2\pi i} z^{m+1}\left[\frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi(w) + \text{reg.}\right]$$

$$= \oint\frac{dw}{2\pi i} w^{n+h-1}\left[h(m+1)w^m\Phi(w) + w^{m+1}\partial\Phi(w)\right]$$

$$= \sum_{k\in\mathbb{Z}}\oint\frac{dw}{2\pi i}\left[h(m+1)w^{n+m-k-1} - (k+h)w^{n+m-k-1}\right]\Phi_k$$

$$= \left[(h-1)m - n\right]\Phi_{m+n}$$

$$\Rightarrow \boxed{[L_m, \phi_n] = [m(h-1)-n]\,\phi_{m+n}}$$

4.2 [BBS] 3.8

$\phi_n|0\rangle = 0$, $n > -h$, w/ $|0\rangle$ defining the conformal vacuum

We now denote by $|h\rangle$ the highest-weight state, w/

$$L_0|h\rangle = h|h\rangle$$

Meanwhile, we define the asymptotic state $|h\rangle = \phi(0)|0\rangle$ $:= |h'\rangle$

The consistency can be checked by applying

$$[L_m, \phi] = h(m+1)w^m\phi(w) + w^{m+1}\partial\phi(w)$$

$$\Rightarrow L_0\phi(0)|0\rangle - \phi(0)L_0|0\rangle = h\phi(0)|0\rangle = h|h'\rangle$$

Considering that $|0\rangle$ is supposed to be inv. under global conformal trnsf's, i.e. $L_0|0\rangle = 0$, $L_1|0\rangle = L_{-1}|0\rangle = 0$.

$$\Rightarrow L_0|h'\rangle = h|h'\rangle \Rightarrow |h'\rangle = |h\rangle \text{ is compatible}$$

Now we show the physical meaning of $|h\rangle$.

In order to have $\phi(z)|0\rangle$ well-defined as $z \to 0$, we require

$$\phi_n|0\rangle = 0 \text{ for } n > -h.$$

$$\Rightarrow \phi(0)|0\rangle = \lim_{z\to 0}\sum_{n=-h}^{\infty} z^{n-h}\phi_{-n}|0\rangle = \phi_{-h}|0\rangle$$

$$\Rightarrow \boxed{|\phi\rangle = \phi_{-h}|0\rangle = |h\rangle}$$

i.e. $|\phi\rangle$ is the highest-weight state.

Meanwhile, $L_n|h\rangle = L_n\phi_{-h}|0\rangle = \left\{\phi_{-h}L_n + [n(h-1)+h]\,\phi_{n-h}\right\}|0\rangle$
$= 0$ for $n > 0$

while $L_n|0\rangle = 0$ should be adapted in order to make $T(z)|0\rangle$ well-defined as $z \to 0$. Thus $|h\rangle$ gives the ground state of the Hamiltonian $H_{L_0} \propto L_0$

4.3 [BBS] 3.9

(i) For field $\phi_i(z)$ that is holomorphic,

$$\phi_i(z) = \sum_{n\in\mathbb{Z}} \frac{\phi_n}{z^{n+h_i}} \quad , \quad \phi_n|0\rangle = 0 \text{ for } n > -h_i \, . \quad |h_i\rangle = \phi_{-h_i}|0\rangle$$

$$[L_m, \phi_n] = [m(h_i-1)-n]\phi_{m+n}$$

$$[L_{-1}, \phi_n] = (-h_i+1+n)\phi_{-1-n}$$

$$\Rightarrow \phi_{-n-1}|0\rangle = \frac{1}{n+1-h_i} L_{-1}\phi_{-n}|0\rangle$$

$$\Rightarrow \phi_{-n}|0\rangle = \frac{1}{(n-h_i)!}(L_{-1})^{n-h_i}\phi_{-h_i}|0\rangle \quad \text{for } n > h_i$$

$$= \frac{1}{(n-h_i)!}(L_{-1})^{n-h_i}|h_i\rangle$$

$$\cdot [L_0, L_{-1}]|h\rangle = L_{-1}|h\rangle \Rightarrow L_0(L_{-1}|h\rangle) = (h+1)|h\rangle$$

$$\Rightarrow L_{-1}|h\rangle = c_0|h+1\rangle$$

$$\Rightarrow \langle h|L_1 L_{-1}|h\rangle = c_0^2 \Rightarrow c_0 = \sqrt{2h} \quad \text{since } L_1|h\rangle = 0$$

So far we obtained
$$\begin{cases} L_{-1}|h\rangle = \sqrt{2h}\,|h+1\rangle \\ L_1|h\rangle = 0 \\ L_0|h\rangle = h|h\rangle \end{cases}$$

Similarly,
$$L_{-1}|h+1\rangle = c_1|h+2\rangle = \sqrt{2(2h+1)}\,|h+2\rangle$$
$$L_1|h+1\rangle = c_1'|h\rangle = \sqrt{2h}\,|h\rangle$$

$$L_1|h+1\rangle = L_1 L_{-1}|h\rangle \cdot \frac{1}{\sqrt{2h}} = \frac{1}{\sqrt{2h}}(L_{-1}L_1 + 2L_0)|h\rangle = \sqrt{2h}\,|h\rangle$$

$$\Rightarrow c_1^2 = \langle h+1|L_1 L_{-1}|h+1\rangle = c'^2 + 2(h+1) = 2(2h+1)$$

And
$$L_{-1}|h+s\rangle = c_s|h+s+1\rangle$$
$$L_1|h+s\rangle = c_s'|h+s-1\rangle$$

$$L_1|h+s\rangle = L_1 L_{-1}|h+s-1\rangle * \frac{1}{c_{s-1}} = \frac{1}{c_{s-1}}(L_{-1}L_1 + 2L_0)|h+s-1\rangle$$

$$= \frac{1}{c_{s-1}}\left[L_{-1}(c_{s-1}'|h+s-2\rangle) + 2(h+s-1)|h+s-1\rangle\right]$$

$$= \frac{1}{c_{s-1}}\left[c_{s-1}' c_{s-2} + 2(h+s-1)\right]|h+s-1\rangle$$

$$\Rightarrow c_s' c_{s-1} - c_{s-1}' c_{s-2} = 2(h+s-1)$$

$$\Rightarrow c_s' c_{s-1} = 2(h+s-1) + 2(h+s-2) + \cdots + 2h = \frac{2(2h+s-1)s}{2} = s\cdot(2h+s-1)$$

$$\Rightarrow \boxed{c_s' c_{s-1} = s(2h+s-1)}$$

$$\Rightarrow C_s^2 = \langle h+s | L_1 L_{-1} | h+s \rangle$$
$$= \langle h+s | L_{-1} L_1 + 2L_0 | h+s \rangle = C_{s'}^2 + 2(h+s)$$

$$\Rightarrow \boxed{C_s^2 - C_{s'}^2 = 2(h+s)}$$

$$\Rightarrow \begin{cases} C_s = \sqrt{(2h+s)(s+1)} \\ C_{s'} = \sqrt{s(2h+s-1)} \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} L_{-1} | h+s \rangle = \sqrt{(2h+s)(s+1)} \; | h+s+1 \rangle \\ L_1 \; | h+s \rangle = \sqrt{s(2h+s-1)} \; | h+s-1 \rangle \end{cases}}$$

$$\Rightarrow \boxed{L_{-1}^{n-h} | h \rangle = \sqrt{(n-h)! \frac{(n+h-1)!}{(2h-1)!}} \, | n \rangle}$$

$$\Rightarrow \langle 0 | \phi_n^{(i)} \phi_{-m}^{(j)} | 0 \rangle \qquad n > h_i, \; m > h_j$$
$$= \langle h_i | L_1^{n-h_i} \frac{1}{(n-h_i)!} \frac{1}{(m-h_j)!} L_{-1}^{m-h_j} | h_j \rangle$$
$$= \frac{1}{(n-h_i)!(m-h_j)!} \sqrt{(n-h_i)! \frac{(n+h_i-1)!}{(2h_i-1)!}} \sqrt{(m-h_j)! \frac{(m+h_j-1)!}{(2h_j-1)!}} \; \underbrace{{}_i\langle n | m \rangle_j}_{\delta_{ij}\delta_{m,n}}$$

$$\Rightarrow \langle 0 | \phi_i(z) \phi_j(w) | 0 \rangle = \sum_{\substack{n \geq h_i \\ m \leq -h_j}} \frac{1}{z^{n+h_i}} \frac{1}{w^{m+h_j}} \langle 0 | \phi_n^{(i)} \phi_m^{(j)} | 0 \rangle$$
$$= \delta_{ij} \sum_{n \geq h_i} \frac{(n+h_i-1)!}{(n-h_i)!(2h_i-1)!} \frac{1}{z^{n+h_i}} \frac{1}{w^{-n+h_i}} \qquad (h_i \to h \text{ for simplicity})$$
$$= \delta_{ij} \sum_{n=0}^{\infty} \frac{(n+2h-1)!}{n!(2h-1)!} \cdot \frac{1}{z^{2h}} \left(\frac{w}{z}\right)^n$$
$$= \delta_{ij} \frac{1}{z^{2h}} \frac{1}{(1-\frac{w}{z})^{2h}} = \frac{\delta_{ij}}{(z-w)^{2h}}$$

$$\Rightarrow \boxed{\langle 0 | \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) | 0 \rangle = \frac{\delta_{ij}}{(z_1 - z_2)^{2h_i} (\bar{z} - \bar{z}_2)^{2\bar{h}_i}}}$$

which gives the correct 2-point function.

(ii) 3-point function

$$\langle 0 | \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) \phi_k(z_3, \bar{z}_3) | 0 \rangle$$

We now ~~take~~ a detour to derive a more generic differential equation for correlators.

(12)

$$T(z)A(w) := \sum_{n\in\mathbb{Z}} (z-w)^{-n-2} \underline{L_n(w)A(w)} \; (L_nA)(w).$$

An arbitrary field

$$= \cdots + \frac{h_A A(w)}{(z-w)^2} + \frac{\partial A(w)}{z-w} + \cdots$$

$$\Rightarrow L_{-n}|h\rangle = L_{-n}\phi(0)|0\rangle := (L_{-n}\phi)(0)|0\rangle$$

Hence we can define the descendant field associated w/ the state

$$L_{-n}|h\rangle \Rightarrow \phi^{(-n)}(w) := (L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint_w dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w)$$

$$\Rightarrow \langle \phi^{(-n)}(w) X\rangle \qquad X = \phi_1(w_1)\cdots\phi_N(w_N) \qquad n\geq 1$$

$$= \frac{1}{2\pi i}\oint_w dz\,(z-w)^{1-n}\langle T(z)\phi(w)X\rangle$$

$$= \frac{1}{2\pi i}\oint_{\{w_i\}} dz\,(z-w)^{1-n}\sum_i\left[\frac{1}{w_i-z}\partial_{w_i}\langle\phi(w)X\rangle + \frac{h_i}{(w_i-z)^2}\langle\phi(w)X\rangle\right]$$

$$= \mathcal{L}_{-n}\langle\phi(w)X\rangle$$

$$\Rightarrow \mathcal{L}_{-n} = \sum_i\left[\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right]$$

At level 2 in a Verma module, a generic linear combination reads

$$|\chi\rangle = (L_{-2} + \alpha L_{-1}^2)|h\rangle$$

We now enforce the condition $L_1|\chi\rangle = L_2|\chi\rangle = 0$ in order to make $|\chi\rangle$ a null state ($\Rightarrow L_n|\chi\rangle = 0$ for all $n\geq 3$)

$$L_1|\chi\rangle = ([L_1, L_{-2}] + \alpha[L_1, L_{-1}^2])|h\rangle$$

$$[L_1, L_{-2}] = 3L_{-1}$$

$$[L_1, L_{-1}^2] = L_{-1}[L_1, L_{-1}] + [L_1, L_{-1}]L_{-1} = 2L_{-1}L_0 + 2L_0L_{-1}$$

$$= 4L_{-1}L_0 + 2L_{-1}$$

$$\Rightarrow L_1|\chi\rangle = (3 + 2\alpha + 4\alpha h)L_{-1}|h\rangle = 0 \Rightarrow \boxed{3 + 2\alpha + 4\alpha h = 0}$$

$$\Rightarrow \alpha = -\frac{3}{2(2h+1)}$$

$$\Rightarrow \mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2 = 0$$

$$\boxed{\Rightarrow \left\{\sum_{i=1}^{2}\left[\frac{h_i}{(w-w_i)^2} + \frac{1}{w-w_i}\frac{\partial}{\partial w_i}\right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial w^2}\right\}\underbrace{\langle\phi(w)\phi(w_1)\phi(w_2)|0\rangle}_{f} = 0}$$

$$\Rightarrow \boxed{\frac{h_1}{(w-w_1)^2} f + \frac{1}{w-w_1}\partial_1 f + \frac{h_2}{(w-w_2)^2} f + \frac{1}{w-w_2}\partial_2 f - \frac{3}{2(2h+1)}\frac{\partial^2 f}{\partial w^2} = 0} \quad (*)$$

Back to the calculation of the 3-point correlator,

Under the constraints enforced by <u>global conformal transformations</u>.

i) Translation Invariance & rotation invariace

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = f\left(|x_1-x_2|, |x_2-x_3|, |x_3-x_1|\right)$$

ii) Scale invariance.

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = \frac{C_{123}^{(abc)}}{|x_1-x_2|^a |x_2-x_3|^b |x_3-x_1|^c}$$

$$a+b+c = \Delta_1+\Delta_2+\Delta_3$$

If spin is zero.

iii) <u>SCT</u>

$$|x_i-x_j| \to \frac{|x_i-x_j|}{\gamma_i^{\frac{1}{2}}\gamma_j^{\frac{1}{2}}}$$

$$\gamma_i = (1-2b\cdot x_i + b^2 x_i^2)$$

$$\Rightarrow \frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \cdot \frac{C_{123}^{(abc)}}{\frac{x_{12}^a}{(\gamma_1\gamma_2)^{\frac{a}{2}}}\frac{x_{23}^b}{(\gamma_2\gamma_3)^{\frac{b}{2}}}\frac{x_{31}^c}{(\gamma_3\gamma_1)^{\frac{c}{2}}}}.$$

$$\Rightarrow \begin{cases} a+c = 2\Delta_1 \\ a+b = 2\Delta_2 \\ b+c = 2\Delta_3 \end{cases} \Rightarrow \begin{cases} a = \Delta_1+\Delta_2-\Delta_3 \\ b = \Delta_2+\Delta_3-\Delta_1 \\ c = \Delta_3+\Delta_1-\Delta_2 \end{cases}$$

Transfer into complex coordinates in 2-D, and introduce nonzero spins, which are supposed to be conserved under rotation

$$\Rightarrow \langle 0|\phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2)\phi_k(z_3,\bar z_3)|0\rangle$$

$$= C_{ijk}\ \frac{1}{(z_1-z_2)^{h_i+h_j-h_k}(z_2-z_3)^{h_j+h_k-h_i}(z_3-z_1)^{h_k+h_i-h_j}} \times \frac{1}{(\bar z_1-\bar z_2)^{\bar h_i+\bar h_j-\bar h_k}(\bar z_2-\bar z_3)^{\bar h_j+\bar h_k-\bar h_i}(\bar z_3-\bar z_1)^{\bar h_k+\bar h_i-\bar h_j}}$$

Applying $(*)$ we obtain an extra relation between h_i, h_j & h_k or $\bar h_i, \bar h_j$ & $\bar h_k$

$$\boxed{2(2h_i+1)(h_i+2h_k-h_j) = 3(h_i-h_j+h_k)(h_i-h_j+h_k+1)} \quad (**)$$

If the minimal model is considered

Otherwise, the correlator should be trivially zero. by the global conformal symm.
Hence the 3-point function can be completely determined up to an overall coefficient C_{ijk}, when (**) is satisfied. Only if there is a solution ($c \geq 25$ or $c \leq 1$)
Generally other sources are required to determine C_{ijk}, for instance via the conformal bootstrap.

4.4 [BBS] 3.10

(i) $\langle \phi | [L_n, L_{-n}] | \phi \rangle = \langle \phi | L_n L_{-n} | \phi \rangle - \langle \phi | L_{-n} L_n | \phi \rangle$

$= \langle h | 2n L_0 + \frac{c}{12}(n^3 - n) | h \rangle$

$= 2nh + \frac{c}{12}(n^3 - n)$

For $n > 0$, $L_n |\phi\rangle = 0 \Rightarrow \langle \phi | L_n L_{-n} | \phi \rangle \geq 0$

$= 2nh + \frac{c}{12}(n^3 - n)$

Set $n = 1 \Rightarrow 2h \geq 0 \Rightarrow \boxed{h \geq 0}$

If $c < 0$, for sufficiently large $n \geq N$, $2nh + \frac{c}{12}(n^3 - n) < 0$, giving nonunitary representations $\Rightarrow \boxed{c \geq 0}$

(ii) If $|\phi\rangle = |0\rangle \Rightarrow \Phi_{-h}|0\rangle = |0\rangle$

$[L_m, \Phi_{-h}]|0\rangle = [m(h-1) + h]\Phi_{m-h}|0\rangle$

$L_m|0\rangle - \Phi_{-h} L_m |0\rangle = [m(h-1)+h]\Phi_{m-h}|0\rangle$

Set $m = 0 \Rightarrow 0 = h\Phi_{-h}|0\rangle = h|0\rangle \Rightarrow \boxed{h = 0}$

Similarly for the anti-holomorphic part $\Rightarrow \boxed{\bar{h} = 0}$

If $h = 0 \Rightarrow [L_m, \Phi_{-h}]|0\rangle = -m\Phi_{m-h}|0\rangle$

Set $m = 0 \Rightarrow L_0 \Phi_{-h}|0\rangle = 0 \Rightarrow \Phi_{-h}|0\rangle = |0\rangle \Rightarrow |\phi(z)\rangle = |0\rangle$

since $L_0|\lambda\rangle = 0$ iff $|\lambda\rangle = |0\rangle$

where $|\lambda\rangle = \sum_{N=0}^{\infty} c_N |N+h\rangle$, $L_0|N+h\rangle = (N+h)|N+h\rangle$

Similarly for the anti-holomorphic part. $\Rightarrow |\phi(\bar{z})\rangle = |0\rangle$

$\Rightarrow \boxed{h = \bar{h} = 0 \Leftrightarrow |\phi\rangle = 0}$