Zeros of Cylindrical Bessel and Neumann Functions

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For every real order ν, cylindrical Bessel and Neumann functions have an infinite number of zeros on the positive real axis. The real zeros on the positive real axis can be found by solving for the roots of

$$
\begin{aligned}
J_\nu\left(j_{\nu,m}\right) &= 0 \\
Y_\nu\left(y_{\nu,m}\right) &= 0\,.
\end{aligned}
\tag{1}
$$

Here, $j_{\nu,m}$ represents the m^{th} root of the cylindrical Bessel function of order ν and $y_{\nu,m}$ represents the m^{th} root of the cylindrical Neumann function of order ν.

Various methods are used to compute initial estimates for $j_{\nu,m}$ and $y_{\nu,m}$, and these will be described in detail below. After finding the initial estimate of a given root, its precision is subsequently refined to the desired level using Newton-Raphson iteration from `Boost.Math`'s root-finding utilities combined with the functions `cyl_bessel_j()` and `cyl_neumann()`, also from `Boost.Math`.

Newton iteration requires both $J_\nu(x)$ (or $Y_\nu(x)$) as well as its derivative. The derivatives of $J_\nu(x)$ and $Y_\nu(x)$ with respect to x are given by Eq. 9.1.3 in [1]. In particular,

$$
\begin{aligned}
\frac{d}{dx}J_\nu(x) &= J_{\nu-1}(x) - \frac{\nu}{x}\,J_\nu(x) \\
\frac{d}{dx}Y_\nu(x) &= Y_{\nu-1}(x) - \frac{\nu}{x}\,Y_\nu(x)\,.
\end{aligned}
\tag{2}
$$

Enumeration of the rank of a root (in other words the index of a root) begins with one and counts up, in other words $m = 1, 2, 3, \ldots$, etc. The value of the first root is always greater than zero.

For certain special parameters, cylindrical Bessel functions and cylindrical Neumann functions have a root at the origin. For example, $J_\nu(x)$ has a root at the origin for every positive order $\nu > 0$ and for every negative integer order $\nu = -n$, with $n \in \mathbb{N}^+$ and $n \neq 0$. In addition, $Y_\nu(x)$ has a root at the origin for every negative half-integer order $\nu = -n/2$, with $n \in \mathbb{N}^+$ and $n \neq 0$. For these special parameter values, the origin with a value of $x = 0$ is provided as the 0^{th} root generated by `cyl_bessel_j_zero()` and `cyl_neumann_zero()`.

When calculating initial estimates for the roots of Bessel functions, a distinction is made between positive order and negative order, and different methods are used for these. In addition, different algorithms are used for the first root ($m = 1$) and for subsequent roots with higher rank ($m \geq 2$). Furthermore, estimates of the roots for Bessel functions with order above and below a cutoff at $\nu = 2.2$ are calculated with different methods.

Calculations of the estimates of $j_{\nu,1}$ and $y_{\nu,1}$ with $0 \leq \nu < 2.2$ use empirically tabulated values. The coefficients for these have been generated by a computer algebra system.

Calculations of the estimates of $j_{\nu,1}$ and $y_{\nu,1}$ with $\nu \geq 2.2$ use Eqs. 9.5.14 and 9.5.15 in [1]. In particular,

$$\begin{aligned} j_{\nu,1} \sim\ & \nu + 1.85575\,71\nu^{\frac{1}{3}} + 1.03315\,0\nu^{-\frac{1}{3}} - 0.00397\nu^{-1} \\ & - 0.0908\nu^{-\frac{5}{3}} + 0.043\nu^{-\frac{7}{3}} + \cdots , \end{aligned} \tag{3}$$

and

$$\begin{aligned} y_{\nu,1} \sim\ & \nu + 0.93157\,68\nu^{\frac{1}{3}} + 0.26035\,1\nu^{-\frac{1}{3}} + 0.01198\nu^{-1} \\ & - 0.0060\nu^{-\frac{5}{3}} - 0.001\nu^{-\frac{7}{3}} + \cdots , \end{aligned} \tag{4}$$

Calculations of the estimates of $j_{\nu,m}$ and $y_{\nu,m}$ with rank $m \geq 2$ and $0 \leq \nu < 2.2$ use McMahon's approximation, as described in Sect. 9.5 of [1] and Eq. 9.5.12 therein. In particular,

$$\begin{aligned} j_{\nu,m},\, y_{\nu,m} \sim\ & \beta - \frac{\mu - 1}{8\beta} - \frac{4(\mu-1)(7\mu-31)}{3(8\beta)^3} \\ & - \frac{32(\mu-1)(83\mu^2 - 982\mu + 3779)}{15(8\beta)^5} \\ & - \frac{64(\mu-1)(6949\mu^3 - 1\,53855\mu^2 + 15\,85743\mu - 62\,77237)}{105(8\beta)^7} \\ & - \cdots , \end{aligned} \tag{5}$$

where $\mu = 4\nu^2$ and $\beta = \left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi$ for $j_{\nu,m}$ and $\beta = \left(m + \frac{1}{2}\nu - \frac{3}{4}\right)\pi$ for $y_{\nu,m}$.

Calculations of the estimates of $j_{\nu,m}$ and $y_{\nu,m}$ with $\nu \geq 2.2$ use one term in the asymptotic expansion given in Eq. 9.5.22 and top line of Eq. 9.5.26 combined with Eq. 9.3.39, all in [1]. The latter two equations are expressed for argument x greater than one. In summary,

$$j_{\nu,m} \sim \nu x(-\zeta) + \frac{f_1(-\zeta)}{\nu} , \tag{6}$$

where $-\zeta = \nu^{-2/3} a_m$ and a_m is the absolute value of the m^{th} root of $Ai(x)$ on the negative real axis. Here, $x = x(-\zeta)$ is the inverse of the function

$$\frac{2}{3}(-\zeta)^{3/2} = \sqrt{x^2 - 1} - \cos^{-1}\left(\frac{1}{x}\right). \tag{7}$$

Furthermore,

$$f_1(-\zeta) = \frac{1}{2}\, x(-\zeta)\big\{h(-\zeta)\big\}^2 b_0(-\zeta)\,, \tag{8}$$

where

$$h(-\zeta) = \left\{\frac{4(-\zeta)}{x^2 - 1}\right\}^{\frac{1}{4}}, \tag{9}$$

and

$$b_0(-\zeta) = -\frac{5}{48\zeta^2} + \frac{1}{(-\zeta)^{\frac{1}{2}}}\left\{\frac{5}{24(x^2 - 1)^{3/2}} + \frac{1}{8(x^2 - 1)^{1/2}}\right\}. \tag{10}$$

When solving for $x(-\zeta)$ in Eq. 7 above, the right-hand-side is expanded to order–(x^2) in a Taylor series for large x. This results in

$$\frac{2}{3}(-\zeta)^{3/2} \approx x + \frac{1}{2x} - \frac{\pi}{2}\,. \tag{11}$$

The positive root of the resulting quadratic equation is used to find an initial estimate $x(-\zeta)$. This initial estimate is subsequently refined with several steps of Newton-Raphson iteration in Eq. 7.

Estimates of the roots of cylindrical Bessel functions of negative order on the positive real axis are found using interlacing relations. For example, the m^{th} root of the cylindrical Bessel function $j_{-\nu,m}$ is bracketed by the m^{th} root and the $(m+1)^{\mathrm{th}}$ root of the Bessel function of corresponding positive integer order. In other words,

$$j_{n_\nu,m} < j_{-\nu,m} < j_{n_\nu,m+1} \tag{12}$$

where $m > 1$ and n_ν represents the integral floor of the absolute value of $|-\nu|$.

Similar bracketing relations are used to find estimates of the roots of Neumann functions of negative order, whereby a discontinuity at every negative half-integer order needs to be handled.

Bracketing relations do not hold for the first root of cylindrical Bessel functions and cylindrical Neumann functions with negative order. Therefore, iterative algorithms combined with root-finding via bisection are used to localize $j_{-\nu,1}$ and $y_{-\nu,1}$.

References

[1] M. Abramowitz and I. A. Stegun: *Handbook of Mathematical Functions, 9^th Printing*, (Dover Publications, New York, 1972)