

COMPLEXITY OF DEEP COMPUTATIONS VIA TOPOLOGY OF FUNCTION SPACES

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ABSTRACT. We study complexity of deep computations. We use topology of function spaces, specifically, the classification Rosenthal compacta, to identify new complexity classes. We use the language of model theory, specifically, the concept of the independence from Shelah’s classification theory, to translate between topology and computation.

0. INTRODUCTION

In this paper we study limit behavior of real-valued computations as the value of certain parameters of the computation model tend towards infinity, or towards zero, or towards some other fixed value, e.g., the depth of a neural network tending to infinity, or the time interval between layers of the network tending toward zero. Recently, particular cases of this situation have attracted considerable attention in deep learning research (e.g., Neural Ordinary Differential Equations [CRBD], Physics-Informed Neural Networks [RPK19], deep equilibrium models [BKK], etc.). In this paper, we combine ideas of topology and model theory to study these limit phenomena from a unified viewpoint.

Informed by model theory, to each computation in a given computation model, we associate a continuous real-valued function, called the *type* of the computation, that describes the logical properties of this computation with respect to the rest of the model. This allows us to view computations in any given computational model as elements of a space of real-valued functions, which is called the *space of types* of the model. The idea of embedding models of theories into their type spaces is central in model theory. The embedding of computations into spaces of types allows us to utilize the vast theory of topology of function spaces, known as C_p -theory, to obtain results about complexity of topological limits of computations. As we shall indicate next, recent classification results for spaces of functions provide an elegant and powerful machinery to classify computations according to their levels of “tameness” or “wildness”, with the former corresponding roughly to polynomial approximability and the latter to exponential approximability. The viewpoint of spaces of types, which we have borrowed from model theory, thus becomes a “Rosetta stone” that allows us to interconnect various classification programs: In topology, the classification of Rosenthal compacta pioneered by Todorčević [Tod99]; in logic, the classification of theories developed by Shelah [She90]; and in statistical learning, the notion PAC learning and VC dimension pioneered by Vapkins and Chervonenkis [VC74, VC71].

³⁶ In a previous paper [ADIW24], we introduced the concept of limits of computations, which we called *ultracomputations* (given they arise as ultrafilter limits of

³⁸ standard computations) and *deep computations* (following usage in machine learning [BKK]). There is a technical difference between both designations, but in this
³⁹ paper, to simplify the nomenclature, we will ignore the difference and use only the
⁴⁰ term “deep computation”.

⁴¹ In [ADIW24], we proved a new “tame vs wild” (i.e., polynomial vs exponential)
⁴² dichotomy for complexity of deep computations by invoking a classical result of
⁴³ Grothendieck from late 50s [Gro52]. Under our model-theoretic Rosetta stone,
⁴⁴ polynomial approximability in the sense of computation becomes identified with the
⁴⁵ notion of continuous extendability in the sense of topology, and with the notions of
⁴⁶ *stability* and *type definability* in the sense of model theory.
⁴⁷

⁴⁸ In this paper, we follow a more general approach, i.e., we view deep computations
⁴⁹ as pointwise limits of continuous functions. In topology, real-valued functions that
⁵⁰ arise as the pointwise limit of a sequence of continuous are called *functions of the*
⁵¹ *first Baire class*, or *Baire class 1* functions, or *Baire-1* for short; Baire class 1 form
⁵² a step above simple continuity in the hierarchy of functions studied in real analysis
⁵³ (Baire class 0 functions being continuous functions). Intuitively, Baire-1 functions
⁵⁴ represent functions with “controlled” discontinuities, so they are crucial in topology
⁵⁵ and set theory.

⁵⁶ We prove a new “tame vs wild” Ramsey-theoretic dichotomy for complexity of
⁵⁷ general deep computations by invoking a famous paper by Bourgain, Fremlin and
⁵⁸ Talagrand from the late 70s [BFT78], and a new trichotomy for the class of “tame”
⁵⁹ deep computations by invoking an equally celebrated result of Todorčević, from the
⁶⁰ late 90s, for functions of the first Baire class [Tod99].

⁶¹ Todorčević’s trichotomy regards *Rosenthal compacta*; these are special classes of
⁶² topological spaces, defined as compact spaces that can be embedded (homeomor-
⁶³ phically identified as a subset) within the space of Baire class 1 functions on some
⁶⁴ Polish (separable, complete metric) space, under the pointwise convergence topol-
⁶⁵ ogy. Rosenthal compacta exhibit “topological tameness,” meaning that they behave
⁶⁶ in relatively controlled ways, and since the late 70’s, they have played a crucial role
⁶⁷ for understanding complexity of structures of functional analysis, especially, Banach
⁶⁸ spaces. Todorčević’s trichotomy has been utilized to settle longstanding problems
⁶⁹ in topological dynamics and topological entropy [GM22].

⁷⁰ Through our Rosetta stone, Rosenthal compacta in topology correspond to the
⁷¹ important concept of “No Independence Property” (known as “NIP”) in model
⁷² theory, identified by Shelah [She71, She90], and to the concept of Probably Ap-
⁷³ proximately Correct learning (known as “PAC learnability”) in statistical learning
⁷⁴ theory identified by Valiant [Val84].

⁷⁵ Going beyond Todorčević’s trichotomy, we invoke a more recent heptachotomy
⁷⁶ for Rosenthal compacta obtained by Argyros, Dodos and Kanellopoulos [ADK08].
⁷⁷ Argyros, Dodos and Kanellopoulos identified the fundamental “prototypes” of sepa-
⁷⁸ rable Rosenthal compacta, and proved that any non-metrizable separable Rosenthal
⁷⁹ compactum must contain a “canonical” embedding of one of these prototypes. They
⁸⁰ showed that if a separable Rosenthal compactum is not hereditarily separable, it
⁸¹ must contain an uncountable discrete subspace of the size of the continuum.

⁸² We believe that the results presented in this paper show practitioners of com-
⁸³ putation, or topology, or descriptive set theory, or model theory, how classification
⁸⁴ invariants used in their field translate into classification invariants of other fields.
⁸⁵ However, in the interest of accessibility, we do not assume previous familiarity with

high-level topology or model theory, or computing. The only technical prerequisite of the paper is undergraduate-level topology. The necessary topological background beyond undergraduate topology is covered in section 1.

Throughout the paper, we focus on classical computation; however, by refining the model-theoretic tools, the results presented here can be extended to quantum computation and open quantum systems. This extension will be addressed in a forthcoming paper.

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In this section we give preliminaries from general topology and function space theory. We include some of the proofs for completeness, but the reader familiar with these topics may skip them.

Recall that a subset of a topological space is F_σ if it is a countable union of closed sets, and G_δ if it is a countable intersection of closed sets. Note that in a metrizable space, every open set is F_σ ; equivalently, every closed set is G_δ .

121 A *Polish space* is a separable and completely metrizable topological space. The
 122 most important examples are the reals \mathbb{R} , the Cantor space $2^{\mathbb{N}}$ (the set of all infinite
 123 binary sequences, endowed with the product topology), and the Baire space $\mathbb{N}^{\mathbb{N}}$ (the
 124 set of all infinite sequences of naturals, also with the product topology). Countable
 125 products of Polish spaces are Polish; this includes spaces like $\mathbb{R}^{\mathbb{N}}$, the space of
 126 sequences of real numbers.

In this paper, we shall discuss subspaces, and so there is a pertinent subtlety of the definitions worth mentioning: *completely metrizable space* is not the same as *complete metric space*; for an illustrative example, the interval $(0, 1)$ with the metric

130 inherited from the reals not complete, but it is Polish since that is homeomorphic
 131 to the real line. Being Polish is a topological property.

132 The following result is a cornerstone of descriptive set theory, closely tied to the
 133 work of Wacław Sierpiński and Kazimierz Kuratowski, with proofs often built upon
 134 their foundations and formalized later, notably, involving Stefan Mazurkiewicz's
 135 work on complete metric spaces.

136 **Fact 1.1.** *A subset A of a Polish space X is itself Polish in the subspace topology
 137 if and only if it is a G_δ set. In particular, closed subsets and open subsets of Polish
 138 spaces are also Polish spaces.*

139 Given two topological spaces X and Y we denote by $C_p(X, Y)$ the set of all con-
 140 tinuous functions $f : X \rightarrow Y$ endowed with the topology of pointwise convergence.
 141 When $Y = \mathbb{R}$, we denote this collection simply as $C_p(X)$. A natural question is, how
 142 do topological properties of X translate to $C_p(X)$ and vice versa? These questions,
 143 and in general the study of these spaces, are the concern of C_p -theory, an active
 144 field of research in general topology which was pioneered by A. V. Arhangel'skiĭ
 145 and his students in the 1970's and 1980's. This field has found many applications in
 146 model theory and functional analysis. Recent surveys on the topics include [HT23]
 147 and [Tka11].

148 A *Baire class 1* function between topological spaces is a function that can be
 149 expressed as the pointwise limit of a sequence of continuous functions. If X and Y
 150 are topological spaces, the Baire class 1 functions $f : X \rightarrow Y$ endowed with the
 151 topology of pointwise convergence is denoted $B_1(X, Y)$. As above, in the special
 152 case $Y = \mathbb{R}$ we denote $B_1(X, Y)$ as $B_1(X)$. Clearly, $C_p(X, Y) \subseteq B_1(X, Y)$. The
 153 Baire hierarchy of functions was introduced by French mathematician René-Louis
 154 Baire in his 1899 doctoral thesis, *Sur les fonctions de variables réelles*. His work
 155 moved away from the 19th-century preoccupation with "pathological" functions
 156 toward a constructive classification based on pointwise limits.

157 A topological space X is *perfectly normal* if it is normal and every closed subset
 158 of X is a G_δ (equivalently, every open subset of X is a G_δ). Note that every
 159 metrizable space is perfectly normal.

160 The following fact was established by Baire in thesis. A proof can be found in
 161 Section 10 of [Tod97].

162 **Fact 1.2 (Baire).** *If X is perfectly normal, then the following conditions are equiv-
 163 alent for a function $f : X \rightarrow \mathbb{R}$:*

- 164 • *f is a Baire class 1 function, that is, $f \in B_1(X)$.*
- 165 • *$f^{-1}[U]$ is an F_σ subset of X whenever $U \subseteq \mathbb{R}$ is open.*
- 166 • *f is a pointwise limit of continuous functions.*
- 167 • *For every closed $F \subseteq X$, the restriction $f|_F$ has a point of continuity.*

168 Moreover, if X is Polish and $f \notin B_1(X)$, then there exists countable $D_0, D_1 \subseteq X$
 169 and reals $a < b$ such that

$$D_0 \subseteq f^{-1}(-\infty, a], \quad D_1 \subseteq f^{-1}[b, \infty), \quad \overline{D_0} = \overline{D_1}.$$

170 A subset L of a topological space X is *relatively compact* in X if the closure
 171 of L in X is compact. Relatively compact subsets of $B_1(X)$ (for X Polish space)
 172 have been objects of interest for researchers in Analysis and Topological Dynamics.

173 We begin with the following well-known result. Recall that a set $A \subseteq \mathbb{R}^X$ of real-
 174 valued functions is *pointwise bounded* if for every $x \in X$ there is $M_x > 0$ such that
 175 $|f(x)| < M_x$ for all $f \in A$. We include a proof for the reader's convenience:

176 **Lemma 1.3.** *Let X be a Polish space and $A \subseteq B_1(X)$ be pointwise bounded. The
 177 following are equivalent:*

- 178 (i) A is relatively compact in $B_1(X)$.
- 179 (ii) A is relatively countably compact in $B_1(X)$, i.e., every countable subset of
 180 A has an accumulation point in $B_1(X)$.
- 181 (iii) $\overline{A} \subseteq B_1(X)$, where \overline{A} denotes the closure in \mathbb{R}^X .

182 *Proof.* Since A is pointwise bounded, for each $x \in X$, fix $M_x > 0$ such that $|f(x)| \leq$
 183 M_x for every $f \in A$.

184 (i) \Rightarrow (ii) holds in general.

185 (ii) \Rightarrow (iii) Assume that A is relatively countably compact in $B_1(X)$ and that
 186 $f \in \overline{A} \setminus B_1(X)$. By Fact 1.2, there are countable $D_0, D_1 \subseteq X$ with $\overline{D}_0 = \overline{D}_1$, and
 187 $a < b$ such that $D_0 \subseteq f^{-1}(-\infty, a]$ and $D_1 \subseteq f^{-1}[b, \infty)$. We claim that there is a
 188 sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in D_0 \cup D_1$. Indeed,
 189 use the countability to enumerate $D_0 \cup D_1$ as $\{x_n\}_{n \in \mathbb{N}}$. Then for each positive n
 190 find $f_n \in A$ with $|f_n(x_i) - f(x_i)| < \frac{1}{n}$ for all $i \leq n$. The claim follows.

191 By relative countable compactness of A , there is an accumulation point $g \in$
 192 $B_1(X)$ of $\{f_n\}_{n \in \mathbb{N}}$. It is straightforward to show that since f and g agree on
 193 $D_0 \cup D_1$, g does not have a point of continuity on the closed set $\overline{D}_0 = \overline{D}_1$, which
 194 contradicts Fact 1.2.

195 (iii) \Rightarrow (i) Suppose that $\overline{A} \subseteq B_1(X)$. Then $\overline{A} \cap B_1(X) = \overline{A}$ is a closed subset of
 196 $\prod_{x \in X} [-M_x, M_x]$; Tychonoff's theorem states that the product of compact spaces
 197 is always compact, and since closed subsets of compact spaces are compact, \overline{A} must
 198 be compact, as desired. \square

199 **1.1. From Rosenthal's dichotomy to the Bourgain-Fremlin-Talagrand dichotomy to Shelah's NIP.** The fundamental idea that connects the rich theory
 200 here presented to real-valued computations is the concept of an *approximation*. In
 201 the reals, points of closure from some subset can always be approximated by points
 202 inside the set, via a convergent sequence. For more complicated spaces, such as
 203 $C_p(X)$, this fails in remarkable ways. To see an example, consider the Cantor space
 204 $X = 2^\mathbb{N}$, and for each $n \in \mathbb{N}$ define $p_n : X \rightarrow \{0, 1\}$ by $p_n(x) = x(n)$ for each $x \in X$.
 205 Then p_n is continuous for each n , but one can show (see Chapter 1.1 of [Tod97]
 206 for details) that the only continuous functions in the closure of $\{p_n\}_{n \in \mathbb{N}}$ are the
 207 functions p_n themselves; moreover, none of the subsequences of $\{p_n\}_{n \in \mathbb{N}}$ converge.
 208 In some sense, this example is the worst possible scenario for convergence. The
 209 topological space obtained from this closure is well-known: it is the *Stone-Čech*
 210 *compactification* of the discrete space of natural numbers, or $\beta\mathbb{N}$ for short, and it
 211 is an important object of study in general topology.

212 The following theorem, established by Haskell Rosenthal in 1974, is fundamental
 213 in functional analysis, and describes a sharp division in the behavior of sequences
 214 within a Banach space:

215 **Theorem 1.4** (Rosenthal's Dichotomy, [Ros74]). *If X is Polish and $\{f_n\} \subseteq C_p(X)$
 216 is pointwise bounded, then either $\{f_n\}_{n \in \mathbb{N}}$ contains a convergent subsequence or a
 217 subsequence whose closure (in \mathbb{R}^X) is homeomorphic to $\beta\mathbb{N}$.*

219 In other words, a pointwise bounded set of continuous functions either contains
 220 a convergent subsequence, or a subsequence whose closure is essentially the same as
 221 the example mentioned in the previous paragraphs (the “wildest” possible scenario).
 222 Note that in the preceding example, the functions are trivially pointwise bounded
 223 in \mathbb{R}^X as the functions can only take values 0 and 1.

224 The genesis of Theorem 1.4 was Rosenthal’s ℓ_1 theorem, which states that the
 225 only reason why Banach space can fail to have an isomorphic copy of ℓ_1 (the space
 226 of absolutely summable sequences) is the presence of a bounded sequence with no
 227 weakly Cauchy subsequence. The theorem is famous for connecting diverse areas
 228 of mathematics: Banach space geometry, Ramsey theory, set theory, and topology
 229 of function spaces.

230 As we transition from $C_p(X)$ to the larger space $B_1(X)$, we find a similar di-
 231 chotomy. Either every point of closure of the set of functions will be a Baire class
 232 1 function, or there is a sequence inside the set that behaves in the wildest pos-
 233 sible way. The theorem is usually not phrased as a dichotomy but rather as an
 234 equivalence:

235 **Theorem 1.5** (The BFT Dichotomy. Bourgain-Fremlin-Talagrand, [BFT78, The-
 236 oreom 4G]). *Let X be a Polish space and $A \subseteq C_p(X)$ be pointwise bounded. The
 237 following are equivalent:*

- 238 (i) *A is relatively compact in $B_1(X)$, i.e., $\overline{A} \subseteq B_1(X)$.*
 238 (ii) *For every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and every $a < b$ there is $I \subseteq \mathbb{N}$ such that*

$$\bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} f_n^{-1}[b, \infty) = \emptyset.$$

239 **Definition 1.6.** We shall say that a set $A \subseteq \mathbb{R}^X$ has the *Independence Property*, or
 240 IP for short, if it satisfies the following condition: There exists every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$
 241 and $a < b$ such that for every pair of disjoint sets $E, F \subseteq \mathbb{N}$, we have

$$\bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

242 If A satisfies the negation of this condition, we will say that A *satisfies NIP*, or
 243 that has the NIP.

Remark 1.7. Note that if X is compact and $A \subseteq C_p(X)$, then A has the NIP if and
 only if for every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and for every $a < b$ there is $I \subseteq \mathbb{N}$ such that

$$\bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} f_n^{-1}[b, \infty) = \emptyset.$$

244 To summarize, the particular case of Theorem 1.8 when for X compact can be
 245 stated in the following way:

246 **Theorem 1.8.** *Let X be a compact Polish space. Then, for every pointwise bounded
 247 $A \subseteq C_p(X)$, one and exactly one of the following two conditions must hold:*

- 248 (i) $\overline{A} \subseteq B_1(X)$.
 249 (ii) A has NIP.

250 The Independence Property was first isolated by Saharon Shelah in model theory
 251 as a dividing line between theories whose models are “tame” (corresponding to
 252 NIP) theories of models are “wild” (corresponding to IP). See [She71, Definition
 253 4.1], [She90].

254 **1.2. NIP as universal dividing line between polynomial and exponential**
 255 **complexity.** The particular case of the BSF Dichotomy (Theorem 1.8) when A
 256 consists of $\{0, 1\}$ -valued (i.e., {Yes, No}-valued) strings was discovered indepen-
 257 dently, around 1971-1972 in many foundational contexts related to polynomial
 258 (“tame”) vs exponential (“wild”) complexity: In model theory, by Saharon She-
 259 lah [She71],[She90], in combinatorics, by Norbert Sauer [Sau72], and Shelah [She72,
 260 She90], and in statistical learning, by Vladimir Vapnik and Alexey Chervonenkis [VC71,
 261 VC74].

262 **In model theory:** Shelah’s classification theory is a foundational program
 263 in mathematical logic devised to categorize first-order theories based on
 264 the complexity and structure of their models. A theory T is considered
 265 classifiable in Shelah’s sense if the number of non-isomorphic models of T
 266 of a given cardinality can be described by a bounded number of numerical
 267 invariants. In contrast, a theory T is unclassifiable if the number of models
 268 of T of a given cardinality is the maximum possible number. This number
 269 is directly impacted by the number of “types” over of parameters in models
 270 of T ; a controlled number of types is a characteristic of a classifiable theory.

271 In Shelah’s classification program [She90], theories without the indepen-
 272 dence property (called NIP theories, or dependent theories) have a well-
 273 behaved, “tame” structure; the number of types over a set of parameters
 274 of size κ of such a theory is of polynomially or similar “slow” growth on κ .
 275 Theories with the Independence Property (called IP theories), in contrast,
 276 are considered “intractable” or “wild”. A theory with the independence
 277 property produces the maximum possible number of types over a set of
 278 parameters; for a set of parameters of cardinality κ , the theory will have
 279 2^{2^κ} -many distinct types.

280 **In combinatorics:** Sauer [Sau72] and Shelah [She72] proved the following:
 281 If $\mathcal{F} = \{S_0, S_1, \dots\}$ is a family of subsets of some infinite set S , then
 282 either for every $n \in \mathbb{N}$, there is a set $A \subseteq S$ with $|A| = n$ such that
 283 $|\{S_i \cap A) : i \in \mathbb{N}\}| = 2^n$ (yielding exponential complexity), or there exists
 284 $N \in \mathbb{N}$ such that $A \subseteq S$ with $|A| \geq N$, one has

$$|\{S_i \cap A) : i \in \mathbb{N}\}| \leq \sum_{i=0}^{N-1} \binom{|A|}{i} \approx O(|A|^N)$$

285 for every $A \subseteq S$ such that $|A| \geq N$ (yielding polynomial complexity). This
 286 answered a question of Erdős.

287 **In machine learning:** Readers familiar with statistical learning may rec-
 288 ognize the Sauer-Shelah lemma as the dichotomy discovered and proved
 289 slightly earlier (1971) by Vapnik and Chervonenkis [VC71, VC74] to ad-
 290 dress the problem of uniform convergence in statistics. The least integer
 291 N given by the preceding paragraph, when it exists, is called the *VC-*
 292 *dimension* of \mathcal{F} . This is a core concept in machine learning. If such an
 293 integer N does not exist, we say that the VC-dimension of \mathcal{F} is infinite. The
 294 lemma provides upper bounds on the number of data points (sample size m)
 295 needed to learn a concept class with VC dimension $d \in \mathbb{N}$ by showing this
 296 number grows polynomially with m and d (namely, $\sum_{i=0}^d \binom{m}{i} \approx O(m^d)$),
 297 not exponentially. The Fundamental Theorem of Statistical Learning states

298 that a hypothesis class is PAC-learnable (PAC stands for “Probably Ap-
299 proximately Correct”) if and only if its VC dimension is finite.

300 **1.3. Rosenthal compacta.** The comprehensiveness of Theorem 1.8, attested by
301 the examples outlined in the preceding section, led to the following definition (in-
302 troduced by Godefroy [God80]):

303 **Definition 1.9.** A Rosenthal compactum is a compact Hausdorff topological space
304 K that can be topologically embedded as a compact subset into the space of all
305 functions of the first Baire class on some Polish space X , equipped with the topology
306 of pointwise convergence.

307 Rosenthal compacta are characterized by significant topological and dynamical
308 tameness properties. They play a significant role in functional analysis, measure
309 theory, dynamical systems, descriptive set theory, and model theory. In this
310 paper, we introduce their applicability in deep computation. For this, we shall first
311 focus on countable languages, which is the theme of the next section.

312 **1.4. The special case $B_1(X, \mathbb{R}^{\mathcal{P}})$ with \mathcal{P} countable.** Our goal now is to charac-
313 terize relatively compact subsets of $B_1(X, Y)$ for the particular case when $Y = \mathbb{R}^{\mathcal{P}}$
314 with \mathcal{P} countable. Given $P \in \mathcal{P}$ we denote the projection map onto the P -coordinate
315 by $\pi_P : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R}$. From a high-level topological interpretation, the subsequent
316 lemma states that, in this context, the spaces \mathbb{R} and $\mathbb{R}^{\mathcal{P}}$ are really not that differ-
317 ent, and that if we understand the Baire class 1 functions of one space, then we
318 also understand the functions of both.

319 **Lemma 1.10.** *Let X be a Polish space and \mathcal{P} be a countable set. Then, $f \in$
320 $B_1(X, \mathbb{R}^{\mathcal{P}})$ if and only if $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$.*

Proof. Only one implication needs a proof. Suppose that $\pi_P \circ f \in B_1(X)$ for all
 $P \in \mathcal{P}$. Let V be a basic open subset of $\mathbb{R}^{\mathcal{P}}$. That is, there exists a finite $\mathcal{P}' \subseteq \mathcal{P}$
such that $V = \bigcap_{P \in \mathcal{P}'} \pi_P^{-1}[U_P]$ where U_P is open in \mathbb{R} . Then,

$$f^{-1}[V] = \bigcap_{P \in \mathcal{P}'} (\pi_P \circ f)^{-1}[U_P]$$

321 is an F_{σ} set. Since \mathcal{P} is countable, $\mathbb{R}^{\mathcal{P}}$ is second countable so every open set U in
322 $\mathbb{R}^{\mathcal{P}}$ is a countable union of basic open sets. Hence, $f^{-1}[U]$ is F_{σ} . \square

323 Below we consider \mathcal{P} with the discrete topology. For each $f : X \rightarrow \mathbb{R}^{\mathcal{P}}$ denote
324 $\hat{f}(P, x) := \pi_P \circ f(x)$ for all $(P, x) \in \mathcal{P} \times X$. Similarly, for each $g : \mathcal{P} \times X \rightarrow \mathbb{R}$ denote
325 $\check{g}(x)(P) := g(P, x)$. Given $A \subseteq (\mathbb{R}^{\mathcal{P}})^X$, we denote \hat{A} as the set of all \hat{f} such that
326 $f \in A$. Note that the map $(\mathbb{R}^{\mathcal{P}})^X \rightarrow \mathbb{R}^{\mathcal{P} \times X}$ given by $f \mapsto \hat{f}$ is a homeomorphism
327 and its inverse is given by $g \mapsto \check{g}$.

328 **Lemma 1.11.** *Let X be a Polish space and \mathcal{P} be countable. Then, $f \in B_1(X, \mathbb{R}^{\mathcal{P}})$
329 if and only if $\hat{f} \in B_1(\mathcal{P} \times X)$.*

Proof. (\Rightarrow) By Lemma 1.10, given an open set of reals U , we have $f^{-1}[\pi_P^{-1}[U]]$ is
 F_{σ} for every $P \in \mathcal{P}$. Given that \mathcal{P} is a discrete countable space, we observe that

$$\hat{f}^{-1}[U] = \bigcup_{P \in \mathcal{P}} (\{P\} \times f^{-1}[\pi_P^{-1}[U]])$$

330 is an F_{σ} as well.

(\Leftarrow) By lemma 1.10 it suffices to show that $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$. Fix an open $U \subseteq \mathbb{R}$. Write $\hat{f}^{-1}[U] = \bigcup_{n \in \mathbb{N}} F_n$ where F_n is closed in $\mathcal{P} \times X$. Then,

$$(\pi_P \circ f)^{-1}[U] = \bigcup_{n \in \mathbb{N}} \{x \in X : (P, x) \in F_n\}$$

which is F_σ . □

Given $A \subseteq Y^X$ and $K \subseteq X$ we write $A|_K := \{f|_K : f \in A\}$, i.e., the set of all restrictions of functions in A to K . The following Theorem is a slightly more general version of Theorem 1.8.

Theorem 1.12. *Assume that \mathcal{P} is countable, X is a Polish space, and $A \subseteq C_p(X, \mathbb{R}^\mathcal{P})$ is such that $\pi_P \circ A$ is pointwise bounded for all $P \in \mathcal{P}$. The following are equivalent for every compact $K \subseteq X$:*

- (1) $\overline{A|_K} \subseteq B_1(K, \mathbb{R}^\mathcal{P})$.
- (2) $\pi_P \circ A|_K$ has the NIP for every $P \in \mathcal{P}$.

Proof. (1) \Rightarrow (2). Let $P \in \mathcal{P}$. Fix $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$. By (1), we have $\overline{A|_K} \subseteq B_1(K, \mathbb{R}^\mathcal{P})$. Applying the homeomorphism $f \mapsto \hat{f}$ and using lemma 1.11 we get $\overline{\hat{A}|_{\mathcal{P} \times K}} \subseteq B_1(\mathcal{P} \times K)$. By Theorem 1.8, there is $I \subseteq \mathbb{N}$ such that

$$(\mathcal{P} \times K) \cap \bigcap_{n \in I} \hat{f}_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} \hat{f}_n^{-1}[b, \infty) = \emptyset$$

Hence,

$$K \cap \bigcap_{n \in I} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n \notin I} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

By the compactness of K , there are finite $E \subseteq I$ and $F \subseteq \mathbb{N} \setminus I$ such that

$$K \cap \bigcap_{n \in E} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n \in F} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

Thus, $\pi_P \circ A|_L$ has the NIP.

(2) \Rightarrow (1) Fix $f \in \overline{A|_K}$. By lemma 1.10 it suffices to show that $\pi_P \circ f \in B_1(K)$ for all $P \in \mathcal{P}$. By (2), $\pi_P \circ A|_K$ has the NIP. Hence, by Theorem 1.8 we have $\pi_P \circ A|_K \subseteq B_1(K)$. But then $\pi_P \circ f \in \overline{\pi_P \circ A|_K} \subseteq B_1(K)$. □

Lastly, a simple but significant result that helps understand the operation of restricting a set of functions to a specific subspace of the domain space X , of course in the context of the NIP, is that we may always assume that said subspace is closed. Concretely, whether we take its closure or not has no effect on the NIP:

Lemma 1.13. *Assume that X is Hausdorff and that $A \subseteq C_p(X)$. The following are equivalent for every $L \subseteq X$:*

- (i) A_L has the NIP.
- (ii) $A|\overline{L}$ has the NIP.

Proof. It suffices to show that (i) \Rightarrow (ii). Suppose that (ii) does not hold, i.e., that there are $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$ such that for all finite disjoint $E, F \subseteq \mathbb{N}$:

$$\overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

Pick $a' < b'$ such that $a < a' < b' < b$. Then, for any finite disjoint $E, F \subseteq \mathbb{N}$ we can choose

$$x \in \overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a') \cap \bigcap_{n \in F} f_n^{-1}(b', \infty)$$

By definition of closure:

$$L \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a'] \cap \bigcap_{n \in F} f_n^{-1}[b', \infty) \neq \emptyset.$$

352 This contradicts (i). □

353 2. COMPOSITIONAL COMPUTATION STRUCTURES.

354 In this section, we connect function spaces with floating point computation. We
355 start by summarizing some basic concepts from [ADIW24].

356 A *computation states structure* is a pair (L, \mathcal{P}) , where L is a set whose elements we
357 call *states* and \mathcal{P} is a collection of real-valued functions on L that we call *predicates*.
358 For a state $v \in L$, *type* of v is the indexed family

$$\text{tp}(v) = (P(v))_{P \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}.$$

359 For each $P \in \mathcal{P}$, we call real value $P(v)$ the P -th *feature* of v . A *transition* of a
360 computation states structure (L, \mathcal{P}) is a map $f : L \rightarrow L$.

361 Intuitively, L is the set of states of a computation, and the predicates $P \in \mathcal{P}$
362 as accepted computational primitives. We think of each state $v \in L$ is uniquely
363 characterized by its type, so in practice, we identify L with a subset of $\mathbb{R}^{\mathcal{P}}$. A
364 typical case will be when $L = \mathbb{R}^{\mathbb{N}}$ or $L = \mathbb{R}^n$ for some positive integer n and there
365 is a predicate $P_i(v) = v_i$ for each of the coordinates v_i of v . We regard the space of
366 types as a topological space, endowed with the topology of pointwise convergence
367 inherited from $\mathbb{R}^{\mathcal{P}}$. In particular, for each $P \in \mathcal{P}$, the projection map $v \mapsto P(v)$ is
368 continuous.

369 **Definition 2.1.** Given a computation states structure (L, \mathcal{P}) , any element of $\mathbb{R}^{\mathcal{P}}$
370 in the image of L under the map $v \mapsto \text{tp}(v)$ will be called a *realized type*. The
371 topological closure of the set of realized types in $\mathbb{R}^{\mathcal{P}}$ (endowed with the point-
372 wise convergence topology) will be called the *space of types* of (L, \mathcal{P}) , denoted \mathcal{L} .
373 Elements of $\mathcal{L} \setminus L$ will be called *unrealized types*.

374 In traditional model theory the space of types of a structure is viewed as a sort of
375 compactification of the structure. However, the space \mathcal{L} is not necessarily compact.
376 To bypass this obstacle, we follow the idea introduced in [ADIW24] of covering \mathcal{L}
377 by “thin” compact subspaces that we call *shards*. We give the formal definition
378 next.

379 **Definition 2.2.** A *sizer* is a tuple $r_{\bullet} = (r_P)_{P \in \mathcal{P}}$ of positive real numbers indexed
380 by \mathcal{P} . Given a sizer r_{\bullet} , we define the r_{\bullet} -*shard* as:

$$L[r_{\bullet}] = L \cap \prod_{P \in \mathcal{P}} [-r_P, r_P].$$

381 For a sizer r_{\bullet} , the r_{\bullet} -*type shard* is defined as $\mathcal{L}[r_{\bullet}] = \overline{L[r_{\bullet}]}$. We define \mathcal{L}_{sh} , is
382 the union of all type-shards.

383 **Definition 2.3.** a *Compositional Computation Structure* (CCS) is a triple (L, \mathcal{P}, Γ) ,
384 where

- 385 • (L, \mathcal{P}) is a computation states structure
 386 • $\Gamma \subseteq L^L$ is a semigroup under composition.

387 We say that $\Delta \subseteq \Gamma$ is *R-confined* if $\gamma|_{L[r_\bullet]} : L[r_\bullet] \rightarrow L[r_\bullet]$ for every $r_\bullet \in R$
 388 and $\gamma \in \Delta$. Elements in Δ are called *real-valued computations* (in this article we
 389 will refer to them simply as *computations*) and elements in $\overline{\Delta} \subseteq \mathcal{L}_{sh}$ are called
 390 (real-valued) *deep computations* or *ultracomputations*.

391 A tenet of our approach is that a map $f : L \rightarrow \mathcal{L}$ is to be considered “effectively
 392 computable” if, for each $Q \in \mathcal{P}$, the output feature $Q \circ f : L \rightarrow \mathbb{R}$ is a *definable*
 393 predicate in the following sense:

394 Given any arbitrary $\varepsilon > 0$ and any $K \subseteq L$ wherein every input feature $P(v)$
 395 remains bounded in magnitude there is an ε -approximating continuous “algebraic”
 396 operator $\varphi(P_1, \dots, P_n)$ of finitely many input features P_1, \dots, P_n , such that the
 397 following holds: for all $v \in K$, the output feature $Q(f(v))$ is ε -approximated by
 398 $\varphi(P_1(v), \dots, P_n(v))$. By “algebraic”, we mean that the approximating operator φ
 399 uses only the algebra (operations of $\mathbb{R}^\mathcal{P}$, i.e., vector addition, vector multiplication,
 400 and scalar addition).

401 It is shown in [ADIW24]) that:

- 402 (1) For a definable $f : L \rightarrow \mathcal{L}$, the approximating operators φ may be taken to
 403 be *polynomials* of the input features, and
 404 (2) Definable transforms $f : L \rightarrow \mathcal{L}$ are precisely those that extend to contin-
 405 uous $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$ (this is the property of *extendibility* mentioned above).

406 This motivates the following definition.

407 **Definition 2.4.** We say that a CCS (L, \mathcal{P}, Γ) satisfies the *Extendability Axiom* if
 408 for all $\gamma \in \Gamma$, there is $\tilde{\gamma} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$ such that for every sizer r_\bullet there is a sizer s_\bullet
 409 such that $\tilde{\gamma}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$ is continuous.

410 For an illustrative example, we can frame Newton’s polynomial root approxima-
 411 tion method in the context of a CCS (see Example 5.6 of [ADIW24] for details) as
 412 follows. Begin by considering the extended complex numbers $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with
 413 the usual Riemann sphere topology that makes it into a compact space (where
 414 unbounded sequences converge to ∞). In fact, not only is this space compact
 415 but it is covered by the shard given by the sizer $(1, 1, 1)$ (the unit sphere is con-
 416 tained in the cube $[-1, 1]^3$). The space $\hat{\mathbb{C}}$ is homeomorphic to the usual unit sphere
 417 $S^2 := \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ of \mathbb{R}^3 , by means of the stereographic pro-
 418 jection and its inverse $\hat{\mathbb{C}} \rightarrow S^2$. This function is regarded as a triple of predi-
 419 cates $x, y, z : \hat{\mathbb{C}} \rightarrow [-1, 1]$ where each will map an extended complex number to
 420 its corresponding real coordinate on the cube $[-1, 1]^3$. Now fix the cubic com-
 421 plex polynomial $p(s) := s^3 - 1$, and consider the map which performs one step
 422 in Newton’s method at a particular (extended) complex number s , for finding a
 423 root of p , $\gamma_p : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. The explicit inner workings of γ_p are irrelevant for this
 424 example, except for the fact that it is a continuous mapping. It follows that
 425 $(S^2, \{x, y, z\}, \{\gamma_p^k : k \in \mathbb{N}\})$ is a CCS. The idea is that repeated applications of
 426 $\gamma_p(s), \gamma_p \circ \gamma_p(s), \gamma_p \circ \gamma_p \circ \gamma_p(s), \dots$ would approximate a root of p provided s was a
 427 good enough initial guess.

428 For a deeper discussion about this axiom, we refer the reader to [ADIW24].

429 3. CLASSIFYING DEEP COMPUTATIONS

430 **3.1. NIP, Rosenthal compacta, and deep computations.** Under what conditions
 431 are deep computations Baire class 1, and thus well-behaved according to our
 432 framework, on type-shards? The following Theorem says that, under the assumption
 433 that \mathcal{P} is countable, the space of deep computations is a Rosenthal compactum
 434 (when restricted to shards) if and only if the set of computations has the NIP,
 435 feature by feature. Hence, we can import the theory of Rosenthal compacta into
 436 this framework of deep computations.

437 **Theorem 3.1.** *Let (L, \mathcal{P}, Γ) be a compositional computational structure (Definition ??) satisfying the Extendability Axiom (Definition 2.4) with \mathcal{P} countable. Let
 438 R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. The following
 439 are equivalent.*

- 441 (1) $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$.
 442 (2) $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ has the NIP for all $P \in \mathcal{P}$ and $r_\bullet \in R$; that is, for all $P \in \mathcal{P}$,
 $r_\bullet \in R$, $a < b$, $\{\gamma_n\}_{n \in \mathbb{N}} \subseteq \Delta$ there are finite disjoint $E, F \subseteq \mathbb{N}$ such that

$$L[r_\bullet] \cap \bigcap_{n \in E} (\pi_P \circ \gamma_n)^{-1}(-\infty, a] \cap \bigcap_{n \in F} (\pi_P \circ \gamma_n)^{-1}[b, \infty) = \emptyset.$$

443 Moreover, if any (hence all) of the preceding conditions hold, then every deep
 444 computation $f \in \overline{\Delta}$ can be extended to a Baire-1 function on shards, i.e., there is
 445 $\tilde{f} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$ such that $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$. In particular, on
 446 each shard every deep computation is the pointwise limit of a countable sequence of
 computations.

447 *Proof.* Since \mathcal{P} is countable, $\mathcal{L}[r_\bullet] \subseteq \mathbb{R}^{\mathcal{P}}$ is Polish. Also, the Extendability Axiom
 448 implies that $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ is a pointwise bounded set of continuous functions for all
 449 $P \in \mathcal{P}$. Hence, Theorem 1.12 and Lemma 1.13 prove the equivalence of (1) and (2).
 450 If (1) holds and $f \in \overline{\Delta}$, then write $f = \underline{\text{Ulim}_i \gamma_i}$ as an ultralimit. Define $\tilde{f} := \underline{\text{Ulim}_i \tilde{\gamma}_i}$.
 451 Hence, for all $r_\bullet \in R$ we have $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in \overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$. That every
 452 deep computation is a pointwise limit of a countable sequence of computations
 453 follows from the fact that for a Polish space X every compact subset of $B_1(X)$ is
 454 Fréchet-Urysohn (that is, a space where topological closures coincide with sequential
 455 closures, see Theorem 3F in [BFT78] or Theorem 4.1 in [Deb13]). \square

456 **3.2. The Todorčević trichotomy and levels of PAC learnability.** Given a
 457 countable set Δ of computations satisfying the NIP on features and shards (con-
 458 dition (2) of Theorem 3.1) we have that $\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ (for a fixed sizer r_\bullet) is a separable
 459 *Rosenthal compactum* (compact subset of $B_1(P \times \mathcal{L}[r_\bullet])$). The work of Todorčević
 460 ([Tod99]) and Argyros, Dodos, Kanellopoulos ([ADK08]) culminates in a trichotomy
 461 theorem for separable Rosenthal Compacta. In this section, inspired by the work
 462 of Glasner and Megrelishvili ([GM22]), we study ways in which this classification
 463 allows us obtain different levels of PAC-learnability (NIP).

464 Recall that a topological space X is *hereditarily separable* (HS) if every subspace
 465 is separable and that X is *first countable* if every point in X has a countable local
 466 basis. Every separable metrizable space is hereditarily separable and it is a result
 467 of R. Pol that every hereditarily separable Rosenthal compactum is first countable
 468 (see section 10 of [Deb13]). This suggests the following definition:

469 **Definition 3.2.** Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendability Axiom and R
470 be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be an R -confined countable set of
471 computations satisfying the NIP on shards and features (condition (2) in Theorem
472 3.1). We say that Δ is:

- 473 (i) NIP₁ if $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}}$ is first countable for every $r_\bullet \in R$.
- 474 (ii) NIP₂ if $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}}$ is hereditarily separable for every $r_\bullet \in R$.
- 475 (iii) NIP₃ if $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}}$ is metrizable for every $r_\bullet \in R$.

476 Observe that NIP₃ \Rightarrow NIP₂ \Rightarrow NIP₁ \Rightarrow NIP. A natural question that would con-
477 tinue this work is to find examples of CCS that separate these levels of NIP. In
478 [Tod99], Todorčević isolates three canonical examples of Rosenthal compacta that
479 witness the failure of the converse implications above.

480 We now present some separable and non-separable examples of Rosenthal com-
481 pacta:

482 Examples 3.3.

- 483 (1) *Alexandroff compactification of a discrete space of size continuum.* For
484 each $a \in 2^{\mathbb{N}}$ consider the map $\delta_a : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ given by $\delta_a(x) = 1$ if $x = a$ and
485 $\delta_a(x) = 0$ otherwise. Let $A(2^{\mathbb{N}}) = \{\delta_a : a \in 2^{\mathbb{N}}\} \cup \{0\}$, where 0 is the zero
486 map. Notice that $A(2^{\mathbb{N}})$ is a compact subset of $B_1(2^{\mathbb{N}})$, in fact $\{\delta_a : a \in 2^{\mathbb{N}}\}$
487 is a discrete subspace of $B_1(2^{\mathbb{N}})$ and its pointwise closure is precisely $A(2^{\mathbb{N}})$.
488 Hence, this is a Rosenthal compactum which is not first countable. Notice
489 that this space is also not separable.
- 490 (2) *Extended Alexandroff compactification.* For each finite binary sequence $s \in$
491 $2^{<\mathbb{N}}$, let $v_s : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by $v_s(x) = 1$ if x extends s and $v_s(x) = 0$
492 otherwise. Let $\hat{A}(2^{\mathbb{N}})$ be the pointwise closure of $\{v_s : s \in 2^{<\mathbb{N}}\}$, i.e.,
493 $\hat{A}(2^{\mathbb{N}}) = A(2^{\mathbb{N}}) \cup \{v_s : s \in 2^{<\mathbb{N}}\}$. Note that this space is a separable
494 Rosenthal compactum which is not first countable.
- 495 (3) *Split Cantor.* Let $<$ be the lexicographic order in the space of infinite
496 binary sequences, i.e., $2^{\mathbb{N}}$. For each $a \in 2^{\mathbb{N}}$ let $f_a^- : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by
497 $f_a^-(x) = 1$ if $x < a$ and $f_a^-(x) = 0$ otherwise. Let $f_a^+ : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given
498 by $f_a^+(x) = 1$ if $x \leq a$ and $f_a^+(x) = 0$ otherwise. The split Cantor is the
499 space $S(2^{\mathbb{N}}) = \{f_a^- : a \in 2^{\mathbb{N}}\} \cup \{f_a^+ : a \in 2^{\mathbb{N}}\}$. This is a separable Rosenthal
500 compactum. One example of a countable dense subset is the set of all f_a^+
501 and f_a^- where a is an infinite binary sequence that is eventually constant.
502 Moreover, it is hereditarily separable but it is not metrizable.
- 503 (4) *Alexandroff Duplicate.* Let K be any compact metric space and consider
504 the Polish space $X = C(K) \sqcup K$, i.e., the disjoint union of $C(K)$ (with its
505 supremum norm topology) and K . For each $a \in K$ define $g_a^0, g_a^1 : X \rightarrow \mathbb{R}$
506 as follows:

$$g_a^0(x) = \begin{cases} x(a), & x \in C(K) \\ 0, & x \in K \end{cases}$$

$$g_a^1(x) = \begin{cases} x(a), & x \in C(K) \\ \delta_a(x), & x \in K \end{cases}$$

503 Let $D(K) = \{g_a^0 : a \in K\} \cup \{g_a^1 : a \in K\}$. Notice that $D(K)$ is a first
504 countable Rosenthal compactum. It is not separable if K is uncountable.
505 The interesting case will be when $K = 2^{\mathbb{N}}$.

- (5) *Extended Alexandroff Duplicate of the split Cantor.* For each finite binary sequence $t \in 2^{<\mathbb{N}}$ let $a_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 0's and let $b_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 1's. Define $h_t : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ by

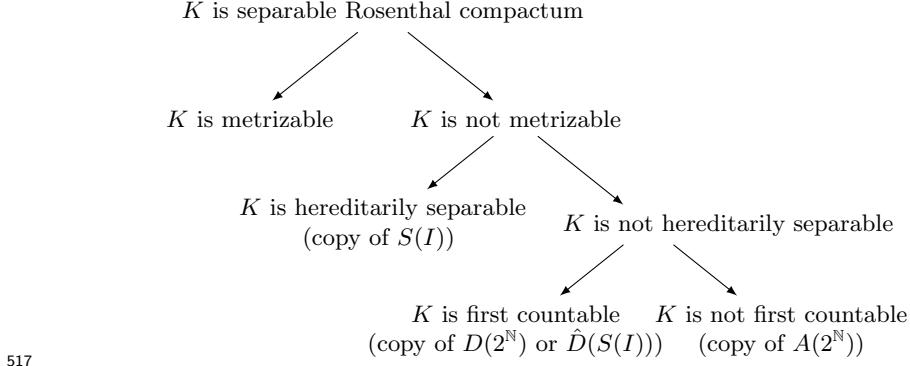
$$h_t(x) = \begin{cases} 0, & x < a_t \\ 1/2, & a_t \leq x \leq b_t \\ 1, & b_t < x \end{cases}$$

Let $\hat{D}(S(2^{\mathbb{N}}))$ be the pointwise closure of the set $\{h_t : t \in 2^{<\mathbb{N}}\}$. Hence, $\hat{D}(S(2^{\mathbb{N}}))$ is a separable first countable Rosenthal compactum which is not hereditarily separable. In fact, it contains an uncountable discrete subspace (see Theorem 5 in [Tod99]).

Theorem 3.4 (Todorčević's Trichotomy, [Tod99], Theorem 3 in [ADK08]). *Let K be a separable Rosenthal Compactum.*

- (i) *If K is hereditarily separable but non-metrizable, then $S(2^{\mathbb{N}})$ embeds into K .*
- (ii) *If K is first countable but not hereditarily separable, then either $D(2^{\mathbb{N}})$ or $\hat{D}(S(2^{\mathbb{N}}))$ embeds into K .*
- (iii) *If K is not first countable, then $A(2^{\mathbb{N}})$ embeds into K .*

We thus have the following classification:



The definitions provided here for NIP_i ($i = 1, 2, 3$) are topological. This raises the following question:

Question 3.5. Is there a non-topological characterization for NIP_i , $i = 1, 2, 3$?

3.3. The Argyros-Dodos-Kanellopoulos heptachotomy, and approximability of deep computation by minimal classes. In the three separable three cases given in 3.3, namely, $(A(2^{\mathbb{N}}), S(2^{\mathbb{N}})$ and $\hat{D}(S(2^{\mathbb{N}}))$), the countable dense subsets are indexed by the binary tree $2^{<\mathbb{N}}$. This choice of index is useful for two reasons:

- (1) Our emphasis is computational. Elements of $2^{<\mathbb{N}}$ represent finite bitstrings, i.e., standard computations, while Rosenthal compacta represent deep computations, i.e., limits of finite computations. Mathematically, deep computations are pointwise limits of standard computations; however, computationally, we are interested in the manner (and the efficiency) in which the approximations can occur.

- 532 (2) The Ramsey theory of perfect subsets of the Cantor space $2^{\mathbb{N}}$ can be im-
 533 ported to analyze the behavior of the accumulation points. Since $2^{<\mathbb{N}}$ is
 534 countable, we can always choose this index for the countable dense subsets.
 535 This is done in [ADK08].

536 **Definition 3.6.** Let X be a Polish space.

- 537 (1) If I is a countable and $\{f_i : i \in I\} \subseteq \mathbb{R}^X$, $\{g_i : i \in I\} \subseteq \mathbb{R}^X$ are two
 538 pointwise families by I . We say that $\{f_i : i \in I\}$ and $\{g_i : i \in I\}$ are
 539 *equivalent* if and only if the map $f_i \mapsto g_i$ is extended to a homeomorphism
 540 from $\overline{\{f_i : i \in I\}}$ to $\overline{\{g_i : i \in I\}}$.
 541 (2) If $\{f_t : t \in 2^{<\mathbb{N}}\}$ is a pointwise bounded family, we say that $\{f_t : t \in 2^{<\mathbb{N}}\}$
 542 is *minimal* if and only if for every dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$,
 543 $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to $\{f_t : t \in 2^{<\mathbb{N}}\}$.

544 One of the main results in [ADK08] is that, up to equivalence, there are seven
 545 minimal families of Rosenthal compacta and that for every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$ there is a dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$
 546 is equivalent to one of the minimal families. We shall describe the minimal families
 547 next. We follow the same notation as in [ADK08]. For any node $t \in 2^{<\mathbb{N}}$, let us
 548 denote by $t^\frown 0^\infty$ ($t^\frown 1^\infty$) the infinite binary sequence starting with t and continuing
 549 will all 0's (respectively, all 1's). Fix a regular dyadic subtree $R = \{s_t : t \in 2^{<\mathbb{N}}\}$
 550 of $2^{<\mathbb{N}}$ (i.e., a dyadic subtree such that every level of R is contained in a level of
 551 $2^{<\mathbb{N}}$) with the property that for all $s, s' \in R$, $s^\frown 0^\infty \neq s'^\frown 0^\infty$ and $s^\frown 1^\infty \neq s'^\frown 1^\infty$.
 552 Given $t \in 2^{<\mathbb{N}}$, let v_t be the characteristic function of the set $\{x \in 2^{\mathbb{N}} : x \text{ extends } t\}$.
 553 Let $<$ be the lexicographic order in $2^{\mathbb{N}}$. Given $a \in 2^{\mathbb{N}}$, let $f_a^+ : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ be the
 554 characteristic function of $\{x \in 2^{\mathbb{N}} : a \leq x\}$ and let $f_a^- : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ be the
 555 characteristic function of $\{x \in 2^{\mathbb{N}} : a < x\}$. Given two maps $f, g : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ we
 556 denote by $(f, g) : 2^{\mathbb{N}} \sqcup 2^{\mathbb{N}} \rightarrow \mathbb{R}$ the function which is f on the first copy of $2^{\mathbb{N}}$ and
 557 g on the second copy of $2^{\mathbb{N}}$.

- 559 (1) $D_1 = \{\frac{1}{|t|+1}v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_1} = A(2^{\mathbb{N}})$.
 560 (2) $D_2 = \{s_t^\frown 0^\infty : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_2} = 2^{\leq N}$.
 561 (3) $D_3 = \{f_{s_t^\frown 0^\infty}^+ : t \in 2^{<\mathbb{N}}\}$. This is a discrete in $\overline{D_3} = S(2^{\mathbb{N}})$.
 562 (4) $D_4 = \{f_{s_t^\frown 1^\infty}^- : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_4} = S(2^{\mathbb{N}})$.
 563 (5) $D_5 = \{v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_5} = \hat{A}(2^{\mathbb{N}})$.
 564 (6) $D_6 = \{(v_{s_t}, s_t^\frown 0^\infty) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_6} = \hat{D}(2^{\mathbb{N}})$.
 565 (7) $D_7 = \{(v_{s_t}, f_{s_t^\frown 0^\infty}^+) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_7} = \hat{D}(S(2^{\mathbb{N}}))$

566 **Theorem 3.7** (Heptacotomy of minimal families, Theorem 2 in [ADK08]). *Let*
 567 *X be Polish. For every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$, there exists*
 568 *$i = 1, 2, \dots, 7$ and a regular dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to D_i . Moreover, all D_i are minimal and mutually non-equivalent.*

571 **3.4. Bourgain-Fremlin-Talagrand, NIP, and essential computability of**
 572 **deep computations.** We now turn to the question: what happens when \mathcal{P} is
 573 uncountable? Notice that the countability assumption is crucial in the proof of
 574 Theorem 1.12 essentially because it makes $\mathbb{R}^{\mathcal{P}}$ a Polish space. For the uncountable
 575 case, we may lose Baire-1 definability so we shall replace $B_1(X)$ by a larger class.
 576 Recall that the *raison d'être* of the class of Baire-1 functions is to have a class that

577 is contains the continuous functions but is closed under pointwise limits, and that (578 Fact 1.2) for perfectly normal X , a function f is in $B_1(X, Y)$ if and only if $f^{-1}[U]$ 579 is an F_σ subset of X for every open $U \subseteq Y$. This motivates the following definition:

580 **Definition 3.8.** Given a Hausdorff space X and a measurable space (Y, Σ) , we say
581 that $f : X \rightarrow Y$ is *universally measurable* (with respect to Σ) if $f^{-1}(E)$ is Borel
582 for every $E \in \Sigma$, i.e., $f^{-1}(E)$ is μ -measurable for every Radon probability measure
583 μ on X . When $Y = \mathbb{R}$ we will always take $\Sigma = \mathcal{B}(\mathbb{R})$, the Borel σ -algebra of \mathbb{R} .
584 In this case, a function $f : X \rightarrow \mathbb{R}$ is universally measurable if and only if $f^{-1}(U)$
585 is μ -measurable for every Radon probability measure μ on X and every open set
586 $U \subseteq \mathbb{R}$.

587 Intuitively, a function is universally measurable if it is “measurable no matter
588 which reasonable way you try to measure things on its domain”. The concept
589 of universal measurability emerged from work of Kallianpur and Sazonov, in the
590 late 1950’s and 1960s, , with later developments by Blackwell, Darst, and others,
591 building on earlier ideas of Gnedenko and Kolmogorov from the 1950s. See [Pap02,
592 Chapters 1 and 2].

593 Following [BFT78], the collection of all universally measurable real-valued func-
594 tions will be denoted by $M_r(X)$. In the context of deep computations, we will be
595 interested in transition maps from a state space $L \subseteq \mathbb{R}^{\mathcal{P}}$ to itself. There are two
596 natural σ -algebras one can consider in the product space $\mathbb{R}^{\mathcal{P}}$: the Borel σ -algebra,
597 i.e., the σ -algebra generated by open sets in $\mathbb{R}^{\mathcal{P}}$, and the cylinder σ -algebra, i.e.,
598 the σ -algebra generated by the sub-basic open sets in $\mathbb{R}^{\mathcal{P}}$. Note that when \mathcal{P} is
599 countable, both σ -algebras coincide but in general the cylinder σ -algebra is strictly
600 smaller. We will use the cylinder σ -algebra to define universally measurable maps
601 $f : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R}^{\mathcal{P}}$. The reason for this choice is the following characterization:

602 **Lemma 3.9.** Let X be a Hausdorff space and $Y = \prod_{i \in I} Y_i$ be any product of
603 measurable spaces (Y_i, Σ_i) for $i \in I$. Let Σ_Y be the cylinder σ -algebra generated by
604 the measurable spaces (Y_i, Σ_i) . Let $f : X \rightarrow Y$. The following are equivalent:

- 605 (i) $f : X \rightarrow Y$ is universally measurable (with respect to Σ_Y).
- 606 (ii) $\pi_i \circ f : X \rightarrow Y_i$ is universally measurable (with respect to Σ_i) for all $i \in I$.

607 *Proof.* (i) \Rightarrow (ii) is clear since the projection maps π_i are measurable and the com-
608 position of measurable functions is measurable. To prove (ii) \Rightarrow (i), suppose that
609 $C = \prod_{i \in I} C_i$ is a measurable cylinder and let J be the finite set of $i \in I$ such that
610 $C_i \neq Y_i$. Then, $C = \bigcap_{i \in J} \pi_i^{-1}(C_i)$, so $f^{-1}(C) = \bigcap_{i \in J} (\pi_i \circ f)^{-1}(C_i)$ is a universally
611 measurable set by assumption. \square

612 The preceding lemma says that a transition map is universally measurable if and
613 only if it is universally measurable on all its features. In other words, we can check
614 measurability of a transition just by checking measurability feature by feature. We
615 will denote by $M_r(X, \mathbb{R}^{\mathcal{P}})$ the collection of all universally measurable functions
616 $f : X \rightarrow \mathbb{R}^{\mathcal{P}}$ (with respect to the cylinder σ -algebra), endowed with the topology
617 of pointwise convergence.

618 We now wish to define the concept of a deep computation being computable
619 except a set of arbitrarily small measure “no matter which reasonable way you try
620 to measure things on its domain” (see the remarks following definition). This is
621 definition below. To motivate the definition, we need to recall two facts:

- 622 (1) Littlewoood's second principle states that every Lebesgue measurable function
 623 is "nearly continuous". The formal version of this, which is Luzin's
 624 theorem, states that if (X, Σ, μ) a Radon measure space and Y be a second-
 625 countable topological space (e.g., $Y = \mathbb{R}^{\mathcal{P}}$ with \mathcal{P} countable) equipped with
 626 a Borel algebra, then any given $f : X \rightarrow Y$ is measurable if and only if for
 627 every $E \in \Sigma$ and every $\varepsilon > 0$ there exists a closed $F \subseteq E$ such that the
 628 restriction of f to F is continuous.
 629 (2) Computability of deep computations can is characterized in terms of con-
 630 tinuous extendibility of computations. This is at the core of [ADIW24].

631 These facts motivate the following definition:

632 **Definition 3.10.** Let (L, \mathcal{P}, Γ) be a CCS. We say that a transition $f : L \rightarrow L$
 633 is *universally essentially computable* if and only if there exists $\tilde{f} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$
 634 extending f such that for every sizer r_{\bullet} there is a sizer s_{\bullet} such that the restriction
 635 $\tilde{f}|_{\mathcal{L}[r_{\bullet}]} : \mathcal{L}[r_{\bullet}] \rightarrow \mathcal{L}[s_{\bullet}]$ is universally measurable, i.e., $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_{\bullet}]} : \mathcal{L}[r_{\bullet}] \rightarrow [-s_P, s_P]$
 636 is μ -measurable for every Radon probability measure μ on $\mathcal{L}[r_{\bullet}]$.

637 For a measure μ on aX , the set of all μ -measurable functions will denoted by
 638 $\mathcal{M}^0(X, \mu)$.

639 We will need the following result about NIP and universally measurable func-
 640 tions:

641 **Theorem 3.11** (Bourgain-Fremlin-Ta set lagrand, Theorem 2F in [BFT78]). *Let*
 642 *X be a Hausdorff space and $A \subseteq C(X)$ be pointwise bounded. The following are*
 643 *equivalent:*

- 644 (i) $\overline{A} \subseteq M_r(X)$.
- 645 (ii) *For every compact $K \subseteq X$, $A|_K$ has the NIP.*
- 646 (iii) *For every Radon measure μ on X , A is relatively countably compact in*
 647 $\mathcal{M}^0(X, \mu)$, i.e., *every countable subset of A has an accumulation point in*
 648 $\mathcal{M}^0(X, \mu)$.

649 Theorem 1.8 immedately yields the following.

650 **Theorem 3.12.** *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendability Axiom. Let R*
 651 *be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. If $\pi_P \circ \Delta|_{L[r_{\bullet}]}$ has*
 652 *the NIP for all $P \in \mathcal{P}$ and all $r_{\bullet} \in R$, then every deep computation is universally*
 653 *essentially computable.*

654 *Proof.* By the Extendability Axiom, Theorem 1.8 and lemma 1.13 we have that
 655 $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_{\bullet}]} \subseteq M_r(\mathcal{L}[r_{\bullet}])$ for all $r_{\bullet} \in R$ and $P \in \mathcal{P}$. Let $f \in \overline{\Delta}$ be a deep
 656 computation. Write $f = \mathcal{U}\lim_i \gamma_i$ as an ultralimit of computations in Δ . De-
 657 fine $\tilde{f} := \mathcal{U}\lim_i \tilde{\gamma}_i$. Then, for all $r_{\bullet} \in R$ and $P \in \mathcal{P}$ $\pi_P \circ \tilde{\gamma}_i|_{\mathcal{L}[r_{\bullet}]} \in M_r(\mathcal{L}[r_{\bullet}])$ for all
 658 i , so $\pi_P \circ f|_{\mathcal{L}[r_{\bullet}]} \in \pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_{\bullet}]} \subseteq M_r(\mathcal{L}[r_{\bullet}])$. \square

659 **Question 3.13.** Under the same assumptions of the preceding theorem, suppose
 660 that every deep computation of Δ is universally essentially computable. Must
 661 $\pi_P \circ \Delta|_{L[r_{\bullet}]}$ have the NIP for all $P \in \mathcal{P}$ and all $r_{\bullet} \in R$?

662 **3.5. Talagrand stability, NIP, and essential computability of deep compu-
 663 tations.** There is another notion closely related to NIP, introduced by Talagrand
 664 in [Tal84] while studying Pettis integration. Suppose that X is a compact Haus-
 665 dorff space and $A \subseteq \mathbb{R}^X$. Let μ be a Radon probability measure on X . Given a
 666 μ -measurable set $E \subseteq X$, a positive integer k and real numbers $a < b$. we write:

$$D_k(A, E, a, b) = \bigcup_{f \in A} \{x \in E^{2k} : f(x_{2i}) \leq a, f(x_{2i+1}) \geq b \text{ for all } i < k\}$$

667 We say that A is *Talagrand μ -stable* if and only if for every μ -measurable set
668 $E \subseteq X$ of positive measure and for every $a < b$ there is $k \geq 1$ such that

$$(\mu^{2k})^*(D_k(A, E, a, b)) < (\mu(E))^{2k},$$

669 where μ^* denotes the outer measure (we work with outer since the sets $D_k(A, E, a, b)$
670 need not be μ -measurable). This is certainly the case when A is a countable set of
671 continuous (or μ -measurable) functions.

672 The following lemma establishes that Talagrand stability is a way to ensure that
673 deep computations are definable by measurable functions. We include a proof for
674 the reader's convenience.

675 **Lemma 3.14.** *If A is Talagrand μ -stable, then \overline{A} is also Talagrand μ -stable and
676 $\overline{A} \subseteq \mathcal{M}^0(X, \mu)$.*

677 *Proof.* First, observe that a subset of a μ -stable set is μ -stable. To show that \overline{A} is
678 μ -stable, observe that $D_k(\overline{A}, E, a, b) \subseteq D_k(A, E, a', b')$ where $a < a' < b' < b$ and E
679 is a μ -measurable set with positive measure. It suffices to show that $\overline{A} \subseteq \mathcal{M}^0(X, \mu)$.
680 Suppose that there exists $f \in \overline{A}$ such that $f \notin \mathcal{M}^0(X, \mu)$. By a characterization
681 of measurable functions (see 413G in [Fre03]), there exists a μ -measurable set E
682 of positive measure and $a < b$ such that $\mu^*(P) = \mu^*(Q) = \mu(E)$ where $P = \{x \in$
683 $E : f(x) \leq a\}$ and $Q = \{x \in E : f(x) \geq b\}$. Then, for any $k \geq 1$: $(P \times Q)^k \subseteq$
684 $D_k(\{f\}, E, a, b)$ so $(\mu^{2k})^*(D_k(\{f\}, E, a, b)) = (\mu^*(P)\mu^*(Q))^k = (\mu(E))^{2k}$. Thus,
685 $\{f\}$ is not μ -stable, but we argued before that a subset of a μ -stable set must be
686 μ -stable. \square

687 We say that A is *universally Talagrand stable* if A is Talagrand μ -stable for every
688 Radon probability measure μ on X . An argument similar to the proof of 3.11, yields
689 the following:

690 **Theorem 3.15.** *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendability Axiom. If
691 $\pi_P \circ \Delta|_{L[r_\bullet]}$ is universally Talagrand stable for all $P \in \mathcal{P}$ and all sizers r_\bullet , then
692 every deep computation is universally essentially computable.*

693 It is then natural to ask: what is the relationship between Talagrand stability
694 and the NIP? The following dichotomy will be useful.

695 **Lemma 3.16** (Fremlin's Dichotomy, 463K in [Fre03]). *If (X, Σ, μ) is a perfect
696 σ -finite measure space (in particular, for X compact and μ a Radon probability
697 measure on X) and $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued measurable functions
698 on X , then either:*

- 699 (i) $\{f_n : n \in \mathbb{N}\}$ has a subsequence that converges μ -almost everywhere, or
- 700 (ii) $\{f_n : n \in \mathbb{N}\}$ has a subsequence with no μ -measurable accumulation point
701 in \mathbb{R}^X .

702 The preceding lemma can be considered as a measure-theoretic version of Rosen-
703 thal's Dichotomy. Combining this dichotomy with the Theorem 3.11 we get the
704 following result:

705 **Theorem 3.17.** *Let X be a Hausdorff space and $A \subseteq C(X)$ be pointwise bounded.
706 The following are equivalent:*

- 707 (i) $\overline{A} \subseteq M_r(X)$.
 708 (ii) For every compact $K \subseteq X$, $A|_K$ has the NIP.
 709 (iii) For every Radon measure μ on X , A is relatively countably compact in
 710 $\mathcal{M}^0(X, \mu)$, i.e., every countable subset of A has an accumulation point in
 711 $\mathcal{M}^0(X, \mu)$.
 712 (iv) For every Radon measure μ on X and every sequence $\{f_n : n \in \mathbb{N}\} \subseteq A$,
 713 there is a subsequence that converges μ -almost everywhere.

714 *Proof.* Notice that the equivalence (i)-(iii) is Theorem 3.11. Notice that the equivalence
 715 of (iii) and (iv) is Fremlin's Dichotomy (Theorem 3.16). \square

716 Finally, it is natural to ask what the connection is between Talagrand stability
 717 and NIP.

718 **Proposition 3.18.** *Let X be a compact Hausdorff space and $A \subseteq C(X)$ be point-
 719 wise bounded. If A is universally Talagrand stable, then A has the NIP.*

720 *Proof.* By Theorem 3.11, it suffices to show that A is relatively countably compact
 721 in $\mathcal{M}^0(X, \mu)$ for all Radon probability measure μ on X . Since A is Talagrand
 722 μ -stable for any such μ , we have $\overline{A} \subseteq \mathcal{M}^0(X, \mu)$. In particular, A is relatively
 723 countably compact in $\mathcal{M}^0(X, \mu)$. \square

724 **Question 3.19.** Is the converse true?

725 The following two results suggest that the precise connection between Talagrand
 726 stability and NIP may be sensitive to set-theoretic axioms (even assuming count-
 727 ability of A).

728 **Theorem 3.20** (Talagrand, Theorem 9-3-1(a) in [Tal84]). *Let X be a compact
 729 Hausdorff space and $A \subseteq M_r(X)$ be countable and pointwise bounded. Assume that
 730 $[0, 1]$ is not the union of $< \mathfrak{c}$ closed measure zero sets. If A has the NIP, then A is
 731 universally Talagrand stable.*

732 **Theorem 3.21** (Fremlin, Shelah, [FS93]). *It is consistent that there exists a count-
 733 able pointwise bounded set of Lebesgue measurable functions with the NIP which is
 734 not Talagrand stable with respect to Lebesgue measure.*

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