Approaching QCD from N=4 SYM

by N.Gromov

Solving finite difference equations

Polynomial solution (easy)

At one loop the anomalous dimention of an important class of twist two operators $\mathrm{tr}(ZD^SZ)$ (S is even) can be found from the Baxter equation

$$T(u)Q(u)+(u+i/2)^2Q(u+i)+(u-i/2)^2Q(u-i)=0$$

where for the physical solution $Q(u)$ is simply a polynomial of degree S. Once $Q(u)$ is found the conformal dimention of the operator is given by

$$\Delta=2+S+2ig^2\partial_u\log\frac{Q(u+i/2)}{Q(u-i/2)}+\mathcal{O}(g^4)$$

• Find $T(u)$ assuming it is of the form $T(u)=\alpha+\beta u^2$.

→ Replace $Q(u)$ by u^S+Cu^{S-2} using patterns

→ Use `Series` to find first two terms of the large u expansion of the Baxter equation

→ Find α and β from the requirement that both terms vanish.
Hint: `LogicalExpand` is your friend
you should find

$$\left\{\alpha\to\frac{1}{2}\left(2S^2+2S+1\right),\beta\to-2\right\}$$

• Guess the general result for the energy as a function of S. For that you can do the following steps

→ Replace $Q(u)$ by $P(u)$ - a generic polynomial of degree S:

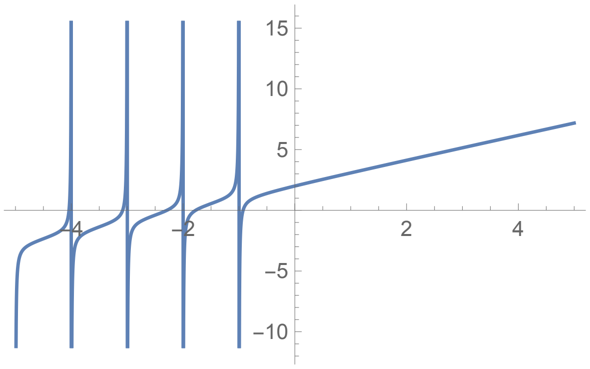
```
P[u_]=Sum[a[n]u^n,{n,0,S0}]/.a[0]->1;
```

→ Again use `Series` and `LogicalExpand` to write and solve the system of equations for each of the coefficients for some specific S. For $S=10$ you should find

$$-\frac{47297536\,u^{10}}{56260575}+\frac{8090368\,u^8}{535815}-\frac{27474304\,u^6}{382725}+\frac{1124492512\,u^4}{11252115}-\frac{28878652\,u^2}{893025}+1$$

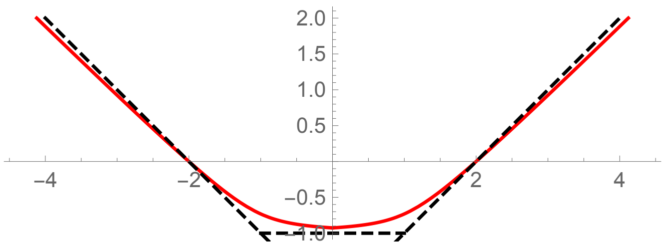
→ Compute the energy Δ for $S=2,4,\ldots,20$. Create a `Table` $\{S,\Delta\}$ for these values of S

→ The function which makes the magic is called `FindSequenceFunction`. Use it to guess the general result for $\Delta(S)$. Plot the result for $S=-5,5$



→ In relation to the talk of Pedro consider the decompactification limit $S \to \infty$. Observe the $\log(S)$ scaling described by the cusp anomalous dimention

→ Reproduce $S(\Delta)$ plot below for $g = 1/10$



Solving Baxter for any S

• Instead of guessing we will try to solve the Baxter equation without fixing S to some particular value. As it is a finite difference equation it is hard to solve it directly with Mathematica. Instead we convert it into the a usual differential equation using Melin transform.
This is similar to a Fourier transformation, which prescribed to replace the initial function $Q(u)$ by

$$Q(u) = \int_0^\infty w^{-iu-1} f(w) dw$$

the nice feature of the transformation is that the shift by i and multiplication by u produce the following transformation of f

$$Q(u+i) \to w f(w) \ \ , \ \ Q(u-i) \to 1/w f(w)$$

$$uQ(u) \to -iw\partial_w f(w)$$

→ Create a new function Mellin which can act on the expressions of the type $u^n Q(u+im)$ transforming them to $(-iw\partial_w)^n w^m f(w)$. It should be enough to use a few simple replacements with patterns. Test it for the following examples

```
Melin[Q[u - I]]
```

$$\frac{f(w)}{w}$$

```
Melin[u^2 Q[u]]
```

$$w^2 (-f''(w)) - w f'(w)$$

```
Melin[u Q[u + I]]
```

$$-i w^2 f'(w) - i w f(w)$$

→ Apply it to our Baxter equation to get

$$-w (w-1)^2 f''(w) - 2 w (w-1) f'(w) + f(w) \left(S^2 + S - \frac{(w-1)^2}{4 w}\right)$$

Note that Mathematica cannot solve this equation in a reasonable way

```
DSolve[mbax == 0, f, w]
```

$$\{\{f \to \text{DifferentialRoot}(\{y, x\} \mapsto \{8 (x-1) y'(x) x^2 + 4 (x-1)^2 y''(x) x^2 + (x^2 - 4 S^2 x - 4 S x - 2 x + 1) y(x) = 0, y(2) = c_1, y'(2) = c_2\}, \ll\gg)\}\}$$

$\rightarrow$ We have to help a bit by geting rid of the first derivative: $f(w) = g\left(\frac{1}{1-w}\right)$ i.e. we change the coordinate $w = \frac{z-1}{z}$

$$\frac{g(z)\left(4S^2(z-1)z+4S(z-1)z-1\right)}{4(z-1)z}-(z-1)z\,g''(z)$$

Now it can be solved

```
DSolve[ee == 0, g[z], z]
```

$$\left\{\left\{g(z)\to c_1\sqrt{1-z}\,\sqrt{z}\,P_S(2z-1)+c_2\sqrt{1-z}\,\sqrt{z}\,Q_S(2z-1)\right\}\right\}$$

Another method to solve this equation, which may work in a more general situation, is the following. Plug the series $g\rightarrow\sqrt{z}\sqrt{1-z}\sum_{n=0}^{\infty}c_n z^n$ and finding a recursive equation on c_n

$$c(n)\left(-n^2-n+S(S+1)\right)+(n+1)^2\,c(n+1)$$

See how to solve this recursive equation using `RSolve` :

$$\frac{c_1\,(1-S)_{n-1}\,(S+2)_{n-1}}{\left((2)_{n-1}\right)^2}$$

from which we recover the function itself

```
gz = Sum[% z^n √z √1 - z , {n, 0, ∞}] /. C[1] → 1
```

$$-\frac{\sqrt{1-z}\,\sqrt{z}\;{}_2F_1(-S,S+1;1;z)}{S(S+1)}$$

Check that this solution is consistent with the general solution by `DSolve`.

Try `Zeta[3]` trick from Pedro's lecture to compute the transformation back to $Q(u)$. Up to an irrelevant periodic function you should find

$${}_3F_2\left(-S,S+1,\frac{1}{2}-i\,u;1,1;1\right)$$

We can compare it now with the previous solution by setting S to some particular value and compare with the previous result

```
% /. S → 12 // Expand
ℚ[12]
% / %% // Simplify
```

$$\frac{96577\,u^{12}}{17107200}-\frac{558467\,u^{10}}{3110400}+\frac{99058609\,u^8}{58060800}-\frac{143126893\,u^6}{24883200}+\frac{2561060957\,u^4}{398131200}-\frac{12272016991\,u^2}{6812467200}+\frac{53361}{1048576}$$

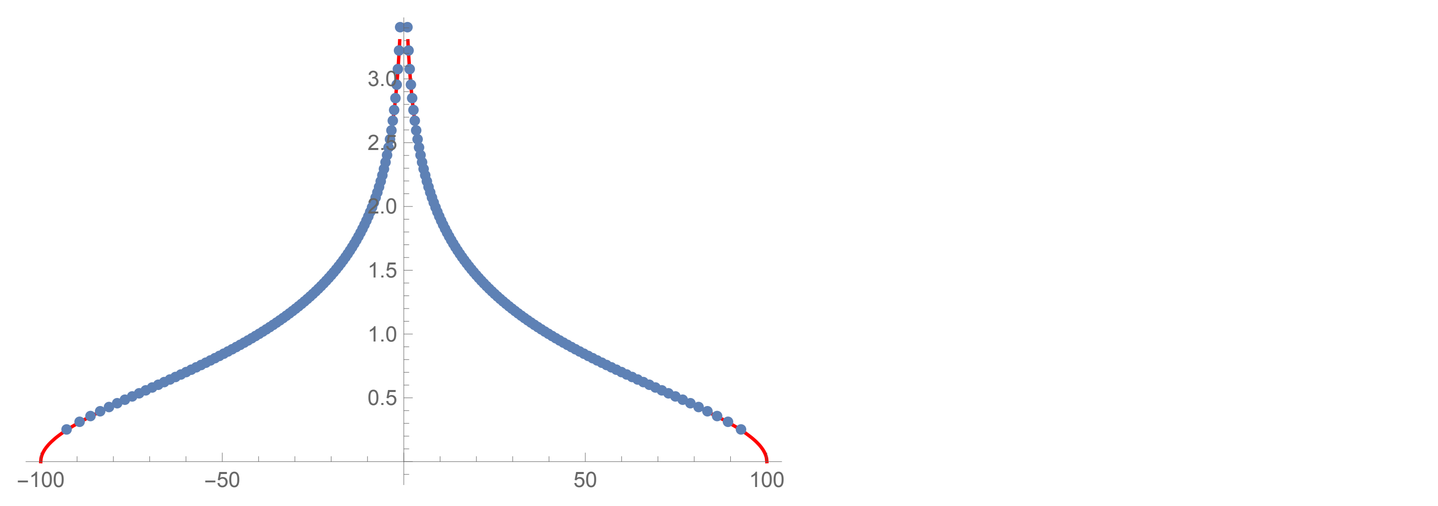
$$\frac{395579392\,u^{12}}{3565848825}-\frac{163391488\,u^{10}}{46309725}+\frac{25359003904\,u^8}{756392175}-\frac{747764992\,u^6}{6615675}+\frac{5853853616\,u^4}{46309725}-\frac{98176135928\,u^2}{2773437975}+1$$

$$\frac{1048576}{53361}$$

Extras

• Try to compute the energy using the $Q_0(u) = {}_3F_2(-S, S+1, 1/2 - iu; 1, 1; 1)$. Show that for non-integer S result is infinite

• Find zeros of $Q(u)$ for $S = 200$, plot the density of the roots



Compare with the analytical result $\rho(u) = \frac{2}{\pi}\tanh^{-1}\left(\sqrt{1 - \frac{u^2}{S^2}}\right)$ of Korchemsky

• The energy is not finite becouse there are two linear independed solutions $Q_0(u)$ and $Q_0(-u)$. Only a particular combination will give the correct energy

$$Q(u) = [Q_0(-u) + Q_0(u)]\cosh^2(\pi u) - \frac{1}{2} i\,[Q_0(-u) - Q_0(u)]\cot\left(\frac{\pi S}{2}\right)\sinh(2\pi u)$$

Check that this combination leads indeed to the finite energy for any S

• The BFKL regime to the leading order is described by the Kotikov-Lipatov result

$$\frac{S+1}{-4g^2} = \psi(\frac{1}{2} - \frac{\Delta}{2}) + \psi(\frac{1}{2} + \frac{\Delta}{2}) - 2\psi(1) + \mathcal{O}(g^2)$$

Expand Δ in powers of g, extracting the leading singularity in the limit $S \to -1$ at each order of perturbation theory ($\psi(x)$ is `PolyGamma[0,x]` in `Mathematica`)

you shoud find that the transcedentality (sum of arguments of zeta functions) is the same for all terms and is equal to number of loops -1. For example at 10 loops you should find

```
Coefficient[res, g, 20] g^20
```

$$g^{20}\left(-\frac{4\,194\,304\,\zeta(9)}{(S+1)^{10}} - \frac{201\,326\,592\,\zeta(3)^3}{(S+1)^{10}}\right)$$