```
<< Local`QFTToolKit2`;
Get[NotebookDirectory[] <>
  "1204.0328 ParticlePhysicsFromAlmostCommutativeSpacetime.2.GWSmodel.out"]
$defStdMdl = {};

{Temporary}

"Notational definitions"
"Note that in the text the symbols may reference
  different Hilbert spaces. This has caused confusion in some of the
  calculations.  To address this problem we will try to label the
  variables by subscripts to designate the applicable Hilbert space.
  NOTE: Need to do notational change for .1,.2 notebooks."

rghtA[a_] := Superscript[a, o]
cl[a_] := ⟨a⟩cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := |a|;
it[a_] := Style[a, Italic]
iD := it[D]
iA := it[A]
iI := it["I"]
C∞ := C"∞"
Bx_ := T[B, "d", {x}]
("∇"S)n_ := T["∇"S, "d", {n}]

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
    ""];
accumStdMdl[item_] := Block[{}, $defStdMdl = tuAppendUniq[item][$defStdMdl];
    ""];
selectStdMdl[heads_, with_: {}, all_: Null] :=
  tuRuleSelect[$defStdMdl][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectGWS[heads_, with_: {}, all_: Null] := tuRuleSelect[$defGWS][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectDef[heads_, with_: {}, all_: Null] := tuRuleSelect[$defall][Flatten[{heads}]] //
    Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;

Clear[expandDC];
expandDC[sub_: {}, scalar_: {}] :=
 tuRepeat[{sub, tuOpDistribute[Dot], tuOpSimplify[Dot, scalar],
   tuOpDistribute[CircleTimes]}, tuCircleTimesSimplify]
Clear[expandCom]
expandCom[subs_: {}][exp_] := Block[{tmp = exp},
   tmp = tmp //. tuCommutatorExpand // expandDC[];
   tmp = tmp /. toxDot //. Flatten[{subs}];
   tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
   tmp
  ];
(**)
$sgeneral := {
  T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
  T[γ, "d", {5}].T[γ, "d", {5}] → 1,
```

```
  ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
  CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  T["▽", "d", {_}][1n_] → 0, a_ . 1n_ → a, 1n_ . a_ → a}
$sgeneral // ColumnBar

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] :→ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "dd", {μ_, ν_}] :→ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] :→ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] :→ CommutatorP[b, a] /; OrderedQ[{b, a}],
    tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
  };
$symmetries // ColumnBar

εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1,}}},
  {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
 ]
εRule[6]
```

Notational definitions

Note that in the text the symbols may reference different Hilbert spaces. This has caused confusion in some of the calculations. To address this problem we will try to label the variables by subscripts to designate the applicable Hilbert space. NOTE: Need to do notational change for .1,.2 notebooks.

$\gamma_5 \to \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$
$\gamma_5 . \gamma_5 \to 1$
$(\gamma_5)^\dagger \to \gamma_5$
$\{\gamma_5,\ \gamma^\mu\}_+ \to 0$
$\nabla_\_\,[1_{n\_}] \to 0$
$(a\_) . 1_{n\_} \to a$
$1_{n\_} . (a\_) \to a$

$tt : g^{\mu\_\,\nu\_} :\to \text{tuIndexSwap}[\{\mu,\ \nu\}][tt]\ /;\ \text{OrderedQ}[\{\nu,\ \mu\}]$
$tt : F^{\mu\_\,\nu\_} :\to -\text{tuIndexSwap}[\{\mu,\ \nu\}][tt]\ /;\ \text{OrderedQ}[\{\nu,\ \mu\}]$
$tt : F_{\mu\_\,\nu\_} :\to -\text{tuIndexSwap}[\{\mu,\ \nu\}][tt]\ /;\ \text{OrderedQ}[\{\nu,\ \mu\}]$
$[a\_,\ b\_]_- :\to -[b,\ a]_-\ /;\ \text{OrderedQ}[\{b,\ a\}]$
$\{a\_,\ b\_\}_+ :\to \{b,\ a\}_+\ /;\ \text{OrderedQ}[\{b,\ a\}]$
$tt : \gamma^\mu . \gamma_5 :\to \text{Reverse}[tt]$

$\{\varepsilon \to 1,\ \varepsilon' \to 1,\ \varepsilon'' \to -1\}$

6. The Standard Model

6.1 The Finite space

```
PR["● The algebra: left-right symmetric algebra ",
  𝒜LR, " and obtain ", 𝒜F ⊂ 𝒜LR, " with Dirac operator ", 𝒟F,
  NL, "The space: ", $sSM = {KOdim -> 6,
    𝒜F[CG["ℂ⊕ℍ⊕M3[3x3 ℂ matrices]"]],
    ℋl[CG[ℂ^4[{νR, eR, νL, eL}]]],
    ℋq[CG[ℂ^4[{uR, dR, uL, dL}]]] ⊗ ℂ^3[CG["color"]],
    ℋF → (ℋl ⊕ ℋl̄ ⊕ ℋq ⊕ ℋq̄)^"⊕3[generation]",
    a ∈ 𝒜F,
    a → {λ, q[CG[M2[ℂ]]], m[CG[M3[ℂ]]]},
    al → {λ, q, m}_ℋl, selectGWS[al],
    aq → {λ, q, m}_ℋq,
    ($ = selectGWS[al] /. l → q; $[[2]] = $[[2]] ⊗ 13[CG["color"]]; $),

    al̄ → {λ, q, m}_ℋl̄, al̄. l̄ → λ 14 . l̄,
    aq̄ → {λ, q, m}_ℋq̄, aq̄. q̄ → λ (14 ⊗ m).q̄,
    {CG["fermionic{fL,fR} grading"],
     γF.fL → fL,
     γF.fR → - fR
    },
    {CG["fermionic Charge conjugation(single generation, no color)"],
     JF.f_ :→
      If[FreeQ[f, OverBar], f̄, f[[1]]] /; tuMemberQ[f, selectStdMdl[basisSM][[2]]]
    },

    𝒟F → {{S, ct[T]}, {T, Conjugate[S]}},
    Sl ->
     Normal[SparseArray[{{1, 3} -> Yν, {2, 4} -> Ye, {3, 1} → ct[Yν], {4, 2} → ct[Ye]}]],
    Sq ⊗ 13 → Normal[SparseArray[{{1, 3} -> Yu, {2, 4} -> Yd,
          {3, 1} → ct[Yu], {4, 2} → ct[Yd]}]] ⊗ 13,
    {Yν, Ye, Yu, Yd} ∈ M3[CG["3 generation mass matrix, symmetric"]],
    T . νR → YR[CG["3×3 symmetric Majorana generation mass matrix"]].ν̄R,
    T.f :→ 0 /; f =!= νR,
    νR → Table[{T[νR, "d", {i}]}, {i, 3}][CG["with generations"]]
   }; $sSM // MatrixForms // ColumnBar, accumStdMdl[$sSM],
  NL,
  CO["Note: ", {al, al̄, aq, aq̄}, " only operate on their respective ℋilbert spaces."]
 ];
```

- The algebra: left-right symmetric algebra $\mathcal{A}_{LR}$ and obtain $\mathcal{A}_F \subset \mathcal{A}_{LR}$ with Dirac operator $\mathcal{D}_F$

The space:

KOdim → 6

$\mathcal{A}_F[\mathbb{C}\oplus\mathbb{H}\oplus M_3$[3x3 $\mathbb{C}$ matrices]]

$\mathcal{H}_l[\mathbb{C}^4[\{\nu_R, e_R, \nu_L, e_L\}]]$

$\mathcal{H}_q[\mathbb{C}^4[\{u_R, d_R, u_L, d_L\}]]\otimes\mathbb{C}^3$[color]

$\mathcal{H}_F \to (\mathcal{H}_l \oplus \mathcal{H}_{\bar{l}} \oplus \mathcal{H}_q \oplus \mathcal{H}_{\bar{q}})^{\oplus 3[\text{generation}]}$

$a \in \mathcal{A}_F$

$a \to \{\lambda, q[M_2[\mathbb{C}]], m[M_3[\mathbb{C}]]\}$

$a_l \to \{\lambda, q, m\}_{\mathcal{H}_l}$

$$a_l \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$$

$a_q \to \{\lambda, q, m\}_{\mathcal{H}_q}$

$$a_q \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix} \otimes 1_3[\text{color}]$$

$a_{\bar{l}} \to \{\lambda, q, m\}_{\mathcal{H}_{\bar{l}}}$

$a_{\bar{l}}.\bar{l} \to \lambda\, 1_4.\bar{l}$

$a_{\bar{q}} \to \{\lambda, q, m\}_{\mathcal{H}_{\bar{q}}}$

$a_{\bar{q}}.\bar{q} \to \lambda\, (1_4 \otimes m).\bar{q}$

{fermionic{f_L,f_R} grading, $\gamma_F.f_L \to f_L$, $\gamma_F.f_R \to -f_R$}

{fermionic Charge conjugation(single generation, no color),

$J_F.(f\_) \mapsto$ If[FreeQ[f, OverBar], $\bar{f}$, f⟦1⟧] /; tuMemberQ[f, selectStdMdl[basisSM]⟦2⟧]}

$$\mathcal{D}_F \to \begin{pmatrix} S & T^\dagger \\ T & S^* \end{pmatrix}$$

$$S_l \to \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ (Y_\nu)^\dagger & 0 & 0 & 0 \\ 0 & (Y_e)^\dagger & 0 & 0 \end{pmatrix}$$

$$S_q \otimes 1_3 \to \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ (Y_u)^\dagger & 0 & 0 & 0 \\ 0 & (Y_d)^\dagger & 0 & 0 \end{pmatrix} \otimes 1_3$$

$\{Y_\nu, Y_e, Y_u, Y_d\} \in M_3$[3 generation mass matrix, symmetric]

$T.\nu_R \to Y_R$[3×3 symmetric Majorana generation mass matrix]$.\bar{\nu}_R$

$T.f \mapsto 0$ /; f =!= ν_R

$$\nu_R \to \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix} \text{[with generations]}$$

Note: $\{a_l, a_{\bar{l}}, a_q, a_{\bar{q}}\}$ only operate on their respective $\mathcal{H}$ilbert spaces.

```
PR["Hilbert space basis: ",
 $ = {$smbasis =
    {l → {νR, eR, νL, eL}, l̄ -> {ν̄R, ēR, ν̄L, ēL}, q -> {uR, dR, uL, dL}, q̄ -> {ūR, d̄R, ūL, d̄L}},
   {q, q̄} → {qcolor, q̄color}, color → {1, 2, 3}, generations → {1, 2, 3}};
 $ // ColumnForms[#, 1] &, accumStdMdl[$],
 NL, "(8[l,l̄]+3[color]*8[q,q̄])*3[generations]->96 dimensions",
 NL, "Dirac operator: ",
 tuRuleSelect[$sSM][{𝒟F}][[1]], " is a 96 x 96 matrix operator."
]
```

```
Hilbert space basis:
    l → νR
        eR
        νL
        eL
    l̄ → ν̄R
        ēR
        ν̄L
        ēL
    q → uR
        dR
        uL
        dL
    q̄ → ūR
        d̄R
        ūL
        d̄L
    q  → qcolor
    q̄    q̄color
    color → 1
            2
            3
    generations → 1
                  2
                  3
(8[l,l̄]+3[color]*8[q,q̄])*3[generations]->96 dimensions
Dirac operator: 𝒟F → {{S, T†}, {T, S*}} is a 96 x 96 matrix operator.
```

Proposition 6.1

```
PR["Proposition 6.1.  The data ", $ = FSM → (#F & /@ {𝒜, ℋ, iD, γ, J}),
 " define a real even finite space of KO-dimension 6.", accumStdMdl[$]
]
```

```
Proposition 6.1.  The data FSM → {𝒜F, ℋF, DF, γF, JF}
  define a real even finite space of KO-dimension 6.
```

6.2 The gauge theory

The gauge group

```
PR["Manifold(p.69) ", M×F_SM,
  NL, "Define sub-algebra: ", $subalg = $ = { 𝒜̃_{F J_F} ⊂ $sSM[[1]],
     a ∈ 𝒜̃_{F J_F}, a.J_F → J_F.ct[a], {λ → cc[λ], α → λ, β → 0, m → λ 1_3}, a ≈ λ[CG[ℝ]]};
  ColumnBar[$], accumStdMdl[$],
  Imply, $subalg[[1, 1]] ≃ ℝ,
  Imply, "LieAlgebra", yield, {h_F → u[ $[[1, 1]] ], u[ $[[1, 1]] ] → {0}},
  line,
  NL, "●Examine the statement that ", $subalg[[2 ;; 3]],
  imply, turuleSelect[$subalg] /@ {λ, α, β, m} // Flatten // ColumnBar,
  NL, "We have: ",
  NL, "Algebra form: ", $a = b → (selectStdMdl[a_1] // Last);
  $a // MatrixForms,
  NL, "Real form: ",
  $s = selectGWS[J_{F_4}]; $s // MatrixForms,
  NL, "Subalgebra relationship: ",
  $ = selectDef[rghtA[b]] /. {rghtA[b] → b, F → F_4},
  Yield, $ = $ /. Dot → xDot /. $s /. $a;
  Yield, $ =
   OrderedxDotMultiplyAll[][$] /. {cc .a_ → Conjugate[a].cc} // tuConjugateSimplify[] //
     (# /. cc.cc → 1) & // tuOpSimplifyF[Dot];
  $ = $ //. rr : Rule[__] :> Thread[rr] // Flatten // DeleteDuplicates //
    (# /. Rule → Equal &), CK,
  Imply, $ = tuRuleSolve[$, {λ, β, Conjugate[α], Conjugate[λ]}];
  Framed[$],
  CR[" ", λ^* → λ, " not indicated."]
 ];
```

Manifold(p.69) $M\times F_{SM}$

Define sub-algebra: $\tilde{\mathcal{A}}_{F\,J_F}\subset(\text{KOdim}\to 6)$
$a\in\tilde{\mathcal{A}}_{F\,J_F}$
$a.J_F\to J_F.a^\dagger$
$\{\lambda\to\lambda^*,\ \alpha\to\lambda,\ \beta\to 0,\ m\to\lambda\,1_3\}$
$a\approx\lambda[\mathbb{R}]$

$\Rightarrow\ \tilde{\mathcal{A}}_{F\,J_F}\simeq\mathbb{R}$

$\Rightarrow$ LieAlgebra $\longrightarrow\{\mathfrak{h}_F\to\mathfrak{u}[\tilde{\mathcal{A}}_{F\,J_F}],\ \mathfrak{u}[\tilde{\mathcal{A}}_{F\,J_F}]\to\{0\}\}$

●Examine the statement that $\{a\in\tilde{\mathcal{A}}_{F\,J_F},\ a.J_F\to J_F.a^\dagger\}\Rightarrow$ $\lambda\to\lambda^*$, $\alpha\to\lambda$, $\beta\to 0$, $m\to\lambda\,1_3$

We have:

Algebra form: $b\to\begin{pmatrix}\lambda&0&0&0\\0&\lambda^*&0&0\\0&0&\alpha&\beta\\0&0&-\beta^*&\alpha^*\end{pmatrix}$

Real form: $J_{F_4}\to\begin{pmatrix}0&0&cc&0\\0&0&0&cc\\cc&0&0&0\\0&cc&0&0\end{pmatrix}$

Subalgebra relationship: $b\to J_{F_4}.b^\dagger.(J_{F_4})^\dagger$

$\to$

$\to\{\lambda=\alpha,\ 0=-\beta^*,\ \text{True},\ 0=\beta,\ \lambda^*=\alpha^*,\ \alpha=\lambda,\ \beta=0,\ -\beta^*=0,\ \alpha^*=\lambda^*\}\longleftarrow$CHECK

$\Rightarrow\ \boxed{\{\lambda\to\alpha,\ \beta\to 0,\ \alpha^*\to\alpha^*\}}$ $\lambda^*\to\lambda$ not indicated.

Proposition 6.2

```
PR[
 NL, "Prop.6.2:  The local gauge group: ", {𝒢[F_SM] ≃ mod[(U[1]×SU[2]×U[3]), {1, -1}]},
 NL, "Demand unimodularity: ", Det[u]_ℋF → 1, imply, (λ Det[m])^12 → 1, " for ",
 u ∈ U[1]×SU[2]×U[3],
 NL, CR["Why 12? Possible rational: "], Det[u]_ℋF → Det[λ] Det[q] Det[m] → 1,
 and, {Det[q] → 1, Det[λ] → λ},
 and, "there is 2 x 2 x 3 possible phases freedoms in Det[u]_ℋF.",
 NL, "Let ", U → u.J.u.ct[J] ↔ 𝒢[F_SM],
 NL, "The subgroup: ",
 $ = S𝒢[F_SM] → {U → u.J.u.ct[J] ∈ 𝒢[F_SM], u → {λ, q, m}, (λ Det[m])^12 → 1};
 $ // ColumnForms,
 NL, "The condition ", $[[2, 3]] ⇒ mod[Det[m] ≃ cc[λ], μ12] ,
 NL, "True gauge group of the SM: ",
 𝒢SM → mod[U[1]×SU[2]×SU[3], μ6]
]
```

Prop.6.2: The local gauge group: $\{\mathcal{G}[F_{SM}] \simeq \mathrm{mod}[U[1]\times SU[2]\times U[3], \{1, -1\}]\}$

Demand unimodularity: $\mathrm{Det}[u]_{\mathcal{H}_F} \to 1 \Rightarrow \lambda^{12}\,\mathrm{Det}[m]^{12} \to 1$ for $u \in U[1]\times SU[2]\times U[3]$

Why 12? Possible rational: $\mathrm{Det}[u]_{\mathcal{H}_F} \to \mathrm{Det}[m]\,\mathrm{Det}[q]\,\mathrm{Det}[\lambda] \to 1$ and $\{\mathrm{Det}[q] \to 1, \mathrm{Det}[\lambda] \to \lambda\}$

and there is 2 x 2 x 3 possible phases freedoms in $\mathrm{Det}[u]_{\mathcal{H}_F}$.

Let $U \to u.J.u.J^{\dagger} \leftrightarrow \mathcal{G}[F_{SM}]$

The subgroup: $S\mathcal{G}[F_{SM}] \to \left|\begin{array}{l} U \to u.J.u.J^{\dagger} \in \mathcal{G}[F_{SM}] \\ u \to \left|\begin{array}{l}\lambda \\ q \\ m\end{array}\right. \\ \lambda^{12}\,\mathrm{Det}[m]^{12} \to 1 \end{array}\right.$

The condition $(\lambda^{12}\,\mathrm{Det}[m]^{12} \to 1) \Rightarrow \mathrm{mod}[\mathrm{Det}[m] \simeq \lambda^{*}, \mu_{12}]$

True gauge group of the SM: $\mathcal{G}_{SM} \to \mathrm{mod}[U[1]\times SU[2]\times SU[3], \mu_6]$

Proposition 6.3

```
PR["Prop 6.3: The unimodular gauge group ", S𝒢[F_SM] ≈ 𝒢SM × μ12
]
```

Prop 6.3: The unimodular gauge group $S\mathcal{G}[F_{SM}] \approx \mathcal{G}_{SM}\times\mu_{12}$

6.2.2 The gauge fields and the Higgs field

```
PR["Calculate ", {T[A, "d", {μ}], ϕ},
 " From 2.13 and 2.14 ", (*define in and get from $defall*)
 {$e213 = T[γ, "u", {μ}] ⊗ T[A, "d", {μ}] →
     a CommutatorM[slash[𝒟] ⊗ 1_F, b] → - I T[γ, "u", {μ}] ⊗ (a tuDDown["∂"][b, μ]),
   $e214 = T[γ, "d", {5}] ⊗ ϕ -> a CommutatorM[T[γ, "d", {5}] ⊗ 𝒟_F, b] ->
      T[γ, "d", {5}] ⊗ (a CommutatorM[𝒟_F, b])
  } // ColumnBar,

 Imply, "Higgs field ", $e61 = $ = {
    ϕ_ℋ1 → {{0, ct[Y]}, {Y, 0}},
    ϕ_ℋ_T → 0,
    ϕ_ℋq → {{0, ct[X]}, {X, 0}} ⊗ 1_3[CG["color"]],
    ϕ_ℋ_q → 0 ,
    {ϕ1, ϕ2} ∈ CG[ℂ],
    Y → {{Y_ν ϕ1, -Y_e Conjugate[ϕ2]}, {Y_ν ϕ2, Y_e Conjugate[ϕ1]}},
    X → {{Y_u ϕ1, -Y_d Conjugate[ϕ2]}, {Y_u ϕ2, Y_d Conjugate[ϕ1]}},
    Φ → Inactivate[𝒟_F2 + {{ϕ, 0}, {0, 0}} + J_F.{{ϕ, 0}, {0, 0}}.ct[J_F], Plus] →
      {{S + ϕ, ct[T]}, {T, Conjugate[S + ϕ]}}
   }; $ // Column // MatrixForms // Framed, accumStdMdl[$], CG[" (6.1,6.2)"],
 NL, "with same GWS ", selectGWS[Tensor[it[A], _, _], {Λ, Q}]
 (*Symbol for A inconsistent.*)
]
```

Calculate $\{A_\mu, \phi\}$ From 2.13 and 2.14 $\left|\begin{array}{l} \gamma^\mu \otimes A_\mu \to a\,[(\not{D}) \otimes 1_F, b]_- \to -i\,\gamma^\mu \otimes (a\,\underline{\partial}_\mu[b]) \\ \gamma_5 \otimes \phi \to a\,[\gamma_5 \otimes \mathcal{D}_F, b]_- \to \gamma_5 \otimes (a\,[\mathcal{D}_F, b]_-) \end{array}\right.$

⇒ Higgs field

$$\boxed{\begin{array}{l}
\phi_{\mathcal{H}_1} \to \begin{pmatrix} 0 & Y^\dagger \\ Y & 0 \end{pmatrix} \\
\phi_{\mathcal{H}_T} \to 0 \\
\phi_{\mathcal{H}_q} \to \begin{pmatrix} 0 & X^\dagger \\ X & 0 \end{pmatrix} \otimes 1_3[\text{color}] \\
\phi_{\mathcal{H}_{\bar{q}}} \to 0 \\
\{\phi_1, \phi_2\} \in \mathbb{C} \\
Y \to \begin{pmatrix} Y_\nu \phi_1 & -(\phi_2)^* Y_e \\ Y_\nu \phi_2 & (\phi_1)^* Y_e \end{pmatrix} \\
X \to \begin{pmatrix} Y_u \phi_1 & -(\phi_2)^* Y_d \\ Y_u \phi_2 & (\phi_1)^* Y_d \end{pmatrix} \\
\Phi \to \mathcal{D}_{F_2} + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F.\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}.(J_F)^\dagger \to \begin{pmatrix} S+\phi & T^\dagger \\ T & (S+\phi)^* \end{pmatrix}
\end{array}}$$

(6.1,6.2)

with same GWS $A_\mu \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\}$

```
PR["•The field from the term ",
  -I (a tuDDown["∂"][b, μ]), " for ", $s = {a → m, b → m', {m, m'} ∈ ℋq},

  Yield,
  $ = {T[V', "d", {μ}] → -I m tuDDown["∂"][m', μ], T[V', "d", {μ}][ℋq], {m, m'} ∈ M3[ℂ]};
  $ // ColumnBar, accumStdMdl[$];
  NL, "If ", $[[1]], " hermitian ⇒ ", $[[1, 1]] ∈ I u[3], imply, $[[1, 1]] ∈ U[3],
  NL, "Impose unimodularity condition to get SU[3] gauge field. ",
  CR["Why is I included?"],
  Imply, TrℋF[T[A, "d", {μ}]] → 0,
  yield, TrℋI[T[Λ, "d", {μ}] 14] + Trℋq[14 ⊗ T[V', "d", {μ}]] → 0,
  imply, Tr[T[V', "d", {μ}]] → -T[Λ, "d", {μ}],
  NL, "Define a traceless SU[3] gauge field: ",
  $ = T[V, "d", {μ}] → -T[V', "d", {μ}] - 13 T[Λ, "d", {μ}] / 3;
  $ // Framed, accumStdMdl[$];

  NL, "Then the gauge field becomes: ", T[A, "d", {μ}],
  Yield, $e63a = $ = {T[Aℋl, "d", {μ}] →
       DiagonalMatrix[{T[Λ, "d", {μ}], -T[Λ, "d", {μ}], T[Q, "d", {μ}]}],
     T[AℋI, "d", {μ}] → 14 T[Λ, "d", {μ}],
     T[Aℋq, "d", {μ}] →
       DiagonalMatrix[{T[Λ, "d", {μ}], -T[Λ, "d", {μ}], T[Q, "d", {μ}]}] ⊗ 13,
     T[Aℋq, "d", {μ}] → -14 ⊗ (Conjugate[T[V, "d", {μ}]] + 13 T[Λ, "d", {μ}] / 3),
     T[Λ, "d", {μ}] ∈ U[1],
     T[Q, "d", {μ}] ∈ SU[2]
    }; $ // Column // MatrixForms // Framed,
  NL, "Action on fermions of field: ",
  $e63b = T[B, "d", {μ}] -> T[A, "d", {μ}] - JF.T[A, "d", {μ}].inv[JF],
  Yield, $e63 = $ = {T[Bℋl, "d", {μ}] →
       DiagonalMatrix[{0, -2 T[Λ, "d", {μ}], T[Q, "d", {μ}] - T[Λ, "d", {μ}] 12}],
     T[Bℋq, "d", {μ}] → DiagonalMatrix[{4 / 3 T[Λ, "d", {μ}] 13 + T[V, "d", {μ}],
         -2 / 3 T[Λ, "d", {μ}] 13 + T[V, "d", {μ}],

         (T[Q, "d", {μ}] + 1 / 3 T[Λ, "d", {μ}] 12) ⊗ 13 + 12 ⊗ T[V, "d", {μ}]}]};
  $ // Column // MatrixForms // Framed, accumStdMdl[{$e63, $e63a, $e63b}]
 ];
PR["Hypercharge assignments(coefficient of Λ's): ", $hypercharge = Association[
   {νR → 0, eR → -2, νL → -1, eL → 2, uR → 4 / 3, dR → -2 / 3, uL → 1 / 3, dL → 1 / 3}],
 NL, CR["How are ", T[Λ, "d", {μ}], " coefficient determined?"]
]
```

•The field from the term $-i\, a\, \partial_\mu[b]$ for $\{a \to m,\ b \to m',\ \{m, m'\} \in \mathcal{H}_{\bar{q}}\}$

$$\to \left| \begin{array}{l} V'_\mu \to -i\, m\, \partial_\mu[m'] \\ V'_\mu[\mathcal{H}_{\bar{q}}] \\ \{m, m'\} \in M_3[\mathbb{C}] \end{array} \right.$$

If $V'_\mu \to -i\, m\, \partial_\mu[m']$ hermitian $\Rightarrow$ $V'_\mu \in i\, \mathfrak{u}[3]$ $\Rightarrow$ $V'_\mu \in U[3]$

Impose unimodularity condition to get SU[3] gauge field. Why is $\bar{I}$ included?

$\Rightarrow$ $\mathrm{Tr}_{\mathcal{H}_F}[A_\mu] \to 0$ $\longrightarrow$ $\mathrm{Tr}_{\mathcal{H}_{\bar{I}}}[1_4\, \Lambda_\mu] + \mathrm{Tr}_{\mathcal{H}_{\bar{q}}}[1_4 \otimes V'_\mu] \to 0$ $\Rightarrow$ $\mathrm{Tr}[V'_\mu] \to -\Lambda_\mu$

Define a traceless SU[3] gauge field: $\boxed{V_\mu \to -\frac{1}{3} 1_3\, \Lambda_\mu - V'_\mu}$

Then the gauge field becomes: A_μ

$$\to \boxed{\begin{array}{l} A_{\mathcal{H}_1\,\mu} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix} \\ A_{\mathcal{H}_{\bar{I}}\,\mu} \to 1_4\, \Lambda_\mu \\ A_{\mathcal{H}_q\,\mu} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix} \otimes 1_3 \\ A_{\mathcal{H}_{\bar{q}}\,\mu} \to -1_4 \otimes \left((V_\mu)^* + \frac{1}{3} 1_3\, \Lambda_\mu\right) \\ \Lambda_\mu \in U[1] \\ Q_\mu \in SU[2] \end{array}}$$

Action on fermions of field: $B_\mu \to -J_F . A_\mu . J_F^{-1} + A_\mu$

$$\to \boxed{\begin{array}{l} B_{\mathcal{H}_1\,\mu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu - 1_2\,\Lambda_\mu \end{pmatrix} \\ B_{\mathcal{H}_q\,\mu} \to \begin{pmatrix} V_\mu + \frac{4}{3} 1_3\, \Lambda_\mu & 0 & 0 \\ 0 & V_\mu - \frac{2}{3} 1_3\, \Lambda_\mu & 0 \\ 0 & 0 & 1_2 \otimes V_\mu + \left(Q_\mu + \frac{1}{3} 1_2\, \Lambda_\mu\right) \otimes 1_3 \end{pmatrix} \end{array}}$$

Hypercharge assignments(coefficient of Λ's):

$$\left\langle \left| \nu_R \to 0,\ e_R \to -2,\ \nu_L \to -1,\ e_L \to 2,\ u_R \to \frac{4}{3},\ d_R \to -\frac{2}{3},\ u_L \to \frac{1}{3},\ d_L \to \frac{1}{3} \right| \right\rangle$$

How are Λ_μ coefficient determined?

```
PR["■Check Calculation of B's. For: ", #[[1]] & /@ $e63 // First,
 NL, "Using ", {$e63b, $sJ =
   {tuRuleSelect[$sr][J_F8][[1]] /. F8 → F, inv[J_F] → J_F, inv[aa : cc | 0] → aa, cc^2 → 1}},
 NL, "•For form 8x8 : ", $s = {$e63a[[1, 1]], $e63a[[2, 1]]},

 NL, "Expand elements of: ",
 $sQ = T[Q, "d", {μ}] -> Table[q_{i,j}, {i, 2}, {j, 2}],
 $s1 = $e63a[[1]] /. $sQ // MapAt[ArrayFlatten[#] &, #, 2] &;
 $s2 = $e63a[[2]] /. 1_4 → DiagonalMatrix[Table[1, {4}]];
 $sA8 =
  $e63a[[1, 1]] → ({{$e63a[[1, 1]], 0}, {0, $e63a[[2, 1]]}} /. $s1 /. $s2 // ArrayFlatten);
 $sA8 // MatrixForms,
 NL, "Compute ",
 $ = $e63b /. Plus → Inactive[Plus] /. Tensor[a_, i_, j_] :→ Tensor[a_{H_1}, i, j],
 Yield, $ = $ // expandCom[{$sJ, $sA8}];
 Yield, $ = $ /. cc . a_ :→ cc[a].cc /; FreeQ[a, cc] /. cc.cc → 1;
 $ // MatrixForms;
 Yield, $Bhl = $ = $ /. 1 → 1⊕1̄ // tuConjugateSimplify[{cc, T[Λ, "d", {μ}]}] // Activate;
 $ // MatrixForms // Framed, accumStdMdl[$], OK
]
```

■Check Calculation of B's. For: $B_{\mathcal{H}_1\mu}$

Using $\{B_\mu \to -J_F.A_\mu.J_F^{-1} + A_\mu,$
$\{J_F \to \{\{0,0,0,0,cc,0,0,0\},\{0,0,0,0,0,cc,0,0\},\{0,0,0,0,0,0,cc,0\},$
$\{0,0,0,0,0,0,0,cc\},\{cc,0,0,0,0,0,0,0\},\{0,cc,0,0,0,0,0,0\},$
$\{0,0,cc,0,0,0,0,0\},\{0,0,0,cc,0,0,0,0\}\}, J_F^{-1} \to J_F, (aa:cc\,|\,0)^{-1} \to aa, cc^2 \to 1\}\}$

•For form 8x8 : $\{A_{\mathcal{H}_1\mu}, A_{\mathcal{H}_{\bar{1}}\mu}\}$

Expand elements of: $Q_\mu \to \{\{q_{1,1}, q_{1,2}\},\{q_{2,1}, q_{2,2}\}\}$ $A_{\mathcal{H}_1\mu} \to \begin{pmatrix} \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{1,1} & q_{1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{2,1} & q_{2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu \end{pmatrix}$

Compute $B_{\mathcal{H}_1\mu} \to -J_F.A_{\mathcal{H}_1\mu}.J_F^{-1} + A_{\mathcal{H}_1\mu}$

→

→

→ $\boxed{B_{\mathcal{H}_{1\oplus\bar{1}}\mu} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{1,1}-\Lambda_\mu & q_{1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{2,1} & q_{2,2}-\Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{1,1})^*+\Lambda_\mu & -(q_{1,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{2,1})^* & -(q_{2,2})^*+\Lambda_\mu \end{pmatrix}}$ OK

```
PR["■Check Calculation of B's. For: ", #[[1]] & /@ $e63 // Last,
  "With ", $0 = $ = {$e63b, $sJ =
      {tuRuleSelect[$sr][J_F8][[1]] /. F8 → F, inv[J_F] → J_F, inv[cc : cc | 0] → cc, cc^2 → 1}};
  $ // MatrixForms // ColumnBar,
  NL, "•Select one copy of the finite portions: ",
  $s = selectStdMdl[Tensor[A_, _, _], #] & /@ {Q, q̄};
  $s // ColumnBar,
  NL, "Expand so q,q̄ versions of A_ℋ are 4x4 so action of J_F matrix is unambiguous: ",
  $a1 = T[A, "d", {μ}] → $e63a[[3, 1]] ⊕ $e63a[[4, 1]],
  $sQ = T[Q, "d", {μ}] -> Table[q_i,j, {i, 2}, {j, 2}];
  NL, $s1 = $e63a[[3]] /. $sQ /. ll : List[List[__], __] ⧴ ArrayFlatten[ll];
  NL, $s2 = $e63a[[4]] /. -1_4 ⊗ a_ :> DiagonalMatrix[Table[-a, {4}]];
  NL,
  $sA8 =
   $a1[[1]] → ({{$e63a[[3, 1]], 0}, {0, $e63a[[4, 1]]}} /. Plus → Inactive[Plus] /. $s1 /.
         $s2 /. a_ ⊗ 1_3 → a 1_3 /. Plus → Inactive[Plus]) /.
    ll : List[List[__], __] ⧴ ArrayFlatten[ll];
  $sA8 // MatrixForms, "POFF",
  Yield, $ = $0[[1]] /. Plus → Inactive[Plus] /. $sA8; $ // MatrixForms,
  Yield, $ = $ // expandCom[{$sJ, $sA8}] // Activate; $ // MatrixForms,

  Yield, $ = $ /. cc . a_ ⧴ Conjugate[a].cc /; FreeQ[a, cc] /. cc.cc → 1 // expandDC[];
  $ // MatrixForms, CK,
  "PONdd",
  Yield, $BHq = $ = $ /. B → B_ℋq⊕ℋq̄ // tuConjugateSimplify[{cc, T[Λ, "d", {μ}], 1_3}] //
     tuOpSimplifyF[Dot, {1_3}];
  $ // MatrixForms // Framed, CG["(6.3)"], accumStdMdl[$],
  NL, CR["Note: ", T[V, "d", {μ}] ∈ M_3[ℂ], " so the notation is ambiguous."]
 ];
```

■Check Calculation of B's. For: $B_{\mathcal{H}_{q\,\mu}}$ With

$$B_\mu \to -J_F . A_\mu . J_F^{-1} + A_\mu$$

$$\{J_F \to \begin{pmatrix} 0 & 0 & 0 & 0 & cc & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & cc & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & cc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & cc & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & cc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cc & 0 & 0 & 0 & 0 \end{pmatrix},\ J_F^{-1} \to J_F,\ (cc : cc \mid 0)^{-1} \to cc,\ cc^2 \to 1\}$$

•Select one copy of the finite portions:

$$A_{\mathcal{H}_{q\,\mu}} \to \{\{\Lambda_\mu, 0, 0\}, \{0, -\Lambda_\mu, 0\}, \{0, 0, Q_\mu\}\} \otimes 1_3$$
$$A_{\mathcal{H}_{\overline{q}\,\mu}} \to -1_4 \otimes \left((V_\mu)^* + \frac{1}{3} 1_3 \Lambda_\mu\right)$$

Expand so q, $\overline{q}$ versions of $A_\mathcal{H}$ are 4x4 so action of J_F matrix is unambiguous:

$$A_\mu \to A_{\mathcal{H}_{q\,\mu}} \oplus A_{\mathcal{H}_{\overline{q}\,\mu}}$$

$$A_\mu \to$$

$$\begin{pmatrix} 1_3 \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1_3 \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_3\, q_{1,1} & 1_3\, q_{1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_3\, q_{2,1} & 1_3\, q_{2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(V_\mu)^* + -\frac{1}{3} 1_3 \Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(V_\mu)^* + -\frac{1}{3} 1_3 \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(V_\mu)^* + -\frac{1}{3} 1_3 \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(V_\mu)^* + -\frac{1}{3} 1_3 \Lambda_\mu \end{pmatrix}$$

....

→

$$B_{\mathcal{H}_q \oplus \mathcal{H}_{\overline{q}\,\mu}} \to \begin{pmatrix} V_\mu + \frac{4}{3} 1_3 \Lambda_\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & V_\mu - \frac{2}{3} 1_3 \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_3\, q_{1,1} + V_\mu + \frac{1}{3} 1_3 \Lambda_\mu & 1_3\, q_{1,2} & 0 & 0 \\ 0 & 0 & 1_3\, q_{2,1} & 1_3\, q_{2,2} + V_\mu + \frac{1}{3} 1_3 \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & -(V_\mu)^* - \frac{4}{3} 1_3 \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -(V_\mu)^* + \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(6.3)

Note: $V_\mu \in M_3[\mathbb{C}]$ so the notation is ambiguous.

Proposition 6.4

```
PR["●Prop.6.4. The action of the gauge
   group SG[M×F_SM] on the fluctuated Dirac operator: ",
 iD_A → slash[iD] ⊗ 1_F + T[γ, "u", {μ}] ⊗ T[B, "d", {μ}] + T[γ, "d", {5}] ⊗ Φ,
 NL, "is implemented by: ",
 $p64 = $ = {T[Λ, "d", {μ}] -> T[Λ, "d", {μ}] - I λ.tuDDown["∂"][Conjugate[λ], μ],
    T[Q, "d", {μ}] → q. T[Q, "d", {μ}].ct[q] - I q.tuDDown["∂"][ct[q], μ],
    Conjugate[T[V, "d", {μ}]] →
     m. Conjugate[T[V, "d", {μ}]] .ct[m] - I m.tuDDown["∂"][ct[m], μ],
    {{ϕ1 + 1}, {ϕ2}} → Conjugate[λ] q .{{ϕ1 + 1}, {ϕ2}},
    λ ∈ C^"∞"[M, U[1]],
    q ∈ C^"∞"[M, SU[2]],
    m ∈ C^"∞"[M, SU[3]]
   }; $ // Column // MatrixForms // Framed,
 line,
 NL, "The proof examines the action of ",
 u → {λ, q, m} ∈ C^"∞"[M, U[1] × SU[2] × SU[3]],
 NL, "as in Proposition 5.3 for ", selectGWS[Tensor[it[A], _, _]],
 Yield, $ = {T[Q, "d", {μ}] → q. T[Q, "d", {μ}].ct[q] ,
   Conjugate[T[V, "d", {μ}]] → m. Conjugate[T[V, "d", {μ}]] .ct[m] ,
   -I u.tuDDown["∂"][ct[u], μ][{ν_R, u_R, H_I}] → -I λ.tuDDown["∂"][Conjugate[λ], μ],
   -I u.tuDDown["∂"][ct[u], μ][{e_R, d_R}] → I λ.tuDDown["∂"][Conjugate[λ], μ],
   -I u.tuDDown["∂"][ct[u], μ][{ν_L, e_L, u_L, d_L}] → -I q.tuDDown["∂"][ct[q], μ],
   -I u.tuDDown["∂"][ct[u], μ][{H_q}] → -I m.tuDDown["∂"][ct[m], μ]
  }; $ // Column
]
```

●Prop.6.4. The action of the gauge group $S\mathcal{G}[M\times F_{SM}]$ on the fluctuated Dirac operator: $D_A \to (\not{D})\otimes 1_F + \gamma_5\otimes\Phi + \gamma^\mu\otimes B_\mu$

is implemented by:

$$\boxed{\begin{array}{l}
\Lambda_\mu \to -i\,\lambda.\underline{\partial}_\mu[\lambda^*] + \Lambda_\mu \\
Q_\mu \to -i\,q.\underline{\partial}_\mu[q^\dagger] + q.Q_\mu.q^\dagger \\
(V_\mu)^* \to -i\,m.\underline{\partial}_\mu[m^\dagger] + m.(V_\mu)^*.m^\dagger \\
\begin{pmatrix} 1+\phi_1 \\ \phi_2 \end{pmatrix} \to \lambda^*\,q.\begin{pmatrix} 1+\phi_1 \\ \phi_2 \end{pmatrix} \\
\lambda \in C^\infty[M, U[1]] \\
q \in C^\infty[M, SU[2]] \\
m \in C^\infty[M, SU[3]]
\end{array}}$$

The proof examines the action of $u \to \{\lambda, q, m\} \in C^\infty[M, U[1]\times SU[2]\times SU[3]]$
as in Proposition 5.3 for $A_\mu \to -i\,u.\underline{\partial}_\mu[u^\dagger] + u.A_\mu.u^\dagger$

$$\to \begin{array}{l}
Q_\mu \to q.Q_\mu.q^\dagger \\
(V_\mu)^* \to m.(V_\mu)^*.m^\dagger \\
-i\,u.\underline{\partial}_\mu[u^\dagger][\{\nu_R, u_R, \mathcal{H}_I\}] \to -i\,\lambda.\underline{\partial}_\mu[\lambda^*] \\
-i\,u.\underline{\partial}_\mu[u^\dagger][\{e_R, d_R\}] \to i\,\lambda.\underline{\partial}_\mu[\lambda^*] \\
-i\,u.\underline{\partial}_\mu[u^\dagger][\{\nu_L, e_L, u_L, d_L\}] \to -i\,q.\underline{\partial}_\mu[q^\dagger] \\
-i\,u.\underline{\partial}_\mu[u^\dagger][\{\mathcal{H}_q\}] \to -i\,m.\underline{\partial}_\mu[m^\dagger]
\end{array}$$

6.3 The spectral action - bosonic part of $\mathcal{L}_{SM}$

Lemma 6.5

```
PR["●Lemma 6.5. ",
 $l65 = Tr[T[F_Hq, "dd", {μ, ν}]. T[F_Hq, "uu", {μ, ν}]] →
   24 (10 / 3 T[Λ, "dd", {μ, ν}] T[Λ, "uu", {μ, ν}] + Tr[
```

```
      T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]] + Tr[T[V, "dd", {μ, ν}] T[V, "uu", {μ, ν}]]),
 line,
 NL, "Proof:",
 next, "The leptonic sector is as in
   Lemma 5.4 multiplied by 3 for the number of generations.",
 next, "The quark sector: ",

 next, "Calculate F's. Using: ",
 NL, "Using ", $e63[[2]] // MatrixForms,
 NL, "• Canonical form: ",
 $ = selectDef[Tensor[F, _, _]] /. Tensor[F_, i_, j_] → Tensor[F_{ℋq}, i, j], "POFF",

 $ = $ /. Plus → Inactive[Plus],
 $ = $ // expandCom[($e63 // tuAddPatternVariable[μ])];
 $ = $ // tuDerivativeExpand[{1_}] // Activate // Expand;
 Yield, $ // MatrixForms, CK,

 $ = $ // tuCircleTimesGather[] // tuOpSimplifyF[Dot, {Tensor[Λ, _, _]}] // Simplify;
 Yield, $ // MatrixForms, "PONdd",
 NL, "Using ",
 NL, "• ",
 $s = (selectDef[Tensor[F, _, _]] /. tuCommutatorExpand // Reverse // (# /. F | B → V &) //
     Expand),
 $ = $ /. $s // Simplify;
 NL, "• ",
 $s = (selectDef[Tensor[F, _, _]] /. tuCommutatorExpand // Reverse // (# /. F | B → Q &) //
     Expand),
 $ = $ /. $s // Simplify;
 NL, "• ", $s = (selectGWS[Tensor[B, _, _]] /. tuCommutatorExpand // Reverse //
      tuAddPatternVariable[{μ, ν}] // (# /. B → Λ &)),
 $ = $ /. $s /. T[Λ, "dd", {ν, μ}] → -T[Λ, "dd", {μ, ν}];
 Yield, $ // MatrixForms // Framed,

line,
 $sF = {$, $ // tuIndicesRaise[{μ, ν}]};
 next, "Compute ", $ = $165[[1]],
 Yield, $ = $ /. Tr → xTr /. toxDot /. $sF // tuMatrixOrderedMultiply //
   tuOpSimplifyF[xDot, {Tensor[Λ, _, _]}];
 Yield, xtmp = $ = $ /. toDot // expandDC[] // tuOpSimplifyF[Dot, {Tensor[Λ, _, _]}] //
     tuCircleTimesGather[];
 $ // MatrixForms,
 NL, "Compute xTr[] ",
 $ = $ //. xx : xTr[a_] :> Thread[xx] /. xTr[0] → 0 /. aa : CircleTimes[a_, 1n_] :→
      tuOpDistributeF[CircleTimes][aa] //. tuOpDistribute[xTr] // Tr[#] &;
 $ // ColumnSumExp;
 NL, "• The Q and V are members of SU[2] and SU[3],
   respectively; hence their Tr[]'s are zero, as well as their
   products. The Tr[] of single Q,V's and Λ will be zero as well.",
 NL, "• Use Rule: ", $s = xTr[a_] :→ 0 /; (tuExtractPattern[Tensor[Q | V | Λ, _, _]][{a}] //
      tuHasAllQ[#, {V, Q}] || tuHasAllQ[#, {V, Λ}] || tuHasAllQ[#, {Λ, Q}] &),
 xtmp = $ = $ /. $s;
 Yield, $ // ColumnSumExp;
 NL, "Use Rules: ", $s = {xTr[1n_ ⊗ a_] → n xTr[a],
   xTr[a_ ⊗ 1n_] → n xTr[a], xTr[a_ 1n_] → n xTr[a], xTr → Tr, Dot → Times},
 Yield, $ = $ //. $s // tuTrSimplify[] // tuIndexDummyOrdered;
 $165[[1]] -> $ // ColumnSumExp // Framed,
 NL, CR["Need to add contribution from l̄,l,q̄  to get complete result."]
,
```

●Lemma 6.5. $\mathrm{Tr}[F_{\mathcal{H}_q\,\mu\nu}.F_{\mathcal{H}_q}{}^{\mu\nu}] \to 24\,(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}] + \mathrm{Tr}[V_{\mu\nu}V^{\mu\nu}])$

Proof:
◆The leptonic sector is as in
Lemma 5.4 multiplied by 3 for the number of generations.
◆The quark sector:
◆Calculate F's. Using:

Using $B_{\mathcal{H}_q\,\mu} \to \begin{pmatrix} V_\mu + \frac{4}{3} 1_3\,\Lambda_\mu & 0 & 0 \\ 0 & V_\mu - \frac{2}{3} 1_3\,\Lambda_\mu & 0 \\ 0 & 0 & 1_2 \otimes V_\mu + (Q_\mu + \frac{1}{3} 1_2\,\Lambda_\mu)\otimes 1_3 \end{pmatrix}$

• Canonical form: $F_{\mathcal{H}_q\,\mu\nu} \to \mathrm{i}\,[B_{\mathcal{H}_q\,\mu}, B_{\mathcal{H}_q\,\nu}]_- - \underline{\partial}_\nu[B_{\mathcal{H}_q\,\mu}] + \underline{\partial}_\mu[B_{\mathcal{H}_q\,\nu}]$

....
Using
• $\mathrm{i}\,V_\mu.V_\nu - \mathrm{i}\,V_\nu.V_\mu - \underline{\partial}_\nu[V_\mu] + \underline{\partial}_\mu[V_\nu] \to V_{\mu\nu}$
• $\mathrm{i}\,Q_\mu.Q_\nu - \mathrm{i}\,Q_\nu.Q_\mu - \underline{\partial}_\nu[Q_\mu] + \underline{\partial}_\mu[Q_\nu] \to Q_{\mu\nu}$
• $-\underline{\partial}_{\nu\_}[\Lambda_{\mu\_}] + \underline{\partial}_{\mu\_}[\Lambda_{\nu\_}] \to \Lambda_{\mu\nu}$

→ $\boxed{F_{\mathcal{H}_q\,\mu\nu} \to \begin{pmatrix} V_{\mu\nu} + \frac{4}{3} 1_3\,\Lambda_{\mu\nu} & 0 & 0 \\ 0 & V_{\mu\nu} - \frac{2}{3} 1_3\,\Lambda_{\mu\nu} & 0 \\ 0 & 0 & 1_2 \otimes V_{\mu\nu} + (Q_{\mu\nu} + \frac{1}{3} 1_2\,\Lambda_{\mu\nu})\otimes 1_3 \end{pmatrix}}$

◆Compute $\mathrm{Tr}[F_{\mathcal{H}_q\,\mu\nu}.F_{\mathcal{H}_q}{}^{\mu\nu}]$
→

→ $\mathrm{xTr}[\begin{pmatrix} V_{\mu\nu}.V^{\mu\nu} + \frac{4}{3} 1_3\,V^{\mu\nu}\Lambda_{\mu\nu} + \frac{4}{3} 1_3\,V_{\mu\nu}\Lambda^{\mu\nu} + \frac{16}{9} 1_3\,\Lambda_{\mu\nu}\Lambda^{\mu\nu} & 0 \\ 0 & V_{\mu\nu}.V^{\mu\nu} - \frac{2}{3} 1_3\,V^{\mu\nu}\Lambda_{\mu\nu} - \frac{2}{3} 1_3\,V_{\mu\nu}\Lambda^{\mu\nu} + \frac{4}{9} 1_3\,\Lambda_{\mu\nu}\Lambda \\ 0 & 0 \end{pmatrix}$

Compute xTr[]
• The Q and V are members of SU[2] and SU[3],
respectively; hence their Tr[]'s are zero, as well as their
products. The Tr[] of single Q,V's and Λ will be zero as well.
• Use Rule: xTr[a_] :→ 0 /;
(tuHasAllQ[#1, {V, Q}] || tuHasAllQ[#1, {V, Λ}] || tuHasAllQ[#1, {Λ, Q}] &)[
tuExtractPattern[Tensor[Q | V | Λ, _, _]][{a}]]
→
Use Rules:
$\{\mathrm{xTr}[1_{n\_}\otimes a\_] \to n\,\mathrm{xTr}[a],\ \mathrm{xTr}[a\_\otimes 1_{n\_}] \to n\,\mathrm{xTr}[a],\ \mathrm{xTr}[a\_\,1_{n\_}] \to n\,\mathrm{xTr}[a],\ \mathrm{xTr} \to \mathrm{Tr},\ \mathrm{Dot} \to \mathrm{Times}\}$

→ $\boxed{\mathrm{Tr}[F_{\mathcal{H}_q\,\mu\nu}.F_{\mathcal{H}_q}{}^{\mu\nu}] \to \sum\left[\begin{array}{|l} 3\,\mathrm{Tr}[Q_\mu{}^\nu Q^\mu{}_\nu] \\ 4\,\mathrm{Tr}[V_\mu{}^\nu V^\mu{}_\nu] \\ \frac{22}{3}\mathrm{Tr}[\Lambda_\mu{}^\nu \Lambda^\mu{}_\nu] \end{array}\right]}$

Need to add contribution from $\bar{l}$,l,$\bar{q}$ to get complete result.

Lemma 6.6

```
PR["●Lemma 6.6 ",
 $166 = $ = {Tr[Φ^2] → 4 a Abs[H']^2 + 2 c,
    Tr[Φ^4] → 4 b Abs[H']^4 + 4 beAbs[H']^2 + 2 d,
    H' → {ϕ1 + 1, ϕ2},
    a → Tr[ct[Yν].Yν + ct[Ye].Ye + 3 ct[Yu].Yu + 3 ct[Yd].Yd],
    b → Tr[(ct[Yν].Yν)^2 + (ct[Ye].Ye)^2 + 3 (ct[Yu].Yu)^2 + 3 (ct[Yd].Yd)^2],
    c → Tr[ct[YR].YR],
    d -> Tr[(ct[YR].YR)^2],
    e -> Tr[ct[YR].YR.ct[Yν].Yν]
   };
 $ // ColumnBar,
 line,
 NL, "Proof: Compute: ", $0 = $ = $166[[1]],

 NL, "Given ", $sΦ = $e61; $sΦ // MatrixForms,
 Yield, $ = tuRuleSelect[$sΦ][Φ][[1]]; $ // MatrixForms,
 Yield, $0 = $[[1]] → $[[2, 2]];
 (*****)
 line,
 NL, "What does this look like for basis (without generations and color): ", (*
 $basisSM={basisSM->Flatten[#[[2]]&/@selectStdMdl/@{l,q,l̄,q̄}],
   basisSM[CG["without generation(3) and color(3 for u,d) indices"]]},*)
 $basisSM = selectStdMdl[basisSM],
 line,
 NL, "Determine: ", $Slq = $ = Slq -> Sl ⊕ Sq,
 NL, "where ",
 $sS = {selectStdMdl[Sl], First[#] & /@ selectStdMdl[Sq ⊗ _]},
 $ = $ /. $sS; $ // MatrixForms;
 $Slq = $ = $ /. a_ ⊕ b_ :> ArrayFlatten[{{a, 0}, {0, b}}]; $ // MatrixForms,
 accumStdMdl[{$}]
]
selectStdMdl[T.νR]
selectStdMdl[T.f]
```

●Lemma 6.6

$$\begin{aligned}&\mathrm{Tr}[\Phi^2]\to 2\,c+4\,a\,\mathrm{Abs}[H']^2\\&\mathrm{Tr}[\Phi^4]\to 2\,d+4\,b\,\mathrm{Abs}[H']^4+4\,\mathrm{beAbs}[H']^2\\&H'\to\{1+\phi_1,\ \phi_2\}\\&a\to\mathrm{Tr}[3\,(Y_d)^\dagger.Y_d+(Y_e)^\dagger.Y_e+3\,(Y_u)^\dagger.Y_u+(Y_\nu)^\dagger.Y_\nu]\\&b\to\mathrm{Tr}[3\,((Y_d)^\dagger.Y_d)^2+((Y_e)^\dagger.Y_e)^2+3\,((Y_u)^\dagger.Y_u)^2+((Y_\nu)^\dagger.Y_\nu)^2]\\&c\to\mathrm{Tr}[(Y_R)^\dagger.Y_R]\\&d\to\mathrm{Tr}[((Y_R)^\dagger.Y_R)^2]\\&e\to\mathrm{Tr}[(Y_R)^\dagger.Y_R.(Y_\nu)^\dagger.Y_\nu]\end{aligned}$$

Proof: Compute: $\mathrm{Tr}[\Phi^2]\to 2\,c+4\,a\,\mathrm{Abs}[H']^2$

Given

$$\{\phi_{\mathcal{H}_1}\to\begin{pmatrix}0 & Y^\dagger\\ Y & 0\end{pmatrix},\ \phi_{\mathcal{H}_{\bar{1}}}\to 0,\ \phi_{\mathcal{H}_q}\to\begin{pmatrix}0 & X^\dagger\\ X & 0\end{pmatrix}\otimes 1_3[\text{color}],\ \phi_{\mathcal{H}_{\bar{q}}}\to 0,\ \{\phi_1,\ \phi_2\}\in\mathbb{C},\ Y\to\begin{pmatrix}Y_\nu\,\phi_1 & -(\phi_2)^*\,Y_e\\ Y_\nu\,\phi_2 & (\phi_1)^*\,Y_e\end{pmatrix},$$
$$X\to\begin{pmatrix}Y_u\,\phi_1 & -(\phi_2)^*\,Y_d\\ Y_u\,\phi_2 & (\phi_1)^*\,Y_d\end{pmatrix},\ \Phi\to\mathcal{D}_{F_2}+\begin{pmatrix}\phi & 0\\ 0 & 0\end{pmatrix}+J_F.\begin{pmatrix}\phi & 0\\ 0 & 0\end{pmatrix}.(J_F)^\dagger\to\begin{pmatrix}S+\phi & T^\dagger\\ T & (S+\phi)^*\end{pmatrix}\}$$

$$\to\ \Phi\to\mathcal{D}_{F_2}+\begin{pmatrix}\phi & 0\\ 0 & 0\end{pmatrix}+J_F.\begin{pmatrix}\phi & 0\\ 0 & 0\end{pmatrix}.(J_F)^\dagger\to\begin{pmatrix}S+\phi & T^\dagger\\ T & (S+\phi)^*\end{pmatrix}$$

→

What does this look like for basis (without generations and color): Last[{}]

Determine: $S_{1q}\to S_1\oplus S_q$

where $\{S_1\to\{\{0,0,Y_\nu,0\},\{0,0,0,Y_e\},\{(Y_\nu)^\dagger,0,0,0\},\{0,(Y_e)^\dagger,0,0\}\},$
$S_q\to\{\{0,0,Y_u,0\},\{0,0,0,Y_d\},\{(Y_u)^\dagger,0,0,0\},\{0,(Y_d)^\dagger,0,0\}\}\}$

$$S_{1q}\to\begin{pmatrix}0&0&Y_\nu&0&0&0&0&0\\0&0&0&Y_e&0&0&0&0\\(Y_\nu)^\dagger&0&0&0&0&0&0&0\\0&(Y_e)^\dagger&0&0&0&0&0&0\\0&0&0&0&0&0&Y_u&0\\0&0&0&0&0&0&0&Y_d\\0&0&0&0&(Y_u)^\dagger&0&0&0\\0&0&0&0&0&(Y_d)^\dagger&0&0\end{pmatrix}$$

$T.\nu_R\to Y_R[3\times 3$ symmetric Majorana generation mass matrix$].\overline{\nu_R}$

$T.f :\to 0\ /;\ f =!= \nu_R$

```
PR["• SM basis Construction of Φ: ",
  NL, "Using relationships: ",
  $ = $sΦ; $ // MatrixForms,
  next, "Construct {l,q} version of: ",
  Imply, $ = selectStdMdl[ϕ_ℋ_l],
  Yield, $l = $ /. selectStdMdl[Y] // MapAt[ArrayFlatten[#] &, #, 2] &;
  $q = selectStdMdl[ϕ_ℋ_q] /. a_ ⊗ b_ → a /. selectStdMdl[X] //
    MapAt[ArrayFlatten[#] &, #, 2] &,
  Yield, $ = {{ϕ, 0}, {0, 0}} -> {{$l[[2]], 0}, {0, $q[[2]]}} //
    MapAt[ArrayFlatten[#] &, #, 2] &;
  $ // MatrixForms,
  $ϕ = $ = $[[1]] -> {{$[[2]], 0}, {0, DiagonalMatrix[Table[0, {8}]]}} //
     MapAt[ArrayFlatten[#] &, #, 2] &;
  $ // MatrixForms
 ];
PR["Check calculation of: ", $0 = J_F.{{ϕ, 0}, {0, 0}}.ct[J_F],
  NL, "Construct: ",
  $ = DiagonalMatrix[Table[cc, {8}], 8] + DiagonalMatrix[Table[cc, {8}], -8];
  $ // MatrixForm;
  $j = $ = J_F → $; $ // MatrixForms,
  Imply,
  $ = J_F.{{ϕ, 0}, {0, 0}}.ct[J_F] /. Dot → xDot /. $j /. $ϕ // OrderedxDotMultiplyAll[];
  $ // MatrixForms;
  Yield,
  $JphJ = $ = $0 -> $ // tuRepeat[{Conjugate[cc] → cc, cc.cc → 1, Conjugate[cc].cc → 1,
       cc . a_ :> Conjugate[a].cc /; a =!= cc}, tuConjugateSimplify[{}]];
  $ // MatrixForms
 ];
```

- SM basis Construction of Φ:

Using relationships:

$$\left\{\phi_{\mathcal{H}_1} \to \begin{pmatrix} 0 & Y^\dagger \\ Y & 0 \end{pmatrix},\ \phi_{\mathcal{H}_T} \to 0,\ \phi_{\mathcal{H}_q} \to \begin{pmatrix} 0 & X^\dagger \\ X & 0 \end{pmatrix} \otimes 1_3[\mathrm{color}],\ \phi_{\mathcal{H}_{\bar q}} \to 0,\ \{\phi_1, \phi_2\} \in \mathbb{C},\ Y \to \begin{pmatrix} Y_\nu\, \phi_1 & -(\phi_2)^* Y_e \\ Y_\nu\, \phi_2 & (\phi_1)^* Y_e \end{pmatrix},\right.$$

$$\left. X \to \begin{pmatrix} Y_u\, \phi_1 & -(\phi_2)^* Y_d \\ Y_u\, \phi_2 & (\phi_1)^* Y_d \end{pmatrix},\ \Phi \to \mathcal{D}_{F_2} + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F . \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} . (J_F)^\dagger \to \begin{pmatrix} S+\phi & T^\dagger \\ T & (S+\phi)^* \end{pmatrix} \right\}$$

◆Construct {l,q} version of:

⇒ $\phi_{\mathcal{H}_1} \to \{\{0, Y^\dagger\}, \{Y, 0\}\}$

→ $\phi_{\mathcal{H}_q} \to \{\{0, 0, (Y_u\, \phi_1)^*, (Y_u\, \phi_2)^*\},$
$\{0, 0, -(Y_d)^* \phi_2, (Y_d)^* \phi_1\}, \{Y_u\, \phi_1, -(\phi_2)^* Y_d, 0, 0\}, \{Y_u\, \phi_2, (\phi_1)^* Y_d, 0, 0\}\}$

$$\to \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \to \left(\begin{array}{cccccccc}
0 & 0 & (Y_\nu\, \phi_1)^* & (Y_\nu\, \phi_2)^* & 0 & 0 & 0 & 0 \\
0 & 0 & -(Y_e)^* \phi_2 & (Y_e)^* \phi_1 & 0 & 0 & 0 & 0 \\
Y_\nu\, \phi_1 & -(\phi_2)^* Y_e & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_\nu\, \phi_2 & (\phi_1)^* Y_e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (Y_u\, \phi_1)^* & (Y_u\, \phi_2)^* \\
0 & 0 & 0 & 0 & 0 & 0 & -(Y_d)^* \phi_2 & (Y_d)^* \phi_1 \\
0 & 0 & 0 & 0 & Y_u\, \phi_1 & -(\phi_2)^* Y_d & 0 & 0 \\
0 & 0 & 0 & 0 & Y_u\, \phi_2 & (\phi_1)^* Y_d & 0 & 0
\end{array}\right) \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \to$$

$$\left(\begin{array}{cccccccccccccccc}
0 & 0 & (Y_\nu\, \phi_1)^* & (Y_\nu\, \phi_2)^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(Y_e)^* \phi_2 & (Y_e)^* \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_\nu\, \phi_1 & -(\phi_2)^* Y_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_\nu\, \phi_2 & (\phi_1)^* Y_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (Y_u\, \phi_1)^* & (Y_u\, \phi_2)^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(Y_d)^* \phi_2 & (Y_d)^* \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y_u\, \phi_1 & -(\phi_2)^* Y_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y_u\, \phi_2 & (\phi_1)^* Y_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Check calculation of: $J_F.\{\{\phi, 0\}, \{0, 0\}\}.(J_F)^\dagger$

Construct: $$J_F \to \begin{pmatrix}
0&0&0&0&0&0&0&0&cc&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&cc&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&cc&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&cc&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&cc&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&cc&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&cc&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&cc\\
cc&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&cc&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&cc&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&cc&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&cc&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&cc&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&cc&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&cc&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$\Rightarrow$

$$\to J_F.\begin{pmatrix}\phi&0\\0&0\end{pmatrix}.(J_F)^\dagger \to \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&Y_\nu.\phi_1&Y_\nu.\phi_2&0&\\
0&0&0&0&0&0&0&0&0&0&-Y_e.(\phi_2)^*&Y_e.(\phi_1)^*&0&\\
0&0&0&0&0&0&0&0&(Y_\nu)^*.(\phi_1)^*&-\phi_2.(Y_e)^*&0&0&0&\\
0&0&0&0&0&0&0&0&(Y_\nu)^*.(\phi_2)^*&\phi_1.(Y_e)^*&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&(Y_u)^*.(\phi_1)^*&-\phi_2.\\
0&0&0&0&0&0&0&0&0&0&0&0&(Y_u)^*.(\phi_2)^*&\phi_1.
\end{pmatrix}$$

```
PR[next, "Construct 16x16: ", $0 = selectStdMdl[{𝒟F}, {T, S}],
 NL, "Given: ", $ = {$Slq = selectStdMdl[S1q], $sT = selectStdMdl[T.νR]};
 $ // MatrixForms,
 Imply, $sT = T → ({Normal[SparseArray[{{1, 1} -> YR}, {8, 8}]]} // ArrayFlatten // First);
 $ // MatrixForms,
 NL, "Inserting into: ", $ = $0,
 Yield, $DF = $ = $ /. ($Slq /. S1q → S) /. $sT // MapAt[ArrayFlatten[#] &, #, 2] &;
 $ // MatrixForms, accumStdMdl[$]
]
```

◆Construct 16x16: $\mathcal{D}_F \to \{\{S, T^{\dagger}\}, \{T, S^{*}\}\}$

Given: $\{S_{1q} \to \begin{pmatrix} 0 & 0 & Y_\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_e & 0 & 0 & 0 & 0 \\ (Y_\nu)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (Y_e)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_d \\ 0 & 0 & 0 & 0 & (Y_u)^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (Y_d)^\dagger & 0 & 0 \end{pmatrix},$

$T.\nu_R \to Y_R$[3×3 symmetric Majorana generation mass matrix]$.\overline{\nu_R}\}$

$\Rightarrow \{S_{1q} \to \begin{pmatrix} 0 & 0 & Y_\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_e & 0 & 0 & 0 & 0 \\ (Y_\nu)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (Y_e)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_d \\ 0 & 0 & 0 & 0 & (Y_u)^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (Y_d)^\dagger & 0 & 0 \end{pmatrix},$

$T.\nu_R \to Y_R$[3×3 symmetric Majorana generation mass matrix]$.\overline{\nu_R}\}$

Inserting into: $\mathcal{D}_F \to \{\{S, T^{\dagger}\}, \{T, S^{*}\}\}$

$\to \mathcal{D}_F \to$

$$\begin{pmatrix}
0 & 0 & Y_\nu & 0 & 0 & 0 & 0 & 0 & (Y_R)^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Y_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(Y_\nu)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (Y_e)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Y_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (Y_u)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (Y_d)^\dagger & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_\nu)^* & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_e)^* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_\nu)^{\dagger *} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_e)^{\dagger *} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_u)^* & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_d)^* \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_u)^{\dagger *} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (Y_d)^{\dagger *} & 0 & 0
\end{pmatrix}$$

```
PR["Substitute into: ", $ = selectStdMdl[Φ],
 $ = $[[1]] → $[[2, 1]]; $ // MatrixForms,
 Yield, $Φ = $ = $ /. ($DF /. F → F2) /. $JphJ /. $ϕ // Activate;
 $ // MatrixForms, accumStdMdl[$]
]
```

Substitute into:

$\Phi \to \mathcal{D}_{F_2} + \{\{\phi, 0\}, \{0, 0\}\} + J_F.\{\{\phi, 0\}, \{0, 0\}\}.(J_F)^\dagger \to \{\{S + \phi, T^\dagger\}, \{T, (S + \phi)^*\}\}$

$\Phi \to \mathcal{D}_{F_2} + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F.\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}.(J_F)^\dagger$

$$\to \Phi \to \left(\begin{array}{ccccccc}
0 & 0 & (Y_\nu \phi_1)^* + Y_\nu & (Y_\nu \phi_2)^* & 0 & 0 & 0 \\
0 & 0 & -(Y_e)^* \phi_2 & Y_e + (Y_e)^* \phi_1 & 0 & 0 & 0 \\
(Y_\nu)^\dagger + Y_\nu \phi_1 & -(\phi_2)^* Y_e & 0 & 0 & 0 & 0 & 0 \\
Y_\nu \phi_2 & (Y_e)^\dagger + (\phi_1)^* Y_e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (Y_u \phi_1)^* + Y \\
0 & 0 & 0 & 0 & 0 & 0 & -(Y_d)^* \phi_2 \\
0 & 0 & 0 & 0 & (Y_u)^\dagger + Y_u \phi_1 & -(\phi_2)^* Y_d & 0 \\
0 & 0 & 0 & 0 & Y_u \phi_2 & (Y_d)^\dagger + (\phi_1)^* Y_d & 0 \\
Y_R & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right.$$

```
PR["Compute ", $0 = Tr[Φ.Φ],
 NL, "• with scalars: ", $scal = {ϕ1, ϕ2},
 NL, "• symmetry of Y's: ", $sY = {Transpose[(yy : Yn_)] → yy, ct[(yy : Yn_)] → cc[yy]},
 NL, "• defining ",
 $ = {{1 + ϕ1}, {ϕ2}};
 $sH = Abs[H']^2 -> ct[$].$,
 $sH = tuRuleSolve[$sH, cc[ϕ2] ϕ2][[1]];
 $sH2 = #^2 & /@ $sH // Expand;
 (**)
 Yield, $TrPP = $ = xTr[ct[Φ].Φ] /. toxDot /. $Φ // tuConjugateTransposeExpand //
      tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
 $ // ColumnSumExp;
 $ = $ // tuTrEvaluate[{}] // (# /. xTr → Tr &); $ // ColumnSumExp;
 $ = $ // tuConjugateTransposeSimplify[{}, $scal]; $ // ColumnSumExp;
 $ = $ /. $sY // Simplify; $ // ColumnSumExp;
 $0a = $ =
   $ /. tt : Tr[a_] ⧴ tuTrSimplify[{ϕ1, ϕ2}][tt] /. tt : Tr[a_] :> tuTrCanonicalOrder[tt] /.
     $sH // Collect[#, Tr[_], Simplify] &;
 $ // ColumnSumExp,
 NL, CR["If ", $s = ϕ1 → 0],
 Yield, $ = $ /. $s; $ // ColumnSumExp,
 NL, CR["we get the result in the Lemma."],
 NL, "• Similarly, An examination of the ϕ1 terms ",
 NL, "with ", $ = Im[a] → (a - cc[a]) / 2;
 Yield, $cc = tuRuleSolve[$, cc[a]] /. a → ϕ1,
 Yield, $ = $0a /. $cc // Collect[#, Tr[_], Simplify] &; $ // ColumnSumExp,
 Yield, $s = Map[#[$] &, (tuTermSelect /@ {cc[Ye].cc[Ye], Ye.Ye})] // Flatten // Column,
 Imply, "Let ", $s = {Im[ϕ1] → ϕ1, cc[Ye_].cc[Ye_] -> Ye.Ye},
 Yield, $ = $ /. $s; $ // ColumnSumExp,
 NL,
 CR["If {ϕ1 pure imaginary, Y's ∈ℝ} the lemma is also satisfied.  This contraint on
    ϕ1 and Y's seems to be missing in the text. "],

 note, " The u,d terms need factors of 3 to account for the 3-color space."
]
```

Compute Tr[Φ.Φ]
• with scalars: $\{\phi_1, \phi_2\}$
• symmetry of Y's: $\{yy : Y_{n\_}^{T} \to yy,\ (yy : Y_{n\_})^{\dagger} \to yy^{*}\}$
• defining $\mathrm{Abs}[H']^2 \to \{\{(1 + (\phi_1)^*)\,(1 + \phi_1) + (\phi_2)^*\,\phi_2\}\}$

$$\to \sum\left[\begin{array}{|l}
4\,\phi_1\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*] \\
-4\,(-\mathrm{Abs}[H']^2 + (\phi_1)^* + \phi_1)\,\mathrm{Tr}[(Y_d)^*.Y_d] \\
4\,\phi_1\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*] \\
-4\,(-\mathrm{Abs}[H']^2 + (\phi_1)^* + \phi_1)\,\mathrm{Tr}[(Y_e)^*.Y_e] \\
2\,\mathrm{Tr}[(Y_R)^*.Y_R] \\
4\,(\phi_1)^*\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*] \\
-4\,(-\mathrm{Abs}[H']^2 + (\phi_1)^* + \phi_1)\,\mathrm{Tr}[(Y_u)^*.Y_u] \\
4\,(\phi_1)^*\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*] \\
-4\,(-\mathrm{Abs}[H']^2 + (\phi_1)^* + \phi_1)\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu] \\
4\,(\phi_1)^*\,\mathrm{Tr}[Y_d.Y_d] \\
4\,(\phi_1)^*\,\mathrm{Tr}[Y_e.Y_e] \\
4\,\phi_1\,\mathrm{Tr}[Y_u.Y_u] \\
4\,\phi_1\,\mathrm{Tr}[Y_\nu.Y_\nu]
\end{array}\right]$$

If $\phi_1 \to 0$

$$\to \sum\left[\begin{array}{|l}
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_d)^*.Y_d] \\
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_e)^*.Y_e] \\
2\,\mathrm{Tr}[(Y_R)^*.Y_R] \\
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_u)^*.Y_u] \\
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu]
\end{array}\right]$$

we get the result in the Lemma.
• Similarly, An examination of the ϕ_1 terms with
$\to \{(\phi_1)^* \to -2\,\mathrm{Im}[\phi_1] + \phi_1\}$

$$\to \sum\left[\begin{array}{|l}
4\,\phi_1\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*] \\
4\,(\mathrm{Abs}[H']^2 + 2\,\mathrm{Im}[\phi_1] - 2\,\phi_1)\,\mathrm{Tr}[(Y_d)^*.Y_d] \\
4\,\phi_1\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*] \\
4\,(\mathrm{Abs}[H']^2 + 2\,\mathrm{Im}[\phi_1] - 2\,\phi_1)\,\mathrm{Tr}[(Y_e)^*.Y_e] \\
2\,\mathrm{Tr}[(Y_R)^*.Y_R] \\
4\,(-2\,\mathrm{Im}[\phi_1] + \phi_1)\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*] \\
4\,(\mathrm{Abs}[H']^2 + 2\,\mathrm{Im}[\phi_1] - 2\,\phi_1)\,\mathrm{Tr}[(Y_u)^*.Y_u] \\
4\,(-2\,\mathrm{Im}[\phi_1] + \phi_1)\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*] \\
4\,(\mathrm{Abs}[H']^2 + 2\,\mathrm{Im}[\phi_1] - 2\,\phi_1)\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu] \\
4\,(-2\,\mathrm{Im}[\phi_1] + \phi_1)\,\mathrm{Tr}[Y_d.Y_d] \\
4\,(-2\,\mathrm{Im}[\phi_1] + \phi_1)\,\mathrm{Tr}[Y_e.Y_e] \\
4\,\phi_1\,\mathrm{Tr}[Y_u.Y_u] \\
4\,\phi_1\,\mathrm{Tr}[Y_\nu.Y_\nu]
\end{array}\right]$$

$$\to \begin{array}{l}
4\,\phi_1\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*] \\
-8\,\mathrm{Im}[\phi_1]\,\mathrm{Tr}[Y_e.Y_e] \\
4\,\phi_1\,\mathrm{Tr}[Y_e.Y_e]
\end{array}$$

⇒ Let $\{\mathrm{Im}[\phi_1] \to \phi_1,\ (Y_{e\_})^*.(Y_{e\_})^* \to Y_e.Y_e\}$

$$\to \sum\left[\begin{array}{|l}
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_d)^*.Y_d] \\
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_e)^*.Y_e] \\
2\,\mathrm{Tr}[(Y_R)^*.Y_R] \\
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_u)^*.Y_u] \\
4\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu]
\end{array}\right]$$

If {ϕ_1 pure imaginary, Y's ∈ℝ} the lemma is also satisfied.
This contraint on ϕ_1 and Y's seems to be missing in the text.
⌘ The u,d terms need factors of 3 to account for the 3-color space.

```
PR[next, "Compute ", $0 = Tr[Φ.Φ.Φ.Φ],
 Yield, $ = xTr[ct[Φ].Φ.ct[Φ].Φ] /. toxDot /. $Φ // tuConjugateTransposeExpand //
     tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[];
 $ // ColumnSumExp;
 $ = $ // tuTrEvaluate[{}] // (# /. xTr → Tr &); $ // ColumnSumExp;
 $ = $ // tuConjugateTransposeSimplify[{}, $scal]; $ // ColumnSumExp;
 $ = $ /. $sY // Simplify; $ // ColumnSumExp;
 $0a =
  $ = $ /. tt : Tr[a_] ⧴ tuTrSimplify[{ϕ1, ϕ2}][tt] /. tt : Tr[a_] :> tuTrCanonicalOrder[
        tt] //. $sH2 //. $sH // Collect[#, Tr[_], Simplify] &;
 $ // ColumnSumExp,
 NL, CR["With the previous conditions, i.e., ", $s = {cc[ϕ1] → -ϕ1, cc[tt : Y_] → tt}],
 Yield,
 $ = $0a //. $s /. tt : Tr[_] :> tuTrCanonicalOrder[tt] // Collect[#, Tr[_], Simplify] &;
 $ // ColumnSumExp,
 note, " The u,d terms need factors of 3 to account for the 3-color space."
]
```

◆Compute $\mathrm{Tr}[\Phi.\Phi.\Phi.\Phi]$

$$
\to \sum\left[\begin{array}{l}
4\,\phi_1^2\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.(Y_d)^*.(Y_d)^*]\\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.(Y_d)^*.Y_d]\\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.(Y_u)^*.(Y_u)^*]\\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_d)^*.(Y_d)^*.Y_d.Y_d]\\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_d)^*.(Y_u)^*.Y_u.Y_d]\\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_d)^*.Y_d.(Y_d)^*.Y\\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_d)^*.Y_d.Y_d.Y_d]\\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_d)^*.Y_d.Y_u.(Y_u)^*]\\
4\,\phi_1^2\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.(Y_e)^*.(Y_e)^*]\\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.(Y_e)^*.Y_e]\\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.(Y_\nu)^*.(Y_\nu)^*]\\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_e)^*.(Y_e)^*.Y_e.Y_e]\\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_e)^*.(Y_\nu)^*.Y_\nu.Y_e]\\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_e)^*.Y_e.(Y_e)^*.Y\\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_e)^*.Y_e.Y_e.Y_e]\\
8\,(\mathrm{Abs}[H']^2-(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[(Y_e)^*.Y_e.Y_\nu.(Y_\nu)^*]\\
4\,(\phi_1)^*\,\mathrm{Tr}[(Y_R)^*.(Y_\nu)^*.(Y_\nu)^*.Y_R]\\
4\,\mathrm{Tr}[(Y_R)^*.(Y_\nu)^*.Y_\nu.Y_R]\\
2\,\mathrm{Tr}[(Y_R)^*.Y_R.(Y_R)^*.Y_R]\\
4\,(\phi_1)^*\,\mathrm{Tr}[(Y_R)^*.Y_R.(Y_\nu)^*.(Y_\nu)^*]\\
-4\,(1-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_R)^*.Y_R.(Y_\nu)^*.Y_\nu]\\
4\,\mathrm{Tr}[(Y_R)^*.Y_R.Y_\nu.(Y_\nu)^*]\\
4\,\phi_1\,\mathrm{Tr}[(Y_R)^*.Y_R.Y_\nu.Y_\nu]\\
-4\,(1-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_R)^*.Y_\nu.(Y_\nu)^*.Y_R]\\
4\,\phi_1\,\mathrm{Tr}[(Y_R)^*.Y_\nu.Y_\nu.Y_R]\\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*.(Y_u)^*.(Y_u)^*]\\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*.(Y_u)^*.Y_u]\\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_u)^*.(Y_u)^*.Y_u.Y_u]\\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_u)^*.Y_u.(Y_u)^*.Y\\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_u)^*.Y_u.Y_u.Y_u]\\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*.(Y_\nu)^*.(Y_\nu)^*]\\
-8\,(\phi_1)^*\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*.(Y_\nu)^*.Y_\nu]\\
8\,(-1+\mathrm{Abs}[H']^2+(\phi_1)^*\,(-1+\phi_1)-\phi_1)\,\mathrm{Tr}[(Y_\nu)^*.(Y_\nu)^*.Y_\nu.Y_\nu]\\
4\,(2+\mathrm{Abs}[H']^4+(\phi_1)^{*2}+2\,\phi_1+\phi_1^2+2\,(\phi_1)^*\,(1+\phi_1)-2\,\mathrm{Abs}[H']^2\,(1+(\phi_1)^*+\phi_1))\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu.(Y_\nu)^*.Y\\
-8\,\phi_1\,(-\mathrm{Abs}[H']^2+(\phi_1)^*+\phi_1)\,\mathrm{Tr}[(Y_\nu)^*.Y_\nu.Y_\nu.Y_\nu]\\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[Y_d.Y_d.Y_d.Y_d]\\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[Y_d.Y_d.Y_u.Y_u]\\
4\,(\phi_1)^{*2}\,\mathrm{Tr}[Y_e.Y_e.Y_e.Y_e]\\
8\,(-\mathrm{Abs}[H']^2+(1+(\phi_1)^*)\,(1+\phi_1))\,\mathrm{Tr}[Y_e.Y_e.Y_\nu.Y_\nu]\\
4\,\phi_1^2\,\mathrm{Tr}[Y_u.Y_u.Y_u.Y_u]\\
4\,\phi_1^2\,\mathrm{Tr}[Y_\nu.Y_\nu.Y_\nu.Y_\nu]
\end{array}\right.
$$

With the previous conditions, i.e., $\{(\phi_1)^* \to -\phi_1,\ (tt:Y\_)^* \to tt\}$

$$
\to \sum\left[\begin{array}{l}
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_d.Y_d.Y_d.Y_d]\\
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_e.Y_e.Y_e.Y_e]\\
2\,\mathrm{Tr}[Y_R.Y_R.Y_R.Y_R]\\
8\,\mathrm{Abs}[H']^2\,\mathrm{Tr}[Y_R.Y_R.Y_\nu.Y_\nu]\\
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_u.Y_u.Y_u.Y_u]\\
4\,\mathrm{Abs}[H']^4\,\mathrm{Tr}[Y_\nu.Y_\nu.Y_\nu.Y_\nu]
\end{array}\right]
$$

⌘ The u,d terms need factors of 3 to account for the 3-color space.

Lemma 6.7

```
PR["Lemma 6.7: ",
 $167 = $ = {Tr[tuDDown[𝒟][Φ, μ] tuDUp[𝒟][Φ, μ]] → 4 a Abs[tuDDown[𝒟̃][H', μ]]^2,
    H' → {ϕ1 + 1, ϕ2}, tuDDown[𝒟̃][H', μ] →
     tuDDown["∂"][H', μ] + I T[Q, "du", {μ, a}] T[σ, "u", {a}] H' - I T[Λ, "d", {μ}] H'};
 accumStdMdl[$];
 $ // Column
]
```

Lemma 6.7:
$\mathrm{Tr}[\mathcal{D}_\mu[\Phi]\,\mathcal{D}^\mu[\Phi]] \to 4\,a\,\mathrm{Abs}[\tilde{\mathcal{D}}_\mu[H']]^2$
$H' \to \{1+\phi_1, \phi_2\}$
$\tilde{\mathcal{D}}_\mu[H'] \to -i\,\Lambda_\mu\,H' + i\,Q_\mu{}^a\,\sigma^a\,H' + \partial_\mu[H']$

Proposition 6.8 : The spectral action of AC − manifold $M \times F_{SM}$

```
PR["Proposition 6.8: The spectral action of AC-manifold M×F_SM is ",
  $ = {Tr[f[D_A / Λ]] → xIntegral[√Abs[g]
       ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}], H'], x ∈ M],
    ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}], H'] →
     96 ℒ_M[T[g, "dd", {μ, ν}]] + ℒ_A[T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}]] +
      ℒ_H[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'],
    ℒ_A[T[Λ, "d", {μ}], T[Q, "d", {μ}], T[V, "d", {μ}]] → f[0]/π^2
      (10/3 T[Λ, "dd", {μ, ν}] T[Λ, "uu", {μ, ν}] + Tr[T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]] +
        Tr[T[V, "dd", {μ, ν}] T[V, "uu", {μ, ν}]]),
    ℒ_A[CG["kinetic terms of the gauge fields"]],
    ℒ_H[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] →
     b f[0] Λ^2/(2 π^2) Abs[H']^4 + (-2 a f2 Λ^2 + e f[0])/π^2 Abs[H']^2 - c f2 Λ^2 / π^2 +
      df[0]/(4 π^2) + a f[0]/(12 π^2) s Abs[H']^2 + c f[0]/(24 π^2) s + a f[0]/(2 π^2) Abs[tuDDown[𝒟̃][H', μ]]^2,
    ℒ_H[CG["Higgs potential"]]
   }; $ // ColumnBar, accumStdMdl[{$}]
 ];
```

Proposition 6.8: The spectral action of AC-manifold $M \times F_{SM}$ is

$\mathrm{Tr}[f[\frac{D_A}{\Lambda}]] \to \int_{x\in M} \sqrt{\mathrm{Abs}[g]}\;\mathcal{L}[g_{\mu\nu}, \Lambda_\mu, Q_\mu, V_\mu, H']$

$\mathcal{L}[g_{\mu\nu}, \Lambda_\mu, Q_\mu, V_\mu, H'] \to \mathcal{L}_A[\Lambda_\mu, Q_\mu, V_\mu] + \mathcal{L}_H[g_{\mu\nu}, \Lambda_\mu, Q_\mu, H'] + 96\,\mathcal{L}_M[g_{\mu\nu}]$

$\mathcal{L}_A[\Lambda_\mu, Q_\mu, V_\mu] \to \frac{f[0]\,(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}] + \mathrm{Tr}[V_{\mu\nu}V^{\mu\nu}])}{\pi^2}$

$\mathcal{L}_A$[kinetic terms of the gauge fields]

$\mathcal{L}_H[g_{\mu\nu}, \Lambda_\mu, Q_\mu, H'] \to$

$\frac{df[0]}{4\pi^2} + \frac{c\,s\,f[0]}{24\pi^2} + \frac{a\,s\,\mathrm{Abs}[H']^2\,f[0]}{12\pi^2} + \frac{b\,\Lambda^2\,\mathrm{Abs}[H']^4\,f[0]}{2\pi^2} + \frac{a\,\mathrm{Abs}[\tilde{\mathcal{D}}_\mu[H']]^2\,f[0]}{2\pi^2} - \frac{c\,\Lambda^2\,f_2}{\pi^2} + \frac{\mathrm{Abs}[H']^2\,(e\,f[0] - 2\,a\,\Lambda^2\,f_2)}{\pi^2}$

$\mathcal{L}_H$[Higgs potential]

6.3.1 Coupling constants and unification.

```
PR["6.3.1 Coupling constants and unification. SU[3] gauge field: ",
 $ = {T[V, "d", {μ}] -> T[V, "du", {μ, i}] T[λ, "d", {i}],
   T[λ, "d", {i}][CG["Gell-Mann matrices"]],
   T[V, "du", {μ, i}][CG[ℝ]]
  }; $ // ColumnBar, accumStdMdl[{$}]
  NL, "Coupling constants rescaling: ",
 $e631 = $ = {T[Λ, "d", {μ}] → 1/2 g₁ T[B, "d", {μ}],
     T[Q, "du", {μ, a}] → 1/2 g₂ T[W, "du", {μ, a}],
     T[V, "du", {μ, i}] → 1/2 g₃ T[G, "du", {μ, i}],
     $[[1]]
   }; $ // ColumnBar,
 NL, "With the relations: ", $ = {Tr[T[σ, "u", {a}] T[σ, "u", {b}]] → 2 T[δ, "uu", {a, b}],
   Tr[T[λ, "u", {a}] T[λ, "u", {b}]] → 2 T[δ, "uu", {i, j}]};
 $ // ColumnBar,
 Yield, $ = ℒA[T[B, "d", {μ}], T[W, "d", {μ}], T[G, "d", {μ}]] →
   f[0]/(2 π^2) (5/3 g₁^2 T[B, "dd", {μ, ν}] T[B, "uu", {μ, ν}] + g₂^2 T[W, "dd", {μ, ν}]
       T[W, "uu", {μ, ν}] + g₃^2 T[G, "dd", {μ, ν}] T[G, "uu", {μ, ν}]), accumStdMdl[{$}],
 NL, "Natural normalization: ", $e66 = $ = {f[0]/(2 π^2) g₃^2 → 1/4,
    f[0]/(2 π^2) g₂^2 → 1/4, 5 f[0]/(6 π^2) g₁^2 → 1/4
   }; $ // Column // Framed,
 Yield, $ = tuEliminate[$, {f[0]}] // Simplify; $ // Framed,
 back, "Relationship between coupling constants at unification.",
 accumStdMdl[{$, $e631, $e66}]
]
```

6.3.1 Coupling constants and unification. SU[3] gauge field:

$V_\mu \to V_\mu{}^i \lambda_i$
λ_i[Gell-Mann matrices]
$V_\mu{}^i$[$\mathbb{R}$]

Coupling constants rescaling: $\Lambda_\mu \to \frac{1}{2} g_1 B_\mu$, $Q_\mu{}^a \to \frac{1}{2} g_2 W_\mu{}^a$, $V_\mu{}^i \to \frac{1}{2} g_3 G_\mu{}^i$, $V_\mu \to V_\mu{}^i \lambda_i$

With the relations: $\mathrm{Tr}[\sigma^a \sigma^b] \to 2\,\delta^{a b}$, $\mathrm{Tr}[\lambda^a \lambda^b] \to 2\,\delta^{i j}$

$$\to \mathcal{L}_A[B_\mu, W_\mu, G_\mu] \to \frac{f[0]\,(\frac{5}{3} g_1^2 B_{\mu\nu} B^{\mu\nu} + g_3^2 G_{\mu\nu} G^{\mu\nu} + g_2^2 W_{\mu\nu} W^{\mu\nu})}{2\pi^2}$$

Natural normalization:
$$\boxed{\begin{array}{l} \frac{f[0]\, g_3^2}{2\pi^2} \to \frac{1}{4} \\ \frac{f[0]\, g_2^2}{2\pi^2} \to \frac{1}{4} \\ \frac{5 f[0]\, g_1^2}{6\pi^2} \to \frac{1}{4} \end{array}}$$

$\to \boxed{5 g_1^2 == 3 g_3^2 \,\&\&\, 5 g_1^2 == 3 g_2^2}$ ⟵Relationship between coupling constants at unification.

Theorem 6.9

```
PR["Theorem 6.9. Spectral action on ACM M×F_SM: ",
  Yield,
  $t69 =
   S_B → xIntegral[(48 f_4 Λ^4/π^2 - c f_2 Λ^2 / π^2 + df[0] / (4 π^2) + (c f[0] / (24 π^2) - 4 f_2 Λ^2 / π^2) s -
        3 f[0]/(10 π^2) T[C, "dddd", {μ, ν, ρ, σ}] T[C, "uuuu", {μ, ν, ρ, σ}] +
        T[B, "dd", {μ, ν}] T[B, "uu", {μ, ν}] / 4 + T[W, "udd", {a, μ, ν}]
         T[W, "uuu", {a, μ, ν}] / 4 + T[G, "udd", {i, μ, ν}] T[G, "uuu", {i, μ, ν}] / 4 +
        b π^2/(2 a^2 f[0]) Abs[H]^4 - (2 a f_2 Λ^2 - e f[0]) / (a f[0]) Abs[H]^2 + s Abs[H]^2 / 12 +
        Abs[tuDDown[D̃][H, μ]] ^2 / 2) √Abs[g] , x ∈ M], accumStdMdl[$t69]
];
```

Theorem 6.9. Spectral action on ACM M×F$_{SM}$:

$$\to S_B \to \int_{x\in M} \sqrt{\mathrm{Abs}[g]} \left(\frac{1}{12} s\,\mathrm{Abs}[H]^2 + \frac{1}{2}\mathrm{Abs}[\tilde{\mathcal{D}}_\mu[H]]^2 + \frac{df[0]}{4\pi^2} + \frac{b\,\pi^2\,\mathrm{Abs}[H]^4}{2 a^2 f[0]} - \frac{c\,\Lambda^2 f_2}{\pi^2} - \frac{\mathrm{Abs}[H]^2\,(-e\, f[0] + 2 a \Lambda^2 f_2)}{a\, f[0]} + s\left(\frac{c\, f[0]}{24\pi^2} - \frac{4\Lambda^2 f_2}{\pi^2}\right) + \frac{48\Lambda^4 f_4}{\pi^2} + \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{3 f[0]\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}}{10\pi^2} + \frac{1}{4} G^i{}_{\mu\nu} G^{i\mu\nu} + \frac{1}{4} W^a{}_{\mu\nu} W^{a\mu\nu}\right)$$

6.4 Fermionic action

```
PR["For fermions need Anticommuting Dirac spinors: ", $ = {ν, e, u, d};
  $ = {T[#, "u", {λ}] & /@ Flatten[{#, OverBar[#]} & /@ $] /.
     tt : Tensor[a_, _, _] :> tuIndexAdd[2, c][tt] /; tuFreeQ[tt, {ν, e}],
    λ[CG["generation"]] → {1, 2, 3}, c[CG["color"]] → {r, g, b}};
  $ // ColumnBar, accumStdMdl[$],
  NL, "Grassman basis vector: ",
  $basisG = $ = {ξ̃[CG["Grassman vector"]] ∈ T[ℋ, "du", {cl, "+"}],
     ξ̃ → ($ = T[ν, "du", {L, λ}] ⊗ T[ν, "du", {L, λ}] +
          T[ν, "du", {R, λ}] ⊗ T[ν, "du", {R, λ}] + T[ν̄, "du", {R, λ}] ⊗ T[ν̄, "du", {L, λ}] +
          T[ν̄, "du", {L, λ}] ⊗ T[ν̄, "du", {R, λ}]) +
       ($ /. ν → e)
       + ($ /. ν → d /. tt : Tensor[_, _, _] :> tuIndexAdd[2, c][tt])
       + ($ /. ν → u /. tt : Tensor[_, _, _] :> tuIndexAdd[2, c][tt]),
     T[ℋ, "du", {cl, "+"}] ∈ {ℋM × ℋF, γ.ξ̃ -> ξ̃.γ}
    }; $ // ColumnSumExp, accumStdMdl[$],
  CR["The text notation is confusing: The OverBar on the
     F-space basis refers to its anti-particle, not its Conjugate."]
 ];
```

For fermions need Anticommuting Dirac spinors: $\{\nu^{\lambda}, \bar{\nu}^{\lambda}, e^{\lambda}, \bar{e}^{\lambda}, u^{\lambda c}, \bar{u}^{\lambda c}, d^{\lambda c}, \bar{d}^{\lambda c}\}$

λ[generation] $\to \{1, 2, 3\}$

c[color] $\to \{r, g, b\}$

Grassman basis vector:

$$\{\tilde{\xi}[\text{Grassman vector}] \in \mathcal{H}_{cl}{}^{+},\ \tilde{\xi} \to \sum\left[\begin{array}{l} d_L{}^{c\lambda} \otimes d_L{}^{c\lambda} \\ d_R{}^{c\lambda} \otimes d_R{}^{c\lambda} \\ e_L{}^{\lambda} \otimes e_L{}^{\lambda} \\ e_R{}^{\lambda} \otimes e_R{}^{\lambda} \\ u_L{}^{c\lambda} \otimes u_L{}^{c\lambda} \\ u_R{}^{c\lambda} \otimes u_R{}^{c\lambda} \\ \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} \\ \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} \\ \bar{d}_L{}^{c\lambda} \otimes \bar{d}_R{}^{c\lambda} \\ \bar{d}_R{}^{c\lambda} \otimes \bar{d}_L{}^{c\lambda} \\ \bar{e}_L{}^{\lambda} \otimes \bar{e}_R{}^{\lambda} \\ \bar{e}_R{}^{\lambda} \otimes \bar{e}_L{}^{\lambda} \\ \bar{u}_L{}^{c\lambda} \otimes \bar{u}_R{}^{c\lambda} \\ \bar{u}_R{}^{c\lambda} \otimes \bar{u}_L{}^{c\lambda} \\ \bar{\nu}_L{}^{\lambda} \otimes \bar{\nu}_R{}^{\lambda} \\ \bar{\nu}_R{}^{\lambda} \otimes \bar{\nu}_L{}^{\lambda} \end{array}\right],\ \mathcal{H}_{cl}{}^{+} \in \{\mathcal{H}_M \times \mathcal{H}_F,\ \gamma.\tilde{\xi} \to \tilde{\xi}.\gamma\}\}$$

The text notation is confusing: The OverBar on the
F-space basis refers to its anti-particle, not its Conjugate.

Gauge fields Transformed

```
PR["For physical gauge fields(5.21): ", $e521 // ColumnBar,
 NL, "Define(6.7-10): ",
 $e67 = $ = {T[Q, "du", {μ, 1}] + I T[Q, "du", {μ, 2}] → g₂ / √2 T[W, "d", {μ}],
    T[Q, "du", {μ, 1}] - I T[Q, "du", {μ, 2}] → g₂ / √2 ct[T[W, "d", {μ}]],
    T[Q, "du", {μ, 3}] - T[Λ, "d", {μ}] → g₂ / (2 c_w) T[Z, "d", {μ}],
    T[Λ, "d", {μ}] → s_w g₂ T[A, "d", {μ}] / 2 - s_w^2 g₂ T[Z, "d", {μ}] / (2 c_w),
    -T[Q, "du", {μ, 3}] - T[Λ, "d", {μ}] →
     -s_w g₂ T[A, "d", {μ}] + g₂ / (2 c_w) (1 - 2 c_w^2) T[Z, "d", {μ}],
    T[Q, "du", {μ, 3}] + T[Λ, "d", {μ}] / 3 → (2 / 3) s_w g₂ T[A, "d", {μ}] -
      g₂ / (6 c_w) (1 - 4 c_w^2) T[Z, "d", {μ}],
    -T[Q, "du", {μ, 3}] + T[Λ, "d", {μ}] / 3 → -(1 / 3) s_w g₂ T[A, "d", {μ}] -
      g₂ / (6 c_w) (1 + 2 c_w^2) T[Z, "d", {μ}],
    H → √(a f[0]) / π {ϕ₁ + 1, ϕ₂},
    H → {v + h + I T[ϕ, "u", {0}], I √2 ϕ⁻},
    T[ϕ, "u", {0}] ∈ ℝ,
    ϕ⁻ ∈ ℂ,
    Y_x[CG["anti-hermitian mass matrix of x"]],
    Y_x → -I √(a f[0]) / (π v) m_x,
    m_x[CG["Hermitian matrix"]],
    Y_R → -I m_R,
    m_R[CG["Majorana mass matrix hermitian symmetric"]]
   }; $ // ColumnBar,
 NL, "Derived relationships: ",
 $ = tuRuleSelect[$e67][H];
 $ = tuRuleSubtract[$] // Thread; $ // Column;
 $e67a = $ = tuRuleSolve[$, {ϕ₁ + 1, ϕ₂}];
 $ // ColumnBar, accumStdMdl[{$, $e521, $e67, $e67a}]

]
```

For physical gauge fields(5.21):

$$W_\mu \to \frac{W_\mu{}^1 + i\, W_\mu{}^2}{\sqrt{2}}$$
$$(W_\mu)^* \to \frac{W_\mu{}^1 - i\, W_\mu{}^2}{\sqrt{2}}$$
$$Z_\mu \to -s_w\, B_\mu + c_w\, W_\mu{}^3$$
$$A_\mu \to c_w\, B_\mu + s_w\, W_\mu{}^3$$

Define(6.7-10):

$$Q_\mu{}^1 + i\, Q_\mu{}^2 \to \frac{g_2\, W_\mu}{\sqrt{2}}$$
$$Q_\mu{}^1 - i\, Q_\mu{}^2 \to \frac{(W_\mu)^\dagger\, g_2}{\sqrt{2}}$$
$$Q_\mu{}^3 - \Lambda_\mu \to \frac{g_2\, Z_\mu}{2\, c_w}$$
$$\Lambda_\mu \to \frac{1}{2} g_2\, s_w\, A_\mu - \frac{g_2\, s_w^2\, Z_\mu}{2\, c_w}$$
$$-Q_\mu{}^3 - \Lambda_\mu \to -g_2\, s_w\, A_\mu + \frac{(1 - 2\, c_w^2)\, g_2\, Z_\mu}{2\, c_w}$$
$$Q_\mu{}^3 + \frac{\Lambda_\mu}{3} \to \frac{2}{3} g_2\, s_w\, A_\mu - \frac{(1 - 4\, c_w^2)\, g_2\, Z_\mu}{6\, c_w}$$
$$-Q_\mu{}^3 + \frac{\Lambda_\mu}{3} \to -\frac{1}{3} g_2\, s_w\, A_\mu - \frac{(1 + 2\, c_w^2)\, g_2\, Z_\mu}{6\, c_w}$$
$$H \to \left\{\frac{\sqrt{a\, f[0]}\, (1 + \phi_1)}{\pi},\ \frac{\sqrt{a\, f[0]}\, \phi_2}{\pi}\right\}$$
$$H \to \{h + v + i\, \phi^0,\ i\, \sqrt{2}\, \phi^-\}$$
$$\phi^0 \in \mathbb{R}$$
$$\phi^- \in \mathbb{C}$$

Y_x[anti-hermitian mass matrix of x]

$$Y_x \to -\frac{i\, \sqrt{a\, f[0]}\, m_x}{\pi\, v}$$

m_x[Hermitian matrix]

$$Y_R \to -i\, m_R$$

m_R[Majorana mass matrix hermitian symmetric]

Derived relationships:

$$1 + \phi_1 \to \frac{\pi\, (h + v)}{\sqrt{a\, f[0]}} + \frac{i\, \pi\, \phi^0}{\sqrt{a\, f[0]}}$$
$$\phi_2 \to \frac{i\, \sqrt{2}\, \pi\, \phi^-}{\sqrt{a\, f[0]}}$$

Theorem 6.10

```
PR["Theorem 6.10. Fermionic action: ",
  $t610 = $ = {S_F → IntegralOp[{{x ∈ M}}, √Abs[g] (ℒkin + ℒgf + ℒHf + ℒR)],
     ℒkin → ($ = -I BraKet[J_M.ē, T[γ, "u", {μ}] . tuDDown["∇"^S][e, μ] ])
       + ($ /. e → ν)
       + ($ /. e → u)
       + ($ /. e → d),
     ℒgf → s_W g2 T[A, "d", {μ}]
        (($ = -BraKet[J_M.ē, T[γ, "u", {μ}].e]) - (2 / 3) ($ /. e → u) + (1 / 3) ($ /. e → d))
       + g2 T[Z, "d", {μ}] / (4 c_W) (
         BraKet[J_M.ν̄, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).ν]
          + BraKet[J_M.ē, T[γ, "u", {μ}].(4 s_W^2 - 1 - T[γ, "d", {5}]).e]
          + BraKet[J_M.ū, T[γ, "u", {μ}].(-8 / 3 s_W^2 + 1 + T[γ, "d", {5}]).u]
          + BraKet[J_M.d̄, T[γ, "u", {μ}].(4 / 3 s_W^2 - 1 - T[γ, "d", {5}]).d]
        )
       + g2 T[W, "d", {μ}] / (2 √2 ) (
         BraKet[J_M.ē, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).ν]
          + BraKet[J_M.d̄, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).u])
       + g2 ct[T[W, "d", {μ}]] / (2 √2 ) (
         BraKet[J_M.ν̄, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).e]
          + BraKet[J_M.ū, T[γ, "u", {μ}].(1 + T[γ, "d", {5}]).d]
        )
       + g3 T[G, "du", {μ, i}] / 2 (
         BraKet[J_M.ū, T[γ, "u", {μ}].T[λ, "d", {i}].u]
          + BraKet[J_M.d̄, T[γ, "u", {μ}].T[λ, "d", {i}].d]
        ),
     ℒgf[CG["gauage-fermion coupling"]],
     ℒHf → I (1 + h / v) (($ = BraKet[J_M.ν̄, m_ν.ν]) + ($ /. ν → e) + ($ /. ν → u) + ($ /. ν → d))
       + T[ϕ, "u", {0}] / v
        (($ = BraKet[J_M.ν̄, T[γ, "d", {5}].m_ν.ν]) - ($ /. ν → e) + ($ /. ν → u) - ($ /. ν → d))
       + ϕ^- / (√2 v) (($ = BraKet[J_M.ē, m_e.(1 + T[γ, "d", {5}]).ν]) -
          ($ /. {m_e → m_ν, tt : T[γ, "d", {5}] → -tt}))
       + ϕ^+ / (√2 v) (($ = BraKet[J_M.ν̄, m_ν.(1 + T[γ, "d", {5}]).e]) -
          ($ /. {m_ν → m_e, tt : T[γ, "d", {5}] → -tt}))
       + ϕ^- / (√2 v) (($ = BraKet[J_M.d̄, m_d.(1 + T[γ, "d", {5}]).u]) -
          ($ /. {m_d → m_u, tt : T[γ, "d", {5}] → -tt}))
       + ϕ^+ / (√2 v) (($ = BraKet[J_M.ū, m_u.(1 + T[γ, "d", {5}]).d]) -
          ($ /. {m_u → m_d, tt : T[γ, "d", {5}] → -tt})),
     ℒHf[CG["Yukawa coupling of Higgs-fermion field"]],
     ℒR → ($ = I BraKet[J_M.ν_R, m_R.ν_R]) + ($ /. ν_R -> ν̄_L),
     ℒR[CG["Majorana mass"]]
    }; $ // ColumnSumExp // Column,
  line,
  NL, "Proof: by applying ",
  $aferm = $ = {S_F → BraKet[J . ξ̃, 𝒟_A.ξ̃] / 2,
     𝒟_A → slash[𝒟] ⊗ 1_F + T[γ, "u", {μ}] ⊗ T[B, "d", {μ}] + T[γ, "d", {5}] ⊗ Φ,
     BraKet[ξ, ψ] → xIntegral[√g BraKet[ξ, ψ], x ∈ M]}; $ // ColumnBar,
  accumStdMdl[{$t610, $}]
 ];
```

Theorem 6.10. Fermionic action:

$$S_F \to \int_{\{x\in M\}} \left[\sum\left[\begin{array}{|l}\mathcal{L}_{gf}\\ \mathcal{L}_{Hf}\\ \mathcal{L}_{kin}\\ \mathcal{L}_R\end{array}\right]\sqrt{\mathrm{Abs}[g]}\right]$$

$$\mathcal{L}_{kin} \to \sum\left[\begin{array}{|l}-i\left\langle J_M.\overline{d}\mid\gamma^\mu.\underline{\nabla}^S{}_\mu[d]\right\rangle\\ -i\left\langle J_M.\overline{e}\mid\gamma^\mu.\underline{\nabla}^S{}_\mu[e]\right\rangle\\ -i\left\langle J_M.\overline{u}\mid\gamma^\mu.\underline{\nabla}^S{}_\mu[u]\right\rangle\\ -i\left\langle J_M.\overline{\nu}\mid\gamma^\mu.\underline{\nabla}^S{}_\mu[\nu]\right\rangle\end{array}\right]$$

$$\mathcal{L}_{gf} \to \sum\left[\begin{array}{|l}\frac{\left(\left\langle J_M.\overline{u}\mid\gamma^\mu.(1+\gamma_5).d\right\rangle+\left\langle J_M.\overline{\nu}\mid\gamma^\mu.(1+\gamma_5).e\right\rangle\right)(W_\mu)^\dagger g_2}{2\sqrt{2}}\\ \left(-\frac{1}{3}\left\langle J_M.\overline{d}\mid\gamma^\mu.d\right\rangle-\left\langle J_M.\overline{e}\mid\gamma^\mu.e\right\rangle+\frac{2}{3}\left\langle J_M.\overline{u}\mid\gamma^\mu.u\right\rangle\right)g_2\, s_w\, A_\mu\\ \frac{1}{2}\left(\left\langle J_M.\overline{d}\mid\gamma^\mu.\lambda_i.d\right\rangle+\left\langle J_M.\overline{u}\mid\gamma^\mu.\lambda_i.u\right\rangle\right)g_3\, G_\mu{}^i\\ \frac{\left(\left\langle J_M.\overline{d}\mid\gamma^\mu.(1+\gamma_5).u\right\rangle+\left\langle J_M.\overline{e}\mid\gamma^\mu.(1+\gamma_5).\nu\right\rangle\right)g_2\, W_\mu}{2\sqrt{2}}\\ \frac{\left(\left\langle J_M.\overline{d}\mid\gamma^\mu.\left(-1+\frac{4 s_w^2}{3}-\gamma_5\right).d\right\rangle+\left\langle J_M.\overline{e}\mid\gamma^\mu.(-1+4 s_w^2-\gamma_5).e\right\rangle+\left\langle J_M.\overline{u}\mid\gamma^\mu.\left(1-\frac{8 s_w^2}{3}+\gamma_5\right).u\right\rangle+\left\langle J_M.\overline{\nu}\mid\gamma^\mu.(1+\gamma_5).\nu\right\rangle\right)g_2\, Z_\mu}{4 c_w}\end{array}\right]$$

$\mathcal{L}_{gf}$[gauage-fermion coupling]

$$\mathcal{L}_{Hf} \to \sum\left[\begin{array}{|l}i\left(1+\frac{h}{v}\right)\left(\left\langle J_M.\overline{d}\mid m_d.d\right\rangle+\left\langle J_M.\overline{e}\mid m_e.e\right\rangle+\left\langle J_M.\overline{u}\mid m_u.u\right\rangle+\left\langle J_M.\overline{\nu}\mid m_\nu.\nu\right\rangle\right)\\ \frac{\left(\left\langle J_M.\overline{d}\mid m_d.(1+\gamma_5).u\right\rangle-\left\langle J_M.\overline{d}\mid m_u.(1-\gamma_5).u\right\rangle\right)\phi^-}{\sqrt{2}\, v}\\ \frac{\left(\left\langle J_M.\overline{e}\mid m_e.(1+\gamma_5).\nu\right\rangle-\left\langle J_M.\overline{e}\mid m_\nu.(1-\gamma_5).\nu\right\rangle\right)\phi^-}{\sqrt{2}\, v}\\ \frac{\left(-\left\langle J_M.\overline{u}\mid m_d.(1-\gamma_5).d\right\rangle+\left\langle J_M.\overline{u}\mid m_u.(1+\gamma_5).d\right\rangle\right)\phi^+}{\sqrt{2}\, v}\\ \frac{\left(-\left\langle J_M.\overline{\nu}\mid m_e.(1-\gamma_5).e\right\rangle+\left\langle J_M.\overline{\nu}\mid m_\nu.(1+\gamma_5).e\right\rangle\right)\phi^+}{\sqrt{2}\, v}\\ \frac{\left(-\left\langle J_M.\overline{d}\mid\gamma_5.m_d.d\right\rangle-\left\langle J_M.\overline{e}\mid\gamma_5.m_e.e\right\rangle+\left\langle J_M.\overline{u}\mid\gamma_5.m_u.u\right\rangle+\left\langle J_M.\overline{\nu}\mid\gamma_5.m_\nu.\nu\right\rangle\right)\phi^0}{v}\end{array}\right]$$

$\mathcal{L}_{Hf}$[Yukawa coupling of Higgs-fermion field]

$$\mathcal{L}_R \to \sum\left[\begin{array}{|l}i\left\langle J_M.\nu_R\mid m_R.\nu_R\right\rangle\\ i\left\langle J_M.\overline{\nu_L}\mid m_R.\overline{\nu_L}\right\rangle\end{array}\right]$$

$\mathcal{L}_R$[Majorana mass]

Proof: by applying $\begin{array}{|l}S_F \to \frac{1}{2}\left\langle J.\tilde{\xi}\mid\mathcal{D}_A.\tilde{\xi}\right\rangle\\ \mathcal{D}_A \to (\not{D})\otimes 1_F+\gamma_5\otimes\Phi+\gamma^\mu\otimes B_\mu\\ \left\langle\xi\mid\psi\right\rangle \to \int_{x\in M}\sqrt{g}\left\langle\xi\mid\psi\right\rangle\end{array}$

$\mathcal{L}_{kin}$

```
PR["For the basis: ", selectStdMdl[ξ̃],
  NL, "and with the ", {χ̃, ψ̃}, " symmetry of: ",
  $symJM = BraKet[J_M.χ̃, slash[𝒟].ψ̃] -> BraKet[J_M.ψ̃, slash[𝒟].ξ̃],
  line, ƒ
   NL, "● Examine the Expressions containing: ",
  $sJ = {J.ξ̃ → (J_M ⊗ J_F).ξ̃, 𝒟_A.ξ̃ -> (slash[𝒟] ⊗ 1_F).ξ̃}; $sJ // ColumnBar,
  NL, "Reproduce the kinetic terms(Theorem 6.10): ", $s = ℒ_kin;
  $ = selectStdMdl[$s] // Framed
 ];
```

For the basis:

$\tilde{\xi} \to d_L{}^{c\,\lambda} \otimes d_L{}^{c\,\lambda} + d_R{}^{c\,\lambda} \otimes d_R{}^{c\,\lambda} + e_L{}^{\lambda} \otimes e_L{}^{\lambda} + e_R{}^{\lambda} \otimes e_R{}^{\lambda} + u_L{}^{c\,\lambda} \otimes u_L{}^{c\,\lambda} + u_R{}^{c\,\lambda} \otimes u_R{}^{c\,\lambda} + \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} + \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} + \overline{d}_L{}^{c\,\lambda} \otimes \overline{d}_R{}^{c\,\lambda} + \overline{d}_R{}^{c\,\lambda} \otimes \overline{d}_L{}^{c\,\lambda} + \overline{e}_L{}^{\lambda} \otimes \overline{e}_R{}^{\lambda} + \overline{e}_R{}^{\lambda} \otimes \overline{e}_L{}^{\lambda} + \overline{u}_L{}^{c\,\lambda} \otimes \overline{u}_R{}^{c\,\lambda} + \overline{u}_R{}^{c\,\lambda} \otimes \overline{u}_L{}^{c\,\lambda} + \overline{\nu}_L{}^{\lambda} \otimes \overline{\nu}_R{}^{\lambda} + \overline{\nu}_R{}^{\lambda} \otimes \overline{\nu}_L{}^{\lambda}$

and with the $\{\tilde{\chi}, \tilde{\psi}\}$ symmetry of: $\langle J_M.\tilde{\chi} \mid (\not{\mathcal{D}}).\tilde{\psi}\rangle \to \langle J_M.\tilde{\psi} \mid (\not{\mathcal{D}}).\tilde{\xi}\rangle$

ƒ● Examine the Expressions containing:

$J.\tilde{\xi} \to (J_M \otimes J_F).\tilde{\xi}$

$\mathcal{D}_A.\tilde{\xi} \to ((\not{\mathcal{D}}) \otimes 1_F).\tilde{\xi}$

Reproduce the kinetic terms(Theorem 6.10):

$$\boxed{\mathcal{L}_{kin} \to -i\,\langle J_M.\overline{d} \mid \gamma^\mu.\underline{\nabla}^S{}_\mu[d]\rangle - i\,\langle J_M.\overline{e} \mid \gamma^\mu.\underline{\nabla}^S{}_\mu[e]\rangle - i\,\langle J_M.\overline{u} \mid \gamma^\mu.\underline{\nabla}^S{}_\mu[u]\rangle - i\,\langle J_M.\overline{\nu} \mid \gamma^\mu.\underline{\nabla}^S{}_\mu[\nu]\rangle}$$

```
PR["■Evaluate ", $ = selectStdMdl[SF],
  yield, $00 = $ = $ /. selectStdMdl[𝒟A] /. $sJ[[1]],
  NL, "Use basis ", $s = selectStdMdl[ξ̃],
  Yield, $ = $[[2]] /. $s; $,
  NL, "Distribute and expand operators: ",
  Yield,
  xtmp = $ = $ /. tuOpDistribute[Dot] /. tuOpDistribute[BraKet] /. tuBraKetSimplify[] //
     tuCircleTimesOp[];
  NL, "Expand CircleTimes, apply definitions for JF and orthogonality: ",
  $s = {BraKet[ CircleTimes[a_, b_], CircleTimes[c_, d_]] → CircleTimes[BraKet[a, c],
      BraKet[b, d]], CO["Separate {M,F}-spaces"],
    JF.a_ ⧴ (a /. Tensor[s_, i_, j_] ⧴ If[FreeQ[s, OverBar],
         Tensor[s̄, i, j], Tensor[s[[1]], i, j]] ), CO["Charge conjugation"],
    c_ ⊗ BraKet[a_, a_] ⧴ c, CO["Simplify Identity"],
    c_ ⊗ BraKet[a_, b_] ⧴ 0 /; FreeQ[{a, b}, Dot] && a =!= b,
    CO["F-basis Orthogonality"],
    1F .a_ → a, CO["Remove identity symbol"]
   }; $s // ColumnBar, CK,
  Yield, $0 = $ = $ //. tuRule[$s];

  line,
  NL, "Examine ", $s = c_ BraKet[_, Dot[slash[𝒟], _]] , " terms.",
  Yield, $1 = $ = $0 // Expand // tuExtractPattern[$s ]; $ // Sort // Column, CK,
   NL, "Use symmetry ", $symJM,
  " to order BraKet[]s and combine {R,L} basis, the sum of these terms: ",
  $s = {BraKet[JM.a_, slash[𝒟].b_] :> BraKet[JM.b, slash[𝒟].a] /; FreeQ[a, OverBar],
    tt : Tensor[a_, u_, d_] ⧴ PL.tuIndexDelete[L][tt] /; ! FreeQ[d, L],
    tt : Tensor[a_, u_, d_] :> PR.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
   }; $s // ColumnBar,
  Yield,
  $ = Apply[Plus, $] //. $s;
  Yield, $2 = $ = $ // Expand; Framed[$]
 ];
PR["•Using the relationships: ", $s = {
    JM.P_L_ → PL.JM, CO["(JM ,P's Commute)"],
    slash[𝒟] . P_l_ .a_ ⧴ If[l === L, PR.slash[𝒟] .a, PL.slash[𝒟] .a],
    CO[slash[𝒟], "Changes chirality"],
    JM . Tensor[ā_, b_, c_] ⧴ Tensor[a, b, c], CO["(Charge Conjugation)"],
    BraKet[P_l_ . a_ , P_l_.slash[𝒟] . a_] ⧴ BraKet[a, Pl.slash[𝒟] .a],
    CO["(Chiral orthogonal)"],
    BraKet[a_, PL.slash[𝒟] . a_] + BraKet[a_, PR.slash[𝒟] .a_] ⧴ BraKet[a, slash[𝒟] .a]};
  accumStdMdl[$s], $s // Column, $scc = tuRule[$s];
  Yield, $ = $ //. $scc,
  NL, "Reinsert JM: ", $s = BraKet[Tensor[a_, b_, c_], slash[𝒟] . Tensor[a_, b_, c_]] :>
    BraKet[JM.Tensor[ā, b, c], slash[𝒟] . Tensor[a, b, c]],
  $ = $ /. $s; Framed[$aferm[[1, 2]] -> $]
 ];
$pass = Expand[$0] - Apply[Plus, $1];
```

■Evaluate $S_F \to \frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_A.\tilde{\xi}\rangle \longrightarrow S_F \to \frac{1}{2}\langle (J_M \otimes J_F).\tilde{\xi} \mid ((\not{D}) \otimes 1_F + \gamma_5 \otimes \Phi + \gamma^\mu \otimes B_\mu).\tilde{\xi}\rangle$

Use basis $\tilde{\xi} \to d_L{}^{c\lambda} \otimes d_L{}^{c\lambda} + d_R{}^{c\lambda} \otimes d_R{}^{c\lambda} + e_L{}^{\lambda} \otimes e_L{}^{\lambda} + e_R{}^{\lambda} \otimes e_R{}^{\lambda} + u_L{}^{c\lambda} \otimes u_L{}^{c\lambda} + u_R{}^{c\lambda} \otimes u_R{}^{c\lambda} + \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} + \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} + \bar{d}_L{}^{c\lambda} \otimes \bar{d}_R{}^{c\lambda} + \bar{d}_R{}^{c\lambda} \otimes \bar{d}_L{}^{c\lambda} + \bar{e}_L{}^{\lambda} \otimes \bar{e}_R{}^{\lambda} + \bar{e}_R{}^{\lambda} \otimes \bar{e}_L{}^{\lambda} + \bar{u}_L{}^{c\lambda} \otimes \bar{u}_R{}^{c\lambda} + \bar{u}_R{}^{c\lambda} \otimes \bar{u}_L{}^{c\lambda} + \bar{\nu}_L{}^{\lambda} \otimes \bar{\nu}_R{}^{\lambda} + \bar{\nu}_R{}^{\lambda} \otimes \bar{\nu}_L{}^{\lambda}$

$$\to \frac{1}{2}\Big\langle (J_M \otimes J_F).(d_L{}^{c\lambda} \otimes d_L{}^{c\lambda} + d_R{}^{c\lambda} \otimes d_R{}^{c\lambda} + e_L{}^{\lambda} \otimes e_L{}^{\lambda} + e_R{}^{\lambda} \otimes e_R{}^{\lambda} + u_L{}^{c\lambda} \otimes u_L{}^{c\lambda} + u_R{}^{c\lambda} \otimes u_R{}^{c\lambda} + \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} + \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} + \bar{d}_L{}^{c\lambda} \otimes \bar{d}_R{}^{c\lambda} + \bar{d}_R{}^{c\lambda} \otimes \bar{d}_L{}^{c\lambda} + \bar{e}_L{}^{\lambda} \otimes \bar{e}_R{}^{\lambda} + \bar{e}_R{}^{\lambda} \otimes \bar{e}_L{}^{\lambda} + \bar{u}_L{}^{c\lambda} \otimes \bar{u}_R{}^{c\lambda} + \bar{u}_R{}^{c\lambda} \otimes \bar{u}_L{}^{c\lambda} + \bar{\nu}_L{}^{\lambda} \otimes \bar{\nu}_R{}^{\lambda} + \bar{\nu}_R{}^{\lambda} \otimes \bar{\nu}_L{}^{\lambda}) \mid ((\not{D}) \otimes 1_F + \gamma_5 \otimes \Phi + \gamma^\mu \otimes B_\mu).(d_L{}^{c\lambda} \otimes d_L{}^{c\lambda} + d_R{}^{c\lambda} \otimes d_R{}^{c\lambda} + e_L{}^{\lambda} \otimes e_L{}^{\lambda} + e_R{}^{\lambda} \otimes e_R{}^{\lambda} + u_L{}^{c\lambda} \otimes u_L{}^{c\lambda} + u_R{}^{c\lambda} \otimes u_R{}^{c\lambda} + \nu_L{}^{\lambda} \otimes \nu_L{}^{\lambda} + \nu_R{}^{\lambda} \otimes \nu_R{}^{\lambda} + \bar{d}_L{}^{c\lambda} \otimes \bar{d}_R{}^{c\lambda} + \bar{d}_R{}^{c\lambda} \otimes \bar{d}_L{}^{c\lambda} + \bar{e}_L{}^{\lambda} \otimes \bar{e}_R{}^{\lambda} + \bar{e}_R{}^{\lambda} \otimes \bar{e}_L{}^{\lambda} + \bar{u}_L{}^{c\lambda} \otimes \bar{u}_R{}^{c\lambda} + \bar{u}_R{}^{c\lambda} \otimes \bar{u}_L{}^{c\lambda} + \bar{\nu}_L{}^{\lambda} \otimes \bar{\nu}_R{}^{\lambda} + \bar{\nu}_R{}^{\lambda} \otimes \bar{\nu}_L{}^{\lambda})\Big\rangle$$

Distribute and expand operators:

→

Expand CircleTimes, apply definitions for J_F and orthogonality:

```
⟨a_⊗b_ | c_⊗d_⟩ → ⟨a | c⟩⊗⟨b | d⟩
Separate {M,F}-spaces
JF.(a_) :→ (a /. Tensor[s_, i_, j_] :→ If[FreeQ[s, OverBar], Tensor[s̄, i, j], Tensor[s⟦1⟧, i, j]])
Charge conjugation
c_⊗⟨a_ | a_⟩ :→ c
Simplify Identity
c_⊗⟨a_ | b_⟩ :→ 0 /; FreeQ[{a, b}, Dot] && a =!= b
F-basis Orthogonality
1F.(a_) → a
Remove identity symbol
```

⟵CHECK

→

Examine $\langle \_ \mid (\not{D}).(\_)\rangle\, c\_$ terms.

→ ⟵CHECK

Use symmetry $\langle J_M.\tilde{\chi} \mid (\not{D}).\tilde{\psi}\rangle \to \langle J_M.\tilde{\psi} \mid (\not{D}).\tilde{\xi}\rangle$ to order BraKet[]s and combine {R,L} basis, the sum of these terms:

```
⟨JM.(a_) | (D̸).(b_)⟩ :→ ⟨JM.b | (D̸).a⟩ /; FreeQ[a, OverBar]
tt : Tensor[a_, u_, d_] :→ PL.tuIndexDelete[L][tt] /; ! FreeQ[d, L]
tt : Tensor[a_, u_, d_] :→ PR.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
```

→

→ $\boxed{0}$

•Using the relationships:

```
JM.PL_ → PL.JM
(JM ,P's Commute)
(D̸).P1_.(a_) :→ If[1 === L, PR.(D̸).a, PL.(D̸).a]
D̸
Changes chirality
JM.Tensor[a̅_, b_, c_] :→ Tensor[a, b, c]
(Charge Conjugation)
⟨P1_.(a_) | P1_.(D̸).(a_)⟩ :→ ⟨a | P1.(D̸).a⟩
(Chiral orthogonal)
⟨a_ | PL.(D̸).(a_)⟩ + ⟨a_ | PR.(D̸).(a_)⟩ :→ ⟨a | (D̸).a⟩
```

→ 0

Reinsert J_M: $\langle \text{Tensor}[a\_, b\_, c\_] \mid (\not{D}).\text{Tensor}[a\_, b\_, c\_]\rangle :\to \langle J_M.\text{Tensor}[\bar{a}, b, c] \mid (\not{D}).\text{Tensor}[a, b, c]\rangle$ $\boxed{\frac{1}{2}\langle J.\tilde{\xi} \mid \mathcal{D}_A.\tilde{\xi}\rangle \to 0}$

We check these calculations with the standard Peskin-Schroder chirality operations on Dirac spinors

```
PR["■Examine standard spinor and chirality relationship: ",
 $spin = q → ({#} & /@ {ψL, ψL, ψR, ψR}) /. a_i_ :→ T[a, "d", {i}],
 Yield, $ = T[γ, "u", {5}].# & /@ $spin;
 yield, $[[2]] = $[[2]] /. tuGammaExpand; $,
 NL, "Using: ", $s = {U̅_ -> ConjugateTranspose[U].T[γ, "u", {0}],
   PL → (14 + T[γ, "u", {5}]) / 2, PR → (14 - T[γ, "u", {5}]) / 2};
 $s // ColumnBar,
 NL, "Calculate: ", $ = q̅,
 yield, $ = $ /. $spin;
 yield, $qbar = $ = $ /. $s /. tuGammaExpand; $,
 NL, "Calculate: ", $ = q̅.q,
 yield, $ = $ /. $spin;
 yield, $ = $ /. $s /. tuGammaExpand; $,
 NL, "Calculate: ", $ = (PL.q)‾. PL.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,
 NL, "Calculate: ", $ = (PL.q)‾. PR.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,
 NL, "Calculate: ", $ = (PR.q)‾. PR.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,
 NL, "Calculate: ", $ = (PR.q)‾. PL.q,
 yield, $ = $ /. toxDot /. $spin;
 yield, $ = $ //. $s /. tuGammaExpand /. toDot // Expand; $ // MatrixForms,

 NL, "For spinor components: ",
 $spin = q → ({#} & /@ {aL, bL, cR, dR}),
 NL, " ", $ = q̅,
 yield, $ = $ /. $spin;
 yield, $qbar = $ = $ /. $s /. tuGammaExpand; $,
 NL, "Charge conjugation LR-Rule[]: ",
 {T[q̅_, "d", {L}] :→ Conjugate[T[q, "d", L]],
  T[q̅_, "d", {R}] :→ -Conjugate[T[q, "d", R]]
 },
 NL, CO["Note the sign change for R-terms. This sign
   makes a difference in the outcome of the previous calculation."]
]
```

■Examine standard spinor and chirality relationship: $q \to \{\{\psi_L\}, \{\psi_L\}, \{\psi_R\}, \{\psi_R\}\}$
$\to \;\longrightarrow\; \gamma^5.q \to \{\{\psi_R\}, \{\psi_R\}, \{\psi_L\}, \{\psi_L\}\}$

Using: $\left| \begin{array}{l} \overline{U\_} \to U^\dagger.\gamma^0 \\ P_L \to \frac{1}{2}(1_4 + \gamma^5) \\ P_R \to \frac{1}{2}(1_4 - \gamma^5) \end{array} \right.$

Calculate: $\bar{q} \longrightarrow \;\longrightarrow \{\{(\psi_L)^*, (\psi_L)^*, -(\psi_R)^*, -(\psi_R)^*\}\}$
Calculate: $\bar{q}.q \longrightarrow \;\longrightarrow \{\{2\,(\psi_L)^*\,\psi_L - 2\,(\psi_R)^*\,\psi_R\}\}$
Calculate: $\overline{P_L.q}.P_L.q \longrightarrow \;\longrightarrow (\,0\,)$
Calculate: $\overline{P_L.q}.P_R.q \longrightarrow \;\longrightarrow (\,(\psi_L)^*\,\psi_L + (\psi_R)^*\,\psi_L - (\psi_L)^*\,\psi_R - (\psi_R)^*\,\psi_R\,)$
Calculate: $\overline{P_R.q}.P_R.q \longrightarrow \;\longrightarrow (\,0\,)$
Calculate: $\overline{P_R.q}.P_L.q \longrightarrow \;\longrightarrow (\,(\psi_L)^*\,\psi_L - (\psi_R)^*\,\psi_L + (\psi_L)^*\,\psi_R - (\psi_R)^*\,\psi_R\,)$
For spinor components: $q \to \{\{a_L\}, \{b_L\}, \{c_R\}, \{d_R\}\}$
$\bar{q} \longrightarrow \;\longrightarrow \{\{(a_L)^*, (b_L)^*, -(c_R)^*, -(d_R)^*\}\}$
Charge conjugation LR-Rule[]: $\{\overline{q\_}_L :\to T[q, d, L]^*, \overline{q\_}_R :\to -T[q, d, R]^*\}$
Note the sign change for R-terms. This sign
makes a difference in the outcome of the previous calculation.

Evaluate terms with B_μ == Gauge terms

```
PR["• Examine ", $s = T[B, "d", {μ}], " terms: ",
  $1 = $ = $pass // Expand // tuExtractPattern[c_ ⊗ BraKet[_, $s . _]];
  NL, "From (6.3) ", $s, " does not mix ",
  {ℋ1, ℋq}, " spaces.  Cross, Bμ.νR-terms are zero,
    {eR,uR}-terms are eigenvector of Bμ(no color),
Bμ does not mix chirality{L,R}, or {l,q},{l,l̄},{q,q̄}: ",
  Yield,
  $sdu = {a_ ⊗ BraKet[b_, $s . c_] :→ 0 /; (tuFreeQ[b, {d, u}] && ! tuFreeQ[c, {d, u}]) ||
       (tuFreeQ[c, {d, u}] && ! tuFreeQ[b, {d, u}]) ||
       (! tuFreeQ[c, {T[ν, "du", {R, λ}]}]),
    a_ ⊗ BraKet[b_, $s. c_] :> a ⊗ (-2 T[Λ, "d", {μ}] BraKet[b, c]) /;
      ! FreeQ[c, T[e, "du", {R, λ}]],

    a_ ⊗ BraKet[b_, $s. c_] :> a ⊗ ((4/3 T[Λ, "d", {μ}] + T[V, "d", {μ}]) BraKet[b, c]) /;
      ! tuFreeQ[c, {T[u, "du", {R, λ}]}],
    a_ ⊗ BraKet[b_, $s. c_] :> a ⊗ ((-2/3 T[Λ, "d", {μ}] + T[V, "d", {μ}]) BraKet[b, c]) /;
      ! tuFreeQ[c, {T[d, "du", {R, λ}]}],
    a_ ⊗ BraKet[b_, c_] :→ 0 /; ! FreeQ[c, T[e, "du", {R, λ}] |
         Conjugate[T[e, "du", {R, λ}]]] && b ≠ c,
    a_ ⊗ BraKet[b_, $s. c_] :→ 0 /; (! FreeQ[c, R] && ! FreeQ[b, L]) ||
       (! FreeQ[b, R] && ! FreeQ[c, L])
   }; $sdu // ColumnBar,
  NL, CR["Need way of generating these Rule[]s."],
  NL, "Orthogonality Rule[]s ", $s = {BraKet[b_, c_] :→ 0 /; (FreeQ[c, Dot] && b =!= c),
     BraKet[b_, c_] :→ 1 /; (FreeQ[c, Dot] && b === c),
     a_ ⊗ b_ :→ 0 /; (a == 0 || b == 0)
   }; $s // ColumnBar,
  Yield, $pass1 = $ = $ //. $sdu //. $s // DeleteCases[#, 0] &
 ];
```

```
• Examine Bμ terms:
From (6.3) Bμ does not mix {ℋ1, ℋq}
  spaces.  Cross, Bμ.νR-terms are zero, {eR,uR}-terms
   are eigenvector of Bμ(no color),
Bμ does not mix chirality{L,R}, or {l,q},{l,l̄},{q,q̄}:
  | a_⊗⟨b_ | Bμ.(c_)⟩ :→ 0 /; (tuFreeQ[b, {d, u}] && ! tuFreeQ[c, {d, u}]) ||
  |    (tuFreeQ[c, {d, u}] && ! tuFreeQ[b, {d, u}]) || ! tuFreeQ[c, {T[ν, du, {R, λ}]}]
  | a_⊗⟨b_ | Bμ.(c_)⟩ :→ a⊗(-2 T[Λ, d, {μ}] ⟨b | c⟩) /; ! FreeQ[c, T[e, du, {R, λ}]]
→ | a_⊗⟨b_ | Bμ.(c_)⟩ :→ a⊗((4/3 T[Λ, d, {μ}] + T[V, d, {μ}]) ⟨b | c⟩) /; ! tuFreeQ[c, {T[u, du, {R, λ}]}]
  | a_⊗⟨b_ | Bμ.(c_)⟩ :→ a⊗((-2/3 T[Λ, d, {μ}] + T[V, d, {μ}]) ⟨b | c⟩) /; ! tuFreeQ[c, {T[d, du, {R, λ}]}]
  | a_⊗⟨b_ | c_⟩ :→ 0 /; ! FreeQ[c, T[e, du, {R, λ}] | (eR^λ)*] && b ≠ c
  | a_⊗⟨b_ | Bμ.(c_)⟩ :→ 0 /; (! FreeQ[c, R] && ! FreeQ[b, L]) || (! FreeQ[b, R] && ! FreeQ[c, L])
Need way of generating these Rule[]s.
                      | ⟨b_ | c_⟩ :→ 0 /; FreeQ[c, Dot] && b =!= c
Orthogonality Rule[]s | ⟨b_ | c_⟩ :→ 1 /; FreeQ[c, Dot] && b === c
                      | a_⊗b_ :→ 0 /; a == 0 || b == 0
→ {}
```

```
PR["● Generate B.x Rule[]s from matrix definitions:
• B for leptons in Tensor notation: ",
  $sB = selectStdMdl[T[B_{H_{1⊕1̄}}, "d", {μ}]] /. q_{i_,j_} → T[q, "ddd", {μ, i, j}];
  $sB // MatrixForms,
  NL, "Basis: ",
  $basis = $smbasis /. \overline{a_{i_}} → T[ā, "du", {i, λ}] /. a_{i_} → T[a, "du", {i, λ}];
  $v = {#} & /@ ({l, l̄} /. $basis // Flatten); $v // ColumnBar, "POFF",
  NL, $sB[[1]].basis,
  yield, $vt = xDot[$sB[[2]], $v] // OrderedxDotMultiplyAll[] // Flatten;
  $vt // ColumnBar,
  Yield, $ = T[B, "d", {μ}].# & /@ Flatten[$v], "PONdd",
  Yield, $sBl = Thread[$ → $vt]; $sBl // ColumnBar
 ];
PR["• B for colorless quarks in Tensor notation:",
 NL, "Convert to Tensor notation: ",
 $sBq = selectStdMdl[T[B_{H_q⊕H_q̄}, "d", {μ}]] /. q_{i_,j_} → T[q, "ddd", {μ, i, j}];
 $sBq // MatrixForms,
 NL, "Basis: ",
 Yield, $v = {#} & /@ ({q, q̄} /. $basis // Flatten); $v // ColumnBar,
 NL, $sBq[[1]].basis,
 Yield, $vt = xDot[$sBq[[2]], $v] // OrderedxDotMultiplyAll[] // Flatten;
 $vt // ColumnBar,
 Yield, $ = T[B, "d", {μ}].# & /@ Flatten[$v],
 Yield, $sBq = Thread[$ → $vt]; $sBq // ColumnBar
]
```

● Generate B.x Rule[]s from matrix definitions:
• B for leptons in Tensor notation:

$$B_{\mathcal{H}_{1\oplus\bar{1}}\,\mu} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,1\,1}-\Lambda_\mu & q_{\mu\,1\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu\,2\,1} & q_{\mu\,2\,2}-\Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,1\,1})^*+\Lambda_\mu & -(q_{\mu\,1\,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu\,2\,1})^* & -(q_{\mu\,2\,2})^*+\Lambda_\mu \end{pmatrix}$$

Basis:

$$\begin{array}{|l} \{\nu_R{}^\lambda\} \\ \{e_R{}^\lambda\} \\ \{\nu_L{}^\lambda\} \\ \{e_L{}^\lambda\} \\ \{\bar{\nu}_R{}^\lambda\} \\ \{\bar{e}_R{}^\lambda\} \\ \{\bar{\nu}_L{}^\lambda\} \\ \{\bar{e}_L{}^\lambda\} \end{array}$$

....

$$\to \begin{array}{|l} B_\mu.\nu_R{}^\lambda \to 0 \\ B_\mu.e_R{}^\lambda \to -2\,\Lambda_\mu.e_R{}^\lambda \\ B_\mu.\nu_L{}^\lambda \to q_{\mu\,1\,2}.e_L{}^\lambda + (q_{\mu\,1\,1}-\Lambda_\mu).\nu_L{}^\lambda \\ B_\mu.e_L{}^\lambda \to q_{\mu\,2\,1}.\nu_L{}^\lambda + (q_{\mu\,2\,2}-\Lambda_\mu).e_L{}^\lambda \\ B_\mu.\bar{\nu}_R{}^\lambda \to 0 \\ B_\mu.\bar{e}_R{}^\lambda \to 2\,\Lambda_\mu.\bar{e}_R{}^\lambda \\ B_\mu.\bar{\nu}_L{}^\lambda \to -(q_{\mu\,1\,2})^*.\bar{e}_L{}^\lambda + (-(q_{\mu\,1\,1})^*+\Lambda_\mu).\bar{\nu}_L{}^\lambda \\ B_\mu.\bar{e}_L{}^\lambda \to -(q_{\mu\,2\,1})^*.\bar{\nu}_L{}^\lambda + (-(q_{\mu\,2\,2})^*+\Lambda_\mu).\bar{e}_L{}^\lambda \end{array}$$

• B for colorless quarks in Tensor notation:

Convert to Tensor notation: $B_{\mathcal{H}_q \oplus \mathcal{H}_{q\,\mu}} \to ($

$$\begin{matrix} V_\mu + \frac{4}{3} 1_3 \Lambda_\mu & 0 & 0 & 0 \\ 0 & V_\mu - \frac{2}{3} 1_3 \Lambda_\mu & 0 & 0 \\ 0 & 0 & 1_3\, q_{\mu\,1\,1} + V_\mu + \frac{1}{3} 1_3 \Lambda_\mu & 1_3\, q_{\mu\,1\,2} \\ 0 & 0 & 1_3\, q_{\mu\,2\,1} & 1_3\, q_{\mu\,2\,2} + V_\mu + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Basis:

$$\to \begin{array}{l} \{u_R{}^\lambda\} \\ \{d_R{}^\lambda\} \\ \{u_L{}^\lambda\} \\ \{d_L{}^\lambda\} \\ \{\overline{u}_R{}^\lambda\} \\ \{\overline{d}_R{}^\lambda\} \\ \{\overline{u}_L{}^\lambda\} \\ \{\overline{d}_L{}^\lambda\} \end{array}$$

$B_{\mathcal{H}_q \oplus \mathcal{H}_{q\,\mu}}$.basis

$$\to \begin{array}{l} (\frac{4}{3} 1_3.\Lambda_\mu + V_\mu).u_R{}^\lambda \\ (-\frac{2}{3} 1_3.\Lambda_\mu + V_\mu).d_R{}^\lambda \\ (1_3.q_{\mu\,1\,1} + \frac{1}{3} 1_3.\Lambda_\mu + V_\mu).u_L{}^\lambda + 1_3.q_{\mu\,1\,2}.d_L{}^\lambda \\ (1_3.q_{\mu\,2\,2} + \frac{1}{3} 1_3.\Lambda_\mu + V_\mu).d_L{}^\lambda + 1_3.q_{\mu\,2\,1}.u_L{}^\lambda \\ (-(V_\mu)^* - \frac{4}{3} 1_3.\Lambda_\mu).\overline{u}_R{}^\lambda \\ (-(V_\mu)^* + \frac{2}{3} 1_3.\Lambda_\mu).\overline{d}_R{}^\lambda \\ (-(V_\mu)^* - (q_{\mu\,1\,1})^*.1_3 - \frac{1}{3} 1_3.\Lambda_\mu).\overline{u}_L{}^\lambda - (q_{\mu\,1\,2})^*.1_3.\overline{d}_L{}^\lambda \\ (-(V_\mu)^* - (q_{\mu\,2\,2})^*.1_3 - \frac{1}{3} 1_3.\Lambda_\mu).\overline{d}_L{}^\lambda - (q_{\mu\,2\,1})^*.1_3.\overline{u}_L{}^\lambda \end{array}$$

$\to \{B_\mu.u_R{}^\lambda, B_\mu.d_R{}^\lambda, B_\mu.u_L{}^\lambda, B_\mu.d_L{}^\lambda, B_\mu.\overline{u}_R{}^\lambda, B_\mu.\overline{d}_R{}^\lambda, B_\mu.\overline{u}_L{}^\lambda, B_\mu.\overline{d}_L{}^\lambda\}$

$$\to \begin{array}{l} B_\mu.u_R{}^\lambda \to (\frac{4}{3} 1_3.\Lambda_\mu + V_\mu).u_R{}^\lambda \\ B_\mu.d_R{}^\lambda \to (-\frac{2}{3} 1_3.\Lambda_\mu + V_\mu).d_R{}^\lambda \\ B_\mu.u_L{}^\lambda \to (1_3.q_{\mu\,1\,1} + \frac{1}{3} 1_3.\Lambda_\mu + V_\mu).u_L{}^\lambda + 1_3.q_{\mu\,1\,2}.d_L{}^\lambda \\ B_\mu.d_L{}^\lambda \to (1_3.q_{\mu\,2\,2} + \frac{1}{3} 1_3.\Lambda_\mu + V_\mu).d_L{}^\lambda + 1_3.q_{\mu\,2\,1}.u_L{}^\lambda \\ B_\mu.\overline{u}_R{}^\lambda \to (-(V_\mu)^* - \frac{4}{3} 1_3.\Lambda_\mu).\overline{u}_R{}^\lambda \\ B_\mu.\overline{d}_R{}^\lambda \to (-(V_\mu)^* + \frac{2}{3} 1_3.\Lambda_\mu).\overline{d}_R{}^\lambda \\ B_\mu.\overline{u}_L{}^\lambda \to (-(V_\mu)^* - (q_{\mu\,1\,1})^*.1_3 - \frac{1}{3} 1_3.\Lambda_\mu).\overline{u}_L{}^\lambda - (q_{\mu\,1\,2})^*.1_3.\overline{d}_L{}^\lambda \\ B_\mu.\overline{d}_L{}^\lambda \to (-(V_\mu)^* - (q_{\mu\,2\,2})^*.1_3 - \frac{1}{3} 1_3.\Lambda_\mu).\overline{d}_L{}^\lambda - (q_{\mu\,2\,1})^*.1_3.\overline{u}_L{}^\lambda \end{array}$$

```
PR["• Reduce previous expression (ignoring color index)",
  $ = tuIndexDeleteAll[c][$pass1] //. $sBl //. $sBq // tuDistributeOp[BraKet[_, _]];
  NL, "define scalars:",
  $scalar = {Tensor[xq | Λ | xV, _, _], cc[Tensor[xq | Λ | xV, _, _]]},
  Yield,
  $ = $ // tuRepeat[
     {tuBraKetSimplify[$scalar], tuConjugateDistribute}, expandDC[{}, $scalar]];
  NL, "Remove unnecessary 1₃: ", $s = 1₃ . a_ :→
    a /; MatchQ[a, Tensor[_, _, _] | cc[Tensor[_, _, _]]],
  $ = $ //. $s;
  NL, "Finite space basis orthogonality condition ",
  $s = {BraKet[a_ , b_] :→ 0 /; UnsameQ[a, b] && FreeQ[a, Dot] && FreeQ[b, Dot],
    BraKet[a_ , T[V, "d", {μ}]. b_] :→ 0 /; UnsameQ[a, b] && FreeQ[a, Dot] && FreeQ[b, Dot]
   },
  NL, CR[T[V, "d", {μ}], "Does not mix finite basis!"],
  Yield,
  $ = $ /. $s /. a_ ⊗ b_ :→ 0 /; b == 0 /. BraKet[a_, a_] → 1;
  $tmp0 = $ = Apply[Plus, $];
  $tmp = $ = $tmp0 //. a_⊗b_ + a_⊗c_ → a⊗(b + c); $ // ColumnSumExp
 ]
```

• Reduce previous expression (ignoring color index)
define scalars: $\{\text{Tensor}[xq \mid \Lambda \mid xV, \_, \_],\ \text{Tensor}[xq \mid \Lambda \mid xV, \_, \_]^*\}$
→
Remove unnecessary 1_3: $1_3.(a\_) :\to a$ /; MatchQ[a, Tensor[_, _, _] | cc[Tensor[_, _, _]]]
Finite space basis orthogonality condition
$\{\langle a\_ \mid b\_\rangle :\to 0$ /; a =!= b && FreeQ[a, Dot] && FreeQ[b, Dot],
$\langle a\_ \mid V_\mu.(b\_)\rangle :\to 0$ /; a =!= b && FreeQ[a, Dot] && FreeQ[b, Dot]}
V_μDoes not mix finite basis!
→ 0

```
PR["• Revert q's to SU[2] Q's (ℝ) so we can relate this to
   physical gauge parameters via: ", $ = selectGWS[T[Q, "d", {_}], {σ}];
 $sq = Table[T[q, "ddd", {μ, i, j}], {i, 2}, {j, 2}] -> $[[2]] /. xSum → Sum /.
     tuPauliExpand //. rr : Rule[__] :→ Thread[rr] // Flatten;
 $sq // ColumnBar,
 Yield, $tmp = $ = $tmp0(*/.$sq*) // tuConjugateSimplify[{Tensor[Q, _, _]}];
 $ // ColumnSumExp
]
```

• Revert q's to SU[2] Q's (ℝ) so we can

relate this to physical gauge parameters via:
$q_{\mu\,1\,1} \to Q_\mu{}^3$
$q_{\mu\,1\,2} \to Q_\mu{}^1 - i\,Q_\mu{}^2$
$q_{\mu\,2\,1} \to Q_\mu{}^1 + i\,Q_\mu{}^2$
$q_{\mu\,2\,2} \to -Q_\mu{}^3$

→
0

```
(*May not need this block*)
PR["Order JM BraKet terms(apply symmetry): ",
  $s = {BraKet[JM. a_, T[γ, "u", {μ}]. b_] :→
     BraKet[JM. b, T[γ, "u", {μ}]. a] /; OrderedQ[{b, a}]},
  Yield,
  $ = $tmp //. $s // tuCircleTimesSimplify;
  NL, "Combine LR BraKet where possible: ",
  $s = BraKet[a1_, b1_] ⊗ c1_ + BraKet[a2_, b2_] ⊗ c1_ :→ (BraKet[a1, b1] /. L → X) ⊗
      ( c1 + c1) /; (MatchQ[a1 /. L → R, a2] && MatchQ[b1 /. L → R, b2]) ||
      (MatchQ[a1 /. R → L, a2] && MatchQ[b1 /. R → L, b2]),
  NL, "Collect F-space terms: ",
  Yield, $ = $ //. c_ a_ ⊗ b_ → a ⊗ (c b) //. ( a_ ⊗ b_ ) + (a_ ⊗ c_) -> a ⊗ ( b + c);
  $tmp1 =
   $ = $ //. BraKet[a1_, b1_] ⊗ c1_ + BraKet[a2_, b2_] ⊗ c1_ :→ (BraKet[a1, b1] /. L → X) ⊗
         ( c1) /; (MatchQ[a1 /. L → R, a2] && MatchQ[b1 /. L → R, b2]) ||
         (MatchQ[a1 /. R → L, a2] && MatchQ[b1 /. R → L, b2]) // tuIndexDeleteAll[X];
  $ // ColumnSumExp
 ];
```

```
Order JM BraKet terms(apply symmetry):
 {⟨JM.(a_) | γ^μ.(b_)⟩ :→ ⟨JM.b | γ^μ.a⟩ /; OrderedQ[{b, a}]}
→
Combine LR BraKet where possible: ⟨a1_ | b1_⟩⊗c1_ + ⟨a2_ | b2_⟩⊗c1_ :→
  (⟨a1 | b1⟩ /. L → X)⊗(c1 + c1) /; (MatchQ[a1 /. L → R, a2] && MatchQ[b1 /. L → R, b2]) ||
   (MatchQ[a1 /. R → L, a2] && MatchQ[b1 /. R → L, b2])
Collect F-space terms:
→ 0
```

```
$e67;
selectStdMdl[T[V, "d", {μ}], {G}];
selectStdMdl[T[V, "du", {_, _}], {}] // tuAddPatternVariable[i];
PR["Conversion to physical parameters: ",
 $s = {selectStdMdl[T[Λ, "d", {μ}]],
   selectStdMdl[T[Q, "du", {μ, 3}] + _],
   selectStdMdl[Tensor[Q, _, _] + I _], selectStdMdl[Tensor[Q, _, _] - I _],
   selectStdMdl[T[V, "d", {_}], {}]
  };
 $s = {tuRuleSolve[$s, T[Q, "du", {μ, 3}]], $s, cc /@ selectStdMdl[Tensor[Q, _, _] + I _],
    cc /@ selectStdMdl[Tensor[Q, _, _] - I _]} // Flatten, and,
 $sV = selectStdMdl[T[V, "du", {_, _}], {}] // tuAddPatternVariable[i],
 Yield,
 $ = $tmp1 /. $s /. $s // Simplify;
 $ // ColumnSumExp;
 NL, "Define scalars for expanding equation: ",
 $scalar = {Tensor[q | Λ | V, _, _], cc[Tensor[q | Λ | V, _, _]]},
 Yield, $ = $ // tuRepeat[
    {tuBraKetSimplify[$scalar], tuConjugateDistribute}, expandDC[{}, $scalar]];
 $ = $ /. $sV;
 $ = $ // tuCircleTimesExpand;

 NL, "Order BraKet basis vector with symmetry Rule: ",
 $s =
  BraKet[JM . ( aa : Tensor[a̅_, i1_, i2_]), T[γ, "u", {μ}] . (bb : Tensor[b_, j1_, j2_])] :>
   BraKet[JM . Tensor[b̅, j1, j2], T[γ, "u", {μ}] . Tensor[a, i1, i2]] /;
    OrderedQ[{b, a}],
```

```
Yield, $ = $ /. $s;
NL, "Define BraKet consolidation Rule: ",
$sLR =
 BraKet[a1_, b1_] ⊗ c1_ + BraKet[a2_, b2_] ⊗ c1_ :→ (BraKet[a1, b1] /. L → X) ⊗ ( c1) /;
   (MatchQ[a1 /. L → R, a2] && MatchQ[b1 /. L → R, b2]) ||
    (MatchQ[a1 /. R → L, a2] && MatchQ[b1 /. R → L, b2]),
$ = $ //. $sLR // tuIndexDeleteAll[X];
NL, "Collect common Gauge terms: ",

(*W*)
$s = $ // tuTermSelect[W] // Apply[Plus, #] &;
$s =
 $s → ($s //. tuOpDistribute[CircleTimes] //. tuOpSimplify[CircleTimes, {sw, g2, cc[
          g2]}] /. xct[ww : Tensor[W, _, _]] → ww /. cc[g2] → g2 // Simplify), CK,
$ = $ /. $s;

(*A*)
$s = $ // tuTermSelect[A] // Apply[Plus, #] &;
$s = $s → ($s //. tuOpSimplify[CircleTimes, {sw, g2, xTensor[A, _, _]}] //.
      tuOpCollect[CircleTimes] // Simplify);
$ = $ /. $s;
(*G*)
$s = $ // tuTermSelect[G] // Apply[Plus, #] &;
$s = $s //. tuOpCollect[CircleTimes] //. $sLR //. tuOpDistribute[CircleTimes];
$s = $s -> tuRepeat[{tuOpSimplify[CircleTimes, {sw, cw, g_, Tensor[G, _, _]}],
      tuOpCollect[CircleTimes]}, Simplify][$s];
$ = $ /. $s;
(*Z*)
$s = $ // tuTermSelect[Z] // Apply[Plus, #] &;
$s = $s -> tuRepeat[{tuOpSimplify[CircleTimes, {sw, cw, g_, Tensor[G, _, _]}],
      tuOpCollect[CircleTimes], $sLR, cw^2 → 1 - sw^2}, Simplify][$s];
$ = $ /. $s;
$ // ColumnSumExp // Framed,
NL, CR[
  "There are a number of differences between this and the text: •The A coefficient
    is twice the text value, •The BraKet[]d coefficients of are
    combined using the chirality operator, •The Hermiticity of W's
    are unclear, •The symmetry of BraKet[JM,] may be incorrect.
    Their method of combining M and F spaces is unclear. "]
]
```

Conversion to physical parameters:

$$\left\{Q_\mu{}^3 \to \frac{4\, c_w\, g_2\, s_w\, A_\mu - g_2\, Z_\mu + 4\, c_w^2\, g_2\, Z_\mu - 2\, c_w\, \Lambda_\mu}{6\, c_w},\ \Lambda_\mu \to \frac{1}{2} g_2\, s_w\, A_\mu - \frac{g_2\, s_w^2\, Z_\mu}{2\, c_w},\right.$$

$$Q_\mu{}^3 + \frac{\Lambda_\mu}{3} \to \frac{2}{3} g_2\, s_w\, A_\mu - \frac{(1-4\, c_w^2)\, g_2\, Z_\mu}{6\, c_w},\ Q_\mu{}^1 + i\, Q_\mu{}^2 \to \frac{g_2\, W_\mu}{\sqrt{2}},\ Q_\mu{}^1 - i\, Q_\mu{}^2 \to \frac{(W_\mu)^\dagger\, g_2}{\sqrt{2}},\ V_\mu \to V_\mu{}^i\, \lambda_i,$$

$$\left.(Q_\mu{}^1)^* - i\,(Q_\mu{}^2)^* \to \frac{(g_2\, W_\mu)^*}{\sqrt{2}},\ (Q_\mu{}^1)^* + i\,(Q_\mu{}^2)^* \to \frac{((W_\mu)^\dagger\, g_2)^*}{\sqrt{2}}\right\} \text{ and } V_\mu{}^{i\_} \to \frac{1}{2} g_3\, G_\mu{}^i$$

→

Define scalars for expanding equation:

```
{Tensor[q | Λ | V, _, _], Tensor[q | Λ | V, _, _]*}
```

→

Order BraKet basis vector with symmetry Rule:

```
⟨J_M.(aa : Tensor[a̅_, i1_, i2_]) | γ^μ.(bb : Tensor[b_, j1_, j2_])⟩ :→
  ⟨J_M.Tensor[b̅, j1, j2] | γ^μ.Tensor[a, i1, i2]⟩ /; OrderedQ[{b, a}]
```

→

Define BraKet consolidation Rule: `⟨a1_ | b1_⟩⊗c1_ + ⟨a2_ | b2_⟩⊗c1_ :→`

```
  (⟨a1 | b1⟩ /. L → X)⊗c1 /; (MatchQ[a1 /. L → R, a2] && MatchQ[b1 /. L → R, b2]) ||
    (MatchQ[a1 /. R → L, a2] && MatchQ[b1 /. R → L, b2])
```

Collect common Gauge terms: $0 \to 0$ ⟵CHECK $\boxed{0}$

There are a number of differences between this and the text: •The A coefficient is twice the text value, •The BraKet[]d coefficients of are combined using the chirality operator, •The Hermiticity of W's are unclear, •The symmetry of BraKet[J_M,] may be incorrect. Their method of combining M and F spaces is unclear.

```
PR["●Evaluate Yukawa coupling of Higgs to fermions. The ",
 $sΦ = Φ, " expression: ",
 NL, $ = $00[[2]] /. {slash[𝒟] → 0, T[B, "d", {_}] → 0} /. tuOpSimplify[CircleTimes],

 NL, "•Extract Φ terms: ",
 $1 = $ = $pass // Expand // tuExtractPattern[c_ ⊗ BraKet[_, $sΦ . _] ];
 NL, "•Use matrix values for ", $sΦ,
 NL, "•Use ", $b = selectStdMdl[basisSM], $b = $b[[2]];
 Yield, $s1 = $sΦ . $b // Thread;
 $ = $sΦ . $b /. $Φ;
 $s1 = $s1 → $ // Thread;
 NL, "•Compute using conjugate basis: ",
 $b = Conjugate[$b],
 Yield, $s2 = $sΦ.$b // Thread;
 $ = $sΦ.$b /. $Φ;
 Yield, $s2 = Append[$s1, Thread[$s2 → $]] // Flatten; $s2 // Column;
 NL, "•Remove generation and color indices (ignored).",
 Imply, $ = $1 /. tt : (Tensor[_, _, _]) :> Fold[tuIndexDelete[#2][#1] &, tt, {λ, c}] ;
 NL, "Change LR Tensor indices to be compatible with basis subscripts. ",
 $1 = $ = $ /. Tensor[a̅_, {i_}, {j_}] ⧴ a̅ⱼ /; MatchQ[a, u | d | e | ν] /.
     Tensor[a_, {i_}, {j_}] ⧴ aⱼ /; MatchQ[a, u | d | e | ν] //. $s2;
 Yield, $ = $ //. tuOpDistribute[BraKet] //. tuOpDistribute[CircleTimes] //.
     tuOpSimplify[BraKet, {ϕ_, cc[ϕ_], Y_, cc[Y_], ct[Y_]}] //
    tuConjugateTransposeSimplify[{ϕ_}, {ϕ_, Y_}] //
   tuOpSimplifyF[BraKet, {ϕ_, cc[ϕ_], Y_, cc[Y_], ct[Y_]}];
 NL, "lepton-quark orthogonal,i.e., Y's do not mix leptons or quarks: ",

 $s = {a_ ⊗ ((cc__ : 1) BraKet[b_ , c_ ]) ⧴ 0 /; disjointQ[b, c], a_ ⊗ BraKet[b_ , c_] ⧴
     a ⊗ BraKet[c, b] /; (! FreeQ[c, Conjugate] && FreeQ[b, OverBar])},
 Yield, $ = $ //. tuRule[$s];
 Yield, $ = Apply[Plus, $]; $ // ColumnSumExp;
 NL, "Substitute: ",
 $s = {selectStdMdl /@ {Y_R, Y_x},
     tuRuleSolve[$e67a, {ϕ1, ϕ2}]} // Flatten // tuAddPatternVariable[{x}],
 NL, "Reals, Scalars, Hermitian: ", $real = { h, √_ , a, f[0], v, T[ϕ, "u", {0}] },
 $scalar = {h, √_ , a, f[0], v, T[ϕ, "u", {0}]}, $hermit = {m_}, "POFF",
 Yield, $pass4 = $ //. $s // tuConjugateTransposeSimplify[$real, $scalar, $hermit]
]
PR["Simplify terms of finite space: ",
 $s = {BraKet[a_, a_] → 1, BraKet[cc[a_], a_] → 1, BraKet[a_, cc[a_]] → 1,
   c_ ⊗ BraKet[a_, b_] ⧴ 0 /; disjointQ[a, b, {OverBar}]
  }, "xPOFF",
 Yield, $pass4 = $pass4 //. $s
]
```

●Evaluate Yukawa coupling of Higgs to fermions. The Φ expression:

$\frac{1}{2}\left\langle (J_M\otimes J_F).\tilde{\xi} \mid (\gamma_5\otimes\Phi).\tilde{\xi}\right\rangle$

•Extract Φ terms:
•Use matrix values for Φ
•Use Last[{}]
→
•Compute using conjugate basis: Last[{}][[2]]*
→
→
•Remove generation and color indices (ignored).
⇒
Change LR Tensor indices to be compatible with basis subscripts.
→
lepton-quark orthogonal,i.e., Y's do not mix leptons or quarks:

{a_⊗(⟨b_ | c_⟩ (cc__ : 1)) :→ 0 /; disjointQ[b, c],
a_⊗⟨b_ | c_⟩ :→ a⊗⟨c | b⟩ /; ! FreeQ[c, Conjugate] && FreeQ[b, OverBar]}

→
→
Substitute:

$\{Y_R \to -i\, m_R,\ Y_{x\_} \to -\frac{i\sqrt{a\,f[0]}\ m_x}{\pi\, v},\ \phi_1 \to \frac{h\,\pi+\pi\, v-\sqrt{a\,f[0]}}{\sqrt{a\,f[0]}}+\frac{i\,\pi\,\phi^0}{\sqrt{a\,f[0]}},\ \phi_2 \to \frac{i\sqrt{2}\,\pi\,\phi^-}{\sqrt{a\,f[0]}}\}$

Reals, Scalars, Hermitian: $\{h, \sqrt{\_}, a, f[0], v, \phi^0\}\{h, \sqrt{\_}, a, f[0], v, \phi^0\}\{m\_\}$

Simplify terms of finite space:

{⟨a_ | a_⟩ → 1, ⟨a_* | a_⟩ → 1, ⟨a_ | a_*⟩ → 1, c_⊗⟨a_ | b_⟩ :→ 0 /; disjointQ[a, b, {OverBar}]}xPOFF
→
{}

```
PR["Compare with text ν-ν calculation: ",
 $t = ν;
 $remain = $ = $pass4 //. tuOpDistribute[CircleTimes];
 $ = Apply[Plus, $ // tuTermSelect[$t, {e}]];
 Yield, $ = $ //. tuOpCollect[CircleTimes]; $ // ColumnSumExp,
 NL, "Order M-BraKet via symmetry: ",
 $s = {BraKet[JM.a_, T[γ, "d", {5}].b_] :>
    BraKet[JM.b, T[γ, "d", {5}].a] /; disjointQ[a, b, {OverBar}] && ! FreeQ[b, OverBar]},
 Yield,
 $ = $ /. $s; $ // ColumnSumExp
]
```

Compare with text ν-ν calculation:
→ 0
Order M-BraKet via symmetry:

{⟨J_M.(a_) | γ_5.(b_)⟩ :→ ⟨J_M.b | γ_5.a⟩ /; disjointQ[a, b, {OverBar}] && ! FreeQ[b, OverBar]}
→
0

```
PR["■For the ", $t = T[ϕ, "u", {0}], " terms: ",
  $ = Select[$0, (! FreeQ[#, $t]) &] // Simplify,
  $remain = $remain - $ // Expand;
  NL, "•Order product and add PR|L: ",
  $s = {BraKet[JM.a_, T[γ, "d", {5}].b_] :>
      BraKet[JM.b, T[γ, "d", {5}].a] /; FreeQ[a, OverBar] && ! FreeQ[b, OverBar],
     tt : Tensor[a_, u_, d_] :> PL.tuIndexDelete[L][tt] /; ! FreeQ[d, L],
     tt : Tensor[a_, u_, d_] :> PR.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
    }; $s // Column,
  Yield, $ = $ //. $s
 ];


PR["Apply relationships: ",
  $s = {
    JM.PL_ → PL.JM, CO["(JM Commutes)"],
    BraKet[ a_, T[γ, "d", {5}] . PL. b_] :> BraKet[a, PR. T[γ, "d", {5}] .b],
    BraKet[ a_, T[γ, "d", {5}] . PR . b_] :> BraKet[a, PL. T[γ, "d", {5}] .b],
    CO["PLR conversion non-standard ??"],
    BraKet[ P11_. a__, P12_ . b__] :> 0 /; 11 =!= 12, CO["Chiral orthogonality"],
     Conjugate[mm : m_] :> mm /; ! FreeQ[{mm}, m], CO["m's Real"],
    c___ BraKet[PL. a_, PL . T[γ, "d", {5}] . b_] +
      c___ BraKet[PR . a_, PR . T[γ, "d", {5}] . b_] :>
     c BraKet[a, T[γ, "d", {5}] .b], CO["Combine P's"],
    a_ ⊗ 1 → a, CO["Simplify notation"]
   }; $s // Column,
  NL, $t, yield, $ = $ //. DeleteCases[$s, CO[_]]; Framed[$]
 ];
```

■For the ϕ^0 terms: 1
•Order product and add $P_{R|L}$:

```
⟨JM.(a_) | γ5.(b_)⟩ :> ⟨JM.b | γ5.a⟩ /; FreeQ[a, OverBar] && ! FreeQ[b, OverBar]
tt : Tensor[a_, u_, d_] :> PL.tuIndexDelete[L][tt] /; ! FreeQ[d, L]
tt : Tensor[a_, u_, d_] :> PR.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
→
1
```

Apply relationships:

```
JM.PL_ → PL.JM
(JM Commutes)
⟨a_ | γ5.PL.(b_)⟩ :> ⟨a | PR.γ5.b⟩
⟨a_ | γ5.PR.(b_)⟩ :> ⟨a | PL.γ5.b⟩
PLR conversion non-standard ??
⟨P11_.(a__) | P12_.(b__)⟩ :> 0 /; 11 =!= 12
Chiral orthogonality
(mm : m_)* :> mm /; ! FreeQ[{mm}, m]
m's Real
⟨PL.(a_) | PL.γ5.(b_)⟩ c___ + ⟨PR.(a_) | PR.γ5.(b_)⟩ c___ :> c ⟨a | γ5.b⟩
Combine P's
a_⊗1 → a
Simplify notation
```

ϕ^0 ⟶ $\boxed{1}$

```
PR[
  "■For the ", $t = ϕ⁻, " terms",
  $ = Select[$0, (! FreeQ[#, $t]) &] // Simplify;
  $remain = $remain - $ // Expand;
  NL, "•Order product and add P_{R|L}: ",
  $s = {BraKet[J_M.a_, T[γ, "d", {5}].b_] :> BraKet[J_M.b, T[γ, "d", {5}].a] /;
       FreeQ[a, OverBar] && ! FreeQ[b, OverBar], CO["product symmetry"],
     tt : Tensor[a_, u_, d_] :→ P_L.tuIndexDelete[L][tt] /; ! FreeQ[d, L],
     tt : Tensor[a_, u_, d_] :> P_R.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
   }; $s // Column, "xPOFF",
  Yield, $ = $ //. DeleteCases[$s, CO[_]]
 ];
PR["Apply relationships: ",
  $s = {
    J_M.P_L_ → P_L.J_M, CO["(J_M Commutes)"],
    BraKet[ a_, T[γ, "d", {5}] . P_L. b_] :→ BraKet[a, P_R. T[γ, "d", {5}] .b],
    BraKet[ a_, T[γ, "d", {5}] . P_R. b_] :→ BraKet[a, P_L. T[γ, "d", {5}] .b],
    BraKet[ P_11_ . a__, P_12_ . b__] :→ 0 /; 11 =!= 12, CO["Chiral orthogonality"],
     Conjugate[mm : m_] :> mm /; ! FreeQ[{mm}, m], CO["m's Real"](*,
    c___  BraKet[P_L. a_,P_L . T[γ,"d",{5}] . b_]+
      c___ BraKet[P_R . a_,P_R . T[γ,"d",{5}] . b_]:→
     c BraKet[a,T[γ,"d",{5}] .b],CO["Combine P's"]*)
   }; $s // Column,
  Yield, $ = $ //. DeleteCases[$s, CO[_]];
  NL, "Apply: ", $s = {BraKet[p1__ . a_, p2__.b_] :→ BraKet[
      p1 . (a /. {d -> u, ν → e}), p2.(b /. {u -> d, e → ν})], CO["Product ordering"]},
  NL, $t, yield, $ = $ //. DeleteCases[$s, CO[_]] // Simplify; Framed[$]
 ];
```

```
■For the ϕ⁻ terms
•Order product and add P_{R|L}:
 ⟨J_M.(a_) | γ5.(b_)⟩ :→ ⟨J_M.b | γ5.a⟩ /; FreeQ[a, OverBar] && ! FreeQ[b, OverBar]
 product symmetry                                                          xPOFF
 tt : Tensor[a_, u_, d_] :→ P_L.tuIndexDelete[L][tt] /; ! FreeQ[d, L]
 tt : Tensor[a_, u_, d_] :→ P_R.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
→
 1
```

```
                        J_M.P_L_ → P_L.J_M
                        (J_M Commutes)
                        ⟨a_ | γ5.P_L.(b_)⟩ :→ ⟨a | P_R.γ5.b⟩
Apply relationships:    ⟨a_ | γ5.P_R.(b_)⟩ :→ ⟨a | P_L.γ5.b⟩
                        ⟨P_11_.(a__) | P_12_.(b__)⟩ :→ 0 /; 11 =!= 12
                        Chiral orthogonality
                        (mm : m_)* :→ mm /; ! FreeQ[{mm}, m]
                        m's Real
→
Apply: {⟨(p1__).(a_) | (p2__).(b_)⟩ :→ ⟨p1.a | p2.b⟩, Product ordering}
ϕ⁻ ⟶ [1]
```

```
PR[
  "■For the ", $t = ϕ⁺, " terms",
  $ = Select[$0, (! FreeQ[#, $t]) &] // Simplify;
  $remain = $remain - $ // Expand;
  NL, "•Order product and add P_R|L: ",
  $s = {BraKet[J_M.a_, T[γ, "d", {5}].b_] :>
      BraKet[J_M.b, T[γ, "d", {5}].a] /; FreeQ[a, OverBar] && ! FreeQ[b, OverBar],
     tt : Tensor[a_, u_, d_] :→ P_L.tuIndexDelete[L][tt] /; ! FreeQ[d, L],
     tt : Tensor[a_, u_, d_] :> P_R.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
    }; $s // Column, "POFF",
  Yield, $ = $ //. $s
 ];
PR["Apply relationships: ",
  $s = {
    J_M.P_L_ → P_L.J_M, CO["(J_M Commutes)"],
    BraKet[ a_, T[γ, "d", {5}] . P_L. b_] :→ BraKet[a, P_R. T[γ, "d", {5}] .b],
    BraKet[ a_, T[γ, "d", {5}] . P_R. b_] :→ BraKet[a, P_L. T[γ, "d", {5}] .b],
    BraKet[ P_l1_ . a__, P_l2_ . b__] ⊗ c_ :→ 0 /; l1 =!= l2, CO["Chiral orthogonality"],
     Conjugate[mm : m_] :> mm /; ! FreeQ[{mm}, m], CO["m's Real"](*,
    c___  BraKet[P_L. a_,P_L . T[γ,"d",{5}] . b_]+
      c___ BraKet[P_R . a_,P_R . T[γ,"d",{5}] . b_]:→
     c BraKet[a,T[γ,"d",{5}] .b],CO["Combine P's"]*)
    }; $s // Column,
  Yield, $ = $ //. DeleteCases[$s, CO[_]];
  NL, "Apply: ", $s = {BraKet[p1__ . a_, p2__.b_] :→ BraKet[
       p1 . (a /. {d -> u, ν → e}), p2.(b /. {u -> d, e → ν})], CO["Product ordering"]},
  NL, $t, yield, $ = $ //. DeleteCases[$s, CO[_]] // Simplify; Framed[$]
 ];
```

```
■For the ϕ⁺ terms
•Order product and add P_R|L:
 ⟨J_M.(a_) | γ5.(b_)⟩ :→ ⟨J_M.b | γ5.a⟩ /; FreeQ[a, OverBar] && ! FreeQ[b, OverBar]
 tt : Tensor[a_, u_, d_] :→ P_L.tuIndexDelete[L][tt] /; ! FreeQ[d, L]
 tt : Tensor[a_, u_, d_] :→ P_R.tuIndexDelete[R][tt] /; ! FreeQ[d, R]
```

```
                        J_M.P_L_ → P_L.J_M
                        (J_M Commutes)
                        ⟨a_ | γ5.P_L.(b_)⟩ :→ ⟨a | P_R.γ5.b⟩
                        ⟨a_ | γ5.P_R.(b_)⟩ :→ ⟨a | P_L.γ5.b⟩
Apply relationships:    ⟨P_l1_.(a__) | P_l2_.(b__)⟩⊗c_ :→ 0 /; l1 =!= l2
                        Chiral orthogonality
                        (mm : m_)* :→ mm /; ! FreeQ[{mm}, m]
                        m's Real
→
Apply: {⟨(p1__).(a_) | (p2__).(b_)⟩ :→ ⟨p1.a | p2.b⟩, Product ordering}
ϕ⁺ ⟶ [1]
```

```
PR["■For the ", $t = mR, " terms",
  $ = Select[$0, (! FreeQ[#, $t]) &] // Simplify; $ // ColumnSumExp,
  $remain = $remain - $ // Expand;
  NL, "Test unit basis: ", $s = {
    Conjugate[Tensor[a̅_, b_, c_]] -> Tensor[a, b, c],
    Conjugate[Tensor[a_, b_, c_]] :> Tensor[a̅, b, c] /; FreeQ[a, OverBar],
    c_ ⊗ BraKet[a_, a_] :→ c, CO["unit basis"]
   },
  Yield, $ = $ /. DeleteCases[$s, CO[_]]
 ];
```

```
■For the mR terms1
Test unit basis: {Tensor[a̅_, b_, c_]* → Tensor[a, b, c],
  Tensor[a_, b_, c_]* :→ Tensor[a̅, b, c] /; FreeQ[a, OverBar], c_ ⊗ ⟨a_ | a_⟩ :→ c, unit basis}
→ 1
```

```
$ = $remain // Collect[#, BraKet[_, _], Simplify] &;
$s = {
   Conjugate[mm : m_] :> mm /; ! FreeQ[{mm}, m], CO["m's Real"],
  BraKet[JM.a_, T[γ, "d", {5}].b_] :> BraKet[JM.b, T[γ, "d", {5}].a] /; FreeQ[a, OverBar],
  CO["Product ordering"],
  f[0] → 0(*,
  c___  BraKet[PL. a_,PL . T[γ,"d",{5}] . b_]+
    c___ BraKet[PR . a_,PR . T[γ,"d",{5}] . b_]:→
   c BraKet[a,T[γ,"d",{5}] .b],CO["Combine P's"]*)
 }
$ = $ /. DeleteCases[$s, CO[_]] // Simplify
```

```
{(mm : m_)* :→ mm /; ! FreeQ[{mm}, m], m's Real,
 ⟨JM.(a_) | γ5.(b_)⟩ :→ ⟨JM.b | γ5.a⟩ /; FreeQ[a, OverBar], Product ordering, f[0] → 0}
```

```
{}
```

```
tuSaveAllVariables[]
```