

COMPLEXITY OF DEEP COMPUTATIONS VIA TOPOLOGY OF FUNCTION SPACES

EDUARDO DUEÑEZ¹ JOSÉ IOVINO¹ TONATIUH MATOS-WIEDERHOLD²
LUCIANO SALVETTI² FRANKLIN D. TALL²

¹Department of Mathematics, University of Texas at San Antonio
²Department of Mathematics, University of Toronto

ABSTRACT. We study complexity of deep computations. We use topology of function spaces, specifically, the classification Rosenthal compacta, to identify new complexity classes. We use the language of model theory, specifically, the concept of the independence from Shelah's classification theory, to translate between topology and computation.

INTRODUCTION

In this paper we study limit behavior of real-valued computations as the value of certain parameters of the computation model tend towards infinity, or towards zero, or towards some other fixed value, e.g., the depth of a neural network tending to infinity, or the time interval between layers of the network tending toward zero. Recently, particular cases of this situation have attracted considerable attention in deep learning research (e.g., Neural Ordinary Differential Equations [CRBD], Physics-Informed Neural Networks [RPK19], deep equilibrium models [BKK], among others). In this paper, we combine ideas of topology, measure theory, and model theory to study these limit phenomena from a unified viewpoint.

Informed by model theory, to each computation in a given computation model, we associate a continuous real-valued function, called the *type* of the computation, that describes the logical properties of this computation with respect to the rest of the model. This allows us to view computations in any given computational model as elements of a space of real-valued functions, which is called the *space of types* of the model. The idea of embedding models of theories into their type spaces is central in model theory. The embedding of computations into spaces of types allows us to utilize the vast theory of topology of function spaces, known as C_p -theory, to obtain results about complexity of topological limits of computations. As we shall indicate next, recent classification results for spaces of functions provide an elegant and powerful machinery to classify computations according to their levels of “tameness” or “wildness”, with the former corresponding roughly to polynomial approximability and the latter to exponential approximability. The viewpoint of spaces of types, which we have borrowed from model theory, thus becomes a “Rosetta stone” that allows us to interconnect various classification programs: In topology, the classification of Rosenthal compacta pioneered by Todorčević [Tod99]; in logic, the classification of theories developed by Shelah [She90]; and in statistical

Date: December 30, 2025.

2000 *Mathematics Subject Classification.* 54H30, 68T27, 68T07, 03C98, 03D15, 05D10, 54D80.

Key words and phrases. Deep computations, deep equilibrium models, physics-informed networks, computational complexity, independence property, NIP, infinite Ramsey theory, Baire class 1, Rosenthal compacta, Bourgain-Fremlin-Talagrand.

³⁴ learning, the notion PAC learning and VC dimension pioneered by Vapnik and
³⁵ Chervonenkis [VC74, VC71].

³⁶ In a previous paper [ADIW24], we introduced the concept of limits of computations,
³⁷ which we called *ultracomputations* (given they arise as ultrafilter limits of
³⁸ standard computations) and *deep computations* (following usage in machine learn-
³⁹ ing [BKK]). There is a technical difference between both designations, but in this
⁴⁰ paper, to simplify the nomenclature, we will ignore the difference and use only the
⁴¹ term “deep computation”.

⁴² In [ADIW24], we proved a new “tame vs wild” (i.e., polynomial vs exponential)
⁴³ dichotomy for complexity of deep computations by invoking a classical result of
⁴⁴ Grothendieck from late 50s [Gro52]. Under our model-theoretic Rosetta stone,
⁴⁵ polynomial approximability in the sense of computation becomes identified with the
⁴⁶ notion of continuous extendability in the sense of topology, and with the notions of
⁴⁷ *stability* and *type definability* in the sense of model theory.

⁴⁸ In this paper, we follow a more general approach, i.e., we view deep computations
⁴⁹ as pointwise limits of continuous functions. In topology functions that arise as the
⁵⁰ pointwise limit of a sequence of continuous are called *functions of the first Baire*
⁵¹ *class*, or *Baire class 1* functions, or *Baire-1* for short; Baire class 1 form a step above
⁵² simple continuity in the hierarchy of functions studied in real analysis (Baire class
⁵³ 0 functions being continuous functions). Intuitively, Baire-1 functions represent
⁵⁴ functions with “controlled” discontinuities, so they are crucial in topology and set
⁵⁵ theory.

⁵⁶ We prove a new “tame vs wild” Ramsey-theoretic dichotomy for complexity of
⁵⁷ general deep computations by invoking a famous paper by Bourgain, Fremlin and
⁵⁸ Talagrand from the late 70s [BFT78], and a new trichotomy for the class of “tame”
⁵⁹ deep computations by invoking an equally celebrated result of Todorčević, from the
⁶⁰ late 90s, for functions of the first Baire class [Tod99].

⁶¹ Todorčević’s trichotomy regards *Rosenthal compacta*; these are special classes of
⁶² topological spaces, defined as compact spaces that can be embedded (homeomor-
⁶³ phically identified as a subset) within the space of Baire class 1 functions on some
⁶⁴ Polish (separable, complete metric) space, under the pointwise convergence topol-
⁶⁵ ogy. Rosenthal compacta exhibit “topological tameness,” meaning that they behave
⁶⁶ in relatively controlled ways, and since the late 70’s, they have played a crucial role
⁶⁷ for understanding complexity of structures of functional analysis, especially, Banach
⁶⁸ spaces. Todorčević’s trichotomy has been utilized to settle longstanding problems
⁶⁹ in topological dynamics and topological entropy [GM22].

⁷⁰ Through our Rosetta stone, Rosenthal compacta in topology correspond to the
⁷¹ important concept of “No Independence Property” (known as “NIP”) in model
⁷² theory, identified by Shelah [She71, She90], and to the concept of Probably Ap-
⁷³ proximately Correct learning (known as “PAC learnability”) in statistical learning
⁷⁴ theory identified by Valiant [Val84].

⁷⁵ Going beyond Todorčević’s trichotomy, we invoke a more recent heptachotomy
⁷⁶ for Rosenthal compacta obtained by Argyros, Dodos and Kanellopoulos [ADK08].
⁷⁷ Argyros, Dodos and Kanellopoulos identified seven fundamental “prototypes” of
⁷⁸ separable Rosenthal compacta, and proved that any non-metrizable separable Rosen-
⁷⁹ thal compactum must contain a “canonical” embedding of one of these prototypes.
⁸⁰ They showed that if a separable Rosenthal compactum is not hereditarily separable,
⁸¹ it must contain an uncountable discrete subspace of the size of the continuum.

82 We believe that the results presented in this paper show practitioners of com-
 83 putation, or topology, or descriptive set theory, or model theory, how classification
 84 invariants used in their field translate into classification invariants of other fields.
 85 However, in the interest of accessibility, we do not assume previous familiarity with
 86 high-level topology or model theory, or computing. The only technical prerequisite
 87 of the paper is undergraduate-level topology and measure theory. The necessary
 88 topological background beyond undergraduate topology is covered in section 1.

89 In section 1, we present the basic topological and combinatorial preliminaries,
 90 and in section 2, we introduce the structural/model-theoretic viewpoint (no previ-
 91 ous exposure to model theory is needed). Section 3 is devoted to the classification
 92 of deep computations. The final section, section 4, presents the probabilistic view-
 93 point.

94 Throughout the paper, we focus on classical computation; however, by refining
 95 the model-theoretic tools, the results presented here can be extended to quantum
 96 computation and open quantum systems. This extension will be addressed in a
 97 forthcoming paper.

98

CONTENTS

| | | |
|-----|--|----|
| 99 | Introduction | 1 |
| 100 | 1. General topological preliminaries: From continuity to Baire class 1 | 4 |
| 101 | 1.1. From Rosenthal’s dichotomy to the Bourgain-Fremlin-Talagrand | |
| 102 | dichotomy to Shelah’s NIP | 5 |
| 103 | 1.2. NIP as universal dividing line between polynomial and exponential | |
| 104 | complexity | 7 |
| 105 | 1.3. Rosenthal compacta | 8 |
| 106 | 1.4. The special case $B_1(X, \mathbb{R}^{\mathcal{P}})$ with \mathcal{P} countable. | 8 |
| 107 | 2. Compositional computation structures: A structural approach to | |
| 108 | floating-point computation | 10 |
| 109 | 3. Classifying deep computations | 12 |
| 110 | 3.1. NIP, Rosenthal compacta, and deep computations | 12 |
| 111 | 3.2. The Todorčević trichotomy and levels of PAC learnability | 13 |
| 112 | 3.3. The Argyros-Dodos-Kanellopoulos heptachotomy, and approximability | |
| 113 | of deep computation by minimal classes | 15 |
| 114 | 4. Measure-theoretic versions of NIP and universal Monte Carlo | |
| 115 | computability of deep computations | 16 |
| 116 | 4.1. A measure-theoretic version of NIP | 16 |
| 117 | 4.2. Universal Monte Carlo computability of deep computations | 17 |
| 118 | 4.3. Bourgain-Fremlin-Talagrand, NIP, and universal Monte Carlo | |
| 119 | computability of deep computations | 18 |
| 120 | 4.4. Talagrand stability, Fremlin’s dichotomy, NIP, and universal Monte | |
| 121 | Carlo computability of deep computations | 18 |
| 122 | References | 20 |

In this section we present the preliminaries from general topology and function space theory. We include some of the proofs for completeness, but the reader familiar with these topics may skip them.

Recall that a subset of a topological space is F_σ if it is a countable union of closed sets, and G_δ if it is a countable intersection of closed sets. Note that in a metrizable space, every open set is F_σ ; equivalently, every closed set is G_δ .

131 A *Polish space* is a separable and completely metrizable topological space. The
 132 most important examples are the reals \mathbb{R} , the Cantor space $2^{\mathbb{N}}$ (the set of all infinite
 133 binary sequences, endowed with the product topology), and the Baire space $\mathbb{N}^{\mathbb{N}}$ (the
 134 set of all infinite sequences of naturals, also with the product topology). Countable
 135 products of Polish spaces are Polish; this includes spaces like $\mathbb{R}^{\mathbb{N}}$, the space of
 136 sequences of real numbers.

In this paper, we shall often discuss subspaces, and so there is a pertinent subtlety of the definitions worth mentioning: *completely metrizable space* is not the same as *complete metric space*; for an illustrative example, the interval $(0, 1)$ with the metric inherited from the reals is not complete, but it is Polish since it is homeomorphic to the real line. Being Polish is a topological property.

142 The following result is a cornerstone of descriptive set theory, closely tied to the
143 work of Waclaw Sierpiński and Kazimierz Kuratowski, with proofs often built upon
144 their foundations and formalized later, notably, involving Stefan Mazurkiewicz's
145 work on complete metric spaces.

Fact 1.1. A subset A of a Polish space X is itself Polish in the subspace topology if and only if it is a G_δ set. In particular, closed subsets and open subsets of Polish spaces are also Polish spaces.

Given two topological spaces X and Y we denote by $C_p(X, Y)$ the set of all continuous functions $f : X \rightarrow Y$ endowed with the topology of pointwise convergence. When $Y = \mathbb{R}$, we denote this collection simply as $C_p(X)$. A natural question is, how do topological properties of X translate into $C_p(X)$ and vice versa? These questions, and in general the study of these spaces, are the concern of C_p -theory, an active field of research in general topology which was pioneered by A. V. Arhangel'skiĭ and his students in the 1970's and 1980's. This field has found many applications in model theory and functional analysis. Recent surveys on the topics include [HT23] and [Tka11].

158 A Baire class 1 function between topological spaces is a function that can be
 159 expressed as the pointwise limit of a sequence of continuous functions. If X and Y
 160 are topological spaces, the Baire class 1 functions $f : X \rightarrow Y$ endowed with the
 161 topology of pointwise convergence is denoted $B_1(X, Y)$. As above, in the special
 162 case $Y = \mathbb{R}$ we denote $B_1(X, Y)$ as $B_1(X)$. Clearly, $C_p(X, Y) \subseteq B_1(X, Y)$.
 163 The Baire hierarchy of functions was introduced by René-Louis Baire in his 1899
 164 doctoral thesis, *Sur les fonctions de variables réelles*. His work moved away from
 165 the 19th-century preoccupation with “pathological” functions toward a constructive
 166 classification based on pointwise limits.

167 A topological space X is *perfectly normal* if it is normal and every closed subset
 168 of X is a G_δ (equivalently, every open subset of X is a G_δ). Note that every
 169 metrizable space is perfectly normal.

170 The following fact was established by Baire in thesis. A proof can be found in
 171 Section 10 of [Tod97].

172 **Fact 1.2** (Baire). *If X is perfectly normal, then the following conditions are equiv-*
 173 *alent for a function $f : X \rightarrow \mathbb{R}$:*

- 174 • *f is a Baire class 1 function, that is, f is a pointwise limit of continuous*
- 175 functions..
- 176 • *$f^{-1}[U]$ is an F_σ subset of X whenever $U \subseteq Y$ is open.*
- 177 • *For every closed $F \subseteq X$, the restriction $f|_F$ has a point of continuity.*

178 Moreover, if X is Polish and $f \notin B_1(X)$, then there exists countable $D_0, D_1 \subseteq X$
 179 and reals $a < b$ such that

$$D_0 \subseteq f^{-1}(-\infty, a], \quad D_1 \subseteq f^{-1}[b, \infty), \quad \overline{D_0} = \overline{D_1}.$$

180 A subset L of a topological space X is *relatively compact* in X if the closure
 181 of L in X is compact. Relatively compact subsets of $B_1(X)$ (for X Polish) have
 182 been objects of interest for researchers in Analysis and Topological Dynamics. We
 183 begin with the following well-known result. Recall that a set $A \subseteq \mathbb{R}^X$ of real-
 184 valued functions is *pointwise bounded* if for every $x \in X$ there is $M_x > 0$ such that
 185 $|f(x)| < M_x$ for all $f \in A$. We include a proof for the reader's convenience:

186 **Lemma 1.3.** *Let X be a Polish space and $A \subseteq B_1(X)$ be pointwise bounded. The*
 187 *following are equivalent:*

- 188 (i) *A is relatively compact in $B_1(X)$.*
- 189 (ii) *A is relatively countably compact in $B_1(X)$, i.e., every countable subset of*
 A has an accumulation point in $B_1(X)$.
- 191 (iii) *$\overline{A} \subseteq B_1(X)$, where \overline{A} denotes the closure in \mathbb{R}^X .*

192 *Proof.* Since A is pointwise bounded, for each $x \in X$, fix $M_x > 0$ such that $|f(x)| \leq$
 193 M_x for every $f \in A$.

194 (i) \Rightarrow (ii) holds in general.

195 (ii) \Rightarrow (iii) Assume that A is relatively countably compact in $B_1(X)$ and that
 196 $f \in \overline{A} \setminus B_1(X)$. By Fact 1.2, there are countable $D_0, D_1 \subseteq X$ with $\overline{D_0} = \overline{D_1}$, and
 197 $a < b$ such that $D_0 \subseteq f^{-1}(-\infty, a]$ and $D_1 \subseteq f^{-1}[b, \infty)$. We claim that there is a
 198 sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in D_0 \cup D_1$. Indeed,
 199 use the countability to enumerate $D_0 \cup D_1$ as $\{x_n\}_{n \in \mathbb{N}}$. Then for each positive n
 200 find $f_n \in A$ with $|f_n(x_i) - f(x_i)| < \frac{1}{n}$ for all $i \leq n$. The claim follows.

201 By relative countable compactness of A , there is an accumulation point $g \in$
 202 $B_1(X)$ of $\{f_n\}_{n \in \mathbb{N}}$. It is straightforward to show that since f and g agree on
 203 $D_0 \cup D_1$, g does not have a point of continuity on the closed set $\overline{D_0} = \overline{D_1}$, which
 204 contradicts Fact 1.2.

205 (iii) \Rightarrow (i) Suppose that $\overline{A} \subseteq B_1(X)$. Then $\overline{A} \cap B_1(X) = \overline{A}$ is a closed subset of
 206 $\prod_{x \in X} [-M_x, M_x]$; Tychonoff's theorem states that the product of compact spaces
 207 is always compact, and since closed subsets of compact spaces are compact, \overline{A} must
 208 be compact, as desired. \square

209 **1.1. From Rosenthal's dichotomy to the Bourgain-Fremlin-Talagrand di-**
 210 **chotomy to Shelah's NIP.** In metrizable spaces, points of closure of some subset
 211 can always be approximated by points inside the set, via a convergent sequence.
 212 For more complicated spaces, such as $C_p(X)$, this fails in remarkable ways. To
 213 see an example, consider the Cantor space $X = 2^\mathbb{N}$, and for each $n \in \mathbb{N}$ define

214 $p_n : X \rightarrow \{0, 1\}$ by $p_n(x) = x(n)$ for each $x \in X$. Then p_n is continuous for each
215 n , but one can show (see Chapter 1.1 of [Tod97] for details) that the only continuous
216 functions in the closure of $\{p_n\}_{n \in \mathbb{N}}$ are the functions p_n themselves; moreover,
217 none of the subsequences of $\{p_n\}_{n \in \mathbb{N}}$ converge. In some sense, this example is the
218 worst possible scenario for convergence. The topological space obtained from this
219 closure is well-known: it is the *Stone-Čech compactification* of the discrete space of
220 natural numbers, or $\beta\mathbb{N}$ for short, and it is an important object of study in general
221 topology.

222 The following theorem, established by Haskell Rosenthal in 1974, is fundamental
223 in functional analysis, and describes a sharp division in the behavior of sequences
224 in a Banach spaces. I

225 **Theorem 1.4** (Rosenthal's Dichotomy, [Ros74]). *If X is Polish and $\{f_n\} \subseteq C_p(X)$
226 is pointwise bounded, then either $\{f_n\}_{n \in \mathbb{N}}$ contains a convergent subsequence or a
227 subsequence whose closure (in \mathbb{R}^X) is homeomorphic to $\beta\mathbb{N}$.*

228 In other words, a pointwise bounded set of continuous functions either contains
229 a convergent subsequence, or a subsequence whose closure is essentially the same as
230 the example mentioned in the previous paragraphs (the “wildest” possible scenario).
231 Note that in the preceding example, the functions are trivially pointwise bounded
232 in \mathbb{R}^X as the functions can only take values 0 and 1.

233 The genesis of Theorem 1.4 was Rosenthal's ℓ_1 theorem, which states that the
234 only reason why Banach space can fail to have an isomorphic copy of ℓ_1 (the space
235 of absolutely summable sequences) is the presence of a bounded sequence with no
236 weakly Cauchy subsequence. The theorem is famous for connecting diverse areas
237 of mathematics, namely, Banach space geometry, Ramsey theory, set theory, and
238 topology of function spaces.

239 As we move from $C_p(X)$ to the larger space $B_1(X)$, we find a similar dichotomy.
240 Either every point of closure of the set of functions will be a Baire class 1 function,
241 or there is a sequence inside the set that behaves in the wildest possible way. The
242 theorem is usually not phrased as a dichotomy, but rather as an equivalence:

243 **Theorem 1.5** (“The BFT Dichotomy”). Bourgain-Fremlin-Talagrand [BFT78,
244 Theorem 4G]). *Let X be a Polish space and $A \subseteq C_p(X)$ be pointwise bounded.
245 The following are equivalent:*

- 246 (i) *A is relatively compact in $B_1(X)$, i.e., $\overline{A} \subseteq B_1(X)$.*
- 246 (ii) *For every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and every $a < b$ there is $I \subseteq \mathbb{N}$ such that*

$$\bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} f_n^{-1}[b, \infty) = \emptyset.$$

247 **Definition 1.6.** We shall say that a set $A \subseteq \mathbb{R}^X$ satisfies the *Independence Prop-*
248 *erty*, or IP for short, if it satisfies the following condition: There exists every
249 $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$ such that for every pair of disjoint sets $E, F \subseteq \mathbb{N}$, we
250 have

$$\bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

251 If A satisfies the negation of this condition, we will say that A *satisfies NIP*, or
252 that has the NIP.

Remark 1.7. Note that if X is compact and $A \subseteq C_p(X)$, then A satisfies the NIP if and only if for every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and for every $a < b$ there is $I \subseteq \mathbb{N}$ such that

$$\bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} f_n^{-1}[b, \infty) = \emptyset.$$

253 To summarize, the particular case of Theorem 1.5 for X compact can be stated
254 in the following way:

255 **Theorem 1.8.** *Let X be a compact Polish space. Then, for every pointwise bounded
256 $A \subseteq C_p(X)$, one and exactly one of the following two conditions must hold:*
257 (i) $\overline{A} \subseteq B_1(X)$.
258 (ii) A has NIP.

259 The Independence Property was first isolated by Saharon Shelah in model theory
260 as a dividing line between theories whose models are “tame” (corresponding to
261 NIP) theories of models are “wild” (corresponding to IP). See [She71, Definition
262 4.1],[She90]. We will discuss this dividing line in more detail in the next section.

263 **1.2. NIP as universal dividing line between polynomial and exponential
264 complexity.** The particular case of the BFT dichotomy (Theorem 1.5) when A
265 consists of $\{0, 1\}$ -valued (i.e., {Yes, No}-valued) strings was discovered indepen-
266 dently, around 1971-1972 in many foundational contexts related to polynomial
267 (“tame”) vs exponential (“wild”) complexity: In model theory, by Saharon She-
268 lah [She71],[She90], in combinatorics, by Norbert Sauer [Sau72], and Shelah [She72,
269 She90], and in statistical learning, by Vladimir Vapnik and Alexey Chervonenkis [VC71,
270 VC74].

271 **In model theory:** Shelah’s classification theory is a foundational program
272 in mathematical logic devised to categorize first-order theories based on
273 the complexity and structure of their models. A theory T is considered
274 classifiable in Shelah’s sense if the number of non-isomorphic models of T
275 of a given cardinality can be described by a bounded number of numerical
276 invariants. In contrast, a theory T is unclassifiable if the number of models
277 of T of a given cardinality is the maximum possible number. A key fact
278 is that the number of models of T is directly impacted by the number of
279 “types” over of parameters in models of T ; a controlled number of types is
280 a characteristic of a classifiable theory.

281 In Shelah’s classification program [She90], theories without the inde-
282 pendence property (called NIP theories, or dependent theories) have a well-
283 behaved, “tame” structure; the number of types over a set of parameters
284 of size κ of such a theory is of polynomially or similar “slow” growth on κ .
285 In contrast, Theories with the Independence Property (called IP theories)
286 are considered “intractable” or “wild”. A theory with the Independence
287 Property produces the maximum possible number of types over a set of
288 parameters; for a set of parameters of cardinality κ , the theory will have
289 2^{2^κ} -many distinct types.

290 **In combinatorics:** Sauer [Sau72] and Shelah [She72] proved the following:
291 If $\mathcal{F} = \{S_0, S_1, \dots\}$ is a family of subsets of some infinite set S , then
292 either for every $n \in \mathbb{N}$, there is either a set $A \subseteq S$ with $|A| = n$ such that
293 $|\{S_i \cap A : i \in \mathbb{N}\}| = 2^n$ (yielding exponential complexity), or there exists

294 $N \in \mathbb{N}$ such that for every $A \subseteq S$ with $|A| \geq N$, one has

$$|\{S_i \cap A) : i \in \mathbb{N}\}| \leq \sum_{i=0}^{N-1} \binom{|A|}{i} \approx O(|A|^N)$$

295 (yielding polynomial complexity). This answered a question of Erdős.

296 **In machine learning:** Readers familiar with statistical learning may rec-
297 ognize the Sauer-Shelah lemma as the dichotomy discovered and proved
298 slightly earlier (1971) by Vapnik and Chervonenkis [VC71, VC74] to ad-
299 dress the problem of uniform convergence in statistics. The least integer
300 N given by the preceding paragraph, when it exists, is called the *VC-*
301 *dimension* of \mathcal{F} . This is a core concept in machine learning. If such an
302 integer N does not exist, we say that the VC-dimension of \mathcal{F} is infinite. The
303 lemma provides upper bounds on the number of data points (sample size m)
304 needed to learn a concept class with VC dimension $d \in \mathbb{N}$ by showing this
305 number grows polynomially with m and d (namely, $\sum_{i=0}^d \binom{m}{i} \approx O(m^d)$),
306 not exponentially. The Fundamental Theorem of Statistical Learning states
307 that a hypothesis class is PAC-learnable (PAC stands for “Probably Ap-
308 proximately Correct”) if and only if its VC dimension is finite.

309 **1.3. Rosenthal compacta.** The comprehensiveness of Theorem 1.5, attested by
310 the examples outlined in the preceding section, led to the following definition (iso-
311 lated by Gilles Godefroy [God80]):

312 **Definition 1.9.** A Rosenthal compactum is a compact Hausdorff topological space
313 K that can be topologically embedded as a compact subset into the space of all
314 functions of the first Baire class on some Polish space X , equipped with the topology
315 of pointwise convergence.

316 Rosenthal compacta are characterized by significant topological and dynamical
317 tameness properties. They play an important role in functional analysis, measure
318 theory, dynamical systems, descriptive set theory, and model theory. In this paper,
319 we introduce their applicability in deep computation. For this, we shall first focus
320 on countable languages, which is the theme of the next subsection.

321 **1.4. The special case $B_1(X, \mathbb{R}^{\mathcal{P}})$ with \mathcal{P} countable.** Our goal now is to charac-
322 terize relatively compact subsets of $B_1(X, Y)$ for the particular case when $Y = \mathbb{R}^{\mathcal{P}}$
323 with \mathcal{P} countable. Given $P \in \mathcal{P}$ we denote the projection map onto the P -coordinate
324 by $\pi_P : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R}$. From a high-level topological interpretation, the next lemma
325 states that, in this context, the spaces \mathbb{R} and $\mathbb{R}^{\mathcal{P}}$ are really not that different,
326 and that if we understand the Baire class 1 functions of one space, then we also
327 understand the functions of both.

328 **Lemma 1.10.** Let X be a Polish space and \mathcal{P} be a countable set. Then, $f \in$
329 $B_1(X, \mathbb{R}^{\mathcal{P}})$ if and only if $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$.

Proof. Only one implication needs a proof. Suppose that $\pi_P \circ f \in B_1(X)$ for all
327 $P \in \mathcal{P}$. Let V be a basic open subset of $\mathbb{R}^{\mathcal{P}}$. That is, there exists a finite $\mathcal{P}' \subseteq \mathcal{P}$
such that $V = \bigcap_{P \in \mathcal{P}'} \pi_P^{-1}[U_P]$ where U_P is open in \mathbb{R} . Then,

$$f^{-1}[V] = \bigcap_{P \in \mathcal{P}'} (\pi_P \circ f)^{-1}[U_P]$$

is an F_σ set. Since \mathcal{P} is countable, $\mathbb{R}^\mathcal{P}$ is second countable so every open set U in $\mathbb{R}^\mathcal{P}$ is a countable union of basic open sets. Hence, $f^{-1}[U]$ is F_σ . \square

Below we consider \mathcal{P} with the discrete topology. For each $f : X \rightarrow \mathbb{R}^\mathcal{P}$ denote $\hat{f}(P, x) := \pi_P \circ f(x)$ for all $(P, x) \in \mathcal{P} \times X$. Similarly, for each $g : \mathcal{P} \times X \rightarrow \mathbb{R}$ denote $\check{g}(x)(P) := g(P, x)$. Given $A \subseteq (\mathbb{R}^\mathcal{P})^X$, we denote \hat{A} as the set of all \hat{f} such that $f \in A$. Note that the map $(\mathbb{R}^\mathcal{P})^X \rightarrow \mathbb{R}^{\mathcal{P} \times X}$ given by $f \mapsto \hat{f}$ is a homeomorphism and its inverse is given by $g \mapsto \check{g}$.

Lemma 1.11. *Let X be a Polish space and \mathcal{P} be countable. Then, $f \in B_1(X, \mathbb{R}^\mathcal{P})$ if and only if $\hat{f} \in B_1(\mathcal{P} \times X)$.*

Proof. (\Rightarrow) By Lemma 1.10, given an open set of reals U , we have $f^{-1}[\pi_P^{-1}[U]]$ is F_σ for every $P \in \mathcal{P}$. Given that \mathcal{P} is a discrete countable space, we observe that

$$\hat{f}^{-1}[U] = \bigcup_{P \in \mathcal{P}} (\{P\} \times f^{-1}[\pi_P^{-1}[U]])$$

is an F_σ as well.

(\Leftarrow) By lemma 1.10 it suffices to show that $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$. Fix an open $U \subseteq \mathbb{R}$. Write $\hat{f}^{-1}[U] = \bigcup_{n \in \mathbb{N}} F_n$ where F_n is closed in $\mathcal{P} \times X$. Then,

$$(\pi_P \circ f)^{-1}[U] = \bigcup_{n \in \mathbb{N}} \{x \in X : (P, x) \in F_n\}$$

which is F_σ . \square

Given $A \subseteq Y^X$ and $K \subseteq X$ we write $A|_K := \{f|_K : f \in A\}$, i.e., the set of all restrictions of functions in A to K . The following Theorem is a slightly more general version of Theorem 1.5.

Theorem 1.12. *Assume that \mathcal{P} is countable, X is a Polish space, and $A \subseteq C_p(X, \mathbb{R}^\mathcal{P})$ is such that $\pi_P \circ A$ is pointwise bounded for all $P \in \mathcal{P}$. The following are equivalent for every compact $K \subseteq X$:*

- (1) $\overline{A|_K} \subseteq B_1(K, \mathbb{R}^\mathcal{P})$.
- (2) $\pi_P \circ A|_K$ satisfies the NIP for every $P \in \mathcal{P}$.

Proof. (1) \Rightarrow (2). Let $P \in \mathcal{P}$. Fix $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$. By (1), we have $\overline{A|_K} \subseteq B_1(K, \mathbb{R}^\mathcal{P})$. Applying the homeomorphism $f \mapsto \hat{f}$ and using lemma 1.11 we get $\hat{A}|_{\mathcal{P} \times K} \subseteq B_1(\mathcal{P} \times K)$. By Theorem 1.5, there is $I \subseteq \mathbb{N}$ such that

$$(\mathcal{P} \times K) \cap \bigcap_{n \in I} \hat{f}_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} \hat{f}_n^{-1}[b, \infty) = \emptyset$$

Hence,

$$K \cap \bigcap_{n \in I} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n \notin I} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

By the compactness of K , there are finite $E \subseteq I$ and $F \subseteq \mathbb{N} \setminus I$ such that

$$K \cap \bigcap_{n \in E} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n \in F} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

Thus, $\pi_P \circ A|_L$ satisfies the NIP.

(2) \Rightarrow (1) Fix $f \in \overline{A|_K}$. By lemma 1.10 it suffices to show that $\pi_P \circ f \in B_1(K)$ for all $P \in \mathcal{P}$. By (2), $\pi_P \circ A|_K$ satisfies the NIP. Hence, by Theorem 1.5 we have $\overline{\pi_P \circ A|_K} \subseteq B_1(K)$. But then, $\pi_P \circ f \in \overline{\pi_P \circ A|_K} \subseteq B_1(K)$. \square

353 Lastly, a simple but useful lemma that helps understand when we restrict a set
 354 of functions to a specific subspace of the domain space, we may always assume that
 355 the subspace is closed, as replacing the subspace by its closure has no effect on NIP.

Lemma 1.13. Assume that X is Hausdorff and that $A \subseteq C_p(X)$. The following are equivalent for every $L \subseteq X$:

- (i) A_L satisfies the NIP.
(ii) $A|_{\overline{L}}$ satisfies the NIP.

Proof. It suffices to show that (i) \Rightarrow (ii). Suppose that (ii) does not hold, i.e., that there are $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$ such that for all finite disjoint $E, F \subseteq \mathbb{N}$:

$$\overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

Pick $a' < b'$ such that $a < a' < b' < b$. Then, for any finite disjoint $E, F \subseteq \mathbb{N}$ we can choose

$$x \in \overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a') \cap \bigcap_{n \in F} f_n^{-1}(b', \infty)$$

By definition of closure:

$$L \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a'] \cap \bigcap_{n \in F} f_n^{-1}[b', \infty) \neq \emptyset.$$

360 This contradicts (i).

2. COMPOSITIONAL COMPUTATION STRUCTURES: A STRUCTURAL APPROACH TO FLOATING-POINT COMPUTATION

In this section, we connect function spaces with floating point computation. We start by summarizing some basic concepts from [ADIW24].

365 A *computation states structure* is a pair (L, \mathcal{P}) , where L is a set whose elements we
 366 call *states* and \mathcal{P} is a collection of real-valued functions on L that we call *predicates*.
 367 For a state $v \in L$, *type* of v is defined as the indexed family

$$\text{tp}(v) = (P(v))_{P \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}.$$

³⁶⁸ For each $P \in \mathcal{P}$, we call real value $P(v)$ the P -th feature of v . A transition of a
³⁶⁹ computation states structure (L, \mathcal{P}) is a map $f : L \rightarrow L$.

370 Intuitively, L is the set of states of a computation, and the predicates $P \in \mathcal{P}$
 371 are primitives that are given and accepted as computable. We think of each state
 372 $v \in L$ as being uniquely characterized by its type $\text{tp}(v)$. Thus, in practice, we
 373 identify L with a subset of $\mathbb{R}^{\mathcal{P}}$. A typical case will be when $L = \mathbb{R}^{\mathbb{N}}$ or $L = \mathbb{R}^n$
 374 for some positive integer n and there is a predicate $P_i(v) = v_i$ for each of the
 375 coordinates v_i of v . We regard the space of types as a topological space, endowed
 376 with the topology of pointwise convergence inherited from $\mathbb{R}^{\mathcal{P}}$. In particular, for
 377 each $P \in \mathcal{P}$, the projection map $v \mapsto P(v)$ is continuous.

378 Definition 2.1. Given a computation states structure (L, \mathcal{P}) , any element of $\mathbb{R}^{\mathcal{P}}$
 379 in the image of L under the map $v \mapsto \text{tp}(v)$ will be called a *realized type*. The
 380 topological closure of the set of realized types in $\mathbb{R}^{\mathcal{P}}$ (endowed with the point-
 381 wise convergence topology) will be called the *space of types* of (L, \mathcal{P}) , denoted \mathcal{L} .
 382 Elements of $\mathcal{L} \setminus L$ will be called *unrealized types*.

In traditional, compact-valued, model theory, the space of types of a structure is viewed as a sort of compactification of the structure, and the compactness of type spaces plays a central role. However, here we are dealing with real-valued structures, and the space \mathcal{L} defined above is not necessarily compact. To bypass this obstacle, we follow the idea introduced in [ADIW24] of covering \mathcal{L} by “thin” compact subspaces that we call *shards*. The formal definition of shard is next.

Definition 2.2. A *sizer* is a tuple $r_\bullet = (r_P)_{P \in \mathcal{P}}$ of positive real numbers indexed by \mathcal{P} . Given a sizer r_\bullet , we define the r_\bullet -*shard* as:

$$L[r_\bullet] = L \cap \prod_{P \in \mathcal{P}} [-r_P, r_P].$$

For a sizer r_\bullet , the r_\bullet -*type shard* is defined as $\mathcal{L}[r_\bullet] = \overline{L[r_\bullet]}$. We define \mathcal{L}_{sh} , as the union of all type-shards.

Definition 2.3. A *Compositional Computation Structure* (CCS) is a triple (L, \mathcal{P}, Γ) , where

- (L, \mathcal{P}) is a computation states structure, and
- $\Gamma \subseteq L^L$ is a semigroup under composition.

The elements of the semigroup Γ are called the *computations* of the structure (L, \mathcal{P}, Γ) .

If $\Delta \subseteq \Gamma$, we say that $\Delta \subseteq \Gamma$ is *R-confined* if $\gamma|_{L[r_\bullet]} : L[r_\bullet] \rightarrow L[r_\bullet]$ for every $r_\bullet \in R$ and $\gamma \in \Delta$. Elements in $\overline{\Delta} \subseteq \mathcal{L}_{\text{sh}}$ are called (real-valued) *deep computations* or *ultracomputations*.

A tenet of our approach is that a map $f : L \rightarrow \mathcal{L}$ is to be considered “effectively computable” if, for each $Q \in \mathcal{P}$, the output feature $Q \circ f : L \rightarrow \mathbb{R}$ is a *definable* predicate in the following sense: Given any arbitrary $\varepsilon > 0$ and any $K \subseteq L$ wherein every input feature $P(v)$ remains bounded in magnitude there is an ε -approximating continuous “algebraic” operator $\varphi(P_1, \dots, P_n)$ of finitely many input predicates $P_1, \dots, P_n \in \mathcal{P}$, such that the following holds: for all $v \in K$, the output feature $Q(f(v))$ is ε -approximated by $\varphi(P_1(v), \dots, P_n(v))$. By “algebraic”, we mean that, *aside from the primitively computable P_1, \dots, P_n , the approximating operator $\varphi(P_1, \dots, P_n)$ uses only the also primitively computable operations of $\mathbb{R}^{\mathcal{P}}$* as an algebra, i.e., vector addition, vector multiplication, and scalar addition.

It is shown in [ADIW24]) that:

- (1) For a definable $f : L \rightarrow \mathcal{L}$, the approximating operators φ may be taken to be *polynomials* of the input features, and
- (2) Definable transforms $f : L \rightarrow \mathcal{L}$ are precisely those that extend to continuous $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$

This motivates the following definition.

Definition 2.4. We say that a CCS (L, \mathcal{P}, Γ) satisfies the *Extendability Axiom* if for all $\gamma \in \Gamma$, there is $\tilde{\gamma} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$ such that for every sizer r_\bullet there is a sizer s_\bullet such that $\tilde{\gamma}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$ is continuous. We refer to $\tilde{\gamma}$ as a *free extension* of γ .

By the preceding remarks, the Extendability Axiom says that the elements of the semigroup Γ are definable. For the rest of the paper, fix for each $\gamma \in \Gamma$ a free extension $\tilde{\gamma}$ of γ . For any $\Delta \subseteq \Gamma$, let $\tilde{\Delta}$ denote $\{\tilde{\gamma} : \gamma \in \Delta\}$.

425 For a detailed discussion of the Extendability Axiom, we refer the reader to [ADIW24].

426 For an illustrative example, we can frame Newton's polynomial root approximation
 427 method in the context of a CCS (see Example 5.6 of [ADIW24] for details) as
 428 follows. Begin by considering the extended complex numbers $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with
 429 the usual Riemann sphere topology that makes it into a compact space (where
 430 unbounded sequences converge to ∞). In fact, not only is this space compact,
 431 but it is covered by the shard given by the sizer $(1, 1, 1)$ (the unit sphere is con-
 432 tained in the cube $[-1, 1]^3$). The space $\hat{\mathbb{C}}$ is homeomorphic to the usual unit sphere
 433 $S^2 := \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ of \mathbb{R}^3 , by means of the stereographic pro-
 434 jection and its inverse $\hat{\mathbb{C}} \rightarrow S^2$. This function is regarded as a triple of predi-
 435 cates $x, y, z : \hat{\mathbb{C}} \rightarrow [-1, 1]$ where each will map an extended complex number to
 436 its corresponding real coordinate on the cube $[-1, 1]^3$. Now fix the cubic com-
 437 plex polynomial $p(s) := s^3 - 1$, and consider the map which performs one step
 438 in Newton's method at a particular (extended) complex number s , for finding a
 439 root of p , $\gamma_p : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. The explicit inner workings of γ_p are irrelevant for this
 440 example, except for the fact that it is a continuous mapping. It follows that
 441 $(S^2, \{x, y, z\}, \{\gamma_p^k : k \in \mathbb{N}\})$ is a CCS. The idea is that repeated applications of
 442 $\gamma_p(s), \gamma_p \circ \gamma_p(s), \gamma_p \circ \gamma_p \circ \gamma_p(s), \dots$ would approximate a root of p provided s was a
 443 good enough initial guess.

444 3. CLASSIFYING DEEP COMPUTATIONS

445 **3.1. NIP, Rosenthal compacta, and deep computations.** Under what condi-
 446 tions are deep computations Baire class 1, and thus well-behaved according to our
 447 framework, on type-shards? The following theorem says that, under the assump-
 448 tion that \mathcal{P} is countable, the space of deep computations is a Rosenthal compactum
 449 (when restricted to shards) if and only if the set of computations satisfies the NIP
 450 feature by feature. Hence, we can import the theory of Rosenthal compacta into
 451 this framework of deep computations.

452 **Theorem 3.1.** *Let (L, \mathcal{P}, Γ) be a compositional computational structure (Defini-
 453 tion 2.3) satisfying the Extendability Axiom (Definition 2.4) with \mathcal{P} countable. Let
 454 R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. The following
 455 are equivalent.*

- 456 (1) $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$.
 457 (2) $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ satisfies the NIP for all $P \in \mathcal{P}$ and $r_\bullet \in R$; that is, for all
 $P \in \mathcal{P}$, $r_\bullet \in R$, $a < b$, $\{\gamma_n\}_{n \in \mathbb{N}} \subseteq \Delta$ there are finite disjoint $E, F \subseteq \mathbb{N}$ such
 that

$$L[r_\bullet] \cap \bigcap_{n \in E} (\pi_P \circ \gamma_n)^{-1}(-\infty, a] \cap \bigcap_{n \in F} (\pi_P \circ \gamma_n)^{-1}[b, \infty) = \emptyset.$$

458 Moreover, if any (hence all) of the preceding conditions hold, then every deep
 459 computation $f \in \overline{\tilde{\Delta}}$ can be extended to a Baire-1 function on shards, i.e., there is
 460 $\tilde{f} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$ such that $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$. In particular, on
 461 each shard every deep computation is the pointwise limit of a countable sequence of
 462 computations.

463 *Proof.* Since \mathcal{P} is countable, $\mathcal{L}[r_\bullet] \subseteq \mathbb{R}^\mathcal{P}$ is Polish. Also, the Extendability Axiom
 464 implies that $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ is a pointwise bounded set of continuous functions for all
 465 $P \in \mathcal{P}$. Hence, Theorem 1.12 and Lemma 1.13 prove the equivalence of (1) and (2).

If (1) holds and $f \in \overline{\Delta}$, then write $f = \mathcal{U}\lim_i \gamma_i$ as an ultralimit. Define $\tilde{f} := \mathcal{U}\lim_i \tilde{\gamma}_i$. Hence, for all $r_\bullet \in R$ we have $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in \overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$. That every deep computation is a pointwise limit of a countable sequence of computations follows from the fact that for a Polish space X every compact subset of $B_1(X)$ is Fréchet-Urysohn (that is, a space where topological closures coincide with sequential closures, see Theorem 3F in [BFT78] or Theorem 4.1 in [Deb13]). \square

3.2. The Todorčević trichotomy and levels of PAC learnability. Given a countable set Δ of computations satisfying the NIP on features and shards (condition (2) of Theorem 3.1) we have that $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ (for a fixed sizer r_\bullet) is a separable *Rosenthal compactum* (see Definition 1.9). Todorčević proved a remarkable trichotomy for Rosenthal compacta [Tod99], and later Argyros, Dodos, Kanellopoulos [ADK08] proved an heptachotomy that refined Todorčević's classification. In this section, inspired by the work of Glasner and Megrelishvili [GM22], we study ways in which this classification allows us obtain different levels of PAC-learnability and NIP.

Recall that a topological space X is *hereditarily separable* if every subspace is separable, and that X is *first countable* if every point in X has a countable local basis. Every separable metrizable space is hereditarily separable, and R. Pol proved that every hereditarily separable Rosenthal compactum is first countable (see section 10 of [Deb13]). This suggests the following definition:

Definition 3.2. Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendability Axiom and R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be an R -confined countable set of computations satisfying the NIP on shards and features (condition (2) in Theorem 3.1). We say that Δ is:

- (i) NIP₁ if $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ is first countable for every $r_\bullet \in R$.
- (ii) NIP₂ if $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ is hereditarily separable for every $r_\bullet \in R$.
- (iii) NIP₃ if $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ is metrizable for every $r_\bullet \in R$.

Observe that NIP₃ \Rightarrow NIP₂ \Rightarrow NIP₁ \Rightarrow NIP. A natural question that would continue this work is to find examples of CCS that separate these levels of NIP. In [Tod99], Todorčević isolates three canonical examples of Rosenthal compacta that witness the failure of the converse implications above.

We now present some separable and non-separable examples of Rosenthal compacta.

Examples 3.3.

- (1) *Alexandroff compactification of a discrete space of size continuum.* For each $a \in 2^\mathbb{N}$ consider the map $\delta_a : 2^\mathbb{N} \rightarrow \mathbb{R}$ given by $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. Let $A(2^\mathbb{N}) = \{\delta_a : a \in 2^\mathbb{N}\} \cup \{0\}$, where 0 is the zero map. Notice that $A(2^\mathbb{N})$ is a compact subset of $B_1(2^\mathbb{N})$, in fact $\{\delta_a : a \in 2^\mathbb{N}\}$ is a discrete subspace of $B_1(2^\mathbb{N})$ and its pointwise closure is precisely $A(2^\mathbb{N})$. Hence, this is a Rosenthal compactum which is not first countable. Notice that this space is also not separable.
- (2) *Extended Alexandroff compactification.* For each finite binary sequence $s \in 2^{<\mathbb{N}}$, let $v_s : 2^\mathbb{N} \rightarrow \mathbb{R}$ be given by $v_s(x) = 1$ if x extends s and $v_s(x) = 0$ otherwise. Let $\hat{A}(2^\mathbb{N})$ be the pointwise closure of $\{v_s : s \in 2^{<\mathbb{N}}\}$, i.e.,

509 $\hat{A}(2^{\mathbb{N}}) = A(2^{\mathbb{N}}) \cup \{v_s : s \in 2^{<\mathbb{N}}\}$. Note that this space is a separable
 510 Rosenthal compactum which is not first countable.

- 511 (3) *Split Cantor.* Let $<$ be the lexicographic order in the space of infinite
 512 binary sequences, i.e., $2^{\mathbb{N}}$. For each $a \in 2^{\mathbb{N}}$ let $f_a^- : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by
 513 $f_a^-(x) = 1$ if $x < a$ and $f_a^-(x) = 0$ otherwise. Let $f_a^+ : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given
 514 by $f_a^+(x) = 1$ if $x \leq a$ and $f_a^+(x) = 0$ otherwise. The split Cantor is the
 515 space $S(2^{\mathbb{N}}) = \{f_a^- : a \in 2^{\mathbb{N}}\} \cup \{f_a^+ : a \in 2^{\mathbb{N}}\}$. This is a separable Rosenthal
 516 compactum. One example of a countable dense subset is the set of all f_a^+
 517 and f_a^- where a is an infinite binary sequence that is eventually constant.
 518 Moreover, it is hereditarily separable, but it is not metrizable.
- 519 (4) *Alexandroff Duplicate.* Let K be any compact metric space and consider
 520 the Polish space $X = C(K) \sqcup K$, i.e., the disjoint union of $C(K)$ (with its
 521 supremum norm topology) and K . For each $a \in K$ define $g_a^0, g_a^1 : X \rightarrow \mathbb{R}$
 522 as follows:

$$g_a^0(x) = \begin{cases} x(a), & x \in C(K) \\ 0, & x \in K \end{cases}$$

$$g_a^1(x) = \begin{cases} x(a), & x \in C(K) \\ \delta_a(x), & x \in K \end{cases}$$

519 Let $D(K) = \{g_a^0 : a \in K\} \cup \{g_a^1 : a \in K\}$. Notice that $D(K)$ is a first
 520 countable Rosenthal compactum. It is not separable if K is uncountable.
 521 The interesting case will be when $K = 2^{\mathbb{N}}$.

- 522 (5) *Extended Alexandroff Duplicate of the split Cantor.* For each finite binary
 523 sequence $t \in 2^{<\mathbb{N}}$ let $a_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending
 524 with 0's and let $b_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with
 525 1's. Define $h_t : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ by

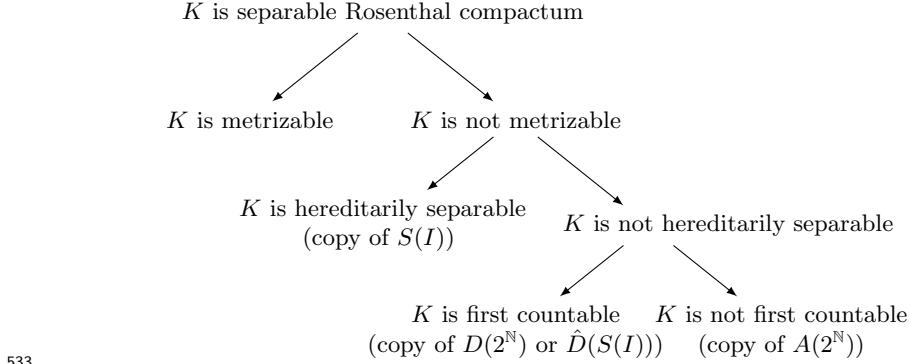
$$h_t(x) = \begin{cases} 0, & x < a_t \\ 1/2, & a_t \leq x \leq b_t \\ 1, & b_t < x \end{cases}$$

526 Let $\hat{D}(S(2^{\mathbb{N}}))$ be the pointwise closure of the set $\{h_t : t \in 2^{<\mathbb{N}}\}$. Hence,
 527 $\hat{D}(S(2^{\mathbb{N}}))$ is a separable first countable Rosenthal compactum which is not
 528 hereditarily separable. In fact, it contains an uncountable discrete subspace
 529 (see Theorem 5 in [Tod99]).

530 **Theorem 3.4** (Todorčević's Trichotomy, [Tod99], Theorem 3 in [ADK08]). *Let K
 531 be a separable Rosenthal Compactum.*

- 532 (i) *If K is hereditarily separable but non-metrizable, then $S(2^{\mathbb{N}})$ embeds into K .*
 533 (ii) *If K is first countable but not hereditarily separable, then either $D(2^{\mathbb{N}})$ or
 534 $\hat{D}(S(2^{\mathbb{N}}))$ embeds into K .*
 535 (iii) *If K is not first countable, then $A(2^{\mathbb{N}})$ embeds into K .*

536 We thus have the following classification:



533

534 The definitions provided here for NIP_i ($i = 1, 2, 3$) are topological. This raises
535 the following question:

536 **Question 3.5.** Is there a non-topological characterization for NIP_i , $i = 1, 2, 3$?

537 **3.3. The Argyros-Dodos-Kanellopoulos heptachotomy, and approxima-**
538 **tionability of deep computation by minimal classes.** In the three separable three
539 cases given in 3.3, namely, $(\hat{A}(2^N), S(2^N)$ and $\hat{D}(S(2^N))$), the countable dense sub-
540 sets are indexed by the binary tree $2^{<\mathbb{N}}$. This choice of index is useful for two
541 reasons:

- 542 (1) Our emphasis is computational. Elements of $2^{<\mathbb{N}}$ represent finite bitstrings,
543 i.e., standard computations, while Rosenthal compacta represent deep com-
544 putations, i.e., limits of finite computations. Mathematically, deep computa-
545 tions are pointwise limits of standard computations. However, computa-
546 tionally, we are interested in the manner (and the efficiency) in which the
547 approximations can occur.
- 548 (2) The Ramsey theory of perfect subsets of the Cantor space $2^{\mathbb{N}}$ can be im-
549 ported to analyze the behavior of the accumulation points. Since $2^{<\mathbb{N}}$ is
550 countable, we can always choose this index for the countable dense subsets.
551 This is done in [ADK08].

552 **Definition 3.6.** Let X be a Polish space.

- 553 (1) If I is a countable and $\{f_i : i \in I\} \subseteq \mathbb{R}^X$, $\{g_i : i \in I\} \subseteq \mathbb{R}^X$ are two
554 pointwise families by I , we say that $\{f_i : i \in I\}$ and $\{g_i : i \in I\}$ are
555 *equivalent* if and only if the map $f_i \mapsto g_i$ is extended to a homeomorphism
556 from $\overline{\{f_i : i \in I\}}$ to $\overline{\{g_i : i \in I\}}$.
- 557 (2) If $\{f_t : t \in 2^{<\mathbb{N}}\}$ is a pointwise bounded family, we say that $\{f_t : t \in 2^{<\mathbb{N}}\}$
558 is *minimal* if and only if for every dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$,
559 $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to $\{f_t : t \in 2^{<\mathbb{N}}\}$.

560 One of the main results in [ADK08] is that, up to equivalence, there are seven
561 minimal families of Rosenthal compacta and that for every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$ there is a dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$
562 is equivalent to one of the minimal families. We shall describe the seven minimal
563 families next. We follow the same notation as in [ADK08]. For any node $t \in 2^{<\mathbb{N}}$,
564 let us denote by $t^\frown 0^\infty$ ($t^\frown 1^\infty$) the infinite binary sequence starting with t and
565 continuing with all 0's (respectively, all 1's). Fix a regular dyadic subtree $R = \{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ (i.e., a dyadic subtree such that every level of R is contained

568 in a level of $2^{<\mathbb{N}}$) with the property that for all $s, s' \in R$, $s^\frown 0^\infty \neq s' \frown 0^\infty$ and
 569 $s^\frown 1^\infty \neq s' \frown 1^\infty$. Given $t \in 2^{<\mathbb{N}}$, let v_t be the characteristic function of the set
 570 $\{x \in 2^\mathbb{N} : x \text{ extends } t\}$. Let $<$ be the lexicographic order in $2^\mathbb{N}$. Given $a \in 2^\mathbb{N}$,
 571 let $f_a^+ : 2^\mathbb{N} \rightarrow \{0, 1\}$ be the characteristic function of $\{x \in 2^\mathbb{N} : a \leq x\}$ and let
 572 $f_a^- : 2^\mathbb{N} \rightarrow \{0, 1\}$ be the characteristic function of $\{x \in 2^\mathbb{N} : a < x\}$. Given two
 573 maps $f, g : 2^\mathbb{N} \rightarrow \mathbb{R}$ we denote by $(f, g) : 2^\mathbb{N} \sqcup 2^\mathbb{N} \rightarrow \mathbb{R}$ the function which is f on
 574 the first copy of $2^\mathbb{N}$ and g on the second copy of $2^\mathbb{N}$.

- 575 (1) $D_1 = \{\frac{1}{|t|+1} v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_1} = A(2^\mathbb{N})$.
- 576 (2) $D_2 = \{s_t^\frown 0^\infty : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_2} = 2^{\leq N}$.
- 577 (3) $D_3 = \{f_{s_t^\frown 0^\infty}^+ : t \in 2^{<\mathbb{N}}\}$. This is a discrete in $\overline{D_3} = S(2^\mathbb{N})$.
- 578 (4) $D_4 = \{f_{s_t^\frown 1^\infty}^- : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_4} = S(2^\mathbb{N})$.
- 579 (5) $D_5 = \{v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_5} = \hat{A}(2^\mathbb{N})$.
- 580 (6) $D_6 = \{(v_{s_t}, s_t^\frown 0^\infty) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_6} = \hat{D}(2^\mathbb{N})$.
- 581 (7) $D_7 = \{(v_{s_t}, f_{s_t^\frown 0^\infty}^+) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_7} = \hat{D}(S(2^\mathbb{N}))$

582 **Theorem 3.7** (Heptacotomy of minimal families, Theorem 2 in [ADK08]). *Let*
 583 *X be Polish. For every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$, there exists*
 584 *$i = 1, 2, \dots, 7$ and a regular dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to D_i . Moreover, all D_i are minimal and mutually non-equivalent.*

587 4. MEASURE-THEORETIC VERSIONS OF NIP AND UNIVERSAL MONTE CARLO 588 COMPUTABILITY OF DEEP COMPUTATIONS

589 The countability assumption on \mathcal{P} played a crucial role in the proof of Theorem
 590 1.12, as it makes $\mathbb{R}^\mathcal{P}$ a Polish space. In this section, we do not assume that \mathcal{P} is
 591 countable. We replace deterministic computability by measure-theoretic ('Monte
 592 Carlo') computability.

593 **4.1. A measure-theoretic version of NIP.** Recall that the *raison d'être* of the
 594 class of Baire-1 functions is to have a class that contains the continuous functions
 595 but is closed under pointwise limits, and that for perfectly normal X , a function
 596 f is in $B_1(X, Y)$ if and only if $f^{-1}[U]$ is an F_σ subset of X for every open $U \subseteq Y$
 597 (see Fact 1.2). This motivates the following definition:

598 **Definition 4.1.** Given a Hausdorff space X and a measurable space (Y, Σ) , we say
 599 that $f : X \rightarrow Y$ is *universally measurable* (with respect to Σ) if $f^{-1}(E)$ is Borel for
 600 every $E \in \Sigma$, i.e., $f^{-1}(E)$ is μ -measurable for every Radon measure μ on X . When
 601 $Y = \mathbb{R}$ we will always take $\Sigma = \mathcal{B}(\mathbb{R})$, the Borel σ -algebra of \mathbb{R} .

602 **Remark 4.2.** A function $f : X \rightarrow \mathbb{R}$ is universally measurable if and only if $f^{-1}(U)$
 603 is μ -measurable for every Radon probability measure μ on X and every open set
 604 $U \subseteq \mathbb{R}$.

605 Intuitively, a function is universally measurable if it is "measurable no matter
 606 which reasonable way you try to measure things on its domain". The concept of
 607 universal measurability emerged from work of Kallianpur and Sazonov, in the late
 608 1950's and 1960s, with later developments by Blackwell, Darst, and others, building
 609 on earlier ideas of Gnedenko and Kolmogorov from the 1950s. See [Pap02, Chapters
 610 1 and 2].

611 **Notation 4.3.** Following [BFT78], the collection of all universally measurable real-
 612 valued functions will be denoted by $M_r(X)$.

613 In the context of deep computations, we will be interested in transition maps of
 614 a state space $L \subseteq \mathbb{R}^P$ into itself. There are two natural σ -algebras one can consider
 615 in the product space \mathbb{R}^P : the Borel σ -algebra, i.e., the σ -algebra generated by open
 616 sets in \mathbb{R}^P , and the cylinder σ -algebra, i.e., the σ -algebra generated by the sub-basic
 617 open sets in \mathbb{R}^P . Note that when P is countable, both σ -algebras coincide, but in
 618 general the cylinder σ -algebra is strictly smaller. We will use the cylinder σ -algebra
 619 to define universally measurable maps $f : \mathbb{R}^P \rightarrow \mathbb{R}^P$. The reason for this choice is
 620 the following characterization:

621 **Lemma 4.4.** Let X be a Hausdorff space and $Y = \prod_{i \in I} Y_i$ be any product of
 622 measurable spaces (Y_i, Σ_i) for $i \in I$. Let Σ_Y be the cylinder σ -algebra generated by
 623 the measurable spaces (Y_i, Σ_i) . Let $f : X \rightarrow Y$. The following are equivalent:

- 624 (i) $f : X \rightarrow Y$ is universally measurable (with respect to Σ_Y).
- 625 (ii) $\pi_i \circ f : X \rightarrow Y_i$ is universally measurable (with respect to Σ_i) for all $i \in I$.

626 *Proof.* (i) \Rightarrow (ii) is clear since the projection maps π_i are measurable and the com-
 627 position of measurable functions is measurable. To prove (ii) \Rightarrow (i), suppose that
 628 $C = \prod_{i \in I} C_i$ is a measurable cylinder and let J be the finite set of $i \in I$ such that
 629 $C_i \neq Y_i$. Then, $C = \bigcap_{i \in J} \pi_i^{-1}(C_i)$, so $f^{-1}(C) = \bigcap_{i \in J} (\pi_i \circ f)^{-1}(C_i)$ is a universally
 630 measurable set by assumption. \square

631 The preceding lemma says that a transition map is universally measurable if and
 632 only if it is universally measurable on all its features; in other words, we can check
 633 measurability of a transition just by checking measurability feature by feature. We
 634 will denote by $M_r(X, \mathbb{R}^P)$ the collection of all universally measurable functions
 635 $f : X \rightarrow \mathbb{R}^P$ (with respect to the cylinder σ -algebra), endowed with the topology
 636 of pointwise convergence.

637 We will need the following result about NIP and universally measurable func-
 638 tions:

639 **Theorem 4.5** (Bourgain-Fremlin-Talagrand, Theorem 2F in [BFT78]). Let X be a
 640 Hausdorff space and $A \subseteq C(X)$ be pointwise bounded. The following are equivalent:

- 641 (i) $\overline{A} \subseteq M_r(X)$.
- 642 (ii) For every compact $K \subseteq X$, $A|_K$ satisfies the NIP.
- 643 (iii) For every Radon measure μ on X , A is relatively countably compact in
 644 $\mathcal{M}^0(X, \mu)$, i.e., every countable subset of A has an accumulation point in
 645 $\mathcal{M}^0(X, \mu)$.

646 **4.2. Universal Monte Carlo computability of deep computations.** We now
 647 wish to define the concept of a deep computation being computable except a set
 648 of arbitrarily small measure “no matter which reasonable way you try to measure
 649 things on its domain” (see the remarks following definition 4.1). This is the concept
 650 of *universal Monte Carlo computability* defined below (Definition 4.6). To motivate
 651 the definition, we need to recall two facts:

- 652 (1) Littlewood's second principle states that every Lebesgue measurable func-
 653 tion is “nearly continuous”. The formal version of this, which is Luzin's
 654 theorem, states that if (X, Σ, μ) a Radon measure space and Y be a second-
 655 countable topological space (e.g., $Y = \mathbb{R}^P$ with P countable) equipped with

a Borel algebra, then any given $f : X \rightarrow Y$ is measurable if and only if for every $E \in \Sigma$ and every $\varepsilon > 0$ there exists a closed $F \subseteq E$ such that the restriction $f|F$ is continuous.

(2) Computability of deep computations can be characterized in terms of continuous extendibility of computations. This is at the core of [ADIW24].

These two facts motivate the following definition:

Definition 4.6. Let (L, \mathcal{P}, Γ) be a CCS. We say that a transition $f : L \rightarrow L$ is *universally Monte Carlo computable* if and only if there exists $\tilde{f} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$ extending f such that for every sizer r_\bullet there is a sizer s_\bullet such that the restriction $\tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$ is universally measurable, i.e., $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow [-s_P, s_P]$ is μ -measurable for every Radon probability measure μ on $\mathcal{L}[r_\bullet]$.

4.3. Bourgain-Fremlin-Talagrand, NIP, and universal Monte Carlo computability of deep computations. Theorem 4.5 immediately yields the following.

Theorem 4.7. Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendability Axiom. Let R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. If $\pi_P \circ \Delta|_{L[r_\bullet]}$ satisfies the NIP for all $P \in \mathcal{P}$ and all $r_\bullet \in R$, then every deep computation is universally Monte Carlo computable.

Proof. By the Extendability Axiom, Theorem 4.5 and lemma 1.13 we have that $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$ and $P \in \mathcal{P}$. Let $f \in \overline{\Delta}$ be a deep computation. Write $f = \mathcal{U} \lim_i \gamma_i$ as an ultralimit of computations in Δ . Define $\tilde{f} := \mathcal{U} \lim_i \tilde{\gamma}_i$. Then, for all $r_\bullet \in R$ and $P \in \mathcal{P}$ $\pi_P \circ \tilde{\gamma}_i|_{\mathcal{L}[r_\bullet]} \in M_r(\mathcal{L}[r_\bullet])$ for all i , so $\pi_P \circ f|_{\mathcal{L}[r_\bullet]} \in \pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$. \square

Question 4.8. Under the same assumptions of the preceding theorem, suppose that every deep computation of Δ is universally Monte Carlo computable. Must $\pi_P \circ \Delta|_{L[r_\bullet]}$ have the NIP for all $P \in \mathcal{P}$ and all $r_\bullet \in R$?

4.4. Talagrand stability, Fremlin's dichotomy, NIP, and universal Monte Carlo computability of deep computations. There is another notion closely related to NIP, introduced by Talagrand in [Tal84] while studying Pettis integration. Suppose that X is a compact Hausdorff space and $A \subseteq \mathbb{R}^X$. Let μ be a Radon probability measure on X . Given a μ -measurable set $E \subseteq X$, a positive integer k and real numbers $a < b$. we write:

$$D_k(A, E, a, b) = \bigcup_{f \in A} \{x \in E^{2k} : f(x_{2i}) \leq a, f(x_{2i+1}) \geq b \text{ for all } i < k\}$$

We say that A is *Talagrand μ -stable* if and only if for every μ -measurable set $E \subseteq X$ of positive measure and for every $a < b$ there is $k \geq 1$ such that

$$(\mu^{2k})^*(D_k(A, E, a, b)) < (\mu(E))^{2k},$$

where μ^* denotes the outer measure (we work with outer since the sets $D_k(A, E, a, b)$ need not be μ -measurable). This is certainly the case when A is a countable set of continuous (or μ -measurable) functions.

Notation 4.9. For a measure μ on a set X , the set of all μ -measurable functions will denoted by $\mathcal{M}^0(X, \mu)$.

695 The following lemma establishes that Talagrand stability is a way to ensure that
 696 deep computations are definable by measurable functions.

697 **Lemma 4.10.** *If A is Talagrand μ -stable, then \bar{A} is also Talagrand μ -stable and
 698 $\bar{A} \subseteq \mathcal{M}^0(X, \mu)$.*

699 *Proof.* First, observe that a subset of a μ -stable set is μ -stable. To show that \bar{A} is
 700 μ -stable, observe that $D_k(\bar{A}, E, a, b) \subseteq D_k(A, E, a', b')$ where $a < a' < b' < b$ and E
 701 is a μ -measurable set with positive measure. It suffices to show that $\bar{A} \subseteq \mathcal{M}^0(X, \mu)$.
 702 Suppose that there exists $f \in \bar{A}$ such that $f \notin \mathcal{M}^0(X, \mu)$. By a characterization
 703 of measurable functions (see 413G in [Fre03]), there exists a μ -measurable set E
 704 of positive measure and $a < b$ such that $\mu^*(P) = \mu^*(Q) = \mu(E)$ where $P = \{x \in E : f(x) \leq a\}$ and $Q = \{x \in E : f(x) \geq b\}$. Then, for any $k \geq 1$: $(P \times Q)^k \subseteq D_k(\{f\}, E, a, b)$ so $(\mu^{2k})^*(D_k(\{f\}, E, a, b)) = (\mu^*(P)\mu^*(Q))^k = (\mu(E))^{2k}$. Thus,
 705 $\{f\}$ is not μ -stable, but we argued before that a subset of a μ -stable set must be
 706 μ -stable. \square

707 We say that A is *universally Talagrand stable* if A is Talagrand μ -stable for every
 708 Radon probability measure μ on X . An argument similar to the proof of 4.5, yields
 709 the following:

710 **Theorem 4.11.** *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendability Axiom. If
 711 $\pi_P \circ \Delta|_{L[r_\bullet]}$ is universally Talagrand stable for all $P \in \mathcal{P}$ and all sizers r_\bullet , then
 712 every deep computation is universally Monte Carlo computable.*

713 It is then natural to ask: what is the relationship between Talagrand stability
 714 and the NIP? The following dichotomy will be useful.

715 **Lemma 4.12** (Fremlin's Dichotomy, 463K in [Fre03]). *If (X, Σ, μ) is a perfect
 716 σ -finite measure space (in particular, for X compact and μ a Radon probability
 717 measure on X) and $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued measurable functions
 718 on X , then one (and only one) of the following conditions holds:*

- 719 (i) $\{f_n : n \in \mathbb{N}\}$ has a subsequence that converges μ -almost everywhere,
- 720 (ii) $\{f_n : n \in \mathbb{N}\}$ has a subsequence with no μ -measurable accumulation point
 721 in \mathbb{R}^X .

722 The preceding lemma can be considered as a measure-theoretic version of Rosen-
 723 thal's dichotomy. Combining this dichotomy with Theorem 4.5, we get the following
 724 result:

725 **Theorem 4.13.** *Let X be a Hausdorff space and $A \subseteq C(X)$ be pointwise bounded.
 726 The following are equivalent:*

- 727 (i) $\bar{A} \subseteq M_r(X)$.
- 728 (ii) For every compact $K \subseteq X$, $A|_K$ satisfies the NIP.
- 729 (iii) For every Radon measure μ on X , A is relatively countably compact in
 730 $\mathcal{M}^0(X, \mu)$, i.e., every countable subset of A has an accumulation point in
 731 $\mathcal{M}^0(X, \mu)$.
- 732 (iv) For every Radon measure μ on X and every sequence $\{f_n : n \in \mathbb{N}\} \subseteq A$,
 733 there is a subsequence that converges μ -almost everywhere.

734 *Proof.* Notice that the equivalence (i)-(iii) is Theorem 4.5. Notice that the equivalence
 735 of (iii) and (iv) is Fremlin's Dichotomy (Theorem 4.12). \square

Finally, it is natural to ask what the connection is between Talagrand stability and NIP.

Proposition 4.14. *Let X be a compact Hausdorff space and $A \subseteq C(X)$ be pointwise bounded. If A is universally Talagrand stable, then A satisfies the NIP.*

Proof. By Theorem 4.5, it suffices to show that A is relatively countably compact in $\mathcal{M}^0(X, \mu)$ for all Radon probability measure μ on X . Since A is Talagrand μ -stable for any such μ , we have $\bar{A} \subseteq \mathcal{M}^0(X, \mu)$. In particular, A is relatively countably compact in $\mathcal{M}^0(X, \mu)$. \square

Question 4.15. Is the converse true?

The following two results suggest that the precise connection between Talagrand stability and NIP may be sensitive to set-theoretic axioms (even assuming countability of A).

Theorem 4.16 (Talagrand, Theorem 9-3-1(a) in [Tal84]). *Let X be a compact Hausdorff space and $A \subseteq M_r(X)$ be countable and pointwise bounded. Assume that $[0, 1]$ is not the union of $< \mathfrak{c}$ closed measure zero sets. If A satisfies the NIP, then A is universally Talagrand stable.*

Theorem 4.17 (Fremlin, Shelah, [FS93]). *It is consistent that there exists a countable pointwise bounded set of Lebesgue measurable functions with the NIP which is not Talagrand stable with respect to Lebesgue measure.*

REFERENCES

- 758 [ADIW24] Samson Alva, Eduardo Dueñez, Jose Iovino, and Claire Walton. Approximability of deep equilibria. *arXiv preprint arXiv:2409.06064*, 2024. Last revised 20 May 2025, version 3.
- 761 [ADK08] Spiros A. Argyros, Pandelis Dodos, and Vassilis Kanellopoulos. A classification of separable Rosenthal compacta and its applications. *Dissertationes Mathematicae*, 449:1–55, 2008.
- 764 [Ark91] A. V. Arkhangel'skii. *Topological Function Spaces*. Springer, New York, 1st edition, 1991.
- 766 [BD19] Shai Ben-David. Understanding machine learning through the lens of model theory. *The Bulletin of Symbolic Logic*, 25(3):319–340, 2019.
- 768 [BFT78] J. Bourgain, D. H. Fremlin, and M. Talagrand. Pointwise compact sets of Baire-measurable functions. *American Journal of Mathematics*, 100(4):845–886, 1978.
- 770 [BKK] Shaojie Bai, J. Zico Kolter, and Vladlen Koltun. Deep equilibrium models. Preprint. <https://arxiv.org/abs/1909.01377>, <https://implicit-layers-tutorial.org/>.
- 772 [CRBD] Ricky T. Q. Chen, Yulia Rubanova, Jesse Bettencourt, and David Duvenaud. Neural ordinary differential equations. Preprint. <https://arxiv.org/abs/1806.07366>.
- 774 [Deb13] Gabriel Debs. Descriptive aspects of Rosenthal compacta. In *Recent Progress in General Topology III*, pages 205–227. Springer, 2013.
- 776 [Fre00] David H. Fremlin. *Measure Theory, Volume 1: The Irreducible Minimum*. Torres Fremlin, Colchester, UK, 2000. Second edition 2011.
- 778 [Fre01] David H. Fremlin. *Measure Theory, Volume 2: Broad Foundations*. Torres Fremlin, Colchester, UK, 2001.
- 780 [Fre03] David H. Fremlin. *Measure Theory, Volume 4: Topological Measure Spaces*. Torres Fremlin, Colchester, UK, 2003.
- 782 [FS93] David H. Fremlin and Saharon Shelah. Pointwise compact and stable sets of measurable functions. *Journal of Symbolic Logic*, 58(2):435–455, 1993.
- 784 [GM22] Eli Glasner and Michael Megrelishvili. Todorčević' trichotomy and a hierarchy in the class of tame dynamical systems. *Transactions of the American Mathematical Society*, 375(7):4513–4548, 2022.

- 787 [God80] Gilles Godefroy. Compacts de Rosenthal. *Pacific J. Math.*, 91(2):293–306, 1980.
 788 [Gro52] A. Grothendieck. Critères de compacité dans les espaces fonctionnels généraux. *Amer. J. Math.*, 74:168–186, 1952.
 789 [HT23] Clovis Hamel and Franklin D. Tall. C_p -theory for model theorists. In Jose Iovino, editor, *Beyond First Order Model Theory, Volume II*, chapter 5, pages 176–213. Chapman and Hall/CRC, 2023.
 790 [Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995.
 791 [Kha20] Karim Khanaki. Stability, *nip*, and *nsop*; model theoretic properties of formulas via
 792 topological properties of function spaces. *Mathematical Logic Quarterly*, 66(2):136–
 793 149, 2020.
 794 [Pap02] E. Pap, editor. *Handbook of measure theory. Vol. I, II*. North-Holland, Amsterdam,
 795 2002.
 796 [Ros74] Haskell P. Rosenthal. A characterization of Banach spaces containing l^1 . *Proc. Nat.
 797 Acad. Sci. U.S.A.*, 71:2411–2413, 1974.
 798 [RPK19] M. Raissi, P. Perdikaris, and G. E. Karniadakis. Physics-informed neural networks: a
 799 deep learning framework for solving forward and inverse problems involving nonlinear
 800 partial differential equations. *J. Comput. Phys.*, 378:686–707, 2019.
 801 [Sau72] N. Sauer. On the density of families of sets. *J. Combinatorial Theory Ser. A*, 13:145–
 802 147, 1972.
 803 [She71] Saharon Shelah. Stability, the f.c.p., and superstability; model theoretic properties of
 804 formulas in first order theory. *Ann. Math. Logic*, 3(3):271–362, 1971.
 805 [She72] Saharon Shelah. A combinatorial problem; stability and order for models and theories
 806 in infinitary languages. *Pacific J. Math.*, 41:247–261, 1972.
 807 [She78] Saharon Shelah. *Unstable Theories*, volume 187 of *Studies in Logic and the Foundations
 808 of Mathematics*. North-Holland, 1978.
 809 [She90] Saharon Shelah. *Classification theory and the number of nonisomorphic models*, vol-
 810 ume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Pub-
 811 lishing Co., Amsterdam, second edition, 1990.
 812 [Sim15a] Pierre Simon. *A Guide to NIP Theories*, volume 44 of *Lecture Notes in Logic*. Cam-
 813 bridge University Press, 2015.
 814 [Sim15b] Pierre Simon. Rosenthal compacta and NIP formulas. *Fundamenta Mathematicae*,
 815 231(1):81–92, 2015.
 816 [Tal84] Michel Talagrand. *Pettis Integral and Measure Theory*, volume 51 of *Memoirs of
 817 the American Mathematical Society*. American Mathematical Society, Providence, RI,
 818 USA, 1984. Includes bibliography (pp. 220–224) and index.
 819 [Tal87] Michel Talagrand. The Glivenko-Cantelli problem. *The Annals of Probability*,
 820 15(3):837–870, 1987.
 821 [Tka11] Vladimir V. Tkachuk. *A C_p -Theory Problem Book: Topological and Function Spaces*.
 822 Problem Books in Mathematics. Springer, 2011.
 823 [Tod97] Stevo Todorcevic. *Topics in Topology*, volume 1652 of *Lecture Notes in Mathematics*.
 824 Springer Berlin, Heidelberg, 1997.
 825 [Tod99] Stevo Todorcevic. Compact subsets of the first Baire class. *Journal of the American
 826 Mathematical Society*, 12(4):1179–1212, 1999.
 827 [Val84] Leslie G. Valiant. A theory of the learnable. *Communications of the ACM*,
 828 27(11):1134–1142, 1984.
 829 [VC71] Vladimir N Vapnik and Alexey Ya Chervonenkis. On the uniform convergence of rel-
 830 ative frequencies of events to their probabilities. *Theory of Probability & Its Applica-
 831 tions*, 16(2):264–280, 1971.
 832 [VC74] Vladimir N Vapnik and Alexey Ya Chervonenkis. *Theory of Pattern Recognition*.
 833 Nauka, Moscow, 1974. German Translation: Theorie der Zeichenerkennung, Akademie-
 834 Verlag, Berlin, 1979.
 835 [WYP22] Sifan Wang, Xinling Yu, and Paris Perdikaris. When and why PINNs fail to train: a
 836 neural tangent kernel perspective. *J. Comput. Phys.*, 449:Paper No. 110768, 28, 2022.
 837
 838
 839
 840