

COMPLEXITY OF DEEP COMPUTATIONS VIA TOPOLOGY OF FUNCTION SPACES

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ABSTRACT. This paper revisits and extends a bridge between functional analysis and model theory, emphasizing its relevance to the theoretical foundations of machine learning. We show that the compactness behavior of families of Baire class 1 functions mirrors the learnability conditions in the sense of *Probably Approximately Correct* (PAC) learning, and that the failure of compactness corresponds to the presence of infinite *Vapnik-Chervonenkis* (VC) dimension. From this perspective, Rosenthal compacta emerge as the natural topological counterpart of PAC-learnable concept classes, while NIP vs. IP structures capture the precise boundary between analytical regularity and combinatorial intractability. These parallels suggest a unified framework linking compactness, definability, and learnability, exemplifying how the topology of function spaces encodes the algorithmic and epistemic limits of prediction.

1. INTRODUCTION

In this paper we study limit behavior of real-valued computations as the value of certain parameters of the computation model tend towards infinity, or towards zero, or towards some other fixed value, e.g., the depth of a neural network tending to infinity, or the time interval between layers of the network tending toward zero. Recently, particular cases of this situation have attracted considerable attention in deep learning research (e.g., Neural Ordinary Differential Equations [CRBD], Physics-Informed Neural Networks [RPK19], deep equilibrium models [BKK], etc.). In this paper, we combine ideas of topology and model theory to study these limit phenomena from a unified viewpoint.

Informed by model theory, to each computation in a given computation model, we associate a continuous real-valued function, called the *type* of the computation, that describes the logical properties of this computation with respect to the rest of the model. This allows us to view computations in any given computational model as elements of a space of real-valued functions, which is called the *space of types* of the model. The idea of embedding models of theories into their type spaces is central in model theory. The embedding of computations into spaces of types allows us to utilize the vast theory of topology of function spaces, known as C_p -theory, to obtain results about complexity of topological limits of computations. As we shall indicate next, recent classification results for spaces of functions provide an elegant and powerful machinery to classify computations according to their levels of “tameness” or “wildness”, with the former corresponding roughly to polynomial approximability and the latter to exponential approximability. The viewpoint of spaces of types, which we have borrowed from model theory, thus becomes a “Rosetta stone” that allows us to interconnect various classification programs: In

³² topology, the classification of Rosenthal compacta pioneered by Todorčević [Tod99];
³³ in logic, the classification of theories developed by Shelah [She90]; and in statistical
³⁴ learning, the notion PAC learning and VC dimension pioneered by Vapkins and
³⁵ Chervonenkis [VC74, VC71].

³⁶ In a previous paper [ADIW24], we introduced the concept of limits of computations,
³⁷ which we called *ultracomputations* (given they arise as ultrafilter limits of
³⁸ standard computations) and *deep computations* (following usage in machine learn-
³⁹ ing [BKK]). There is a technical difference between both designations, but in this
⁴⁰ paper, to simplify the nomenclature, we will ignore the difference and use only the
⁴¹ term “deep computation”.

⁴² In [ADIW24], we proved a new “tame vs wild” (i.e., polynomial vs exponential)
⁴³ dichotomy for complexity of deep computations by invoking a classical result of
⁴⁴ Grothendieck from late 50s [Gro52]. Under our model-theoretic Rosetta stone,
⁴⁵ polynomial approximability in the sense of computation becomes identified with the
⁴⁶ notion of continuous extendability in the sense of topology, and with the notions of
⁴⁷ *stability* and *type definability* in the sense of model theory.

⁴⁸ In this paper, we follow a more general approach, i.e., we view deep computations
⁴⁹ as pointwise limits of continuous functions. In topology, real-valued functions that
⁵⁰ arise as the pointwise limit of a sequence of continuous are called *functions of the*
⁵¹ *first Baire class*, or *Baire class 1* functions, or *Baire-1* for short; Baire class 1 form
⁵² a step above simple continuity in the hierarchy of functions studied in real analysis
⁵³ (Baire class 0 functions being continuous functions). Intuitively, Baire-1 functions
⁵⁴ represent functions with “controlled” discontinuities, so they are crucial in topology
⁵⁵ and set theory.

⁵⁶ We prove a new “tame vs wild” Ramsey-theoretic dichotomy for complexity of
⁵⁷ general deep computations by invoking a famous paper by Bourgain, Fremlin and
⁵⁸ Talagrand from the late 70s [BFT78], and a new trichotomy for the class of “tame”
⁵⁹ deep computations by invoking an equally celebrated result of Todorčević, from the
⁶⁰ late 90s, for functions of the first Baire class [Tod99].

⁶¹ Todorčević’s trichotomy regards *Rosenthal compacta*; these are special classes of
⁶² topological spaces, defined as compact spaces that can be embedded (homeomor-
⁶³ phically identified as a subset) within the space of Baire class 1 functions on some
⁶⁴ Polish (separable, complete metric) space, under the pointwise convergence topol-
⁶⁵ ogy. Rosenthal compacta exhibit “topological tameness,” meaning they behave in
⁶⁶ relatively controlled ways, and since the late 70’s, they have played a crucial role
⁶⁷ for understanding complexity of structures of functional analysis, especially, Banach
⁶⁸ spaces. Todorčević’s trichotomy has been utilized to settle longstanding problems
⁶⁹ in topological dynamics and topological entriopy [GM22].

⁷⁰ Through our Rosetta stone, Rosenthal compacta in topology correspond to the
⁷¹ important concept of “No Independence Property” (known as “NIP”) in model
⁷² theory, identified by Shelah [She71, She90], and to the concept of Probably Ap-
⁷³ proximately Correct learning (known as “PAC learnability”) in statistical learning
⁷⁴ theory identified by Valiant [Val84].

⁷⁵ Going beyond Todorčević’s trichotomy, we invoke a more recent heptachotomy
⁷⁶ for Rosenthal compacta obtained by Argyros, Dodos and Kanellopoulos [ADK08].
⁷⁷ Argyros, Dodos and Kanellopoulos identified the fundamental “prototypes” of sepa-
⁷⁸ rable Rosenthal compacta, and proved that any non-metrizable separable Rosenthal
⁷⁹ compactum must contain a “canonical” embedding of one of these prototypes. They

80 showed that if a separable Rosenthal compactum is not hereditarily separable, it
 81 must contain an uncountable discrete subspace of the size of the continuum.

82 We believe that the results presented in this paper show practitioners of com-
 83 putation, or topology, or descriptive set theory, or model theory, how classification
 84 invariants used in their field translate into classification invariants of other fields.
 85 However, in the interest of accessibility, we do not assume previous familiarity with
 86 high-level topology or model theory, or computing. The only technical prerequisite
 87 of the paper is undergraduate-level topology. The necessary topological background
 88 beyond undergraduate topology is covered in section 2.

89 Throughout the paper, we focus on classical computation; however, by refining
 90 the model-theoretic tools, the results presented here can be extended to quantum
 91 computation and open quantum systems. This extension will be addressed in a
 92 forthcoming paper.

93 2. GENERAL TOPOLOGICAL PRELIMINARIES

94 In this section we give preliminaries from general topology and function space
 95 theory. We include some of the proofs for completeness, but the reader familiar
 96 with these topics may skip them.

97 Recall that a subset of a topological space is F_σ if it is a countable union of
 98 closed sets, and G_δ if it is a countable intersection of closed sets. Note that in a
 99 metrizable space, every open set is F_σ ; equivalently, every closed set is G_δ .

100 A *Polish space* is a separable and completely metrizable topological space. The
 101 most important examples are the reals \mathbb{R} , the Cantor space $2^\mathbb{N}$ (the set of all infinite
 102 binary sequences, endowed with the product topology), and the Baire space $\mathbb{N}^\mathbb{N}$ (the
 103 set of all infinite sequences of naturals, also with the product topology). Countable
 104 products of Polish spaces are Polish; this includes spaces like $\mathbb{R}^\mathbb{N}$, the space of
 105 sequences of real numbers.

106 In this paper, we shall discuss subspaces, and so there is a pertinent subtlety of
 107 the definitions worth mentioning: *completely metrizable space* is not the same as
 108 *complete metric space*; for an illustrative example, the interval $(0, 1)$ with the metric
 109 inherited from the reals is not complete, but it is Polish since that is homeomorphic
 110 to the real line. Being Polish is a topological property.

111 The following result is a cornerstone of descriptive set theory, closely tied to the
 112 work of Waclaw Sierpiński and Kazimierz Kuratowski, with proofs often attributed
 113 to or built upon their foundations and formalized later, notably involving Stefan
 114 Mazurkiewicz's work on complete metric spaces.

115 **Fact 2.1.** *A subset A of a Polish space X is itself Polish in the subspace topology
 116 if and only if it is a G_δ set. In particular, closed subsets and open subsets of Polish
 117 spaces are also Polish spaces.*

118 Given two topological spaces X and Y we denote by $C_p(X, Y)$ the set of all
 119 continuous functions $f : X \rightarrow Y$ endowed with the topology of pointwise convergence.
 120 When $Y = \mathbb{R}$, we denote this collection simply as $C_p(X)$. A natural question is, how
 121 do topological properties of X translate to $C_p(X)$ and vice versa? These questions,
 122 and in general the study of these spaces, are the concern of C_p -theory, an active
 123 field of research in general topology which was pioneered by A. V. Arhangel'skiĭ
 124 and his students in the 1970's and 1980's. This field has found many applications in
 125 model theory and functional analysis. Recent surveys on the topics include [HT23]
 126 and [Tka11].

127 A *Baire class 1* function between topological spaces is a function that can be
 128 expressed as the pointwise limit of a sequence of continuous functions. If X and Y
 129 are topological spaces, the Baire class 1 functions $f : X \rightarrow Y$ endowed with the
 130 topology of pointwise convergence is denoted $B_1(X, Y)$. As above, in the special
 131 case $Y = \mathbb{R}$ we denote $B_1(X, Y)$ as $B_1(X)$. Clearly, $C_p(X, Y) \subseteq B_1(X, Y)$. The Baire
 132 hierarchy of functions was introduced by French mathematician René-Louis Baire
 133 in his 1899 doctoral thesis, *Sur les fonctions de variables réelles*. His work moved
 134 away from the 19th-century preoccupation with "pathological" functions toward a
 135 constructive classification based on pointwise limits.

136 A topological space X is *perfectly normal* if it is normal and every closed subset of
 137 X is a G_δ (equivalently, every open subset of X is a G_δ). Note that every metrizable
 138 space is perfectly normal.

139 The following fact was established by Baire in thesis. A proof can be found in
 140 Section 10 of [Tod97].

141 **Fact 2.2** (Baire). *If X is perfectly normal, then the following conditions are equivalent for a function $f : X \rightarrow \mathbb{R}$:*

- 143 • *f is a Baire class 1 function, that is, $f \in B_1(X)$.*
- 144 • *$f^{-1}[U]$ is an F_σ subset of X whenever $U \subseteq \mathbb{R}$ is open.*
- 145 • *f is a pointwise limit of continuous functions.*
- 146 • *For every closed $F \subseteq X$, the restriction $f|_F$ has a point of continuity.*

147 Moreover, if X is Polish and $f \notin B_1(X)$, then there exists countable $D_0, D_1 \subseteq X$ and
 148 reals $a < b$ such that

$$D_0 \subseteq f^{-1}(-\infty, a], \quad D_1 \subseteq f^{-1}[b, \infty), \quad \overline{D_0} = \overline{D_1}.$$

149 A subset L of a topological space X is *relatively compact* in X if the closure of
 150 L in X is compact. Relatively compact subsets of $B_1(X)$ (for X Polish space) have
 151 been objects of interest for researchers in Analysis and Topological Dynamics. We
 152 begin with the following well-known result. Recall that a set $A \subseteq \mathbb{R}^X$ of real-
 153 valued functions is *pointwise bounded* if for every $x \in X$ there is $M_x > 0$ such that
 154 $|f(x)| < M_x$ for all $f \in A$. We include a proof for the reader's convenience:

155 **Lemma 2.3.** *Let X be a Polish space and $A \subseteq B_1(X)$ be pointwise bounded. The
 156 following are equivalent:*

- 157 (i) *A is relatively compact in $B_1(X)$.*
- 158 (ii) *A is relatively countably compact in $B_1(X)$, i.e., every countable subset of
 159 A has an accumulation point in $B_1(X)$.*
- 160 (iii) *$\overline{A} \subseteq B_1(X)$, where \overline{A} denotes the closure in \mathbb{R}^X .*

161 *Proof.* Since A is pointwise bounded, for each $x \in X$, fix $M_x > 0$ such that $|f(x)| \leq
 162 M_x$ for every $f \in A$.

163 (i) \Rightarrow (ii) holds in general.

164 (ii) \Rightarrow (iii) Assume that A is relatively countably compact in $B_1(X)$ and that
 165 $f \in \overline{A} \setminus B_1(X)$. By Fact 2.2, there are countable $D_0, D_1 \subseteq X$ with $\overline{D_0} = \overline{D_1}$, and
 166 $a < b$ such that $D_0 \subseteq f^{-1}(-\infty, a]$ and $D_1 \subseteq f^{-1}[b, \infty)$. We claim that there is a
 167 sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in D_0 \cup D_1$. Indeed,
 168 use the countability to enumerate $D_0 \cup D_1$ as $\{x_n\}_{n \in \mathbb{N}}$. Then for each positive n
 169 find $f_n \in A$ with $|f_n(x_i) - f(x_i)| < \frac{1}{n}$ for all $i \leq n$. The claim follows.

170 By relative countable compactness of A , there is an accumulation point $g \in
 171 B_1(X)$ of $\{f_n\}_{n \in \mathbb{N}}$. It is straightforward to show that since f and g agree on $D_0 \cup D_1$,

172 g does not have a point of continuity on the closed set $\overline{D_0} = \overline{D_1}$, which contradicts
 173 Fact 2.2.

174 (iii) \Rightarrow (i) Suppose that $\overline{A} \subseteq B_1(X)$. Then $\overline{A} \cap B_1(X) = \overline{A}$ is a closed subset of
 175 $\prod_{x \in X} [-M_x, M_x]$; Tychonoff's theorem states that the product of compact spaces
 176 is always compact, and since closed subsets of compact spaces are compact, \overline{A} must
 177 be compact, as desired. \square

178 **2.1. From Rosenthal's dichotomy to NIP.** The fundamental idea that
 179 connects the rich theory here presented to real-valued computations is the concept
 180 of an *approximation*. In the reals, points of closure from some subset can always
 181 be approximated by points inside the set, via a convergent sequence. For more
 182 complicated spaces, such as $C_p(X)$, this fails in a remarkably intriguing way. Let
 183 us show an example that is actually the protagonist of a celebrated result. Con-
 184 sider the Cantor space $X = 2^{\mathbb{N}}$ and let $p_n(x) = x(n)$ define a continuous mapping
 185 $X \rightarrow \{0, 1\}$. Then one can show (see Chapter 1.1 of [Tod97] for details) that, per-
 186 haps surprisingly, the only continuous functions in the closure of $\{p_n\}_{n \in \mathbb{N}}$ are the
 187 functions p_n themselves; moreover, none of the subsequences of $\{p_n\}_{n \in \mathbb{N}}$ converge.
 188 In some sense, this example is the worst possible scenario for convergence. The
 189 topological space obtained from this closure is well-known. Topologists refer to it
 190 as the Stone-Čech compactification of the discrete space of natural numbers, or $\beta\mathbb{N}$
 191 for short, and it is an important object of study in general topology.

192 **Theorem 2.4** (Rosenthal's Dichotomy). *If X is Polish and $\{f_n\} \subseteq C_p(X)$ is point-
 193 wise bounded, then either $\{f_n\}_{n \in \mathbb{N}}$ contains a convergent subsequence or a subse-
 194 quence whose closure (in \mathbb{R}^X) is homeomorphic to $\beta\mathbb{N}$.*

195 In other words, a pointwise bounded set of continuous functions will either con-
 196 tain a subsequence that converges or a subsequence whose closure is essentially
 197 the same as the example mentioned in the previous paragraphs (the worst possible
 198 scenario). Note that in the preceding example, the functions are trivially pointwise
 199 bounded in \mathbb{R}^X as the functions can only take values 0 and 1.

200 If we intend to generalize our results from $C_p(X)$ to the bigger space $B_1(X)$, we
 201 find a similar dichotomy. Either every point of closure of the set of functions will
 202 be a Baire class 1 function, or there is a sequence inside the set that behaves in the
 203 worst possible way (which in this context, is the IP!). The theorem is usually not
 204 phrased as a dichotomy but rather as an equivalence (with the NIP instead):

205 **Theorem 2.5** (Bourgain-Fremlin-Talagrand, Theorem 4G in [BFT78]). *Let X be
 206 a Polish space and $A \subseteq C_p(X)$ be pointwise bounded. The following are equivalent:*

- 207 (i) A is relatively compact in $B_1(X)$, i.e., $\overline{A} \subseteq B_1(X)$.
- 207 (ii) For every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and for every $a < b$ there is $I \subseteq \mathbb{N}$ such that

$$\bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} f_n^{-1}[b, \infty) = \emptyset.$$

208 Our goal now is to characterize relatively compact subsets of $B_1(X, Y)$ when
 209 $Y = \mathbb{R}^P$ with P countable. Given $P \in \mathcal{P}$ we denote the *projection map* onto the
 210 P -coordinate by $\pi_P : \mathbb{R}^P \rightarrow \mathbb{R}$. From a high-level topological interpretation, the
 211 subsequent lemma states that, in this context, the spaces \mathbb{R} and \mathbb{R}^P are really not
 212 that different, and that if we understand the Baire class 1 functions of one space,
 213 then we also understand the functions of both. In fact, \mathbb{R} and any other Polish
 214 space is embeddable as a closed subspace of \mathbb{R}^P .

215 **Lemma 2.6.** *Let X be a Polish space and \mathcal{P} be a countable set. Then, $f \in B_1(X, \mathbb{R}^{\mathcal{P}})$*
216 *if and only if $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$.*

Proof. Only one implication needs a proof. Suppose that $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$. Let V be a basic open subset of $\mathbb{R}^{\mathcal{P}}$. That is, there exists a finite $\mathcal{P}' \subseteq \mathcal{P}$ such that $V = \bigcap_{P \in \mathcal{P}'} \pi_P^{-1}[U_P]$ where U_P is open in \mathbb{R} . Finally,

$$f^{-1}[V] = \bigcap_{P \in \mathcal{P}'} (\pi_P \circ f)^{-1}[U_P]$$

217 is an F_σ set. Since \mathcal{P} is countable, $\mathbb{R}^{\mathcal{P}}$ is second countable so every open set U in
218 $\mathbb{R}^{\mathcal{P}}$ is a countable union of basic open sets. Hence, $f^{-1}[U]$ is F_σ . \square

219 Below we consider \mathcal{P} with the discrete topology. For each $f : X \rightarrow \mathbb{R}^{\mathcal{P}}$ denote
220 $\hat{f}(P, x) := \pi_P \circ f(x)$ for all $(P, x) \in \mathcal{P} \times X$. Similarly, for each $g : \mathcal{P} \times X \rightarrow \mathbb{R}$ denote
221 $\check{g}(x)(P) := g(P, x)$. Given $A \subseteq (\mathbb{R}^{\mathcal{P}})^X$, we denote \hat{A} as the set of all \hat{f} such that
222 $f \in A$.

223 The map $(\mathbb{R}^{\mathcal{P}})^X \rightarrow \mathbb{R}^{\mathcal{P} \times X}$ given by $f \mapsto \hat{f}$ is a homeomorphism and its inverse is
224 given by $g \mapsto \check{g}$.

225 **Lemma 2.7.** *Let X be a Polish space and \mathcal{P} be countable. Then, $f \in B_1(X, \mathbb{R}^{\mathcal{P}})$ if*
226 *and only if $\hat{f} \in B_1(\mathcal{P} \times X)$.*

Proof. (\Rightarrow) Given an open set of reals U , we have that for every $P \in \mathcal{P}$, $f^{-1}[\pi_P^{-1}[U]]$ is F_σ by Lemma 2.6. Given that \mathcal{P} is a discrete countable space, we observe that

$$\hat{f}^{-1}[U] = \bigcup_{P \in \mathcal{P}} (\{P\} \times f^{-1}[\pi_P^{-1}[U]])$$

is also an F_σ set. (\Leftarrow) By lemma 2.6 it suffices to show that $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$. Fix an open $U \subseteq \mathbb{R}$. Write $\hat{f}^{-1}[U] = \bigcup_{n \in \mathbb{N}} F_n$ where F_n is closed in $\mathcal{P} \times X$. Then,

$$(\pi_P \circ f)^{-1}[U] = \bigcup_{n \in \mathbb{N}} \{x \in X : (P, x) \in F_n\}$$

227 which is F_σ . \square

228 We now direct our attention to a notion of the NIP that is more general than
229 the one from the introduction. It can be interpreted as a sort of continuous version
230 of the one presented in the preceding section.

Definition 2.8. We say that $A \subseteq \mathbb{R}^X$ has the *Non-Independence Property* (NIP) if and only if for every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and for every $a < b$ there are finite disjoint sets $E, F \subseteq \mathbb{N}$ such that

$$\bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) = \emptyset.$$

Note that if X is compact and $A \subseteq C_p(X)$, then A has the NIP if and only if for every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and for every $a < b$ there is $I \subseteq \mathbb{N}$ such that

$$\bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} f_n^{-1}[b, \infty) = \emptyset.$$

231 Given $A \subseteq Y^X$ and $K \subseteq X$ we write $A|_K := \{f|_K : f \in A\}$, i.e., the set of
232 all restrictions of functions in A to K . The following Theorem is a slightly more
233 general version of Theorem 2.5.

234 **Theorem 2.9.** Assume that \mathcal{P} is countable, X is a Polish space, and $A \subseteq C_p(X, \mathbb{R}^{\mathcal{P}})$
235 is such that $\pi_P \circ A$ is pointwise bounded for all $P \in \mathcal{P}$. The following are equivalent
236 for every compact $K \subseteq X$:

- 237 (1) $\overline{A|_K} \subseteq B_1(K, \mathbb{R}^{\mathcal{P}})$.
238 (2) $\pi_P \circ A|_K$ has the NIP for every $P \in \mathcal{P}$.

Proof. (1) \Rightarrow (2). Let $P \in \mathcal{P}$. Fix $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$. By (1) we have that $\overline{A|_K} \subseteq B_1(K, \mathbb{R}^{\mathcal{P}})$. Applying the homeomorphism $f \mapsto \hat{f}$ and using lemma 2.7 we get $\overline{\hat{A}|_{\mathcal{P} \times K}} \subseteq B_1(\mathcal{P} \times K)$. By Theorem 2.5, there is $I \subseteq \mathbb{N}$ such that

$$(\mathcal{P} \times K) \cap \bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \notin I} f_n^{-1}[b, \infty) = \emptyset$$

Hence,

$$K \cap \bigcap_{n \in I} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n \notin I} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

By compactness, there are finite $E \subseteq I$ and $F \subseteq \mathbb{N} \setminus I$ such that

$$K \cap \bigcap_{n \in E} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n \in F} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

239 Thus, $\pi_P \circ A|_L$ has the NIP.

240 (2) \Rightarrow (1) Fix $f \in \overline{A|_K}$. By lemma 2.6 it suffices to show that $\pi_P \circ f \in B_1(K)$
241 for all $P \in \mathcal{P}$. By (2), $\pi_P \circ A|_K$ has the NIP. Hence, by Theorem 2.5 we have
242 $\pi_P \circ A|_K \subseteq B_1(K)$. But then $\pi_P \circ f \in \overline{\pi_P \circ A|_K} \subseteq B_1(K)$. \square

243 Lastly, a simple but significant result that helps understand the operation of
244 restricting a set of functions to a specific subspace of the domain space X , of course
245 in the context of the NIP, is that we may always assume that said subspace is
246 closed. Concretely, whether we take its closure or not has no effect on the NIP:

247 **Lemma 2.10.** Assume that X is Hausdorff and that $A \subseteq C_p(X)$. The following
248 are equivalent for every $L \subseteq X$:

- 249 (i) A_L has the NIP.
250 (ii) $A|_{\bar{L}}$ has the NIP.

Proof. It suffices to show that (i) \Rightarrow (ii). Suppose that (ii) does not hold, i.e., that there are $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$ such that for all finite disjoint $E, F \subseteq \mathbb{N}$:

$$\bar{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

Pick $a' < b'$ such that $a < a' < b' < b$. Then, for any finite disjoint $E, F \subseteq \mathbb{N}$ we can choose

$$x \in \bar{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a') \cap \bigcap_{n \in F} f_n^{-1}(b', \infty)$$

By definition of closure:

$$L \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a') \cap \bigcap_{n \in F} f_n^{-1}(b', \infty) \neq \emptyset.$$

251 This contradicts (i). \square

252 3. NIP IN THE CONTEXT OF COMPOSITIONAL COMPUTATION STRUCTURES

253 In this section, we study what the NIP tell us in the context of deep computations as defined in [ADIW24]. We say a structure (L, \mathcal{P}, Γ) is a *Compositional Computation Structure* (CCS) if $L \subseteq \mathbb{R}^{\mathcal{P}}$ is a subspace of $\mathbb{R}^{\mathcal{P}}$, with the pointwise convergence topology, and $\Gamma \subseteq L^L$ is a semigroup under composition. The motivation for CCS comes from (continuous) model theory, where \mathcal{P} is a fixed collection of predicates and L is a (real-valued) structure. Every point in L is identified with its “type”, which is the tuple of all values the point takes on the predicates from \mathcal{P} , i.e., an element of $\mathbb{R}^{\mathcal{P}}$. In this context, elements of \mathcal{P} are called *features*. In the discrete model theory framework, one views the space of complete-types as a sort of compactification of the structure L . In this context, we don’t want to consider only points in L (realized types) but in its closure \bar{L} (possibly unrealized types). The problem is that the closure \bar{L} is not necessarily compact, an assumption that turns out to be very useful in the context of continuous model theory. To bypass this problem in a framework for deep computations, Alva, Dueñez, Iovino and Walton introduced in [ADIW24] the concept of *shards*, which essentially consists in covering (a large fragment) of the space \bar{L} by compact, and hence pointwise-bounded, subspaces (shards). We shall give the formal definition next.

270 A *sizer* is a tuple $r_{\bullet} = (r_p)_{p \in \mathcal{P}}$ of positive real numbers indexed by \mathcal{P} . Given a sizer r_{\bullet} , we define the r_{\bullet} -*shard* as:

$$L[r_{\bullet}] = L \cap \prod_{p \in \mathcal{P}} [-r_p, r_p]$$

272 For an illustrative example, we can frame Newton’s polynomial root approximation method in the context of a CCS (see Example 5.6 of [ADIW24] for details) as follows. Begin by considering the extended complex numbers $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with the usual Riemann sphere topology that makes it into a compact space (where unbounded sequences converge to ∞). In fact, not only is this space compact but it is covered by the shard given by the sizer $(1, 1, 1)$ (the unit sphere is contained in the cube $[-1, 1]^3$). The space $\hat{\mathbb{C}}$ is homeomorphic to the usual unit sphere $S^2 := \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ of \mathbb{R}^3 , by means of the stereographic projection and its inverse $\hat{\mathbb{C}} \rightarrow S^2$. This function is regarded as a triple of predicates $x, y, z : \hat{\mathbb{C}} \rightarrow [-1, 1]$ where each will map an extended complex number to its corresponding real coordinate on the cube $[-1, 1]^3$. Now fix the cubic complex polynomial $p(s) := s^3 - 1$, and consider the map which performs one step in Newton’s method at a particular (extended) complex number s , for finding a root of p , $\gamma_p : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. The explicit inner workings of γ_p are irrelevant for this example, except for the fact that it is a continuous mapping. It follows that $(S^2, \{x, y, z\}, \{\gamma_p^k : k \in \mathbb{N}\})$ is a CCS. The idea is that repeated applications of $\gamma_p(s), \gamma_p \circ \gamma_p(s), \gamma_p \circ \gamma_p \circ \gamma_p(s), \dots$ would approximate a root of p provided s was a good enough initial guess.

290 The r_{\bullet} -type-shard is defined as $\mathcal{L}[r_{\bullet}] = \overline{L[r_{\bullet}]}$ and \mathcal{L}_{sh} is the union of all type-shards. Notice that \mathcal{L}_{sh} is not necessarily equal to $\mathcal{L} = \bar{L}$, unless \mathcal{P} is countable (see [ADIW24]). A *transition* is a map $f : L \rightarrow L$, in particular, every element in the semigroup Γ is a transition (these are called *realized computations*). In practice, one would like to work with “definable” computations, i.e., ones that can be described by a computer. In this topological framework, being continuous is an expected requirement. However, as in the case of complete-types in model theory,

we will work with “unrealized computations”, i.e., maps $f : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$. Note that continuity of a computation does not imply that it can be continuously extended to \mathcal{L}_{sh} .

Suppose that a transition map $f : L \rightarrow \mathcal{L}$ can be extended continuously to a map $\mathcal{L} \rightarrow \mathcal{L}$. Then, the Stone-Weierstrass theorem implies that the feature $\pi_P \circ f$ (here P is a fixed predicate, and the feature is hence continuous) can be uniformly approximated by polynomials on the compact set $\mathcal{L}[r_\bullet]$. Theorem 2.2 in [ADIW24] formalizes the converse of this fact, in the sense that transitions maps that are not continuously extendable in this fashion cannot be obtained from simple constructions involving predicates. Under this framework, the features $\pi_P \circ f$ of such transitions f are not approximable by polynomials, and so they are understood as “non-computable” since, again, we expect the operations computers carry out to be determined by elementary algebra corresponding to polynomials (namely addition and multiplication). Therefore it is crucial we assume some extendibility conditions.

We say that the CCS (L, \mathcal{P}, Γ) satisfies the *Extendibility Axiom* if for all $\gamma \in \Gamma$, there is $\tilde{\gamma} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$ such that for every sizer r_\bullet there is an s_\bullet such that $\tilde{\gamma}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$ is continuous. For a deeper discussion about this axiom, we refer the reader to [ADIW24].

A collection R of sizers is called *exhaustive* if $\mathcal{L}_{sh} = \bigcup_{r_\bullet \in R} \mathcal{L}[r_\bullet]$. We say that $\Delta \subseteq \Gamma$ is R -*confined* if $\gamma|_{L[r_\bullet]} : L[r_\bullet] \rightarrow L[r_\bullet]$ for every $r_\bullet \in R$ and $\gamma \in \Delta$. Elements in Δ are called *real-valued computations* (in this article we will refer to them simply as *computations*) and elements in $\overline{\Delta} \subseteq \mathcal{L}_{sh}^L$ are called (real-valued) *deep computations* or *ultracomputations*. By $\tilde{\Delta}$ we denote the set of all extensions $\tilde{\gamma}$ for $\gamma \in \Delta$. For a more complete description of this framework, we refer the reader to [ADIW24].

3.1. NIP and Baire-1 definability of deep computations. Under what conditions are deep computations Baire class 1, and thus well-behaved according to our framework, on type-shards? The next Theorem says that, again under the assumption that \mathcal{P} is countable, the space of deep computations is a Rosenthal compactum (when restricted to shards) if and only if the set of computations has the NIP on features. Hence, we can import the theory of Rosenthal compacta into this framework of deep computations.

Theorem 3.1. *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendibility Axiom with \mathcal{P} countable. Let R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. The following are equivalent.*

- (1) $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$.
- (2) $\pi_P \circ \Delta|_{L[r_\bullet]}$ has the NIP for all $P \in \mathcal{P}$ and $r_\bullet \in R$, that is, for all $P \in \mathcal{P}$, $r_\bullet \in R$, $a < b$, $\{\gamma_n\}_{n \in \mathbb{N}} \subseteq \Delta$ there are finite disjoint $E, F \subseteq \mathbb{N}$ such that

$$L[r_\bullet] \cap \bigcap_{n \in E} (\pi_P \circ \gamma_n)^{-1}(-\infty, a) \cap \bigcap_{n \in F} (\pi_P \circ \gamma_n)^{-1}[b, \infty) = \emptyset.$$

Moreover, if any (hence all) of the preceding conditions hold, then every deep computation $f \in \overline{\Delta}$ can be extended to a Baire-1 function on shards, i.e., there is $\tilde{f} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$ such that $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$. In particular, on each shard every deep computation is the pointwise limit of a countable sequence of computations.

Proof. Since \mathcal{P} is countable, then $\mathcal{L}[r_\bullet] \subseteq \mathbb{R}^\mathcal{P}$ is Polish. Also, the Extendibility Axiom implies that $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ is a pointwise bounded set of continuous functions

for all $P \in \mathcal{P}$. Hence, Theorem 2.9 and Lemma 2.10 prove the equivalence of (1) and (2). If (1) holds and $f \in \overline{\Delta}$, then write $f = \underline{\mathcal{U}\lim_i} \gamma_i$ as an ultralimit. Define $\tilde{f} := \underline{\mathcal{U}\lim_i} \tilde{\gamma}_i$. Hence, for all $r_\bullet \in R$ we have $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in \overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$. That every deep computation is a pointwise limit of a countable sequence of computations follows from the fact that for a Polish space X every compact subset of $B_1(X)$ is Fréchet-Urysohn (that is, a space where topological closures coincide with sequential closures, see Theorem 3F in [BFT78] or Theorem 4.1 in [Deb13]). \square

Given a countable set Δ of computations satisfying the NIP on features and shards (condition (2) of Theorem 3.1) we have that $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ (for a fixed sizer r_\bullet) is a separable *Rosenthal compactum* (compact subset of $B_1(P \times \mathcal{L}[r_\bullet])$). The work of Todorčević ([Tod99]) and Argyros, Dodos, Kanellopoulos ([ADK08]) culminates in a trichotomy theorem for separable Rosenthal Compacta. Inspired by the work of Glasner and Megrelishvili ([GM22]), we are interested to see how this allows us to classify and obtain different levels of PAC-learnability (NIP).

Recall that a topological space X is *hereditarily separable* (HS) if every subspace is separable and that X is *first countable* if every point in X has a countable local basis. Every separable metrizable space is hereditarily separable and it is a result of R. Pol that every hereditarily separable Rosenthal compactum is first countable (see section 10 in [Deb13]). This suggests the following definition:

Definition 3.2. Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendibility Axiom and R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be an R -confined countable set of computations satisfying the NIP on shards and features (condition (2) in Theorem 3.1). We say that Δ is:

- (i) NIP_1 if $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ is first countable for every $r_\bullet \in R$.
- (ii) NIP_2 if $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ is hereditarily separable for every $r_\bullet \in R$.
- (iii) NIP_3 if $\overline{\tilde{\Delta}}|_{\mathcal{L}[r_\bullet]}$ is metrizable for every $r_\bullet \in R$.

Observe that $NIP_3 \Rightarrow NIP_2 \Rightarrow NIP_1 \Rightarrow NIP$. A natural question that would continue this work is to find examples of CCS that separate these levels of NIP. In [Tod99], Todorčević isolates 3 canonical examples of Rosenthal compacta that witness the failure of the converse implications above. We now present some separable and non-separable examples of Rosenthal compacta:

- (1) *Alexandroff compactification of a discrete space of size continuum.* For each $a \in 2^{\mathbb{N}}$ consider the map $\delta_a : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ given by $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. Let $A(2^{\mathbb{N}}) = \{\delta_a : a \in 2^{\mathbb{N}}\} \cup \{0\}$, where 0 is the zero map. Notice that $A(2^{\mathbb{N}})$ is a compact subset of $B_1(2^{\mathbb{N}})$, in fact $\{\delta_a : a \in 2^{\mathbb{N}}\}$ is a discrete subspace of $B_1(2^{\mathbb{N}})$ and its pointwise closure is precisely $A(2^{\mathbb{N}})$. Hence, this is a Rosenthal compactum which is not first countable. Notice that this space is also not separable.
- (2) *Extended Alexandroff compactification.* For each finite binary sequence $s \in 2^{<\mathbb{N}}$, let $v_s : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by $v_s(x) = 1$ if x extends s and $v_s(x) = 0$ otherwise. Let $\hat{A}(2^{\mathbb{N}})$ be the pointwise closure of $\{v_s : s \in 2^{<\mathbb{N}}\}$, i.e., $\hat{A}(2^{\mathbb{N}}) = A(2^{\mathbb{N}}) \cup \{v_s : s \in 2^{<\mathbb{N}}\}$. Note that this space is a separable Rosenthal compactum which is not first countable.
- (3) *Split Cantor.* Let $<$ be the lexicographic order in the space of infinite binary sequences, i.e., $2^{\mathbb{N}}$. For each $a \in 2^{\mathbb{N}}$ let $f_a^- : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by

384 $f_a^-(x) = 1$ if $x < a$ and $f_a^-(x) = 0$ otherwise. Let $f_a^+ : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given
 385 by $f_a^+(x) = 1$ if $x \leq a$ and $f_a^+(x) = 0$ otherwise. The split Cantor is the
 386 space $S(2^{\mathbb{N}}) = \{f_a^- : a \in 2^{\mathbb{N}}\} \cup \{f_a^+ : a \in 2^{\mathbb{N}}\}$. This is a separable Rosenthal
 387 compactum. One example of a countable dense subset is the set of all f_a^+
 388 and f_a^- where a is an infinite binary sequence that is eventually constant.
 389 Moreover, it is hereditarily separable but it is not metrizable.

- (4) *Alexandroff Duplicate.* Let K be any compact metric space and consider the Polish space $X = C(K) \sqcup K$, i.e., the disjoint union of $C(K)$ (with its supremum norm topology) and K . For each $a \in K$ define $g_a^0, g_a^1 : X \rightarrow \mathbb{R}$ as follows:

$$g_a^0(x) = \begin{cases} x(a), & x \in C(K) \\ 0, & x \in K \end{cases}$$

$$g_a^1(x) = \begin{cases} x(a), & x \in C(K) \\ \delta_a(x), & x \in K \end{cases}$$

390 Let $D(K) = \{g_a^0 : a \in K\} \cup \{g_a^1 : a \in K\}$. Notice that $D(K)$ is a first
 391 countable Rosenthal compactum. It is not separable if K is uncountable.
 392 The interesting case will be when $K = 2^{\mathbb{N}}$.

- (5) *Extended Alexandroff Duplicate of the split Cantor.* For each finite binary sequence $t \in 2^{<\mathbb{N}}$ let $a_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 0's and let $b_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 1's. Define $h_t : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ by

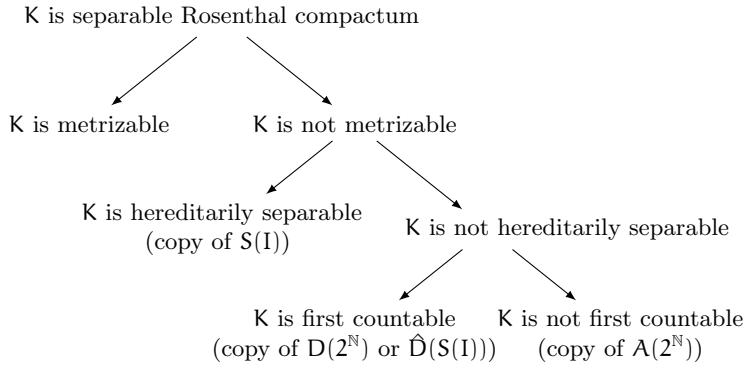
$$h_t(x) = \begin{cases} 0, & x < a_t \\ 1/2, & a_t \leq x \leq b_t \\ 1, & b_t < x \end{cases}$$

393 Let $\hat{D}(S(2^{\mathbb{N}}))$ be the pointwise closure of the set $\{h_t : t \in 2^{<\mathbb{N}}\}$. Hence,
 394 $\hat{D}(S(2^{\mathbb{N}}))$ is a separable first countable Rosenthal compactum which is not
 395 hereditarily separable. In fact, it contains an uncountable discrete subspace
 396 (see Theorem 5 in [Tod99]).

397 **Theorem 3.3** (Todorčević's Trichotomy, [Tod99], Theorem 3 in [ADK08]). *Let K
 398 be a separable Rosenthal Compactum.*

- 399 (i) *If K is hereditarily separable but non-metrizable, then $S(2^{\mathbb{N}})$ embeds into K .*
- 400 (ii) *If K is first countable but not hereditarily separable, then either $D(2^{\mathbb{N}})$ or
 $\hat{D}(S(2^{\mathbb{N}}))$ embeds into K .*
- 402 (iii) *If K is not first countable, then $A(2^{\mathbb{N}})$ embeds into K .*

403 In other words, we have the following classification:



404

405 Lastly, the definitions provided here for NIP_i ($i = 1, 2, 3$) are topological.

406 **Question 3.4.** Is there a non-topological characterization for NIP_i , $i = 1, 2, 3$?

407 More can be said about the nature of the embeddings in Todorčević's Trichotomy.
 408 Given a separable Rosenthal compactum K , there is typically more than one countable dense subset of K . We can view a separable Rosenthal compactum as the accumulation points of a countable family of pointwise bounded real-valued functions.
 409 The choice of the countable families is not important when a bijection between
 410 them can be lifted to a homeomorphism of their closures. To be more precise:
 411

412 **Definition 3.5.** Given a Polish space X , a countable set I and two pointwise
 413 bounded families $\{f_i : i \in I\} \subseteq \mathbb{R}^X$, $\{g_i : i \in I\} \subseteq \mathbb{R}^X$ indexed by I . We say that
 414 $\{f_i : i \in I\}$ and $\{g_i : i \in I\}$ are *equivalent* if and only if the map $f_i \mapsto g_i$ is extended
 415 to a homeomorphism from $\overline{\{f_i : i \in I\}}$ to $\overline{\{g_i : i \in I\}}$.

416 Notice that in the separable examples discussed before $(\hat{A}(2^N), S(2^N)$ and $\hat{D}(S(2^N))$)
 417 the countable dense subsets are indexed by the binary tree $2^{<\mathbb{N}}$. This choice of index
 418 is useful because the Ramsey theory of perfect subsets of the Cantor space $2^{\mathbb{N}}$
 419 can be imported to analyze the behavior of the accumulation points. Since $2^{<\mathbb{N}}$ is
 420 countable, we can always choose this index for the countable dense subsets. This
 421 is done in [ADK08].

422 **Definition 3.6.** Given a Polish space X and a pointwise bounded family $\{f_t : t \in 2^{<\mathbb{N}}\}$. We say that $\{f_t : t \in 2^{<\mathbb{N}}\}$ is *minimal* if and only if for every dyadic subtree
 423 $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$, $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to $\{f_t : t \in 2^{<\mathbb{N}}\}$.

424 One of the main results in [ADK08] is that there are (up to equivalence) seven
 425 minimal families of Rosenthal compacta and that for every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$ there is a dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$
 426 is equivalent to one of the minimal families. We shall describe the minimal families
 427 next. We will follow the same notation as in [ADK08]. For any node $t \in 2^{<\mathbb{N}}$, we
 428 denote by $t^\frown 0^\infty$ ($t^\frown 1^\infty$) the infinite binary sequence starting with t and ending
 429 in 0's (1's). Fix a regular dyadic subtree $R = \{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ (i.e., a dyadic
 430 subtree such that every level of R is contained in a level of $2^{<\mathbb{N}}$) with the property
 431 that for all $s, s' \in R$, $s^\frown 0^\infty \neq s'^\frown 0^\infty$ and $s^\frown 1^\infty \neq s'^\frown 1^\infty$. Given $t \in 2^{<\mathbb{N}}$, let
 432 v_t be the characteristic function of the set $\{x \in 2^{\mathbb{N}} : x \text{ extends } t\}$. Let $<$ be the
 433 lexicographic order in $2^{\mathbb{N}}$. Given $a \in 2^{\mathbb{N}}$, let $f_a^+ : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ be the characteristic
 434 function of $\{x \in 2^{\mathbb{N}} : a \leq x\}$ and let $f_a^- : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ be the characteristic function of

438 $\{x \in 2^{\mathbb{N}} : a < x\}$. Given two maps $f, g : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ we denote by $(f, g) : 2^{\mathbb{N}} \sqcup 2^{\mathbb{N}} \rightarrow \mathbb{R}$
 439 the function which is f on the first copy of $2^{\mathbb{N}}$ and g on the second copy of $2^{\mathbb{N}}$.

- 440 (1) $D_1 = \{\frac{1}{|t|+1}v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_1} = A(2^{\mathbb{N}})$.
- 441 (2) $D_2 = \{s_t^\frown 0^\infty : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_2} = 2^{\leq\mathbb{N}}$.
- 442 (3) $D_3 = \{f_{s_t^\frown 0^\infty}^+ : t \in 2^{<\mathbb{N}}\}$. This is a discrete in $\overline{D_3} = S(2^{\mathbb{N}})$.
- 443 (4) $D_4 = \{f_{s_t^\frown 1^\infty}^- : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_4} = S(2^{\mathbb{N}})$.
- 444 (5) $D_5 = \{v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_5} = \hat{A}(2^{\mathbb{N}})$.
- 445 (6) $D_6 = \{(v_{s_t}, s_t^\frown 0^\infty) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_6} = \hat{D}(2^{\mathbb{N}})$.
- 446 (7) $D_7 = \{(v_{s_t}, x_{s_t^\frown 0^\infty}^+) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_7} = \hat{D}(S(2^{\mathbb{N}}))$

447 **Theorem 3.7** (Heptacotomy of minimal families, Theorem 2 in [ADK08]). *Let
 448 X be Polish. For every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$, there exists $i =$
 449 $1, 2, \dots, 7$ and a regular dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$
 450 is equivalent to D_i . Moreover, all D_i are minimal and mutually non-equivalent.*

451 **3.2. NIP and definability by universally measurable functions.** We now
 452 turn to the question: what happens when \mathcal{P} is uncountable? Notice that the
 453 countability assumption is crucial in the proof of Theorem 2.9 essentially because it
 454 makes $\mathbb{R}^{\mathcal{P}}$ a Polish space. For the uncountable case, we may lose Baire-1 definability
 455 so we shall replace $B_1(X)$ by a bigger class. Recall that the purpose of studying the
 456 class of Baire-1 functions is that a pointwise limit of continuous functions is not
 457 necessarily continuous. In [BFT78], J. Bourgain, D.H. Fremlin and M. Talagrand
 458 characterized the Non-Independence Property of a set of continuous functions with
 459 various notions of compactness in function spaces containing $C(X)$, such as $B_1(X)$.
 460 In this section we will replace $B_1(X)$ with the larger space $M_r(X)$ of universally
 461 measurable functions. The development of this section is based on Theorem 2F in
 462 [BFT78]. We now give the relevant definitions. Readers with little familiarity with
 463 measure theory can review the appendix for standard definitions appearing in this
 464 subsection.

465 Given a Hausdorff space X and a measurable space (Y, Σ) , we say that $f : X \rightarrow Y$
 466 is *universally measurable* (with respect to Σ) if $f^{-1}(E)$ is universally measurable
 467 for every $E \in \Sigma$, i.e., $f^{-1}(E)$ is μ -measurable for every Radon probability measure
 468 μ on X . When $Y = \mathbb{R}$ we will always take $\Sigma = \mathcal{B}(\mathbb{R})$, the Borel σ -algebra of \mathbb{R} .
 469 In that case, a function $f : X \rightarrow \mathbb{R}$ is universally measurable if and only if $f^{-1}(U)$
 470 is μ -measurable for every Radon probability measure μ on X and every open set
 471 $U \subseteq \mathbb{R}$. Following [BFT78], the collection of all universally measurable real-valued
 472 functions will be denoted by $M_r(X)$. In the context of deep computations, we will
 473 be interested in transition maps from a state space $L \subseteq \mathbb{R}^{\mathcal{P}}$ to itself. There are two
 474 natural σ -algebras one can consider in the product space $\mathbb{R}^{\mathcal{P}}$: the Borel σ -algebra,
 475 i.e., the σ -algebra generated by open sets in $\mathbb{R}^{\mathcal{P}}$; and the cylinder σ -algebra, i.e.,
 476 the σ -algebra generated by Borel cylinder sets or equivalently basic open sets in
 477 $\mathbb{R}^{\mathcal{P}}$. Note that when \mathcal{P} is countable, both σ -algebras coincide but in general the
 478 cylinder σ -algebra is strictly smaller. We will use the cylinder σ -algebra to define
 479 universally measurable maps $f : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R}^{\mathcal{P}}$. The reason for this choice is because of
 480 the following characterization:

481 **Lemma 3.8.** *Let X be a Hausdorff space and $Y = \prod_{i \in I} Y_i$ be any product of
 482 measurable spaces (Y_i, Σ_i) for $i \in I$. Let Σ_Y be the cylinder σ -algebra generated by
 483 the measurable spaces (Y_i, Σ_i) . Let $f : X \rightarrow Y$. The following are equivalent:*

- 484 (i) $f : X \rightarrow Y$ is universally measurable (with respect to Σ_Y).
 485 (ii) $\pi_i \circ f : X \rightarrow Y_i$ is universally measurable (with respect to Σ_i) for all $i \in I$.

486 *Proof.* (i) \Rightarrow (ii) is clear since the projection maps π_i are measurable and the com-
 487 position of measurable functions is measurable. To prove (ii) \Rightarrow (i), suppose that
 488 $C = \prod_{i \in I} C_i$ is a measurable cylinder and let J be the finite set of $i \in I$ such that
 489 $C_i \neq Y_i$. Then, $C = \bigcap_{i \in J} \pi_i^{-1}(C_i)$ so $f^{-1}(C) = \bigcap_{i \in J} (\pi_i \circ f)^{-1}(C_i)$ is a universally
 490 measurable set by assumption. \square

491 The previous lemma says that a transition map is universally measurable if and
 492 only if it is universally measurable on all its features. In other words, we can check
 493 measurability of a transition just by checking measurability in all its features. We
 494 will denote by $M_r(X, \mathbb{R}^P)$ the collection of all universally measurable functions
 495 $f : X \rightarrow \mathbb{R}^P$ (with respect to the cylinder σ -algebra), endowed with the topology of
 496 pointwise convergence.

497 **Definition 3.9.** Let (L, \mathcal{P}, Γ) be a CCS. We say that a transition $f : L \rightarrow L$ is
 498 *universally measurable shard-definable* if and only if there exists $\tilde{f} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$
 499 extending f such that for every sizer r_\bullet there is a sizer s_\bullet such that the restriction
 500 $\tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$ is universally measurable, i.e. $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow [-s_P, s_P]$
 501 is μ -measurable for every Radon probability measure μ on $\mathcal{L}[r_\bullet]$.

502 We will need the following result about NIP and universally measurable func-
 503 tions:

504 **Theorem 3.10** (Bourgain-Fremlin-Talagrand, Theorem 2F in [BFT78]). *Let X be a
 505 Hausdorff space and $A \subseteq C(X)$ be pointwise bounded. The following are equivalent:*

- 506 (i) $\overline{A} \subseteq M_r(X)$.
 507 (ii) For every compact $K \subseteq X$, $A|_K$ has the NIP.
 508 (iii) For every Radon measure μ on X , A is relatively countably compact in
 509 $\mathcal{L}^0(X, \mu)$, i.e., every countable subset of A has an accumulation point in
 510 $\mathcal{L}^0(X, \mu)$.

511 Theorem 2.5 immediately yields the following.

512 **Theorem 3.11.** *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendibility Axiom. Let R
 513 be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. If $\pi_P \circ \Delta|_{L[r_\bullet]}$ has
 514 the NIP for all $P \in \mathcal{P}$ and all $r_\bullet \in R$, then every deep computation is universally
 515 measurable shard-definable.*

516 *Proof.* By the Extendibility Axiom, Theorem 2.5 and lemma 2.10 we have that
 517 $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$ and $P \in \mathcal{P}$. Let $f \in \overline{\Delta}$ be a deep computation.
 518 Write $f = \mathcal{U} \lim_i \gamma_i$ as an ultralimit of computations in Δ . Define $\tilde{f} := \mathcal{U} \lim_i \tilde{\gamma}_i$.
 519 Then, for all $r_\bullet \in R$ and $P \in \mathcal{P}$ $\pi_P \circ \tilde{\gamma}_i|_{\mathcal{L}[r_\bullet]} \in M_r(\mathcal{L}[r_\bullet])$ for all i so $\pi_P \circ f|_{\mathcal{L}[r_\bullet]} \in$
 520 $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$. \square

521 **Question 3.12.** Under the same assumptions of the previous Theorem, suppose
 522 that every deep computation of Δ is universally measurable shard-definable. Must
 523 $\pi_P \circ \Delta|_{L[r_\bullet]}$ have the NIP for all $P \in \mathcal{P}$ and all $r_\bullet \in R$?

524 **3.3. Talagrand stability and definability by universally measurable func-**
 525 **tions.** There is another notion closely related to NIP, introduced by Talagrand
 526 in [Tal84] while studying Pettis integration. Suppose that X is a compact Haus-
 527 dorff space and $A \subseteq \mathbb{R}^X$. Let μ be a Radon probability measure on X . Given a
 528 μ -measurable set $E \subseteq X$, a positive integer k and real numbers $a < b$. we write:

$$D_k(A, E, a, b) = \bigcup_{f \in A} \{x \in E^{2k} : f(x_{2i}) \leq a, f(x_{2i+1}) \geq b \text{ for all } i < k\}$$

529 We say that A is *Talagrand μ -stable* if and only if for every μ -measurable
 530 set $E \subseteq X$ of positive measure and for every $a < b$ there is $k \geq 1$ such that
 531 $(\mu^{2k})^*(D_k(A, E, a, b)) < (\mu(E))^{2k}$. Notice that we work with the outer measure
 532 because it is not necessarily true that the sets $D_k(A, E, a, b)$ are μ -measurable.
 533 This is certainly the case when A is a countable set of continuous (or μ -measurable)
 534 functions.

535 The following lemma establishes that Talagrand stability is a way to ensure that
 536 deep computations are definable by measurable functions. We include the proof for
 537 the reader's convenience.

538 **Lemma 3.13.** *If A is Talagrand μ -stable, then \bar{A} is also Talagrand μ -stable and*
 539 $\bar{A} \subseteq \mathcal{L}^0(X, \mu)$.

540 *Proof.* First, observe that a subset of a μ -stable set is μ -stable. To show that \bar{A}
 541 is μ -stable, observe that $D_k(\bar{A}, E, a, b) \subseteq D_k(A, E, a', b')$ where $a < a' < b' <$
 542 b and E is a μ -measurable set with positive measure. It suffices to show that
 543 $\bar{A} \subseteq \mathcal{L}^0(X, \mu)$. Suppose that there exists $f \in \bar{A}$ such that $f \notin \mathcal{L}^0(X, \mu)$. By a
 544 characterization of measurable functions (see 413G in [Fre03]), there exists a μ -
 545 measurable set E of positive measure and $a < b$ such that $\mu^*(P) = \mu^*(Q) = \mu(E)$
 546 where $P = \{x \in E : f(x) \leq a\}$ and $Q = \{x \in E : f(x) \geq b\}$. Then, for any $k \geq 1$:
 547 $(P \times Q)^k \subseteq D_k(\{f\}, E, a, b)$ so $(\mu^{2k})^*(D_k(\{f\}, E, a, b)) = (\mu^*(P)\mu^*(Q))^k = (\mu(E))^{2k}$.
 548 Thus, $\{f\}$ is not μ -stable, but we argued before that a subset of a μ -stable set must
 549 be μ -stable. \square

550 We say that A is *universally Talagrand stable* if A is Talagrand μ -stable for
 551 every Radon probability measure μ on X . A similar argument as before, yields the
 552 following:

553 **Theorem 3.14.** *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendibility Axiom. If*
 554 $\pi_P \circ \Delta|_{L[r_\bullet]}$ *is universally Talagrand stable for all $P \in \mathcal{P}$ and all sizers r_\bullet , then*
 555 *every deep computation is universally measurable sh-definable.*

556 It is then natural to ask: what is the relationship between Talagrand stability
 557 and the NIP? The following dichotomy will be useful.

558 **Lemma 3.15** (Fremlin's Dichotomy, 463K in [Fre03]). *If (X, Σ, μ) is a perfect σ -*
 559 *finite measure space (in particular, for X compact and μ a Radon probability measure*
 560 *on X) and $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued measurable functions on X , then*
 561 *either:*

- 562 (i) $\{f_n : n \in \mathbb{N}\}$ has a subsequence that converges μ -almost everywhere, or
- 563 (ii) $\{f_n : n \in \mathbb{N}\}$ has a subsequence with no μ -measurable accumulation point in
 564 \mathbb{R}^X .

565 The preceding lemma can be considered as the measure theoretic version of
 566 Rosenthal's Dichotomy. Combining this dichotomy with the Theorem 3.10 we get
 567 the following result:

568 **Theorem 3.16.** *Let X be a Hausdorff space and $A \subseteq C(X)$ be pointwise bounded.
 569 The following are equivalent:*

- 570 (i) $\overline{A} \subseteq M_r(X)$.
- 571 (ii) *For every compact $K \subseteq X$, $A|_K$ has the NIP.*
- 572 (iii) *For every Radon measure μ on X , A is relatively countably compact in
 $L^0(X, \mu)$, i.e., every countable subset of A has an accumulation point in
 $L^0(X, \mu)$.*
- 575 (iv) *For every Radon measure μ on X and every sequence $\{f_n : n \in \mathbb{N}\} \subseteq A$,
 there is a subsequence that converges μ -almost everywhere.*

577 *Proof.* Notice that the equivalence (i)-(iii) is Theorem 3.10. Notice that the equivalence of (iii) and (iv) is Fremlin's Dichotomy Theorem. \square

579 **Lemma 3.17.** *Let X be a compact Hausdorff space and $A \subseteq C(X)$ be pointwise bounded. If A is universally Talagrand stable, then A has the NIP.*

581 *Proof.* By Theorem 3.10, it suffices to show that A is relatively countably compact
 582 in $L^0(X, \mu)$ for all Radon probability measure μ on X . Since A is Talagrand μ -stable
 583 for any such μ , then $\overline{A} \subseteq L^0(X, \mu)$. In particular, A is relatively countably compact
 584 in $L^0(X, \mu)$. \square

585 **Question 3.18.** Is the converse true?

586 There is a delicate point in this question, as it may be sensitive to set-theoretic
 587 axioms (even assuming countability of A).

588 **Theorem 3.19** (Talagrand, Theorem 9-3-1(a) in [Tal84]). *Let X be a compact
 589 Hausdorff space and $A \subseteq M_r(X)$ be countable and pointwise bounded. Assume that
 590 $[0, 1]$ is not the union of $< c$ closed measure zero sets. If A has the NIP, then A is
 591 universally Talagrand stable.*

592 **Theorem 3.20** (Fremlin, Shelah, [FS93]). *It is consistent that there exists a countable
 593 pointwise bounded set of Lebesgue measurable functions with the NIP which is
 594 not Talagrand stable with respect to Lebesgue measure.*

595 APPENDIX: MEASURE THEORY

596 Given a set X , a collection Σ of subsets of X is called a σ -algebra if Σ contains
 597 X and is closed under complements and countable unions. Hence, for example, a
 598 σ -algebra is also closed under countable intersections. Intuitively, a σ -algebra is
 599 a collection of sets in which we can define a σ -additive measure. We call sets in
 600 a σ -algebra Σ measurable sets and the pair (X, Σ) a measurable space. If X is a
 601 topological space, there is a natural σ -algebra of subsets of X , namely the Borel
 602 σ -algebra $\mathcal{B}(X)$, i.e., the smallest σ -algebra containing all open subsets of X . Given
 603 two measurable spaces (X, Σ_X) and (Y, Σ_Y) , we say that a function $f : X \rightarrow Y$ is
 604 measurable if and only if $f^{-1}(E) \in \Sigma_X$ for every $E \in \Sigma_Y$. In particular, we say that
 605 $f : X \rightarrow \mathbb{R}$ is measurable if $f^{-1}(E) \in \Sigma_X$ for all $E \in \mathcal{B}(\mathbb{R})$ (equivalently, E open in
 606 \mathbb{R}).

Given a measurable space (X, Σ) , a σ -additive measure is a non-negative function $\mu : \Sigma \rightarrow \mathbb{R}$ with the property that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} \mu(A_n)$ whenever $\{A_n : n \in \mathbb{N}\} \subseteq \Sigma$ is pairwise disjoint. We call (X, Σ, μ) a *measure space*. A σ -additive measure is called a *probability measure* if $\mu(X) = 1$. A measure μ is *complete* if for every $A \subseteq B \in \Sigma$, $\mu(B) = 0$ implies $A \in \Sigma$. In words, subsets of measure-zero sets are always measurable (and hence, by the monotonicity of μ , have measure zero as well). A measure μ is σ -finite if $X = \bigcup_{n=1}^{\infty} X_n$ where $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$ (i.e., X can be decomposed into countably many finite measure sets). A measure μ is *perfect* if for every measurable $f : X \rightarrow \mathbb{R}$ and every measurable set E with $\mu(E) > 0$, there exists a compact $K \subseteq f(E)$ such that $\mu(f^{-1}(K)) > 0$. We say that a property $\phi(x)$ about $x \in X$ holds μ -almost everywhere if $\mu(\{x \in X : \phi(x) \text{ does not hold}\}) = 0$.

A special example of the preceding concepts is that of a *Radon measure*. If X is a Hausdorff topological space, then a measure μ on the Borel sets of X is called a *Radon measure* if

- for every open set U , $\mu(U)$ is the supremum of $\mu(K)$ over all compact $K \subseteq U$, that is, the measure of open sets may be approximated via compact sets; and
- every point of X has a neighborhood $U \ni x$ for which $\mu(U)$ is finite.

Perhaps the most famous example of a Radon measure on \mathbb{R} is the Lebesgue measure of Borel sets. If X is finite, $\mu(A) := |A|$ (the cardinality of A) defines a Radon measure on X . Every Radon measure is perfect (see 451A, 451B and 451C in [Fre03]).

While not immediately obvious, sets can be measurable according to one measure, but non-measurable according to another. Given a measure space (X, Σ, μ) we say that a set $E \subseteq X$ is μ -measurable if there are $A, B \in \Sigma$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. The set of all μ -measurable sets is a σ -algebra containing Σ and it is denoted by Σ_{μ} . A set $E \subseteq X$ is *universally measurable* if it is μ -measurable for every Radon probability measure on X . It follows that Borel sets are universally measurable. We say that $f : X \rightarrow \mathbb{R}$ is μ -measurable if $f^{-1}(E) \in \Sigma_{\mu}$ for all $E \in \mathcal{B}(\mathbb{R})$ (equivalently, E open in \mathbb{R}). The set of all μ -measurable functions is denoted by $\mathcal{L}^0(X, \mu)$.

Recall that if $\{X_i : i \in I\}$ is a collection of topological spaces indexed by some set I , then the product space $X := \prod_{i \in I} X_i$ is endowed with the topology generated by *cylinders*, that is, sets of the form $\prod_{i \in I} U_i$ where each U_i is open in X_i , and $U_i = X_i$ except for finitely many indices $i \in I$. If each space is measurable, say we pair X_i with a σ -algebra Σ_i , then there are multiple ways to interpret the product space X as a measurable space, but the interpretation we care about in this paper is the so called *cylinder σ -algebra*, as used in Lemma 3.8. Namely, let Σ be the σ -algebra generated by sets of the form

$$\prod_{i \in I} C_i, \quad C_i \in \Sigma_i, \quad C_i = X_i \text{ for all but finitely many } i \in I.$$

We remark that when I is uncountable and $\Sigma_i = \mathcal{B}(X_i)$ for all $i \in I$, then Σ is, in general, strictly **smaller** than $\mathcal{B}(X)$.

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