I. CONFORMAL DATA BY DIAGONALIZING A QUANTUM SPIN CHAIN HAMILTONIAN [EASY/MEDIUM]

[Problem 1 for Guifre Vidal's lectures for Mathematica Summer School]

A. Introduction

Consider the Hamiltonian for the critical quantum Ising model

$$H = -\sum_{r=1}^{N} (\sigma_{z,r} + \sigma_{x,r}\sigma_{x,r+1}) \tag{1}$$

where

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

are spin-1/2 Pauli matrices, r labels the N sites of the quantum spin chain, and we assume periodic boundary conditions, so that site $r = N+1$ is actually $r = 1$.

This Hamiltonian is invariant under translations, $THT^\dagger = H$, where T is a one-site translation operator, which maps site r to site $r+1$. The eigenvectors $|E_n\rangle$ of H,

$$H|E_n\rangle = E_n|E_n\rangle, \tag{3}$$

can therefore be chosen to have well-defined momentum K_n,

$$T|E_n\rangle = e^{i\frac{2\pi}{N}K_n}|E_n\rangle, \tag{4}$$

for some $K_n \in \{-(N/2-1), \cdots, -1, 0, 1, \cdots N/2\}$, where for simplicity we have assumed N to be even. It turns out that at a quantum critical point, the low energy spectra of H is organized according to the scaling dimensions $\Delta_n \equiv h_n + \bar{h}_n$ and conformal spins $s_n \equiv h_n - \bar{h}_n$ of the scaling operators ϕ_n of the theory, where h_n and $\bar{h}_n$ are the (holomorphic and antiholomorphic) conformal dimensions of ϕ_n [This is a manifestation of the operator-state correspondence in a CFT]. Specifically,

$$E_n = A + \frac{B}{N}\Delta_n + O(\frac{1}{N^2}), \qquad K_n = s_n, \tag{5}$$

where A and B are non-universal constants (they depend on the normalization of H). Taking into account that the *identity* primary field $\mathbb{I}$ (which corresponds to the ground state $|E_0\rangle$) has $\Delta_{\mathbb{I}} = 0$, we find that

$$E_n - E_0 = \frac{B}{N}\Delta_n + O(\frac{1}{N^2}). \tag{6}$$

In addition, in every CFT, the identity primary field has the stress tensor as a second descendant, with $\Delta = 2$ and $s = 2$ for the holomorphic part (and $s = -2$ for the antiholomorphic part). Let $|E_{\tilde{n}}\rangle$ be the corresponding eigenstate of H. Then,

$$\Delta_n = 2\frac{E_n - E_0}{E_{\tilde{n}} - E_0} + O(\frac{1}{N}). \tag{7}$$

That is, we have obtained an estimate of Δ_n from E_n (and E_0 and $E_{\tilde{n}}$)

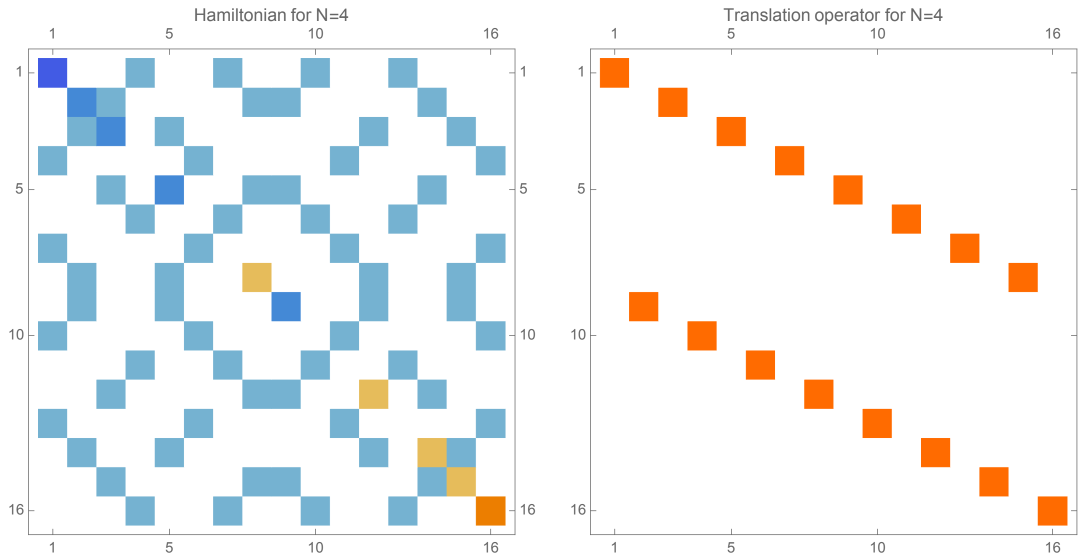

FIG. 1: `MatrixPlot` of the Hamiltonian and Translation operator for four sites.

B. Exact diagonalization. Scaling dimensions and conformal spins

For $N = 6, 8, 10, 12$:

- Build $2^N \times 2^N$ matrices for the Hamiltonian operator H and for the translation operator T.

  Hint: Tensor products can be easily implemented in Mathematica using `KroneckerProduct`. Check it on two small matrices. Using it you can easily generate the Hamiltonian. To generate the translation operator you have at least two simple options: one is to generate it as a matrix directly (after all it follows a very simple pattern as illustrated in figure 1); another option is to create it as a sequence of permutations of neighbouring spins $T = P_{12}P_{23}\dots P_{N-1N}$. If you use this second approach you might find the representation $P = \frac{1}{2}(\mathbb{I} + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$ useful. If so, make sure to understand why this representation is correct. Either way, as a basic check of the translation operator you obtain, you can check that T^N is the identity. Note: When dealing with huge matrices with lots of zeros (such as these ones) it is quite convenient to use `SparseArray`'s. As a check, you can visualize these matrices with `MatrixPlot`. For $N = 4$ you should get something as in figure 1.

- Diagonalize H so as to obtain the energy eigenvectors $|E_n\rangle$. `ListPlot` the lowest 12 energy levels.

  Hint: You can use the built-in functions `Eigenvalues` and/or `Eigensystem`.

- For the 12 lowest energies $\{E_0, E_1, \cdots, E_{11}\}$, identify the corresponding momenta $\{K_0, K_1, \cdots, K_{11}\}$.

  Hint: There are at least two natural ways of achieving the simultaneous diagonalization of H and T. One option is to diagonalize first the Hamiltonian and then, within each degenerate subspace, diagonalize the translation operator. This option is easy and fast but might be slightly inconvenient to automatize. Another option is to use the following trick: We can always diagonalize $(H + c\mathbb{I}) \cdot T$ with c being a constant shift such that all energies are positive. The eigenvectors of this operator diagonalize simultaneously both the Hamiltonian and the translation operator. From its eigenvalues we straightforwardly extract the energies (from the absolute values) as well as the corresponding momenta (from the phases) at once. The disadvantage of this second method is that since this new operator is less symmetric (and non-Hermitian), its diagonalization can be slightly slower.

- From (E_n, K_n) for $n = 0, 1, \cdots 11$, estimate the scaling dimensions and conformal spin (Δ_n, s_n) of the first 12 scaling operators $\{\phi_0, \phi_1, \cdots, \phi_{11}\}$ of the Ising CFT. Fig. 2 shows how the result should look like in the limit of large N, where the $O(1/N)$ corrections vanish.

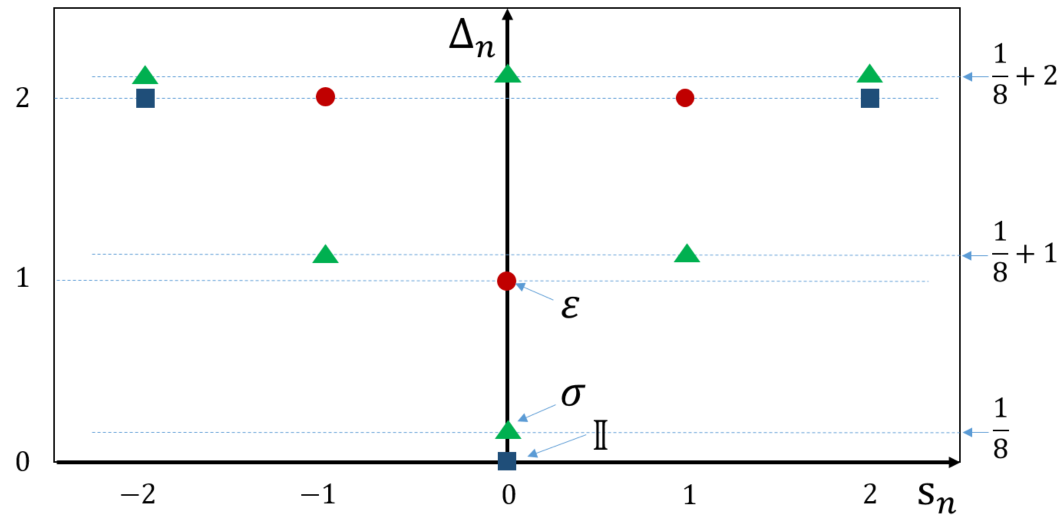

FIG. 2: Scaling dimensions Δ_n and conformal spin s_n for the 1+1 critical Ising model. Only the 12 lowest scaling dimensions have been plotted. Notice that the scaling operators ϕ_n are organized in three conformal towers, corresponding to the *identity* $\mathbb{I}$, *spin* σ, and *energy density* ε primary fields, which have scaling dimensions 0, 1/8, and 1, respectively.