Home assignment 2

Numerical Optimization and its Applications - Spring 2019

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1 The efficiency of different iterative methods for solving a linear system

(a) Following are the implementation for the four methods:
Jacobi:

```
from numpy import diag, matmul, array
from numpy.linalg import inv, norm


def weighted_jacobi(A, b, x_0, maxIter, epsilon, w):
    D = diag(diag(A))
    x = x_0
    res = [norm(matmul(A, x) - b)]
    for i in range(maxIter):
        x = x + w * matmul(inv(D), b - matmul(A, x))
        res.append(norm(matmul(A, x) - b))
        if res[-1] / norm(b) < epsilon:
            break
    return x, array(res)
```

Gauss-Seidel:

```
from numpy import matmul, tril, array
from numpy.linalg import inv, norm


def weighted_gauss_seidel(A, b, x_0, maxIter, epsilon, w):
    L_D = tril(A, k=0)
    x = x_0
    res = [norm(matmul(A, x) - b)]
    for i in range(maxIter):
        x = x + w * matmul(inv(L_D), b - matmul(A, x))
        res.append(norm(matmul(A, x) - b))
        if res[-1] / norm(b) < epsilon:
            break
    return x, array(res)
```

Steepest Descent:

```python
from numpy import matmul, array, dot, copy
from numpy.linalg import norm
from py_files.utils import is_pos_def


def steepest_decent(A, b, x_0, max_iter, epsilon):
    if not is_pos_def(A):
        print("matrix is not SPD, can't solve using steepest decent...")
        return None, None
    x = copy(x_0)
    r = b - matmul(A, x)
    all_r = [norm(r)]
    for k in range(max_iter):
        alph = dot(r, r) / dot(r, matmul(A, r))
        x = x + alph * r
        # res.append(norm(matmul(A, x) - b))
        r = b - matmul(A, x)
        all_r.append(norm(r))
        if norm(r) / norm(b) < epsilon:
            break
    return x, array(all_r)
```

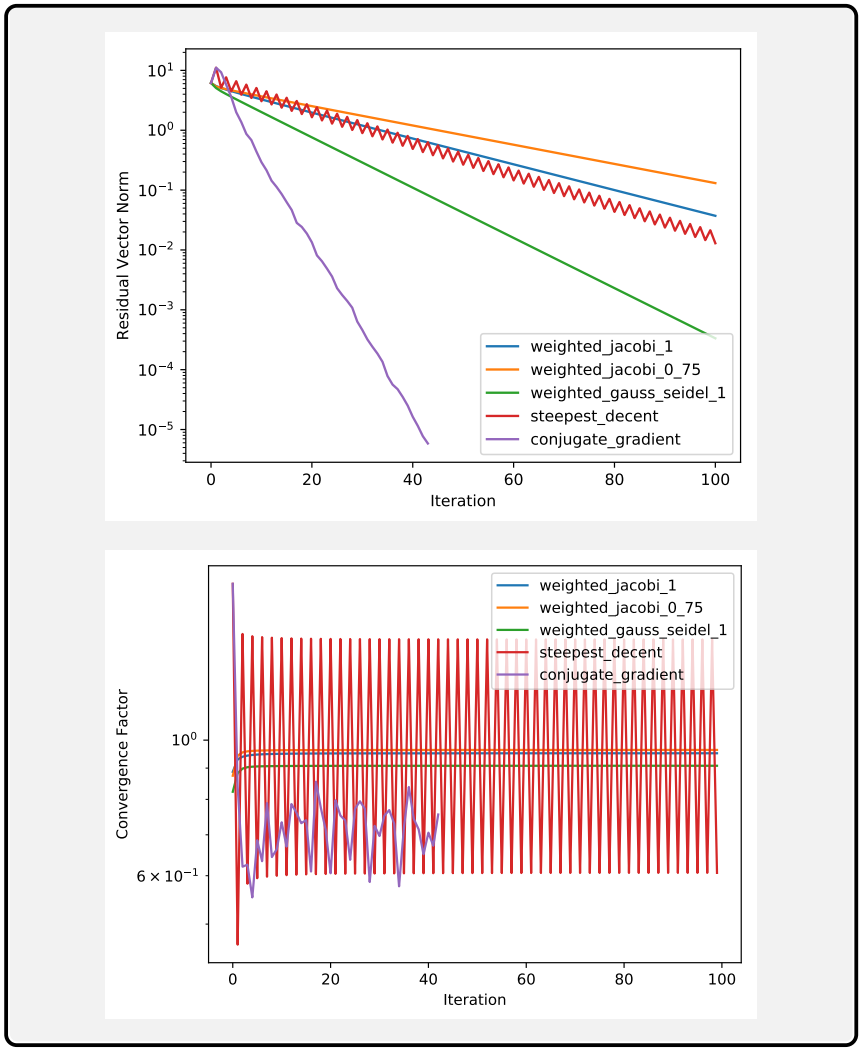
Conjugate Gradient

```python
from numpy import matmul, array, dot, copy
from numpy.linalg import norm
from py_files.utils import is_pos_def


def conjugate_gradient(A, b, x_0, max_iter, epsilon):
    if not is_pos_def(A):
        print("matrix is not SPD, can't solve using steepest decent...")
        return None, None
    x = copy(x_0)
    r = b - matmul(A, x)
    p = r
    all_r = [norm(r)]
    for k in range(max_iter):
        alph = dot(r, p) / dot(p, matmul(A, p))
        x = x + alph * p
        # res.append(norm(matmul(A, x) - b))
        r_prev = copy(r)
        r = b - matmul(A, x)
        all_r.append(norm(r))
        if norm(r) / norm(b) < epsilon:
            break
        beta = dot(r, r) / dot(r_prev, r_prev)
        p = r + beta * p
    return x, array(all_r)
```

(b) Following are the system and parameters definition, methods calls, residual vector norm and convergence factor plotting:

```python
import numpy as np
from scipy.sparse import spdiags
import matplotlib.pyplot as plt
from py_files.part_1_gauss_seidel import weighted_gauss_seidel
from py_files.part_1_jacobi import weighted_jacobi
from py_files.part_1_sd import steepest_decent
from py_files.part_1_cg import conjugate_gradient

n = 100
```

```
# TODO: not sure if .toarray() is the right approach
A = spdiags(np.array([-np.ones(n), 2.1 * np.ones(n), -np.ones(n)]),
            np.array([-1, 0, 1]), n, n).toarray()
x_0 = np.zeros(n)
b = np.random.rand(n)
maxIter = 100
epsilon = 1e-6

res = dict()
x, res['weighted_jacobi_1'] = weighted_jacobi(A, b, x_0, maxIter, epsilon, 1)
#print('weighted_jacobi_1 result:', x)
x, res['weighted_jacobi_0_75'] = weighted_jacobi(A, b, x_0, maxIter,epsilon,0.75)
#print('weighted_jacobi_0_75 result:', x)
x,res['weighted_gauss_seidel_1']=weighted_gauss_seidel(A,b,x_0,maxIter,epsilon,1)
#print('weighted_gauss_seidel_1 result:', x)
x, res['steepest_decent'] = steepest_decent(A, b, x_0, maxIter, epsilon)
#print('steepest_decent result:', x)
x, res['conjugate_gradient'] = conjugate_gradient(A, b, x_0, maxIter, epsilon)
#print('conjugate_gradient result:', x)

convergence_factor = dict()
for alg_res in res:
    convergence_factor[alg_res] = res[alg_res][1:] / res[alg_res][:-1]

plt.figure()
for alg_res in res:
    plt.semilogy(res[alg_res], label=alg_res)
plt.legend()
plt.xlabel("Iteration")
plt.ylabel("Residual Vector Norm")

plt.savefig("myplot1.pdf", bbox_inches="tight")
print(r"\saveandshowplot{myplot1.pdf}")

plt.figure()
for alg_con in convergence_factor:
    plt.semilogy(convergence_factor[alg_con], label=alg_con)
plt.legend()
plt.xlabel("Iteration")
plt.ylabel("Convergence Factor")

plt.savefig("myplot2.pdf", bbox_inches="tight")
print(r"\saveandshowplot{myplot2.pdf}")
```



2 Convergence properties

(a)

Lemma 1.

$$0 < \alpha < \frac{2}{\lambda_{max}} \Rightarrow \rho(I - \alpha A) < 1$$

Proof. A is symmetric positive definite matrix, thus:

$$0 < \lambda_{min} \leq \cdots \leq \lambda_{max}$$

therefore, we get that:

$$\rho(I - \alpha A) = max(|1 - \alpha\lambda_{min}|, |1 - \alpha\lambda_{max}|)$$

$|1 - \alpha\lambda_{max}|$, then we get that:

$$-1 < 1 - \alpha\lambda_{max} < 1 \Rightarrow |1 - \alpha\lambda_{max}| < 1$$

And,

$$-1 < 1 - 2\frac{\lambda_{min}}{\lambda_{max}} < 1 - \alpha\lambda_{min} < 1 \Rightarrow |1 - \alpha\lambda_{min}| < 1$$

thus,

$$max(|1 - \alpha\lambda_{min}|, |1 - \alpha\lambda_{max}|) < 1$$

so we get that:

$$\rho(I - \alpha A) < 1$$

□

In our case:

$$\alpha = \frac{1}{||A||}$$

we know that for any induced norm:

$$||A|| > \rho(A) = \lambda_{max}$$

thus,

$$\frac{1}{||A||} < \frac{1}{\lambda_{max}} < \frac{2}{\lambda_{max}}$$

therefore, by **Lemma 1** we get that

$$\rho(I - \alpha A) \leq 1$$

and the method converges.

(b) In the case A is indefinite, we a negative eigenvalue, therefore:

$$\rho(I - \alpha A) \geq |1 - \alpha\lambda_{min}|$$

$\lambda_{min} < 0$, by definition, therefore

$$\rho(I - \alpha A) \geq |1 - \alpha\lambda_{min}| > 1$$

(c) (i)

$$f(\mathbf{x}) = \frac{1}{2}||\mathbf{x}^* - \mathbf{x}||_A^2$$

$$
\begin{aligned}
f(\mathbf{x}^{(k)}) &= \frac{1}{2}||\mathbf{x}^* - \mathbf{x}^{(\mathbf{k})}||_A^2 \\
&= \frac{1}{2}||\mathbf{e}^{(k)}||_A^2 \\
&= \frac{1}{2}((\mathbf{e}^{(k)})^T A\mathbf{e}^{(k)}) \\
&= \frac{1}{2}\langle \mathbf{e}^{(k)}, A\mathbf{e}^{(k)} \rangle
\end{aligned}
$$

$$
\begin{aligned}
f(\mathbf{x}^{(k+1)}) &= f(\mathbf{x}^{(k)}) + \alpha \mathbf{r}^{(k)} \\
&= \frac{1}{2}\langle \mathbf{e}^{(k)}, A\mathbf{e}^{(k)} \rangle - \alpha \langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle + \frac{1}{2}\alpha^2 \langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle \\
&= \frac{1}{2}\langle \mathbf{e}^{(k)}, A\mathbf{e}^{(k)} \rangle - \frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle} + \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2 \langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle^2} \quad * \alpha = \frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle} \\
&= \frac{1}{2}\langle \mathbf{e}^{(k)}, A\mathbf{e}^{(k)} \rangle - \frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle} + \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle} \\
&= \frac{1}{2}\langle \mathbf{e}^{(k)}, A\mathbf{e}^{(k)} \rangle - \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle} \\
&= f(\mathbf{x}^{(k)}) - \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle}
\end{aligned}
$$

We get that:

$$
f(\mathbf{x}^{(k+1)}) = f(\mathbf{x}^{(k)}) - \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle}
$$

A is symmetric positive definite matrix, thus

$$
\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle} > 0
$$

and therefore,

$$
f(\mathbf{x}^{(k+1)}) = f(\mathbf{x}^{(k)}) - \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle} < f(\mathbf{x}^{(k)})
$$

(ii) From previous section:

$$
f(\mathbf{x}^{(k+1)}) = C^{(k)} f(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) - \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle}
$$

thus,

$$
C^{(k)} = 1 - \frac{1}{2}\frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle f(\mathbf{x}^{(k)})}
$$

finally,

$$
C^{(k)} = 1 - \frac{\langle \mathbf{r}^{(k)}, A\mathbf{e}^{(k)} \rangle^2}{\langle \mathbf{r}^{(k)}, A\mathbf{r}^{(k)} \rangle \langle \mathbf{e}^{(k)}, A\mathbf{e}^{(k)} \rangle}
$$

(iii) t

(iv) t

3 GMRES(1) method

(a)

$$||\mathbf{r}^{(k+1)}||_2 = ||\mathbf{b} - A\mathbf{x}^{(k+1)}||_2$$

we define the following scalar function $g(\alpha)$:

$$\begin{aligned} g(\alpha) &\triangleq f(\mathbf{x}^{(k)}) + \alpha\mathbf{r}^{(k)} \\ &= \frac{1}{2}||\mathbf{b} - A\mathbf{x}^{(k)} - \alpha A\mathbf{r}^{(k)}||_2 \\ &= \frac{1}{2}||\mathbf{r}^{(k)} - \alpha A\mathbf{r}^{(k)}||_2 \\ &= \frac{1}{2}(\mathbf{r}^{(k)})^T\mathbf{r}^{(k)} - \alpha(\mathbf{r}^{(k)})^T A\mathbf{r}^{(k)} + \frac{1}{2}\alpha^2(A\mathbf{r}^{(k)})^T A\mathbf{r}^{(k)} \end{aligned}$$

And the minimization of g with respect to α is done by:

$$g'(\alpha) = -(\mathbf{r}^{(k)})^T A\mathbf{r}^{(k)} + \alpha(A\mathbf{r}^{(k)})^T A\mathbf{r}^{(k)} = 0$$

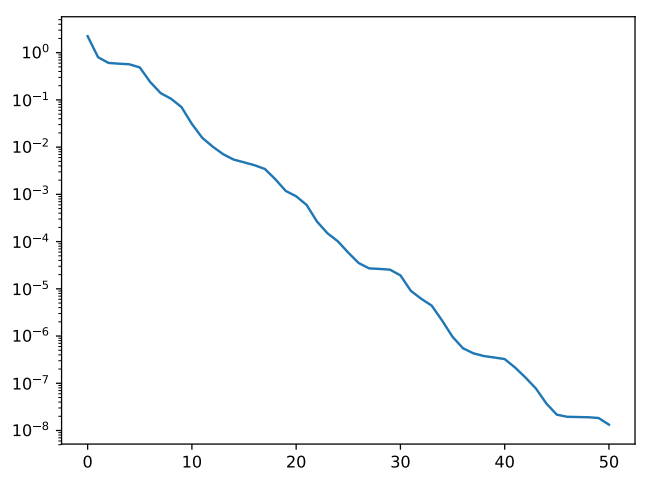
$$\Rightarrow \alpha_{opt} = \frac{(\mathbf{r}^{(k)})^T A\mathbf{r}^{(k)}}{(A\mathbf{r}^{(k)})^T A\mathbf{r}^{(k)}} = \frac{(\mathbf{r}^{(k)})^T A\mathbf{r}^{(k)}}{(\mathbf{r}^{(k)})^T A^T A\mathbf{r}^{(k)}}$$

(b) (non-mandatory)

(c) a:

```python
from numpy import matmul, array, dot, copy, transpose, vectorize
from numpy.linalg import norm
from py_files.utils import is_pos_def
import matplotlib.pyplot as plt


def steepest_decent(A, b, x_0, max_iter, epsilon):
    if not is_pos_def(A):
        print("matrix is not SPD, can't solve using steepest decent...")
        return None, None
    x = copy(x_0)
    r = b - matmul(A, x)
    all_r = [norm(r)]
    for k in range(max_iter):
        alph = dot(r, matmul(A, r)) / matmul(r, matmul(transpose(A),matmul(A, r)))
        x = x + alph * r
        r = b - matmul(A, x)
        all_r.append(norm(r))
        if norm(r) / norm(b) < epsilon:
            break
    return x, array(all_r)


A = array([
    [5, 4, 4, -1, 0],
```

```
    [3, 12, 4, -5, -5],
    [-4, 2, 6, 0, 3],
    [4, 5, -7, 10, 2],
    [1, 2, 5, 3, 10]
])
b = array([1, 1, 1, 1, 1])
x_0 = array([0, 0, 0, 0, 0])
x, all_r = steepest_decent(A, b, x_0, 50, 0.00000000001)

plt.figure()
plt.semilogy(all_r)
plt.savefig("myplot3.pdf", bbox_inches="tight")
print(r"\saveandshowplot{myplot3.pdf}")
```



(d) t

(e) t

4 Convexity

(a) i. e^{ax} is convex:

$$(e^{ax})'' = a^2 e^{ax} \geq 0 \quad \forall x$$

ii. $-log(x)$ is convex:

$$(-log(x))'' = \frac{1}{x^2} > 0 \quad \forall x > 0$$

iii. $log(x)$ is concave:

$$(log(x))'' = -\frac{1}{x^2} < 0 \quad \forall x > 0$$

iv. $|x|^a, \quad a \geq 1$ is convex:

$$\begin{aligned} f(\alpha x + (1-\alpha)y) &= |\alpha x + (1-\alpha)y|^a \\ &\leq (|\alpha x| + |(1-\alpha)y|)^a \qquad \text{*triangle inequality} \\ &= (\alpha|x| + (1-\alpha)|y|)^a \\ &\leq \alpha|x|^a + (1-\alpha)|y|^a \qquad *\,\alpha \leq 1 \\ &= \alpha f(x) + (1-\alpha)f(y) \end{aligned}$$

v. x^3 is none of those:
Not convex:
for $x = -1, y = 0, \alpha = 0.5$:

$$f(\alpha x + (1-\alpha)y) = f(0.5(-1) + (1-0.5)0) = f(-0.5) = (-0.5)^3 = -0.125$$

$$\alpha f(x) + (1-\alpha)f(y) = 0.5f(-1) + (1-0.5)f(0) = 0.5(-1)^3 = -0.5$$

we get that

$$f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y)$$

Not concave:
for $x = 0, y = 1, \alpha = 0.5$:

$$f(\alpha x + (1-\alpha)y) = f(0.5(0) + (1-0.5)1) = f(0.5) = (0.5)^3 = 0.125$$

$$\alpha f(x) + (1-\alpha)f(y) = 0.5f(0) + (1-0.5)f(1) = 0.5(1)^3 = 0.5$$

we get that

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$$

(b) Let

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + \mathbf{c}$$

computing the Hessian of $f(\mathbf{x})$

$$\nabla f(\mathbf{x}) = 2A\mathbf{x} + \mathbf{b}$$

$$\nabla^2 f(\mathbf{x}) = J(\nabla f(\mathbf{x})) = 2A$$

$f(\mathbf{x})$ is a convex function over a convex region Ω if and only if $2A \succeq 0$, is positive semi definite.

(c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in a convex domain Ω. We'll show the following:

$$f \quad \text{is convex} \Leftrightarrow f(y) > f(x) + \nabla f(x)^T(y-x) \quad , \forall x, y \in \Omega$$

$\Rightarrow$:
f is convex. Then, according to the fundamental definition of convex functions, the following inequality condition must be satisfied:

$$\begin{aligned}
&f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \qquad \forall x, y \in \Omega \wedge \alpha \in [0,1] \\
&\Rightarrow f(x + \alpha(y-x)) \leq f(x) + \alpha(f(y) - f(x)) \\
&\Rightarrow f(x + \alpha(y-x)) - f(x) \leq \alpha(f(y) - f(x)) \\
&\Rightarrow \frac{f(x + \alpha(y-x)) - f(x)}{\alpha} \leq f(y) - f(x) \\
&\Rightarrow f(y) \geq f(x) + \frac{f(x + \alpha(y-x)) - f(x)}{\alpha}
\end{aligned}$$

Now, let

$$g(\alpha) = f(x + \alpha(y-x))$$

therefore

$$f(y) \geq f(x) + \frac{g(\alpha) - g(0)}{\alpha} \quad * g(0) = f(x)$$

Now taking the limit as $\alpha \to 0$ we get

$$\begin{aligned}
&f(y) \geq f(x) + \lim_{\alpha \to 0} \frac{g(\alpha) - g(0)}{\alpha} \\
&\Rightarrow f(y) \geq f(x) + g'(0)
\end{aligned}$$

In order to find $g'(0)$, we'll compute the more general $g'(t)$:

$$g'(t) = \nabla_x f(x + t(y-x))^T (y-x)$$

assigning $t = 0$, we get

$$g'(0) = \nabla_x f(x)^T (y-x)$$

finally, by substituting $g'(0)$ we get

$$f(y) \geq f(x) + \nabla_x f(x)^T (y-x)$$

therefore

$$f \quad \text{is convex} \Rightarrow f(y) > f(x) + \nabla f(x)^T (y-x) \quad , \forall x, y \in \Omega$$

$\Leftarrow$:
Let

$$f(y) > f(x) + \nabla_x f(x)^T (y-x) \quad , \forall x, y \in \Omega$$

Now, consider

$$z = \alpha x + (1-\alpha)y \quad \forall \alpha \in [0,1]$$

Notice that, since Ω is a convex domain, $z \in \Omega$. Therefore:

$$f(x) > f(z) + \nabla_z f(z)^T (x - z) \tag{1}$$

$$f(y) > f(z) + \nabla_z f(z)^T (y - z) \tag{2}$$

Now multiplying the inequalities in Equations (1) and (2) with α and $(1 - \alpha)$ respectively and adding the results we get:

$$\alpha f(x) + (1 - \alpha) f(y) > f(z) + \nabla_z f(z)^T (\alpha x + (1 - \alpha) y - z)$$

by substituting $z = \alpha x + (1 - \alpha) y$, we get

$$\alpha f(x) + (1 - \alpha) f(y) > f(\alpha x + (1 - \alpha) y) + \nabla_z f(z)^T (\alpha x + (1 - \alpha) y - \alpha x + (1 - \alpha) y)$$
$$\Rightarrow \alpha f(x) + (1 - \alpha) f(y) > f(\alpha x + (1 - \alpha) y)$$

Observe that this is exactly the inequality that $f(x)$ must satisfy in order to be considered as a convex function, hence

$$f \quad \text{is convex} \Leftarrow f(y) > f(x) + \nabla f(x)^T (y - x) \quad , \forall x, y \in \Omega$$

and combining the 2 sides, we showed that

$$f \quad \text{is convex} \Leftrightarrow f(y) > f(x) + \nabla f(x)^T (y - x) \quad , \forall x, y \in \Omega$$

5 Non Linear Optimization

(a) Following is the implementation for the function that, given data matrix X and labels, computes the logistic regression objective, its gradient, and its Hessian matrix:

```python
from numpy import matmul, exp, log, asarray, diag, outer


def sigmoid(x):
    return 1 / (1 + exp(-x))


def reg_obj_grad_hes(X, y, w):
    m = len(y)
    c_1 = y
    c_2 = 1-y
    sig = sigmoid(matmul(X.T,w))
    obj = -(1/m)*(matmul(c_1.T,log(sig))+matmul(c_2.T,log(1-sig)))
    grad = (1/m)*matmul(X, sig-c_1)
    # TODO: maybe outer(sig,(1-sig)) should be replaced to diag(sig*(1-sig))
    hes = (1/m)*matmul(matmul(X, diag(sig*(1-sig))), X.T)
    return obj, grad, hes
```

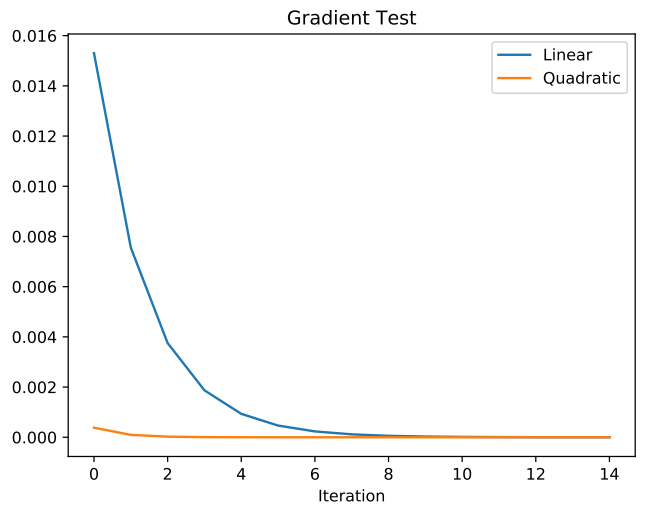
(b) Implementation of Gradient and Jacobian verification:

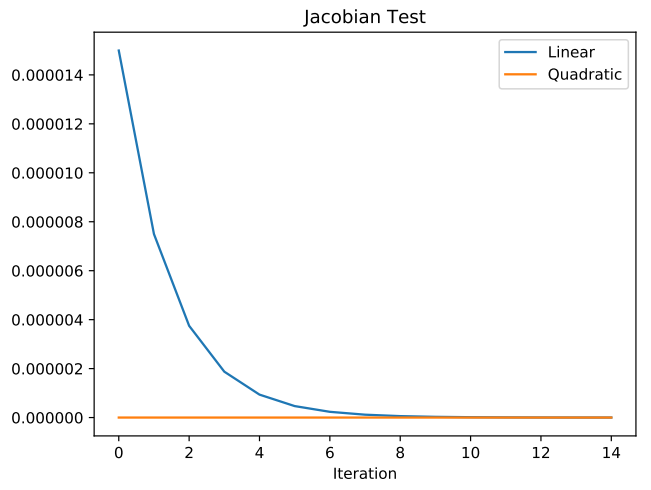
```python
from numpy.random import rand
from numpy import matmul
from numpy.linalg import norm
from py_files.part_5_a import reg_obj_grad_hes


def gradient_test_reg_obj_grad_hes(X, y, w, epsilon, iterations):
    d = rand(w.shape[0])
    d = d/d.sum()
    res1 = []
    res2 = []
    obj_0, grad_0, _ = reg_obj_grad_hes(X, y, w)
    for _ in range(iterations):
        obj, grad, _ = reg_obj_grad_hes(X, y, w+(epsilon*d))
        res1.append(abs(obj-obj_0))
        res2.append(abs(obj-obj_0-epsilon*matmul(d.T,grad_0)))
        epsilon *= 0.5
    return res1, res2


def jacobian_test_reg_obj_grad_hes(X, y, w, epsilon, iterations):
    d = rand(w.shape[0])
    d = d/d.sum()
    res1 = []
    res2 = []
    obj_0, grad_0, hes_0 = reg_obj_grad_hes(X, y, w)
    for _ in range(iterations):
        obj, grad, _ = reg_obj_grad_hes(X, y, w+(epsilon*d))
        res1.append(norm(grad-grad_0))
        res2.append(norm(grad-grad_0-matmul(hes_0, epsilon*d.T)))
        epsilon *= 0.5
    return res1, res2
```

Testing our gradient and hessiam implementation:

```python
from numpy import asarray, array
from numpy.random import rand
from mnist import MNIST
import matplotlib.pyplot as plt
from py_files.part_5_b import gradient_test_reg_obj_grad_hes
from py_files.part_5_b import jacobian_test_reg_obj_grad_hes

X = array([[1, 0, 1], [0, 1, 1]]).T
y = array([0, 1])
w = array([1, 1, 0])

res1, res2 = gradient_test_reg_obj_grad_hes(X, y, w, 1e-1, 15)

plt.figure()
plt.plot(res1, label="Linear")
plt.plot(res2, label="Quadratic")
plt.legend()
plt.title("Gradient Test")
plt.xlabel("Iteration")

plt.savefig("myplot4.pdf", bbox_inches="tight")
print(r"\saveandshowplot{myplot4.pdf}")

res1, res2 = jacobian_test_reg_obj_grad_hes(X, y, w, 1e-4, 15)

plt.figure()
plt.plot(res1, label="Linear")
plt.plot(res2, label="Quadratic")
plt.legend()
```

```
plt.title("Jacobian Test")
plt.xlabel("Iteration")

plt.savefig("myplot5.pdf", bbox_inches="tight")
print(r"\saveandshowplot{myplot5.pdf}")
```





(c) Implementation of Gradient Descent, and exact Newton methods:

```python
from numpy import matmul, array, dot, copy, asarray, mean, round, eye
from numpy.linalg import norm, inv
from numpy.random import rand
from mnist import MNIST
import matplotlib.pyplot as plt
from py_files.utils import is_pos_def
from py_files.part_5_a import reg_obj_grad_hes, sigmoid


def steepest_decent(X, y, w_0, max_iter, epsilon, alph):
    w = copy(w_0)
    f = []
    for _ in range(max_iter):
        obj, grad, _ = reg_obj_grad_hes(X, y, w)
        w = w - alph * grad
        f.append(obj)
        if len(f) > 1 and f[-2] - obj < epsilon:
            break
    return w, f


def newton(X, y, w_0, max_iter, epsilon):
    w = copy(w_0)
    f = []
    for _ in range(max_iter):
        obj, grad, hes = reg_obj_grad_hes(X, y, w)
        w = w - matmul(inv(hes+eye(hes.shape[0])*epsilon), grad)
        f.append(obj)
        if len(f) > 1 and f[-2] - obj < epsilon:
            break
    return w, f


def accuracy(X, y, w):
    return mean(y == round(sigmoid(matmul(X.T, w))))
```

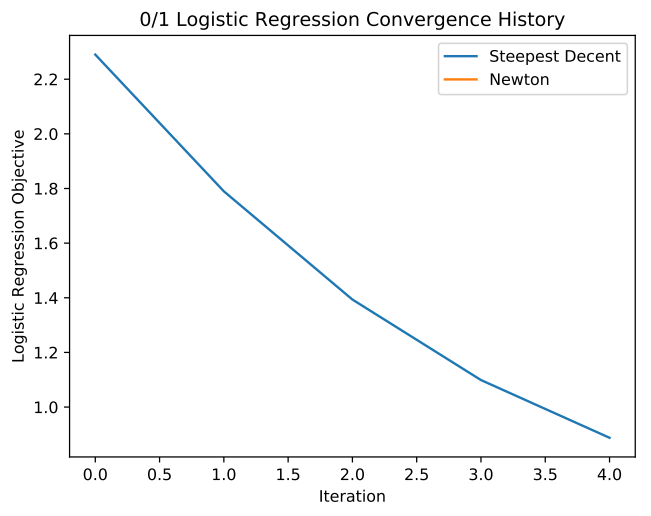
Running over the MNIST data sets for the digits 0,1 and 8,9 and shwing the results:

```python
from numpy import matmul, array, dot, copy, asarray, sum
from numpy.linalg import norm, inv
from numpy.random import rand
from mnist import MNIST
import matplotlib.pyplot as plt
from py_files.utils import is_pos_def
from py_files.part_5_c import steepest_decent, newton, accuracy

# loading data


try:
    mndata = MNIST('py_files/data')
    mndata.gz = True
    images_train, labels_train = mndata.load_training()
    images_test, labels_test = mndata.load_testing()
except FileNotFoundError:
    mndata = MNIST('data')
    mndata.gz = True
    images_train, labels_train = mndata.load_training()
    images_test, labels_test = mndata.load_testing()

# parameters definition
max_population = 30000
max_iter = 5 # TODO: change back to 1000
```

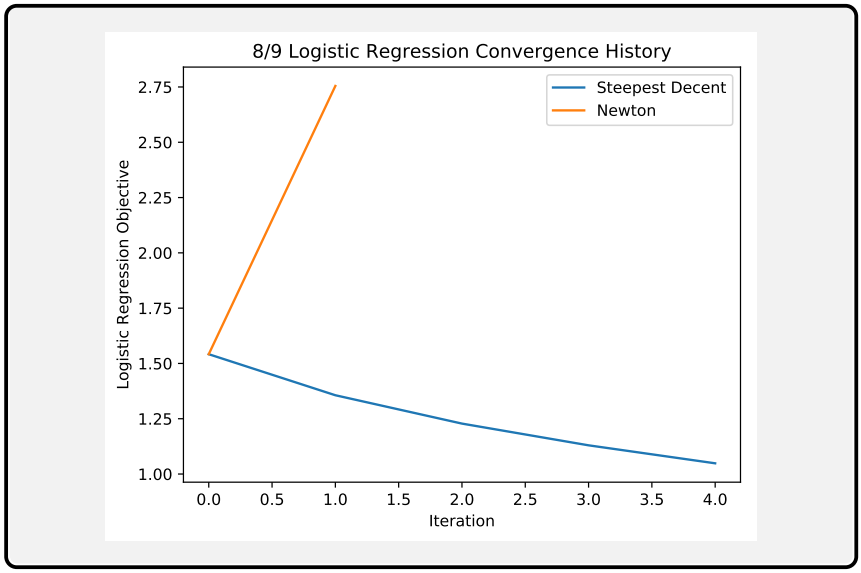
```
epsilon = 1e-2
alph = 1e-1

# 0/1

# train data pre processing
images_train = asarray(images_train)
labels_train = asarray(labels_train)
X_train = images_train[labels_train <= 1].T
X_train = X_train / X_train.max()
y_train = labels_train[labels_train <= 1].T

# test data pre processing
images_test = asarray(images_test)
labels_test = asarray(labels_test)
X_test = images_test[labels_test <= 1].T
X_test = X_test / X_test.max()
y_test = labels_test[labels_test <= 1].T

# initializing weights
w_0 = rand(X_train.shape[0]) * 2 - 1

# running sd an newton
w_sd, f_sd = steepest_decent(X_train, y_train, w_0, max_iter, epsilon, alph)
w_n, f_n = newton(X_train, y_train, w_0, max_iter, epsilon)

# computing sd accuracy
print(r"Steepest Decent Accuracy:")
print(r"Train:", accuracy(X_train, y_train, w_sd))
print(r"Test:", accuracy(X_test, y_test, w_sd))

# computing newton accuracy
print(r"Newton Accuracy:")
print(r"Train:", accuracy(X_train, y_train, w_n))
print(r"Test:", accuracy(X_test, y_test, w_n))

# plotting convergence history
plt.figure()
plt.plot(f_sd, label="Steepest Decent")
plt.plot(f_n, label="Newton")
plt.legend()
plt.title("0/1 Logistic Regression Convergence History")
plt.ylabel("Logistic Regression Objective")
plt.xlabel("Iteration")

plt.savefig("myplot6.pdf", bbox_inches="tight")
print(r"\saveandshowplot{myplot6.pdf}")

# 8/9

# train data pre processing
images_train = asarray(images_train)
labels_train = asarray(labels_train)
X_train = images_train[labels_train >= 8].T
X_train = X_train / X_train.max()
y_train = labels_train[labels_train >= 8].T
y_train -= 8

# test data pre processing
images_test = asarray(images_test)
labels_test = asarray(labels_test)
X_test = images_test[labels_test >= 8].T
X_test = X_test / X_test.max()
y_test = labels_test[labels_test >= 8].T
y_test -= 8
```

```
# initializing weights
w_0 = rand(X_train.shape[0]) * 2 - 1

# running sd an newton
w_sd, f_sd = steepest_decent(X_train, y_train, w_0, max_iter, epsilon, alph)
w_n, f_n = newton(X_train, y_train, w_0, max_iter, epsilon)

# computing sd accuracy
print(r"Steepest Decent Accuracy:")
print(r"Train:", accuracy(X_train, y_train, w_sd))
print(r"Test:", accuracy(X_test, y_test, w_sd))

# computing newton accuracy
print(r"Newton Accuracy:")
print(r"Train:", accuracy(X_train, y_train, w_n))
print(r"Test:", accuracy(X_test, y_test, w_n))

# plotting convergence history
plt.figure()
plt.plot(f_sd, label="Steepest Decent")
plt.plot(f_n, label="Newton")
plt.legend()
plt.title("8/9 Logistic Regression Convergence History")
plt.ylabel("Logistic Regression Objective")
plt.xlabel("Iteration")

plt.savefig("myplot7.pdf", bbox_inches="tight")
print(r"\saveandshowplot{myplot7.pdf}")
```

Steepest Decent Accuracy: Train: 0.679352546388 Test: 0.668085106383 Newton Accuracy: Train: 0.996841689696 Test: 0.99621749409



Steepest Decent Accuracy: Train: 0.700338983051 Test: 0.721633888048 Newton Accuracy: Train: 0.52093220339 Test: 0.525466464952


8/9 Logistic Regression Convergence History
Steepest Decent
Newton
Logistic Regression Objective
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Iteration