COMPLEXITY OF DEEP COMPUTATIONS VIA TOPOLOGY OF FUNCTION SPACES

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Abstract. We study complexity of deep computations. We use topology of function spaces, specifically, the classification Rosenthal compacta, to identify new complexity classes. We use the language of model theory, specifically, the concept of the independence from Shelah's classification theory, to translate between topology and computation.

0. Introduction

In this paper we study limit behavior of real-valued computations as the value of certain parameters of the computation model tend towards infinity, or towards zero, or towards some other fixed value, e.g., the depth of a neural network tending to infinity, or the time interval between layers of the network tending toward zero. Recently, particular cases of this situation have attracted considerable attention in deep learning research (e.g., Neural Ordinary Differential Equations [CRBD], Physics-Informed Neural Networks [RPK19], deep equilibrium models [BKK], etc). In this paper, we combine ideas of topology and model theory to study these limit phenomena from a unified viewpoint.

Informed by model theory, to each computation in a given computation model, we associate a continuous real-valued function, called the *type* of the computation, that describes the logical properties of this computation with respect to the rest of the model. This allows us to view computations in any given computational model as elements of a space of real-valued functions, which is called the *space of types* of the model. The idea of embedding models of theories into their type spaces is central in model theory. The embedding of computations into spaces of types allows us to utilize the vast theory of topology of function spaces, known as C_p-theory, to obtain results about complexity of topological limits of computations. As we shall indicate next, recent classification results for spaces of functions provide an elegant and powerful machinery to classify computations according to their levels of "tameness" or "wildness", with the former corresponding roughly to polynomial approximability and the latter to exponential approximability. The viewpoint of spaces of types, which we have borrowed from model theory, thus becomes a "Rosetta stone" that allows us to interconnect various classification programs: In topology, the classification of Rosenthal compacta pioneered by Todorčević [Tod99]; in logic, the classification of theories developed by Shelah [She90]; and in statistical learning, the notion PAC learning and VC dimension pioneered by Vapkins and Chervonenkis [VC74, VC71].

In a previous paper [ADIW24], we introduced the concept of limits of computations, which we called *ultracomputations* (given they arise as ultrafilter limits of standard computations) and *deep computations* (following usage in machine learning [BKK]). There is a technical difference between both designations, but in this paper, to simplify the nomenclature, we will ignore the difference and use only the term “deep computation”.

In [ADIW24], we proved a new “tame vs wild” (i.e., polynomial vs exponential) dichotomy for complexity of deep computations by invoking a classical result of Grothendieck from late 50s [Gro52]. Under our model-theoretic Rosetta stone, polynomial approximability in the sense of computation becomes identified with the notion of continuous extendability in the sense of topology, and with the notions of *stability* and *type definability* in the sense of model theory.

In this paper, we follow a more general approach, i.e., we view deep computations as pointwise limits of continuous functions. In topology, real-valued functions that arise as the pointwise limit of a sequence of continuous are called *functions of the first Baire class*, or *Baire class 1* functions, or *Baire-1* for short; Baire class 1 form a step above simple continuity in the hierarchy of functions studied in real analysis (Baire class 0 functions being continuous functions). Intuitively, Baire-1 functions represent functions with “controlled” discontinuities, so they are crucial in topology and set theory.

We prove a new “tame vs wild” Ramsey-theoretic dichotomy for complexity of general deep computations by invoking a famous paper by Bourgain, Fremlin and Talagrand from the late 70s [BFT78], and a new trichotomy for the class of “tame” deep computations by invoking an equally celebrated result of Todorčević, from the late 90s, for functions of the first Baire class [Tod99].

Todorčević’s trichotomy regards *Rosenthal compacta*; these are special classes of topological spaces, defined as compact spaces that can be embedded (homeomorphically identified as a subset) within the space of Baire class 1 functions on some Polish (separable, complete metric) space, under the pointwise convergence topology. Rosenthal compacta exhibit “topological tameness,” meaning that they behave in relatively controlled ways, and since the late 70’s, they have played a crucial role for understanding complexity of structures of functional analysis, especially, Banach spaces. Todorčević’s trichotomy has been utilized to settle longstanding problems in topological dynamics and topological entropy [GM22].

Through our Rosetta stone, Rosenthal compacta in topology correspond to the important concept of “No Independence Property” (known as “NIP”) in model theory, identified by Shelah [She71, She90], and to the concept of Probably Approximately Correct learning (known as “PAC learnability”) in statistical learning theory identified by Valiant [Val84].

Going beyond Todorčević’s trichotomy, we invoke a more recent heptachotomy for Rosenthal compacta obtained by Argyros, Dodos and Kanellopoulos [ADK08]. Argyros, Dodos and Kanellopoulos identified the fundamental “prototypes” of separable Rosenthal compacta, and proved that any non-metrizable separable Rosenthal compactum must contain a “canonical” embedding of one of these prototypes. They showed that if a separable Rosenthal compactum is not hereditarily separable, it must contain an uncountable discrete subspace of the size of the continuum.

We believe that the results presented in this paper show practitioners of computation, or topology, or descriptive set theory, or model theory, how classification invariants used in their field translate into classification invariants of other fields. However, in the interest of accessibility, we do not assume previous familiarity with high-level topology or model theory, or computing. The only technical prerequisite of the paper is undergraduate-level topology. The necessary topological background beyond undergraduate topology is covered in section 1.

Throughout the paper, we focus on classical computation; however, by refining the model-theoretic tools, the results presented here can be extended to quantum computation and open quantum systems. This extension will be addressed in a forthcoming paper.

Contents

1. General topological preliminaries: From continuity to Baire class 1

In this section we give preliminaries from general topology and function space theory. We include some of the proofs for completeness, but the reader familiar with these topics may skip them.

Recall that a subset of a topological space is F_σ if it is a countable union of closed sets, and G_δ if it is a countable intersection of closed sets. Note that in a metrizable space, every open set is F_σ; equivalently, every closed set is G_δ.

A *Polish space* is a separable and completely metrizable topological space. The most important examples are the reals $\mathbb{R}$, the Cantor space $2^{\mathbb{N}}$ (the set of all infinite binary sequences, endowed with the product topology), and the Baire space $\mathbb{N}^{\mathbb{N}}$ (the set of all infinite sequences of naturals, also with the product topology). Countable products of Polish spaces are Polish; this includes spaces like $\mathbb{R}^{\mathbb{N}}$, the space of sequences of real numbers.

In this paper, we shall discuss subspaces, and so there is a pertinent subtlety of the definitions worth mentioning: *completely metrizable space* is not the same as *complete metric space*; for an illustrative example, the interval $(0, 1)$ with the metric inherited from the reals not complete, but it is Polish since that is homeomorphic to the real line. Being Polish is a topological property.

The following result is a cornerstone of descriptive set theory, closely tied to the work of Wacław Sierpiński and Kazimierz Kuratowski, with proofs often built upon their foundations and formalized later, notably, involving Stefan Mazurkiewicz's work on complete metric spaces.

Fact 1.1. *A subset A of a Polish space X is itself Polish in the subspace topology if and only if it is a G_δ set. In particular, closed subsets and open subsets of Polish spaces are also Polish spaces.*

Given two topological spaces X and Y we denote by $C_p(X, Y)$ the set of all continuous functions $f : X \to Y$ endowed with the topology of pointwise convergence. When $Y = \mathbb{R}$, we denote this collection simply as $C_p(X)$. A natural question is, how do topological properties of X translate to $C_p(X)$ and vice versa? These questions, and in general the study of these spaces, are the concern of C_p-theory, an active field of research in general topology which was pioneered by A. V. Arhangel'skiĭ and his students in the 1970's and 1980's. This field has found many applications in model theory and functional analysis. Recent surveys on the topics include [HT23] and [Tka11].

A *Baire class 1* function between topological spaces is a function that can be expressed as the pointwise limit of a sequence of continuous functions. If X nd Y are topological spaces, the Baire class 1 functions $f : X \to Y$ endowed with the topology of pointwise convergence is denoted $B_1(X, Y)$. As above, in the special case $Y = \mathbb{R}$ we denote $B_1(X, Y)$. as $B_1(X)$. Clearly, $C_p(X, Y) \subseteq B_1(X, Y)$. The Baire hierarchy of functions was introduced by French mathematician René-Louis Baire in his 1899 doctoral thesis, *Sur les fonctions de variables réelles*. His work moved away from the 19th-century preoccupation with "pathological" functions toward a constructive classification based on pointwise limits.

A topological space X is *perfectly normal* if it is normal and every closed subset of X is a G_δ (equivalently, every open subset of X is a G_δ). Note that every metrizable space is perfectly normal.

The following fact was established by Baire in thesis. A proof can be found in Section 10 of [Tod97].

Fact 1.2 (Baire)**.** *If X is perfectly normal, then the following conditions are equivalent for a function $f : X \to \mathbb{R}$:*

- *f is a Baire class 1 function, that is, $f \in B_1(X)$.*
- *$f^{-1}[U]$ is an F_σ subset of X whenever $U \subseteq Y$ is open.*
- *f is a pointwise limit of continuous functions.*
- *For every closed $F \subseteq X$, the restriction $f|_F$ has a point of continuity.*

Moreover, if X is Polish and $f \notin B_1(X)$, then there exists countable $D_0, D_1 \subseteq X$ and reals $a < b$ such that

$$D_0 \subseteq f^{-1}(-\infty, a], \quad D_1 \subseteq f^{-1}[b, \infty), \quad \overline{D_0} = \overline{D_1}.$$

A subset L of a topological space X is *relatively compact* in X if the closure of L in X is compact. Relatively compact subsets of $B_1(X)$ (for X Polish space) have been objects of interest for researchers in Analysis and Topological Dynamics.

We begin with the following well-known result. Recall that a set $A \subseteq \mathbb{R}^X$ of real-valued functions is *pointwise bounded* if for every $x \in X$ there is $M_x > 0$ such that $|f(x)| < M_x$ for all $f \in A$. We include a proof for the reader's convenience:

Lemma 1.3. *Let X be a Polish space and $A \subseteq B_1(X)$ be pointwise bounded. The following are equivalent:*

(i) *A is relatively compact in $B_1(X)$.*
(ii) *A is relatively countably compact in $B_1(X)$, i.e., every countable subset of A has an accumulation point in $B_1(X)$.*
(iii) *$\overline{A} \subseteq B_1(X)$, where $\overline{A}$ denotes the closure in $\mathbb{R}^X$.*

Proof. Since A is pointwise bounded, for each $x \in X$, fic $M_x > 0$ such that $|f(x)| \leq M_x$ for every $f \in A$.

(i)$\Rightarrow$(ii) holds in general.

(ii)$\Rightarrow$(iii) Assume that A is relatively countably compact in $B_1(X)$ and that $f \in \overline{A} \setminus B_1(X)$. By Fact 1.2, there are countable $D_0, D_1 \subseteq X$ with $\overline{D_0} = \overline{D_1}$, and $a < b$ such that $D_0 \subseteq f^{-1}(-\infty, a]$ and $D_1 \subseteq f^{-1}[b, \infty)$. We claim that there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq A$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in D_0 \cup D_1$. Indeed, use the countability to enumerate $D_0 \cup D_1$ as $\{x_n\}_{n\in\mathbb{N}}$. Then for each positive n find $f_n \in A$ with $|f_n(x_i) - f(x_i)| < \frac{1}{n}$ for all $i \leq n$. The claim follows.

By relative countable compactness of A, there is an accumulation point $g \in B_1(X)$ of $\{f_n\}_{n\in\mathbb{N}}$. It is straightforward to show that since f and g agree on $D_0 \cup D_1$, g does not have a point of continuity on the closed set $\overline{D_0} = \overline{D_1}$, which contradicts Fact 1.2.

(iii)$\Rightarrow$(i) Suppose that $\overline{A} \subseteq B_1(X)$. Then $\overline{A} \cap B_1(X) = \overline{A}$ is a closed subset of $\prod_{x\in X}[-M_x, M_x]$; Tychonoff's theorem states that the product of compact spaces is always compact, and since closed subsets of compact spaces are compact, $\overline{A}$ must be compact, as desired. □

1.1. From Rosenthal's dichotomy to the Bourgain-Fremlin-Talagrand dichotomy to Shelah's NIP.

The fundamental idea that connects the rich theory here presented to real-valued computations is the concept of an *approximation*. In the reals, points of closure from some subset can always be approximated by points inside the set, via a convergent sequence. For more complicated spaces, such as $C_p(X)$, this fails in remarkable ways. To see an example, consider the Cantor space $X = 2^{\mathbb{N}}$, and for each $n \in \mathbb{N}$ define $p_n : X \to \{0, 1\}$ by $p_n(x) = x(n)$ for each $x \in X$. Then p_n is continuous for each n, but one can show (see Chapter 1.1 of [Tod97] for details) that the only continuous functions in the closure of $\{p_n\}_{n\in\mathbb{N}}$ are the functions p_n themselves; moreover, none of the subsequences of $\{p_n\}_{n\in\mathbb{N}}$ converge. In some sense, this example is the worst possible scenario for convergence. The topological space obtained from this closure is well-known: it is the *Stone-Čech compactification* of the discrete space of natural numbers, or $\beta\mathbb{N}$ for short, and it is an important object of study in general topology.

The following theorem, established by Haskell Rosenthal in 1974, is fundamental in functional analysis, and describes a sharp division in the behavior of sequences within a Banach space:

Theorem 1.4 (Rosenthal's Dichotomy, [Ros74])**.** *If X is Polish and $\{f_n\} \subseteq C_p(X)$ is pointwise bounded, then either $\{f_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence or a subsequence whose closure (in $\mathbb{R}^X$) is homeomorphic to $\beta\mathbb{N}$.*

In other words, a pointwise bounded set of continuous functions either contains a convergent subsequence, or a subsequence whose closure is essentially the same as the example mentioned in the previous paragraphs (the "wildest" possible scenario). Note that in the preceding example, the functions are trivially pointwise bounded in $\mathbb{R}^X$ as the functions can only take values 0 and 1.

The genesis of Theorem 1.4 was Rosenthal's ℓ_1 theorem, which states that the only reason why Banach space can fail to have an isomorphic copy of ℓ_1 (the space of absolutely summable sequences) is the presence of a bounded sequence with no weakly Cauchy subsequence. The theorem is famous for connecting diverse areas of mathematics: Banach space geometry, Ramsey theory, set theory, and topology of function spaces.

As we transition from $C_p(X)$ to the larger space $B_1(X)$, we find a similar dichotomy. Either every point of closure of the set of functions will be a Baire class 1 function, or there is a sequence inside the set that behaves in the wildest possible way. The theorem is usually not phrased as a dichotomy but rather as an equivalence:

Theorem 1.5 (The BFT Dichotomy. Bourgain-Fremlin-Talagrand, [BFT78, Theorem 4G])**.** *Let X be a Polish space and $A \subseteq C_p(X)$ be pointwise bounded. The following are equivalent:*

(i) A is relatively compact in $B_1(X)$, i.e., $\overline{A} \subseteq B_1(X)$.
(ii) For every $\{f_n\}_{n\in\mathbb{N}} \subseteq A$ and every $a < b$ there is $I \subseteq \mathbb{N}$ such that

$$\bigcap_{n\in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n\notin I} f_n^{-1}[b, \infty) = \emptyset.$$

Definition 1.6. We shall say that a set $A \subseteq \mathbb{R}^X$ has the *Independence Property*, or IP for short, if it satisfies the following condition: There exists every $\{f_n\}_{n\in\mathbb{N}} \subseteq A$ and $a < b$ such that for every pair of disjoint sets $E, F \subseteq \mathbb{N}$, we have

$$\bigcap_{n\in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n\in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

If A satisfies the negation of this condition, we will say that *A satisfies NIP*, or that has the NIP.

Remark 1.7. Note that if X is compact and $A \subseteq C_p(X)$, then A has the NIP if and only if for every $\{f_n\}_{n\in\mathbb{N}} \subseteq A$ and for every $a < b$ there is $I \subseteq \mathbb{N}$ such that

$$\bigcap_{n\in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n\notin I} f_n^{-1}[b, \infty) = \emptyset.$$

To summarize, the particular case of Theorem 1.8 when for X compact can be stated in the following way:

Theorem 1.8. *Let X be a compact Polish space. Then, for every pointwise bounded $A \subseteq C_p(X)$, one and exactly one of the following two conditions must hold:*

(i) $\overline{A} \subseteq B_1(X)$.
(ii) A has NIP.

The Independence Property was first isolated by Saharon Shelah in model theory as a dividing line between theories whose models are "tame" (corresponding to NIP) theories of models are "wild" (corresponding to IP). See [She71, Definition 4.1],[She90].

1.2. **NIP as universal dividing line between polynomial and exponential complexity.** The particular case of the BSF Dichotomy (Theorem 1.8) when A consists of $\{0,1\}$-valued (i.e., $\{\text{Yes},\text{No}\}$-valued) strings was discovered independently, around 1971-1972 in many foundational contexts related to polynomial ("tame") vs exponential ("wild") complexity: In model theory, by Saharon Shelah [She71],[She90],in combinatorics, by Norbert Sauer [Sau72], and Shelah [She72, She90], and in statistical learning, by Vladimir Vapnik and Alexey Chervonenkis [VC71, VC74].

In model theory: Shelah's classification theory is a foundational program in mathematical logic devised to categorize first-order theories based on the complexity and structure of their models. A theory T is considered classifiable in Shelah's sense if the number of non-isomorphic models of T of a given cardinality can be described by a bounded number of numerical invariants. In contrast, a theory T is unclassifiable if the number of models of T of a given cardinality is the maximum possible number. This number is directly impacted by the number of "types" over of parameters in models of T; a controlled number of types is a characteristic of a classifiable theory.

In Shelah's classification program [She90], theories without the independence property (called NIP theories, or dependent theories) have a well-behaved, "tame" structure; the number of types over a set of parameters of size κ of such a theory is of polynomially or similar "slow" growth on κ. Theories with the Independence Property (called IP theories), in contrast, are considered "intractable" or "wild". A theory with the independence property produces the maximum possible number of types over a set of parameters; for a set of parameters of cardinality κ, the theory will have 2^{2^κ}-many distinct types.

In combinatorics: Sauer [Sau72] and Shelah [She72] proved the following: If $\mathscr{F} = \{S_0, S_1, \dots\}$ is a family of subsets of some infinite set S, then either for every $n \in \mathbb{N}$, there is a set $A \subseteq S$ with $|A| = n$ such that $|\{S_i \cap A) : i \in \mathbb{N}\}| = 2^n$ (yielding exponential complexity), or there exists $N \in \mathbb{N}$ such that $A \subseteq S$ with $|A| \geq N$, one has

$$|\{S_i \cap A) : i \in \mathbb{N}\}| \leq \sum_{i=0}^{N-1} \binom{|A|}{i} \approx O(|A|^N)$$

for every $A \subseteq S$ such that $|A| \geq N$ (yielding polynomial complexity). This answered a question of Erdős.

In machine learning: Readers familiar with statistical learning may recognize the Sauer-Shelah lemma as the dichotomy discovered and proved slightly earlier (1971) by Vapknis and Chervonenkis [VC71, VC74] to address the problem of uniform convergence in statistics. The least integer N given by the preceding paragraph, when it exists, is called the *VC-dimension* of $\mathscr{F}$. This is a core concept in machine learning. If such an integer N does not exist, we say that the VC-dimension of $\mathscr{F}$ is infinite. The lemma provides upper bounds on the number of data points (sample size m) needed to learn a concept class with VC dimension $d \in \mathbb{N}$ by showing this number grows polynomially with m and d (namely, $\sum_{i=0}^{d} \binom{m}{i} \approx O(m^d)$), not exponentially. The Fundamental Theorem of Statistical Learning states that a hypothesis class is PAC-learnable (PAC stands for "Probably Approximately Correct") if and only if its VC dimension is finite.

1.3. **Rosenthal compacta.** The comprehensiveness of Theorem 1.8, attested by the examples outlined in the preceding section, led to the following definition (isolated by Godefroy [God80]):

Definition 1.9. A Rosenthal compactum is a compact Hausdorff topological space K that can be topologically embedded as a compact subset into the space of all functions of the first Baire class on some Polish space X, equipped with the topology of pointwise convergence.

Rosenthal compacta are characterized by significant topological and dynamical tameness properties. They play a significant role in functional analysis, measure theory, dynamical systems, descriptive set theory, and model theory. In this paper, we introduce their applicability in deep computation. For this, we shall first focus on countable languages, which is the theme of the next section.

1.4. **The special case $B_1(X, \mathbb{R}^{\mathcal{P}})$ with $\mathcal{P}$ countable.** Our goal now is to characterize relatively compact subsets of $B_1(X, Y)$ for the particular case when $Y = \mathbb{R}^{\mathcal{P}}$ with $\mathcal{P}$ countable. Given $P \in \mathcal{P}$ we denote the projection map onto the P-coordinate by $\pi_P : \mathbb{R}^{\mathcal{P}} \to \mathbb{R}$. From a high-level topological interpretation, the subsequent lemma states that, in this context, the spaces $\mathbb{R}$ and $\mathbb{R}^{\mathcal{P}}$ are really not that different, and that if we understand the Baire class 1 functions of one space, then we also understand the functions of both.

Lemma 1.10. *Let X be a Polish space and $\mathcal{P}$ be a countable set. Then, $f \in B_1(X, \mathbb{R}^{\mathcal{P}})$ if and only if $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$.*

Proof. Only one implication needs a proof. Suppose that $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$. Let V be a basic open subset of $\mathbb{R}^{\mathcal{P}}$. That is, there exists a finite $\mathcal{P}' \subseteq \mathcal{P}$ such that $V = \bigcap_{P \in \mathcal{P}'} \pi_P^{-1}[U_P]$ where U_P is open in $\mathbb{R}$. Then,

$$f^{-1}[V] = \bigcap_{P \in \mathcal{P}'} (\pi_P \circ f)^{-1}[U_P]$$

is an F_σ set. Since $\mathcal{P}$ is countable, $\mathbb{R}^{\mathcal{P}}$ is second countable so every open set U in $\mathbb{R}^{\mathcal{P}}$ is a countable union of basic open sets. Hence, $f^{-1}[U]$ is F_σ. □

Below we consider $\mathcal{P}$ with the discrete topology. For each $f : X \to \mathbb{R}^{\mathcal{P}}$ denote $\hat{f}(P, x) := \pi_P \circ f(x)$ for all $(P, x) \in \mathcal{P} \times X$. Similarly, for each $g : \mathcal{P} \times X \to \mathbb{R}$ denote $\check{g}(x)(P) := g(P, x)$. Given $A \subseteq (\mathbb{R}^{\mathcal{P}})^X$, we denote $\hat{A}$ as the set of all $\hat{f}$ such that $f \in A$. Note that the map $\left(\mathbb{R}^{\mathcal{P}}\right)^X \to \mathbb{R}^{\mathcal{P} \times X}$ given by $f \mapsto \hat{f}$ is a homeomorphism and its inverse is given by $g \mapsto \check{g}$.

Lemma 1.11. *Let X be a Polish space and $\mathcal{P}$ be countable. Then, $f \in B_1(X, \mathbb{R}^{\mathcal{P}})$ if and only if $\hat{f} \in B_1(\mathcal{P} \times X)$.*

Proof. ($\Rightarrow$) By Lemma 1.10, g iven an open set of reals U, we have $f^{-1}[\pi_P^{-1}[U]]$ is F_σ for every $P \in \mathcal{P}$. Given that $\mathcal{P}$ is a discrete countable space, we observe that

$$\hat{f}^{-1}[U] = \bigcup_{P \in \mathcal{P}} \left(\{P\} \times f^{-1}[\pi_P^{-1}[U]]\right)$$

is an F_σ as well.

($\Leftarrow$) By lemma 1.10 it suffices to show that $\pi_P \circ f \in B_1(X)$ for all $P \in \mathcal{P}$. Fix an open $U \subseteq \mathbb{R}$. Write $\hat{f}^{-1}[U] = \bigcup_{n\in\mathbb{N}} F_n$ where F_n is closed in $\mathcal{P} \times X$. Then,

$$(\pi_P \circ f)^{-1}[U] = \bigcup_{n\in\mathbb{N}} \{x \in X : (P, x) \in F_n\}$$

which is F_σ. $\square$

Given $A \subseteq Y^X$ and $K \subseteq X$ we write $A|_K := \{f|_K : f \in A\}$, i.e., the set of all restrictions of functions in A to K. The following Theorem is a slightly more general version of Theorem 1.8.

Theorem 1.12. *Assume that $\mathcal{P}$ is countable, X is a Polish space, and $A \subseteq C_p(X, \mathbb{R}^{\mathcal{P}})$ is such that $\pi_P \circ A$ is pointwise bounded for all $P \in \mathcal{P}$. The following are equivalent for every compact $K \subseteq X$:*

1. *$\overline{A|_K} \subseteq B_1(K, \mathbb{R}^{\mathcal{P}})$.*
2. *$\pi_P \circ A|_K$ has the NIP for every $P \in \mathcal{P}$.*

Proof. (1)$\Rightarrow$(2). Let $P \in \mathcal{P}$. Fix $\{f_n\}_{n\in\mathbb{N}} \subseteq A$ and $a < b$. By (1), we have $\overline{A|_K} \subseteq B_1(K, \mathbb{R}^{\mathcal{P}})$. Applying the homeomorphism $f \mapsto \hat{f}$ and using lemma 1.11 we get $\overline{\hat{A}|_{\mathcal{P}\times K}} \subseteq B_1(\mathcal{P} \times K)$. By Theorem 1.8, there is $I \subseteq \mathbb{N}$ such that

$$(\mathcal{P} \times K) \cap \bigcap_{n\in I} \hat{f_n}^{-1}(-\infty, a] \cap \bigcap_{n\notin I} \hat{f_n}^{-1}[b, \infty) = \emptyset$$

Hence,

$$K \cap \bigcap_{n\in I} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n\notin I} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

By the compactness of K, there are finite $E \subseteq I$ and $F \subseteq \mathbb{N}\backslash I$ such that

$$K \cap \bigcap_{n\in E} (\pi_P \circ f_n)^{-1}(-\infty, a] \cap \bigcap_{n\in F} (\pi_P \circ f_n)^{-1}[b, \infty) = \emptyset$$

Thus, $\pi_P \circ A|_L$ has the NIP.

(2)$\Rightarrow$(1) Fix $f \in \overline{A|_K}$. By lemma 1.10 it suffices to show that $\pi_P \circ f \in B_1(K)$ for all $P \in \mathcal{P}$. By (2), $\pi_P \circ A|_K$ has the NIP. Hence, by Theorem 1.8 we have $\overline{\pi_P \circ A|_K} \subseteq B_1(K)$. But then $\pi_P \circ f \in \overline{\pi_P \circ A|_K} \subseteq B_1(K)$. $\square$

Lastly, a simple but significant result that helps understand the operation of restricting a set of functions to a specific subspace of the domain space X, of course in the context of the NIP, is that we may always assume that said subspace is closed. Concretely, whether we take its closure or not has no effect on the NIP:

Lemma 1.13. *Assume that X is Hausdorff and that $A \subseteq C_p(X)$. The following are equivalent for every $L \subseteq X$:*

(i) *A_L has the NIP.*
(ii) *$A|_{\overline{L}}$ has the NIP.*

Proof. It suffices to show that (i)$\Rightarrow$(ii). Suppose that (ii) does not hold, i.e., that there are $\{f_n\}_{n\in\mathbb{N}} \subseteq A$ and $a < b$ such that for all finite disjoint $E, F \subseteq \mathbb{N}$:

$$\overline{L} \cap \bigcap_{n\in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n\in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

Pick $a' < b'$ such that $a < a' < b' < b$. Then, for any finite disjoint $E, F \subseteq \mathbb{N}$ we can choose

$$x \in \overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a') \cap \bigcap_{n \in F} f_n^{-1}(b', \infty)$$

By definition of closure:

$$L \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a'] \cap \bigcap_{n \in F} f_n^{-1}[b', \infty) \neq \emptyset.$$

This contradicts (i). □

2. Compositional computation structures.

In this section, we connect function spaces with floating point computation. We start by summarizing some basic concepts from [ADIW24].

A *computation states structure* is a pair $(L, \mathcal{P})$, where L is a set whose elements we call *states* and $\mathcal{P}$ is a collection of real-valued functions on L that we call *predicates*. For a state $v \in L$, *type* of v is the indexed family

$$\mathrm{tp}(v) = (P(v))_{P \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}.$$

For each $P \in \mathcal{P}$, we call real value $P(v)$ the P-th *feature* of v. A *transition* of a computation states structure $(L, \mathcal{P})$ is a map $f : L \to L$.

Intuitively, L is the set of states of a computation, and the predicates $P \in \mathcal{P}$ are primitives that are given and accepted as computational. We think of each state $v \in L$ as being uniquely characterized by its type $\mathrm{tp}(v)$; thus, in practice, we identify L with a subset of f $\mathbb{R}^{\mathcal{P}}$. A typical case will be when $L = \mathbb{R}^{\mathbb{N}}$ or $L = \mathbb{R}^n$ for some positive integer n and there is a predicate $P_i(v) = v_i$ for each of the coordinates v_i of v. We regard the space of types as a topological space, endowed with the topology of pointwise convergence inherited from $\mathbb{R}^{\mathcal{P}}$. In particular, for each $P \in \mathcal{P}$, the projection map $v \mapsto P(v)$ is continuous.

Definition 2.1. Given a computation states structure $(L, \mathcal{P})$, any element of $\mathbb{R}^{\mathcal{P}}$ in the image of L under the map $v \mapsto \mathrm{tp}(v)$ will be called a *realized type*. The topological closure of the set of realized types in $\mathbb{R}^{\mathcal{P}}$ (endowed with the pointwise convergence topology) will be called the *space of types* of $(L, \mathcal{P})$, denoted $\mathcal{L}$. Elements of $\mathcal{L} \setminus L$ will be called *unrealized types*.

In traditional model theory, the space of types of a structure is viewed as a sort of compactification of the structure. However, the space $\mathcal{L}$ is not necessarily compact. To bypass this obstacle, we follow the idea introduced in [ADIW24] of covering $\mathcal{L}$ by "thin" compact subspaces that we call *shards*. The formal definition of shard is next.

Definition 2.2. A *sizer* is a tuple $r_\bullet = (r_P)_{P \in \mathcal{P}}$ of positive real numbers indexed by $\mathcal{P}$. Given a sizer $r_\bullet$, we define the *$r_\bullet$-shard* as:

$$L[r_\bullet] = L \cap \prod_{P \in \mathcal{P}} [-r_P, r_P].$$

For a sizer $r_\bullet$, the *$r_\bullet$-type shard* is defined as $\mathcal{L}[r_\bullet] = \overline{L[r_\bullet]}$. We define $\mathcal{L}_{sh}$, as the union of all type-shards.

Definition 2.3. A *Compositional Computation Structure* (CCS) is a triple $(L, \mathcal{P}, \Gamma)$, where

- $(L, \mathcal{P})$ is a computation states structure
- $\Gamma \subseteq L^L$ is a semigroup under composition.

The elements of the Γ is called the *computations* of the structure $(L, \mathcal{P}, \Gamma)$.

If $\Delta \subseteq \Gamma$, we say that $\Delta \subseteq \Gamma$ is *R-confined* if $\gamma|_{L[r_\bullet]} : L[r_\bullet] \to L[r_\bullet]$ for every $r_\bullet \in R$ and $\gamma \in \Delta$. Elements in $\overline{\Delta} \subseteq \mathcal{L}_{sh}$ are called (real-valued) *deep computations* or *ultracomputations*.

A tenet of our approach is that a map $f : L \to \mathcal{L}$ is to be considered "effectively computable" if, for each $Q \in \mathcal{P}$, the output feature $Q \circ f : L \to \mathbb{R}$ is a *definable* predicate in the following sense:

Given any arbitrary $\varepsilon > 0$ and any $K \subseteq L$ wherein every input feature $P(v)$ remains bounded in magnitude there is an ε-approximating continuous "algebraic" operator $\varphi(P_1, \ldots, P_n)$ of finitely many input features $P_1, \ldots, P_n$, such that the following holds: for all $v \in K$, the output feature $Q(f(v))$ is ε-approximated by $\varphi(P_1(v), \ldots, P_n(v))$. By "algebraic", we mean that the approximating operator $\varphi(P_1, \ldots, P_n)$ uses only,in addition to the primitives $P_1, \ldots, P_n$, the algebra operations of $\mathbb{R}^{\mathcal{P}}$, i.e., vector addition, vector multiplication, and scalar addition.

It is shown in [ADIW24]) that:

(1) For a definable $f : L \to \mathcal{L}$, the approximating operators φ may be taken to be *polynomials* of the input features, and
(2) Definable transforms $f : L \to \mathcal{L}$ are precisely those that extend to continuous $\tilde{f} : \mathcal{L} \to \mathcal{L}$ (this is the property of *extendibility* mentioned above).

This motivates the following definition.

Definition 2.4. We say that a CCS $(L, \mathcal{P}, \Gamma)$ satisfies the *Extendability Axiom* if for all $\gamma \in \Gamma$, there is $\tilde{\gamma} : \mathcal{L}_{sh} \to \mathcal{L}_{sh}$ such that for every sizer $r_\bullet$ there is a sizer $s_\bullet$ such that $\tilde{\gamma}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \to \mathcal{L}[s_\bullet]$ is continuous. We refer to $\tilde{\gamma}$ as a *free* extension of γ.

By the preceding remarks, the Extendability Axiom says that the elements of the semigroup Γ are definable.

For an illustrative example, we can frame Newton's polynomial root approximation method in the context of a CCS (see Example 5.6 of [ADIW24] for details) as follows. Begin by considering the extended complex numbers $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with the usual Riemann sphere topology that makes it into a compact space (where unbounded sequences converge to ∞). In fact, not only is this space compact but it is covered by the shard given by the sizer $(1, 1, 1)$ (the unit sphere is contained in the cube $[-1, 1]^3$). The space $\hat{\mathbb{C}}$ is homeomorphic to the usual unit sphere $S^2 := \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ of $\mathbb{R}^3$, by means of the stereographic projection and its inverse $\hat{\mathbb{C}} \to S^2$. This function is regarded as a triple of predicates $x, y, z : \hat{\mathbb{C}} \to [-1, 1]$ where each will map an extended complex number to its corresponding real coordinate on the cube $[-1, 1]^3$. Now fix the cubic complex polynomial $p(s) := s^3 - 1$, and consider the map which performs one step in Newton's method at a particular (extended) complex number s, for finding a root of p, $\gamma_p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The explicit inner workings of γ_p are irrelevant for this example, except for the fact that it is a continuous mapping. It follows that $(S^2, \{x, y, z\}, \{\gamma_p^k : k \in \mathbb{N}\})$ is a CCS. The idea is that repeated applications of $\gamma_p(s), \gamma_p \circ \gamma_p(s), \gamma_p \circ \gamma_p \circ \gamma_p(s), \ldots$ would approximate a root of p provided s was a good enough initial guess.

For a deeper discussion about this axiom, we refer the reader to [ADIW24].

3. Classifying deep computations

3.1. **NIP, Rosenthal compacta, and deep computations.** Under what conditions are deep computations Baire class 1, and thus well-behaved according to our framework, on type-shards? The followiing Theorem says that, under the assumption that $\mathcal{P}$ is countable, the space of deep computations is a Rosenthal compactum (when restricted to shards) if and only if the set of computations has the NIP, feature by feature. Hence, we can import the theory of Rosenthal compacta into this framework of deep computations.

Theorem 3.1. *Let $(L, \mathcal{P}, \Gamma)$ be a compositional computational structure (Definition 2.3) satisfying the Extendability Axiom (Definition 2.4) with $\mathcal{P}$ countable. Let R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R-confined. The following are equivalent.*

(1) $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ *for all* $r_\bullet \in R$.

(2) $\pi_P \circ \Delta|_{L[r_\bullet]}$ *has the NIP for all* $P \in \mathcal{P}$ *and* $r_\bullet \in R$*; that is, for all* $P \in \mathcal{P}$, $r_\bullet \in R$, $a < b$, $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \Delta$ *there are finite disjoint* $E, F \subseteq \mathbb{N}$ *such that*

$$L[r_\bullet] \cap \bigcap_{n\in E} (\pi_P \circ \gamma_n)^{-1}(-\infty, a] \cap \bigcap_{n\in F} (\pi_P \circ \gamma_n)^{-1}[b, \infty) = \emptyset.$$

Moreover, if any (hence all) of the preceding conditions hold, then every deep computation $f \in \overline{\Delta}$ can be extended to a Baire-1 function on shards, i.e., there is $\tilde{f} : \mathcal{L}_{sh} \to \mathcal{L}_{sh}$ such that $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$. In particular, on each shard every deep computation is the pointwise limit of a countable sequence of computations.

Proof. Since $\mathcal{P}$ is countable, $\mathcal{L}[r_\bullet] \subseteq \mathbb{R}^{\mathcal{P}}$ is Polish. Also, the Extendability Axiom implies that $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ is a pointwise bounded set of continuous functions for all $P \in \mathcal{P}$. Hence, Theorem 1.12 and Lemma 1.13 prove the equivalence of (1) and (2). If (1) holds and $f \in \overline{\Delta}$, then write $f = \mathcal{U}\lim_i \gamma_i$ as an ultralimit. Define $\tilde{f} := \mathcal{U}\lim_i \tilde{\gamma}_i$. Hence, for all $r_\bullet \in R$ we have $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in \overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$. That every deep computation is a pointwise limit of a countable sequence of computations follows from the fact that for a Polish space X every compact subset of $B_1(X)$ is Fréchet-Urysohn (that is, a space where topological closures coincide with sequential closures, see Theorem 3F in [BFT78] or Theorem 4.1 in [Deb13]). □

3.2. **The Todorčević trichotomy and levels of PAC learnability.** Given a countable set Δ of computations satisfying the NIP on features and shards (condition (2) of Theorem 3.1) we have that $\overline{\tilde{\Delta}_{\mathcal{L}[r_\bullet]}}$ (for a fixed sizer $r_\bullet$) is a separable *Rosenthal compactum* (compact subset of $B_1(P \times \mathcal{L}[r_\bullet])$. The work of Todorčević ([Tod99]) and Argyros, Dodos, Kanellopoulos ([ADK08]) culminates in a trichotomy theorem for separable Rosenthal Compacta. In this section, inspired by the work of Glasner and Megrelishvili ([GM22]), we study ways in which this classification allows us obtain different levels of PAC-learnability (NIP).

Recall that a topological space X is *hereditarily separable* (HS) if every subspace is separable and that X is *first countable* if every point in X has a countable local basis. Every separable metrizable space is hereditarily separable and it is a result of R. Pol that every hereditarily separable Rosenthal compactum is first countable (see section 10 of [Deb13]). This suggests the following definition:

Definition 3.2. Let $(L, \mathcal{P}, \Gamma)$ be a CCS satisfying the Extendability Axiom and R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be an R-confined countable set of computations satisfying the NIP on shards and features (condition (2) in Theorem 3.1). We say that Δ is:

(i) NIP_1 if $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}}$ is first countable for every $r_\bullet \in R$.
(ii) NIP_2 if $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}}$ is hereditarily separable for every $r_\bullet \in R$.
(iii) NIP_3 if $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}}$ is metrizable for every $r_\bullet \in R$.

Observe that $\mathrm{NIP}_3 \Rightarrow \mathrm{NIP}_2 \Rightarrow \mathrm{NIP}_1 \Rightarrow \mathrm{NIP}$. A natural question that would continue this work is to find examples of CCS that separate these levels of NIP. In [Tod99], Todorčević isolates three canonical examples of Rosenthal compacta that witness the failure of the converse implications above.

We now present some separable and non-separable examples of Rosenthal compacta:

Examples 3.3.

(1) *Alexandroff compactification of a discrete space of size continuum.* For each $a \in 2^{\mathbb{N}}$ consider the map $\delta_a : 2^{\mathbb{N}} \to \mathbb{R}$ given by $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. Let $A(2^{\mathbb{N}}) = \{\delta_a : a \in 2^{\mathbb{N}}\} \cup \{0\}$, where 0 is the zero map. Notice that $A(2^{\mathbb{N}})$ is a compact subset of $B_1(2^{\mathbb{N}})$, in fact $\{\delta_a : a \in 2^{\mathbb{N}}\}$ is a discrete subspace of $B_1(2^{\mathbb{N}})$ and its pointwise closure is precisely $A(2^{\mathbb{N}})$. Hence, this is a Rosenthal compactum which is not first countable. Notice that this space is also not separable.
(2) *Extended Alexandroff compactification.* For each finite binary sequence $s \in 2^{<\mathbb{N}}$, let $v_s : 2^{\mathbb{N}} \to \mathbb{R}$ be given by $v_s(x) = 1$ if x extends s and $v_s(x) = 0$ otherwise. Let $\hat{A}(2^{\mathbb{N}})$ be the pointwise closure of $\{v_s : s \in 2^{<\mathbb{N}}\}$, i.e., $\hat{A}(2^{\mathbb{N}}) = A(2^{\mathbb{N}}) \cup \{v_s : s \in 2^{<\mathbb{N}}\}$. Note that this space is a separable Rosenthal compactum which is not first countable.
(3) *Split Cantor.* Let $<$ be the lexicographic order in the space of infinite binary sequences, i.e., $2^{\mathbb{N}}$. For each $a \in 2^{\mathbb{N}}$ let $f_a^- : 2^{\mathbb{N}} \to \mathbb{R}$ be given by $f_a^-(x) = 1$ if $x < a$ and $f_a^-(x) = 0$ otherwise. Let $f_a^+ : 2^{\mathbb{N}} \to \mathbb{R}$ be given by $f_a^+(x) = 1$ if $x \le a$ and $f_a^+(x) = 0$ otherwise. The split Cantor is the space $S(2^{\mathbb{N}}) = \{f_a^- : a \in 2^{\mathbb{N}}\} \cup \{f_a^+ : a \in 2^{\mathbb{N}}\}$. This is a separable Rosenthal compactum. One example of a countable dense subset is the set of all f_a^+ and f_a^- where a is an infinite binary sequence that is eventually constant. Moreover, it is hereditarily separable but it is not metrizable.
(4) *Alexandroff Duplicate.* Let K be any compact metric space and consider the Polish space $X = C(K) \sqcup K$, i.e., the disjoint union of $C(K)$ (with its supremum norm topology) and K. For each $a \in K$ define $g_a^0, g_a^1 : X \to \mathbb{R}$ as follows:

$$g_a^0(x) = \begin{cases} x(a), & x \in C(K) \\ 0, & x \in K \end{cases}$$

$$g_a^1(x) = \begin{cases} x(a), & x \in C(K) \\ \delta_a(x), & x \in K \end{cases}$$

Let $D(K) = \{g_a^0 : a \in K\} \cup \{g_a^1 : a \in K\}$. Notice that $D(K)$ is a first countable Rosenthal compactum. It is not separable if K is uncountable. The interesting case will be when $K = 2^{\mathbb{N}}$.

(5) *Extended Alexandroff Duplicate of the split Cantor.* For each finite binary sequence $t \in 2^{<\mathbb{N}}$ let $a_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 0's and let $b_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 1's. Define $h_t : 2^{\mathbb{N}} \to \mathbb{R}$ by

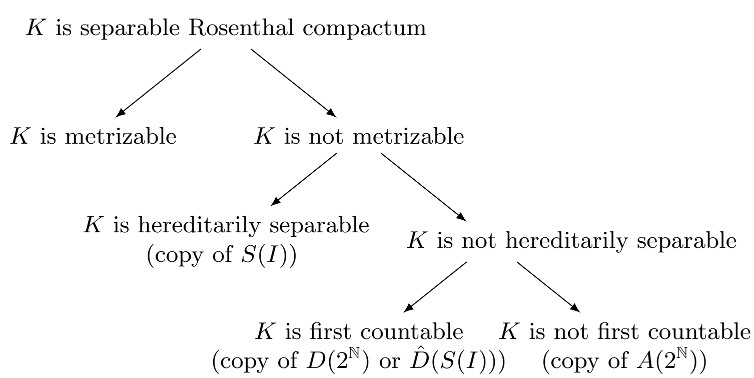
$$h_t(x) = \begin{cases} 0, & x < a_t \\ 1/2, & a_t \leq x \leq b_t \\ 1, & b_t < x \end{cases}$$

Let $\hat{D}(S(2^{\mathbb{N}}))$ be the pointwise closure of the set $\{h_t : t \in 2^{<\mathbb{N}}\}$. Hence, $\hat{D}(S(2^{\mathbb{N}}))$ is a separable first countable Rosenthal compactum which is not hereditarily separable. In fact, it contains an uncountable discrete subspace (see Theorem 5 in [Tod99]).

Theorem 3.4 (Todorčević's Trichotomy, [Tod99], Theorem 3 in [ADK08])**.** *Let K be a separable Rosenthal Compactum.*

(i) If K is hereditarily separable but non-metrizable, then $S(2^{\mathbb{N}})$ embeds into K.
(ii) If K is first countable but not hereditarily separable, then either $D(2^{\mathbb{N}})$ or $\hat{D}(S(2^{\mathbb{N}}))$ embeds into K.
(iii) If K is not first countable, then $A(2^{\mathbb{N}})$ embeds into K.

We thus have the following classification:



The definitions provided here for NIP_i $(i = 1, 2, 3)$ are topological. This raises the following question:

Question 3.5. Is there a non-topological characterization for NIP_i, $i = 1, 2, 3$?

3.3. **The Argyros-Dodos-Kanellopoulos heptachotomy, and approximability of deep computation by minimal classes.** In the three separable three cases given in 3.3, namely, ($\hat{A}(2^{\mathbb{N}})$, $S(2^{\mathbb{N}})$ and $\hat{D}(S(2^{\mathbb{N}}))$), the countable dense subsets are indexed by the binary tree $2^{<\mathbb{N}}$. This choice of index is useful for two reasons:

(1) Our emphasis is computational. Elements of $2^{<\mathbb{N}}$ represent finite bitstrings, i.e., standard computations, while Rosenthal compacta represent deep computations, i.e., limits of finite computations. Mathematically, deep computations are pointwise limits of standard computations; however, computationally, we are interested in the manner (and the efficiency) in which the approximations can occur.
(2) The Ramsey theory of perfect subsets of the Cantor space $2^{\mathbb{N}}$ can be imported to analyze the behavior of the accumulation points. Since $2^{<\mathbb{N}}$ is countable, we can always choose this index for the countable dense subsets. This is done in [ADK08].

Definition 3.6. Let X be a Polish space.

(1) If I is a countable and l$\{f_i : i \in I\} \subseteq \mathbb{R}^X$, $\{g_i : i \in I\} \subseteq \mathbb{R}^X$ are two pointwise families by I. We say that $\{f_i : i \in \ I\}$ and $\{g_i : i \in I\}$ are *equivalent* if and only if the map $f_i \mapsto g_i$ is extended to a homeomorphism from $\overline{\{f_i : i \in I\}}$ to $\overline{\{g_i : i \in I\}}$.
(2) If $\{f_t : t \in 2^{<\mathbb{N}}\}$ is a pointwise bounded family, we say that $\{f_t : t \in 2^{<\mathbb{N}}\}$ is *minimal* if and only if for every dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$, $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to $\{f_t : t \in 2^{<\mathbb{N}}\}$.

One of the main results in [ADK08] is that, up to equivalence, there are seven minimal families of Rosenthal compacta and that for every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$ there is a dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to one of the minimal families. We shall describe the minimal families next. We follow the same notation as in [ADK08]. For any node $t \in 2^{<\mathbb{N}}$, let us denote by $t^\frown 0^\infty$ ($t^\frown 1^\infty$) the infinite binary sequence starting with t and continuing will all 0's (respectively, all 1's). Fix a regular dyadic subtree $R = \{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ (i.e., a dyadic subtree such that every level of R is contained in a level of $2^{<\mathbb{N}}$) with the property that for all $s, s' \in R$, $s^\frown 0^\infty \neq s'^\frown 0^\infty$ and $s^\frown 1^\infty \neq s'^\frown 1^\infty$. Given $t \in 2^{<\mathbb{N}}$, let v_t be the characteristic function of the set $\{x \in 2^{\mathbb{N}} : x \text{ extends } t\}$. Let $<$ be the lexicographic order in $2^{\mathbb{N}}$. Given $a \in 2^{\mathbb{N}}$, let $f_a^+ : 2^{\mathbb{N}} \to \{0,1\}$ be the characteristic function of $\{x \in 2^{\mathbb{N}} : a \leq x\}$ and let $f_a^- : 2^{\mathbb{N}} \to \{0,1\}$ be the characteristic function of $\{x \in 2^{\mathbb{N}} : a < x\}$. Given two maps $f, g : 2^{\mathbb{N}} \to \mathbb{R}$ we denote by $(f,g) : 2^{\mathbb{N}} \sqcup 2^{\mathbb{N}} \to \mathbb{R}$ the function which is f on the first copy of $2^{\mathbb{N}}$ and g on the second copy of $2^{\mathbb{N}}$.

(1) $D_1 = \{\frac{1}{|t|+1} v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_1} = A(2^{\mathbb{N}})$.
(2) $D_2 = \{s_t^\frown 0^\infty : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_2} = 2^{\leq N}$.
(3) $D_3 = \{f^+_{s_t^\frown 0^\infty} : t \in 2^{<\mathbb{N}}\}$. This is a discrete in $\overline{D_3} = S(2^{\mathbb{N}})$.
(4) $D_4 = \{f^-_{s_t^\frown 1^\infty} : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_4} = S(2^{\mathbb{N}})$.
(5) $D_5 = \{v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_5} = \hat{A}(2^{\mathbb{N}})$.
(6) $D_6 = \{(v_{s_t}, s_t^\frown 0^\infty) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_6} = \hat{D}(2^{\mathbb{N}})$.
(7) $D_7 = \{(v_{s_t}, x^+_{s_t^\frown 0^\infty}) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_7} = \hat{D}(S(2^{\mathbb{N}}))$

Theorem 3.7 (Heptacotomy of minimal families, Theorem 2 in [ADK08])**.** *Let X be Polish. For every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$, there exists $i = 1, 2, \ldots, 7$ and a regular dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to D_i. Moreover, all D_i are minimal and mutually non-equivalent.*

3.4. **Bourgain-Fremlin-Talagrand, NIP, and essential computability of deep computations.** We now turn to the question: what happens when $\mathcal{P}$ is uncountable? Notice that the countability assumption is crucial in the proof of Theorem 1.12 essentially because it makes $\mathbb{R}^{\mathcal{P}}$ a Polish space. For the uncountable case, we may lose Baire-1 definability so we shall replace $B_1(X)$ by a larger class. Recall that the *raison d'être* of the class of Baire-1 functions is to have a class that is contains the continuous functions but is closed under pointwise limits, and that (Fact 1.2) for perfectly normal X, a function f is in $B_1(X, Y)$ if and only if $f^{-1}[U]$ is an F_σ subset of X for every open $U \subseteq Y$. This motivates the following definition:

Definition 3.8. Given a Hausdorff space X and a measurable space (Y, Σ), we say that $f : X \to Y$ is *universally measurable* (with respect to Σ) if $f^{-1}(E)$ is Borel for every $E \in \Sigma$, i.e., $f^{-1}(E)$ is μ-measurable for every Radon probability measure μ on X. When $Y = \mathbb{R}$ we will always take $\Sigma = \mathcal{B}(\mathbb{R})$, the Borel σ-algebra of $\mathbb{R}$. In this case, a function $f : X \to \mathbb{R}$ is universally measurable if and only if $f^{-1}(U)$ is μ-measurable for every Radon probability measure μ on X and every open set $U \subseteq \mathbb{R}$.

Intuitively, a function is universally measurable if it is "measurable no matter which reasonable way you try to measure things on its domain". The concept of universal measurability emerged from work of Kallianpur and Sazonov, in the late 1950's and 1960s, , with later developments by Blackwell, Darst, and others, building on earlier ideas of Gnedenko and Kolmogorov from the 1950s. See [Pap02, Chapters 1 and 2].

Following [BFT78], the collection of all universally measurable real-valued functions will be denoted by $M_r(X)$. In the context of deep computations, we will be interested in transition maps from a state space $L \subseteq \mathbb{R}^{\mathcal{P}}$ to itself. There are two natural σ-algebras one can consider in the product space $\mathbb{R}^{\mathcal{P}}$: the Borel σ-algebra, i.e., the σ-algebra generated by open sets in $\mathbb{R}^{\mathcal{P}}$, and the cylinder σ-algebra, i.e., the σ-algebra generated by the sub-basic open sets in $\mathbb{R}^{\mathcal{P}}$. Note that when $\mathcal{P}$ is countable, both σ-algebras coincide but in general the cylinder σ-algebra is strictly smaller. We will use the cylinder σ-algebra to define universally measurable maps $f : \mathbb{R}^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}}$. The reason for this choice is the following characterization:

Lemma 3.9. *Let X be a Hausdorff space and $Y = \prod_{i \in I} Y_i$ be any product of measurable spaces (Y_i, Σ_i) for $i \in I$. Let Σ_Y be the cylinder σ-algebra generated by the measurable spaces (Y_i, Σ_i). Let $f : X \to Y$. The following are equivalent:*

(i) *$f : X \to Y$ is universally measurable (with respect to Σ_Y).*
(ii) *$\pi_i \circ f : X \to Y_i$ is universally measurable (with respect to Σ_i) for all $i \in I$.*

Proof. (i)⇒(ii) is clear since the projection maps π_i are measurable and the composition of measurable functions is measurable. To prove (ii)⇒(i), suppose that $C = \prod_{i \in I} C_i$ is a measurable cylinder and let J be the finite set of $i \in I$ such that $C_i \neq Y_i$. Then, $C = \bigcap_{i \in J} \pi_i^{-1}(C_i)$, so $f^{-1}(C) = \bigcap_{i \in J} (\pi_i \circ f)^{-1}(C_i)$ is a universally measurable set by assumption. □

The preceding lemma says that a transition map is universally measurable if and only if it is universally measurable on all its features. In other words, we can check measurability of a transition just by checking measurability feature by feature. We will denote by $M_r(X, \mathbb{R}^{\mathcal{P}})$ the collection of all universally measurable functions

$f : X \to \mathbb{R}^{\mathcal{P}}$ (with respect to the cylinder σ-algebra), endowed with the topology of pointwise convergence.

We now wish to define the concept of a deep computation being computable except a set of arbitrarily small measure "no matter which reasonable way you try to measure things on its domain" (see the remarks following definition). This is definition below. To motivate the definition, we need to recall two facts:

(1) Littlewoood's second principle states that every Lebesgue measurable function is "nearly continuous". The formal version of this, which is Luzin's theorem, states that if (X, Σ, μ) a Radon measure space and Y be a second-countable topological space (e.g., $Y = \mathbb{R}^{\mathcal{P}}$ with $\mathcal{P}$ countable) equipped with a Borel algebra, then any given $f : X \to Y$ is measurable if and only if for every $E \in \Sigma$ and every $\varepsilon > 0$ there exists a closed $F \subseteq E$ such that the restriction of f to F is continuous.
(2) Computability of deep computations can is characterized in terms of continuous extendibility of computations. This is at the core of [ADIW24].

These facts motivate the following definition:

Definition 3.10. Let $(L, \mathcal{P}, \Gamma)$ be a CCS. We say that a transition $f : L \to L$ is *universally essentially computable* if and only if there exists $\tilde{f} : \mathcal{L}_{sh} \to \mathcal{L}_{sh}$ extending f such that for every sizer $r_\bullet$ there is a sizer $s_\bullet$ such that the restriction $\tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \to \mathcal{L}[s_\bullet]$ is universally measurable, i.e., $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \to [-s_P, s_P]$ is μ-measurable for every Radon probability measure μ on $\mathcal{L}[r_\bullet]$.

For a measure μ on aX, the set of all μ-measurable functions will denoted by $\mathscr{M}^0(X, \mu)$.

We will need the following result about NIP and universally measurable functions:

Theorem 3.11 (Bourgain-Fremlin-Ta set lagrand, Theorem 2F in [BFT78])**.** *Let X be a Hausdorff space and $A \subseteq C(X)$ be pointwise bounded. The following are equivalent:*

(i) $\overline{A} \subseteq M_r(X)$.
(ii) *For every compact $K \subseteq X$, $A|_K$ has the NIP.*
(iii) *For every Radon measure μ on X, A is relatively countably compact in $\mathscr{M}^0(X, \mu)$, i.e., every countable subset of A has an accumulation point in $\mathscr{M}^0(X, \mu)$.*

Theorem 1.8 immediately yields the following.

Theorem 3.12. *Let $(L, \mathcal{P}, \Gamma)$ be a CCS satisfying the Extendability Axiom. Let R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R-confined. If $\pi_P \circ \Delta|_{L[r_\bullet]}$ has the NIP for all $P \in \mathcal{P}$ and all $r_\bullet \in R$, then every deep computation is universally essentially computable.*

Proof. By the Extendability Axiom, Theorem 1.8 and lemma 1.13 we have that $\overline{\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq M_r(\mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$ and $P \in \mathcal{P}$. Let $f \in \overline{\Delta}$ be a deep computation. Write $f = \mathcal{U}\lim_i \gamma_i$ as an ultralimit of computations in Δ. Define $\tilde{f} := \mathcal{U}\lim_i \tilde{\gamma}_i$. Then, for all $r_\bullet \in R$ and $P \in \mathcal{P}$ $\pi_P \circ \tilde{\gamma}_i|_{\mathcal{L}[r_\bullet]} \in M_r(\mathcal{L}[r_\bullet])$ for all i, so $\pi_P \circ f|_{\mathcal{L}[r_\bullet]} \in \overline{\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq M_r(\mathcal{L}[r_\bullet])$. ∎

Question 3.13. Under the same assumptions of the preceding theorem, suppose that every deep computation of Δ is universally essentially computable. Must $\pi_P \circ \Delta|_{L[r_\bullet]}$ have the NIP for all $P \in \mathcal{P}$ and all $r_\bullet \in R$?

3.5. **Talagrand stability, NIP, and essential computability of deep computations.** There is another notion closely related to NIP, introduced by Talagrand in [Tal84] while studying Pettis integration. Suppose that X is a compact Hausdorff space and $A \subseteq \mathbb{R}^X$. Let μ be a Radon probability measure on X. Given a μ-measurable set $E \subseteq X$, a positive integer k and real numbers $a < b$. we write:

$$D_k(A, E, a, b) = \bigcup_{f \in A} \{x \in E^{2k} : f(x_{2i}) \leq a,\ f(x_{2i+1}) \geq b \text{ for all } i < k\}$$

We say that A is *Talagrand μ-stable* if and only if for every μ-measurable set $E \subseteq X$ of positive measure and for every $a < b$ there is $k \geq 1$ such that

$$(\mu^{2k})^*(D_k(A, E, a, b)) < (\mu(E))^{2k},$$

where $\mu*$ denotes the outer measure (we work with outer since the sets $D_k(A, E, a, b)$ need not be μ-measurable). This is certainly the case when A is a countable set of continuous (or μ-measurable) functions.

The following lemma establishes that Talagrand stability is a way to ensure that deep computations are definable by measurable functions. We include a proof for the reader's convenience.

Lemma 3.14. *If A is Talagrand μ-stable, then $\overline{A}$ is also Talagrand μ-stable and $\overline{A} \subseteq \mathscr{M}^0(X, \mu)$.*

Proof. First, observe that a subset of a μ-stable set is μ-stable. To show that $\overline{A}$ is μ-stable, observe that $D_k(\overline{A}, E, a, b) \subseteq D_k(A, E, a', b')$ where $a < a' < b' < b$ and E is a μ-measurable set with positive measure. It suffices to show that $\overline{A} \subseteq \mathscr{M}^0(X, \mu)$. Suppose that there exists $f \in \overline{A}$ such that $f \notin \mathscr{M}^0(X, \mu)$. By a characterization of measurable functions (see 413G in [Fre03]), there exists a μ-measurable set E of positive measure and $a < b$ such that $\mu^*(P) = \mu^*(Q) = \mu(E)$ where $P = \{x \in E : f(x) \leq a\}$ and $Q = \{x \in E : f(x) \geq b\}$. Then, for any $k \geq 1$: $(P \times Q)^k \subseteq D_k(\{f\}, E, a, b)$ so $(\mu^{2k})^*(D_k(\{f\}, E, a, b)) = (\mu^*(P)\mu^*(Q))^k = (\mu(E))^{2k}$. Thus, $\{f\}$ is not μ-stable, but we argued before that a subset of a μ-stable set must be μ-stable. □

We say that A is *universally Talagrand stable* if A is Talagrand μ-stable for every Radon probability measure μ on X. An argument similar to the proof of 3.11, yields the following:

Theorem 3.15. *Let $(L, \mathcal{P}, \Gamma)$ be a CCS satisfying the Extendability Axiom. If $\pi_P \circ \Delta|_{L[r_\bullet]}$ is universally Talagrand stable for all $P \in \mathcal{P}$ and all sizers $r_\bullet$, then every deep computation is universally essentially computable.*

It is then natural to ask: what is the relationship between Talagrand stability and the NIP? The following dichotomy will be useful.

Lemma 3.16 (Fremlin's Dichotomy, 463K in [Fre03])**.** *If (X, Σ, μ) is a perfect σ-finite measure space (in particular, for X compact and μ a Radon probability measure on X) and $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued measurable functions on X, then either:*

(i) $\{f_n : n \in \mathbb{N}\}$ *has a subsequence that converges* μ*-almost everywhere, or*
(ii) $\{f_n : n \in \mathbb{N}\}$ *has a subsequence with no* μ*-measurable accumulation point in* $\mathbb{R}^X$.

The preceding lemma can be considered as a measure-theoretic version of Rosenthal's Dichotomy. Combining this dichotomy with the Theorem 3.11 we get the following result:

Theorem 3.17. *Let* X *be a Hausdorff space and* $A \subseteq C(X)$ *be pointwise bounded. The following are equivalent:*

(i) $\overline{A} \subseteq M_r(X)$.
(ii) *For every compact* $K \subseteq X$, $A|_K$ *has the NIP.*
(iii) *For every Radon measure* μ *on* X, A *is relatively countably compact in* $\mathscr{M}^0(X,\mu)$*, i.e., every countable subset of* A *has an accumulation point in* $\mathscr{M}^0(X,\mu)$.
(iv) *For every Radon measure* μ *on* X *and every sequence* $\{f_n : n \in \mathbb{N}\} \subseteq A$*, there is a subsequence that converges* μ*-almost everywhere.*

Proof. Notice that the equivalence (i)-(iii) is Theorem 3.11. Notice that the equivalence of (iii) and (iv) is Fremlin's Dichotomy (Theorem 3.16). □

Finally, it is natural to ask what the connection is between Talagrand stability and NIP.

Proposition 3.18. *Let* X *be a compact Hausdorff space and* $A \subseteq C(X)$ *be pointwise bounded. If* A *is universally Talagrand stable, then* A *has the NIP.*

Proof. By Theorem 3.11, it suffices to show that A is relatively countably compact in $\mathscr{M}^0(X,\mu)$ for all Radon probability measure μ on X. Since A is Talagrand μ-stable for any such μ, we have $\overline{A} \subseteq \mathscr{M}^0(X,\mu)$. In particular, A is relatively countably compact in $\mathscr{M}^0(X,\mu)$. □

Question 3.19. Is the converse true?

The following two results suggest that the precise connection between Talagrand stability and NIP may be sensitive to set-theoretic axioms (even assuming countability of A).

Theorem 3.20 (Talagrand, Theorem 9-3-1(a) in [Tal84])**.** *Let* X *be a compact Hausdorff space and* $A \subseteq M_r(X)$ *be countable and pointwise bounded. Assume that* $[0,1]$ *is not the union of* $< \mathfrak{c}$ *closed measure zero sets. If* A *has the NIP, then* A *is universally Talagrand stable.*

Theorem 3.21 (Fremlin, Shelah, [FS93])**.** *It is consistent that there exists a countable pointwise bounded set of Lebesgue measurable functions with the NIP which is not Talagrand stable with respect to Lebesgue measure.*

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