A Short Note on Oberlin's Elementary Inequality

Yiyu Liang and Tongou Yang

Let $1 < p < q < \infty$. Consider the inequality studied by Oberlin in [1] and [2]:

$$\left(\frac{1}{3}\left[\left(\frac{a+b}{2}\right)^q + \left(\frac{b+c}{2}\right)^q + \left(\frac{a+c}{2}\right)^q\right]\right)^{\frac{1}{q}} \leqslant \left(\frac{a^p+b^p+c^p}{3}\right)^{\frac{1}{p}}. \tag{1}$$

We would like to determine the set of pairs (p, q) such that the above holds for all $a, b, c \geqslant 0$.

The first observation is the duality of the problem.

Lemma 1. *Let $1 < p < q < \infty$. (1) holds with exponents (p, q) if and only if it holds with exponents (q', p').*

Proof. It suffices to note that (1) is a convolution inequality. To see this, define the measure space as $\{0, 1, 2\}$ with the normalized counting measure, and define $\nu := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$. Then (1) becomes

$$\|f * \nu\|_q \leqslant \|f\|_p.$$

$\square$

Let $\phi(a, b, c) := (a + b)^q + (a + c)^q + (b + c)^q$. It suffices to consider the maximum of ϕ subject to $a^p + b^p + c^p = 1$. We have (1) holds if and only if the maximum is less than or equal to

$$\left(2 \times 3^{\frac{1}{q}-\frac{1}{p}}\right)^q,$$

which is the value of ϕ at $a = b = c$. Hence inequality (1) holds if and only if ϕ attains an absolute maximum at $a = b = c$.

Lemma 2. *Let $1 < p < q < \infty$. Assume $p \geqslant 2$ or $2p + q \leqslant 6$. If ϕ attains a constrained maximum at (a, b, c), then at least two of a, b, c must be equal.*

Proof. The method of Lagrange multipliers gives

$$q[(a+b)^{q-1} + (a+c)^{q-1}] = \lambda p a^{p-1}, \tag{2}$$
$$q[(a+b)^{q-1} + (b+c)^{q-1}] = \lambda p b^{p-1}, \tag{3}$$
$$q[(a+c)^{q-1} + (b+c)^{q-1}] = \lambda p c^{p-1}. \tag{4}$$

Eliminating λ, we have (note if $a = 0$, then (2) gives $b = c = 0$, which is absurd)

$$\frac{(a+b)^{q-1} + (a+c)^{q-1}}{a^{p-1}} = \frac{(a+b)^{q-1} + (b+c)^{q-1}}{b^{p-1}} = \frac{(a+c)^{q-1} + (b+c)^{q-1}}{c^{p-1}}.$$

Let $S = (a+b)^{q-1} + (a+c)^{q-1} + (b+c)^{q-1}$. Then the above can be written as

$$\frac{S-(b+c)^{q-1}}{a^{p-1}} = \frac{S-(a+c)^{q-1}}{b^{p-1}} = \frac{S-(a+b)^{q-1}}{c^{p-1}}.$$

Eliminating S, we get

$$[b^{p-1}(b+c)^{q-1} - a^{p-1}(a+c)^{q-1}](c^{p-1} - a^{p-1}) = [c^{p-1}(b+c)^{q-1} - a^{p-1}(a+b)^{q-1}](b^{p-1} - a^{p-1}).$$

Expanding both sides and cancelling the factor a^{p-1}, we get

$$a^{p-1}[(a+c)^{q-1} - (a+b)^{q-1}] + c^{p-1}[(b+c)^{q-1} - (a+c)^{q-1}] = b^{p-1}[(b+c)^{q-1} - (a+b)^{q-1}].$$

Define

$$H(x) = a^{p-1}[(a+x)^{q-1} - (a+b)^{q-1}] + x^{p-1}[(b+x)^{q-1} - (a+x)^{q-1}] - b^{p-1}[(b+x)^{q-1} - (a+b)^{q-1}].$$

If $a = b$, then we are done. Suppose $a < b$. Then $H(a) = H(b) = 0$ and we will show that there cannot be $c > b$ such that $H(c) = 0$. By Rolle's theorem, it suffices to show that there cannot be $c > b$ such that $H'(c) = 0$.

We compute

$$\begin{aligned} H'(x) &= (b+x)^{q-2}[(p+q-2)x^{p-1} + b(p-1)x^{p-2} - (q-1)b^{p-1}] \\ &\quad - (a+x)^{q-2}[(p+q-2)x^{p-1} + a(p-1)x^{p-2} - (q-1)a^{p-1}]. \end{aligned}$$

Observe that the first term on the right is independent of b and the second term is independent of a. Regarding x as a constant first, and define

$$K_x(y) = (y+x)^{q-2}[(p+q-2)x^{p-1} + y(p-1)x^{p-2} - (q-1)y^{p-1}],$$

where $0 \leqslant y \leqslant x$. Thus $H'(x) = K_x(b) - K_x(a)$. It suffices to show that K_x is either increasing or decreasing on $[0, x]$.

By direct computation, we have

$$K_x'(y) = (q-1)(y+x)^{q-3}[(p+q-3)(x^{p-1} - y^{p-1}) + (p-1)(x^{p-2}y - xy^{p-2})].$$

Our goal is to show that $K_x'(y)$ does not change sign for $0 < y < x$.

Hence it suffices to consider the function

$$M(t) := (p+q-3)(1-t^{p-1}) + (p-1)(t - t^{p-2}), \quad 0 < t < 1.$$

We can also write

$$M(t) = (p-1)(t+1)(1-t^{p-2}) + (q-2)(1-t^{p-1}).$$

For $q \leqslant 2$, as $1 < p < q$, $M(t) < 0$ for all $0 < t < 1$. If $p \geqslant 2$, then $M(t) \geqslant (q-2)(1-t^{p-1}) > 0$ for $0 < t < 1$. So the most interesting cases happen when $1 < p < 2 < q$.

In this case, M diverges to $-\infty$ as $t \to 0^+$ because of the dominant factor $-t^{p-2}$. We compute

$$M'(t) = (p-1)[(3-p-q)t^{p-2} + 1 + (2-p)t^{p-3}].$$

$$M''(t) = (p-1)(2-p)[(p+q-3)t^{p-3} + (p-3)t^{p-4}],$$

which is negative for $0 < t < 1$, as $2p + q \leqslant 6$. Thus M' is decreasing on $(0, 1)$. Since $M'(1) = (p-1)(6-2p-q) \geqslant 0$, we have $M'(t) > 0$ for all $0 < t < 1$. Therefore we have shown that in the case

$$2p + q \leqslant 6,$$

the lemma is true. Combining the three cases gives a sufficient condition $p \geqslant 2$ or $2p + q \leqslant 6$.

Remark: Computer-aided plots suggest that if $p < 2$, then $2p + q \leqslant 6$ is also necessary for the lemma to hold, but we have not found a proof for this statement.

□

Theorem 1. *Let $1 < p < q < \infty$. Then*

1. *(1) holds only if both of the following are true:*

$$\frac{1}{q} - \frac{1}{1-\alpha}\frac{1}{p} \geqslant -\frac{\alpha}{1-\alpha} \tag{5}$$

$$\frac{1}{p'} - \frac{1}{1-\alpha}\frac{1}{q'} \geqslant -\frac{\alpha}{1-\alpha}, \tag{6}$$

   where $\alpha := \log 2/\log 3 \approx 0.63$.

2. *Suppose the conclusion in Lemma 2 holds. Then (1) holds only if*

$$q + 3 \leqslant 4p \quad \Longleftrightarrow \quad p' + 3 \leqslant 4q'. \tag{7}$$

   In particular, If $q = 3$, then (1) holds only if $p \geqslant 3/2$.

3. *(1) holds only if $2p + q \leqslant 6$ or $2q' + p' \leqslant 6$. Therefore, to find (p, q) such that (1) holds, we can always assume without loss of generality that the conclusion in Lemma 2 holds.*

4. *If either*

   (a) $2p + q \leqslant 6$ and

$$1 - q + 2\left(\frac{q}{p} - 1\right)^p \leqslant 0, \tag{8}$$

   or

   (b) $2q' + p' \leqslant 6$ and

$$1 - p' + 2\left(\frac{p'}{q'} - 1\right)^{q'} \leqslant 0, \tag{9}$$

   then (1) holds.

   In particular, (1) holds if $q = 3$ and $p = 3/2$.

Proof. 1. The first inequality is by simply comparing the value of ϕ at $a = b = c$ and the value of ϕ at $a = 1, b = 0, c = 0$, respectively. The second inequality follows from duality.

2. By the result of Lemma 2, we consider the function

$$f_0(t) := \frac{[(2t)^q + 2(1+t)^q]^{1/q}}{(2t^p+1)^{1/p}}, \quad t \in [0,\infty).$$

Our goal is to show whether f_0 attains a global maximum at $t = 1$.

Equivalently, we consider $f(t) := f_0(t)^p$. Direct computation shows that $f'(t)$ has the same sign as

$$g(t) := 2^{q-1}t^{q-1} + (1+t)^{q-1}(1 - 2t^{p-1}),$$

which, in turn, has the same sign as

$$h(t) := 2^{q-1} + (1+t)^{q-1}(t^{1-q} - 2t^{p-q}).$$

We have $h(0^+) = \infty$, $h(1) = 0$, $h(\infty) = -\infty$. If f attains a global maximum at $t = 1$, then we necessarily have $f''(1) \leqslant 0$, which, by simple calculation, is equivalent to $h'(1) \leqslant 0$.

Direct computation shows

$$h'(t) = (1+t)^{q-2}[2(q-p)t^{p-q-1} - (q-1)t^{-q} - 2(p-1)t^{p-q}],$$

which has the same sign as the function

$$k(t) := 2(q-p)t^{p-1} - (q-1) - 2(p-1)t^p.$$

Thus $h'(1)$ has the same sign as $k(1) = q - 4p + 3$. Thus necessarily, we must have $q - 4p + 3 \leqslant 0$.

In particular, if $q = 3$ and $p < 3/2$, then $2p + q \leqslant 6$, so the conclusion of Lemma 2 holds, and the above argument implies that (1) holds only if $3 + 3 \leqslant 4p$, which is a contradiction. So if $q = 3$, we must have $p \geqslant 3/2$.

3. From the previous part, if $q = 3$, then (1) holds only if $p \geqslant 3/2$. By duality and monotonicity, (1) fails to hold if $(1/p, 1/q)$ lies in the square $S := [2/3, 1] \times [0, 1/3] \backslash \{(2/3, 1/3)\}$, which contains $\{(p,q) : 1 < p < q < \infty,\ 2p + q > 6 \text{ and } 2q' + p' > 6\}$. Hence (1) holds only if $2p + q \leqslant 6$ or $2q' + p' \leqslant 6$.

   Now given $1 < p < q < \infty$. If $2p + q \leqslant 6$, then the conclusion of Lemma 2 holds. If $2q' + p' \leqslant 6$, then the conclusion of Lemma 2 holds with the pair (q', p'), so it suffices to consider (1) for the dual pair (q', p'). If both of them fail, then the previous paragraph shows that (1) fails.

4. By duality, we consider the case (a) only. It suffices to show that h is decreasing on $(0, \infty)$.

   Following the notations in (a), we have $k(0) = 1 - q < 0$, $k(\infty) = -\infty$. It suffices to show $\max\{k(t) : 0 \leqslant t < \infty\} \leqslant 0$. We have

$$k'(t) = 2(p-1)(q - p - pt).$$

   It is easy to see that k attains the global maximum at $t = (q-p)/p \in (0, \infty)$. The maximum value of k is

$$M := 2\left(\frac{q}{p} - 1\right)^p - (q-1).$$

Hence our assumption was exactly the statement that $M \leqslant 0$.

Notice that q can be viewed as an implicit function of p, which is strictly increasing from 0 to 1.

□

However, the requirement that h is decreasing on $(0,\infty)$ was too strong. We can somehow relax the condition so as to get better bounds on p,q. In particular, we have

Theorem 2. *Let $q = 2$. Then (1) holds if and only if $p \geqslant \log 4/\log 3$.*

Proof. Since we can always assume Lemma 2, it suffices to show that the function

$$u(t) := \frac{3^{\frac{1}{p}-\frac{1}{q}}}{2} \cdot \frac{[(2t)^q + 2(1+t)^q]^{\frac{1}{q}}}{(2t^p+1)^{\frac{1}{p}}} \leqslant 1.$$

Taking logarithm on both sides, the above becomes

$$v(t) := \log 2 + \frac{1}{p}\log(2t^p+1) - \left(\frac{1}{p}-\frac{1}{q}\right)\log 3 - \frac{1}{q}\log[(2t)^q + 2(1+t)^q] \geqslant 0.$$

Notice that $v(0) \geqslant 0$ is equivalent to saying that both (5) and (6) hold. Note also $v(1) = 0$ and $v'(1) = 0$. We have $v''(1) \geqslant 0$ if and only if (7) holds.

By direct computations, we have $v'(t)$ has the same sign as

$$\psi(t) := 2t^{p-1} - \left(\frac{2t}{1+t}\right)^{q-1} - 1.$$

As $t \to \infty$, $\psi(t) \to \infty$ since $p > 1$. Therefore, to show that $v(t) \geqslant 0$ for all $t \geqslant 0$, apart from $v(0) \geqslant 0$, it suffices to show that $v(t_0) \geqslant 0$ whenever $\psi(t_0) = 0$.

For different values of (p,q), the numbers of roots of ψ are different. (8) holds if and only if ψ has a unique root at $t = 1$, in which case v has a unique absolute minimum 0 at $t = 1$.

If (8) fails, then things become more subtle. In the case $q = 2$, $\psi'(t) = 2(p-1)t^{p-2} - 2(1+t)^{-2}$, and $\psi'(1) \geqslant 0$ if and only if $p \geqslant 5/4 = (q+3)/4$. $\psi'(t) \geqslant 0$ for all $t \geqslant 1$ if and only if $w(t) := t^{p-2}(1+t)^2 \geqslant 1/(p-1)$ for all $t \geqslant 1$. But then w is increasing on $[2/p-1,\infty)$, (where $2/p - 1 = q/p - 1 \in (0,1)$) so it attains minimum at $t = 1$, whence $w = 4 \geqslant 1/(p-1)$. This shows that ψ has no roots for $t > 1$.

Hence if $t_0 > 0$ is a root of ψ other than 1, then $t_0 \in (0,1)$. There are exactly two roots $0 < t_0 < t_1 < 1$ due to the growth rates of the functions $2t^{p-1}$ and $2t/(1+t)+1$, and we have t_0 is a local minimum and t_1 is a local maximum. Hence $v(t) \geqslant 0$ if and only if $v(t_0) \geqslant 0$.

If $p = \log 4/\log 3$, we can check that $t_0 = 1/3$, whence $v(t_0) = 0$, so $p = \log 4/\log 3$ works for (1), which is an equality in this case.

On the other hand, we can check using Jensen's inequality that whenever a,b,c are not all equal,

$$p \mapsto \left(\frac{a^p+b^p+c^p}{3}\right)^{\frac{1}{p}}$$

is strictly increasing. With $a = b = 1/3$, $c = 1$, the above shows that (1) is an equality when $p = \log 4/\log 3$; thus if p is smaller, (1) fails. □

References

[1] Daniel M. Oberlin, *A convolution property of the Cantor-Lebesgue measure*, Colloq. Math. **47** (1982), no. 1, 113–117, DOI 10.4064/cm-47-1-113-117. MR679392

[2] ———, *A convolution property of the Cantor-Lebesgue measure. II*, Colloq. Math. **97** (2003), no. 1, 23–28, DOI 10.4064/cm97-1-3. MR2010539