SMALL SETS CONTAINING CONVERGING SEQUENCES

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Abstract. Given a sequence $s_m \searrow 0$ arbitrarily slowly, we construct a sequence $\alpha_m \searrow 0$ and a perfect bounded set C in $\mathbb{R}^n$ of Hausdorff dimension zero, both depending on $\{s_m\}$, such that C contains a similar copy of every sequence in $\mathbb{R}^n$ converging to 0 faster than α_m. Moreover, we give an explicit expression on the rate of decay of α_m. This generalises Máthé's construction [4] and also manifests the main idea of [8].

1. Introduction

Given measurable sets $A, B \subseteq \mathbb{R}^n$, we say A contains a similar copy (without rotation or reflection in this paper) of the configuration (or pattern) B if there is some $\delta > 0$ and $t \in \mathbb{R}^n$ such that $A \supseteq t + \delta B$. The Lebesgue density theorem shows that if $A \subseteq \mathbb{R}^n$ has positive Lebesgue measure, then A contains a similar copy of every finite configuration. Many research work has been done to generalise this classical theorem.

1.1. **Literature review.** A natural question to ask whether every positive Lebesgue measure set $A \subseteq \mathbb{R}^n$ also contains similar copies of some prescribed infinite patterns. The answer is thought to be false by Erdős [1], who conjectured that for any infinite set $B \subseteq \mathbb{R}^n$ there is a set A of positive Lebesgue measure that does not contain any similar copy of B. A simple argument using projections shows that the conjecture in higher dimensions in as difficult as in the case $n = 1$, and another simple argument shows that it suffices to assume B is given by a *zero sequence*, that is, a sequence in $\mathbb{R}$ strictly decreasing to 0. Falconer [2] proved that if B is given by a zero sequence x_m with $x_{m+1}/x_m \to 1$, then there is a set A of positive Lebesgue measure that does not contain any similar copy of B; this includes all sequences x_m of the form m^{-p} where $0 < p < \infty$ and also all sequences of the form R^{-m^p} where $R > 1$, $0 < p < 1$. However, it remains unknown if this is true for the sequence 2^{-m}. The interested reader may refer to [7] for a survey of this problem.

We will not study Erdős's conjecture in this paper. However, it motivates an important observation, namely, the study of sets containing similar copies of convergent sequences amounts to investigating the rate of convergence of the sequences. For example, the author [8] proved the following theorem in $n = 1$, and the main idea of the proof is exactly to study the rate of decay of zero sequences.

Theorem 1.1 (Theorems 1.1 and 1.2 of [8]). *If $A \subseteq \mathbb{R}$ contains a similar copy of any zero sequence, then the closure of A contains an interval. On the other hand, given any zero sequence η_m, there is a compact and nowhere dense set $A \subseteq [0,1]$, depending on η_m, that contains a similar copy of every sequence α_m with $\sup_m |\alpha_m|/\eta_m < \infty$.*

Another attempt to generalise the Lebesgue density theorem is to find small sets that still contain a similar copy of every finite and possibly some infinite configurations. For example, in 1955, Erdős and Kakutani constructed a perfect set $A \subseteq \mathbb{R}$ of zero Lebesgue

2010 *Mathematics Subject Classification.* 28A78, 28A80, 40A05.

measure and Hausdorff dimension one that contains a similar copy of every finite set. More recently, Máthé [4] strengthened this theorem by exhibiting a compact $A \subseteq \mathbb{R}$ of zero Hausdorff dimension that contains a similar copy of every finite configuration on $\mathbb{R}$, although this statement was not mentioned explicitly in [4] but rather pointed out by Keleti later. This direction will be the main topic of our paper.

1.2. Main results.

The main results of this paper are based on a detailed study of Máthé's construction in [4].

Our first theorem is a qualitative result, which is a generalisation of [4] to $\mathbb{R}^n$ and to a general dimension function. (See Definition 2.1 and 2.2 for the terminology. The number 6 is not important and can be replaced by 1, for example.)

Theorem 1.2. *Let h be a dimension function. Then there is a perfect set $C \subseteq [0,6]^n$ with $\mathcal{H}^h(C)=0$ and a zero sequence α_m depending on h, such that C contains a similar copy of every sequence $\beta^{(m)} \subseteq \mathbb{R}^n$ with $\sup_m |\beta^{(m)}|/\alpha_m < \infty$.*

Remark 1. Taking $h(t) = -(\ln t)^{-1}$ (see (2.1) for the precise definition), we get a perfect set C with zero Hausdorff dimension. This recovers Máthé's result on $\mathbb{R}$.

Remark 2. In [6] the authors prove that for every dimension function h there is an F_σ set $E \subseteq \mathbb{R}^n$ with no isolated points and with $\mathcal{H}^h(E)=0$, such that E contains a translate of every countable set. However, their result is in another direction and not stronger than ours as our set C is closed and their set E cannot be closed. See the introduction and also the appendix of [8].

If we are given a concrete dimension function like $h(t)=t^s$ where $0<s\le n$, then by a delicate quantitative analysis, we can obtain an explicit expression on the rate of decay of α_m.

Theorem 1.3. *Let $0<s\le n$. Then there is a perfect set $C \subseteq [0,6]^n$ with $\dim_H(C)\le s$ and a zero sequence α_m depending on s, such that C contains a similar copy of every sequence $\beta^{(m)} \subseteq \mathbb{R}^n$ with $\sup_m |\beta^{(m)}/\alpha_m < \infty$. Furthermore, the sequence α_m can be taken to satisfy*

$$\lim_{m\to\infty} \frac{\ln \alpha_m^{-1}}{m^2 \ln m} = \frac{1}{2}, \quad \text{if } s=n, \tag{1.1}$$

and for some constants $0<C_1<C_2<\infty$ depending only on n,s,

$$C_1 < \frac{\ln \alpha_m^{-1}}{\left(\frac{n}{s}\right)^{\frac{m^2+3m}{2}}} < C_2, \quad \text{if } s<n. \tag{1.2}$$

That is, α_m decays like m^{-m^2} for $s=n$ and essentially like $\exp(-(n/s)^{m^2})$ for $s<n$.

Theorem 1.4. *Let s_m be a sequence strictly decreasing to 0 arbitrarily slowly. Then there is a perfect set $C \subseteq [0,6]^n$ of Hausdorff dimension 0 and a zero sequence α_m, both depending on $\{s_m\}$, such that C contains a similar copy of every sequence $\beta^{(m)} \subseteq \mathbb{R}^n$ with $\sup_m |\beta^{(m)}|/\alpha_m < \infty$. Furthermore, the sequence α_m can be taken to satisfy*

$$\ln \alpha_m^{-1} \sim \prod_{k=1}^{m} \left(\frac{n}{s_k}\right)^{k+1}. \tag{1.3}$$

Remark. As we have seen from the first remark after Theorem 1.2, it already gives a set C of zero Hausdorff dimension. We could have analysed quantitatively for the dimension function $h(t) = -1/(\ln t)$ as in the proof of Theorem 1.3, but the computation is too difficult. Hence, we would rather follow the proof technique of [4] which gives us a set C of zero Hausdorff dimension plus an explicit rate of decay of sequences it contains.

1.3. **Open problem.** The sequence α_m in both Theorems 1.3 and 1.4 decay extremely rapidly. Even when $s = n$, α_m decays like m^{-m^2}, which is much faster than geometric sequences, and we know that geometric sequences are a barrier to the Erdős's similarity conjecture. Hence, one may ask the following question, which is still open to my knowledge even in $n = 1$.

Conjecture 1.5. *Does there exist a compact $A \subseteq [0, 1]$ of zero Lebesgue measure that contains a similar copy of every geometric sequence? If yes, can A have Hausdorff dimension less than one, or even zero?*

Outline of the paper. In Section 2 we introduce the notation and convention used in this paper. In Section 3 we establish Theorem 1.2 assuming a main lemma, which will be proved in Section 5 using a key constructive lemma in Section 4. In Section 6 we restrict ourselves to the usual Hausdorff measures, do a quantitative analysis on the key lemma and use it in Sections 7 and 8 to prove Theorems 1.3 and 1.4.

Acknowledgement. I thank my advisor Malabika Pramanik for her useful suggestions on improving the readability of the paper.

2. Notation

The following notation will be used in this paper.

(1) If $x \subseteq \mathbb{R}^n$ where $n \geq 2$, we reserve the subscript notation x_i for the i-th coordinate of x. With this, we denote by $|x|$ the Euclidean norm of x and $\|x\| = \max_i |x_i|$. A sequence in $\mathbb{R}$ will be denoted by the standard notation x_m, but a sequence in $\mathbb{R}^n$ will be denoted as $x^{(m)}$ to avoid confusion.
(2) We use $A \lesssim B$, $B \gtrsim A$ or $A = O(B)$ to mean that $A \leq CB$ for some constant $C > 0$. The notation $A \sim B$ means that we have both $A \lesssim B$ and $A \gtrsim B$.
(3) We use $A_m \lessapprox B_m$ or $B_m \gtrapprox A_m$ to mean that $A_m \leq B_m(1 + e_m)$ where $e_m \to 0$ as $m \to \infty$. Thus $A_m \lessapprox B_m$ implies $A_m \lesssim B_m$ but not vice versa. The notation $A_m \approx B_m$ means that we have both $A_m \lessapprox B_m$ and $A_m \gtrapprox B_m$ (or equivalently, $A_m/B_m \to 1$.)
(4) A zero sequence is a sequence in $\mathbb{R}$ strictly decreasing to 0. The notation $x_m \nearrow x$ means that x_m increases to x (not necessarily strictly). Similar for $x_m \searrow x$.
(5) We denote by $\#A$, $\operatorname{diam}(A)$, $\mathcal{L}^n(A)$ the cardinality, diameter and the n-dimensional Lebesgue measure of A, respectively.

The following definitions will be used in this paper. They can also be found in standard texts such as [3, 5].

Definition 2.1 (Dimension function)**.** *Let $h : [0, \infty) \to [0, \infty)$. We say it is a dimension function if it is right-continuous, increasing, $h(0) = 0$ and $h(t) > 0$ for $t > 0$.*

For example, for $s>0$, the functions $x \mapsto x^s$ is a dimension function. Another example is the function

$$h(t)=\begin{cases}0, & \text{if } t=0\\ \frac{1}{\ln t^{-1}}, & \text{if } 0<t<\frac{1}{2}\\ \frac{1}{\ln 2}, & \text{if } t\geq \frac{1}{2}\end{cases} \tag{2.1}$$

which has $t^{\varepsilon}=o(h(t))$ as $t\to 0^+$, for any $\varepsilon>0$.

Definition 2.2 (h-Hausdorff measure). *Let h be a dimension function. For a set $A\subseteq\mathbb{R}^n$ and $0<\delta\leq\infty$, we define*

$$\mathcal{H}^h_\delta(A)=\inf\left\{\sum_{i=1}^{\infty} h\left(\operatorname{diam} A_i\right):\bigcup_{i=1}^{\infty}A_i\supseteq A, \operatorname{diam}(A_i)<\delta\right\},$$

which is a decreasing function of δ. We then define the h-Hausdorff measure of A by

$$\mathcal{H}^h(A):=\sup_{0<\delta\leq\infty}\mathcal{H}^h_\delta(A)=\lim_{\delta\to 0^+}\mathcal{H}^h_\delta(A).$$

Definition 2.3 (Hausdorff dimension). *If $s\geq 0$, we define $\mathcal{H}^s_\delta(A)=\mathcal{H}^h_\delta(A)$ and similarly $\mathcal{H}^s(A)=\mathcal{H}^h(A)$ where $h(t)=t^s$. The Hausdorff dimension of A, denoted $\dim_H(A)$, is defined by*

$$\dim_H(A)=\sup\{s\geq 0:\mathcal{H}^s(A)=\infty\}=\inf\{s\geq 0:\mathcal{H}^s(A)=0\}.$$

We will need the following facts. First, if $h(t)=t^s$, then $\mathcal{H}^h_\delta=\mathcal{H}^s_\delta$. Second, $\mathcal{H}^h_\delta$ is subadditive. Third, for any dimension function h and any set $A\subseteq\mathbb{R}^n$, we have $\mathcal{H}^h(A)=0$ if and only if $\mathcal{H}^h_\infty(A)=0$. Lastly, note that if $h(t)=-(\ln t)^{-1}$ as in (2.1), then $\mathcal{H}^h(A)=0$ implies $\dim_H(A)=0$.

3. Proof of Theorem 1.2

3.1. **The main proof.** In this section we prove Theorem 1.2, by slightly modifying the formulation of "Slaloms" in [4]. The motivation of such a terminology will be clear later.

Definition 3.1 (Slaloms). *Let $n\geq 1$. Let $f:\mathbb{N}\to\mathbb{N}_{\geq 2}$. We say $S\subseteq\mathbb{R}^n$ is an f-slalom if there is a sequence of sets $B_k\subseteq(\mathbb{Z}\cap[-f(k)+1,f(k)-1])^n$ with $\#B_k\leq k+1$, such that*

$$S=\sum_{k=1}^{\infty}\frac{B_k}{F(k)}:=\left\{\sum_{k=1}^{\infty}\frac{b_k}{F(k)}:b_k\in B_k\quad\forall k\right\}, \tag{3.1}$$

where (we will use the notation throughout the paper)

$$F(k):=f(1)\cdots f(k).$$

Note that every f-slalom is a compact subset of $[-1,1]^n$.

Theorem 1.2 follows from the following lemmas. Among these, Lemmas 3.3 and 3.4 will be proved right after the proof of the main theorem, and the main Lemma 3.2 will be proved in Section 5.

Lemma 3.2 (Main lemma). *Let h be a dimension function. Then there is a function $f:\mathbb{N}\to\mathbb{N}_{\geq 2}$ and a perfect set $C\subseteq[0,6]^n$ with $\mathcal{H}^h(C)=0$, such that C contains a translate of every f-slalom S.*

Lemma 3.3. *Let $f : \mathbb{N} \to \mathbb{N}_{\geq 2}$. Let $\beta^{(m)} \subseteq \mathbb{R}^n$ be a sequence such that $\|\beta^{(m)}\|$ is decreasing and such that there is some $\delta_0 > 0$ for which $\delta_0\|\beta^{(m)}\|F(m) < 1$ for all m. Then there is an f-slalom S such that for every $-\delta_0 \leq \delta \leq \delta_0$ and every m, we have $\delta\beta^{(m)} \in S$.*

Lemma 3.4 (Rearrangement)**.** *Let α_m be a zero sequence. If $\beta^{(m)}$ is a sequence in $\mathbb{R}^n$ such that $\sup_m |\beta^{(m)}|/\alpha_m < \infty$, then there is a sequence $\gamma^{(m)}$ with $\{\gamma^{(m)}\} = \{\beta^{(m)}\}$, such that $\|\gamma^{(m)}\| \searrow 0$ and $\sup_m\|\gamma^{(m)}\|/\alpha_m < \infty$.*

Proof of Theorem 1.2. Given the dimension function h, let $f : \mathbb{N} \to \mathbb{N}_{\geq 2}$ and $C \subseteq [0, 6]^n$ be as in Lemma 3.2. Take the zero sequence $\alpha_m = F(m)^{-1}$. If $\beta^{(m)} \subseteq \mathbb{R}^n$ is such that $\sup_m |\beta^{(m)}|/\alpha_m < \infty$, then by Lemma 3.4 above, we may assume in addition that $\|\beta^{(m)}\|$ is decreasing, and also there is some $\delta_0 > 0$ such that $\delta_0\|\beta^{(m)}\| < \alpha_m = F(m)^{-1}$. By Lemma 3.3, there is some f-slalom such that for all $-\delta_0 < \delta \leq \delta_0$ we have $\delta\beta^{(m)} \subseteq S$. But by Lemma 3.2, there is $t \in \mathbb{R}^n$ such that $t + S \subseteq C$. Combining the above, we thus have for every $\beta^{(m)} \subseteq \mathbb{R}^n$ with $\sup_m |\beta^{(m)}|/\alpha_m < \infty$, there is some $\delta > 0$ and $t \in \mathbb{R}^n$ such that $t + \delta\alpha_m \in C$ for all m, completing the proof. □

3.2. **Proof of Lemma 3.3.**

Proof. Let $-\delta_0 \leq \delta \leq \delta_0$. By our choice of the sup-norm $\|\cdot\|_\infty$, each $\delta\beta^{(m)}$ lies within $[-1, 1]^n$. For each coordinate i, we will decompose $\delta\beta_i^{(m)}$ in the following way:

If $\delta\beta_i^{(m)} \geq 0$, then it admits a unique $f(k)$-expansion as follows:

$$\delta\beta_i^{(m)} = \sum_{k=1}^{\infty} \frac{\beta_i^{(m,k)}}{F(k)},$$

where $\beta_i^{(m,k)} \in \{0, \ldots, f(k) - 1\}$.

If $\delta\beta_i^{(m)} < 0$, then $-\delta\beta_i^{(m)} > 0$ which also admits a unique $f(k)$-expansion:

$$-\delta\beta_i^{(m)} = \sum_{k=1}^{\infty} \frac{-\beta_i^{(m,k)}}{F(k)},$$

where $-\beta_i^{(m,k)} \in \{0, \ldots, f(k) - 1\}$. In either case, we have for each i

$$\delta\beta_i^{(m)} = \sum_{k=1}^{\infty} \frac{\beta_i^{(m,k)}}{F(k)},$$

where $\beta_i^{(m,k)} \in \{1 - f(k), \ldots, f(k) - 1\}$. Thus, we have the vector-valued expansion

$$\delta\beta^{(m)} = \sum_{k=1}^{\infty} \frac{\beta^{(m,k)}}{F(k)}, \tag{3.2}$$

where $\beta^{(m,k)} \in \{1 - f(k), \ldots, f(k) - 1\}^n$.

By assumption, we have $|\delta|\|\beta^{(m)}\|F(1) < 1$ for all $m \geq 1$. Hence the set B_1 of the first digit tuples of all $\delta\beta^{(m)}$ obeys

$$B_1 := \{\beta^{(m,1)} : m \in \mathbb{N}\} = \{\vec{0}\}.$$

Since $|\delta|\|\beta^{(m)}\|F(2)<1$ for all $m\geq 2$, the set of the second digits of all $\delta\beta^{(m)}$

$$B_2 := \{\beta^{(m,2)} : m \in \mathbb{N}\} = \{\vec{0}\} \cup \{\beta^{(1,2)}\}.$$

Similarly, we can show that for all $k \geq 1$,

$$B_k := \{\beta^{(m,k)} : m \in \mathbb{N}\} = \{\vec{0}\} \cup \{\beta^{(m,k)} : 1 \leq m \leq k-1\}.$$

Thus $\#B_k \leq k$. Also, $B_k \subseteq \{1-f(k), \dots, f(k)-1\}^n$. As a result, $\{\delta\beta^{(m)}\}$ is contained in the f-slalom S given by

$$S = \sum_{k=1}^{\infty} \frac{B_k}{F(k)}.$$

□

3.3. **Proof of Lemma 3.4.**

Proof. Let $S = \{\|\beta^{(m)}\| : m \geq 1\}$. If S is finite then there is some $m_0 \geq 1$ such that $\beta^{(m)} = 0$ for all $m \geq m_0$. Hence we just rearrange $\{\beta^{(m)} : 1 \leq m \leq m_0\}$ so that $\|\beta^{(m)}\|$ is decreasing from 1 to m_0, and let the first m_0 terms of $\gamma^{(m)}$ be this rearrangement. For $m > m_0$ we simply set $\gamma^{(m)} = 0$. It is then trivial to see that $\sup_m \|\gamma^{(m)}\|/\alpha_m < \infty$.

Now we consider the case when S is infinite. In this case, S is bounded and has a unique accumulation point 0. Let I_1 be the set of indices m such that $\|\beta^{(m)}\|$ equals the largest element of S. Let $\nu_1 = \#I_1$ and note that $\nu_1 \leq \max I_1$. For $1 \leq m \leq \nu_1$, define $\gamma^{(m)}$ so that $\{\gamma^{(m)} : 1 \leq m \leq \nu_1\}$ is any permutation of $\{\beta^{(m)} : m \in I_1\}$. For each $k \geq 2$, let I_k be the set of indices m such that $\|\beta^{(m)}\|$ equals the k-th largest element of S. Let

$$\nu_k = \nu_{k-1} + \#I_k$$

and note that $\nu_k \leq \max I_k$. For $\nu_{k-1} < m \leq \nu_k$, define $\gamma^{(m)}$ so that $\{\gamma^{(m)} : \nu_{k-1} < m \leq \nu_k\}$ is any permutation of $\{\beta^{(m)} : m \in I_k\}$.

As S is infinite, this process cannot stop, and so we arrive at a rearrangement $\{\gamma^{(m)}\}$ of $\{\beta^{(m)}\}$ such that $\|\gamma^{(m)}\|$ is decreasing.

It only remains to show $\sup_m \|\gamma^{(m)}\|/\alpha_m < \infty$. For each $m \geq 1$, let $k \geq 1$ be the unique integer such that $\nu_{k-1} < m \leq \nu_k$ (with $\nu_0 := 0$). By construction, $\|\gamma^{(m)}\| = \|\beta^{(j)}\|$ for every $\nu_{k-1} < m \leq \nu_k$, $j \in I_k$. Since α_m is decreasing, it then suffices to show that $\sup_k \|\beta^{(\max I_k)}\|/\alpha_{\nu_k} < \infty$. But since $\max I_k \geq \nu_k$ and α_m is decreasing, we have

$$\sup_k \frac{\|\beta^{(\max I_k)}\|}{\alpha_{\nu_k}} = \sup_k \frac{\|\beta^{(\max I_k)}\|}{\alpha_{\max I_k}} \frac{\alpha_{\max I_k}}{\alpha_{\nu_k}} \leq \sup_k \frac{\|\beta^{(\max I_k)}\|}{\alpha_{\max I_k}} < \infty,$$

where we have used $\sup_m |\beta^{(m)}|/\alpha_m < \infty$ and that $|\cdot|$ and $\|\cdot\|$ are equivalent norms on $\mathbb{R}^n$. □

Thus, for the proof of Theorem 1.2, all that remains is the proof of Lemma 3.2.

4. Key lemma for construction

We start with a qualitative generalisation of Lemma 3.1 of [4] in $\mathbb{R}^n$.

Lemma 4.1 (Key lemma). *For every dimension function h, every $\delta > 0$ and $k \geq 1$, there exists a compact set $K_k = K_k(\delta) \subseteq [-3,3]^n$ and $\varepsilon_k = \varepsilon_k(\delta) > 0$ such that $\mathcal{H}^h_\infty(K_k) \leq \delta$ and for every $B \subseteq \mathbb{R}^n$ with $\#B = k+1$, there is $t \in \mathbb{R}^n$ such that*

$$B + t + [0, \varepsilon_k]^n \subseteq K_k. \tag{4.1}$$

Proof. We will actually show that for every $B \subseteq [-1,1]^n$ with $\#B = k+1$ there exists

$$t \in [-1, 2 - 2^{-k}]^n \subseteq [-1, 2]^n \tag{4.2}$$

for which

$$B + t + [0, \varepsilon_k]^n \subseteq K_k. \tag{4.3}$$

We argue by induction on k. For $k = 1$ and $0 < 2b \leq a \leq 1$ to be determined, we let

$$K_1 = [-a, a]^n \cup \left([-3,3]^n \cap \bigcup_{j_i \in \mathbb{Z}} \prod_{i=1}^n [j_i a - b, j_i a + b] \right). \tag{4.4}$$

Thus K_1 can be covered by a cube of side length $2a$ and at most $\lfloor (6/a) \rfloor^n$ cubes of side length b. Thus

$$\mathcal{H}^h_\infty(K_1) \leq h(2\sqrt{n}a) + \left(\frac{6}{a}\right)^n h\left(2\sqrt{n}b\right), \tag{4.5}$$

which would be less than δ if we first let a be small enough and then $b \leq a/2$ be small enough. Our choice of $\varepsilon_1(\delta)$ will just be b.

We still need to prove (4.3). Denote $B = \{x, y\} \subseteq [-1,1]^n$. We first apply the translation $t' = -x \in [-1,1]^n$ to get $B + t' = \{0, z\}$, where $z := y - x \in [-2,2]^n$.

Now we consider each coordinate z_i. By the division algorithm, there are unique $j_i \in \mathbb{Z}$ and $r_i \in [0, a)$ such that $z_i = j_i a + r_i$. Take $t''_i = a - r_i \in (0, a]$ so that

$$\begin{aligned} z_i + t''_i + [0, b] &= [(j_i + 1)a, (j_i + 1)a + b], \\ z_i + t''_i + [0, b] &\subseteq [-2 + 0 + 0, 2 + a + b] \subseteq [-2, 3]. \end{aligned}$$

Hence we have $z + t'' + [0, b]^n \subseteq K_1$. Also, $0 + t'' \in [0, a]^n \subseteq K_1$. Letting $t = t' + t''$, we have (4.3) with $\varepsilon_1 = b$. Finally, for each i we have $t'_i + t''_i \subseteq [-1, 1 + a] \subseteq [-1, 3/2]$, so we have (4.2). This finishes the proof of the case $k = 1$.

Now assume the conclusion holds for $k \geq 1$ and we want to prove it for $k + 1$. With h and δ given, by the induction hypothesis, find $K_k = K_k(\delta/2)$ and $\varepsilon_k = \varepsilon_k(\delta/2)$ such that $\mathcal{H}^h_\infty(K_k) \leq \delta/2$ and that (4.3) holds for every $B \subseteq [-1,1]^n$ with $\#B = k+1$.

Now let $0 < \varepsilon_{k+1} \leq \varepsilon_k/2$ be so small that

$$\left(\frac{6}{\varepsilon_k(\delta/2)}\right)^n h(2\sqrt{n}\varepsilon_{k+1}) \leq \frac{\delta}{2}. \tag{4.6}$$

We also let

$$K_{k+1} = K_k \cup \left([-3,3]^n \cap \bigcup_{j_i \in \mathbb{Z}} \prod_{i=1}^n [j_i \varepsilon_k - \varepsilon_{k+1}, j_i \varepsilon_k + \varepsilon_{k+1}] \right), \tag{4.7}$$

so by (4.6) and the fact that $\mathcal{H}^h_\infty$ is sub-additive, we have $\mathcal{H}^h_\infty(K_{k+1}) \leq \delta$.

Now we come to (4.3). Let $\#B = k+2$, and let $x \subseteq [-1,1]^n$ be any member of B. Applying the induction hypothesis to $B \backslash \{x\}$, there is some $t' \in [-1, 2 - 2^{-k}]^n$ such that

$$B \backslash \{x\} + t' + [0, \varepsilon_k]^n \subseteq K_k. \tag{4.8}$$

For each coordinate i, by the division algorithm, there are unique $j_i \in \mathbb{Z}$ and $r_i \in [0, \varepsilon_k)$ such that $x_i + t'_i = j_i \varepsilon_k + r_i$.

Let

$$t''_i = \begin{cases} \varepsilon_k - \varepsilon_{k+1} - r_i & \text{if } 0 \leq r_i \leq \varepsilon_k - \varepsilon_{k+1} \\ 0 & \text{if } \varepsilon_k - \varepsilon_{k+1} < r_i < \varepsilon_k \end{cases}. \tag{4.9}$$

This makes $t''_i \in [0, \varepsilon_k - \varepsilon_{k+1}]$. Thus we have

$$x_i + t'_i + t''_i + [0, \varepsilon_{k+1}] \subseteq [-1 - 1 + 0 + 0, 1 + 2 - 2^{-k} + \varepsilon_k] \subseteq [-2, 3],$$

and thus $x + t' + t'' + [0, \varepsilon_{k+1}]^n \subseteq K_{k+1}$ by our choice of t''_i.

Since $t''_i \in [0, \varepsilon_k - \varepsilon_{k+1}]$, we have $[0, \varepsilon_k] \supseteq t''_i + [0, \varepsilon_k - t''_i] \supseteq t''_i + [0, \varepsilon_{k+1}]$. In view of (4.8), we thus have

$$K_k \supseteq B \backslash \{x\} + t' + t'' + [0, \varepsilon_{k+1}]^n.$$

Letting $t = t' + t''$, we have (4.3). Also, (4.2) follows since each i we have

$$t_i = t'_i + t''_i \in [-1, 2 - 2^{-k}] + [0, \varepsilon_k - \varepsilon_{k+1}] \subseteq [-1, 2 - 2^{-1-k}],$$

where we used the fact that $\varepsilon_k \leq 2^{-k}$ by our choice of ε_k. This closes the induction.

□

5. Proof of Lemma 3.2

We are ready to prove Lemma 3.2, that is, given a dimension function h, there is some function $f : \mathbb{N} \to \mathbb{N}_{\geq 2}$ and a perfect set $C \subseteq [0, 6]^n$ with $\mathcal{H}^h(C) = 0$, such that C contains a translate of every f-slalom S.

5.1. **Construction of f and C.** By Lemma 4.1, take $\delta_1 = 1$ and find $K_1 = K_1(\delta_1)$ and $\eta_1 := \varepsilon_1(\delta_1)$ with $\mathcal{H}^h_\infty(K_1) \leq \delta_1$. With this, let $f(1)$ be any integer in $[20/\eta_1, 40/\eta_1]$.

Let $e = (1, \dots, 1) \in \mathbb{R}^n$. Define A_1 as

$$A_1 = \left\{ j \in \mathbb{Z}^n : \frac{j}{f(1)} + \left[0, \frac{12}{f(1)}\right]^n \subseteq K_1 + 3e \right\} \subseteq \{0, 1, \dots, 6f(1) - 1\}^n. \tag{5.1}$$

Obviously, A_1 contains at least 2 points (actually much more). For each $k \geq 2$, let

$$\delta_k = 2^{1-k} \# A_1^{-1} \cdots \# A_{k-1}^{-1} \tag{5.2}$$

and find $K_k = K_k(\delta_k)$ and $\eta_k := \varepsilon_k(\delta_k)$ with $\mathcal{H}^h_\infty(K_k) \leq \delta_k$. With this, pick any

$$f(k) \in \mathbb{N} \cap \left[\frac{20}{\eta_k}, \frac{40}{\eta_k}\right]. \tag{5.3}$$

Then we define A_k as

$$A_k = \left\{ j \in \mathbb{Z}^n : \frac{j}{f(k)} + \left[0, \frac{12}{f(k)}\right]^n \subseteq K_k + 3e \right\} \subseteq \{0, 1, \dots, 6f(k) - 1\}^n. \tag{5.4}$$

Also, $\# A_k \geq 2$. Now we can define the perfect set C as required:

$$C = \sum_{k=1}^\infty \frac{A_k}{F(k)} = \left\{ \sum_{k=1}^\infty \frac{a^{(k)}}{F(k)} : a^{(k)} \in A_k \right\}. \tag{5.5}$$

From the range of A_k and $f(k) \geq 2$, the above series converges absolutely for every choice of $a^{(k)} \in A_k$, and we have $C \subseteq [0, 6]^n$. Also, C is perfect, which is a standard consequence of the base $f(k)$ expansion, since $\# A_k \geq 2$.

5.2. **Containment of slaloms.** We now show that C contains a translate of every f-slalom S. Indeed, let S be an f-slalom with the associated B_k as in Definition 3.1. Since $\#B_k \le k+1$, by Lemma 4.1 applied to $B_k/f(k) \subseteq [-1,1]^n$, there exists $t^{(k)} \in \mathbb{R}^n$ such that $B_k/f(k) + t^{(k)} + [0,\eta_k]^n \subseteq K_k$. Thus $B_k + t^{(k)} f(k) + [0, \eta_k f(k)]^n \subseteq f(k) K_k$. By (5.3), the length of the interval $[t_i^{(k)} f(k), t_i^{(k)} f(k) + \eta_k f(k)]$ is at least 20, so it has a subinterval of the form $[v_i^{(k)}, v_i^{(k)} + 12]$ where $v_i^{(k)} \in \mathbb{Z}$. Letting $u^{(k)} = (v_i^{(k)}) + f(k)e$, we thus have

$$B_k + u^{(k)} + [0,12]^n \subseteq f(k)(K_k + e).$$

Dividing both sides by $f(k)$ and using the definition of A_k in (5.4), we have $B_k + u^{(k)} \subseteq A_k$. Thus, taking

$$t = \sum_{k=1}^{\infty} \frac{u^{(k)}}{F(k)} \in \mathbb{R}^n,$$

we then have

$$S + t = S + \sum_{k=1}^{\infty} \frac{u^{(k)}}{F(k)} = \sum_{k=1}^{\infty} \frac{B_k + u^{(k)}}{F(k)} \subseteq \sum_{k=1}^{\infty} \frac{A_k}{F(k)} = C.$$

5.3. **Zero Hausdorff measure.** We now prove that $\mathcal{H}^h(C) = 0$.

Proof. It suffices to prove $\mathcal{H}^h_\infty(C) = 0$. For $m \ge 1$ let

$$C_m := \sum_{k=m}^{\infty} \frac{A_k}{F(k)} = \frac{A_m}{F(m)} + C_{m+1}. \tag{5.6}$$

Then $C = C_1$ and C is the union of $\#A_1 \cdots \#A_{m-1}$ many translated copies of C_m. First we show that C_m lies in a small neighbourhood of $F(m)^{-1} A_m$. Denote $\|A\| := \sup_{x \in A} \|x\|$ for $A \subseteq \mathbb{R}^n$. In view of the range of A_k, for $p \ge 1$ we have

$$\frac{\|A_{m+p}\|}{\prod_{j=m}^{m+p} f(j)} \le \frac{6}{\prod_{j=m}^{m+p-1} f(j)} \le \frac{6}{f(m) 2^{p-1}},$$

and thus

$$\sum_{p=1}^{\infty} \frac{\|A_{m+p}\|}{\prod_{j=m}^{m+p} f(j)} \le \sum_{p=1}^{\infty} \frac{6}{f(m) 2^{p-1}} = \frac{12}{f(m)}.$$

Using this and

$$C_{m+1} = \sum_{k=m+1}^{\infty} \frac{A_k}{F(k)} = \frac{1}{F(m-1)} \sum_{p=1}^{\infty} \frac{A_{m+p}}{\prod_{j=m}^{m+p} f(j)},$$

we have

$$\|F(m-1) C_{m+1}\| \le \frac{12}{f(m)}.$$

Thus, using (5.6), we have

$$F(m-1) C_m \subseteq \frac{A_m}{f(m)} + \left[0, \frac{12}{f(m)}\right]^n \subseteq K_m + 3e, \tag{5.7}$$

where the last containment follows from (5.4). Using this and the construction of K_m at the beginning of the section, we have

$$\mathcal{H}^h_\infty(C_m) \le \mathcal{H}^h_\infty(F(m-1) C_m) \le \mathcal{H}^h_\infty(K_m) \le \delta_m. \tag{5.8}$$

As mentioned before, C is the union of $\#A_1 \cdots \#A_{m-1}$ many translated copies of C_m, from which we have

$$\mathcal{H}^h_\infty(C) \le \#A_1 \cdots \#A_{m-1}\delta_m \le 2^{1-m}, \tag{5.9}$$

by (5.2). Letting $m \to \infty$ we have $\mathcal{H}^h_\infty(C) = 0$. □

Therefore, the proof of Theorem 1.2 is all complete.

6. A quantitative study of key Lemma 4.1

It remains to prove the quantitative Theorems 1.3 and 1.4. This entails us to clarify the dependence of the parameters that appeared in the proof of Theorem 1.2.

Throughout the section we will denote

$$r = \frac{n}{s} \in [1, \infty). \tag{6.1}$$

6.1. Relation between parameters.

First, given a dimension function h, Lemma 4.1 together with Section 5.1 give rise to sequences δ_k, ε_k, η_k and more importantly $f(k)$ and A_k which depend ultimately on h. The rapidly increasing sequence $f(k)$ and the n-dimensional digit set A_k are then used to construct the required set C.

On the other hand, in the proof of Lemma 3.3, we have defined the sequence $\alpha_k = F(k)^{-1} = f(1)^{-1} \cdots f(k)^{-1}$. Since we would like our set C to contain more patterns, we hope α_k decreases as slowly as possible. As a result, it amounts to the study of the rate of increase of $f(k)$, which then amounts to the study of the dependence of ε_k on δ_k. To this end, we will return to the proof of the key Lemma 4.1, and hope to give a lower bound on $\varepsilon_k(\delta)$ by refining its argument.

6.2. A quantitative version of Lemma 4.1.

Let $h(t) = t^s$ where $0 < s \le n$. With this specific choice of the dimension function, we seek to refine the argument in Lemma 4.1 and find an explicit expression of $\varepsilon_k(\delta)$.

Since we are in $\mathbb{R}^n$ and $h(t) = t^s$, it will be helpful to use the following equivalent definition of the Hausdorff pre-measure:

$$\mathcal{H}^s_\infty(A) = 100^{-n} \inf\left\{ \sum_{i=1}^\infty l(A_i)^s : \bigcup_{i=1}^\infty A_i \supseteq A, A_i \text{ are cubes with side lengths } l(A_i) \right\}. \tag{6.2}$$

This choice may help us get rid of some dimensional constants, and increase the readability of the following arguments.

Lemma 6.1. *Let $h(t) = t^s$ where $0 < s \le n$. Then in Lemma 4.1, we can actually define $\varepsilon_k(\delta)$ in the following inductive way (where $r = n/s$):*

$$\lambda_0 := \text{ a maximiser of } \lambda \mapsto (1-\lambda)(\lambda\delta)^r \text{ over } (0,1), \tag{6.3}$$

$$\varepsilon_1(\delta) := \min\left\{ \frac{\lambda_0^{\frac{1}{s}}\delta^{\frac{1}{s}}}{2}, \lambda_0^{\frac{r}{s}}(1-\lambda_0)^{\frac{1}{s}}\delta^{\frac{1+r}{s}} \right\}, \tag{6.4}$$

$$\lambda_k := \text{ a maximiser of } \lambda \mapsto (1-\lambda)\varepsilon_k^n(\lambda\delta) \text{ over } (0,1), \tag{6.5}$$

$$\varepsilon_{k+1}(\delta) := \min\left\{ \frac{\varepsilon_k(\lambda_k\delta)}{2}, \varepsilon_k^r(\lambda_k\delta)(1-\lambda_k)^{\frac{1}{s}}\delta^{\frac{1}{s}} \right\}. \tag{6.6}$$

The choice of the parameters will be clear in the proof below.

Proof. We first examine the proof of Lemma 4.1 in the case $k = 1$. The inequality (4.5) in the new notation is simplified to

$$\mathcal{H}^s_\infty(K_1) \le 100^{-n} 2^s (a^s + 6^n a^{-n} b^s) \le a^s + a^{-n} b^s,$$

which we would like to be bounded by δ. For a parameter λ_0 to be chosen, it suffices to choose $0 < 2b \le a \le 1$ such that

$$a^s = \lambda_0 \delta, \quad a^{-n} b^s = (1 - \lambda_0)\delta, \tag{6.7}$$

and then let $\varepsilon_1(\delta) = b$ as before. Our goal is to maximise $\varepsilon_1(\delta)$ over $\lambda_0 \in (0,1)$, which can be achieved by the choices (6.3) and (6.4).

Next we examine the proof of Lemma 4.1 in the case $k+1$. With s and δ given, instead of requiring $\mathcal{H}^s_\infty(K_k) \le \delta/2$ and (4.6), we could instead select a parameter λ_k and require $\mathcal{H}^s_\infty(K_k) \le \lambda_k \delta$ and

$$\varepsilon_k(\lambda_k \delta)^{-n} \varepsilon_{k+1}(\delta)^s \le (1 - \lambda_k)\delta. \tag{6.8}$$

We will then maximise $\varepsilon_{k+1}(\delta)$ over $\lambda_k \in (0,1)$, which can be achieved by (6.5) and (6.6). □

6.3. **Explicit expression of $\varepsilon_k(\delta)$.** Since $h(t) = t^s$ is a nice function, we can actually use calculus to find the explicit expression of $\varepsilon_k(\delta)$.

Proposition 6.2. *Let $r = n/s \in [1, \infty)$ and $\delta \le 2^{-s}$. Following the definition of the parameters given by Lemma 6.1, we have $\varepsilon_k(\delta) = c_k \delta^{u_k}$, where*

$$u_k = \frac{1}{s} \sum_{j=0}^{k} r^j, \quad k \ge 0 \tag{6.9}$$

and

$$c_k = \prod_{j=0}^{k-1} \left(\left(\frac{n u_j}{1 + n u_j} \right)^{r u_j} (1 + n u_j)^{-\frac{1}{s}} \right)^{r^j}, \ k \ge 1. \tag{6.10}$$

We also have an estimate on the Lebesgue measure of $K_k(\delta)$: for some dimensional constant $\kappa_n \ge 1$,

$$\mathcal{L}^n(K_k(\delta)) \le \kappa_n \delta^r. \tag{6.11}$$

Proof. We prove by induction on k assuming some choice of κ_n which is to be specified at the end. For $k = 1$, by direct computation, the maximiser of $\lambda \mapsto (1 - \lambda)(\lambda\delta)^r$ is given by $\lambda_0 = \frac{r}{r+1}$. Using $\delta \le 2^{-s}$, we check that the minimum of (6.4) is attained in the latter expression, and thus

$$\varepsilon_1(\delta) = \lambda_0^{\frac{r}{s}} (1 - \lambda_0)^{\frac{1}{s}} \delta^{\frac{1+r}{s}},$$

which is just (6.9) and (6.10) for $k = 1$.

For the Lebesgue measure, using (4.5) with $h(t) = t^n$, $a = \lambda_0^{1/s} \delta^{1/s}$, and $b = \varepsilon_1(\delta)$, we have

$$\mathcal{L}^n(K_1(\delta)) \le (2a)^n + \left(\frac{6}{a} \right)^n (2b)^n \le \kappa_n' \delta^r,$$

for some dimensional constant κ_n'.

Now assume the result holds for $k \geq 1$ and we will prove it for $k+1$. The maximiser λ_k of $\lambda \mapsto (1-\lambda)\lambda^{nu_k}$ is given by

$$\lambda_k = 1 - (1+nu_k)^{-1} = \frac{nu_k}{1+nu_k}. \tag{6.12}$$

Using $\delta \leq 2^{-s}$ and $c_k \leq 1$, the minimum of (6.6) is attained in the latter expression, and thus

$$\varepsilon_{k+1}(\delta) = \varepsilon_k^r(\lambda_k\delta)(1-\lambda_k)^{\frac{1}{s}}\delta^{\frac{1}{s}},$$

which gives (6.10) for $k+1$ using (6.10) for k.

We now come to the Lebesgue measure of $K_{k+1}(\delta)$. Using (4.7) and the induction hypothesis we compute

$$\begin{aligned}\mathcal{L}^n(K_{k+1}(\delta)) &\leq \mathcal{L}^n(K_k(\lambda_k\delta)) + \left(\frac{6}{\varepsilon_k(\lambda_k\delta)}\right)^n (2\varepsilon_{k+1}(\delta))^n \\ &\leq \kappa_n\lambda_k^r\delta^r + \left(\frac{12c_{k+1}}{c_k\lambda_k^{u_k}}\delta^{\frac{r^{k+1}}{s}}\right)^n.\end{aligned}$$

Using (6.10) and $\lambda_k^{nu_k} > e^{-1}$ which follows from the elementary inequality $(1+1/x)^x \in [2,e)$ for $x \geq 1$, the second term above is bounded above by some dimensional constant κ_n'' times

$$(1-\lambda_k)^{r^{k+1}}\delta^{r^{k+1}} \leq (1-\lambda_k)^r\delta^r.$$

Thus, if we choose $\kappa_n = \max\{\kappa_n', \kappa_n'', 1\}$ at the beginning, then

$$\mathcal{L}^n(K_{k+1}(\delta)) \leq \delta^r\left(\kappa_n\lambda_k^r + \kappa_n(1-\lambda_k)^r\right) \leq \kappa_n\delta^r,$$

and so the induction closes. $\square$

We next examine the rate of decay of c_k.

Proposition 6.3. *The constant c_k defined in (6.10) has the following estimate:*

$$c^{u_{k-1}}\prod_{j=0}^{k-1}u_j^{-\frac{r^j}{s}} \leq c_k \leq (ec)^{u_{k-1}}\prod_{j=0}^{k-1}u_j^{-\frac{r^j}{s}}, \tag{6.13}$$

where $c = c(n,s)$ is a constant that goes to 0 as $s \to 0$. As a result, we have

$$\lim_{m\to\infty}\frac{\sum_{k=1}^m \ln c_k^{-1}}{m^2\ln m} = \frac{1}{2}, \qquad \text{if } s = n, \tag{6.14}$$

and

$$\lim_{m\to\infty}\frac{\sum_{k=1}^m \ln c_k^{-1}}{mr^m} = \frac{1}{s\ln r}, \qquad \text{if } s < n. \tag{6.15}$$

Proof. Using (6.9), (6.10) and the inequality $(1+x)^x \in [2,e)$ for $x \geq 1$, we have (6.13).

If $s = n$, then $r = 1$ and thus $u_k = (k+1)/n$. By (6.13), we have

$$\begin{aligned}\ln c_k^{-1} &\leq \frac{k}{n}\ln(ce)^{-1} + \frac{1}{n}\sum_{j=0}^{k-1}\ln\left(\frac{j+1}{n}\right) \\ &\leq \frac{k}{n}\ln(ce)^{-1} - \frac{k\ln n}{n} + \int_1^{k+1}\ln x dx \\ &= \frac{k}{n}\ln(ce)^{-1} - \frac{k\ln n}{n} + (k+1)\ln(k+1) - k,\end{aligned}$$

so

$$\sum_{k=1}^{m} \ln c_k^{-1} \lessapprox C\frac{m^2}{2} + \frac{m^2 \ln m}{2} \approx \frac{m^2 \ln m}{2}.$$

Similarly, we can show that $\sum_{k=1}^{m} \ln c_k^{-1} \gtrapprox \frac{1}{2} m^2 \ln m$. Combining the upper and lower bounds gives (6.14).

If $s < n$, we have the upper bound

$$\begin{aligned}
\ln c_k^{-1} &\le u_{k-1} \ln(ce)^{-1} + \frac{1}{s}\sum_{j=0}^{k-1} r^j \ln u_j \\
&\le \frac{r^k}{n-s} \ln(ce)^{-1} + \frac{1}{s}\sum_{j=0}^{k-1} [(j+1)r^j \ln r - r^j \ln(n-s)] \\
&\lessapprox Cr^k + \frac{1}{s} kr^k.
\end{aligned}$$

Thus

$$\sum_{k=1}^{m} \ln c_k^{-1} \lessapprox Cr^m + \frac{1}{s \ln r} mr^m \approx \frac{1}{s \ln r} mr^m.$$

The lower bound is proved in a similar way. Thus we have (6.15). □

7. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3 using the quantitative analysis developed in Section 6. We still write $r = n/s \ge 1$ throughout the section.

We recall again that α_k depends on $f(k)^{-1} \sim \eta_k = \varepsilon_k(\delta_k)$. An explicit expression for ε_k has been obtained, so we need to specify our refinement of the parameters δ_k.

7.1. **Refining the choice of δ_k.** We will go back to Section 5. Our first refinement will be the the first inequality in (5.8) which was used to show that $\mathcal{H}^h_\infty(C) = 0$. This estimate was rough because we had no further information about h. But now, since we have $h(t) = t^s$, we actually have

$$\begin{aligned}
\mathcal{H}^s_\infty(C_m) &= F(m-1)^{-s}\mathcal{H}^s_\infty\left(F(m-1)C_m\right) \\
&\le F(m-1)^{-s}\mathcal{H}^s_\infty(K_m) \\
&\le F(m-1)^{-s}\delta_m. \qquad (7.1)
\end{aligned}$$

Note that all argument before (5.7) in Section 5.3 is unaffected. Hence, in order that (5.9) still holds, it suffices to require δ_k to obey

$$\lim_{m\to\infty} \#A_1 \cdots \#A_{m-1} F(m-1)^{-s}\delta_m = 0. \qquad (7.2)$$

Definition 7.1. *The refined sequence δ_k is defined as follows.*

(1) If $s = n$, then $\delta_k = 2^{-n}\kappa_n^{-1}$ for any $k \ge 1$.
(2) If $s < n$, then $\delta_1 = 2^{-n}$ and for $k \ge 2$, we let

$$\delta_k = \min\left\{2^{-n}, 2^{1-k}\#A_1^{-1}\cdots\#A_{k-1}^{-1}F(k-1)^s\right\}. \qquad (7.3)$$

Here κ_n is the same dimensional constant in (6.11).

Remark. The minimum in (7.3) is to ensure that all δ_k are small, which is "obvious" but not technically immediate. We will show in Proposition 7.5 below that the minimum is indeed attained at the latter expression when k is large.

It is also important to note that for each k, the terms δ_k, $\#A_k$, η_k, $f(k)$ all depend on s, n only.

Proposition 7.2. *With the new definition of δ_k above, if we define $K_k(\delta_k), \eta_k, f(k), A_k$ and C in the same way as in Section 5, we have $\mathcal{H}^s(C) = 0$ and that C contains a translate of every f-slalom.*

Proof. Since the argument in Section 5.2 is unaffected by the change of δ_k, we see that C still contains a translate of every f-slalom. To prove $\mathcal{H}^s(C) = 0$, by the discussion above, it suffices to prove (7.2), which is immediate in the case $s < n$. If $s = n$, by Proposition 7.3 below, we have

$$\#A_1 \cdots \#A_{m-1} F(m-1)^{-s} \delta_m \leq \kappa_n^{m-1} (2^n \kappa_n)^{-mn} = 2^{-mn^2},$$

and so (7.2) also holds. □

Proposition 7.3. *For all $r \geq 1$, we have the following upper bound for $\#A_k$:*

$$\#A_k \leq \kappa_n \delta_k^r f(k)^n, \tag{7.4}$$

where κ_n is as in (6.11).

Proof. Using the definition of $\#A_k$ in (5.4), we have

$$\begin{aligned} A_k &= \left\{ j \in \mathbb{Z}^n : \frac{j}{f(k)} + \left[0, \frac{12}{f(k)}\right]^n \subseteq K_k + 3e \right\} \\ &\subseteq \left\{ j \in \mathbb{Z}^n : \frac{j}{f(k)} + \left[0, \frac{1}{f(k)}\right]^n \subseteq K_k + 3e \right\} \\ &= \left\{ j \in \mathbb{Z}^n : j + [0,1]^n \subseteq f(k)(K_k + 3e) \right\}, \end{aligned}$$

and hence $\#A_k \leq \mathcal{L}^n(f(k)(K_k + 3e))$. The result then follows from (6.11). □

Now that we have specified the choice of all parameters, Theorem 1.3 will hold once we prove the required rate of decay of α_m.

7.2. **Decay estimate when $s = n$.**

We can now finish the proof of Theorem 1.3 when $s = n$, which amounts to proving (1.1).

Since $r = 1$, we have $u_k = (k+1)/n$. Since $\eta_k = \varepsilon_k(\delta_k)$, using $\delta_k = 2^{-n}\kappa_n^{-1}$ and Proposition 6.2, we have $\eta_k = c_k 2^{-nu_k} \kappa_n^{-u_k}$. Taking logarithm on both sides, we have

$$\ln \eta_k^{-1} = \ln c_k^{-1} + (n \ln 2 + \ln \kappa_n) u_k.$$

Recall that $F(m) = f(1) \cdots f(m)$ and $f(m) \sim \eta_m^{-1}$. Summing over k and using (6.14), we have

$$\ln F(m) \approx \sum_{k=1}^m \ln \eta_k^{-1} + C \approx Cm^2 + Cm + \sum_{k=1}^m \ln c_k^{-1} \approx \frac{m^2 \ln m}{2},$$

which finishes the proof.

7.3. **Decay estimate when $s < n$.**

Now we give a proof of Theorem 1.3 when $s < n$ (so $r > 1$), which amounts to proving (1.2). This requires a more detailed analysis.

7.3.1. *A lower bound for $\#A_k$.*

Proposition 7.4. *Let $0 < s < n$. Then there is some $l_0 = l_0(n,s) \geq 1$ such that for all $k \geq l_0$ we have*

$$\#A_k \geq \kappa \delta_k^r f(k)^n, \tag{7.5}$$

where $\kappa = \kappa(n,s)$ is a constant that goes to 0 as $s \to n$.

Proof. We will return to the proof of Lemma 4.1. By (4.4), (4.7) and (6.7), for all $\delta \leq 2^{-n}$ we have

$$\begin{aligned} K_k(\delta) &\supseteq K_{k-1}(\lambda_{k-1}\delta) \supseteq K_{k-2}(\lambda_{k-2}\lambda_{k-1}\delta) \\ &\supseteq \cdots \supseteq K_1(\lambda_1\lambda_2\cdots\lambda_{k-1}\delta) \\ &\supseteq [-a(\lambda_1\lambda_2\cdots\lambda_{k-1}\delta), a(\lambda_1\lambda_2\cdots\lambda_{k-1}\delta)]^n \\ &= \left[-(\lambda_0\cdots\lambda_{k-1}\delta)^{\frac{1}{s}}, (\lambda_0\cdots\lambda_{k-1}\delta)^{\frac{1}{s}}\right]^n. \end{aligned}$$

Since $1-\lambda_k = (1+nu_k)^{-1}$ and $r > 1$, by (6.9) we see that u_k is a geometric sequence, and so the sequence $1-\lambda_k$ is summable. Thus $\prod_{k=0}^{\infty} \lambda_k > 0$, and hence for some $\kappa' = \kappa'(n,s) > 0$ (which converges to 0 as $s \to n$, since $1-\lambda_k$ is not summable if $r = 1$) we have

$$(\lambda_0\cdots\lambda_{k-1})^{\frac{1}{s}} > \kappa',$$

and thus we have

$$K_k(\delta) \supseteq \left[-\kappa'\delta^{\frac{1}{s}}, \kappa'\delta^{\frac{1}{s}}\right]^n.$$

Hence, for $\delta = \delta_k \leq 2^{-n}$ we have

$$\begin{aligned} \#A_k &= \#\left\{j \in \mathbb{Z}^n : \frac{j}{f(k)} + \left[0, \frac{12}{f(k)}\right]^n \subseteq K_k + 3e\right\} \\ &\geq \#\left\{j \in \mathbb{Z}^n : \frac{j}{f(k)} + \left[0, \frac{12}{f(k)}\right]^n \subseteq \left[-\kappa'\delta_k^{\frac{1}{s}} + 3, \kappa'\delta_k^{\frac{1}{s}} + 3\right]^n\right\} \\ &= \#\left\{j \in \mathbb{Z}^n : j + [0,12]^n \subseteq \left[-\kappa' f(k)\delta_k^{\frac{1}{s}} + 3f(k), \kappa' f(k)\delta_k^{\frac{1}{s}} + 3f(k)\right]^n\right\} \\ &\geq (2\kappa' f(k)\delta_k^{\frac{1}{s}} - 12)^n. \end{aligned}$$

But

$$\frac{2\kappa' f(k)\delta_k^{\frac{1}{s}}}{12} \geq \frac{40\kappa'\delta_k^{\frac{1}{s}}}{12\eta_k} = \frac{40\kappa'\delta_k^{\frac{1}{s}}}{12c_k\delta_k^{u_k}} \geq \frac{3\kappa'}{c_k},$$

which is greater than 3 for k large enough, as $c_k \to 0$. Hence there is some $l_0 \geq 1$ such that for $k \geq l_0$,

$$\#A_k \geq \kappa'^n f(k)^n \delta_k^r.$$

Thus (7.5) follows if we take $\kappa = \kappa'^n$. ∎

7.3.2. *Decay of δ_k.* Recall the choice of δ_k as in (7.3). We show that if $s < n$, then for a large k, the minimum is always attained by the latter expression.

Proposition 7.5. *Let $0 < s < n$, and for $k \geq 2$, denote*

$$\sigma_k = 2^{1-k}\#A_1^{-1}\cdots\#A_{k-1}^{-1}F(k-1)^s.$$

Then there is some $l = l(n,s) \geq l_0$ (as in Proposition 7.4) such that for all $k \geq l$, we have $\sigma_k < 2^{-n}$. Thus for all $k \geq l$ we have $\delta_k = \sigma_k$.

Proof. It suffices to show that $\sigma_k \to 0$. Using the lower bound of A_k (7.5), for $k \geq l_0$ we have

$$\frac{\sigma_{k+1}}{\sigma_k} = \frac{f(k)^s}{2\#A_k} \lesssim \delta_k^{-r} f(k)^{s-n} \sim \delta_k^{(n-s)u_k - r} c_k^{n-s} \leq \delta_k^{r^{k+1}-r-1}.$$

Since $r > 1$, there is some $l_1 \geq l_0$ such that for $k \geq l_1$, we have $r^{k+1} - r - 1 > 0$. For such a k we then have

$$\frac{\sigma_{k+1}}{\sigma_k} \lesssim 2^{-n(r^{k+1}-r-1)},$$

which goes to 0 as $k \to \infty$. Thus the result follows. □

7.3.3. *Explicit decay estimate.* Finally, we prove (1.2), which completes the proof of Theorem 1.3.

Proof of (1.2). Let $k \geq l$ where l is as in Proposition 7.5. Then

$$\delta_k = 2^{1-k} \#A_1^{-1} \cdots \#A_{k-1}^{-1} F(k-1)^s.$$

Taking logarithm on both sides, we have

$$a_k := \ln \delta_k^{-1} = (k-1)\ln 2 + \sum_{j=1}^{k-1} \ln \#A_j - s \ln f(j). \tag{7.6}$$

Using (7.4), (7.5), $f(k) \sim \eta_k = \varepsilon_k(\delta_k) = c_k \delta_k^{u_k}$ where u_k is as in (6.9), we have for $k \geq l$

$$\ln \#A_k - s \ln f(k) = (r^{k+1} - r - 1)\ln \delta_k^{-1} + (n-s)\ln c_k^{-1} + O(1).$$

Then for $k \geq l$, we have

$$a_k = O(k) + \sum_{j=l}^{k-1} (r^{j+1} - r - 1)a_j + (n-s)\ln c_j^{-1}.$$

Denote $v_j = r^{j+1} - r$ and

$$d_k = O(k) + \sum_{j=l}^{k-1} (n-s)\ln c_j^{-1}, \tag{7.7}$$

and so we have

$$a_k = d_k + \sum_{j=l}^{k-1} (v_j - 1)a_j.$$

Subtracting this equation from the $(k+1)$-th equation, we have

$$a_{k+1} = d_{k+1} - d_k + v_k a_k,$$

and thus by iterating this identity, we have

$$a_k = v_l \cdots v_{k-1} a_l + \sum_{j=l+1}^{k} (d_j - d_{j-1}) v_j \cdots v_{k-1}.$$

We can also write

$$a_k = v_l \cdots v_{k-1} \left(a_l + \sum_{j=l+1}^{k} \frac{d_j - d_{j-1}}{v_l \cdots v_{j-1}} \right). \tag{7.8}$$

We claim that

$$\sum_{j=l+1}^{\infty} \frac{d_j - d_{j-1}}{v_l \cdots v_{j-1}} < \infty. \tag{7.9}$$

To prove this, by (7.7) and (6.15) we have

$$d_j - d_{j-1} = O(j) + (n-s)\ln c_j^{-1} \approx (n-s)\frac{1}{s} j r^j.$$

But for all large j, using $v_j = r^{j+1} - r$ we have

$$v_l \cdots v_{j-1} \geq v_{j-2} v_{j-1} = r^2 (r^{j-2} - 1)(r^{j-1} - 1),$$

and thus

$$\frac{d_j - d_{j-1}}{v_l \cdots v_{j-1}} \lesssim j r^{-j},$$

which is summable. This proves (7.9). Hence, by (7.8), we have

$$a_k \sim v_l \cdots v_{k-1} \sim r^{\frac{k^2+k}{2}}.$$

Lastly, recall that $\eta_k = \varepsilon_k(\delta_k) = c_k \delta_k^{u_k}$ and $f(m) \sim \eta_m^{-1}$, so using (6.9) and (6.15) we have

$$\ln F(m) \approx \sum_{k=1}^{m} \ln c_k^{-1} + u_k a_k \sim m r^m + r^m r^{\frac{m^2+m}{2}} \approx r^{\frac{m^2+3m}{2}},$$

which is (1.2). □

8. Proof of Theorem 1.4

In this section we prove Theorem 1.4, and in particular, recover the result of Máthé [4] on the real line. The idea is also to modify the proof of Theorem 1.2, and then do a quantitative analysis as in the last section.

8.1. Refining the choice of δ_k.

Let $s_m \searrow 0$ be a slowly decreasing sequence, so that $s_{m+1}/s_m \to 1$. We may also assume $s_1 \leq 1/2$.

We will then modify the argument in Section 5.1 to suit our needs here.

Let $\delta_1 = 2^{-n}$. By Lemma 4.1 with the dimension function $h(t) = t^{s_1}$, find $K_1 = K_1(\delta_1)$ and $\eta_1 = \varepsilon_1(\delta_1)$ with $\mathcal{H}^{s_1}_\infty(K_1) \leq \delta_1$. With this, let $f(1)$ be any integer in $[20/\eta_1, 40/\eta_1]$, the same as before. Then define $\#A_1$ in the same way as in (5.1).

For each $k \geq 2$, redefine δ_k as

$$\delta_k = \min\left\{2^{-n}, 2^{1-k} \#A_1^{-1} \cdots \#A_{k-1}^{-1} F(k-1)^{s_k}\right\}, \tag{8.1}$$

where $F(k) = f(1) \cdots f(k)$ as before. Then we use Lemma 4.1 with the dimension function $h(t) = t^{s_k}$ to find $K_k = K_k(\delta_k)$ and $\eta_k = \varepsilon_k(\delta_k)$ with $\mathcal{H}^{s_k}_\infty(K_k) \leq \delta_k$. Then define $f(k)$ and A_k the same way as in (5.3) and (5.4), respectively. Define the perfect set $C \subseteq [0,6]^n$ the same way as in (5.5).

It will be important to note that C contains a translate of every f-slalom, since the argument in Section 5.2 remains unchanged.

8.2. **Hausdorff dimension.** We now show that $\dim_H(C) = 0$, by modifying the proof in Section 5.3. Note that all argument before (5.7) remains unchanged, and so it suffices to modify the argument after that.

For any $\varepsilon > 0$, find k such that $s_k < \varepsilon$. Then for all $m \geq k$, the computation corresponding to (5.8) becomes

$$\mathcal{H}^{s_m}_\infty(C_m) \leq F(m-1)^{-s_m} \mathcal{H}^{s_m}_\infty(K_m) \leq \delta_m F(m-1)^{-s_m}.$$

With (8.1), the computation corresponding to (5.9) becomes

$$\begin{aligned}\mathcal{H}^{s_m}_\infty(C) &\leq \#A_1 \cdots \#A_{m-1} \mathcal{H}^{s_m}_\infty(C_m)\\ &\leq \#A_1 \cdots \#A_{m-1} \delta_m F(m-1)^{-s_m}\\ &\leq 2^{1-m}.\end{aligned}$$

Since C is bounded and $s_k < \varepsilon$,

$$\mathcal{H}^{\varepsilon}_\infty(C) \lesssim \mathcal{H}^{s_m}_\infty(C) \leq 2^{1-m}.$$

Letting $m \to \infty$ shows that $\mathcal{H}^{\varepsilon}_\infty(C) = 0$. Since $\varepsilon > 0$ is arbitrary, we have $\dim_H(C) = 0$.

8.3. **Decay estimate.** We now analyse the growth of $F(k) = f(1) \cdots f(k)$ which amounts to the decay of $\eta_k = \varepsilon_k(\delta_k)$. Let $r_k = n/s_k \nearrow \infty$ where $r_1 \geq 2$. Then Proposition 6.2 still holds with all r replaced by r_k. For example

$$u_k = \frac{1}{s_k} \sum_{j=0}^{k} r_k^j = \frac{r_k^{k+1} - 1}{n - s_k} \approx \frac{1}{n} r_k^{k+1},$$

and similar for other terms. The bound (6.11) still holds with the same κ_n.

The upper and lower bounds of $\#A_k$ in (7.4) and (7.5) still hold with r replaced by r_k, with κ' and l_0 being absolute constants since $s_k \leq 1/2$. Proposition 7.5 now holds for some absolute constant l.

Now let $k \geq l$. Proceed almost verbatim as in Section 7.3.3 with all r replaced by r_k, until we arrive at the proof of the counterpart of (7.9), which we restate below:

$$\sum_{j=l+1}^{\infty} \frac{d_j - d_{j-1}}{v_l \cdots v_{j-1}} < \infty. \tag{8.2}$$

Note that now $v_j = r_j^{j+1} - r_j$. To prove the inequality, note that we still have

$$d_j - d_{j-1} = O(j) + (n - s_j) \ln c_j^{-1} \approx (n - s_j) \frac{1}{s_j} j r_j^j = j(r_j - 1) r_j^j,$$

and

$$v_l \cdots v_{j-1} \geq v_{j-2} v_{j-1} \geq (r_{j-2}^{j-1} - r_{j-2})(r_{j-1}^j - r_{j-1}),$$

and so

$$\frac{d_j - d_{j-1}}{v_l \cdots v_{j-1}} \lesssim \frac{j r_j^{j+1}}{r_{j-2}^{j-1} r_{j-1}^j},$$

which is summable by the root test, since by our choice of s_j, we have $r_j / r_{j-1} \to 1$. This proves (8.2). Hence, by the counterpart of (7.8), we still have

$$a_k \sim v_l \cdots v_{k-1} \sim r_l^{l+1} \cdots r_{k-1}^k \sim r_1^2 \cdots r_{k-1}^k.$$

Hence

$$u_m a_m \sim \prod_{k=1}^{m} r_k^{k+1}.$$

Using

$$\ln F(m) \approx \sum_{k=1}^{m} \ln c_k^{-1} + u_k a_k,$$

and following the same proof as of (6.15), we can show that

$$\sum_{k=1}^{m} \ln c_k^{-1} \lesssim \sum_{k=1}^{m} \frac{k r_k^k \ln r_k}{s_k} \leq m^2 r_m^m s_m^{-1} \ln r_m = o(u_m a_m).$$

Hence

$$\ln F(m) \approx \sum_{k=1}^{m} u_k a_k \sim \prod_{k=1}^{m} r_k^{k+1},$$

which completes the proof of (1.4) and consequently Theorem 1.4.

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