```
<< Local`QFTToolKit2`;
Get[$HomeDirectory <> "/Mathematica/NonCommutative/1204.0328
      ParticlePhysicsFromAlmostCommutativeSpacetime.2.redo.out"];

$defGWS = {};

"Notational definitions"
"Note that in the text the symbols may reference
  different Hilbert spaces. This has caused confusion in some of the
  calculations.  To address this problem we will try to label the
  variables by subscripts to designate the applicable Hilbert space.
  NOTE: Need to do notational change for .1,.2 notebooks."

rghtA[a_] := Superscript[a, o]
cl[a_] := 〈a〉cl;
clB[a_] := {a}cl;
ct[a_] := ConjugateTranspose[a];
cc[a_] := Conjugate[a];
star[a_] := Superscript[a, "*"];
cross[a_] := Superscript[a, "×"];
deg[a_] := ¦a¦;
it[a_] := Style[a, Italic]
iA := it[A]
iD := it[D]
iI := it["I"]
C∞ := C"∞"
Bx_ := T[B, "d", {x}]
("∇"S)n_ := T["∇"S, "d", {n}]
$noArg := tuDDown[a_][b_, c_] → a

accumDef[item_] := Block[{}, $defall = tuAppendUniq[item][$defall];
    ""];
accumGWS[item_] := Block[{}, $defGWS = tuAppendUniq[item][$defGWS];
    ""];

selectEM[heads_, with_: {}, all_: Null] := tuRuleSelect[$defEM][Flatten[{heads}]] //
     Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectGWS[heads_, with_: {}, all_: Null] := tuRuleSelect[$defGWS][Flatten[{heads}]] //
     Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;
selectDef[heads_, with_: {}, all_: Null] := tuRuleSelect[$defall][Flatten[{heads}]] //
     Select[#, tuHasAllQ[#, Flatten[{with}]] &] & // If[all === Null, Last[#], #] &;

Clear[expandDC];
expandDC[sub_: {}] := tuRepeat[tuRule[{sub, tuOpDistribute[Dot],
     tuOpSimplify[Dot], tuOpDistribute[CircleTimes], tuOpSimplify[CircleTimes]}]]
Clear[expandCom]
expandCom[subs_: {}][exp_] := Block[{tmp = exp},
    tmp = tmp //. tuCommutatorExpand // expandDC[];
    tmp = tmp /. toxDot /. tuRule[Flatten[{subs}]];
    tmp = tmp // tuMatrixOrderedMultiply // (# /. toDot &) // expandDC[subs];
    tmp
  ];
(**)
$sgeneral := {
  T[γ, "d", {5}] → Product[T[γ, "u", {μ}], {μ, 4}],
  T[γ, "d", {5}].T[γ, "d", {5}] → 1,
  ConjugateTranspose[T[γ, "d", {5}]] -> T[γ, "d", {5}],
```

```
  CommutatorP[T[γ, "d", {5}], T[γ, "u", {μ}]] → 0,
  T["▽", "d", {_}][1n_] → 0, a_ . 1n_ → a, 1n_ . a_ → a}
$sgeneral // ColumnBar

Clear[$symmetries]
$symmetries := {tt : T[g, "uu", {μ_, ν_}] ⧴ tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "uu", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    tt : T[F, "dd", {μ_, ν_}] ⧴ -tuIndexSwap[{μ, ν}][tt] /; OrderedQ[{ν, μ}],
    CommutatorM[a_, b_] ⧴ -CommutatorM[b, a] /; OrderedQ[{b, a}],
    CommutatorP[a_, b_] ⧴ CommutatorP[b, a] /; OrderedQ[{b, a}],
    tt : T[γ, "u", {μ}] . T[γ, "d", {5}] :> Reverse[tt]
  };
$symmetries // ColumnBar

εRule[KOdim_Integer] := Block[{n = Mod[KOdim, 8],
    table =
     {{1, 1, -1, -1, -1, -1, 1, 1}, {1, -1, 1, 1, 1, -1, 1, 1}, {1, , -1, , 1, , -1, }}},
  {ε → table[[1, n + 1]], ε' → table[[2, n + 1]], ε'' → table[[3, n + 1]]}
 ]
εRule[6]
```

Notational definitions

Note that in the text the symbols may reference different Hilbert spaces. This has caused confusion in some of the calculations. To address this problem we will try to label the variables by subscripts to designate the applicable Hilbert space. NOTE: Need to do notational change for .1,.2 notebooks.

$\gamma_5 \to \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$
$\gamma_5 . \gamma_5 \to 1$
$(\gamma_5)^\dagger \to \gamma_5$
$\{\gamma_5, \gamma^\mu\}_+ \to 0$
$\nabla\_[1_{n\_}] \to 0$
$(a\_).1_{n\_} \to a$
$1_{n\_}.(a\_) \to a$

$tt : g^{\mu\_\,\nu\_} \mapsto \text{tuIndexSwap}[\{\mu, \nu\}][tt]$ /; OrderedQ[{ν, μ}]
$tt : F^{\mu\_\,\nu\_} \mapsto -\text{tuIndexSwap}[\{\mu, \nu\}][tt]$ /; OrderedQ[{ν, μ}]
$tt : F_{\mu\_\,\nu\_} \mapsto -\text{tuIndexSwap}[\{\mu, \nu\}][tt]$ /; OrderedQ[{ν, μ}]
$[a\_, b\_]_- \mapsto -[b, a]_-$ /; OrderedQ[{b, a}]
$\{a\_, b\_\}_+ \mapsto \{b, a\}_+$ /; OrderedQ[{b, a}]
$tt : \gamma^\mu . \gamma_5 \mapsto \text{Reverse}[tt]$

$\{\varepsilon \to 1, \varepsilon' \to 1, \varepsilon'' \to -1\}$

# 1204.0328: Particle Physics From Almost Commutative Spacetime

# 5. Glashow-Weinberg-Salam Model

## ■ 5.1 Constructing the finite space $F_{\mathrm{GWS}}$(p.52)

```
PR[
 "The Basis of finite space includes {e,ν}: ",
 $b = {($ = {eR, eL, ēR, ēL}), ($ /. e → ν)} // Flatten,
 NL, "Lepton basis ", $lep = ℋ1[CG[ℂ^4]] ->
   (Select[$b, Head[#] =!= OverBar &] // Sort[#] & // Permute[#, Cycles[{{1, 4}}]] &),
 NL, "AntiLepton basis ", $antilep = ℋ1̄[CG[ℂ^4]] ->
   (Select[$b, Head[#] == OverBar &] // Sort[#] & // Permute[#, Cycles[{{1, 4}}]] &),
 NL, "Compose ", $h2 = ℋF2 -> ℋ1[CG[ℂ^4]] ⊕ ℋ1̄[CG[ℂ^4]],
 NL, "with ",
 $basis = ℋF8 → $h2[[2]] /. {$lep, $antilep} /. CirclePlus[a__] :> Flatten[List[a]]
]
```

```
The Basis of finite space includes {e,ν}: {eR, eL, ēR, ēL, νR, νL, ν̄R, ν̄L}
Lepton basis ℋ1[ℂ^4] → {νR, eR, νL, eL}
AntiLepton basis ℋ1̄[ℂ^4] → {ν̄R, ēR, ν̄L, ēL}
Compose ℋF2 → ℋ1[ℂ^4] ⊕ ℋ1̄[ℂ^4]
with ℋF8 → {νR, eR, νL, eL, ν̄R, ēR, ν̄L, ēL}
```

```
PR["● The Algebra ", iAF, " constructed by Expanding E-M
   algebra, ℂ[a1]⊕ℂ[a2], to accomodate weak interactions → ℂ⊕ℍ",
 NL, $alg = $ = {iAF → ℂ ⊕ ℍ[CG["quarterions"]],
    {q[CG["∈ℍ"]] → α + β j, q → {{α, β}, {-cc[β], cc[α]}}, {α, β} ∈ ℂ},
    qλ[CG["embedding of ℂ in ℍ"]] → {{λ, 0}, {0, cc[λ]}}};
 $ // MatrixForms // ColumnBar,
 NL, "1-Algebra definition: ",
 $alg1 = $ = {a1 ∈ iAF1, a1 → {λ, q},
    a1 → ($ = {{qλ, 0}, {0, q}}),
    a1 → ($ /. tuRule[$alg][[-2 ;; -1]] // ArrayFlatten)
   }; $ // MatrixForms // ColumnBar,
 NL, CR["Not clear how one chooses the algebra and
    the connection between weak interactions and quaterions."],

 NL, "● For ", $h = ℋF4 → ℋ1R ⊕ ℋ1L ⊕ ℋ1̄R ⊕ ℋ1̄L,
 Yield, $alg2 = {($ = a1) → ($ /. tuRuleSelect[$alg1][a1][[-1]]),
   1 ∈ ℋ1, CG["By definition"], a1̄ → DiagonalMatrix[Table[λ, 4]], 1̄ ∈ ℋ1̄,
   a8 → ({{(a1 /. tuRuleSelect[$alg1][a1][[-1]]), 0},
       {0, DiagonalMatrix[Table[λ, 4]]}} // ArrayFlatten)
  }; MatrixForms[$alg2] // ColumnBar
]
```

- The Algebra $\mathcal{A}_F$ constructed by Expanding E-M algebra, $\mathbb{C}[a_1]\oplus\mathbb{C}[a_2]$, to accomodate weak interactions → $\mathbb{C}\oplus\mathbb{H}$

$\mathcal{A}_F \to \mathbb{C}\oplus\mathbb{H}$[quarterions]

$\{q[\in\mathbb{H}] \to \alpha + j\,\beta,\ q \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}\},\ \{\alpha, \beta\} \in \mathbb{C}\}$

$q_\lambda$[embedding of $\mathbb{C}$ in $\mathbb{H}$] $\to \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$

l-Algebra definition:

$a_1 \in \mathcal{A}_{F_1}$

$a_1 \to \{\lambda, q\}$

$a_1 \to \begin{pmatrix} q_\lambda & 0 \\ 0 & q \end{pmatrix}$

$a_1 \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$

Not clear how one chooses the algebra and the connection between weak interactions and quaterions.

- For $\mathcal{H}_{F_4} \to \mathcal{H}_{1_R} \oplus \mathcal{H}_{1_L} \oplus \mathcal{H}_{\bar{1}_R} \oplus \mathcal{H}_{\bar{1}_L}$

$a_1 \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}$

$l \in \mathcal{H}_1$

By definition

$a_{\bar{1}} \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$

→

$\bar{l} \in \mathcal{H}_{\bar{1}}$

$a_8 \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^* & \alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$

```
PR["● Choose ℤ2-grading γF, and real structure JF for KO-dimension 6.",
  NL, "So that: ", $sr = {JF.l -> l̄, JF.l̄ → l,
     γF4 → DiagonalMatrix[{-1, 1, 1, -1}],
     γF8 → (DiagonalMatrix[{-1, 1, 1, -1}] /. 1 → {{1, 0}, {0, 1}} /.
         -1 → {{-1, 0}, {0, -1}} // ArrayFlatten),
     $s = JF4 → SparseArray[{Band[{1, 3}] → cc, Band[{3, 1}] → cc}, {4, 4}] // Normal,
     JF8 → ($s[[2]] /. cc → {{cc, 0}, {0, cc}} // ArrayFlatten)
    };
  MatrixForms[$sr] // Column // Framed,
  NL, CG[cc → "ComplexConjugate", ", F4 refers to ", $h, ", F8 refers to ", $basis],
  accumGWS[{$h2, $h, $basis, $alg, $alg1, $alg2, $sr}]; ""
 ];
```

● Choose $\mathbb{Z}_2$-grading $\gamma_F$, and real structure $J_F$ for KO-dimension 6.

So that:

$J_F.\mathrm{l} \to \bar{\mathrm{l}}$

$J_F.\bar{\mathrm{l}} \to \mathrm{l}$

$$\gamma_{F_4} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma_{F_8} \to \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$J_{F_4} \to \begin{pmatrix} 0 & 0 & cc & 0 \\ 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 \\ 0 & cc & 0 & 0 \end{pmatrix}$$

$$J_{F_8} \to \begin{pmatrix} 0 & 0 & 0 & 0 & cc & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & cc & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & cc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & cc & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & cc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cc & 0 & 0 & 0 & 0 \end{pmatrix}$$

$cc \to$ ComplexConjugate, $F_4$ refers to $\mathcal{H}_{F_4} \to \mathcal{H}_{1_R} \oplus \mathcal{H}_{1_L} \oplus \mathcal{H}_{\bar{1}_R} \oplus \mathcal{H}_{\bar{1}_L}$ , $F_8$ refers to $\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$

## 5.1.1 Finite Dirac Operator

```
PR["● Derive Hermitian Dirac operator in: ", tuRuleSelect[$defGWS][ℋF₂],
  NL, $df = $ = {iDF₂ → {{S , ct[T]}, {T, S '}}, {iDF₂, S, S '}[CG ["Hermitian"]] };
  MatrixForms[$], accumGWS[$df];

  next, "Since ", $s = {JF₂ → {{0, cc}, {cc, 0}}, a_ .cc ⧴ cc . cc[a]};
  $s // MatrixForms,
  Imply, iDF₂ , " the requirement: ", $ = CommutatorM[iDF₂, JF₂] → 0,
  $ = $ // expandCom[{$df, $s}];
  Yield, $ = $ /. $s //. tuOpCollect[]; $ // MatrixForms,
  Imply, $ = $ /. cc.a_ → a;
  $c1 = $ = Thread[Flatten[$[[1]]] → 0];
  $ // ColumnBar,
  $s = tuRuleSolve[$c1, {ct[T], S '}];
  Imply, $df[[1]] = $df[[1]] /. $s;

  $df // MatrixForms // Framed, accumGWS[{$s, $df}];

  next, " In ", tuRuleSelect[$defGWS][ℋF₄], " space, Let ",
  $s = {S -> Table[si,j, {i, 2}, {j, 2}], T -> Table[ti,j, {i, 2}, {j, 2}]};
  $s // MatrixForms, "POFF",
  Yield, $0 = $ = $df[[1, 2]] /. $s // ArrayFlatten; MatrixForm[$],
  Yield, $ht = ct[$]; MatrixForm[$ht],
  Yield, $ = $ → $ht //. rr : Rule[__] ⧴ Thread[rr]; MatrixForms[$], "PON",
  Yield, $s1 = tuRuleSolve[Flatten[$], {s2,1, t2,1}],
  Yield, $df44 = $ = iDF₄ -> $0 /. $s1 /. Conjugate[si_,i_] → si,i;
  MatrixForms[$] // Framed,

  next, " The requirement: ", $ = CommutatorP[iDF, γF] → 0, "POFF",
  Yield, $ = $ /. F → F₄,

  Yield, $ = $ /. $sr /. $df44; MatrixForms[$],
  Yield, $ = $ /. tuCommutatorExpand; MatrixForms[$], "PON",
  yield, $ = $ //. rr : Rule[__] ⧴ Thread[rr] // Flatten // DeleteDuplicates //
    tuRuleSolve[#, {s1,1, s2,2, t1,2}] &,
  Yield, $ = $df44 /. $; MatrixForms[$] // Framed, accumGWS[$], "PON",

  NL, "Using notation ", $s = {s1,2 → Conjugate[Y0], t1,1 → TR, t2,2 → TL},
  Imply, $df44 = $ = $ /. $s;
  MatrixForms[$] // Framed, accumGWS[{$s, $df44}],

  NL, "In the space ", $basis, yield,
  {Y0, TR, TL}, " are symmetric 2x2 matrices.",
  NL, "So in ", $ = {$df[[1]], S → $[[2, 1 ;; 2, 1 ;; 2]], T → $[[2, 3 ;; 4, 1 ;; 2]]};
  $ // MatrixForms, accumGWS[$]
 ];
```

● Derive Hermitian Dirac operator in: $\{\mathcal{H}_{F_2} \to \mathcal{H}_1[\mathbb{C}^4] \oplus \mathcal{H}_{\bar{1}}[\mathbb{C}^4]\}$

$\{\mathcal{D}_{F_2} \to \begin{pmatrix} S & T^\dagger \\ T & S' \end{pmatrix}, \{\mathcal{D}_{F_2}, S, S'\}[\text{Hermitian}]\}$

◆Since $\{J_{F_2} \to \begin{pmatrix} 0 & cc \\ cc & 0 \end{pmatrix}, (a\_).cc :\to cc.cc[a]\}$

⇒ $\mathcal{D}_{F_2}$ the requirement: $[\mathcal{D}_{F_2}, J_{F_2}]_- \to 0$

→ $\begin{pmatrix} cc.(-T + T^{\dagger *}) & cc.(S^* - S') \\ cc.(-S + (S')^*) & cc.(T^* - T^\dagger) \end{pmatrix} \to 0$

⇒ $\left| \begin{array}{l} -T + T^{\dagger *} \to 0 \\ S^* - S' \to 0 \\ -S + (S')^* \to 0 \\ T^* - T^\dagger \to 0 \end{array} \right.$

⇒ $\boxed{\{\mathcal{D}_{F_2} \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix}, \{\mathcal{D}_{F_2}, S, S'\}[\text{Hermitian}]\}}$

◆ In $\{\mathcal{H}_{F_4} \to \mathcal{H}_{1_R} \oplus \mathcal{H}_{1_L} \oplus \mathcal{H}_{\bar{1}_R} \oplus \mathcal{H}_{\bar{1}_L}\}$ space, Let $\{S \to \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{pmatrix}, T \to \begin{pmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{pmatrix}\}$

→ $\{s_{2,1} \to (s_{1,2})^*, t_{2,1} \to t_{1,2}\}$

→ $\boxed{\mathcal{D}_{F_4} \to \begin{pmatrix} s_{1,1} & s_{1,2} & (t_{1,1})^* & (t_{1,2})^* \\ (s_{1,2})^* & s_{2,2} & (t_{1,2})^* & (t_{2,2})^* \\ t_{1,1} & t_{1,2} & s_{1,1} & (s_{1,2})^* \\ t_{1,2} & t_{2,2} & s_{1,2} & s_{2,2} \end{pmatrix}}$

◆ The requirement: $\{\mathcal{D}_F, \gamma_F\}_+ \to 0 \longrightarrow \{s_{1,1} \to 0, s_{2,2} \to 0, t_{1,2} \to 0\}$

→ $\boxed{\mathcal{D}_{F_4} \to \begin{pmatrix} 0 & s_{1,2} & (t_{1,1})^* & 0 \\ (s_{1,2})^* & 0 & 0 & (t_{2,2})^* \\ t_{1,1} & 0 & 0 & (s_{1,2})^* \\ 0 & t_{2,2} & s_{1,2} & 0 \end{pmatrix}}$

Using notation $\{s_{1,2} \to (Y_0)^*, t_{1,1} \to T_R, t_{2,2} \to T_L\}$

⇒ $\boxed{\mathcal{D}_{F_4} \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix}}$

In the space $\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\}$
  $\longrightarrow \{Y_0, T_R, T_L\}$ are symmetric 2x2 matrices.

So in $\{\mathcal{D}_{F_2} \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix}, S \to \begin{pmatrix} 0 & (Y_0)^* \\ Y_0 & 0 \end{pmatrix}, T \to \begin{pmatrix} T_R & 0 \\ 0 & T_L \end{pmatrix}\}$

```
$basis8 = $basis[[2]]
PR["■ How does the restriction: ",
 $req = {T.$basis8[[1]] → Y_R.$basis8[[5]], T.l ⧴ 0 /; FreeQ[l, $basis8[[1]]]};
 $req // ColumnBar,
 " constrain T? ",
 NL, "where ", $t = T → DiagonalMatrix[{T_R, T_L}],
 NL, CO["Allows order-1 condition to be satisfied."],
 Yield, $t = $t /. tt : T_R_ -> Table[t[R]_{i,j}, {i, 2}, {j, 2}];
 $t[[2]] = $t[[2]] // ArrayFlatten;
 Yield, MatrixForms[$t], accumGWS[{$req, $t}];

 NL, "•Hermiticity of ", iD_F, imply, $st = {t[L]_{2,1} -> t[L]_{1,2}, t[R]_{2,1} -> t[R]_{1,2}},
 Yield, $t44 = $t = $t /. $st; $t // MatrixForms, accumGWS[{$st, $t}];

 NL, "In the ", selectGWS[ℋ_F8], " space: ", "POFF",
 Yield, $ = {{0, Conjugate[T]}, {T, 0}},
 Yield, $t = T → ($ /. $t // ArrayFlatten); $t // MatrixForms,
 Yield, $ = T . Transpose[{$basis8}]; $ // MatrixForms,
 Yield, $ = $ → $; $ // MatrixForms,
 Yield, $[[2]] = $[[2]] /. $t; "PON",
 MatrixForms[$],
 NL, "The requirement ", $req, imply,
 "The only non-zero element of T: ", Conjugate[t[R]_{1,1}] // Framed,
 Yield, $t = $t /. tt : t[_]_{i_,j_} ⧴ 0 /; tt =!= t[R]_{1,1}; $t // MatrixForms,
 NL, "also ", $ = y_{2,1} → y_{1,2},
 NL, "Require ℋ_F to be mass eigenstates ",
 $Y = Y_0 → DiagonalMatrix[{Y_ν, Y_e}], accumGWS[{$t, $, $Y}];

 line,
 NL, "Rules for ", selectGWS[ℋ_F8], " space.",
 Yield, $df44[[1]],
 Yield, $sDAgws = $ = {$df44, tt : T_R_ → Table[t[R]_{i,j}, {i, 2}, {j, 2}],
    t[RL_]_{i_,j_} ⧴ 0 /; (i ≠ 1 || j ≠ 1 || RL =!= R), $Y};
 $ // MatrixForms,
 accumGWS[$];
 NL, $ = iD_F8 -> iD_F4 /. tuRuleSelect[$defGWS][iD_F4][[-1]] //. $sDAgws;
 Yield, $[[2]] = $[[2]] /. t[R] → t_R // ArrayFlatten;
 $ // MatrixForms, accumGWS[$]
]
```

$\{\nu_R, e_R, \nu_L, e_L, \overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L}\}$

■ How does the restriction: $\left|\begin{array}{l} T.\nu_R \to Y_R.\bar{\nu}_R \\ T.l :\to 0 \,/;\, \mathrm{FreeQ}[l, \$basis8[[1]]] \end{array}\right.$ constrain T?

where $T \to \{\{T_R, 0\}, \{0, T_L\}\}$

Allows order-1 condition to be satisfied.

→

$$\to T \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} & 0 & 0 \\ t[R]_{2,1} & t[R]_{2,2} & 0 & 0 \\ 0 & 0 & t[L]_{1,1} & t[L]_{1,2} \\ 0 & 0 & t[L]_{2,1} & t[L]_{2,2} \end{pmatrix}$$

•Hermiticity of $\mathcal{D}_F$ ⇒ $\{t[L]_{2,1} \to t[L]_{1,2},\ t[R]_{2,1} \to t[R]_{1,2}\}$

$$\to T \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} & 0 & 0 \\ t[R]_{1,2} & t[R]_{2,2} & 0 & 0 \\ 0 & 0 & t[L]_{1,1} & t[L]_{1,2} \\ 0 & 0 & t[L]_{1,2} & t[L]_{2,2} \end{pmatrix}$$

In the $\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$ space: $T.\begin{pmatrix} \nu_R \\ e_R \\ \nu_L \\ e_L \\ \bar{\nu}_R \\ \bar{e}_R \\ \bar{\nu}_L \\ \bar{e}_L \end{pmatrix} \to \begin{pmatrix} (t[R]_{1,2})^* \bar{e}_R + (t[R]_{1,1})^* \bar{\nu}_R \\ (t[R]_{2,2})^* \bar{e}_R + (t[R]_{1,2})^* \bar{\nu}_R \\ (t[L]_{1,2})^* \bar{e}_L + (t[L]_{1,1})^* \bar{\nu}_L \\ (t[L]_{2,2})^* \bar{e}_L + (t[L]_{1,2})^* \bar{\nu}_L \\ \nu_R\, t[R]_{1,1} + e_R\, t[R]_{1,2} \\ \nu_R\, t[R]_{1,2} + e_R\, t[R]_{2,2} \\ \nu_L\, t[L]_{1,1} + e_L\, t[L]_{1,2} \\ \nu_L\, t[L]_{1,2} + e_L\, t[L]_{2,2} \end{pmatrix}$

The requirement $\{T.\nu_R \to Y_R.\bar{\nu}_R,\ T.l :\to 0 \,/;\, \mathrm{FreeQ}[l, \$basis8[[1]]]\}$

⇒ The only non-zero element of T: $\boxed{(t[R]_{1,1})^*}$

$$\to T \to \begin{pmatrix} 0 & 0 & 0 & 0 & (t[R]_{1,1})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t[R]_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

also $y_{2,1} \to y_{1,2}$

Require $\mathcal{H}_F$ to be mass eigenstates $Y_0 \to \{\{Y_\nu, 0\}, \{0, Y_e\}\}$

---

Rules for $\mathcal{H}_{F_8} \to \{\nu_R, e_R, \nu_L, e_L, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$ space.

→ $\mathcal{D}_{F_4}$

$$\to \left\{\mathcal{D}_{F_4} \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix},\ tt : T_{R\_} \to \begin{pmatrix} t[R]_{1,1} & t[R]_{1,2} \\ t[R]_{2,1} & t[R]_{2,2} \end{pmatrix},\right.$$

$$\left. t[RL\_]_{i\_,j\_} :\to 0 \,/;\, i \neq 1 \,||\, j \neq 1 \,||\, RL =!= R,\ Y_0 \to \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}\right\}$$

$$\to \mathcal{D}_{F_8} \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 & (t_{R1,1})^* & 0 & 0 & 0 \\ 0 & 0 & 0 & (Y_e)^* & 0 & 0 & 0 & 0 \\ Y_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{R1,1} & 0 & 0 & 0 & 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_e \\ 0 & 0 & 0 & 0 & (Y_\nu)^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (Y_e)^* & 0 & 0 \end{pmatrix}$$

(* the representation of l and $\bar{l}$ must
be distinguished in the following calculation.*)

```
PR[CO["General rules: "],
 $conditions = tuRuleSelect[Select[$defall, tuHasAnyQ[#, {γF, JF}] &]][
    {CommutatorM[__], CommutatorP[__], rghtA[_], Dot[_, _]}] // DeleteDuplicates;
 $conditions // ColumnBar;
]
```

```
General rules: Null
```

```
PR["Prop.5.1. ", $ = F_GWS → Map[# /. a_ → a_F &, {iA, ℋ, iD, γ, J}],
 " define a real even KOdim→6 space.",
 imply, KOdim → 6,
 Imply, $se6 = εRule[6],

 line,
 NL, "Recall general conditions: ", $conditions // ColumnBar,
 next, "Check: ", $ = tuRuleSelect[$conditions][CommutatorM[γF, _]][[1]],
 imply, "OK γF diagonal. ",
 next, "Check: ", $ = tuRuleSelect[$conditions][JF.JF][[1]] /. $se6,
 imply, "OK",
 next, "Check: ",
 $ = tuRuleSelect[$conditions][JF.iDF] /. $se6 /. tuOpSimplify[Dot] // First,
 " by construction.",
 next, "Check: ",
 $ = tuRuleSelect[$conditions][JF.γF] /. $se6 /. tuOpSimplify[Dot] // First,
 yield, $ = $ /. tuRuleSelect[$defGWS][{γF, JF}] /. Rule → Equal,
 next, "Check: ", $ = tuRuleSelect[$conditions][CommutatorP[_, _]] // First,
 yield, $ = $ /. tuRuleSelect[$defGWS][{γF, iDF}] /. tuCommutatorExpand /. Rule → Equal,

 next, "Check order-0 condition: ",
 $ = tuRuleSelect[$conditions][CommutatorM[a, _]] // First,
 NL, CR["Need 8×8 space for correct computation."],
 Yield, $ = $ /. tuRuleSelect[$conditions][rghtA[_]] /. {aa : a | b → aa8, F → F8},
 NL, "for algebra's ",
 $s = tuRuleSelect[$defGWS][a8] // Select[#, tuHasAnyQ[#, α] &] & // First;

 $s = {$s, ($s /. aa : λ | α | β → aab /. a → b)}; $s // MatrixForms, "POFF",
 Yield, $ = $ /. tuCommutatorExpand /. Dot → xDot;
 Yield, $ = $ /. $s /. tuRuleSelect[$defGWS][{JF8}]; $ // MatrixForms, CK,
 Yield, $ = $ // tuMatrixOrderedMultiply // (# /. xDot → Dot &),
 NL, "Using: ", $s = {cc . a_ → Conjugate[a].cc, Conjugate[cc] → cc, cc.cc → 1},
 Yield, $ = $ //. $s; $ // MatrixForms, CK, "PONdd",
 Yield, $ = $ // tuRepeat[
     {$s, tuOpSimplify[Dot, {λ, Conjugate[λ], α, β, Conjugate[α], Conjugate[β]}]},
     tuConjugateSimplify[{cc}]] // Simplify;
 $ // MatrixForms, CG["0→OK"],
 next, "Check order-1 condition: ",
 $ = tuRuleSelect[$conditions][CommutatorM[CommutatorM[_, _], _]] // First,
 Yield, $ = $ /. (tuRuleSelect[$conditions][rghtA[_]] // tuAddPatternVariable[b]) /.
    {aa : a | b → aa8, F → F8},
 NL, "for algebra's ",
 $s = tuRuleSelect[$defGWS][a8] // Select[#, tuHasAnyQ[#, α] &] & // First;
 $s = {$s, ($s /. aa : λ | α | β → aab /. a → b)}; $s // MatrixForms, "POFF",
 Yield, $ = $ // expandCom[{$s, tuRuleSelect[$defGWS][{iDF8, JF8}]}];
 $ // MatrixForms, "PONdd",
 NL, "Using: ",
 $s = {cc . Shortest[a_] → Conjugate[a].cc, Conjugate[cc] → cc, cc.cc → 1},
 Yield, $ = $ // tuRepeat[
     {$s, tuOpSimplify[Dot, {λ, Conjugate[λ], α, β, Conjugate[α], Conjugate[β]}]},
     tuConjugateSimplify[{cc}]] // Simplify;
 $ = $ /. Dot → Times;
 $ // MatrixForms, yield, CG["0→OK"]
]
```

Prop.5.1. $F_{GWS} \to \{\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F\}$ define a real even KOdim→6 space. ⇒ KOdim → 6
⇒ $\{\varepsilon \to 1, \varepsilon' \to 1, \varepsilon'' \to -1\}$

Recall general conditions:
$$\left|\begin{array}{l} [\gamma_F, a \in A_F]_- \to 0 \\ [a, b^o]_- \to 0 \\ [[D_F, a]_-, b^o]_- \to 0 \\ \{\gamma_F, D_F\}_+ \to 0 \\ \{J_F, i\}_+ \to 0 \\ b^o \to J_F . b^\dagger . (J_F)^\dagger \\ \gamma_F . \gamma_F \to 1_F \\ J_F . J_F \to \varepsilon \\ J_F . D_F \to \varepsilon' . D_F . J_F \\ J_F . \gamma_F \to \varepsilon'' . \gamma_F . J_F \\ \gamma_F . \mathcal{H} \to \{\{\mathcal{H}^+, 0\}, \{0, \mathcal{H}^-\}\} \\ \gamma_F . \{\{a\_, b\_\}, \{c\_, d\_\}\} \to \{\{a, 0\}, \{0, d\}\} \end{array}\right.$$

◆Check: $[\gamma_F, a \in A_F]_- \to 0$ ⇒ OK $\gamma_F$ diagonal.
◆Check: $J_F . J_F \to 1$ ⇒ OK
◆Check: $J_F . D_F \to D_F . J_F$ by construction.
◆Check: $J_F . \gamma_F \to -\gamma_F . J_F$ ⟶ $J_F . \gamma_F == -\gamma_F . J_F$
◆Check: $\{\gamma_F, D_F\}_+ \to 0$ ⟶ $\gamma_F . D_F + D_F . \gamma_F == 0$
◆Check order-0 condition: $[a, b^o]_- \to 0$
Need 8×8 space for correct computation.
→ $[a_8, J_{F_8} . (b_8)^\dagger . (J_{F_8})^\dagger]_- \to 0$
for algebra's

$$\left\{a_8 \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^* & \alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, b_8 \to \begin{pmatrix} \lambda_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\lambda_b)^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_b & \beta_b & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\beta_b)^* & (\alpha_b)^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b \end{pmatrix}\right\}$$

.......

$$\to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to 0 \; 0 \to \text{OK}$$

◆Check order-1 condition: $[[D_F, a]_-, b^o]_- \to 0$
→ $[[D_{F_8}, a_8]_-, J_{F_8} . (b_8)^\dagger . (J_{F_8})^\dagger]_- \to 0$
for algebra's

$$\left\{a_8 \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^* & \alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, b_8 \to \begin{pmatrix} \lambda_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\lambda_b)^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_b & \beta_b & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\beta_b)^* & (\alpha_b)^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_b \end{pmatrix}\right\}$$

.......
Using: {cc.Shortest[a_] → a*.cc, cc* → cc, cc.cc → 1}

$$\to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to 0 \longrightarrow 0 \to \text{OK}$$

## 5.2 The gauge theory

### 5.2.1 The gauge group (p.54)

```
PR["• The Local gauge group from ", F_GWS,
  NL, "Examine subalgebra ", $0 = $ = {iÃ_{F J_F}, {iA_F → ℂ⊕ℍ, a ∈ iÃ_{F J_F}, a.J_F → J_F.ct[a]}};
  $ // ColumnForms,
  NL, "For the above case: ", $s = {a -> a_8, J_F → J_{F_8}},
  Yield, $ = tuRuleSelect[$][a.J_F] /. $s,
  Yield,
  $ = $ // expandCom[{selectGWS[{a_8, J_{F_8}}, {}, all], cc.a_ → cc[a].cc, cc → 1}] // First;
  $ // MatrixForms,
  Yield, $ = Thread[$] /. rr : Rule[__] ↦ Thread[rr] // Flatten // DeleteDuplicates //
     DeleteCases[#, Rule[a_, a_]] & // tuRule,
  Yield, $ = tuRuleSolve[$, {β, α, λ}, Complexes],
  NL, "Since ", $s = λ ∈ Reals,
  yield, $s1 = Refine[$, Assumptions → $s],
  Imply, $ = selectGWS[a_8] /. $s1 // Refine[#, Assumptions → $s] &;
  $ // MatrixForms,
  imply, $e54 = {$0[[1]] → λ 1_{ℋ_F}, $0[[1]] ≃ ℝ}, CG[" (5.4)"]
 ];
```

• The Local gauge group from $F_{GWS}$

Examine subalgebra $\left|\begin{array}{l}\tilde{A}_{F J_F}\\ \left|\begin{array}{l}A_F\to\mathbb{C}\oplus\mathbb{H}\\ a\in\tilde{A}_{F J_F}\\ a.J_F\to J_F.a^\dagger\end{array}\right.\end{array}\right.$

For the above case: $\{a\to a_8,\ J_F\to J_{F_8}\}$

→ $\{a_8.J_{F_8}\to J_{F_8}.(a_8)^\dagger\}$

→ $\begin{pmatrix}0&0&0&0&\lambda&0&0&0\\0&0&0&0&0&\lambda^*&0&0\\0&0&0&0&0&0&\alpha&\beta\\0&0&0&0&0&0&-\beta^*&\alpha^*\\\lambda&0&0&0&0&0&0&0\\0&\lambda&0&0&0&0&0&0\\0&0&\lambda&0&0&0&0&0\\0&0&0&\lambda&0&0&0&0\end{pmatrix}\to\begin{pmatrix}0&0&0&0&\lambda&0&0&0\\0&0&0&0&0&\lambda&0&0\\0&0&0&0&0&0&\lambda&0\\0&0&0&0&0&0&0&\lambda\\\lambda&0&0&0&0&0&0&0\\0&\lambda^*&0&0&0&0&0&0\\0&0&\alpha&-\beta^*&0&0&0&0\\0&0&\beta&\alpha^*&0&0&0&0\end{pmatrix}$

→ $\{\lambda^*\to\lambda,\ \alpha\to\lambda,\ \beta\to0,\ \beta^*\to0,\ \alpha^*\to\lambda,\ \lambda\to\lambda^*,\ \lambda\to\alpha,\ 0\to-\beta^*,\ 0\to\beta,\ \lambda\to\alpha^*\}$

→ {β → ConditionalExpression[0, Im[λ] == 0], α → ConditionalExpression[λ, Im[λ] == 0]}

Since $\lambda\in$ Reals ⟶ $\{\beta\to0,\ \alpha\to\lambda\}$

⇒ $a_8\to\begin{pmatrix}\lambda&0&0&0&0&0&0&0\\0&\lambda&0&0&0&0&0&0\\0&0&\lambda&0&0&0&0&0\\0&0&0&\lambda&0&0&0&0\\0&0&0&0&\lambda&0&0&0\\0&0&0&0&0&\lambda&0&0\\0&0&0&0&0&0&\lambda&0\\0&0&0&0&0&0&0&\lambda\end{pmatrix}$ ⇒ $\{\tilde{A}_{F J_F}\to\lambda\,1_{\mathcal{H}_F},\ \tilde{A}_{F J_F}\simeq\mathbb{R}\}$ (5.4)

```
PR["", Ω_iD^"1" → {xSum[a_j. CommutatorM[iD, b_j], {j}], a_j | b_j ∈ iA}
]
```

$\Omega^1_D\to\{\sum_{\{j\}}[a_j.[D,b_j]_-],\ a_j\mid b_j\in A\}$

```
PR["• Consider Lie algebra (2.11b) ", hF → u[$e54[[1, 1]]],
 Yield, {u[CG["anti-hermitian"]] ∈ u[iAF], u → {λ, q},
  λ ∈ I ℝ, q → -I xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 3}]},
 imply, Conjugate[λ] → -λ,
 Imply, {hF → u[$e54[[1, 1]]], {λ, Conjugate[λ], α, Conjugate[α]} → 0},
 imply, $lh = hF → {0},

 line,
 NL, "●Prop.5.2:  The local gauge group of FGWS is ",
 $G = 𝒢[FGWS] ≃ xMod[U[1] × SU[2], {1, -1}],
 line,
 NL, "Proof:
The unitary elements: ", $1u = U[iAF] ≃ U[1] × U[ℍ],
 NL, "• For ", {q ∈ ℍ, q → I xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 0, 3}]},
 and, {(q[CG["Unitary"]] ⇔ (Abs[q]^2 → 1)) ⇒ (Det[q] → 1)},
 imply, U[ℍ] ≃ SU[2],
 NL, "• Since ", $e54,
 imply, $s = {ℋF → U[$e54[[1, 1]]], ℋF → {1, -1}},
 Imply, $G,
 yield, $G /. Reverse[$s[[-1]]], CG[" QED"],
 line,
 next, " Since ", $lh,
 " the gauge field ", T[it[A], "d", {μ}],
 CR[" takes values"], " in the Lie subalgebra ",
 $ = {gF → Mod[u[iAF], hF], Mod[u[iAF], hF] -> u[iAF], u[iAF] -> u[1] ⊕ su[2]};
 $ // ColumnBar, CO["From ", $1u, " above"]
]
```

• Consider Lie algebra (2.11b) $h_F \to u[\tilde{A}_{F J_F}]$

→ $\{u[\text{anti-hermitian}] \in u[A_F],\ u \to \{\lambda, q\},\ \lambda \in i\,\mathbb{R},\ q \to -i \sum_{\{i,3\}} [q_i\, \sigma^i]\}$ ⇒ $\lambda^* \to -\lambda$

⇒ $\{h_F \to u[\tilde{A}_{F J_F}],\ \{\lambda, \lambda^*, \alpha, \alpha^*\} \to 0\}$ ⇒ $h_F \to \{0\}$

---

●Prop.5.2:  The local gauge group of $F_{GWS}$ is $\mathcal{G}[F_{GWS}] \simeq \text{xMod}[U[1] \times SU[2], \{1, -1\}]$

---

Proof:
The unitary elements: $U[A_F] \simeq U[1] \times U[\mathbb{H}]$

• For $\{q \in \mathbb{H},\ q \to i \sum_{\{i,0,3\}} [q_i\, \sigma^i]\}$ and

$\{q[\text{Unitary}] \Leftrightarrow (\text{Abs}[q]^2 \to 1) \Rightarrow (\text{Det}[q] \to 1)\}$ ⇒ $U[\mathbb{H}] \simeq SU[2]$

• Since $\{\tilde{A}_{F J_F} \to \lambda\, 1_{\mathcal{H}_F},\ \tilde{A}_{F J_F} \simeq \mathbb{R}\}$ ⇒ $\{\mathcal{H}_F \to U[\tilde{A}_{F J_F}],\ \mathcal{H}_F \to \{1, -1\}\}$

⇒ $\mathcal{G}[F_{GWS}] \simeq \text{xMod}[U[1] \times SU[2], \{1, -1\}]$ ⟶ $\mathcal{G}[F_{GWS}] \simeq \text{xMod}[U[1] \times SU[2], \mathcal{H}_F]$ QED

---

◆ Since $h_F \to \{0\}$ the gauge field $A_\mu$ takes values

in the Lie subalgebra
$g_F \to \text{Mod}[u[A_F], h_F]$
$\text{Mod}[u[A_F], h_F] \to u[A_F]$ From $U[A_F] \simeq U[1] \times U[\mathbb{H}]$ above
$u[A_F] \to u[1] \oplus su[2]$

## 5.2.2 The gauge fields and the Higgs field(p.55)

```
PR["● Derive gauge and Higgs fields (2.13,2.14) ", {T[it[A], "d", {μ}], ϕ},
 NL, "Let ", {a → {λ, q}, b → {λ', q'}, {a, b} ∈ (𝒜 → C"∞"[M, ℂ⊕ℍ])},
 NL, "▪Calculate inner fluctuation (5.2) ",
 $tA = T[it[A], "d", {μ}] → -I a.tuDPartial[b, μ] /. {a → a1, b → b1},
 NL, "•Let ", $s = selectGWS[a1]; $sb = $s /. {a → b, λ → λ', β → β', α → α'};
 $sab = {$s, $sb} // Flatten,
 Imply,
 $tA = $tA /. $sab //. tt : tuDDown["∂"][__] :→ Thread[tt] /. tuDDown["∂"][0, _] → 0 //
    tuConjugateSimplify[{μ}] // tuDerivativeExpand[];
 MatrixForms[$tA],
 NL, "•Hermiticity of ", $tA[[1]],
 imply, {$tA[[2, 1, 1]], $tA[[2, 2, 2]]} ∈ ℝ,

 NL, "•For the lower-right blocks ",
 $ = {$a = qa -> $sab[[1, 2, 3 ;; -1, 3 ;; -1]],
   $b = qb -> $sab[[2, 2, 3 ;; -1, 3 ;; -1]]}; $ // MatrixForms,
 Yield, $ = Thread[Inactive[Dot][$a, $b], Rule] // tuMatrixOrderedMultiply //
   tuOpSimplifyF[dotOps];
 $ // MatrixForms,
 Imply, $ = $tA[[2, 3 ;; -1, 3 ;; -1]] → -I ($[[1]] /. qb → tuDPartial[qb, μ]);
 $ // MatrixForms,
 NL, "Defining ", $ = {T[Λ, "d", {μ}] -> $tA[[2, 1, 1]], T[Q, "d", {μ}] → $[[-1]]};
 $ // ColumnBar, accumGWS[$];
 NL, "we can represent ", $A3 = $ = {T[it[A], "d", {μ}] →
     DiagonalMatrix[{T[Λ, "d", {μ}], -T[Λ, "d", {μ}], T[Q, "d", {μ}]}],
    T[Q, "d", {μ}] → I xSum[T[q, "d", {i}] T[σ, "u", {i}], {i, 3}], T[q, "d", {i}] ∈ ℝ};
 $ // MatrixForms, accumGWS[$]
]
```

● Derive gauge and Higgs fields (2.13,2.14) $\{A_\mu, \phi\}$

Let $\{a \to \{\lambda, q\}, b \to \{\lambda', q'\}, \{a, b\} \in (\mathcal{A} \to C^\infty[M, \mathbb{C}\oplus\mathbb{H}])\}$

▪Calculate inner fluctuation (5.2) $A_\mu \to -i\, a_1 . \underline{\partial}_\mu[b_1]$

•Let $\{a_1 \to \{\{\lambda, 0, 0, 0\}, \{0, \lambda^*, 0, 0\}, \{0, 0, \alpha, \beta\}, \{0, 0, -\beta^*, \alpha^*\}\},$
$b_1 \to \{\{\lambda', 0, 0, 0\}, \{0, (\lambda')^*, 0, 0\}, \{0, 0, \alpha', \beta'\}, \{0, 0, -(\beta')^*, (\alpha')^*\}\}\}$

$$\Rightarrow A_\mu \to \begin{pmatrix} -i\lambda\underline{\partial}_\mu[\lambda'] & 0 & 0 & 0 \\ 0 & -i\lambda^*\underline{\partial}_\mu[\lambda']^* & 0 & 0 \\ 0 & 0 & i\beta\underline{\partial}_\mu[\beta']^* - i\alpha\underline{\partial}_\mu[\alpha'] & -i\beta\underline{\partial}_\mu[\alpha']^* - i\alpha\underline{\partial}_\mu[\beta'] \\ 0 & 0 & i\alpha^*\underline{\partial}_\mu[\beta']^* + i\beta^*\underline{\partial}_\mu[\alpha'] & -i\alpha^*\underline{\partial}_\mu[\alpha']^* + i\beta^*\underline{\partial}_\mu[\beta'] \end{pmatrix}$$

•Hermiticity of $A_\mu \Rightarrow \{-i\lambda\underline{\partial}_\mu[\lambda'], -i\lambda^*\underline{\partial}_\mu[\lambda']^*\} \in \mathbb{R}$

•For the lower-right blocks $\left\{q_a \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, q_b \to \begin{pmatrix} \alpha' & \beta' \\ -(\beta')^* & (\alpha')^* \end{pmatrix}\right\}$

$$\to q_a . q_b \to \begin{pmatrix} \alpha.\alpha' - \beta.(\beta')^* & \alpha.\beta' + \beta.(\alpha')^* \\ -\alpha^*.(\beta')^* - \beta^*.\alpha' & \alpha^*.(\alpha')^* - \beta^*.\beta' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} i\beta\underline{\partial}_\mu[\beta']^* - i\alpha\underline{\partial}_\mu[\alpha'] & -i\beta\underline{\partial}_\mu[\alpha']^* - i\alpha\underline{\partial}_\mu[\beta'] \\ i\alpha^*\underline{\partial}_\mu[\beta']^* + i\beta^*\underline{\partial}_\mu[\alpha'] & -i\alpha^*\underline{\partial}_\mu[\alpha']^* + i\beta^*\underline{\partial}_\mu[\beta'] \end{pmatrix} \to -i\, q_a . \underline{\partial}_\mu[q_b]$$

Defining
$\Lambda_\mu \to -i\lambda\underline{\partial}_\mu[\lambda']$
$Q_\mu \to -i\, q_a . \underline{\partial}_\mu[q_b]$

we can represent $\left\{A_\mu \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix}, Q_\mu \to i \sum_{\{i,3\}} [q_i\, \sigma^i], q_i \in \mathbb{R}\right\}$

```
PR["■From the definition ", ϕ → a . CommutatorM[iD_F, b],
  NL, "For this case ", $ = {$df44, $Y}; MatrixForms[$],
  NL, "Previous calculation show that only the
    upper left quadrant (S) does not commute with the algebra. ",
  Imply, $sD = S → ($df44[[2, 1 ;; 2, 1 ;; 2]] /. $Y // ArrayFlatten);
  MatrixForms[$sD], accumGWS[$sD];
  NL, "• ", $ = ϕ -> a_1 . CommutatorM[S, b_1]; $,
  yield, $ = $ /. $sD /. $sab; MatrixForms[$],
  Yield, $ph = $ = $ /. tuCommutatorExpand // FullSimplify; MatrixForms[$],
  (**)
  NL, "Let ", $i = 1;
  $ph0 = $ph /. (yy : Y_ | cc[Y_]) _ :→ yy ϕ_{$i++};
  $ph0 // MatrixForms,
  NL, "By inspection: ", $sp = {ϕ_4 → cc[ϕ_1], ϕ_8 → cc[ϕ_5], ϕ_3 → -cc[ϕ_2], ϕ_7 → -cc[ϕ_6]},
  $ph0 = $ph0 /. $sp; $ph0 // MatrixForms, CG[" (5.6)"],
  NL, "Hermitian requirement: ", $ = ϕ → ct[ϕ],
  Yield, $ = $ /. $ph0 //. tt : Rule[__] :→ Thread[tt];
  Yield,
  $ = $ // tuConjugateSimplify[] // Flatten // DeleteDuplicates // DeleteCases[#, 0 → 0] &;
  $ // ColumnBar;
  Imply, $ = #[[2]] & /@ tuSolve /@ $ // Flatten; $ // Column;
  $ = Reduce[$, Table[ϕ_i, {i, 8}], Complexes];
  $ = Apply[List, $] /. {Equal → Rule},
  Yield, $e56 = $ph0 = $ph0 /. $;
  $ph0 // MatrixForms, CG[" (5.6)"], accumGWS[$e56];
  NL, "There only 2 independent relationships with equivalent formulas: ",
  Yield, $ = Thread[$ph0[[2]] -> $ph[[2]]] /. rr : Rule[__] :→ Thread[rr];
  Yield, $ph12 = $ = #[[2]] & /@ tuSolve /@ $ /. Equal → Rule // Flatten //
      DeleteCases[#, cc[_] -> _] & // Simplify;
  $ // FramedColumn
 ];
```

■From the definition $\phi \to a.[D_F, b]_-$

For this case $\{D_{F_4} \to \begin{pmatrix} 0 & (Y_0)^* & (T_R)^* & 0 \\ Y_0 & 0 & 0 & (T_L)^* \\ T_R & 0 & 0 & Y_0 \\ 0 & T_L & (Y_0)^* & 0 \end{pmatrix}, Y_0 \to \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}\}$

Previous calculation show that only the
upper left quadrant (S) does not commute with the algebra.

$\Rightarrow$ $S \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}$

• $\phi \to a_1.[S, b_1]_-$ $\longrightarrow$ $\phi \to \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}.[\begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & (\lambda')^* & 0 & 0 \\ 0 & 0 & \alpha' & \beta' \\ 0 & 0 & -(\beta')^* & (\alpha')^* \end{pmatrix}]_-$

$\to$ $\phi \to$

$$\begin{pmatrix} 0 & 0 & \lambda (Y_\nu)^* (\alpha' - \lambda') & \lambda (Y_\nu)^* \beta' \\ 0 & 0 & -\lambda^* (Y_e)^* (\beta')^* & \lambda^* (Y_e)^* ((\alpha')^* - (\lambda')^*) \\ Y_\nu (\beta (\beta')^* + \alpha (-\alpha' + \lambda')) & -Y_e (\beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta') & 0 & 0 \\ Y_\nu (\alpha^* (\beta')^* + \beta^* (\alpha' - \lambda')) & Y_e (\alpha^* (-(\alpha')^* + (\lambda')^*) + \beta^* \beta') & 0 & 0 \end{pmatrix}$$

Let $\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & (Y_e)^* \phi_3 & (Y_e)^* \phi_4 \\ Y_\nu \phi_5 & Y_e \phi_6 & 0 & 0 \\ Y_\nu \phi_7 & Y_e \phi_8 & 0 & 0 \end{pmatrix}$

By inspection: $\{\phi_4 \to (\phi_1)^*, \phi_8 \to (\phi_5)^*, \phi_3 \to -(\phi_2)^*, \phi_7 \to -(\phi_6)^*\}$

$$\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ Y_\nu \phi_5 & Y_e \phi_6 & 0 & 0 \\ -(\phi_6)^* Y_\nu & (\phi_5)^* Y_e & 0 & 0 \end{pmatrix} \quad (5.6)$$

Hermitian requirement: $\phi \to \phi^\dagger$

$\to$

$\to$

$\Rightarrow$ $\{\phi_5 \to (\phi_1)^*, \phi_6 \to -\phi_2\}$

$\to$ $\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 \end{pmatrix}$ (5.6)

There only 2 independent relationships with equivalent formulas:

$\to$

$\to$ $\boxed{\begin{array}{l} \phi_1 \to \lambda (\alpha' - \lambda') \\ \phi_2 \to \lambda \beta' \\ \phi_2 \to \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta' \\ \phi_1 \to \alpha^* (-(\alpha')^* + (\lambda')^*) + \beta^* \beta' \end{array}}$

```
PR["●Note: ϕ's is generally a sum of like terms: ",
 $ = Map[# /. tt : λ' | α' | β' | λ | α | β ⧴ T[tt, "d", {j}] &, $ph12];
 Yield, $ph12p = $ =
   Map[#[[1]] -> xSum[#[[2]], {j}] &, $] /. xSum[a_ -> b_, c_] → xSum[a, c] -> xSum[b, c];
 Column[$],
 NL, CR["Recall that ", ϕ → a . CommutatorM[iD_F, b],
  " is the Higg's like field defined by the algebra and the Dirac
    operator.  What is the effect of different algebras on ϕ?"]
]
```

●Note: $\phi$'s is generally a sum of like terms:

$\phi_1 \to \sum_{\{j\}} [\lambda_j (\alpha'_j - \lambda'_j)]$

$\phi_2 \to \sum_{\{j\}} [\lambda_j \beta'_j]$

→

$\phi_2 \to \sum_{\{j\}} [(\alpha'_j)^* \beta_j - (\lambda'_j)^* \beta_j + \alpha_j \beta'_j]$

$\phi_1 \to \sum_{\{j\}} [(\alpha_j)^* (-(\alpha'_j)^* + (\lambda'_j)^*) + (\beta_j)^* \beta'_j]$

Recall that $\phi \to a.[D_F, b]_-$
is the Higg's like field defined by the algebra and the
Dirac operator.  What is the effect of different algebras on $\phi$?

```
PR["Summary: ",
  NL, $e57 = $ = {CG["On ℋ₁"],
      $A3, T[Λ, "d", {μ}] ∈ ℝ,
      ϕ → {{0, ct[Y]}, {Y, 0}},
      $ph0,
      $ph12,
      T[B_ℋ₁, "d", {μ}] →
       {{0, 0, 0}, {0, -2 T[Λ, "d", {μ}], 0}, {0, 0, T[Q, "d", {μ}] - T[Λ, "d", {μ}] 1₂}},
      CG["On ℋ_Ī"],
      T[B_ℋ_Ī, "d", {μ}] → {{0, 0, 0}, {0, 2 T[Λ, "d", {μ}], 0},
        {0, 0, -T[Λ, "d", {μ}] 1₂ - Conjugate[T[Q, "d", {μ}]]}}
     } // Flatten;
  $ // MatrixForms // ColumnBar, accumGWS[$e57],

  NL, "● Calculate ", $ = $b = selectEM[T[B, "d", {μ}], {A}],
  NL, "In 8x8 representation ", "POFF",
  Yield, $q = T[Q, "d", {μ}] → Table[T[q, "d", {μ}]_i,j, {i, 2}, {j, 2}];
  MatrixForms[$q],
  Yield, $b = $b /. toxDot /. A → iA /. F → F₈ /. Plus → Inactive[Plus],
  Yield, $a = selectGWS[T[iA, "d", {μ}], {Q}] /. it[A] → it[A]_l /. $q;
  Yield, $a = MapAt[ArrayFlatten[#] &, $a, -1]; $a // MatrixForms,
  Yield, $aa = ($a[[1]] /. l → l̄) → (DiagonalMatrix[Table[T[Λ, "d", {μ}], 4]]) // Normal,
  Yield, $aaa = {{$a[[1]], 0}, {0, $aa[[1]]}},
  Yield, $aaa = T[it[A], "d", {μ}] -> ($aaa /. $a /. $aa // ArrayFlatten);
  $aaa // MatrixForms, accumGWS[{$a, $aa, $aaa, $q}];
  $j8 = selectGWS[J_F₈],
  $b =
   $b // expandCom[{$aaa, $j8, Dot[cc, a_] :→ Dot[cc[a], cc], cc[cc] → cc, cc.cc → 1}] //
     tuConjugateSimplify[{T[Λ, "d", {μ}]}] // Activate,
  "PONdd",
  Yield, $e58 = $ = $b // Activate;
  $ // MatrixForms // Framed, CG[" (5.8)"],
  NL, "Coefficients of Λ associated with hyper-charge."; accumGWS[$e58]
 ];
```

Summary:

On $\mathcal{H}_1$

$$A_\mu \to \begin{pmatrix} \Lambda_\mu & 0 & 0 \\ 0 & -\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu \end{pmatrix}$$

$$Q_\mu \to i \sum_{\{i,3\}} [q_i\, \sigma^i]$$

$q_i \in \mathbb{R}$

$\Lambda_\mu \in \mathbb{R}$

$$\phi \to \begin{pmatrix} 0 & Y^\dagger \\ Y & 0 \end{pmatrix}$$

$$\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 \end{pmatrix}$$

$\phi_1 \to \lambda (\alpha' - \lambda')$

$\phi_2 \to \lambda \beta'$

$\phi_2 \to \beta (\alpha')^* - \beta (\lambda')^* + \alpha \beta'$

$\phi_1 \to \alpha^* (-(\alpha')^* + (\lambda')^*) + \beta^* \beta'$

$$B_{\mathcal{H}_1 \mu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 \\ 0 & 0 & Q_\mu - 1_2 \Lambda_\mu \end{pmatrix}$$

On $\mathcal{H}_{\bar{1}}$

$$B_{\mathcal{H}_{\bar{1}} \mu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\Lambda_\mu & 0 \\ 0 & 0 & -(Q_\mu)^* - 1_2 \Lambda_\mu \end{pmatrix}$$

- Calculate $B_\mu \to -J_F . A_\mu . (J_F)^\dagger + A_\mu$

In 8x8 representation

.......

$$\to \boxed{B_\mu \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 1,1} - \Lambda_\mu & q_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 2,1} & q_{\mu 2,2} - \Lambda_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 1,1})^* + \Lambda_\mu & -(q_{\mu 1,2})^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -(q_{\mu 2,1})^* & -(q_{\mu 2,2})^* + \Lambda_\mu \end{pmatrix}} \quad (5.8)$$

```
PR["● Higgs field ",
 $ = Φ → Inactivate[ iD_F + {{ϕ, 0}, {0, 0}} + J_F.{{ϕ, 0}, {0, 0}}.ct[J_F], Plus];
 MatrixForms[$], CG["(5.9)"],
 NL, "In 8x8 representation: ",
 $sϕ = {{ϕ, 0}, {0, 0}} → ArrayFlatten[DiagonalMatrix[{ϕ, 0, 0, 0, 0}] /. $ph0];
 MatrixForms[$sϕ], accumGWS[$sϕ], "POFF",

 $ = $ /. J_F → J_F8;
 $ = $ /. toxDot /. $j8 /. $sϕ; MatrixForms[$],
 Yield,
 $ = $ // tuMatrixOrderedMultiply // (# /. toDot &) //
   tuRepeat[{tuOpSimplify[Dot], Dot[cc, a_] :→ Dot[cc[a], cc], cc[cc] → cc, cc.cc → 1}];
 "PON",
 MatrixForms[$], accumGWS[$];
 NL, "From (5.6) ", $e56 // MatrixForms,
 NL, "Condense into space ", selectGWS[ℋ_F2],
 Yield, $s = {$[[2, -1]] → {{0, 0}, {0, cc[ϕ]}}, $[[2, -2]] → {{ϕ, 0}, {0, 0}}};
 $s // MatrixForms,
 Yield, $ = $ /. $s /. F → F2, CK,
 NL, "From ", $s = selectGWS[iD_F2]; MatrixForms[$s],
 Yield, $ = $ /. $s // Activate;
 MatrixForms[$e59 = $] // Framed, CG[" (5.9)"]; accumGWS[$e59]
]
```

● Higgs field $\Phi \to D_F + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F.\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}.(J_F)^{\dagger}$ (5.9)

In 8x8 representation: $\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$\Phi \to D_F + \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_\nu.(\phi_1)^* & Y_\nu.(\phi_2)^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y_e.\phi_2 & Y_e.\phi_1 \\ 0 & 0 & 0 & 0 & \phi_1.(Y_\nu)^* & -(Y_e)^*.(\phi_2)^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_2.(Y_\nu)^* & (Y_e)^*.(\phi_1)^* & 0 & 0 \end{pmatrix}$$

From (5.6) $\phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 \end{pmatrix}$

Condense into space $\mathcal{H}_{F_2} \to \mathcal{H}_l[\mathbb{C}^4] \oplus \mathcal{H}_{\bar{l}}[\mathbb{C}^4]$

$$\to \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_\nu.(\phi_1)^* & Y_\nu.(\phi_2)^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y_e.\phi_2 & Y_e.\phi_1 \\ 0 & 0 & 0 & 0 & \phi_1.(Y_\nu)^* & -(Y_e)^*.(\phi_2)^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_2.(Y_\nu)^* & (Y_e)^*.(\phi_1)^* & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & \phi^* \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & (Y_\nu)^* \phi_1 & (Y_\nu)^* \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(Y_e)^* (\phi_2)^* & (Y_e)^* (\phi_1)^* & 0 & 0 & 0 & 0 \\ (\phi_1)^* Y_\nu & -Y_e \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\phi_2)^* Y_\nu & Y_e \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

→ $\Phi \to D_{F_2} + \{\{\phi, 0\}, \{0, 0\}\} + \{\{0, 0\}, \{0, \phi^*\}\}$ ⟵CHECK

From $D_{F_2} \to \begin{pmatrix} S & T^* \\ T & S^* \end{pmatrix}$

→ $\boxed{\Phi \to \begin{pmatrix} S + \phi & T^* \\ T & S^* + \phi^* \end{pmatrix}}$

Prop.5.3.

```
PR["●Prop.5.3. The action of the gauge group ",
  𝒢[M×F_GWS][iD_iA → slash[iD]⊗𝕀 + T[γ, "u", {μ}]⊗T[B, "d", {μ}] + T[γ, "d", {5}]⊗Φ],
  NL, "is given by: ",
  $ = {T[Λ, "d", {μ}] -> T[Λ, "d", {μ}] - I λ.tuDPartial[Conjugate[λ], μ],
    T[Q, "d", {μ}] → q.T[Q, "d", {μ}].ct[q] - I q.tuDPartial[ct[q], μ],
    {{ϕ1}, {ϕ2}} → Conjugate[λ].q.{{ϕ1}, {ϕ2}} + (Conjugate[λ].q - 1).{{1}, {0}},
    λ ∈ C^"∞"[M, U[1]], q ∈ C^"∞"[M, SU[2]]
   }; MatrixForms[$e221a = $] // ColumnBar,

  line,
  NL, "■Proof: For the fields (5.7) compute the transformations (2.21): ",
  $e221 = $ = {T[it[A], "d", {μ}] → u.T[it[A], "d", {μ}].ct[u] - I u.tuDPartial[ct[u], μ],
     ϕ -> u.ϕ.ct[u] + u.CommutatorM[iD_F, ct[u]],
     {u → {λ, q}} ∈ C^"∞"[M, U[1]×SU[2]][CG["gauge transformation"]]
    }; $ // ColumnBar, accumGWS[{$e221a, $e221}], accumGWS[{$e221, $e221a}],

  NL, "The 8x8 representation: ",
  $a88 = $ = selectGWS[T[iA, "d", {μ}], {q}]; $ // MatrixForms,
  NL, "and(from ", $a88[[1]], "): ",
  $u = u -> $a88[[2]] /. {T[Λ, "d", {μ}] -> λ, q → uq}; MatrixForms[$u],

  NL, "It is easy to see that ", $0 = u. $a88[[1]] . ct[u],
  imply, T[Q, "d", {μ}] → u.T[Q, "d", {μ}].ct[u],
  " since the block diagonal elements are independant.", "POFF",
  Yield, $1 = $ = $0 -> $u [[2]].$a88[[2]].ct[$u[[2]]];
  MatrixForms[$], "PON",

  NL, "Similarly for ", $0 = $ = I u.tuDPartial[ct[u], μ], "POFF",
  Yield,
  $ = $0 → ($ /. $u //. tt : tuDDown["∂"][_, _] ⧴ Thread[tt] /. tuDDown["∂"][0, _] → 0);
  MatrixForms[$], CK, "PON",
  Imply,
  {$e221a[[2]][CG["over the q's"]], $e221a[[1]][CG["over the λ's"]]} // ColumnBar
 ];
```

●Prop.5.3. The action of the gauge group $\mathcal{G}[M\times F_{GWS}][D_A \to (\not{D})\otimes\mathbb{I} + \gamma_5\otimes\Phi + \gamma^\mu\otimes B_\mu]$

is given by: $\left|\begin{array}{l} \Lambda_\mu \to -\mathbb{i}\,\lambda.\underline{\partial}_\mu[\lambda^*] + \Lambda_\mu \\ Q_\mu \to -\mathbb{i}\,q.\underline{\partial}_\mu[q^\dagger] + q.Q_\mu.q^\dagger \\ \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to (-1+\lambda^*.q).\begin{pmatrix}1\\0\end{pmatrix} + \lambda^*.q.\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \\ \lambda \in C^\infty[M, U[1]] \\ q \in C^\infty[M, SU[2]] \end{array}\right.$

---

■Proof: For the fields (5.7) compute the transformations (2.21):

$\left|\begin{array}{l} A_\mu \to -\mathbb{i}\,u.\underline{\partial}_\mu[u^\dagger] + u.A_\mu.u^\dagger \\ \phi \to u.[D_F, u^\dagger]_- + u.\phi.u^\dagger \\ \{u \to \{\lambda, q\}\} \in C^\infty[M, U[1]\times SU[2]][\text{gauge transformation}] \end{array}\right.$

The 8x8 representation: $A_\mu \to \begin{pmatrix} \Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Lambda_\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 1,1} & q_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{\mu 2,1} & q_{\mu 2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_\mu \end{pmatrix}$

and(from $A_\mu$): $u \to \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & uq_{\mu 1,1} & uq_{\mu 1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & uq_{\mu 2,1} & uq_{\mu 2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$

It is easy to see that $u.A_\mu.u^\dagger \Rightarrow Q_\mu \to u.Q_\mu.u^\dagger$
since the block diagonal elements are independant.
Similarly for $\mathbb{i}\,u.\underline{\partial}_\mu[u^\dagger]$

$\Rightarrow \left|\begin{array}{l} (Q_\mu \to -\mathbb{i}\,q.\underline{\partial}_\mu[q^\dagger] + q.Q_\mu.q^\dagger)[\text{over the q's}] \\ (\Lambda_\mu \to -\mathbb{i}\,\lambda.\underline{\partial}_\mu[\lambda^*] + \Lambda_\mu)[\text{over the }\lambda\text{'s}] \end{array}\right.$

```
PR["Check Higg's field gauge transformation ", $ = $e221[[2]],
 NL, "Use 8×8 representation of ",
 $u = selectGWS[T[iA, "d", {μ}], {q}] /. T[iA, "d", {μ}] → u;
 $d = selectGWS[iD_F8, {}] /. iD_F8 -> iD_F;
 $ϕ = ϕ0[CG["Transformed ϕ"]] -> $sϕ[[2]],
 $ = $ /. ϕ → ϕ0,
 $ = $ // expandCom[tuRule[{$u, $d, $ϕ}]];
 $ = $ /. Dot → Times;
 $[[1]] = $[[1]] /. ϕ → ϕ0;
 $ // MatrixForms;
 next, "Collect terms element by element: ",
 $ = Thread[Flatten /@ $] // DeleteDuplicates // Rest // Simplify;
 $0 = $ = $ /. rr : (aa : a_ (yy : cc[Y_] | Y_) -> b_) :> (1 / yy # & /@ rr) // Simplify // Sort;
 $ // ColumnBar
]
PR["• Examine interrelationships of these terms. ",
 $2 = $ // tuExtractPattern[(aa__: 1) (ϕ0₂ | cc[ϕ0₂]) → _];
 $2 = MapAt[cc[#] & /@ # &, $2, {{1}, {2}}];
 $2 // ColumnBar;
 $2a = tuRuleAdd[$2[[{1, 4}]]] // Simplify;
 $2a = Collect[$2a, {ϕ₂, 1 + ϕ₁}];
 $2b = tuRuleAdd[$2[[{2, 3}]]] // Simplify;
 $2b = Collect[$2b, {ϕ₂, 1 + ϕ₁}];
```

```
  $1 = $ // tuExtractPattern[(aa__ : 1) (ϕ0₁ | cc[ϕ0₁]) → _];
  $1 = MapAt[cc[#] & /@ # &, $1, {{1}, {2}}];
  $1 // ColumnBar;
  $1a = tuRuleSubtract[$1[[{2, 4}]]] // Simplify;
  $1a = Collect[$1a, {ϕ₂, 1 + ϕ₁}];
  $1b = tuRuleSubtract[$1[[{1, 3}]]] // Simplify;
  $1b = Collect[$1b, {ϕ₂, 1 + ϕ₁}];
  NL, "•Assuming that ϕ's are arbitrary so their coefficients are 0: ",
  $ = {Collect[$2a[[2]], {ϕ₁ + 1, ϕ₂}, CCC],
    Collect[$2b[[2]], {ϕ₁ + 1, ϕ₂}, CCC],
    Collect[$1a[[2]], {ϕ₁ + 1, ϕ₂}, CCC],
    Collect[$1b[[2]], {ϕ₁ + 1, ϕ₂}, CCC]};
  $ = tuExtractPattern[CCC[_]][$] //. CCC[a__] → a // DeleteDuplicates;
  $ // ColumnBar;
  $[[4]] = -1 cc[($[[4]])];
  $[[7]] = cc[($[[7]])];
  $ = $ // DeleteDuplicates;
  $ // ColumnBar;
  $s = Select[$, tuHasAllQ[#, {T[q, "d", {μ}]₁,₂, T[q, "d", {μ}]₂,₁}] &&
      tuHasNoneQ[#, {T[q, "d", {μ}]₁,₁, T[q, "d", {μ}]₂,₂}] &];
  $s // ColumnBar;
  $s = Thread[$s → $s];
  $s[[2, 2]] = cc[$s[[2, 2]]];
  $s[[3, 2]] = -cc[$s[[3, 2]]];
  $s // ColumnBar;
  $ = $ /. $s;
  $ = $ // DeleteDuplicates;
  $ // ColumnBar,
  $ = # → 0 & /@ $;
  Yield, $s = $ // tuExtractPattern[Tensor[_, _, _]] // DeleteDuplicates;
  $s = $s /. qq : Tensor[q, _, _] :→ Table[qq_i,j, {i, 2}, {j, 2}] // Flatten;
  $s0 = $s = tuRuleSolve[$, $s[[2 ;; -1]]]; $s // Framed,
  NL, $ = $ /. $s, CG[" 0's→OK"],

  Imply, q[CG["Hermitian, traceless"]] // Framed
 ]
PR["Inserting this q relationships into the original
   set of transform equations and selecting the 2 simplest: ",
  $ = $0 /. $s // Expand;
  $ = tuRuleSelect[$][{ϕ0₁, ϕ0₂}] // Rest,
  next, "Assume unitarity of Λ's: ", $s = cc[T[Λ, "d", {μ}]] T[Λ, "d", {μ}] → 1,
  Yield,
  $ = $ /. $s,
  $[[1]] = (1 + #) & /@ $[[1]];
  $ = $ // Simplify;
  (*
  $s=tuRuleSolve[$s0,{Tensor[q,_,_]₂,₁,Tensor[q,_,_]₂,₂}]*)
  $[[2]] = $[[2]] /. $s0 // Expand // Simplify;
  $ // ColumnBar,
  next, "This can be put into the matrix form: ",
  $ = {{$[[1, 1]]}, {$[[2, 1]]}} →
    T[Λ, "d", {μ}] cc[q].({{$[[1, 1]]}, {$[[2, 1]]}} /. ϕ0 → ϕ);
  $ // MatrixForms // Framed
 ]
```

Check Higg's field gauge transformation $\phi \to u.[D_F, u^\dagger]_- + u.\phi.u^\dagger$

Use 8×8 representation of $\phi0$[Transformed $\phi$] $\to \{\{0, 0, (Y_\nu)^* \phi_1, (Y_\nu)^* \phi_2, 0, 0, 0, 0\}, \{0, 0, -(Y_e)^* (\phi_2)^*, (Y_e)^* (\phi_1)^*, 0, 0, 0, 0\}, \{(\phi_1)^* Y_\nu, -Y_e \phi_2, 0, 0, 0, 0, 0, 0\}, \{(\phi_2)^* Y_\nu, Y_e \phi_1, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0\}\}\phi0 \to u.[D_F, u^\dagger]_- + u.\phi0.u^\dagger$

◆Collect terms element by element:

$$(\phi0_1)^* \to -(q_{\mu 1,1})^* q_{\mu 1,1} - (q_{\mu 1,2})^* q_{\mu 1,2} + (\Lambda_\mu)^* ((1 + (\phi_1)^*) q_{\mu 1,1} + (\phi_2)^* q_{\mu 1,2})$$
$$(\phi0_1)^* \to ((\phi_2)^* (q_{\mu 2,1})^* - (1 + (\phi_1)^*) (q_{\mu 2,2})^* - (\Lambda_\mu)^*) \Lambda_\mu$$
$$-(\phi0_2)^* \to ((\phi_2)^* (q_{\mu 1,1})^* - (1 + (\phi_1)^*) (q_{\mu 1,2})^*) \Lambda_\mu$$
$$(\phi0_2)^* \to -(q_{\mu 1,1})^* q_{\mu 2,1} - (q_{\mu 1,2})^* q_{\mu 2,2} + (\Lambda_\mu)^* ((1 + (\phi_1)^*) q_{\mu 2,1} + (\phi_2)^* q_{\mu 2,2})$$
$$\phi0_1 \to -(q_{\mu 2,1})^* q_{\mu 2,1} - (q_{\mu 2,2})^* q_{\mu 2,2} + (\Lambda_\mu)^* (\phi_2 q_{\mu 2,1} - (1 + \phi_1) q_{\mu 2,2})$$
$$\phi0_1 \to (-(\Lambda_\mu)^* + (q_{\mu 1,1})^* (1 + \phi_1) + (q_{\mu 1,2})^* \phi_2) \Lambda_\mu$$
$$-\phi0_2 \to -(q_{\mu 2,1})^* q_{\mu 1,1} - (q_{\mu 2,2})^* q_{\mu 1,2} + (\Lambda_\mu)^* (\phi_2 q_{\mu 1,1} - (1 + \phi_1) q_{\mu 1,2})$$
$$\phi0_2 \to ((q_{\mu 2,1})^* (1 + \phi_1) + (q_{\mu 2,2})^* \phi_2) \Lambda_\mu$$

• Examine interrelationships of these terms.

•Assuming that $\phi$'s are arbitrary so their coefficients are 0:

$$-(\Lambda_\mu)^* q_{\mu 1,2} + (q_{\mu 2,1})^* \Lambda_\mu$$
$$(\Lambda_\mu)^* q_{\mu 1,1} + (q_{\mu 2,2})^* \Lambda_\mu$$
$$-2 (q_{\mu 2,1})^* q_{\mu 1,1} - 2 (q_{\mu 2,2})^* q_{\mu 1,2}$$
$$-(q_{\mu 1,1})^* q_{\mu 1,1} - (q_{\mu 1,2})^* q_{\mu 1,2} + (q_{\mu 2,1})^* q_{\mu 2,1} + (q_{\mu 2,2})^* q_{\mu 2,2}$$

$$\to \boxed{\left\{q_{\mu 1,1} \to -\frac{(q_{\mu 2,2})^* \Lambda_\mu}{(\Lambda_\mu)^*},\ q_{\mu 1,2} \to \frac{(q_{\mu 2,1})^* \Lambda_\mu}{(\Lambda_\mu)^*}\right\}}$$

$\{0 \to 0, 0 \to 0, 0 \to 0, 0 \to 0\}$ 0's→OK

⇒ $\boxed{\text{q[Hermitian, traceless]}}$

Inserting this q relationships into the original set of transform equations and selecting the 2 simplest:

$$\{\phi0_1 \to (\Lambda_\mu)^* \phi_2 q_{\mu 2,1} - (\Lambda_\mu)^* q_{\mu 2,2} - (\Lambda_\mu)^* \phi_1 q_{\mu 2,2} - (\Lambda_\mu)^* \Lambda_\mu, \phi0_2 \to (q_{\mu 2,1})^* \Lambda_\mu + (q_{\mu 2,1})^* \phi_1 \Lambda_\mu + (q_{\mu 2,2})^* \phi_2 \Lambda_\mu\}$$

◆Assume unitarity of $\Lambda$'s: $(\Lambda_\mu)^* \Lambda_\mu \to 1$

$$\to \{\phi0_1 \to -1 + (\Lambda_\mu)^* \phi_2 q_{\mu 2,1} - (\Lambda_\mu)^* q_{\mu 2,2} - (\Lambda_\mu)^* \phi_1 q_{\mu 2,2}, \phi0_2 \to (q_{\mu 2,1})^* \Lambda_\mu + (q_{\mu 2,1})^* \phi_1 \Lambda_\mu + (q_{\mu 2,2})^* \phi_2 \Lambda_\mu\}$$

$$1 + \phi0_1 \to (\Lambda_\mu)^* (\phi_2 q_{\mu 2,1} - (1 + \phi_1) q_{\mu 2,2})$$
$$\phi0_2 \to ((q_{\mu 2,1})^* (1 + \phi_1) + (q_{\mu 2,2})^* \phi_2) \Lambda_\mu$$

◆This can be put into the matrix form: $\boxed{\begin{pmatrix} 1 + \phi0_1 \\ \phi0_2 \end{pmatrix} \to q^* . \begin{pmatrix} 1 + \phi_1 \\ \phi_2 \end{pmatrix} \Lambda_\mu}$

## 5.3 Spectral Action(bosonic part of the Lagrangian)

```
$p37;
$e57;
$e58;
$F;
PR["● Lemma 5.4: ",
  $l54 = $ = {Tr[T[F, "uu", {μ, ν}] T[F, "dd", {μ, ν}]] → 12 T[Λ, "dd", {μ, ν}]
        T[Λ, "uu", {μ, ν}] + 2 Tr[T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]],
     T[Λ, "dd", {μ, ν}] → tuDPartial[T[Λ, "d", {ν}], μ] - tuDPartial[T[Λ, "d", {μ}], ν],
     T[Q, "dd", {μ, ν}] → tuDPartial[T[Q, "d", {ν}], μ] -
       tuDPartial[T[Q, "d", {μ}], ν] + I CommutatorM[T[Q, "d", {μ}], T[Q, "d", {ν}]]
    }; FramedColumn[$], accumGWS[$l54]
 ];
```

● Lemma 5.4:

$\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 12\, \Lambda_{\mu\nu} \Lambda^{\mu\nu} + 2\, \mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}]$

$\Lambda_{\mu\nu} \to -\underline{\partial}_{\nu}[\Lambda_{\mu}] + \underline{\partial}_{\mu}[\Lambda_{\nu}]$

$Q_{\mu\nu} \to i\, [Q_{\mu}, Q_{\nu}]_{-} - \underline{\partial}_{\nu}[Q_{\mu}] + \underline{\partial}_{\mu}[Q_{\nu}]$

```
PR["■ Proof:  From the definition: ",
 $ = tuRuleSelect[$defall][T[F, "dd", {μ, ν}]][[1]],
 NL, "Use above ",
 $sb = tuRuleSelect[$defGWS][T[B, "d", {μ}]][[-1]] // tuAddPatternVariable[μ],
 Yield, $ = $ /. tuCommutatorExpand /. Plus → Inactive[Plus] /. $sb //.
     tuOpDistribute[tuDDown["∂"], List] // tuDerivativeExpand[] // Activate;
 $ // MatrixForms,
 ,
 NL, "q's are hermitian and tracelist: ",
 $sq = {Conjugate[(qq : T[q, "d", {μ_}])i_,j_] :> qqj,i,
   Conjugate[(qq : T[q, "u", {μ_}])i_,j_] :> qqj,i, Conjugate[qq : q_i_,i_] → qi,i,
   T[q, "d", {μ_}]1,1 + T[q, "d", {μ_}]2,2 → 0,
   T[q, "u", {μ_}]1,1 + T[q, "u", {μ_}]2,2 → 0
  };

 $sq // ColumnBar,
 Yield, $ = $ //. $sq // tuConjugateSimplify[{μ, ν}]; $ // MatrixForms,
 NL, "Tr[] of Product: ", $u = $ // tuIndicesRaise[{μ, ν}];
 $ = Thread[Dot[$ , $u], Rule]; $ // MatrixForms;
 Yield, $trff = $ = Tr /@ $ //. $sq // Expand;
 NL, "Common index substitutions: ",
 $sqsub = {aa : a_ b__ :→ tuIndexSwapUpDown[{μ}][aa] /; ! FreeQ[aa, T[q, "d", {μ}]],
   aa : a_ b__ :→ tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, T[q, "d", {ν}]],
   aa : a_ b__ :→ tuIndexSwap[{ν, μ}][aa] /; ! FreeQ[aa, T[q, "u", {ν}]], aa : a_ b__ :→
    tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, tuDDown["∂"][T[q, "u", {i_}]_,_, ν]],
   aa : a_ tuDDown["∂"][T[q, "u", {i_}]_,_, ν_] :→ tuIndexSwapUpDown[{i}][aa],
   aa : a_ tuDUp["∂"][T[q, "u", {i_}]_,_, ν_] :→ tuIndexSwapUpDown[{i, ν}][aa],
   aa : a__ tuDUp["∂"][T[q, "d", {i_}]_,_, ν_] :→ tuIndexSwapUpDown[{ν}][aa],
   aa : a_ T[q, "d", {μ_}]i_,i_ :> tuIndexSwapUpDown[{μ}][aa],
   aa : a_ T[q, "u", {ν}]i_,i_ :> tuIndexSwap[{μ, ν}][aa]
  },

 next, "The ΛΛ terms: ", $ll = (Apply[Plus, ($trff // tuTermSelect[Λ, q])] // Simplify);
 $ll // Framed,

 next, "The Λq terms: ", $lq = $ = Apply[Plus, ($trff // tuTermSelect[{Λ, q}])];
```

```
yield, $lq = $ = $ //. $sqsub[[1 ;; 4]] /. $sqsub // Simplify //
     (# //. tuOpCollect[tuDDown["∂"]] /. $sq &) // tuDerivativeExpand[];
$ // Framed,
(**)
next, "The qq terms: ", $qq = $ = Apply[Plus, ($trff // tuTermSelect[q, Λ])];
yield, $qq = $ = $ /. $sqsub // Simplify; $ // Framed,
NL, "Too many terms to find text
  relationship directly. Compare with direct computation of ",

$ = tuRuleSelect[$defGWS][T[Q, "dd", {_, _}]][[1]];
$s = $ // tuIndicesRaise[{μ, ν}];
Yield, $ = $ . $s // Thread[#, Rule] &, "POFF",
Yield, $ = $ /. tuCommutatorExpand // expandDC[],
NL, "Use ",
$s = tuRuleSelect[$defGWS][T[Q, "d", {_}]] // Select[#, tuHasAnyQ[#, {2}] &] & // First,
$s = {$s, $s // tuIndicesRaise[μ]} // tuAddPatternVariable[μ];
Yield,
$ = $ /. toxDot //. $s //. tt : (tuDDown["∂"][_, _] | tuDUp["∂"][_, _]) :→ Thread[tt] //
    tuMatrixOrderedMultiply // (# /. xDot → Times &);
"PONdd",
Yield, $qq0 = $ = Tr /@ $ // Simplify,
NL, "Comparing ", 2 $qq0[[1]], " with FF calculation ", imply,
2 $qq0[[2]] == $qq // tuIndicesLower[{μ, ν}] // Simplify // Framed,
CG[" QED"]
]
```

■ Proof: From the definition: $F_{\mu\nu} \to i\,[B_\mu, B_\nu]_- - \underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu]$

Use above

$B_{\mu\_} \to \{\{0,0,0,0,0,0,0,0\}, \{0,-2\Lambda_\mu,0,0,0,0,0,0\}, \{0,0,q_{\mu 1,1}-\Lambda_\mu, q_{\mu 1,2},0,0,0,0\}, \{0,0,q_{\mu 2,1}, q_{\mu 2,2}-\Lambda_\mu,0,0,0,0\}, \{0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,2\Lambda_\mu,0,0\}, \{0,0,0,0,0,0,-(q_{\mu 1,1})^*+\Lambda_\mu, -(q_{\mu 1,2})^*\}, \{0,0,0,0,0,0,-(q_{\mu 2,1})^*, -(q_{\mu 2,2})^*+\Lambda_\mu\}\}$

$$\to F_{\mu\nu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\underline{\partial}_\nu[\Lambda_\mu] - 2\underline{\partial}_\mu[\Lambda_\nu] & 0 \\ 0 & 0 & i(-q_{\mu 2,1}q_{\nu 1,2} + q_{\mu 1,2}q_{\nu 2,1}) - \underline{\partial}_\nu[q_{\mu 1,1}] + \underline{\partial}_\mu[q_{\nu 1,1}] + \underline{\partial}_\nu[\Lambda_\mu] \\ 0 & 0 & i(-q_{\nu 2,1}(q_{\mu 1,1}-\Lambda_\mu) + q_{\nu 2,1}(q_{\mu 2,2}-\Lambda_\mu) + q_{\mu 2,1}(q_{\nu 1,1}-\Lambda_\nu) - q_{\mu 2,1}(q_{\nu 2,2}-\Lambda_\nu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

q's are hermitian and tracelist: 
$$\begin{array}{|l} (qq : q_{\mu\_i\_,j\_})^* :\to qq_{j,i} \\ (qq : q^{\mu\_}{}_{i\_,j\_})^* :\to qq_{j,i} \\ (qq : q_{\_i\_,i\_})^* \to q_{i,i} \\ q_{\mu\_1,1} + q_{\mu\_2,2} \to 0 \\ q^{\mu\_}{}_{1,1} + q^{\mu\_}{}_{2,2} \to 0 \end{array}$$

$$\to F_{\mu\nu} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\underline{\partial}_\nu[\Lambda_\mu] - 2\underline{\partial}_\mu[\Lambda_\nu] & 0 \\ 0 & 0 & -i\,q_{\mu 2,1}q_{\nu 1,2} + i\,q_{\mu 1,2}q_{\nu 2,1} - \underline{\partial}_\nu[q_{\mu 1,1}] + \underline{\partial}_\mu[q_{\nu 1,1}] + \underline{\partial}_\nu[\Lambda_\mu] - \underline{\partial}_\mu[\Lambda_\nu] \\ 0 & 0 & i\,q_{\mu 2,1}q_{\nu 1,1} - i\,q_{\mu 1,1}q_{\nu 2,1} + i\,q_{\mu 2,2}q_{\nu 2,1} - i\,q_{\mu 2,1}q_{\nu 2,2} - \underline{\partial}_\nu[q_{\mu 2,1}] + \underline{\partial}_\mu[q_{\nu 2,1}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Tr[] of Product:

→

Common index substitutions:

{aa : a_b__ :→ tuIndexSwapUpDown[{μ}][aa] /; ! FreeQ[aa, T[q, d, {μ}]],

```
aa : a_ b__ :→ tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, T[q, d, {ν}]],
aa : a_ b__ :→ tuIndexSwap[{ν, μ}][aa] /; ! FreeQ[aa, T[q, u, {ν}]],
aa : a_ b__ :→ tuIndexSwapUpDown[{ν}][aa] /; ! FreeQ[aa, ∂_ν[q^i-_,_]],
aa : a_ ∂_ν_[q^i-_,_] :→ tuIndexSwapUpDown[{i}][aa],
aa : a_ ∂^ν-[q^i-_,_] :→ tuIndexSwapUpDown[{i, ν}][aa],
aa : a__ ∂^ν-[q_i_,_] :→ tuIndexSwapUpDown[{ν}][aa],
aa : a_ q_μ_i_,i_ :→ tuIndexSwapUpDown[{μ}][aa], aa : a_ q^ν_i_,i_ :→ tuIndexSwap[{μ, ν}][aa]}
```

◆The ΛΛ terms: $\boxed{12\,(\underline{\partial}_\nu[\Lambda_\mu] - \underline{\partial}_\mu[\Lambda_\nu])\,(\underline{\partial}^\nu[\Lambda^\mu] - \underline{\partial}^\mu[\Lambda^\nu])}$

◆The Λq terms: ⟶ $\boxed{0}$

◆The qq terms: ⟶

$$\boxed{\begin{aligned}
-2\,(&-q_{\mu 1,1}\, q_{\nu 2,1}\, q^\mu{}_{1,1}\, q^\nu{}_{1,2} + q_{\mu 2,2}\, q_{\nu 2,1}\, q^\mu{}_{1,1}\, q^\nu{}_{1,2} + q_{\mu 1,1}\, q_{\nu 1,1}\, q^\mu{}_{2,1}\, q^\nu{}_{1,2} -\\
&q_{\mu 2,2}\, q_{\nu 1,1}\, q^\mu{}_{2,1}\, q^\nu{}_{1,2} - q_{\mu 1,1}\, q_{\nu 2,2}\, q^\mu{}_{2,1}\, q^\nu{}_{1,2} + q_{\mu 2,2}\, q_{\nu 2,2}\, q^\mu{}_{2,1}\, q^\nu{}_{1,2} + q_{\mu 1,1}\, q_{\nu 2,1}\, q^\mu{}_{2,2}\, q^\nu{}_{1,2} -\\
&q_{\mu 2,2}\, q_{\nu 2,1}\, q^\mu{}_{2,2}\, q^\nu{}_{1,2} - q_{\mu 1,1}\, q_{\nu 1,2}\, q^\mu{}_{1,1}\, q^\nu{}_{2,1} + q_{\mu 2,2}\, q_{\nu 1,2}\, q^\mu{}_{1,1}\, q^\nu{}_{2,1} + q_{\mu 1,1}\, q_{\nu 1,1}\, q^\mu{}_{1,2}\, q^\nu{}_{2,1} -\\
&q_{\mu 2,2}\, q_{\nu 1,1}\, q^\mu{}_{1,2}\, q^\nu{}_{2,1} - q_{\mu 1,1}\, q_{\nu 2,2}\, q^\mu{}_{1,2}\, q^\nu{}_{2,1} + q_{\mu 2,2}\, q_{\nu 2,2}\, q^\mu{}_{1,2}\, q^\nu{}_{2,1} + q_{\mu 1,1}\, q_{\nu 1,2}\, q^\mu{}_{2,2}\, q^\nu{}_{2,1} -\\
&q_{\mu 2,2}\, q_{\nu 1,2}\, q^\mu{}_{2,2}\, q^\nu{}_{2,1} - i\, q_{\mu 1,1}\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q^\mu{}_{1,2}] + i\, q_{\mu 2,2}\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q^\mu{}_{1,2}] + i\, q_{\mu 1,1}\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q^\mu{}_{2,1}] -\\
&i\, q_{\mu 2,2}\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q^\mu{}_{2,1}] + q_{\mu 2,1}\, (q_{\nu 2,2}\, q^\mu{}_{1,2}\, q^\nu{}_{1,1} - 2\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, q^\nu{}_{1,2} - q_{\nu 2,2}\, q^\mu{}_{1,2}\, q^\nu{}_{2,2} +\\
&\quad q_{\nu 1,1}\, q^\mu{}_{1,2}\, (-q^\nu{}_{1,1} + q^\nu{}_{2,2}) + q_{\nu 1,2}\, (2\, q^\mu{}_{2,1}\, q^\nu{}_{1,2} + q^\mu{}_{1,1}\, (q^\nu{}_{1,1} - q^\nu{}_{2,2}) + q^\mu{}_{2,2}\, (-q^\nu{}_{1,1} + q^\nu{}_{2,2})) -\\
&\quad i\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q^\mu{}_{1,1}] + i\, q^\nu{}_{1,1}\, \underline{\partial}_\nu[q^\mu{}_{1,2}] - i\, q^\nu{}_{2,2}\, \underline{\partial}_\nu[q^\mu{}_{1,2}] + i\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q^\mu{}_{2,2}]) +\\
&q_{\mu 1,2}\, (q_{\nu 2,2}\, q^\mu{}_{2,1}\, q^\nu{}_{1,1} - 2\, q_{\nu 1,2}\, q^\mu{}_{2,1}\, q^\nu{}_{2,1} - q_{\nu 2,2}\, q^\mu{}_{2,1}\, q^\nu{}_{2,2} + q_{\nu 1,1}\, q^\mu{}_{2,1}\, (-q^\nu{}_{1,1} + q^\nu{}_{2,2}) +\\
&\quad q_{\nu 2,1}\, (2\, q^\mu{}_{1,2}\, q^\nu{}_{2,1} + q^\mu{}_{1,1}\, (q^\nu{}_{1,1} - q^\nu{}_{2,2}) + q^\mu{}_{2,2}\, (-q^\nu{}_{1,1} + q^\nu{}_{2,2})) + i\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q^\mu{}_{1,1}] -\\
&\quad i\, q^\nu{}_{1,1}\, \underline{\partial}_\nu[q^\mu{}_{2,1}] + i\, q^\nu{}_{2,2}\, \underline{\partial}_\nu[q^\mu{}_{2,1}] - i\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q^\mu{}_{2,2}]) - 2\, i\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, \underline{\partial}_\mu[q^\nu{}_{1,1}] +\\
&2\, i\, q_{\nu 1,2}\, q^\mu{}_{2,1}\, \underline{\partial}_\mu[q^\nu{}_{1,1}] + 2\, \underline{\partial}_\nu[q^\mu{}_{1,1}]\, \underline{\partial}_\mu[q^\nu{}_{1,1}] + 2\, i\, q_{\nu 2,1}\, q^\mu{}_{1,1}\, \underline{\partial}_\mu[q^\nu{}_{1,2}] -\\
&2\, i\, q_{\nu 1,1}\, q^\mu{}_{2,1}\, \underline{\partial}_\mu[q^\nu{}_{1,2}] + 2\, i\, q_{\nu 2,2}\, q^\mu{}_{2,1}\, \underline{\partial}_\mu[q^\nu{}_{1,2}] - 2\, i\, q_{\nu 2,1}\, q^\mu{}_{2,2}\, \underline{\partial}_\mu[q^\nu{}_{1,2}] +\\
&2\, \underline{\partial}_\nu[q^\mu{}_{2,1}]\, \underline{\partial}_\mu[q^\nu{}_{1,2}] - 2\, i\, q_{\nu 1,2}\, q^\mu{}_{1,1}\, \underline{\partial}_\mu[q^\nu{}_{2,1}] + 2\, i\, q_{\nu 1,1}\, q^\mu{}_{1,2}\, \underline{\partial}_\mu[q^\nu{}_{2,1}] -\\
&2\, i\, q_{\nu 2,2}\, q^\mu{}_{1,2}\, \underline{\partial}_\mu[q^\nu{}_{2,1}] + 2\, i\, q_{\nu 1,2}\, q^\mu{}_{2,2}\, \underline{\partial}_\mu[q^\nu{}_{2,1}] + 2\, \underline{\partial}_\nu[q^\mu{}_{1,2}]\, \underline{\partial}_\mu[q^\nu{}_{2,1}] +\\
&2\, i\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, \underline{\partial}_\mu[q^\nu{}_{2,2}] - 2\, i\, q_{\nu 1,2}\, q^\mu{}_{2,1}\, \underline{\partial}_\mu[q^\nu{}_{2,2}] + 2\, \underline{\partial}_\nu[q^\mu{}_{2,2}]\, \underline{\partial}_\mu[q^\nu{}_{2,2}] +\\
&i\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, \underline{\partial}^\nu[q_{\mu 1,1}] - i\, q_{\nu 1,2}\, q^\mu{}_{2,1}\, \underline{\partial}^\nu[q_{\mu 1,1}] - \underline{\partial}_\nu[q^\mu{}_{1,1}]\, \underline{\partial}^\nu[q_{\mu 1,1}] -\\
&i\, q_{\nu 2,1}\, q^\mu{}_{1,1}\, \underline{\partial}^\nu[q_{\mu 1,2}] + i\, q_{\nu 1,1}\, q^\mu{}_{2,1}\, \underline{\partial}^\nu[q_{\mu 1,2}] - i\, q_{\nu 2,2}\, q^\mu{}_{2,1}\, \underline{\partial}^\nu[q_{\mu 1,2}] +\\
&i\, q_{\nu 2,1}\, q^\mu{}_{2,2}\, \underline{\partial}^\nu[q_{\mu 1,2}] - \underline{\partial}_\nu[q^\mu{}_{2,1}]\, \underline{\partial}^\nu[q_{\mu 1,2}] + i\, q_{\nu 1,2}\, q^\mu{}_{1,1}\, \underline{\partial}^\nu[q_{\mu 2,1}] -\\
&i\, q_{\nu 1,1}\, q^\mu{}_{1,2}\, \underline{\partial}^\nu[q_{\mu 2,1}] + i\, q_{\nu 2,2}\, q^\mu{}_{1,2}\, \underline{\partial}^\nu[q_{\mu 2,1}] - i\, q_{\nu 1,2}\, q^\mu{}_{2,2}\, \underline{\partial}^\nu[q_{\mu 2,1}] -\\
&\underline{\partial}_\nu[q^\mu{}_{1,2}]\, \underline{\partial}^\nu[q_{\mu 2,1}] - i\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, \underline{\partial}^\nu[q_{\mu 2,2}] + i\, q_{\nu 1,2}\, q^\mu{}_{2,1}\, \underline{\partial}^\nu[q_{\mu 2,2}] - \underline{\partial}_\nu[q^\mu{}_{2,2}]\, \underline{\partial}^\nu[q_{\mu 2,2}] -\\
&\underline{\partial}_\mu[q^\nu{}_{1,1}]\, \underline{\partial}^\mu[q_{\nu 1,1}] - \underline{\partial}_\mu[q^\nu{}_{2,1}]\, \underline{\partial}^\mu[q_{\nu 1,2}] - \underline{\partial}_\mu[q^\nu{}_{1,2}]\, \underline{\partial}^\mu[q_{\nu 2,1}] - \underline{\partial}_\mu[q^\nu{}_{2,2}]\, \underline{\partial}^\mu[q_{\nu 2,2}])
\end{aligned}}$$

Too many terms to find text relationship directly. Compare with direct computation of

→ $Q_{\mu\nu}\cdot Q^{\mu\nu} \to (i\,[Q_\mu, Q_\nu]_- - \underline{\partial}_\nu[Q_\mu] + \underline{\partial}_\mu[Q_\nu])\cdot(i\,[Q^\mu, Q^\nu]_- - \underline{\partial}^\nu[Q^\mu] + \underline{\partial}^\mu[Q^\nu])$

.......

→

$$\begin{aligned}
\mathrm{Tr}[Q_{\mu\nu}\cdot Q^{\mu\nu}] \to\ & q_{\mu 2,2}\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, q^\nu{}_{1,1} + q_{\mu 1,2}\, q_{\nu 1,1}\, q^\mu{}_{2,1}\, q^\nu{}_{1,1} + q_{\mu 2,2}\, q_{\nu 1,2}\, q^\mu{}_{2,1}\, q^\nu{}_{1,1} - q_{\mu 1,2}\, q_{\nu 2,2}\, q^\mu{}_{2,1}\, q^\nu{}_{1,1} -\\
&q_{\mu 2,2}\, q_{\nu 2,1}\, q^\mu{}_{1,1}\, q^\nu{}_{1,2} + 2\, q_{\mu 1,2}\, q_{\nu 2,1}\, q^\mu{}_{2,1}\, q^\nu{}_{1,2} + q_{\mu 2,2}\, q_{\nu 2,1}\, q^\mu{}_{2,2}\, q^\nu{}_{1,2} - q_{\mu 1,2}\, q_{\nu 1,1}\, q^\mu{}_{1,1}\, q^\nu{}_{2,1} -\\
&q_{\mu 2,2}\, q_{\nu 1,2}\, q^\mu{}_{1,1}\, q^\nu{}_{2,1} + q_{\mu 1,2}\, q_{\nu 2,2}\, q^\mu{}_{1,1}\, q^\nu{}_{2,1} - 2\, q_{\mu 1,2}\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, q^\nu{}_{2,1} + q_{\mu 1,2}\, q_{\nu 1,1}\, q^\mu{}_{2,2}\, q^\nu{}_{2,1} +\\
&q_{\mu 2,2}\, q_{\nu 1,2}\, q^\mu{}_{2,2}\, q^\nu{}_{2,1} - q_{\mu 1,2}\, q_{\nu 2,2}\, q^\mu{}_{2,2}\, q^\nu{}_{2,1} - q_{\mu 2,2}\, q_{\nu 2,1}\, q^\mu{}_{1,2}\, q^\nu{}_{2,2} - q_{\mu 1,2}\, q_{\nu 1,1}\, q^\mu{}_{2,1}\, q^\nu{}_{2,2} -\\
&q_{\mu 2,2}\, q_{\nu 1,2}\, q^\mu{}_{2,1}\, q^\nu{}_{2,2} + q_{\mu 1,2}\, q_{\nu 2,2}\, q^\mu{}_{2,1}\, q^\nu{}_{2,2} + i\, q^\mu{}_{2,1}\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q_{\mu 1,1}] - i\, q^\mu{}_{1,2}\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q_{\mu 1,1}] -\\
&i\, q^\mu{}_{2,1}\, q^\nu{}_{1,1}\, \underline{\partial}_\nu[q_{\mu 1,2}] + i\, q^\mu{}_{1,1}\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q_{\mu 1,2}] - i\, q^\mu{}_{2,2}\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q_{\mu 1,2}] +\\
&i\, q^\mu{}_{2,1}\, q^\nu{}_{2,2}\, \underline{\partial}_\nu[q_{\mu 1,2}] + i\, q^\mu{}_{1,2}\, q^\nu{}_{1,1}\, \underline{\partial}_\nu[q_{\mu 2,1}] - i\, q^\mu{}_{1,1}\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q_{\mu 2,1}] +\\
&i\, q^\mu{}_{2,2}\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q_{\mu 2,1}] - i\, q^\mu{}_{1,2}\, q^\nu{}_{2,2}\, \underline{\partial}_\nu[q_{\mu 2,1}] - i\, q^\mu{}_{2,1}\, q^\nu{}_{1,2}\, \underline{\partial}_\nu[q_{\mu 2,2}] + i\, q^\mu{}_{1,2}\, q^\nu{}_{2,1}\, \underline{\partial}_\nu[q_{\mu 2,2}] -\\
&i\, q^\mu{}_{2,1}\, q^\nu{}_{1,2}\, \underline{\partial}_\mu[q_{\nu 1,1}] + i\, q^\mu{}_{1,2}\, q^\nu{}_{2,1}\, \underline{\partial}_\mu[q_{\nu 1,1}] + i\, q^\mu{}_{2,1}\, q^\nu{}_{1,1}\, \underline{\partial}_\mu[q_{\nu 1,2}] - i\, q^\mu{}_{1,1}\, q^\nu{}_{2,1}\, \underline{\partial}_\mu[q_{\nu 1,2}] +\\
&i\, q^\mu{}_{2,2}\, q^\nu{}_{2,1}\, \underline{\partial}_\mu[q_{\nu 1,2}] - i\, q^\mu{}_{2,1}\, q^\nu{}_{2,2}\, \underline{\partial}_\mu[q_{\nu 1,2}] - i\, q^\mu{}_{1,2}\, q^\nu{}_{1,1}\, \underline{\partial}_\mu[q_{\nu 2,1}] + i\, q^\mu{}_{1,1}\, q^\nu{}_{1,2}\, \underline{\partial}_\mu[q_{\nu 2,1}] -\\
&i\, q^\mu{}_{2,2}\, q^\nu{}_{1,2}\, \underline{\partial}_\mu[q_{\nu 2,1}] + i\, q^\mu{}_{1,2}\, q^\nu{}_{2,2}\, \underline{\partial}_\mu[q_{\nu 2,1}] + i\, q^\mu{}_{2,1}\, q^\nu{}_{1,2}\, \underline{\partial}_\mu[q_{\nu 2,2}] - i\, q^\mu{}_{1,2}\, q^\nu{}_{2,1}\, \underline{\partial}_\mu[q_{\nu 2,2}] -\\
&i\, q_{\mu 1,2}\, q_{\nu 2,1}\, \underline{\partial}^\nu[q^\mu{}_{1,1}] + \underline{\partial}_\nu[q_{\mu 1,1}]\, \underline{\partial}^\nu[q^\mu{}_{1,1}] - \underline{\partial}_\mu[q_{\nu 1,1}]\, \underline{\partial}^\nu[q^\mu{}_{1,1}] - i\, q_{\mu 2,2}\, q_{\nu 2,1}\, \underline{\partial}^\nu[q^\mu{}_{1,2}] +\\
&\underline{\partial}_\nu[q_{\mu 2,1}]\, \underline{\partial}^\nu[q^\mu{}_{1,2}] - \underline{\partial}_\mu[q_{\nu 2,1}]\, \underline{\partial}^\nu[q^\mu{}_{1,2}] + i\, q_{\mu 1,2}\, q_{\nu 1,1}\, \underline{\partial}^\nu[q^\mu{}_{2,1}] + i\, q_{\mu 2,2}\, q_{\nu 1,2}\, \underline{\partial}^\nu[q^\mu{}_{2,1}] -\\
&i\, q_{\mu 1,2}\, q_{\nu 2,2}\, \underline{\partial}^\nu[q^\mu{}_{2,1}] + \underline{\partial}_\nu[q_{\mu 1,2}]\, \underline{\partial}^\nu[q^\mu{}_{2,1}] - \underline{\partial}_\mu[q_{\nu 1,2}]\, \underline{\partial}^\nu[q^\mu{}_{2,1}] + i\, q_{\mu 1,2}\, q_{\nu 2,1}\, \underline{\partial}^\nu[q^\mu{}_{2,2}] +\\
&\underline{\partial}_\nu[q_{\mu 2,2}]\, \underline{\partial}^\nu[q^\mu{}_{2,2}] - \underline{\partial}_\mu[q_{\nu 2,2}]\, \underline{\partial}^\nu[q^\mu{}_{2,2}] + i\, q_{\mu 1,2}\, q_{\nu 2,1}\, \underline{\partial}^\mu[q^\nu{}_{1,1}] - \underline{\partial}_\nu[q_{\mu 1,1}]\, \underline{\partial}^\mu[q^\nu{}_{1,1}] +\\
&\underline{\partial}_\mu[q_{\nu 1,1}]\, \underline{\partial}^\mu[q^\nu{}_{1,1}] + i\, q_{\mu 2,2}\, q_{\nu 2,1}\, \underline{\partial}^\mu[q^\nu{}_{1,2}] - \underline{\partial}_\nu[q_{\mu 2,1}]\, \underline{\partial}^\mu[q^\nu{}_{1,2}] + \underline{\partial}_\mu[q_{\nu 2,1}]\, \underline{\partial}^\mu[q^\nu{}_{1,2}] +\\
&q_{\mu 1,1}\, (q_{\nu 2,1}\, (q^\mu{}_{1,1}\, q^\nu{}_{1,2} - q^\mu{}_{2,2}\, q^\nu{}_{1,2} + q^\mu{}_{1,2}\, (-q^\nu{}_{1,1} + q^\nu{}_{2,2}) + i\, \underline{\partial}^\nu[q^\mu{}_{1,2}] - i\, \underline{\partial}^\mu[q^\nu{}_{1,2}]) +\\
&\quad q_{\nu 1,2}\, (q^\mu{}_{1,1}\, q^\nu{}_{2,1} - q^\mu{}_{2,2}\, q^\nu{}_{2,1} + q^\mu{}_{2,1}\, (-q^\nu{}_{1,1} + q^\nu{}_{2,2}) - i\, \underline{\partial}^\nu[q^\mu{}_{2,1}] + i\, \underline{\partial}^\mu[q^\nu{}_{2,1}])) -\\
&i\, q_{\mu 1,2}\, q_{\nu 1,1}\, \underline{\partial}^\mu[q^\nu{}_{2,1}] - i\, q_{\mu 2,2}\, q_{\nu 1,2}\, \underline{\partial}^\mu[q^\nu{}_{2,1}] + i\, q_{\mu 1,2}\, q_{\nu 2,2}\, \underline{\partial}^\mu[q^\nu{}_{2,1}] - \underline{\partial}_\nu[q_{\mu 1,2}]\, \underline{\partial}^\mu[q^\nu{}_{2,1}] +
\end{aligned}$$

```
∂_μ[q_{ν1,2}] ∂^μ[q^ν_{2,1}] - i q_{μ1,2} q_{ν2,1} ∂^μ[q^ν_{2,2}] - ∂_ν[q_{μ2,2}] ∂^μ[q^ν_{2,2}] + ∂_μ[q_{ν2,2}] ∂^μ[q^ν_{2,2}] +
q_{μ2,1} (q_{ν2,2} (q^μ_{1,1} q^ν_{1,2} - q^μ_{2,2} q^ν_{1,2} + q^μ_{1,2} (-q^ν_{1,1} + q^ν_{2,2}) + i ∂^ν[q^μ_{1,2}] - i ∂^μ[q^ν_{1,2}]) +
   q_{ν1,1} (-q^μ_{1,1} q^ν_{1,2} + q^μ_{2,2} q^ν_{1,2} + q^μ_{1,2} (q^ν_{1,1} - q^ν_{2,2}) - i ∂^ν[q^μ_{1,2}] + i ∂^μ[q^ν_{1,2}]) +
   q_{ν1,2} (-2 q^μ_{2,1} q^ν_{1,2} + 2 q^μ_{1,2} q^ν_{2,1} + i (∂^ν[q^μ_{1,1}] - ∂^ν[q^μ_{2,2}] - ∂^μ[q^ν_{1,1}] + ∂^μ[q^ν_{2,2}])))
Comparing 2 Tr[Q_{μν}.Q^{μν}] with FF calculation ⇒ True QED
```

Lemma 5.5 (p.59)

```
PR["●Lemma 5.5: ",
 $155 = $ = {Tr[Φ^2] → 4 a Abs[H']^2 + 2 c,
    Tr[Φ^4] → 4 b Abs[H']^4 + 8 e Abs[H']^2 + 2 d,
    H' → {ϕ1 + 1, ϕ2},
    a → Abs[Y_ν]^2 + Abs[Y_e]^2,
    b → Abs[Y_ν]^4 + Abs[Y_e]^4,
    c -> Abs[Y_R]^2, d -> Abs[Y_R]^4, e -> Abs[Y_R]^2 Abs[Y_ν]^2
   }; ColumnBar[$],
 $155a = Association[$155];
 Imply, $155x = $ = {
     Abs[H']^2 → (H' . Conjugate[H'] /. $155) // FullSimplify // Reverse,
     t[RL_]_{i_,j_} :→ 0 /; i ≠ 1 || j ≠ 1 || RL == L,
     t[_]_{i,j}[CG["GWS basis"]]
    } /. Re[x_] → (x + Conjugate[x]) / 2; $ // ColumnBar,
 NL, "The requirement ", tuRuleSelect[$defGWS][T.ν_R],
 Yield,
 $ = ($ = ct[T].T) → ($ /. tuRuleSelect[$defGWS][T][[-1]] /. tuRule[$155x] // Simplify);
 MatrixForms[$],
 Yield, $ = $[[2, 1, 1]] -> Abs[Y_R]^2; $ // Framed,
 AppendTo[$155x, $];

 line,
 NL, "Proof: ",
 NL, "Use the 8x8 ℋ_{F_8} representation of: ",
 $ = tuRuleSelect[$defGWS][Φ] // Select[#, tuHasAllQ[#, S] &] & // First;
 $ // MatrixForms,
 NL, "where ", $s = {tuRuleSelect[$defGWS][S] // Select[#, tuHasAllQ[#, ν] &] &,
     tuRuleSelect[$defGWS][ϕ] // Select[#, tuHasAllQ[#, ν] &] & // First,
     tuRuleSelect[$defGWS][T] // Select[#, tuHasAllQ[#, 2] &] & // Last} /.
    tuRule[$155x] // Flatten;
 $s // MatrixForms,
 Yield, $[[2]] = $[[2]] /. $s // ArrayFlatten;
 ($sΦ1 = $) // MatrixForms,

 next, "Compute: ", $01 = $ = Inactive[Tr][Φ.Φ], "POFF",
 Yield, $ = $ /. $sΦ1; MatrixForms[$];
 Yield, $ = $01 -> $ // Activate // FullSimplify,
 Yield, $ = $ //. tuRule[($155x // FullSimplify)] // Simplify,
 Yield, $ = {$, $155[[{-5, -3}]]} // Flatten; $ // ColumnBar,
 Yield, $ = tuEliminate[$, {Abs[Y_e]^2, Abs[Y_ν]^2}] /. And → List; "PONdd",
 Yield, $ = $ // tuRuleSolve[#, Tr[Φ.Φ]] & // First // (# /. $155[[{-5, -3}]] &);
 $ // Framed,
 Yield, $ /. (Reverse /@ $155) // Framed

]
```

●Lemma 5.5:
$$\begin{array}{l} \mathrm{Tr}[\Phi^2] \to 2\,c + 4\,a\,\mathrm{Abs}[H']^2 \\ \mathrm{Tr}[\Phi^4] \to 2\,d + 8\,e\,\mathrm{Abs}[H']^2 + 4\,b\,\mathrm{Abs}[H']^4 \\ H' \to \{1+\phi_1, \phi_2\} \\ a \to \mathrm{Abs}[Y_e]^2 + \mathrm{Abs}[Y_\nu]^2 \\ b \to \mathrm{Abs}[Y_e]^4 + \mathrm{Abs}[Y_\nu]^4 \\ c \to \mathrm{Abs}[Y_R]^2 \\ d \to \mathrm{Abs}[Y_R]^4 \\ e \to \mathrm{Abs}[Y_R]^2\,\mathrm{Abs}[Y_\nu]^2 \end{array}$$

$$\Rightarrow \begin{array}{l} 1 + \mathrm{Abs}[\phi_1]^2 + \mathrm{Abs}[\phi_2]^2 + (\phi_1)^* + \phi_1 \to \mathrm{Abs}[H']^2 \\ t[RL\_]_{i\_,j\_} :\to 0 \;/;\; i \neq 1 \;||\; j \neq 1 \;||\; RL == L \\ t[\_]_{i,j}[\text{GWS basis}] \end{array}$$

The requirement $\{T.\nu_R \to Y_R.\overline{\nu_R}\}$

$$\to T^\dagger.T \to \begin{pmatrix} (t[R]_{1,1})^*\,t[R]_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (t[R]_{1,1})^*\,t[R]_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \boxed{(t[R]_{1,1})^*\,t[R]_{1,1} \to \mathrm{Abs}[Y_R]^2}$$

---

Proof:

Use the 8x8 $\mathcal{H}_{F_8}$ representation of: $\Phi \to \begin{pmatrix} S+\phi & T^* \\ T & S^*+\phi^* \end{pmatrix}$

where

$$\left\{S \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* & 0 \\ 0 & 0 & 0 & (Y_e)^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix},\ \phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 \\ 0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^*\,(\phi_1)^* \\ (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 \\ (\phi_2)^*\,Y_\nu & Y_e\,\phi_1 & 0 & 0 \end{pmatrix},\ T \to \begin{pmatrix} t[R]_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

$$\to \Phi \to \begin{pmatrix} 0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & (Y_\nu)^*\,\phi_2 & (t[R]_{1,1})^* & 0 & \cdots \\ 0 & 0 & -(Y_e)^*\,(\phi_2)^* & (Y_e)^* + (Y_e)^*\,(\phi_1)^* & 0 & 0 & \cdots \\ Y_\nu + (\phi_1)^*\,Y_\nu & -Y_e\,\phi_2 & 0 & 0 & 0 & 0 & \cdots \\ (\phi_2)^*\,Y_\nu & Y_e + Y_e\,\phi_1 & 0 & 0 & 0 & 0 & \cdots \\ t[R]_{1,1} & 0 & 0 & 0 & 0 & 0 & Y_\nu + \\ 0 & 0 & 0 & 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 & (Y_\nu)^* + (Y_\nu)^*\,\phi_1 & -(Y_e\,\phi_2)^* & \cdots \\ 0 & 0 & 0 & 0 & (Y_\nu)^*\,\phi_2 & (Y_e)^* + (Y_e\,\phi_1)^* & \cdots \end{pmatrix}$$

◆Compute: Tr[Φ.Φ]

.......

$$\to \boxed{\mathrm{Tr}[\Phi.\Phi] \to 2\,(\mathrm{Abs}[Y_R]^2 + 2\,(\mathrm{Abs}[Y_e]^2 + \mathrm{Abs}[Y_\nu]^2)\,\mathrm{Abs}[H']^2)}$$

$$\to \boxed{\mathrm{Tr}[\Phi.\Phi] \to 2\,(c + 2\,a\,\mathrm{Abs}[H']^2)}$$

```
$sexp = {Conjugate[a_ b_] → Conjugate[a] Conjugate[b], Abs[a_ b_] → Abs[a] Abs[b],
   a_ Conjugate[a_] → Abs[a]^2, a_^2 Conjugate[a_]^2 → Abs[a]^4}
PR["In the same way Compute: ", $01 = $ = Inactive[Tr][Φ.Φ.Φ.Φ],
  Yield, $ = $ /. $sΦ1 /. tX[R | L]_,_ → Y_R // Activate // Simplify;
  Yield, $ = Expand[$] //. tuRule[$155x] //. $sexp;
  Yield, $ = $01 -> $ //. tuRule[$155x] // tuTrSimplify[{Abs[_]}] // Simplify;
  Yield, $ = $ /.
      tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] /.
     tuRuleSolve[$155x[[1]], $155x[[1, 1, 2]]] // Simplify;
  Yield, $ = $ /. tuRuleSolve[$155x[[1]], $155x[[1, 1, 4]]] /.
     tuRuleSolve[$155x[[1]], $155x[[1, 1, 3]]] // Simplify;
  Yield, $ = $ /. (#^2 & /@ tuRuleSolve[$155x[[1]], $155x[[1, 1, 3]]][[1]] // Expand) /.
     $sexp // Simplify;
  Yield, $ = $ /. $sexp /. tuRuleSolve[$155x[[1]], $155x[[1, 1, 2]]];
  Yield, $ =
   $ /. tuRuleSolve[$155x[[-1]], $155x[[-1, 1, 1]]] // Collect[#, Abs[H'], Simplify] &;
  ColumnSumExp[$];
  Yield,
  $trpppp = $ = $ /. tuRuleSolve[Expand[#^2 & /@ $155x[[-1]]], Abs[t[L]1,1^4]] // Collect[#,
       {Abs[H'], Tr[Abs[Y_R]^2], Conjugate[Y_ν], Y_ν, t[L]1,1, Abs[t[R]1,1]}, Simplify] &;
  ColumnSumExp[$] // Framed,
  Yield, $ /. (Reverse /@ $155) // Framed
  (*t[L]1,1 → in this case.*)
 ];
```

$\{(a\_b\_)^* \to a^* b^*, \text{Abs}[a\_b\_] \to \text{Abs}[a]\,\text{Abs}[b], a\_^* a\_ \to \text{Abs}[a]^2, a\_^{*2} a\_^2 \to \text{Abs}[a]^4\}$

In the same way Compute: Tr[Φ.Φ.Φ.Φ]
→
→
→
→
→
→
→
→

→ $\boxed{\text{Tr}[\Phi.\Phi.\Phi.\Phi] \to \sum\left[\begin{array}{l} 2\,\text{Abs}[t[R]_{1,1}]^4 \\ 8\,\text{Abs}[Y_R]^2\,\text{Abs}[Y_\nu]^2\,\text{Abs}[H']^2 \\ 4\,(\text{Abs}[Y_e]^4 + \text{Abs}[Y_\nu]^4)\,\text{Abs}[H']^4 \end{array}\right]}$

→ $\boxed{\text{Tr}[\Phi.\Phi.\Phi.\Phi] \to 2\,\text{Abs}[t[R]_{1,1}]^4 + 8\,e\,\text{Abs}[H']^2 + 4\,b\,\text{Abs}[H']^4}$

Lemma 5.6

```
PR["●Lemma 5.6. ",
 $156 = $ = {Tr[tuDDown[iD][Φ, μ] tuDUp[iD][Φ, μ]] → 4 a Abs[tuDDown[iD̃][H', μ]]^2,
    tuDDown[iD̃][H', μ] → tuDDown["∂"][H', μ] +
      I xSum[T[Q, "du", {μ, j}] T[σ, "d", {j}], {j, 3}].H' - I T[Λ, "d", {μ}].H',
    $e31 = tuDDown[iD][Φ, μ] → tuDPartial[Φ, μ] + I CommutatorM[T[B, "d", {μ}], Φ],
    tuRuleSelect[$defGWS][Φ] // Select[#, tuHasAllQ[#, S] &] & // First,
    H' → {ϕ1 + 1, ϕ2},
    T[Q, "d", {μ}] -> xSum[T[Q, "du", {μ, j}] T[σ, "d", {j}], {j, 3}],
    T[Q, "du", {μ, j}][CG["ℝ"]]
```

```
    }; ColumnBar[$], accumGWS[$],
  line,
  NL, "■ Proof: ",
  NL, "Recall ", $156[[3, 1]], " is finite part of ", selectDef[tuDDown[iD][Φ, μ]],

  next, " In 8x8 space Calculate ", $ = $156[[3, 2, 1]], "POFF",
  NL, "Use: ", $s = {$sΦ1, $e58}; $s // MatrixForms // ColumnBar,
  Yield, $part[1] = $ = $ // expandCom[$s] // Simplify; $ // MatrixForms, CK,

  "PON",
  next, " Calculate ", $ = $156[[3, 2, 2]], "POFF",
  Yield, $ = $ /. $s //. tt : tuDDown["∂"][a_, b_] :→ Thread[tt];
  Yield, $ = $ // tuDerivativeExpand[{μ, ν}],
  Yield, $ // MatrixForms, "PONdd",

  NL, "Summing: ", "POFF",
  Yield, $d = $ = $e31[[1]] -> $part[1] + $ // Simplify;
  MatrixForms[$], CK, "PONdd",
  note,

  Yield, $u = $d // tuIndicesRaise[{ν, μ}];
  NL, "Compute: ", $u [[1]]. $d[[1]], "POFF",
  NL, "Compute: ", $ = Thread[ $u . $d, Rule] // Simplify, "PONdd",
  NL, "Take Tr[]: ", $ = Tr /@ $; "POFF",
  $ // Framed,
  "PONdd",
  NL, "Discard Dot (since all variabls are scalars) and expand/simplify: ",
  $ = $ /. Dot → Times // Expand;
  $ = $ //. tuConjugateDistribute // tuConjugateSimplify[{μ, ν}] // Expand //
    tuIndexDummyOrdered;
  NL, "Substitute to Abs[]: ", $scc = {a_ cc[a_] → Abs[a]^2,
    a_ (dd : DerivOps)[cc[a_], n_] → dd[Abs[a]^2, n] / 2,
    cc[a_] (dd : DerivOps)[a_, n_] → dd[Abs[a]^2, n] / 2
   }; $s // ColumnBar,
  Yield, $ = $ /. $scc;
  NL, "Simplify using symmetry properties of Q: ",
  $sq = {cc[T[q, "u", {μ}]2,1] -> T[q, "u", {μ}]1,2,
    cc[T[q, "d", {μ}]2,1] -> T[q, "d", {μ}]1,2, T[q, "u", {μ}]2,1 → cc[T[q, "u", {μ}]1,2],
    T[q, "d", {μ}]2,1 → cc[T[q, "d", {μ}]1,2], cc[T[q, "d", {μ}]n_,n_] → T[q, "d", {μ}]n,n,
    cc[T[q, "u", {μ}]n_,n_] → T[q, "u", {μ}]n,n,
    T[q, "u", {μ}]2,2 → -T[q, "u", {μ}]1,1, T[q, "d", {μ}]2,2 → -T[q, "d", {μ}]1,1
   }; $sq // ColumnBar,
  Yield, $ = $ //. $scc //. $sq;
  NL, "Introduce: ", $s = a -> Abs[Yν]^2 + Abs[Ye]^2,
  Yield, $[[2]] = $[[2]] /. tuRuleSolve[$s, Abs[Ye]^2] // tuDerivativeExpand[] // Expand;
  NL, "Collect by terms: ",
  $s = {a, tuDDown["∂"][a, μ], tuDUp["∂"][a, μ], T[Λ, "u", {μ}]},
  Yield, $[[2]] = Collect[$[[2]], $s, Simplify[tuIndexDummyOrdered[Expand[#]]] &];
  $pass = $;
  NL, "Substitute: ",
  $sH = Abs[H']^2 → (1 + ϕ1) cc[(1 + ϕ1)] + ϕ2 cc[ϕ2] // Expand // (# /. $scc &),
  Yield, $ = $pass /. tuRuleSolve[$sH, Abs[ϕ2]^2] // Expand;
  Yield, $[[2]] = Collect[$[[2]], $s, Simplify[tuIndexDummyOrdered[Expand[#]]] &];
  $pass1 = $
 ]
```

●Lemma 5.6.

$$\begin{aligned}
&\mathrm{Tr}[\underline{D}_\mu[\Phi]\,\underline{D}^\mu[\Phi]] \to 4\,a\,\mathrm{Abs}[\tilde{\underline{D}}_\mu[H']]^2\\
&\tilde{\underline{D}}_\mu[H'] \to -i\,\Lambda_\mu.H' + i\sum_{\{j,3\}}[Q_\mu{}^j\,\sigma_j].H' + \underline{\partial}_\mu[H']\\
&\underline{D}_\mu[\Phi] \to i\,[B_\mu,\Phi]_- + \underline{\partial}_\mu[\Phi]\\
&\Phi \to \{\{S+\phi,\ T^*\},\ \{T,\ S^*+\phi^*\}\}\\
&H' \to \{1+\phi_1,\ \phi_2\}\\
&Q_\mu \to \sum_{\{j,3\}}[Q_\mu{}^j\,\sigma_j]\\
&Q_\mu{}^j[\mathbb{R}]
\end{aligned}$$

■ Proof:

Recall $\underline{D}_\mu[\Phi]$ is finite part of $\underline{D}_\mu[\Phi] \to 1_N\otimes(-i\,[\Phi,B_\mu]_-) + 1_N\otimes\underline{\partial}_\mu[\Phi]$

◆ In 8x8 space Calculate $i\,[B_\mu,\Phi]_-$

◆ Calculate $\underline{\partial}_\mu[\Phi]$

.......

Summing:

.......

⌘

→

Compute: $\underline{D}^\mu[\Phi].\underline{D}_\mu[\Phi]$

.......

Take Tr[]:

.......

Discard Dot (since all variabls are scalars) and expand/simplify:

Substitute to Abs[]:

$$\begin{aligned}
&\Phi \to \{\{0,0,(Y_\nu)^*+(Y_\nu)^*\phi_1,(Y_\nu)^*\phi_2,(t[R]_{1,1})^*,0,0,0\},\\
&\quad\{0,0,-(Y_e)^*(\phi_2)^*,(Y_e)^*+(Y_e)^*(\phi_1)^*,0,0,0,0\},\\
&\quad\{Y_\nu+(\phi_1)^*Y_\nu,-Y_e\phi_2,0,0,0,0,0,0\},\{(\phi_2)^*Y_\nu,Y_e+Y_e\phi_1,0,0,0,0,0,0\},\\
&\quad\{t[R]_{1,1},0,0,0,0,0,Y_\nu+(\phi_1)^*Y_\nu,(\phi_2)^*Y_\nu\},\{0,0,0,0,0,0,-Y_e\phi_2,Y_e+Y_e\phi_1\},\\
&\quad\{0,0,0,0,(Y_\nu)^*+(Y_\nu)^*\phi_1,-(Y_e\phi_2)^*,0,0\},\{0,0,0,0,(Y_\nu)^*\phi_2,(Y_e)^*+(Y_e\phi_1)^*,0,0\}\}\\
&B_\mu \to \{\{0,0,0,0,0,0,0,0\},\{0,-2\Lambda_\mu,0,0,0,0,0,0\},\{0,0,q_{\mu 1,1}-\Lambda_\mu,q_{\mu 1,2},0,0,0,0\},\\
&\quad\{0,0,q_{\mu 2,1},q_{\mu 2,2}-\Lambda_\mu,0,0,0,0\},\{0,0,0,0,0,0,0,0\},\{0,0,0,0,0,2\Lambda_\mu,0,0\},\\
&\quad\{0,0,0,0,0,0,-(q_{\mu 1,1})^*+\Lambda_\mu,-(q_{\mu 1,2})^*\},\{0,0,0,0,0,0,-(q_{\mu 2,1})^*,-(q_{\mu 2,2})^*+\Lambda_\mu\}\}
\end{aligned}$$

→

Simplify using symmetry properties of Q:

$$\begin{aligned}
&(q^\mu{}_{2,1})^* \to q^\mu{}_{1,2}\\
&(q_{\mu 2,1})^* \to q_{\mu 1,2}\\
&q^\mu{}_{2,1} \to (q^\mu{}_{1,2})^*\\
&q_{\mu 2,1} \to (q_{\mu 1,2})^*\\
&(q_{\mu n_,n_})^* \to q_{\mu n,n}\\
&(q^\mu{}_{n_,n_})^* \to q^\mu{}_{n,n}\\
&q^\mu{}_{2,2} \to -q^\mu{}_{1,1}\\
&q_{\mu 2,2} \to -q_{\mu 1,1}
\end{aligned}$$

→

Introduce: $a \to \mathrm{Abs}[Y_e]^2+\mathrm{Abs}[Y_\nu]^2$

→

Collect by terms: $\{a,\ \underline{\partial}_\mu[a],\ \underline{\partial}^\mu[a],\ \Lambda^\mu\}$

→

Substitute: $\mathrm{Abs}[H']^2 \to 1+\mathrm{Abs}[\phi_1]^2+\mathrm{Abs}[\phi_2]^2+(\phi_1)^*+\phi_1$

→

→ $\mathrm{Tr}[\underline{D}_\mu[\Phi]\,\underline{D}^\mu[\Phi]] \to$

$$\begin{aligned}
&4\,\mathrm{Abs}[H']^2\,(\underline{\partial}^\mu[Y_e]^*\,\underline{\partial}_\mu[Y_e]+\underline{\partial}^\mu[Y_\nu]^*\,\underline{\partial}_\mu[Y_\nu]) + a\,(\Lambda^\mu\,(4\,i\,\underline{\partial}_\mu[\phi_1]^*+4\,\mathrm{Abs}[H']^2\,\Lambda_\mu-4\,i\,\underline{\partial}_\mu[\phi_1])+\\
&\quad 4\,(-i\,\underline{\partial}^\mu[\phi_2]^*\,q_{\mu 1,2}-i\,\underline{\partial}^\mu[\phi_2]^*\,\phi_1\,q_{\mu 1,2}+\mathrm{Abs}[H']^2\,q_{\mu 1,1}\,q^\mu{}_{1,1}-4\,q^\mu{}_{1,1}\,\Lambda_\mu-\\
&\qquad 4\,\mathrm{Abs}[\phi_1]^2\,q^\mu{}_{1,1}\,\Lambda_\mu+2\,\mathrm{Abs}[H']^2\,q^\mu{}_{1,1}\,\Lambda_\mu-4\,(\phi_1)^*\,q^\mu{}_{1,1}\,\Lambda_\mu-4\,\phi_1\,q^\mu{}_{1,1}\,\Lambda_\mu-2\,(\phi_2)^*\,q^\mu{}_{1,2}\,\Lambda_\mu-\\
&\qquad 2\,(\phi_2)^*\,\phi_1\,q^\mu{}_{1,2}\,\Lambda_\mu+i\,q^\mu{}_{1,1}\,\underline{\partial}_\mu[\phi_1]+i\,(\phi_2)^*\,q^\mu{}_{1,2}\,\underline{\partial}_\mu[\phi_1]+\underline{\partial}^\mu[\phi_1]^*\,(-i\,q_{\mu 1,1}+\underline{\partial}_\mu[\phi_1])+\\
&\qquad (q^\mu{}_{1,2})^*\,(-i\,\underline{\partial}_\mu[\phi_1]^*\,\phi_2+\mathrm{Abs}[H']^2\,q_{\mu 1,2}-(1+(\phi_1)^*)\,(2\,\phi_2\,\Lambda_\mu-i\,\underline{\partial}_\mu[\phi_2]))+\underline{\partial}^\mu[\phi_2]^*\,\underline{\partial}_\mu[\phi_2]))+\\
&\quad 2\,\underline{\partial}^\mu[t[R]_{1,1}]^*\,\underline{\partial}_\mu[t[R]_{1,1}]+2\,(\mathrm{Abs}[\phi_1]\,\underline{\partial}_\mu[\mathrm{Abs}[\phi_1]]+\mathrm{Abs}[\phi_2]\,\underline{\partial}_\mu[\mathrm{Abs}[\phi_2]]+\underline{\partial}_\mu[\phi_1])\,\underline{\partial}^\mu[a]+\\
&\quad 2\,\underline{\partial}_\mu[a]\,(\underline{\partial}^\mu[\phi_1]^*+\mathrm{Abs}[\phi_1]\,\underline{\partial}^\mu[\mathrm{Abs}[\phi_1]]+\mathrm{Abs}[\phi_2]\,\underline{\partial}^\mu[\mathrm{Abs}[\phi_2]])
\end{aligned}$$

```
PR[$ = $pass1;
 "If constant: ", $s = {Y_, t[R]__, a},
 Yield,
 $ = $ // tuDerivativeExpand[$s];
 $ = $ // Collect[#, {a, Abs[H']^2, Tensor[Λ, __]}, Simplify] &;
 $ // ColumnSumExp,
 CR["It is unclear if this expression is equivalent expression on p.61."]
]
```

If constant: {Y_, t[R]__, a}

→ $\mathrm{Tr}[\underline{D}_\mu[\Phi]\,\underline{D}^\mu[\Phi]] \to$

$$a\sum\left[\begin{array}{l} -8\,((1+(\phi_1)^*)\,(q^\mu{}_{1,2})^*\,\phi_2+2\,(1+\mathrm{Abs}[\phi_1]^2+(\phi_1)^*+\phi_1)\,q^\mu{}_{1,1}+(\phi_2)^*\,(1+\phi_1)\,q^\mu{}_{1,2})\,\Lambda_\mu \\ \mathrm{Abs}[H']^2\,(4\,((q^\mu{}_{1,2})^*\,q_{\mu 1,2}+q_{\mu 1,1}\,q^\mu{}_{1,1})+\Lambda_\mu\,(8\,q^\mu{}_{1,1}+4\,\Lambda^\mu)) \\ 4\,i\,\Lambda^\mu\,(\underline{\partial}_\mu[\phi_1]^*-\underline{\partial}_\mu[\phi_1]) \\ -4\,i\,(\underline{\partial}^\mu[\phi_2]^*\,q_{\mu 1,2}+\underline{\partial}^\mu[\phi_2]^*\,\phi_1\,q_{\mu 1,2}+\underline{\partial}^\mu[\phi_1]^*\,(q_{\mu 1,1}+i\,\underline{\partial}_\mu[\phi_1])-q^\mu{}_{1,1}\,\underline{\partial}_\mu[\phi_1]- \\ \quad (\phi_2)^*\,q^\mu{}_{1,2}\,\underline{\partial}_\mu[\phi_1]+i\,\underline{\partial}^\mu[\phi_2]^*\,\underline{\partial}_\mu[\phi_2]+(q^\mu{}_{1,2})^*\,(\underline{\partial}_\mu[\phi_1]^*\,\phi_2-(1+(\phi_1)^*)\,\underline{\partial}_\mu[\phi_2])) \end{array}\right]$$

It is unclear if this expression is equivalent expression on p.61.

```
PR[CO["By transforming the coefficients
    of Y's the derivation becomes more transparent."],
 NL, "If constant: ", $s = {Y_, t[R]__},
 NL, "Evaluate expression for ",
 $ = $d /. Dot → Times //. $sq // tuDerivativeExpand[{Y_, t[R]__}] // Simplify;
 $[[1]], " can be written ",
 $0 = $;

 NL, "Relabel the coefficients of ", $s = Ye,
 Yield,
 $1p0 = $0 // tuExtractPositionPattern[$s_] // Collect[#, T[Λ, "d", {μ}], Simplify] & //
   Collect[#, {$s, (1 + ϕ_)}, Simplify] &;
 (*painful way of manipulating equations*)
 Yield, $1p = $1p0 // tuDerivativeExpand[{ϕ_}],
 Yield, $1 = Last /@ $1p;
 $11 = CoefficientList[$1, $s] // DeleteDuplicates // Most // Rest /@ # & // Flatten //
   Simplify,
 $χ1 = $χ = $11 → -I cc /@ {χ2, χ1} // Thread; $χ // ColumnBar,
 Yield, $s = tuRuleSolve[$χ, {_ - T[Λ, "d", {μ}], _ + T[Λ, "d", {μ}]}];
 Yield, $1p = $1p0 /. $s // Expand // Simplify,
 Yield, $0 = tuReplacePart[$0, $1p];
]
PR["Relabel the coefficients of ", $s = cc[Ye],
 Yield,
 $1p0 = $0 // tuExtractPositionPattern[$s_] // Collect[#, T[Λ, "d", {μ}], Simplify] & //
   Collect[#, {$s, (1 + cc[ϕ_])}, Simplify] &;
 (*painful way of manipulating equations*)
 Yield, $1p = $1p0 // tuDerivativeExpand[{ϕ_}];
 Yield, $1 = Last /@ $1p;
 $11 = CoefficientList[$1, $s] // DeleteDuplicates // Rest /@ # & // Flatten // Simplify;
 $χ2 = $χ = $11 → I {χ2, χ1} // Thread; $χ // ColumnBar,
 Yield, $s = tuRuleSolve[$χ, {_ - T[Λ, "d", {μ}], _ + T[Λ, "d", {μ}]}];
 Yield, $1p = $1p0 /. $s // Expand // Simplify,
 Yield, $0 = tuReplacePart[$0, $1p];
]
PR["Relabel the coefficients of ", $s = Yν,
 Yield,
```

```
 $1p0 = $0 // tuExtractPositionPattern[$s _] // Collect[#, T[Λ, "d", {μ}], Simplify] & //
   Collect[#, {$s, (1 + cc[ϕ_])}, Simplify] &,
 (*painful way of manipulating equations*)
 Yield, $1p = $1p0 // tuDerivativeExpand[{ϕ_}];
 Yield, $1 = Last /@ $1p;
 $11 = CoefficientList[$1, $s];
 Yield, $11 = $11 // DeleteDuplicates // Rest /@ # & // Flatten // Simplify;
 $χ3 = $χ = $11 → I {χ1, -χ2} // Thread; $χ // ColumnBar,
 Yield, $s = tuRuleSolve[$χ, {_ - T[Λ, "d", {μ}], _ + T[Λ, "d", {μ}]}];
 Yield, $1p = $1p0 /. $s // Expand // Simplify,
 Yield, $0 = tuReplacePart[$0, $1p];

]
PR["Relabel the coefficients of ", $s = cc[Yν],
  Yield,
  $1p0 = $0 // tuExtractPositionPattern[$s _] // Collect[#, T[Λ, "d", {μ}], Simplify] & //
    Collect[#, {$s, (1 + ϕ_)}, Simplify] &, (*painful way of manipulating equations*)
  Yield, $1p = $1p0 // tuDerivativeExpand[{ϕ_}];
  Yield, $1 = Last /@ $1p;
  $11 = CoefficientList[$1, $s];
  Yield, $11 = $11 // DeleteDuplicates // Rest /@ # & // Flatten // Simplify;
  $χ4 = $χ = $11 → -I cc /@ {χ1, -χ2} // Thread; $χ // ColumnBar,
  Yield, $s = tuRuleSolve[$χ, {_ - T[Λ, "d", {μ}], _ + T[Λ, "d", {μ}]}];
  Yield, $1p = $1p0 /. $s // Expand // Simplify,
  Yield, $pass2 = $0 = tuReplacePart[$0, $1p] // Expand // Simplify;
 ];
PR["We reproduce the equation on p.61 except for Conjugate[ϕ_] expression. ",
 $0 // MatrixForms,
 NL, CO["This is the expression on p.69 except for Conjugate[ϕ_].
    To derive this expressions for χ's the manipulation had to
    be  controlled in detail.  Is there a general method? "],

 NL, "The χ's are expressed: ",
 $ = {$χ1, $χ2, $χ3, $χ4} // Flatten; $ // ColumnBar;
 Yield, $1 = Select[$, ! FreeQ[#, χ1] &] // DeleteDuplicates // tuRuleSimplify;
 Yield, $2 = tuRuleSolve[$1, {cc[χ1], χ1}] // Simplify;

 Yield, $1 = Select[$, ! FreeQ[#, χ2] &] // DeleteDuplicates // tuRuleSimplify;
 Yield, $ = {$2, tuRuleSolve[$1, {cc[χ2], χ2}] // Simplify};
 $ // Flatten // ColumnBar // Framed
]
PR[
 "Using ", $s = a -> Abs[Yν]^2 + Abs[Ye]^2,
 imply, $s = tuRuleSolve[$s, Abs[Ye]^2],
 NL, "Compute ",
 $ = $pass2 /. tt : χn_ → T[tt, "d", {μ}];
 $1 = $ // tuIndexRaiseAll[μ, μ];
 $ = Thread[ Dot[$ , $1] , Rule];
 $ = Tr /@ $ // Expand;
 Yield, $ = $ /. $scc;
 Yield, $ = $ /. $s // tuConjugateSimplify[{μ}] // (# /. $scc /. $s &) // Expand;
 Yield, $ = $ // tuIndexDummyOrdered // Collect[#, {a}, Simplify] &;
 $ // ColumnSumExp,
 NL, CG["This is equivalent to 2× the expression on p.61 for Tr_H1."]
]
```

By transforming the coefficients of Y's the derivation becomes more transparent.
If constant: {Y_, t[R]__}
Evaluate expression for $D_\mu[\Phi]$ can be written
Relabel the coefficients of $Y_e$

→

→ $\{\{2,3,2\}\to Y_e\,(i\,(1+\phi_1)\,q_{\mu 1,2}-i\,\phi_2\,(q_{\mu 1,1}+\Lambda_\mu)),$
$\{2,4,2\}\to Y_e\,(-i\,(q_{\mu 1,2})^*\,\phi_2-i\,(1+\phi_1)\,(q_{\mu 1,1}-\Lambda_\mu)),$
$\{2,6,7\}\to Y_e\,(i\,(1+\phi_1)\,q_{\mu 1,2}-i\,\phi_2\,(q_{\mu 1,1}+\Lambda_\mu)),$
$\{2,6,8\}\to Y_e\,(-i\,(q_{\mu 1,2})^*\,\phi_2-i\,(1+\phi_1)\,(q_{\mu 1,1}-\Lambda_\mu)),\ \{2,7,6,1,2,1\}\to 0,\ \{2,8,6,1,1\}\to 0\}$

→ $\{i\,((1+\phi_1)\,q_{\mu 1,2}-\phi_2\,(q_{\mu 1,1}+\Lambda_\mu)),\ -i\,((q_{\mu 1,2})^*\,\phi_2+(1+\phi_1)\,(q_{\mu 1,1}-\Lambda_\mu))\}$

$i\,((1+\phi_1)\,q_{\mu 1,2}-\phi_2\,(q_{\mu 1,1}+\Lambda_\mu))\to -i\,(\chi_2)^*$
$-i\,((q_{\mu 1,2})^*\,\phi_2+(1+\phi_1)\,(q_{\mu 1,1}-\Lambda_\mu))\to -i\,(\chi_1)^*$

→

→ $\{\{2,3,2\}\to Y_e\,(-i\,(\chi_2)^*-\partial_\mu[\phi_2]),\ \{2,4,2\}\to Y_e\,(-i\,(\chi_1)^*+\partial_\mu[\phi_1]),$
$\{2,6,7\}\to Y_e\,(-i\,(\chi_2)^*-\partial_\mu[\phi_2]),\ \{2,6,8\}\to Y_e\,(-i\,(\chi_1)^*+\partial_\mu[\phi_1]),$
$\{2,7,6,1,2,1\}\to Y_e\,\partial_\mu[\phi_2],\ \{2,8,6,1,1\}\to Y_e\,\partial_\mu[\phi_1]\}$

→ Null

Relabel the coefficients of $(Y_e)^*$

→

→

→ $-i\,(1+(\phi_1)^*)\,(q_{\mu 1,2})^*+i\,(\phi_2)^*\,(q_{\mu 1,1}+\Lambda_\mu)\to i\,\chi_2$
$i\,((\phi_2)^*\,q_{\mu 1,2}+(1+(\phi_1)^*)\,(q_{\mu 1,1}-\Lambda_\mu))\to i\,\chi_1$

→

→ $\{\{2,2,3\}\to -(Y_e)^*\,(\partial_\mu[\phi_2]^*-i\,\chi_2),$
$\{2,2,4\}\to (Y_e)^*\,(\partial_\mu[\phi_1]^*+i\,\chi_1),\ \{2,7,6,2\}\to i\,(Y_e)^*\,\chi_2,\ \{2,8,6,2\}\to i\,(Y_e)^*\,\chi_1\}$

→ Null

Relabel the coefficients of $Y_\nu$

→ $\{\{2,3,1\}\to Y_\nu\,(\partial_\mu[\phi_1]^*+i\,(\phi_2)^*\,q_{\mu 1,2}+i\,(1+(\phi_1)^*)\,(q_{\mu 1,1}-\Lambda_\mu)),$
$\{2,4,1\}\to Y_\nu\,(i\,(1+(\phi_1)^*)\,(q_{\mu 1,2})^*+\partial_\mu[\phi_2]^*-i\,(\phi_2)^*\,(q_{\mu 1,1}+\Lambda_\mu)),$
$\{2,5,7\}\to Y_\nu\,(\partial_\mu[\phi_1]^*+i\,(\phi_2)^*\,q_{\mu 1,2}+i\,(1+(\phi_1)^*)\,(q_{\mu 1,1}-\Lambda_\mu)),$
$\{2,5,8\}\to Y_\nu\,(i\,(1+(\phi_1)^*)\,(q_{\mu 1,2})^*+\partial_\mu[\phi_2]^*-i\,(\phi_2)^*\,(q_{\mu 1,1}+\Lambda_\mu))\}$

→

→

→ $i\,((\phi_2)^*\,q_{\mu 1,2}+(1+(\phi_1)^*)\,(q_{\mu 1,1}-\Lambda_\mu))\to i\,\chi_1$
$i\,((1+(\phi_1)^*)\,(q_{\mu 1,2})^*-(\phi_2)^*\,(q_{\mu 1,1}+\Lambda_\mu))\to -i\,\chi_2$

→

→ $\{\{2,3,1\}\to Y_\nu\,(\partial_\mu[\phi_1]^*+i\,\chi_1),\ \{2,4,1\}\to Y_\nu\,(\partial_\mu[\phi_2]^*-i\,\chi_2),$
$\{2,5,7\}\to Y_\nu\,(\partial_\mu[\phi_1]^*+i\,\chi_1),\ \{2,5,8\}\to Y_\nu\,(\partial_\mu[\phi_2]^*-i\,\chi_2)\}$

→ Null

Relabel the coefficients of $(Y_\nu)^*$

→ $\{\{2,1,3\}\to (Y_\nu)^*\,(-i\,(q_{\mu 1,2})^*\,\phi_2-i\,(1+\phi_1)\,(q_{\mu 1,1}-\Lambda_\mu)+\partial_\mu[\phi_1]),$
$\{2,1,4\}\to (Y_\nu)^*\,(-i\,(1+\phi_1)\,q_{\mu 1,2}+i\,\phi_2\,(q_{\mu 1,1}+\Lambda_\mu)+\partial_\mu[\phi_2]),$
$\{2,7,5\}\to (Y_\nu)^*\,(-i\,(q_{\mu 1,2})^*\,\phi_2-i\,(1+\phi_1)\,(q_{\mu 1,1}-\Lambda_\mu)+\partial_\mu[\phi_1]),$
$\{2,8,5\}\to (Y_\nu)^*\,(-i\,(1+\phi_1)\,q_{\mu 1,2}+i\,\phi_2\,(q_{\mu 1,1}+\Lambda_\mu)+\partial_\mu[\phi_2])\}$

→

→

→ $-i\,((q_{\mu 1,2})^*\,\phi_2+(1+\phi_1)\,(q_{\mu 1,1}-\Lambda_\mu))\to -i\,(\chi_1)^*$
$-i\,(1+\phi_1)\,q_{\mu 1,2}+i\,\phi_2\,(q_{\mu 1,1}+\Lambda_\mu)\to i\,(\chi_2)^*$

→

→ $\{\{2,1,3\}\to (Y_\nu)^*\,(-i\,(\chi_1)^*+\partial_\mu[\phi_1]),\ \{2,1,4\}\to (Y_\nu)^*\,(i\,(\chi_2)^*+\partial_\mu[\phi_2]),$
$\{2,7,5\}\to (Y_\nu)^*\,(-i\,(\chi_1)^*+\partial_\mu[\phi_1]),\ \{2,8,5\}\to (Y_\nu)^*\,(i\,(\chi_2)^*+\partial_\mu[\phi_2])\}$

→ Null

We reproduce the equation on p.61 except for Conjugate[$\phi$_] expression. $\underline{D}_\mu[\Phi] \to ($ $Y_\nu$ ($\underline{\partial}_\mu$ $Y_\nu$ ($\underline{\partial}_\mu$

This is the expression on p.69 except for Conjugate[$\phi$_].
To derive this expressions for $\chi$'s the manipulation had to be controlled in detail. Is there a general method?
The $\chi$'s are expressed:

→

→

→

→
$$\boxed{\begin{array}{l} (\chi_1)^* \to (q_{\mu 1,2})^* \phi_2 + (1+\phi_1)(q_{\mu 1,1} - \Lambda_\mu) \\ \chi_1 \to (1+(\phi_1)^*) q_{\mu 1,1} + (\phi_2)^* q_{\mu 1,2} - (1+(\phi_1)^*)\Lambda_\mu \\ (\chi_2)^* \to -(1+\phi_1) q_{\mu 1,2} + \phi_2 (q_{\mu 1,1} + \Lambda_\mu) \\ \chi_2 \to -(1+(\phi_1)^*)(q_{\mu 1,2})^* + (\phi_2)^* (q_{\mu 1,1} + \Lambda_\mu) \end{array}}$$

Using $a \to \text{Abs}[Y_e]^2 + \text{Abs}[Y_\nu]^2$ ⇒ $\{\text{Abs}[Y_e]^2 \to a - \text{Abs}[Y_\nu]^2\}$
Compute

→

→

→
$$\text{Tr}[\underline{D}_\mu[\Phi].\underline{D}^\mu[\Phi]] \to 4\, a \sum\left[ \begin{array}{l} (\chi_1{}^\mu)^* (\chi_{1\mu} - i\, \underline{\partial}_\mu[(\phi_1)^*]) \\ (\chi_2{}^\mu)^* (\chi_{2\mu} + i\, \underline{\partial}_\mu[(\phi_2)^*]) \\ i\, \chi_1{}^\mu\, \underline{\partial}_\mu[\phi_1] \\ -i\, \chi_2{}^\mu\, \underline{\partial}_\mu[\phi_2] \\ \underline{\partial}_\mu[\phi_1]\, \underline{\partial}^\mu[(\phi_1)^*] \\ \underline{\partial}_\mu[\phi_2]\, \underline{\partial}^\mu[(\phi_2)^*] \end{array} \right]$$

This is equivalent to 2× the expression on p.61 for $\text{Tr}_{\mathcal{H}_1}$.

Proposition 5.7. The spectral action of the AC-manifold

```
PR["Proposition 5.7. The spectral action of the AC-manifold ",
 $ = M × F_GWS → {C∞[M, ℂ ⊕ ℍ], L^2[M, S] ⊗ (ℂ^4 ⊕ ℂ^4),
    slash[iD] ⊗ 1_F + T[γ, "d", {5}] ⊗ iD_F, T[γ, "d", {5}] ⊗ T[γ, "d", {F}], J_M ⊗ J_F};
 $ // ColumnForms,
 NL, "is ", $p57 = $ = {Tr[f[iD_A / Λ]] → xIntegral[
      ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] Sqrt[Det[g]], x^4],
    ℒ[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] →
     8 ℒ_M[T[g, "dd", {μ, ν}]] + ℒ_A[T[Λ, "d", {μ}], T[Q, "d", {μ}]] +
      ℒ_H[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'],
    ℒ_A[T[Λ, "d", {μ}], T[Q, "d", {μ}]] → f[0] / (12 π^2) (6 T[Λ, "dd", {μ, ν}]
         T[Λ, "uu", {μ, ν}] + Tr[ T[Q, "dd", {μ, ν}] T[Q, "uu", {μ, ν}]]),
    ℒ_H[T[g, "dd", {μ, ν}], T[Λ, "d", {μ}], T[Q, "d", {μ}], H'] →
     b f[0] / (2 π^2) Abs[H']^4 + (-2 a f_2 Λ^2 + e f[0]) / π^2 Abs[H']^2 -
      c f_2 Λ^2 / π^2 + d f[0] / (4 π^2) + a f[0] s Abs[H']^2 / (12 π^2) +
      c f[0] s / (24 π^2) + a f[0] Abs[tuDDown[iD̃][H', μ]]^2 / (2 π^2),
    $p35[[-1]]
   }; $ // ColumnBar,

 line,
 NL, "Prop 3.5", yield, $ = tuRuleSelect[$p35][ℒ_M[_]][[1]], AppendTo[$p57, $];
 NL, "Prop 3.7, Lemma 5.4", yield, $ = {tuRuleSelect[$p37][ℒ_B[_]][[1]], $154};
 $ // ColumnBar, AppendTo[$p57, $];
 NL, "Prop 3.5, Lemma 5.5", yield, $ = tuRule[{tuRuleSelect[$p37][ℒ_H[__]][[1]],
     $155, $156}] // Flatten; $ // ColumnBar, AppendTo[$p57, $];
 accumGWS[prop57 -> $p57]
]
```

**Proposition 5.7.** The spectral action of the AC-manifold

$$M\times F_{GWS}\to\left|\begin{array}{l}C^\infty[M,\ \mathbb{C}\oplus\mathbb{H}]\\ L^2[M,\ S]\otimes(\mathbb{C}^4\oplus\mathbb{C}^4)\\ (\not{D})\otimes 1_F+\text{Tensor}[\gamma,\ \mid\text{Void},\ \mid 5]\otimes\mathcal{D}_F\\ \text{Tensor}[\gamma,\ \mid\text{Void},\ \mid 5]\otimes\text{Tensor}[\gamma,\ \mid\text{Void},\ \mid F]\\ J_M\otimes J_F\end{array}\right.$$

is

$$\left|\begin{array}{l}\text{Tr}[f[\frac{D_A}{\Lambda}]]\to\int\sqrt{\text{Det}[g]}\ \mathcal{L}[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']\,dx^4\\ \mathcal{L}[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']\to\mathcal{L}_A[\Lambda_\mu,\ Q_\mu]+\mathcal{L}_H[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']+8\,\mathcal{L}_M[g_{\mu\nu}]\\ \mathcal{L}_A[\Lambda_\mu,\ Q_\mu]\to\frac{f[0]\,(6\,\Lambda_{\mu\nu}\,\Lambda^{\mu\nu}+\text{Tr}[Q_{\mu\nu}\,Q^{\mu\nu}])}{12\,\pi^2}\\ \mathcal{L}_H[g_{\mu\nu},\ \Lambda_\mu,\ Q_\mu,\ H']\to\\ \quad\frac{d\,f[0]}{4\,\pi^2}+\frac{c\,s\,f[0]}{24\,\pi^2}+\frac{a\,s\,\text{Abs}[H']^2\,f[0]}{12\,\pi^2}+\frac{b\,\text{Abs}[H']^4\,f[0]}{2\,\pi^2}+\frac{a\,\text{Abs}[\tilde{\underline{D}}_\mu[H']]^2\,f[0]}{2\,\pi^2}-\frac{c\,\Lambda^2\,f_2}{\pi^2}+\frac{\text{Abs}[H']^2\,(e\,f[0]-2\,a\,\Lambda^2\,f_2)}{\pi^2}\\ \mathcal{L}_M[g_{\mu\nu}]\to-\frac{\Lambda^2\,f_2}{24\,\pi^2}+\frac{\Lambda^4\,f_4}{2\,\pi^2}+\frac{f[0]\,(\frac{11\,R^*.R^*}{360}-\frac{1}{20}C_{\mu\nu\rho\sigma}\,C^{\mu\nu\rho\sigma}+\frac{\Delta[s]}{30})}{16\,\pi^2}\end{array}\right.$$

---

Prop 3.5 $\longrightarrow$ $\mathcal{L}_M[g_{\mu\nu}]\to-\frac{\Lambda^2\,f_2}{24\,\pi^2}+\frac{\Lambda^4\,f_4}{2\,\pi^2}+\frac{f[0]\,(\frac{11\,R^*.R^*}{360}-\frac{1}{20}C_{\mu\nu\rho\sigma}\,C^{\mu\nu\rho\sigma}+\frac{\Delta[s]}{30})}{16\,\pi^2}$

Prop 3.7, Lemma 5.4 $\longrightarrow$

$$\left|\begin{array}{l}\mathcal{L}_B[B_\mu]\to\frac{f[0]\,\text{Tr}[F_{\mu\nu}\,F^{\mu\nu}]}{24\,\pi^2}\\ \{\text{Tr}[F_{\mu\nu}\,F^{\mu\nu}]\to 12\,\Lambda_{\mu\nu}\,\Lambda^{\mu\nu}+2\,\text{Tr}[Q_{\mu\nu}\,Q^{\mu\nu}],\ \Lambda_{\mu\nu}\to-\underline{\partial}_\nu[\Lambda_\mu]+\underline{\partial}_\mu[\Lambda_\nu],\ Q_{\mu\nu}\to i\,[Q_\mu,\ Q_\nu]_- -\underline{\partial}_\nu[Q_\mu]+\underline{\partial}_\mu[Q_\nu]\}\end{array}\right.$$

Prop 3.5, Lemma 5.5 $\longrightarrow$

$$\left|\begin{array}{l}\mathcal{L}_H[g_{\mu\nu},\ B_\mu,\ \Phi]\to\frac{f[0]\,s[x]\,\text{Tr}[\Phi.\Phi]}{48\,\pi^2}-\frac{\Lambda^2\,f_2\,\text{Tr}[\Phi.\Phi]}{2\,\pi^2}+\frac{f[0]\,\text{Tr}[D_\mu[\Phi].D^\mu[\Phi]]}{8\,\pi^2}+\frac{f[0]\,\text{Tr}[\Phi.\Phi.\Phi.\Phi]}{8\,\pi^2}+\frac{f[0]\,\Delta[\text{Tr}[\Phi.\Phi]]}{24\,\pi^2}\\ \text{Tr}[\Phi^2]\to 2\,c+4\,a\,\text{Abs}[H']^2\\ \text{Tr}[\Phi^4]\to 2\,d+8\,e\,\text{Abs}[H']^2+4\,b\,\text{Abs}[H']^4\\ H'\to\{1+\phi_1,\ \phi_2\}\\ a\to\text{Abs}[Y_e]^2+\text{Abs}[Y_\nu]^2\\ b\to\text{Abs}[Y_e]^4+\text{Abs}[Y_\nu]^4\\ c\to\text{Abs}[Y_R]^2\\ d\to\text{Abs}[Y_R]^4\\ e\to\text{Abs}[Y_R]^2\,\text{Abs}[Y_\nu]^2\\ \text{Tr}[\underline{D}_\mu[\Phi]\ \underline{D}^\mu[\Phi]]\to 4\,a\,\text{Abs}[\tilde{\underline{D}}_\mu[H']]^2\\ \tilde{\underline{D}}_\mu[H']\to-i\,\Lambda_\mu.H'+i\sum_{\{j,3\}}[Q_\mu{}^j\,\sigma_j].H'+\underline{\partial}_\mu[H']\\ \underline{D}_\mu[\Phi]\to i\,[B_\mu,\ \Phi]_-+\underline{\partial}_\mu[\Phi]\\ \Phi\to\{\{S+\phi,\ T^*\},\ \{T,\ S^*+\phi^*\}\}\\ H'\to\{1+\phi_1,\ \phi_2\}\\ Q_\mu\to\sum_{\{j,3\}}[Q_\mu{}^j\,\sigma_j]\end{array}\right.$$

## 5.4 Normalization of kinetic terms

### 5.4.1 Rescaling the Higgs field

```
PR["The canonical kinetic energy term is of form ",
 ℒ → xIntegral[1 / 2 tuDPartial[ψ, μ] tuDPartialu[ψ, μ] √Abs[Det[g]] , x^4],
 NL, "Identify this form within ",
 NL, $ = tuRuleSelect[$defGWS][ℒH[__]] // Select[#, tuHasNoneQ[#, Φ] &] & // Last,
 Yield, $ = $ // tuTermSelect[iD] // Last,
 " as the kinetic energy Lagrangian density for
   Higg's and rescale to canonical form by ",
 $ = H → √ (a f[0] / π^2) H'; $ // Framed, accumGWS[$]
]
```

The canonical kinetic energy term is of form $\mathcal{L} \to \int \frac{1}{2} \sqrt{\text{Abs[Det[g]]}}\, \underline{\partial}_{\mu}[\psi]\, \underline{\partial}^{\mu}[\psi]\, dx^4$

Identify this form within

$$\mathcal{L}_H[g_{\mu\nu}, \Lambda_\mu, Q_\mu, H'] \to \frac{d\, f[0]}{4\pi^2} + \frac{c\, s\, f[0]}{24\pi^2} + \frac{a\, s\, \text{Abs}[H']^2 f[0]}{12\pi^2} + \frac{b\, \text{Abs}[H']^4 f[0]}{2\pi^2} + \frac{a\, \text{Abs}[\underline{\tilde{D}}_\mu[H']]^2 f[0]}{2\pi^2} - \frac{c\, \Lambda^2 f_2}{\pi^2} + \frac{\text{Abs}[H']^2 (e\, f[0] - 2\, a\, \Lambda^2 f_2)}{\pi^2}$$

$\to \frac{a\, \text{Abs}[\underline{\tilde{D}}_\mu[H']]^2 f[0]}{2\pi^2}$ as the kinetic energy Lagrangian density for Higg's and rescale to canonical form by $\boxed{H \to \frac{\sqrt{a\, f[0]}\, H'}{\pi}}$

## 5.4.2 The coupling constants

```
PR["Rescale Gauge fields: ", $gaugeRescaled = $ = {
    T[Λ, "d", {μ}] -> g1[CG["coupling"]] / 2 T[B, "d", {μ}],
    T[Q, "du", {μ, a}] -> T[W, "du", {μ, a}] g2[CG["coupling"]] / 2,
    T[Q, "d", {μ}] -> T[W, "d", {μ}] g2 / 2 ,
    g1[CG["coupling"]],
    g2[CG["coupling"]],
    T[B, "d", {μ}][CG["U[1] hypercharge field"]],
    T[Λ, "dd", {μ, ν}] -> g1 / 2 T[B, "dd", {μ, ν}],
    T[Q, "ddu", {μ, ν, a}] -> g2 / 2 T[W, "ddu", {μ, ν, a}]
   }; $ // ColumnBar, accumGWS[$gaugeRescaled], CR["Why?"],
 NL, "Previously ",
 $pass =
  $ = tuRuleSelect[$defGWS][{T[Q, "dd", {μ, ν}], T[Λ, "dd", {μ, ν}]}] // DeleteDuplicates;
 $ // ColumnBar
]
PR["From Lemma 5.6: ",
 $s = tuRuleSelect[$defGWS][T[Q, "d", {_}]] // Select[#, ! FreeQ[#, xSum] &] & // Last //
    tuAddPatternVariable[μ] // (# /. xSum[a_, _] → a &),
 Yield, $ = $pass /. $s /. CommutatorM[a_, b_] :→ CommutatorM[a, (b /. j → i)] //
    tuCommutatorSimplify[{Tensor[Q, _, _]}] // tuDerivativeExpand[{Tensor[σ, _, _]}],
 Yield, $ = $ /. tuSU2commutation[σ] // tuIndexSwapUpDown[c$];
 NL, "In σ components: ",
 $[[1, 2, 1]] = $[[1, 2, 1]] /. {j → k, c$ → j};
 $ = $ /. Tensor[σ, _, _] → 1 /. tt : T[Q, "dd", {μ, ν}] :→ tuIndexAdd[-1, j][tt];
 $ // ColumnBar
]
PR["The rescaled relationships ",
 $ = $ /. (tuRule[$gaugeRescaled] // tuAddPatternVariable[{a, μ, ν}]) //
   tuDerivativeExpand[{g_}];
 $ = tuRuleSolve[$, {T[W, "ddu", {_, _, _}], T[B, "dd", {_, _}]}] // Expand;
 $ // ColumnBar, accumGWS[$]
]
```

Rescale Gauge fields:

$\Lambda_\mu \to \frac{1}{2} B_\mu\, g_1[\text{coupling}]$

$Q_\mu{}^a \to \frac{1}{2} W_\mu{}^a\, g_2[\text{coupling}]$

$Q_\mu \to \frac{1}{2} g_2\, W_\mu$

$g_1[\text{coupling}]$

$g_2[\text{coupling}]$

$B_\mu[\text{U[1] hypercharge field}]$

$\Lambda_{\mu\nu} \to \frac{1}{2} g_1\, B_{\mu\nu}$

$Q_{\mu\nu}{}^a \to \frac{1}{2} g_2\, W_{\mu\nu}{}^a$

Why?

Previously

$Q_{\mu\nu} \to i\,[Q_\mu, Q_\nu]_- - \underline{\partial}_\nu[Q_\mu] + \underline{\partial}_\mu[Q_\nu]$

$\Lambda_{\mu\nu} \to -\underline{\partial}_\nu[\Lambda_\mu] + \underline{\partial}_\mu[\Lambda_\nu]$

$\Lambda_{\mu\nu} \to \frac{1}{2} g_1\, B_{\mu\nu}$

From Lemma 5.6: $Q_{\mu\_} \to Q_\mu{}^j \sigma_j$

$\to \{Q_{\mu\nu} \to i\,[\sigma_j, \sigma_i]_- Q_\nu{}^i Q_\mu{}^j - \sigma_j \underline{\partial}_\nu[Q_\mu{}^j] + \sigma_j \underline{\partial}_\mu[Q_\nu{}^j],\ \Lambda_{\mu\nu} \to -\underline{\partial}_\nu[\Lambda_\mu] + \underline{\partial}_\mu[\Lambda_\nu],\ \Lambda_{\mu\nu} \to \frac{1}{2} g_1 B_{\mu\nu}\}$

$\to$

In $\sigma$ components: $\left| \begin{array}{l} Q_{\mu\nu}{}^j \to -2\, Q_\nu{}^i Q_\mu{}^k \epsilon_{k\,i}{}^j - \underline{\partial}_\nu[Q_\mu{}^j] + \underline{\partial}_\mu[Q_\nu{}^j] \\ \Lambda_{\mu\nu} \to -\underline{\partial}_\nu[\Lambda_\mu] + \underline{\partial}_\mu[\Lambda_\nu] \\ \Lambda_{\mu\nu} \to \frac{1}{2} g_1 B_{\mu\nu} \end{array} \right.$

The rescaled relationships $\left| \begin{array}{l} W_{\mu\nu}{}^j \to -g_2 W_\nu{}^i W_\mu{}^k \epsilon_{k\,i}{}^j - \underline{\partial}_\nu[W_\mu{}^j] + \underline{\partial}_\mu[W_\nu{}^j] \\ B_{\mu\nu} \to -\underline{\partial}_\nu[B_\mu] + \underline{\partial}_\mu[B_\nu] \end{array} \right.$

```
PR["Evaluate ",
 $ = selectGWS[Tr[_], {F, T[Λ, "dd", {μ, ν}], Q, μ, ν}],
 NL, "Apply ",
 $s = selectGWS[{Tensor[Λ, __]}, {B, μ, ν}];
 $s = {$s, tuIndicesRaise[{μ, ν}][$s]},
 Yield, $ = $ /. $s,
 NL, "Apply ",
 $s = selectGWS[Tensor[Q, __], σ],
 $s = T[σ, "d", {a}] # & /@ $s,
 $s = {$s, tuIndicesRaise[{μ, ν}][$s]} // tuAddPatternVariable[a] // Flatten,
 Yield,
 $ = $ /. tt : Tensor[Q, _, _] :> tuIndexAdd[-1, a][tt] /.
     tt : T[Q, "uuu", {i_, j_, a}] :> ((tt /. a → b) T[σ, "d", {b}]) /.
   tt : T[Q, "ddu", {i_, j_, a}] → tt T[σ, "d", {a}],
 Yield, $ = $ // tuTrSimplify[{Tensor[Q, _, _]}],
 NL, "Apply ",
 $s = Tr[T[σ, "d", {a}] T[σ, "d", {b}]] → 2 T[δ, "dd", {a, b}],
 Yield, $ = $ /. $s;
 $[[2]] = tuIndexContractUpDn[δ, {b}] /@ $[[2]]; $,
 NL, "Apply ", $s = selectGWS[Tensor[Q, __], W];
 $s = {$s, $s // tuIndicesRaise[{μ, ν}] // tuIndicesLower[{a}]} //
   tuAddPatternVariable[{μ, ν, a}],
 Yield, $ = $ /. $s; $ // Framed, CG[" (5.14)"], accumGWS[$]
]
```

Evaluate $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 12\, \Lambda_{\mu\nu} \Lambda^{\mu\nu} + 2\, \mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}]$

Apply $\{\Lambda_{\mu\nu} \to \frac{1}{2} g_1 B_{\mu\nu},\ \Lambda^{\mu\nu} \to \frac{1}{2} g_1 B^{\mu\nu}\}$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3\, g_1^2 B_{\mu\nu} B^{\mu\nu} + 2\, \mathrm{Tr}[Q_{\mu\nu} Q^{\mu\nu}]$

Apply $Q_\mu \to \sum_{\{j,3\}} [Q_\mu{}^j \sigma_j] Q_\mu \sigma_a \to \sigma_a \sum_{\{j,3\}} [Q_\mu{}^j \sigma_j] \{Q_\mu \sigma_{a\_} \to \sigma_a \sum_{\{j,3\}} [Q_\mu{}^j \sigma_j],\ Q^\mu \sigma_{a\_} \to \sigma_a \sum_{\{j,3\}} [Q^{\mu\,j} \sigma_j]\}$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3\, g_1^2 B_{\mu\nu} B^{\mu\nu} + 2\, \mathrm{Tr}[Q_{\mu\nu}{}^a Q^{\mu\nu\,b} \sigma_a \sigma_b]$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3\, g_1^2 B_{\mu\nu} B^{\mu\nu} + 2\, Q_{\mu\nu}{}^a Q^{\mu\nu\,b}\, \mathrm{Tr}[\sigma_a \sigma_b]$

Apply $\mathrm{Tr}[\sigma_a \sigma_b] \to 2\, \delta_{a\,b}$

$\to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3\, g_1^2 B_{\mu\nu} B^{\mu\nu} + 4\, Q_{\mu\nu}{}^a Q^{\mu\nu}{}_a$

Apply $\{Q_{\mu\_\nu\_}{}^{a\_} \to \frac{1}{2} g_2 W_{\mu\nu}{}^a,\ Q^{\mu\_\nu\_}{}_{a\_} \to \frac{1}{2} g_2 W^{\mu\nu}{}_a\}$

$\to \boxed{\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to 3\, g_1^2 B_{\mu\nu} B^{\mu\nu} + g_2^2 W_{\mu\nu}{}^a W^{\mu\nu}{}_a}$ (5.14)

### 5.4.3 Electroweak unification

```
PR["The canonical form of gauge field Kinetic term: ",
 $ = selectGWS[Tr[ T[F, "dd", {μ, ν}] _ ]];
 $0 = ℒ → -1 / 2 $[[1]],
 NL, "Here ", $f = $,
 NL, "Since ",
 $ = {selectGWS[ℒA[__], {f[0]}], selectGWS[Tr[_], {F, T[Λ, "dd", {μ, ν}], Q, μ, ν}]};
 $ // ColumnBar,

 NL, "Eliminate ", $s = tuTermSelect[Q][$][[1]] / 2, " from ",
 NL, $ = {$} // Flatten; $ // ColumnBar,

 Yield, $1 = $ = tuEliminate[$, {$s}] // tuRuleSolve[#, ℒA[__]] & // Last,
 NL, "Canonical form: ", $1 = $0[[2]] → $[[2]],
 Imply, $ = tuRuleSolve[$1, f[0]] // Last; $ // Framed, accumGWS[$],
 Yield, $ = $1 /. $,
 Yield, $ = $1 /. $f // Expand,
 NL, "Imposing conditions(5.16): ",
 $s = {f[0] g1^2 / (8 π^2) → 1 / 4, f[0] g2^2 / (24 π^2) → 1 / 4},
 Yield, $ = $ /. $s; $ // Framed, accumGWS[$],
 Yield, $ = tuEliminate[$s, f[0]] // Simplify;
 ($ = $ /. Equal → Rule) // Framed, accumGWS[$],
 NL, CR["This relationship stems from the imposed condition and may be arbitrary. "]
]
```

The canonical form of gauge field Kinetic term: $\mathcal{L}\to -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$

Here $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to 3\,g_1^2 B_{\mu\nu}B^{\mu\nu}+g_2^2 W_{\mu\nu}{}^{a}W^{\mu\nu}{}_{a}$

Since
$$\mathcal{L}_A[\Lambda_\mu, Q_\mu]\to \frac{f[0]\,(6\Lambda_{\mu\nu}\Lambda^{\mu\nu}+\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}])}{12\pi^2}$$
$$\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to 12\,\Lambda_{\mu\nu}\Lambda^{\mu\nu}+2\,\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}]$$

Eliminate $\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}]$ from
$$\mathcal{L}_A[\Lambda_\mu, Q_\mu]\to \frac{f[0]\,(6\Lambda_{\mu\nu}\Lambda^{\mu\nu}+\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}])}{12\pi^2}$$
$$\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to 12\,\Lambda_{\mu\nu}\Lambda^{\mu\nu}+2\,\mathrm{Tr}[Q_{\mu\nu}Q^{\mu\nu}]$$

→ $\mathcal{L}_A[\Lambda_\mu, Q_\mu]\to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]}{24\pi^2}$

Canonical form: $-\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\to \frac{f[0]\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]}{24\pi^2}$

⇒ $\boxed{f[0]\to -12\pi^2}$

→ $\mathcal{L}_A[\Lambda_\mu, Q_\mu]\to -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$

→ $\mathcal{L}_A[\Lambda_\mu, Q_\mu]\to \frac{f[0]\,g_1^2 B_{\mu\nu}B^{\mu\nu}}{8\pi^2}+\frac{f[0]\,g_2^2 W_{\mu\nu}{}^{a}W^{\mu\nu}{}_{a}}{24\pi^2}$

Imposing conditions(5.16): $\left\{\frac{f[0]\,g_1^2}{8\pi^2}\to\frac{1}{4},\ \frac{f[0]\,g_2^2}{24\pi^2}\to\frac{1}{4}\right\}$

→ $\boxed{\mathcal{L}_A[\Lambda_\mu, Q_\mu]\to \frac{1}{4}B_{\mu\nu}B^{\mu\nu}+\frac{1}{4}W_{\mu\nu}{}^{a}W^{\mu\nu}{}_{a}}$

→ $\boxed{3\,g_1^2\to g_2^2}$

This relationship stems from the imposed condition and may be arbitrary.

```
PR["• Evaluate: ", $ = selectGWS[{tuDDown[iD̃][_, μ]}, {}] /. xSum[a_, _] → a,
 NL, "The scaling for H' drops out and using ",
 $s = tuRule[selectGWS[#, {"coupling"}] & /@ {T[Λ, "d", {_}], T[Q, "du", {_, _}]}] //
   tuAddPatternVariable[{a, μ}],
 Yield, $e515 = $ = $ /. $s /. H' → H;
 $ // Framed, accumGWS[$];
 CG[" (5.15)"], accumGWS[$]
]
```

• Evaluate: $\underline{\tilde{D}}_\mu[H'] \to -\mathbb{i}\,\Lambda_\mu.H' + \mathbb{i}\,(Q_\mu{}^j\,\sigma_j).H' + \underline{\partial}_\mu[H']$

The scaling for H' drops out and using $\{\Lambda_{\mu\_} \to \frac{1}{2} g_1 B_\mu,\ Q_{\mu\_}{}^{a\_} \to \frac{1}{2} g_2 W_\mu{}^a\}$

→ $\boxed{\underline{\tilde{D}}_\mu[H] \to -\mathbb{i}\,(\frac{1}{2} g_1 B_\mu).H + \mathbb{i}\,(\frac{1}{2} g_2 W_\mu{}^j\,\sigma_j).H + \underline{\partial}_\mu[H]}$ (5.15)

## 5.5 The Higgs mechanism

```
PR["● The Higgs portion of the Lagrangian ",
 $ = tuRuleSelect[$defGWS][ℒH[__]] // Select[#, tuHasNoneQ[#, Φ] &] & // Last;
 $ = selectGWS[ℒH[__], H'];
 Yield, $higgsL =
  $ = $ /. tuRuleSolve[selectGWS[H, f[0]], H'] // tuDerivativeExpand[{f[0], a}] //
    tuOpSimplifyF[Abs, {1 / √(a f[0]) }];
 $ // ColumnSumExp,
 NL, "Assuming scalar curvature ", $s = s → 0, ", minimize the Potential wrt H: ",

 $ = ℒHpot → (Apply[Plus, tuTermSelect[H][$]] /. tuDDown[iD̃][_, _] → 0) /. $s,
 accumGWS[$], CK,
 Imply, "The non-zero minima is ",
 $ = 0 -> tuDPartial[$[[2]], Abs[H]] // tuDerivOps2D;
 $ = tuRuleSolve[$, Abs[H]];
 $ = #^2 & /@ $[[2]]; $ // Framed, CG[" (5.18)"], accumGWS[$],

 NL, " which is identified with the vacuum state of the Higgs field ",
 {v, 0} ⇒ ($ = v² → $[[1]]), accumGWS[$],
 line,
 next, "Simplify Higgs potential by unitary transform: ",
 $u = {H -> u.H, u[CG["U[1]×SU[2]"]], u → {{a, -cc[b]}, {b, cc[a]}}, a cc[a] + cc[b] b → 1};
 $u // MatrixForms // ColumnBar,
 NL, "For general Higgs doublet: ", $ = {{h1, h2} → u .{Abs[H], 0}, h1|2[CG[ℂ]]},
 yield, $ = $ /. tuRuleSelect[$u][u]; $ // ColumnForms,
 Imply, "Can express ", $e519 = $ = {H → u[x].{{v + h[x]}, {0}},
    u[x] -> {{a[x], -cc[b[x]]}, {b[x], cc[a[x]]}}, h[x] → Abs[H[x]] - v};
 $ // ColumnBar,
 CR["u[x] transform is the gauge freedom of H."],


 NL, "Re-express ", $0 = selectGWS[ℒHpot],
 NL, "in terms of ", $h2 = $ = Abs[H];
 $ = ($ /. $e519 /. Abs → xAbs /. tuRuleSelect[$e519][u[x]] /.
      xAbs[vv : {a_, b_}] → √(ct[{{a}, {b}}].{{a}, {b}}) //
     tuConjugateTransposeSimplify[{v, h[x]}, {a[x], b[x], h[x], v}]) // Simplify;
 $h2 = $h2 → ($ /. (tuRuleSelect[$u][a cc[a] + b cc[b]] /. {a → a[x], b → b[x]}) // Flatten //
     Last),
 Yield, $ = $0 /. $h2,
 NL, "Substituting ", v², yield, $s = selectGWS[Abs[H]^2],
 yield, $s[[1]] = v^2; $s,
 Yield, $ = tuEliminate[{$, $s}, f2],
 Yield, $ = Solve[$, ℒHpot][[1, 1]] // Collect[#, {b, a, f[0], π, h[x]}] &;
 $ // Framed, CG[" (5.20)"], accumGWS[{$, $e519, $u, $h2}],
 NL, "Note {mass,interaction,cosmological} terms with ", {h[x]², h[x]^(n>2), h[x]^"0"}
]
```

- The Higgs portion of the Lagrangian

$$\rightarrow \mathcal{L}_H[g_{\mu\nu}, \Lambda_\mu, Q_\mu, \frac{H\pi}{\sqrt{a\,f[0]}}] \rightarrow \sum[\begin{array}{l} \frac{1}{12} s\,\text{Abs}[H]^2 \\ \frac{1}{2}\text{Abs}[\tilde{\underline{D}}_\mu[H]]^2 \\ \frac{b\pi^2\,\text{Abs}[H]^4}{2a^2 f[0]} \\ \frac{d\,f[0]}{4\pi^2} \\ \frac{c\,s\,f[0]}{24\pi^2} \\ -\frac{c\Lambda^2 f_2}{\pi^2} \\ \frac{\text{Abs}[H]^2 (e\,f[0]-2a\Lambda^2 f_2)}{a\,f[0]} \end{array}]$$

Assuming scalar curvature $s \rightarrow 0$, minimize the Potential wrt H:

$$\mathcal{L}_{\text{Hpot}} \rightarrow \frac{b\pi^2\,\text{Abs}[H]^4}{2a^2 f[0]} + \frac{\text{Abs}[H]^2 (e\,f[0] - 2a\Lambda^2 f_2)}{a\,f[0]} \longleftarrow \text{CHECK}$$

$\Rightarrow$ The non-zero minima is $\boxed{\text{Abs}[H]^2 \rightarrow \frac{a(-e\,f[0] + 2a\Lambda^2 f_2)}{b\pi^2}}$ (5.18)

which is identified with the vacuum state of the Higgs field
$\{v, 0\} \Rightarrow (v^2 \rightarrow \text{Abs}[H]^2)$

---

◆Simplify Higgs potential by unitary transform: $\begin{array}{l} H \rightarrow u.H \\ u[U[1]\times SU[2]] \\ u \rightarrow \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \\ a\,a^* + b\,b^* \rightarrow 1 \end{array}$

For general Higgs doublet: $\{\{h_1, h_2\} \rightarrow u.\{\text{Abs}[H], 0\}, h_{1|2}[\mathbb{C}]\} \longrightarrow \begin{array}{l} \begin{array}{l} h_1 \\ h_2 \end{array} \rightarrow \begin{array}{l} a\,\text{Abs}[H] \\ b\,\text{Abs}[H] \end{array} \\ h_{1|2}[\mathbb{C}] \end{array}$

$\Rightarrow$ Can express $\begin{array}{l} H \rightarrow u[x].\{\{v + h[x]\}, \{0\}\} \\ u[x] \rightarrow \{\{a[x], -b[x]^*\}, \{b[x], a[x]^*\}\} \\ h[x] \rightarrow -v + \text{Abs}[H[x]] \end{array}$

u[x] transform is the gauge freedom of H.

Re-express $\mathcal{L}_{\text{Hpot}} \rightarrow \frac{b\pi^2\,\text{Abs}[H]^4}{2a^2 f[0]} + \frac{\text{Abs}[H]^2 (e\,f[0] - 2a\Lambda^2 f_2)}{a\,f[0]}$

in terms of $\text{Abs}[H] \rightarrow \sqrt{(v + h[x])^2}$

$$\rightarrow \mathcal{L}_{\text{Hpot}} \rightarrow \frac{b\pi^2 (v + h[x])^4}{2a^2 f[0]} + \frac{(v + h[x])^2 (e\,f[0] - 2a\Lambda^2 f_2)}{a\,f[0]}$$

Substituting $v^2 \longrightarrow \text{Abs}[H]^2 \rightarrow \frac{a(-e\,f[0] + 2a\Lambda^2 f_2)}{b\pi^2} \longrightarrow v^2 \rightarrow \frac{a(-e\,f[0] + 2a\Lambda^2 f_2)}{b\pi^2}$

$$\rightarrow 2a^2 \mathcal{L}_{\text{Hpot}} == \frac{b\pi^2 (-v^4 + 4v^2 h[x]^2 + 4v\,h[x]^3 + h[x]^4)}{f[0]} \;\&\&\; b \neq 0 \;\&\&\; f[0] \neq 0$$

$$\rightarrow \boxed{\mathcal{L}_{\text{Hpot}} \rightarrow \frac{b\pi^2 (-\frac{v^4}{2} + 2v^2 h[x]^2 + 2v\,h[x]^3 + \frac{h[x]^4}{2})}{a^2 f[0]}} \quad (5.20)$$

Note {mass,interaction,cosmological} terms with $\{h[x]^2, h[x]^{n>2}, h[x]^0\}$

## 5.5.1 Massive gauge bosons

```
PR["● The Higgs Lagrangian at mininum potential ",
 $ = $higgsL;
 $ = $ /. s → 0 /. (√# & /@ selectGWS[Abs[H]^2]) // Expand,
 NL, "is the kinetic energy portion: ", $[[2]] = $[[2]] // tuTermSelect[H] // First;
 $ = $ /. ℒH → ℒkin,
 NL, "and must be invariant under unitary gauge transform ",
 $e519 // MatrixForms // ColumnBar,
 NL, "Examine ", $0 = $ = selectGWS[tuDDown[iD̃][H, μ]],
 NL, "Under transform: ",
 $s = {H → u.H, ($pass /. T[W, "d", {μ}] → T[W, "du", {μ, j}] T[σ, "d", {j}])};
 $s // ColumnBar,
 Yield, $[[2]] = $[[2]] /. $s,
 Yield, $ = $ //. tuOpDistribute[Dot] //. tuOpSimplify[Dot, {g_, Tensor[B | σ, __]}],
 NL, "Since ", $s = {u.ct[u] → 1, $$ = ct[u].u → 1,
    tuDPartial[#, μ] & /@ $$ // tuDerivativeExpand[] //
     tuConjugateTransposeSimplify[{μ}] // tuRuleSolve[#, #[[1, 2]]] &
   } // Flatten,
 Yield, $ = $ // expandDC[$s] // tuDerivativeExpand[] // Expand;
 $ // Framed,
 NL, CG[Abs[$[[1]]], " invariant under gauge transform."]
]
```

● The Higgs Lagrangian at mininum potential

$$\mathcal{L}_H\left[g_{\mu\nu},\Lambda_\mu,Q_\mu,\frac{H\pi}{\sqrt{a\,f[0]}}\right]\to\frac{1}{2}\mathrm{Abs}[\tilde{\underline{D}}_\mu[H]]^2+\frac{d\,f[0]}{4\pi^2}-\frac{e^2 f[0]}{2b\pi^2}-\frac{c\Lambda^2 f_2}{\pi^2}+\frac{2ae\Lambda^2 f_2}{b\pi^2}-\frac{2a^2\Lambda^4 f_2^2}{b\pi^2 f[0]}$$

is the kinetic energy portion: $\mathcal{L}_{kin}\left[g_{\mu\nu},\Lambda_\mu,Q_\mu,\frac{H\pi}{\sqrt{a\,f[0]}}\right]\to\frac{1}{2}\mathrm{Abs}[\tilde{\underline{D}}_\mu[H]]^2$

and must be invariant under unitary gauge transform

$$\begin{array}{|l} H\to u[x].\begin{pmatrix} v+h[x]\\ 0\end{pmatrix}\\ u[x]\to\begin{pmatrix} a[x] & -b[x]^*\\ b[x] & a[x]^*\end{pmatrix}\\ h[x]\to -v+\mathrm{Abs}[H[x]]\end{array}$$

Examine $\tilde{\underline{D}}_\mu[H]\to -i\left(\frac{1}{2}g_1 B_\mu\right).H+i\left(\frac{1}{2}g_2 W_\mu{}^j\sigma_j\right).H+\underline{\partial}_\mu[H]$

Under transform:

$$\begin{array}{|l} H\to u.H\\ \{Q_{\mu\nu}\to i\,[Q_\mu,Q_\nu]_- -\underline{\partial}_\nu[Q_\mu]+\underline{\partial}_\mu[Q_\nu],\ \Lambda_{\mu\nu}\to-\underline{\partial}_\nu[\Lambda_\mu]+\underline{\partial}_\mu[\Lambda_\nu],\ \Lambda_{\mu\nu}\to\frac{1}{2}g_1 B_{\mu\nu}\}\end{array}$$

→ $-i\left(\frac{1}{2}g_1 B_\mu\right).H+i\left(\frac{1}{2}g_2 W_\mu{}^j\sigma_j\right).H+\underline{\partial}_\mu[H]\ /.$

$\{H\to u.H,\ \{Q_{\mu\nu}\to i\,[Q_\mu,Q_\nu]_- -\underline{\partial}_\nu[Q_\mu]+\underline{\partial}_\mu[Q_\nu],\ \Lambda_{\mu\nu}\to-\underline{\partial}_\nu[\Lambda_\mu]+\underline{\partial}_\mu[\Lambda_\nu],\ \Lambda_{\mu\nu}\to\frac{1}{2}g_1 B_{\mu\nu}\}\}$

→ $\tilde{\underline{D}}_\mu[H]\to\left(-\frac{1}{2}i\,H\,g_1 B_\mu+\frac{1}{2}i\,W_\mu{}^j.H\,g_2\sigma_j+\underline{\partial}_\mu[H]\ /.\right.$

$\left.\{H\to u.H,\ \{Q_{\mu\nu}\to i\,[Q_\mu,Q_\nu]_- -\underline{\partial}_\nu[Q_\mu]+\underline{\partial}_\mu[Q_\nu],\ \Lambda_{\mu\nu}\to-\underline{\partial}_\nu[\Lambda_\mu]+\underline{\partial}_\mu[\Lambda_\nu],\ \Lambda_{\mu\nu}\to\frac{1}{2}g_1 B_{\mu\nu}\}\}\right)$

Since $\{u.u^\dagger\to 1,\ u^\dagger.u\to 1,\ \underline{\partial}_\mu[u^\dagger].u\to -u^\dagger.\underline{\partial}_\mu[u]\}$

→ $$\boxed{\begin{array}{l}\tilde{\underline{D}}_\mu[H]\to\left(-\frac{1}{2}i\,H\,g_1 B_\mu+\frac{1}{2}i\,W_\mu{}^j.H\,g_2\sigma_j+\underline{\partial}_\mu[H]\ /.\right.\\ \quad\left.\{H\to u.H,\ \{Q_{\mu\nu}\to i\,[Q_\mu,Q_\nu]_- -\underline{\partial}_\nu[Q_\mu]+\underline{\partial}_\mu[Q_\nu],\ \Lambda_{\mu\nu}\to-\underline{\partial}_\nu[\Lambda_\mu]+\underline{\partial}_\mu[\Lambda_\nu],\ \Lambda_{\mu\nu}\to\frac{1}{2}g_1 B_{\mu\nu}\}\}\right)\end{array}}$$

Abs[$\tilde{\underline{D}}_\mu$[H]] invariant under gauge transform.

```
PR["Using ", $s = selectGWS[H, u[x]] /. u[x] → 1 // expandDC[],
 NL, "•Evaluate ", $ = $e515, $real = {v, h[x], g_, Tensor[W | B, _, _], μ};
 Yield, $ = $ // tuIndexSum[{j}, {1, 2, 3}],
 Yield, $ = $ // expandDC[] // (# //. tuOpSimplify[Dot, {g_, Tensor[W | B, _, _]}] &),
 Yield, $[[2]] =
  $[[2]] /. Plus → Inactive[Plus] /. $s /. tuPauliExpand // tuDerivativeExpand[{v}];
 $ // MatrixForms // ColumnSumExp, CK,
 $ = {$, tuIndicesRaise[μ][$]};
 NL, "•Compute ",
 $ = ct[$[[1, 1]]].($[[2, 1]]) → (ct[$[[1, 1]]].$[[2, 1]] /. $) // Activate,
 Yield, $ = $ // tuConjugateSimplify[$real] // tuIndexDummyOrdered // Simplify;
 $[[1]] = Abs[tuDDown[iD̃][H, μ]]^2;
 $[[2]] = Flatten[$[[2]]] // Last;
 ($d2 = $) // ColumnSumExp // Framed,
 NL, CR[ Plus @@ tuTermSelect[{B, W}][Expand[$]] // Simplify,
  " gives the electro-weak mixing angle between the gauge fields."],
 NL, "defined as ", $ = {c_w → Cos[θ_w], Cos[θ_w] → g₂ / √(g₁^2 + g₂^2) ,
    s_w → Sin[θ_w], Sin[θ_w] → g₁ / √(g₁^2 + g₂^2) };
 $ // ColumnBar, accumGWS[{$d2, $}],
 NL, "Given the relation ", $sg = tuRuleSolve[selectGWS[a_ g₁^2], g₂] // Last,
 Imply, $ = tuRuleSelect[$ /. $sg][{Cos[_], Sin[_]}];
 $ = Map[#^2 & /@ # &, $];
 $ // ColumnBar, accumGWS[$],
 CR["at the electroweak unification scale ", Λ_EW, ". Why at this scale?"]
]
```

Using $H \to \{\{v + h[x]\}, \{0\}\}$

•Evaluate $\tilde{\underline{D}}_\mu[H] \to -i\,(\frac{1}{2} g_1 B_\mu).H + i\,(\frac{1}{2} g_2 W_\mu{}^j \sigma_j).H + \underline{\partial}_\mu[H]$

→ $\tilde{\underline{D}}_\mu[H] \to -i\,(\frac{1}{2} g_1 B_\mu).H + i\,(\frac{1}{2} g_2 W_\mu{}^1 \sigma_1 + \frac{1}{2} g_2 W_\mu{}^2 \sigma_2 + \frac{1}{2} g_2 W_\mu{}^3 \sigma_3).H + \underline{\partial}_\mu[H]$

→ $\tilde{\underline{D}}_\mu[H] \to -\frac{1}{2} i\, H\, g_1 B_\mu + i\,(\frac{1}{2}\sigma_1.H\, g_2 W_\mu{}^1 + \frac{1}{2}\sigma_2.H\, g_2 W_\mu{}^2 + \frac{1}{2}\sigma_3.H\, g_2 W_\mu{}^3) + \underline{\partial}_\mu[H]$

→

$$\tilde{\underline{D}}_\mu[H] \to \sum\left[\begin{array}{l} \begin{pmatrix} -\frac{1}{2} i\,(v+h[x])\, g_1 B_\mu \\ 0 \end{pmatrix} \\ i\left(\begin{pmatrix} 0 \\ \frac{1}{2}(v+h[x])\, g_2 W_\mu{}^1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} i\,(v+h[x])\, g_2 W_\mu{}^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(v+h[x])\, g_2 W_\mu{}^3 \\ 0 \end{pmatrix}\right) \\ \begin{pmatrix} \underline{\partial}_\mu[h[x]] \\ 0 \end{pmatrix} \end{array}\right]$$ ⟵CHECK

•Compute $\tilde{\underline{D}}_\mu[H]^\dagger.\tilde{\underline{D}}^\mu[H] \to$

$$\{\{(\frac{1}{2}(v+h[x])\, g_2 W_\mu{}^1 + \frac{1}{2} i\,(v+h[x])\, g_2 W_\mu{}^2)^* \,(\frac{1}{2}(v+h[x])\, g_2 W^{\mu\,1} + \frac{1}{2} i\,(v+h[x])\, g_2 W^{\mu\,2}) + (-\frac{1}{2} i\,(v+h[x])\, g_1 B_\mu + \frac{1}{2} i\,(v+h[x])\, g_2 W_\mu{}^3 + \underline{\partial}_\mu[h[x]])^* \,(-\frac{1}{2} i\,(v+h[x])\, g_1 B^\mu + \frac{1}{2} i\,(v+h[x])\, g_2 W^{\mu\,3} + \underline{\partial}^\mu[h[x]])\}\}$$

→ $$\boxed{\mathrm{Abs}[\tilde{D}_{\underline{\mu}}[H]]^2 \to \frac{1}{4}\sum\left[\begin{array}{l} (v+h[x])^2 g_1^2 B_\mu B^\mu \\ -2\,(v+h[x])^2 g_1 g_2 B^\mu W_\mu{}^3 \\ (v+h[x])^2 g_2^2 (W_\mu{}^1 W^{\mu\,1} + W_\mu{}^2 W^{\mu\,2} + W_\mu{}^3 W^{\mu\,3}) \\ 4\,\partial_{\underline{\mu}}[h[x]]\,\underline{\partial}^\mu[h[x]] \end{array}\right]}$$

$-\frac{1}{2}(v+h[x])^2 g_1 g_2 B^\mu W_\mu{}^3$

gives the electro-weak mixing angle between the gauge fields.

defined as
$$\left|\begin{array}{l} c_w \to \mathrm{Cos}[\theta_w] \\ \mathrm{Cos}[\theta_w] \to \frac{g_2}{\sqrt{g_1^2+g_2^2}} \\ s_w \to \mathrm{Sin}[\theta_w] \\ \mathrm{Sin}[\theta_w] \to \frac{g_1}{\sqrt{g_1^2+g_2^2}} \end{array}\right.$$

Given the relation $g_2 \to \sqrt{3}\, g_1$

⇒ $\left|\begin{array}{l} \mathrm{Cos}[\theta_w]^2 \to \frac{3}{4} \\ \mathrm{Sin}[\theta_w]^2 \to \frac{1}{4} \end{array}\right.$ at the electroweak unification scale $\Lambda_{EW}$. Why at this scale?

```
PR["•From ", $d2 = selectGWS[{Abs[_]^2, iD̃}],
 NL, "we see that ", {T[W, "du", {μ, 1}], T[W, "du", {μ, 2}]}, " are mass eigenstates.",
 NL, "•Defining ",
 $e521 = $ = {T[W, "d", {μ}] → (T[W, "du", {μ, 1}] + I T[W, "du", {μ, 2}]) / √2 ,
    cc[T[W, "d", {μ}]] → (T[W, "du", {μ, 1}] - I T[W, "du", {μ, 2}]) / √2 ,
    T[Z, "d", {μ}] → c_w T[W, "du", {μ, 3}] - s_w T[B, "d", {μ}],
    T[A, "d", {μ}] → s_w T[W, "du", {μ, 3}] + c_w T[B, "d", {μ}]
   }; $ // ColumnBar,
 NL, "Inverting ",
 $s = $ = tuRuleSolve[$e521, {T[W, "du", {μ, 1}], T[W, "du", {μ, 2}], T[W, "du", {μ, 3}],
       T[B, "d", {μ}]}] /. Map[#^2 & /@ # &, selectGWS[#] & /@ {c_w, s_w}] // Simplify;

 $s = {$s, $s // tuIndicesRaise[{μ}]} // Flatten; $s // ColumnBar,
 Imply, $ = $d2 /. $s // Expand // Simplify; $ // ColumnSumExp,
 Yield, $ = $ // tuIndexDummyOrdered // Collect[#, {Tensor[Z, _, _],
      Tensor[A, _, _], Tensor[W, _, _], cc[Tensor[W, _, _]]}, Simplify] &;
 $ // ColumnSumExp,
 NL, "Given ", $s = (selectGWS[#] & /@ {s_w, c_w, Cos[θ_w], Sin[θ_w]}) /. $sg // PowerExpand,
 Imply, $ = $ /. (tuRuleSolve[selectGWS[a_ g_1 ^2], g_1] // Last) //. $s // Simplify;
 $ // ColumnSumExp, CK,
 NL, "So W's, and Z's acquire a mass term, but A's do not.",
 CR["Do A's interact with h's? Consider interaction terms."],
 NL, "Let the masses be ", {M_W → v g_2 / 2, M_Z → v g_2 / (2 c_w)}

]
```

•From $\text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to \frac{1}{4}((v+h[x])^2 g_1^2 B_\mu B^\mu - 2(v+h[x])^2 g_1 g_2 B^\mu W_\mu{}^3 + (v+h[x])^2 g_2^2 (W_\mu{}^1 W^{\mu\,1} + W_\mu{}^2 W^{\mu\,2} + W_\mu{}^3 W^{\mu\,3}) + 4\,\underline{\partial}_\mu[h[x]]\,\underline{\partial}^\mu[h[x]])$

we see that $\{W_\mu{}^1, W_\mu{}^2\}$ are mass eigenstates.

•Defining
$$\begin{array}{|l} W_\mu \to \frac{W_\mu{}^1 + i\,W_\mu{}^2}{\sqrt{2}} \\ (W_\mu)^* \to \frac{W_\mu{}^1 - i\,W_\mu{}^2}{\sqrt{2}} \\ Z_\mu \to -s_w B_\mu + c_w W_\mu{}^3 \\ A_\mu \to c_w B_\mu + s_w W_\mu{}^3 \end{array}$$

Inverting
$$\begin{array}{|l} W_\mu{}^1 \to \frac{(W_\mu)^* + W_\mu}{\sqrt{2}} \\ W_\mu{}^2 \to \frac{i\,((W_\mu)^* - W_\mu)}{\sqrt{2}} \\ W_\mu{}^3 \to s_w A_\mu + c_w Z_\mu \\ B_\mu \to c_w A_\mu - s_w Z_\mu \\ W^{\mu\,1} \to \frac{(W^\mu)^* + W^\mu}{\sqrt{2}} \\ W^{\mu\,2} \to \frac{i\,((W^\mu)^* - W^\mu)}{\sqrt{2}} \\ W^{\mu\,3} \to s_w A^\mu + c_w Z^\mu \\ B^\mu \to c_w A^\mu - s_w Z^\mu \end{array}$$

$$\Rightarrow \text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to \frac{1}{4}\sum[\begin{array}{|l} (v+h[x])^2 g_2^2 (s_w^2 A_\mu A^\mu + (W^\mu)^* W_\mu + (W_\mu)^* W^\mu) \\ 2(v+h[x])^2 g_1 g_2 s_w^2 A_\mu Z^\mu \\ v^2 g_1^2 s_w^2 Z_\mu Z^\mu \\ 2\,v\,h[x]\,g_1^2 s_w^2 Z_\mu Z^\mu \\ h[x]^2 g_1^2 s_w^2 Z_\mu Z^\mu \\ (v+h[x])^2 c_w^2 (g_1^2 A_\mu A^\mu - 2 g_1 g_2 A^\mu Z_\mu + g_2^2 Z_\mu Z^\mu) \\ -(v+h[x])^2 c_w s_w (g_1^2 (A^\mu Z_\mu + A_\mu Z^\mu) - g_2^2 (A^\mu Z_\mu + A_\mu Z^\mu) + 2 g_1 g_2 (A_\mu A^\mu - Z_\mu Z^\mu)) \\ 4\,\underset{-\mu}{\partial}[h[x]]\,\underline{\partial}^\mu[h[x]] \end{array}]$$

$\to \text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to$
$$\sum[\begin{array}{|l} \frac{1}{4}(v+h[x])^2 (c_w g_1 - g_2 s_w)^2 A_\mu A^\mu \\ \frac{1}{2}(W^\mu)^* (v+h[x])^2 g_2^2 W_\mu \\ Z_\mu\,(-\frac{1}{2}(v+h[x])^2 (c_w^2 g_1 g_2 + c_w (g_1^2 - g_2^2) s_w - g_1 g_2 s_w^2) A^\mu + \frac{1}{4}(v+h[x])^2 (c_w g_2 + g_1 s_w)^2 Z^\mu) \\ \underset{-\mu}{\partial}[h[x]]\,\underline{\partial}^\mu[h[x]] \end{array}]$$

Given $\{s_w \to \text{Sin}[\theta_w],\ c_w \to \text{Cos}[\theta_w],\ \text{Cos}[\theta_w] \to \frac{\sqrt{3}}{2},\ \text{Sin}[\theta_w] \to \frac{1}{2}\}$

$$\Rightarrow \text{Abs}[\underline{\tilde{D}}_\mu[H]]^2 \to \frac{1}{18}\sum[\begin{array}{|l} 9\,(W^\mu)^* (v+h[x])^2 g_2^2 W_\mu \\ 6\,(v+h[x])^2 g_2^2 Z_\mu Z^\mu \\ 18\,\underset{-\mu}{\partial}[h[x]]\,\underline{\partial}^\mu[h[x]] \end{array}] \longleftarrow \text{CHECK}$$

So W's, and Z's acquire a mass term, but A's do not.
Do A's interact with h's? Consider interaction terms.

Let the masses be $\{M_W \to \frac{v\,g_2}{2},\ M_Z \to \frac{v\,g_2}{2\,c_w}\}$

```
tuSaveAllVariables[]
```