

# COMPLEXITY OF DEEP COMPUTATIONS VIA TOPOLOGY OF FUNCTION SPACES

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**ABSTRACT.** We use topological methods to study the complexity of deep computations and limit computations. We use topology of function spaces, specifically, the classification of Rosenthal compacta, to identify new complexity classes. We use the language of model theory, specifically, the concept of *independence* from Shelah’s classification theory, to translate between topology and computation. We use the theory of Rosenthal compacta to characterize approximability of deep computations, both deterministically and probabilistically.

## INTRODUCTION

In this paper we study asymptotic behavior of computations, e.g., the depth of a neural network tending to infinity, or the time interval between layers of a time-series network tending toward zero. Recently, particular cases of this concept have attracted considerable attention in deep learning research (e.g., Neural Ordinary Differential Equations [CRBD], Physics-Informed Neural Networks [RPK19], and deep equilibrium models [BKK]). The formal framework introduced here provides a unified setting to study these limit phenomena from a foundational viewpoint.

Informed by model theory, to each computation in a given computation model, we associate a continuous real-valued function, called the *type* of the computation, that describes the logical properties of this computation with respect to the rest of the model. This allows us to view computations in any given computational model as elements of a space of real-valued functions, which is called the *space of types* of the model. The idea of embedding models of theories into their type spaces is central in model theory. In the context of this paper, the embedding of computations into spaces of types allows us to utilize the vast theory of topology of function spaces, known as  $C_p$ -*theory*, to obtain results about complexity of topological limits of computations. As we shall indicate next, recent classification results for spaces of functions provide an elegant and powerful machinery to classify computations according to their levels of “tameness” or “wildness”, with the former corresponding roughly to polynomial approximability and the latter to exponential approximability. The viewpoint of spaces of types, which we have borrowed from

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model theory, thus becomes a “Rosetta stone” that allows us to interconnect various classification programs: In topology, the classification of Rosenthal compacta pioneered by Todorčević [Tod99]; in logic, the classification of theories developed by Shelah [She90]; and in statistical learning, the notions of PAC learning and VC dimension pioneered by Vapnik and Chervonenkis [VC74, VC71].

In a previous paper [ADIW24], we introduced the concept of limits of computations, which we called *ultracomputations* (given they arise as ultrafilter limits of standard computations) and *deep computations* (following usage in machine learning [BKK]). There is a technical difference between both designations, but in this paper, to simplify the nomenclature, we will ignore the difference and use only the term “deep computation”.

In [ADIW24], we proved a new “tame vs wild” (i.e., polynomially approximable vs nonpolynomially approximable) dichotomy for the complexity of deep computations by invoking a classical result of Grothendieck from the 1950s [Gro52]. Under our model-theoretic Rosetta stone, the property of polynomial approximability of computations is identified with continuous extendibility in the sense of topology, and with the notions of *stability* and *type definability* in model theory.

Simplest among deep computations are those arising as pointwise limits of (continuous) computations. In topology, the *first Baire class*, or *Baire class 1* consists of functions (also called simply “*Baire-1*”) arising as pointwise limits of sequences of continuous functions. Intuitively, the Baire-1 class consists of functions with “controlled” discontinuities, and lies just one level of topological complexity above the Baire class 0 which (by definition) consists of continuous functions.

We prove a new “tame vs wild” Ramsey-theoretic dichotomy for complexity of general deep computations by invoking a famous paper by Bourgain, Fremlin and Talagrand from the late 70s [BFT78], and a new trichotomy for the class of “tame” deep computations by invoking an equally celebrated result of Todorčević, from the late 90s, for functions of the first Baire class [Tod99].

Todorčević’s trichotomy concerns *Rosenthal compacta*; these are special classes of topological spaces, defined as compact spaces that can be embedded (homeomorphically identified as a subset) within the space of Baire class 1 functions on some Polish (separable, completely metrizable) space, under the topology of pointwise convergence; that is, the subspace topology inherited from the product topology. Rosenthal compacta exhibit “topological tameness,” meaning that they behave in relatively controlled ways; since the late 70’s, they have played a crucial role in understanding the complexity of structures of functional analysis, especially Banach spaces. Todorčević’s trichotomy has been utilized to settle longstanding problems in topological dynamics and topological entropy [GM22]. It is noteworthy that Todorčević’s proof relies on sophisticated set-theoretic forcing and infinite Ramsey theory. At the time of writing this paper, decades after his original argument, no elementary proof has been found [Tod23, HT19].

Through our Rosetta stone, Rosenthal compacta in topology correspond to the important concept of “Non Independence Property” (known as “NIP”) in model theory, identified by Shelah [She71, She90], and to the concept of Probably Approximately Correct learning (known as “PAC learnability”) in statistical learning theory identified by Valiant [Val84].

Refining Todorčević’s trichotomy, we invoke a more recent heptachotomy for separable Rosenthal compacta obtained by Argyros, Dodos and Kanellopoulos [ADK08];

77 they identify seven fundamental “prototypes” of separable Rosenthal compacta,  
 78 and show that any non-metrizable separable Rosenthal compactum must contain a  
 79 “canonical” embedding of one of these prototypes.

80 We believe that the results presented in this paper show practitioners of com-  
 81 putation, or topology, or descriptive set theory, or model theory, how classification  
 82 invariants used in their field translate into classification invariants of other fields. In  
 83 the interest of accessibility, we do not assume the reader to have previous familiarity  
 84 with advanced topology, model theory, or computing. The only technical prereq-  
 85 uisites to read this paper are undergraduate-level topology and measure theory.  
 86 The necessary topological background beyond undergraduate topology is covered  
 87 in section 1.

88 In section 1, we present basic topological and combinatorial preliminaries, and  
 89 in section 2, we introduce the structural/model-theoretic viewpoint (no previous  
 90 exposure to model theory is needed). Section 3 is devoted to the classification of  
 91 deep computations.

92 Throughout the paper, our results pertain to classical models of computation  
 93 (particularly computations involving real-valued quantities that are known and  
 94 manipulated to a finite degree of precision). The final section, Section 4, intro-  
 95 duces a probabilistic viewpoint, the development of which we intend to pursue in  
 96 future research, extending the present framework to encompass non-deterministic  
 97 and quantum computations.

98

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In this section we present some preliminaries from general topology and function space theory. In the interest of completeness, we include some proofs that may be safely skipped by readers familiar with these topics.

Recall that a subset of a topological space is  $F_\sigma$  if it is a countable union of closed sets, and  $G_\delta$  if it is a countable intersection of closed sets. A space is metrizable if its topology agrees with the topology induced by some metric therein. Two such metrics inducing the same topology may induce quite different properties in the category of metric spaces. For example, the interval  $(0, 1)$  with the usual metric (as a subset) of the reals is not complete; however,  $(0, 1)$  is homeomorphic to the real line, which is complete with respect to the usual metric thereon. In a metrizable space, every open set is  $F_\sigma$ ; equivalently, every closed set is  $G_\delta$ .

A Polish space is a separable and completely metrizable topological space, i.e., admitting some complete metric inducing its topology. Although other (possibly incomplete) metrics may induce the same topology, being Polish is a purely topological property. One of the most important Polish spaces is the real line  $\mathbb{R}$ ; the others include the Cantor space  $2^{\mathbb{N}}$  and the Baire space  $\mathbb{N}^{\mathbb{N}}$ . The class of Polish spaces is closed under countable topological products; in particular, the Cantor space  $2^{\mathbb{N}}$  (the set of all infinite binary sequences, endowed with the product topology), the Baire space  $\mathbb{N}^{\mathbb{N}}$  (the set of all infinite sequences of naturals, also with the product topology), and the space  $\mathbb{R}^{\mathbb{N}}$  of sequences of real numbers are all Polish. Recall that the product topology on these spaces is the topology of pointwise convergence: a sequence converges in the space if and only if it converges at each coordinate index.

**Fact 1.1.** A subset of a Polish space is itself Polish in the subspace topology if and only if it is a  $G_\delta$  set. In particular, closed subsets and open subsets of Polish spaces are also Polish spaces.

<sup>153</sup> For a proof, see [Eng89, 4.3.24].

Given two topological spaces  $X$  and  $Y$  we denote by  $C_p(X, Y)$  the set of all continuous functions  $f : X \rightarrow Y$  endowed with the topology of pointwise convergence. The space  $C_p(X, \mathbb{R})$  of continuous real functions on  $X$  is denoted simply  $C_p(X)$ . A natural question is, how do topological properties of  $X$  translate into  $C_p(X)$  and vice versa? This general question, and the study of these spaces in general, is the concern of  $C_p$ -theory, an active field of research in general topology which was pioneered by A. V. Arhangel'skiĭ and his students in the 1970's and 1980's [Ark92]. This field has found many applications in model theory and functional analysis. For a recent survey, see [Tka11].

163 A *Baire class 1* function between topological spaces is a function that can be  
 164 expressed as the pointwise limit of a sequence of continuous functions. In symbols,  
 165  $f : X \rightarrow Y$  is *Baire class 1* if there is a sequence of continuous functions  $f_n : X \rightarrow Y$   
 166 such that for all  $x \in X$ ,  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . If  $X$  and  $Y$  are topological spaces,

the space of Baire class 1 functions  $f : X \rightarrow Y$  endowed with the topology of pointwise convergence is denoted  $B_1(X, Y)$  (as above,  $B_1(X, \mathbb{R})$  is denoted  $B_1(X)$ ). Clearly,  $C_p(X, Y) \subseteq B_1(X, Y) \subseteq Y^X$  and we give these the topology (called the *topology of pointwise convergence*) inherited from the product topology of  $Y^X$ . The Baire hierarchy of functions was introduced by René-Louis Baire in his 1899 doctoral thesis, *Sur les fonctions de variables réelles*. His work moved away from the 19th-century preoccupation with “pathological” functions toward a constructive classification based on pointwise limits. An elementary fact about Baire class 1 functions is that they are continuous except on a set of first category (also called a *meager* set, a set of first category is the countable union of sets whose closure has empty interior; intuitively, these sets are “topologically small”). Thus, Baire class 1 functions are continuous on a “topologically large” subset of their domain.

A topological space  $X$  is *perfectly normal* if it is normal and every closed subset of  $X$  is a  $G_\delta$  (equivalently, every open subset of  $X$  is a  $F_\sigma$ ). Every metrizable space (hence, every Polish space) is perfectly normal.

A topological space  $X$  is *Baire* if every countable intersection of dense open sets is dense. The Baire Category Theorem states that every compact Hausdorff or completely metrizable space (hence, every Polish space) is Baire.

The following fact was established by Baire in his 1899 thesis. A proof can be found in [Tod97, Section 10].

**Fact 1.2** (Baire). *If  $X$  is perfectly normal, then the following conditions are equivalent for a function  $f : X \rightarrow \mathbb{R}$ :*

- (1)  *$f$  is a Baire class 1 function, that is,  $f$  is a pointwise limit of continuous functions.*
- (2)  *$f^{-1}[U]$  is an  $F_\sigma$  subset of  $X$  whenever  $U \subseteq \mathbb{R}$  is open.*

*If, moreover,  $X$  is Baire, then (1) and (2) are equivalent to:*

- (3) *For every closed  $F \subseteq X$ , the restriction  $f|_F$  has a point of continuity.*

*Moreover, if  $X$  is Polish and  $f \notin B_1(X)$ , then there exist countable  $D_0, D_1 \subseteq X$  and reals  $a < b$  such that*

$$D_0 \subseteq f^{-1}(-\infty, a], \quad D_1 \subseteq f^{-1}[b, \infty), \quad \overline{D_0} = \overline{D_1}.$$

A subset  $L$  of a topological space  $X$  is *relatively compact* in  $X$  if the closure of  $L$  in  $X$  is compact. Relatively compact subsets of  $B_1(X)$  (for  $X$  Polish) are of interest in analysis and topological dynamics. We begin with the following well-known result. Recall that a set  $A \subseteq \mathbb{R}^X$  of real-valued functions is *pointwise bounded* if for every  $x \in X$  there is  $M_x \geq 0$  (a *pointwise bound at  $x$* ) such that  $|f(x)| \leq M_x$  for all  $f \in A$ . We include a proof for the reader’s convenience:

**Lemma 1.3.** *Let  $X$  be a Polish space and  $A \subseteq B_1(X)$  be pointwise bounded. The following are equivalent:*

- (i)  *$A$  is relatively compact in  $B_1(X)$ .*
- (ii)  *$A$  is relatively countably compact in  $B_1(X)$ , i.e., every countable subset of  $A$  has a limit point in  $B_1(X)$ .*
- (iii)  *$\overline{A} \subseteq B_1(X)$ , where  $\overline{A}$  denotes the closure in  $\mathbb{R}^X$ .*

*Proof.* (i) $\Rightarrow$ (ii) Relatively compact subsets of any space are countably compact therein.

(ii) $\Rightarrow$ (iii) Consider any  $f \in \overline{A}$  and any countable subset  $\{x_i\}_{i \in \mathbb{N}} \subseteq X$ . We claim that there is a sequence  $\{f_n\}_{n \in \mathbb{N}} \subseteq A$  such that  $\lim_{n \rightarrow \infty} f_n(x_i) = f(x_i)$  for all

212     $i \in \mathbb{N}$ . Since  $A$  carries the relative product topology, for each  $n \in \mathbb{N}$  there exists  
 213     $f_n \in A$  such that  $|f_n(x_i) - f(x_i)| < \frac{1}{n}$  for all  $i \leq n$ ; the sequence  $\{f_n\}$  is as claimed.  
 214    Seeking a contradiction, assume that  $A$  is relatively countably compact in  $B_1(X)$ ,  
 215    but there exists some  $f \in \overline{A} \setminus B_1(X)$ . By Fact 1.2, there are countable  $D_0, D_1 \subseteq X$   
 216    with  $\overline{D_0} = \overline{D_1}$ , and  $a < b$  such that  $D_0 \subseteq f^{-1}(-\infty, a]$  and  $D_1 \subseteq f^{-1}[b, \infty)$ . Per  
 217    the claim above, let  $\{f_n\}_{n \in \mathbb{N}} \subseteq A$  satisfy  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in D_0 \cup D_1$   
 218    (the latter being a countable set). By relative countable compactness of  $A$ , there  
 219    is a limit point  $g \in B_1(X)$  of  $\{f_n\}_{n \in \mathbb{N}}$ ; clearly,  $f$  and  $g$  agree on  $D_0 \cup D_1$ . Thus  
 220     $g$  takes values  $g(x_i) = f(x_i) \leq a$  as well as values  $g(x_j) = f(x_j) \geq b > a$  on any  
 221    open subset of the closed set  $\overline{D_0} = \overline{D_1}$ , contradicting the implication (1) $\Rightarrow$ (3) in  
 222    Fact 1.2.

223    (iii) $\Rightarrow$ (i) For each  $x \in X$ , let  $M_x \geq 0$  be a pointwise bound for  $A$ . Since  $\overline{A}$   
 224    is a closed subset of the compact space  $\prod_{x \in X} [-M_x, M_x] \subseteq \mathbb{R}^X$ , it follows that  $\overline{A}$   
 225    is compact. By (iii), it is also the closure of  $A$  in  $B_1(X)$ . Thus,  $A$  is relatively  
 226    compact in  $B_1(X)$ .  $\square$

227    **1.1. From Rosenthal's dichotomy to the Bourgain-Fremlin-Talagrand di-  
 228    chotomy to Shelah's NIP.** In metrizable spaces, points of the closure of some  
 229    subset can always be approximated by points in the set proper, via a convergent  
 230    sequence. For more complicated spaces, such as  $C_p$ -spaces, this fails in remarkable  
 231    ways. The  $n$ -th coordinate map  $p_n : 2^\mathbb{N} \rightarrow \{0, 1\}$  on the Cantor space  $X = 2^\mathbb{N}$   
 232    ( $= \{0, 1\}^\mathbb{N}$ ) is continuous for each  $n \in \mathbb{N}$ , and one can show (e.g., [Tod97, Chap-  
 233    ter 1.1]) that  $\{p_n\}_{n \in \mathbb{N}}$  has no convergent subsequences, in  $\mathbb{R}^X$ . In a sense, this  
 234    example exhibits the worst failure of sequential convergence possible. The closure  
 235    of  $\{p_n\}$  in  $\{0, 1\}^X$  (or in  $\mathbb{R}^X$  for that matter) is homeomorphic to the *Stone-Čech*  
 236    *compactification* of the discrete space of natural numbers, usually denoted  $\beta\mathbb{N}$ ,  
 237    which is an important object of study in general topology.

238    The following theorem, proved by Haskell Rosenthal in 1974, is fundamental in  
 239    functional analysis and captures a sharp division in the behavior of sequences in a  
 240    Banach space.

241    **Theorem 1.4** (Rosenthal's Dichotomy, [Ros74]). *If  $X$  is Polish and  $\{f_n\} \subseteq C_p(X)$   
 242    is pointwise bounded, then  $\{f_n\}_{n \in \mathbb{N}}$  has a convergent subsequence, or a subsequence  
 243    whose closure in  $\mathbb{R}^X$  is homeomorphic to  $\beta\mathbb{N}$ .*

244    Rosenthal's Dichotomy states that a pointwise bounded set of continuous func-  
 245    tions contains either a convergent subsequence, or a subsequence whose closure is  
 246    essentially the same as the example mentioned in the previous paragraphs (i.e.,  
 247    “wildest” possible). The genesis of this theorem was Rosenthal's “ $\ell_1$ -Theorem”,  
 248    which states that a Banach space includes an isomorphic copy of  $\ell_1$  (the space of  
 249    absolutely summable sequences), or else every bounded sequence therein is weakly  
 250    Cauchy. The  $\ell_1$ -Theorem connects diverse areas: Banach space geometry, Ramsey  
 251    theory, set theory, and topology of function spaces.

252    As we move from  $C_p(X)$  to the larger space  $B_1(X)$ , a dichotomy paralleling the  
 253     $\ell_1$ -Theorem holds: Either every point of the closure of a set of functions is a Baire  
 254    class 1 function, or there is a sequence in the set behaving in the wildest possible  
 255    way. This result is usually not phrased as a dichotomy, but rather as an equivalence  
 256    as in Theorem 1.5 below.

257 First, we introduce some useful notation. For any set  $A \subseteq \mathbb{R}^X$  and any real  $a$ ,  
 258 define

$$X_{\leq a}^A := \bigcap_{f \in A} f^{-1}(-\infty, a] = \{x \in X : f(x) \leq a \text{ for all } f \in A\},$$

$$X_{\geq a}^A := \bigcap_{f \in A} f^{-1}[a, +\infty) = \{x \in X : f(x) \geq a \text{ for all } f \in A\}.$$

259 (In case  $A = \emptyset$ , we define  $X_{\leq a}^\emptyset = X = X_{\geq a}^\emptyset$ .) For any sequence  $\{f_n\} \subseteq \mathbb{R}^X$  and  
 260  $I \subseteq \mathbb{N}$ , define  $I^C := \mathbb{N} \setminus I$  and  $f_I := \{f_i : i \in I\}$ .

261 **Theorem 1.5** (“The BFT Dichotomy”. Bourgain-Fremlin-Talagrand [BFT78]).  
 262 Let  $X$  be a Polish space and  $A \subseteq C_p(X)$  be pointwise bounded. The following are  
 263 equivalent:

264 (i)  $A$  is relatively compact in  $B_1(X)$ .

265 (ii) For every  $\{f_n\}_{n \in \mathbb{N}} \subseteq A$  and every  $a < b$  there is  $I \subseteq \mathbb{N}$  such that  $X_{\leq a}^{f_I} \cap X_{\geq b}^{f_{I^C}} = \emptyset$ .

267 (As stated above, the BFT Dichotomy is a particular case of the equivalence  
 268 (ii)  $\Leftrightarrow$  (v) in [BFT78, Corollary 4G].)

269 The sets  $X_{\leq a}^{f_I}$  and  $X_{\geq b}^{f_{I^C}}$  appearing in condition Theorem 1.5(ii) are defined,  
 270 respectively, in terms of  $|I|$ -many inequalities of the form  $f_i(x) \leq a$ , and  $|I^C|$ -many  
 271 of the form  $f_j(x) \geq b$ . Thus, at least one of  $X_{\leq a}^{f_I}$  and  $X_{\geq b}^{f_{I^C}}$  is defined by the  
 272 satisfaction of infinitely (countably) many inequalities. For our purposes, it is more  
 273 natural to consider only finitely many inequalities at a time, which motivates the  
 274 definitions below.

275 **Definition 1.6.** We say that a function collection  $A \subseteq \mathbb{R}^X$  has the *finitary No-*  
 276 *Independence Property (NIP)* if, for all sequences  $\{f_n\}_{n \in \mathbb{N}} \subseteq A$  and reals  $a < b$ ,  
 277 there exist finite disjoint sets  $E, F \subseteq \mathbb{N}$  such that  $X_{\leq a}^{f_E} \cap X_{\geq b}^{f_F} = \emptyset$ . We say that  
 278 such  $E, F$  witness *finitary NIP* for  $A$ ,  $\{f_n\}$  and  $a, b$ .

279 A set  $A \subseteq \mathbb{R}^X$  has the *finitary Independence Property (IP)* if it does not have  
 280 finitary NIP, i.e., if there exists a sequence  $\{f_n\}_{n \in \mathbb{N}} \subseteq A$  and reals  $a < b$  such that  
 281 for every pair of finite disjoint sets  $E, F \subseteq \mathbb{N}$ , we have  $X_{\leq a}^{f_E} \cap X_{\geq b}^{f_F} \neq \emptyset$ .

282 If the word “finite” is omitted in the above definitions, we obtain the definitions of  
 283 *countable NIP* (weaker than finitary NIP) and *countable IP* (stronger than finitary  
 284 IP), respectively.

285 If we insist on witnesses  $E, F \subseteq \mathbb{N}$  such that  $F = E^C$ , we call the respective  
 286 properties “BFT-NIP” (even weaker than countable NIP) and “BFT-IP” (even  
 287 stronger than countable IP). Thus, Theorem 1.5 becomes that statement, for point-  
 288 wise bounded function collections  $A \subseteq C_p(X)$ , that  $A$  is relatively compact in  
 289  $B_1(X)$  if and only if  $A$  has BFT-NIP.

290 Unless otherwise unspecified, IP/NIP shall mean *finitary IP/NIP* henceforth.

291 **Proposition 1.7.** If  $X$  is compact and  $A \subseteq C_p(X)$ , then  $A$  has BFT-NIP if and  
 292 only if it has finitary NIP.

293 (No pointwise boundedness is assumed of  $A$ .)

294 *Proof.* Trivially (as per the preceding discussion), finitary NIP implies BFT-NIP.  
 295 Reciprocally, assume that  $X$  is compact and has finitary IP. Fix  $A \subseteq C_p(X)$ ,

296 a sequence  $\{f_n\} \subseteq A$  and reals  $r < s$ . For any  $I, J \subseteq \mathbb{N}$  (almost disjoint in  
 297 applications), write  $X_{I,J}$  for  $X_{\leq r}^{f_I} \cap X_{\geq s}^{f_J}$ . For  $I \subseteq I' \subseteq \mathbb{N}$  and  $J \subseteq J' \subseteq \mathbb{N}$ , we  
 298 have  $X_{I,J} \supseteq X_{I',J'}$ ; moreover,  $\bar{X}_{I,J} = \bigcap_{E \subseteq I, F \subseteq J} X_{E,F}$ , where the index variables  
 299  $E \subseteq I, F \subseteq J$  range over finite subsets of  $I, J$ , respectively. Clearly,  $E, F \subseteq \mathbb{N}$   
 300 witness finitary NIP for  $\{f_n\}$  if and only if  $X^{E,F} = \emptyset$ .

301 Fix  $I \subseteq \mathbb{N}$ . Since  $\{f_n\} \subseteq A \subseteq C_p(X)$  is a sequence of continuous functions, and  
 302  $X$  is compact, the nested family  $\{X_{E,F} : E \subseteq I, F \subseteq I^c\}$  consists of closed, thus  
 303 compact, sets. Since  $A$  has finitary IP by hypothesis, the nested family consists  
 304 of nonempty sets, hence its intersection  $X_{I,I^c} \neq \emptyset$  by compactness. This holds for  
 305 arbitrary  $\{f_n\} \subseteq A$  and  $r < s$ , so  $A$  has BFT-IP.  $\square$

306 **Theorem 1.8.** *Let  $X$  be a compact metrizable (hence Polish) space. For every  
 307 pointwise bounded  $A \subseteq C_p(X)$ , the following properties are all equivalent:*

- 308   (i)  *$A$  is relatively compact in  $B_1(X)$ ;*
- 309   (ii)  *$A$  has BFT-NIP;*
- 310   (iii)  *$A$  has countable NIP;*
- 311   (iv)  *$A$  has finitary NIP.*

312   (The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) hold for arbitrary compact  $X$ .)

313   *Proof.* Corollary of Theorem 1.5 and Proposition 1.7.  $\square$

314   Theorem 1.8 may be stated as the following dichotomy (under the assumptions):  
 315 either  $A$  is relatively compact in  $B_1(X)$ , or  $A$  has IP (in either sense).

316   The Independence Property was first isolated by Saharon Shelah in model theory  
 317 as a dividing line between theories whose models are “tame” (corresponding to NIP)  
 318 and theories whose models are “wild” (corresponding to IP). See [She71, Definition  
 319 4.1], [She90]. We will discuss this dividing line in more detail in the next section.

320   **1.2. NIP as a universal dividing line between polynomial and exponential  
 321 complexity.** The particular case of the BFT dichotomy (Theorem 1.5) when  
 322  $A$  consists of  $\{0, 1\}$ -valued (i.e., {Yes, No}-valued) strings was discovered inde-  
 323 pendently, around 1971-1972 in many foundational contexts related to polyno-  
 324 mial (“tame”) vs exponential (“wild”) complexity: In model theory, by Saharon  
 325 Shelah [She71], [She90], in combinatorics, by Norbert Sauer [Sau72], and She-  
 326 lah [She72], [She90], and in statistical learning, by Vladimir Vapnik and Alexey  
 327 Chervonenkis [VC71], [VC74].

328   **In model theory:** Shelah’s classification theory is a foundational program  
 329 in mathematical logic devised to categorize first-order theories based on  
 330 the complexity and structure of their models. A theory  $T$  is considered  
 331 classifiable in Shelah’s sense if the number of non-isomorphic models of  $T$   
 332 of a given cardinality can be described by a bounded number of numerical  
 333 invariants. In contrast, a theory  $T$  is unclassifiable if the number of models  
 334 of  $T$  of a given cardinality is the maximum possible number. A key fact  
 335 is that the number of models of  $T$  is directly impacted by the number of types  
 336 over sets of parameters in models of  $T$ ; a controlled number of types  
 337 is a characteristic of a classifiable theory.

In Shelah's classification program [She90], theories without the independence property (called NIP theories, or dependent theories) have a well-behaved, "tame" structure; the number of types over a set of parameters of size  $\kappa$  of such a theory is of polynomially or similar "slow" growth on  $\kappa$ .

In contrast, theories with the Independence Property (called IP theories) are considered "intractable" or "wild". A theory with the Independence Property produces the maximum possible number of types over a set of parameters; for a set of parameters of cardinality  $\kappa$ , the theory will have  $2^{2^\kappa}$ -many distinct types.

**In combinatorics:** Sauer [Sau72] and Shelah [She72] proved the following independently: Let  $\mathcal{F}$  be a family of subsets of some set  $S$ . Either: for every  $n \in \mathbb{N}$  there is a set  $A \subseteq S$  with  $|A| = n$  such that  $|\{S_i \cap A : i \in \mathbb{N}\}| = 2^n$  ( $\mathcal{F}$  has "exponential complexity"); or: there exists  $N \in \mathbb{N}$  such that for every  $A \subseteq S$  with  $|A| \geq N$ , one has

$$|\{S_i \cap A : i \in \mathbb{N}\}| \leq \sum_{i=0}^{N-1} \binom{|A|}{i}.$$

( $\mathcal{F}$  has "polynomial complexity"). Clearly, any family  $\mathcal{F}$  of subsets of a *finite* set  $S$  has polynomial complexity. The "polynomial" name is justified: indeed, for fixed  $N > 0$ , as a function of the size  $|A| = m > 0$ , we have

$$\sum_{i=0}^{N-1} \binom{m}{i} \leq \sum_{i=0}^{N-1} \frac{m^i}{i!} \leq \left( \sum_{i=0}^{N-1} \frac{1}{i!} \right) \cdot m^{N-1} < e \cdot m^{N-1} = O(m^N).$$

(More precisely, the order of magnitude is  $O(m^{N-1})$ : polynomial in  $m$  for  $N$  fixed.)

**In machine learning:** Readers familiar with statistical learning may recognize the Sauer-Shelah lemma as the dichotomy discovered and proved slightly earlier (1971) by Vapnik and Chervonenkis [VC71, VC74] to address uniform convergence in statistics. The least integer  $N$  given by the preceding paragraph, when it exists, is called the *VC-dimension* of  $\mathcal{F}$ ; it is a core concept in machine learning. If such an integer  $N$  does not exist, we say that the VC-dimension of  $\mathcal{F}$  is infinite. The lemma provides upper bounds on the number of data points (sample size) needed to learn a concept class of known VC dimension  $d$  up to a given admissible error in the statistical sense. The Fundamental Theorem of Statistical Learning states that a hypothesis class is PAC-learnable (PAC stands for "Probably Approximately Correct") if and only if its VC dimension is finite.

**1.3. Rosenthal compacta.** The universal classification implied by Theorem 1.5, as attested by the examples outlined in the preceding section, led to the following definition (by Gilles Godefroy [God80]):

**Definition 1.9.** A Rosenthal compactum is any topological space realized as a compact subset of the space  $B_1(X) = B_1(X, \mathbb{R})$  (equipped with the topology of pointwise convergence) of all real functions of the first Baire class on some Polish space  $X$ .

A Rosenthal compactum  $K$  is necessarily Hausdorff since it is a topological subspace of the Hausdorff product space  $\mathbb{R}^X$ .

378 Rosenthal compacta possess significant topological and dynamical tameness properties,  
 379 and play an important role in functional analysis, measure theory, dynamical  
 380 systems, descriptive set theory, and model theory. In this paper, we use them to  
 381 study deep computations. For this, we shall first focus on countable languages,  
 382 which is the theme of the next subsection.

383 **1.4. The special case  $B_1(X, \mathbb{R}^{\mathcal{P}})$  with  $\mathcal{P}$  countable.** Fix an arbitrary (at most)  
 384 countable set  $\mathcal{P}$  whose elements  $P \in \mathcal{P}$  will be called *predicate symbols* or *for-*  
 385 *mal predicates*. Our present goal is to characterize relatively compact subsets of  
 386  $B_1(X, \mathbb{R}^{\mathcal{P}})$ , where  $X$  is always assumed to be a perfectly normal space (often a  
 387 Polish space).

388 The set  $\mathcal{P}$  shall be considered discrete whenever regarded as a topological space.  
 389 Since  $C_p(X, \mathbb{R}^{\mathcal{P}}) \subseteq B_1(X, \mathbb{R}^{\mathcal{P}}) \subseteq (\mathbb{R}^{\mathcal{P}})^X$ , the “ambient” space  $(\mathbb{R}^{\mathcal{P}})^X$  is quite relevant.  
 390 The product  $X \times \mathcal{P}$  will be regarded as either a pointset, or as a topological  
 391 product depending on context. We have natural homeomorphic identifications

$$(\mathbb{R}^{\mathcal{P}})^X \cong \mathbb{R}^{X \times \mathcal{P}} \cong (\mathbb{R}^X)^{\mathcal{P}},$$

392 given by

$$\begin{aligned} \mathbb{R}^{X \times \mathcal{P}} &\rightarrow (\mathbb{R}^{\mathcal{P}})^X : \varphi \mapsto \hat{\varphi} \\ \mathbb{R}^{X \times \mathcal{P}} &\rightarrow (\mathbb{R}^X)^{\mathcal{P}} : \varphi \mapsto \varphi^*, \end{aligned}$$

393 where

$$\hat{\varphi}(x) := \varphi(x, \cdot) \in \mathbb{R}^{\mathcal{P}}, \quad \varphi^*(P) := \varphi(\cdot, P) \in \mathbb{R}^X.$$

394 Such identifications view  $X$  and  $\mathcal{P}$  as mere pointsets (the topology of  $X$  in particular  
 395 plays no role).

396 For  $x \in X$ , define the “left projection” map

$$\lambda_x : \mathbb{R}^{X \times \mathcal{P}} \rightarrow \mathbb{R}^{\mathcal{P}} : \varphi \mapsto \lambda_x(\varphi) := \varphi(x, \cdot);$$

397 for  $P \in \mathcal{P}$ , the “right projection” map

$$\rho_P : \mathbb{R}^{X \times \mathcal{P}} \rightarrow \mathbb{R}^X : \varphi \mapsto \varphi(\cdot, P).$$

398 For fixed  $x \in X$  and  $P \in \mathcal{P}$ , we also have canonical projection maps

$$\pi_x : \mathbb{R}^X \rightarrow \mathbb{R} : f \mapsto f(x), \quad \pi_P : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R} : f \mapsto f(P).$$

399 When clear from context, rather than using the specific symbols (“ $\lambda$ ” for left, “ $\rho$ ”  
 400 for right) to denote projections, we may use the generic symbol “ $\pi$ "; thus,  $\pi_x$  may  
 401 mean  $\lambda_x$ , and  $\pi_P$  may mean  $\rho_P$ .

402 The Proposition below reduces the study of  $\mathbb{R}^{\mathcal{P}}$ -valued continuous or Baire-1  
 403 functions on  $X$  to the special case of real-valued ones.

404 **Proposition 1.10.** *The identification  $(\mathbb{R}^{\mathcal{P}})^X \cong \mathbb{R}^{X \times \mathcal{P}} \cong (\mathbb{R}^X)^{\mathcal{P}}$  induces identifi-*  
 405 *cations*

$$C_p(X, \mathbb{R}^{\mathcal{P}}) \cong C_p(X \times \mathcal{P}) \cong C_p(X)^{\mathcal{P}}, \quad B_1(X, \mathbb{R}^{\mathcal{P}}) \cong B_1(X \times \mathcal{P}) \cong B_1(X)^{\mathcal{P}}.$$

406 (The cardinality of  $\mathcal{P}$  plays no role.)

407 *Proof.* The identification of  $C_p$ -spaces follows trivially from the definition of topo-  
 408 logical product and the fact that  $\mathcal{P}$  is discrete: a continuous map  $X \rightarrow \mathbb{R}^{\mathcal{P}}$  is  
 409 precisely a  $\mathcal{P}$ -indexed family of continuous functions  $X \rightarrow \mathbb{R}$ , and these correspond  
 410 to continuous functions  $X \times \mathcal{P} \rightarrow \mathbb{R}$ . The identification of Baire-1 spaces follows

immediately, since it is defined in terms of the purely topological notion of limit (in the ambient space) of sequences of continuous functions.  $\square$

Given  $A \subseteq Y^X$  and  $K \subseteq X$  we write  $A|_K := \{f|_K : f \in A\}$ , i.e., the set of all restrictions of functions in  $A$  to  $K$ . The following Theorem is a slightly more general version of Theorem 1.5.

**Theorem 1.11.** *Assume that  $\mathcal{P}$  is countable,  $X$  is a Polish space, and  $A \subseteq C_p(X, \mathbb{R}^\mathcal{P})$  is pointwise bounded in the sense that  $\pi_P \circ A$  ( $\subseteq C_p(X)$ ) is pointwise bounded for every  $P \in \mathcal{P}$ . The following are equivalent for every compact  $K \subseteq X$ :*

- (i)  $A|_K$  is relatively compact in  $B_1(K, \mathbb{R}^\mathcal{P})$ ;
- (ii)  $\pi_P \circ A|_K$  has NIP for every  $P \in \mathcal{P}$ .

*Proof.* Compact subsets  $K \subseteq X$  are closed, hence also Polish. Therefore, the asserted equivalence follows from Theorems 1.5 and 1.7.  $\square$

Lastly, a simple but useful lemma that helps understand when we restrict a set of functions to a specific subspace of the domain space, we may always assume that the subspace is closed, as replacing the subspace by its closure has no effect on NIP.

**Lemma 1.12.** *Assume that  $X$  is Hausdorff and that  $A \subseteq C_p(X)$ . The following are equivalent for every  $L \subseteq X$ :*

- (i)  $A_L$  satisfies the NIP;
- (ii)  $A|_{\overline{L}}$  satisfies the NIP.

*Proof.* It suffices to show that (i)  $\Rightarrow$  (ii). Suppose that (ii) does not hold, i.e., that there are  $\{f_n\}_{n \in \mathbb{N}} \subseteq A$  and  $a < b$  such that for all finite disjoint  $E, F \subseteq \mathbb{N}$ :

$$\overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

Pick  $a' < b'$  such that  $a < a' < b' < b$ . Then, for any finite disjoint  $E, F \subseteq \mathbb{N}$  we can choose

$$x \in \overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a') \cap \bigcap_{n \in F} f_n^{-1}(b', \infty)$$

By definition of closure:

$$L \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a'] \cap \bigcap_{n \in F} f_n^{-1}[b', \infty) \neq \emptyset.$$

This contradicts (i).  $\square$

## 2. COMPOSITIONAL COMPUTATION STRUCTURES: A STRUCTURAL APPROACH TO ARBITRARY-PRECISION ARITHMETIC

In this section, we connect function spaces with arbitrary-precision arithmetic computations. We start by summarizing some basic concepts from [ADIW24].

A *computation states structure* is a pair  $(L, \mathcal{P})$ , where  $L$  is a set whose elements we call *states* and  $\mathcal{P}$  is a collection of real-valued functions on  $L$  that we call *predicates*. For a state  $v \in L$ , the *type* of a state  $v$  is the indexed family

$$\text{tp}(v) = (P(v))_{P \in \mathcal{P}} \in \mathbb{R}^\mathcal{P}.$$

For each  $P \in \mathcal{P}$ , we call the value  $P(v)$  the  $P$ -th *feature* of  $v$ . A *transition* of a computation states structure  $(L, \mathcal{P})$  is a map  $f : L \rightarrow L$ .

440     Intuitively,  $L$  is the set of states of a computation, and the predicates  $P \in \mathcal{P}$  are  
 441     primitives that are given and accepted as computable. Each state  $v \in L$  is uniquely  
 442     characterized by its type  $\text{tp}(v)$ , so we may identify  $L$  with a subset of  $\mathbb{R}^{\mathcal{P}}$ . Important  
 443     state spaces are  $L = \mathbb{R}^{\mathbb{N}}$  and  $L = \mathbb{R}^n$  for some positive integer  $n$ , endowed with  
 444     predicate  $P_i(v) = v_i$ , one each for the  $i$ -th coordinate of  $v$ . We regard the space of  
 445     types as a topological space, endowed with the topology of pointwise convergence  
 446     induced by the product topology of  $\mathbb{R}^{\mathcal{P}}$ . Via the identification  $v \mapsto \text{tp}(v)$ , the states  
 447     space  $L$  is correspondingly topologized; in particular, for each  $P \in \mathcal{P}$ , the projection  
 448     map  $v \mapsto P(v)$  is continuous.

449     **Definition 2.1.** Given a computation states structure  $(L, \mathcal{P})$ , any element of  $\mathbb{R}^{\mathcal{P}}$  in  
 450     the image of  $L$  under the map  $v \mapsto \text{tp}(v)$  will be called a *realized (state) type*. The  
 451     topological closure of the set of realized types in  $\mathbb{R}^{\mathcal{P}}$  (endowed with the pointwise  
 452     convergence topology) will be called the *space of types* of  $(L, \mathcal{P})$ , denoted  $\mathcal{L}$ ; elements  
 453      $\xi \in \mathcal{L}$  are called *state types*. Elements of  $\mathcal{L} \setminus L$  will be called *unrealized types*.

454     Intuitively, state types capture a notion of “limit state”.

455     As we combine ideas of model theory [Kei03] and topology [BFT78], we are  
 456     interested in families of real-valued functions that are pointwise bounded. This leads  
 457     us to the concepts of *sizer* and *shard* introduced first in introduced in [ADIW24]:

458     **Definition 2.2.** A *sizer* is a family  $r_{\bullet} = (r_P)_{P \in \mathcal{P}}$  of positive real numbers, indexed  
 459     by  $\mathcal{P}$ . Given a sizer  $r_{\bullet}$ , let  $\mathbb{R}^{[r_{\bullet}]} = \prod_{P \in \mathcal{P}} [-r_P, r_P]$  (a compact space), and let the  
 460      $r_{\bullet}$ -*shard* of a states space  $L$  be

$$L[r_{\bullet}] = L \cap \mathbb{R}^{[r_{\bullet}]}.$$

461     For a sizer  $r_{\bullet}$ , the  $r_{\bullet}$ -*type shard* is defined as  $\mathcal{L}[r_{\bullet}] = \overline{L[r_{\bullet}]}$  (a closed, hence  
 462     compact subset of  $\mathbb{R}^{[r_{\bullet}]}$ ).

463     Let also  $\mathcal{L}_{\text{sh}}$  be the union of all type-shards as the sizer  $r_{\bullet}$  varies.

464     In general,  $\mathcal{L}_{\text{sh}} \subseteq \mathcal{L}$ , and the inclusion may be proper. However, equality holds  
 465     in the important special case when  $\mathcal{P}$  is countable (see [ADIW24]).

## 466     2.1. Compositional Computation Structures.

467     **Definition 2.3.** A *Compositional Computation Structure* (CCS) is a triple  $(L, \mathcal{P}, \Gamma)$ ,  
 468     where

- 469       •  $(L, \mathcal{P})$  is a computation states structure, and
- 470       •  $\Gamma \subseteq L^L$  is a semigroup under composition.

471     Elements of the semigroup  $\Gamma$  are called the *computations* of the structure  $(L, \mathcal{P}, \Gamma)$ .  
 472     We assume that the identity map  $\text{id}$  on  $L$  is an element of  $\Gamma$  (which is thus not  
 473     merely a semigroup but a monoid of transformations of  $L$ ).

474     We topologize  $\Gamma$  as a subset of the topological product  $L^L$ , where the “exponent”  
 475      $L$  serves merely as an index set, but the “base”  $L$  is topologized by type;  
 476     consequently, one may identify  $\Gamma$  with a subset of the topological product  $(\mathbb{R}^{\mathcal{P}})^L$ .  
 477     More specifically,  $\Gamma$  is identified with a subset of  $\mathcal{L}^L$ , which is a closed subspace  
 478     of  $(\mathbb{R}^{\mathcal{P}})^L$ . Therefore, we have an inclusion  $\overline{\Gamma} \subseteq \mathcal{L}^L$ . Elements  $\xi \in \overline{\Gamma}$  are called  
 479     (real-valued) *deep computations* or *ultracomputations*.

480     The reason why we require  $\Gamma$  to be a semigroup is because in many practical  
 481     applications we want to perform an iterative process of computations (e.g., see  
 482     subsection 2.3). In these scenarios we need the set of computations to be closed

under composition. This leads to other concepts that are not addressed in this paper but are rather discussed in [ADIW24, Section 5]. However, in other applications we do not need to work on a set of computations that is closed under composition (e.g., see subsection 2.4). Given a set  $\Delta \subseteq L^L$  of computations (not necessarily a semigroup), we can always take the semigroup  $\Gamma$  generated by  $\Delta$ , i.e., the smallest semigroup containing  $\Delta$ .

A collection  $R$  of sizers is *exhaustive* if  $L = \bigcup_{r_\bullet \in R} L[r_\bullet]$  (shards  $L[r_\bullet]$  exhaust  $L$ ). A transformation  $\gamma \in \Gamma$  is *R-confined* if  $\gamma$  restricts to a map  $\gamma|_{L[r_\bullet]} : L[r_\bullet] \rightarrow L[r_\bullet]$  (into  $L[r_\bullet]$  itself) for every  $r_\bullet \in R$ . A subset  $\Delta \subseteq \Gamma$  is *R-confined* if each  $\gamma \in \Delta$  is.

**Proposition 2.4.** *If  $\Delta \subseteq \Gamma$  is confined by an exhaustive sizer collection, then  $\overline{\Delta}$  is a compact subset of  $\mathcal{L}_{\text{sh}}^L$ .*

*Proof.* Assume that  $R$  confines  $\Delta$ . For each  $v \in L$ , let  $r_\bullet^{(v)} \in R$  be a sizer such that  $v \in L[r_\bullet^{(v)}]$ . An arbitrary  $\gamma \in \Delta$  restricts to a map  $\gamma|_{L[r_\bullet^{(v)}]} : L[r_\bullet^{(v)}] \rightarrow L[r_\bullet^{(v)}]$ , so  $\Gamma \subseteq K := \prod_{v \in L} \mathcal{L}[r_\bullet^{(v)}]$ . The space  $K$  is a product of compact spaces, hence compact, so  $\overline{\Gamma}$  is a closed, hence compact subset thereof, and a subset of  $\mathcal{L}_{\text{sh}}^L \supseteq K$  *a fortiori*.  $\square$

For a CCS  $(L, \mathcal{P}, \Gamma)$ , we regard the elements of  $\Gamma$  as “standard” finitary computations, and the elements of  $\overline{\Gamma}$ , i.e., deep computations, as possibly infinitary limits of standard computations. The main goal of this paper is to study the computability, definability and computational complexity of deep computations. Since deep computations are defined through a combination of topological concepts (namely, topological closure) and structural and model-theoretic concepts (namely, models and types), we will import technology from both topology and model theory.

**2.2. Computability and definability of deep computations and the Extendibility Axiom.** Let  $f : L \rightarrow \mathcal{L}$  be a function that maps each input state type  $(P(v))_{P \in \mathcal{P}} \in \mathbb{R}^\mathcal{P}$  to an output state type  $(P \circ f(v))_{P \in \mathcal{P}} \in \mathbb{R}^\mathcal{P}$ .

(1) We will say that  $f$  is *definable* if for each  $Q \in \mathcal{P}$ , the output feature  $Q \circ f : L \rightarrow \mathbb{R}$  is a definable predicate in the following sense: There is an *approximating function*  $\varphi_{Q, K, \varepsilon} : L \rightarrow \mathbb{R}$  that can be built recursively out of a finite number of the (primitively computable) predicates in  $\mathcal{P}$  and by a finite number of iterations of the finitary lattice operations  $\wedge$  ( $=\min$ ) and  $\vee$  ( $=\max$ ), the operations of  $\mathbb{R}^\mathbb{R}$  as a vector algebra (that is, vector addition and multiplication and scalar multiplication) and the operators sup and inf applied on individual variables from  $L$ , and such that

$$|\varphi_{Q, K, \varepsilon}(v) - Q \circ f(v)| \leq \varepsilon, \quad \text{for all } v \in K.$$

*Remark:* What we have defined above is a model-theoretic concept; it is a special case of the concept of *first-order definability* for real-valued predicates in the model theory of real-valued structures first introduced in [Iov94] for model theory of functional analysis and now standard in model theory (see [Kei03]). The  $\wedge$  ( $=\min$ ) and  $\vee$  ( $=\max$ ) operations correspond to the positive Boolean logical connectives “and” and “or”, and the sup and inf operators correspond to the first-order quantifiers,  $\forall$  and  $\exists$ .

(2) We will say that  $f$  is *computable* if it is definable in the sense defined above under (1), but without the use of the sup/inf operators; in other words, if for every choice of  $Q, K, \varepsilon$ , the approximation function  $\varphi_{Q, K, \varepsilon} : L \rightarrow \mathcal{L}$  can

be constructed without any use of sup or inf operators. This is quantifier-free definability (i.e., definability as given by the preceding paragraph, but without use of quantifiers), which, from a logic viewpoint, corresponds to computability (the presence of the quantifiers  $\exists$  and  $\forall$  are the reason behind the undecidability of first-order logic).

It is shown in [ADIW24] that:

- (1) For a definable  $f : L \rightarrow \mathcal{L}$ , the approximating functions  $\varphi_{Q, K, \varepsilon}$  may be taken to be *polynomials* of the input features, and
- (2) Definable transforms  $f : L \rightarrow \mathcal{L}$  are precisely those that extend to continuous  $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$ .

To summarize, a function  $f : L \rightarrow \mathcal{L}$  is computable if and only if it is definable if and only if it is polynomially approximable if and only if it can be extended to a continuous  $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$ . This is the reason for the following definition.

**Definition 2.5.** We say that a CCS  $(L, \mathcal{P}, \Gamma)$  satisfies the *Extendibility Axiom* if for all  $\gamma \in \Gamma$ , there is  $\tilde{\gamma} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$  such that for every sizer  $r_\bullet$  there is a sizer  $s_\bullet$  such that  $\tilde{\gamma}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$  is continuous. We refer to  $\tilde{\gamma}$  as a *free extension* of  $\gamma$ .

For the rest of the paper, fix for each  $\gamma \in \Gamma$  a free extension  $\tilde{\gamma}$  of  $\gamma$ . For any  $\Delta \subseteq \Gamma$ , let  $\tilde{\Delta}$  denote  $\{\tilde{\gamma} : \gamma \in \Delta\}$ .

For a more detailed discussion of the Extendibility Axiom, we refer the reader to [ADIW24].

**2.3. Newton's method as a CCS.** Let  $p(z)$  be a non-constant polynomial with complex coefficients. We say that  $(L_p, \mathcal{P}, \Gamma_p)$  is *Newton's method for  $p(z)$*  if:

- $L_p$  is the set of all  $z \in \mathbb{C}$  such that there exists an open neighborhood  $U$  of  $z$  such that every sequence in  $\{N_p^n : n \in \mathbb{N}\}$  has a subsequence that converges uniformly on compact subsets of  $U$ , where

$$N_p(z) = z - \frac{p(z)}{p'(z)}$$

and  $N_p^n := N_p \circ N_p \circ \dots \circ N_p$  is the nth-iteration of  $N_p$ .

- $\mathcal{P} := \{P_1, P_2, P_3\}$  where

$$\begin{aligned} P_1(z) &= \frac{2\text{Re}(z)}{|z|^2 + 1}, \\ P_2(z) &= \frac{2\text{Im}(z)}{|z|^2 + 1}, \\ P_3(z) &= \frac{|z|^2 - 1}{|z|^2 + 1}. \end{aligned}$$

- $\Gamma_p := \{N_p^n : n \in \mathbb{N}\}$ .

**Remarks 2.6.**

- (1) The set  $L_p$  is known as the *Fatou set* of  $N_p$  (see section 2 in [Bla84]). Its complement  $\mathbb{C} \setminus L_p$ , is known as the *Julia set* of  $N_p$ .
- (2) The map  $z \mapsto (P_1(z), P_2(z), P_3(z))$  is the stereographic projection into the Riemann Sphere  $S^2$ .
- (3) The set  $L_p$  is open and dense in  $\mathbb{C}$  ([Bla84], Corollary 4.6). Hence, its closure  $\mathcal{L}$  in  $\mathbb{R}^P$  is the Riemann sphere  $S^2$  (i.e., the extended complex plane).

- 559 (4) The set  $L_p$  is completely invariant under iterations of  $N_p$ , i.e.,  $N_p(L_p) = L_p$   
 560 and  $N_p^{-1}(L_p) = L_p$ . This implies that all iterations  $N_p^n : L_p \rightarrow L_p$  are  
 561 transition maps.  
 562 (5)  $\Gamma_p$  is the semigroup generated by  $\{N_p\}$ . Thus,  $(L_p, \mathcal{P}, \Gamma_p)$  is a CCS.

Newton's method is an iterative method that is used to approximate a root of  $p(z)$ . The map  $N_p(z)$  defined above is known as *Newton's map*. The method consists of taking an initial guess  $z_0 \in \mathbb{C}$  and iterating the rational map  $N_p$  to obtain a sequence given by

$$z_{n+1} = N_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)}$$

563 For each root  $r \in \mathbb{C}$  of  $p(z)$ , there exists an  $\varepsilon > 0$  such that for any initial guess  $z_0$   
 564 in the  $\varepsilon$ -ball centered at  $r$ , Newton's iteration converges to  $r$  (provided  $p'(r) \neq 0$ )  
 565 and the convergence is quadratic in that case, meaning the error at each step is  
 566 roughly squared, causing the number of correct digits to double, leading to fast  
 567 convergence.

Given a root  $r$  of  $p(z)$ , the set  $B_r = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} N_p^n(z) = r\}$  is an open set called the *basin* of  $r$ . However, Newton's method can fail to converge to any root for some choices of  $z_0$ . For example, consider the polynomial  $p(z) = z^3 - 2z + 2$ . The Newton map is given by

$$N_p(z) = z - \frac{z^3 - 2z + 2}{3z^2 - 2} = \frac{2z^3 - 2}{3z^2 - 2}$$

568 Notice that taking  $z_0 = 0$  as an initial guess will yield the sequence  $0, 1, 0, 1, 0, 1, \dots$   
 569 that oscillates between 0 and 1 but none of them are roots of  $p(z)$ . Another more  
 570 chaotic way Newton's method can fail to converge is when the sequence of iterations  
 571 has no convergent subsequence. The set of such points, i.e. the *Julia set* associated  
 572 to  $N_p$ , is typically a fractal. This can be visualized by adding a dash of color: let  
 573 us give each complex number  $z_0$  a color  $(R, G, B)$  where  $R, G, B \in [0, 1]$  (so that  
 574  $(1, 0, 0)$  is red,  $(0, 1, 0)$  is green,  $(0, 0, 1)$  is blue and  $(0.5, 0, 0.5)$  is a light purple, for  
 575 example). The values of  $R, G$  and  $B$  are determined by looking at the image of said  
 576 number at each stage of the iteration,  $N_p^n(z_0)$ , and computing the current distance  
 577 to each of the roots of  $p(z)$ ; so  $R = 1/d_r$  where  $d_r$  is the positive distance to the  
 578 root which is colored red, and so on. In this way, the roots themselves are colored  
 579 red, green, and blue, and every other point gets a mix of the three colors. As the  
 580 number of iterations increases, each point gets a sharper color, as the sequence of  
 581 images  $\{N_p^n(z_0)\}_{n=1}^\infty$  converges to one of the three roots. At each stage, the complex  
 582 plane looks as if out of focus because the coloring function is continuous. As the  
 583 reader can see in Figure 1, the points at the boundary of each color class form the  
 584 famous Newton's fractal (of which, interestingly, Newton was unaware).

585 Another example of a Newton's fractal is for  $p(z) = z^3 - 1$ . The roots of  $p(z)$   
 586 are the 3rd roots of unity and the Newton map is given by:

$$N_p(z) = z - \frac{z^3 - 1}{3z^2} = \frac{2z^3 + 1}{3z^2}$$

587 In this case, there are three basins of attraction (one for each root) and the  
 588 complement of their union is the Julia set, i.e., the common boundary.

589 **Proposition 2.7.** Let  $p(z)$  be a non-constant polynomial.  $(L_p, \mathcal{P}, \Gamma_p)$  satisfies the  
 590 Extendibility Axiom.

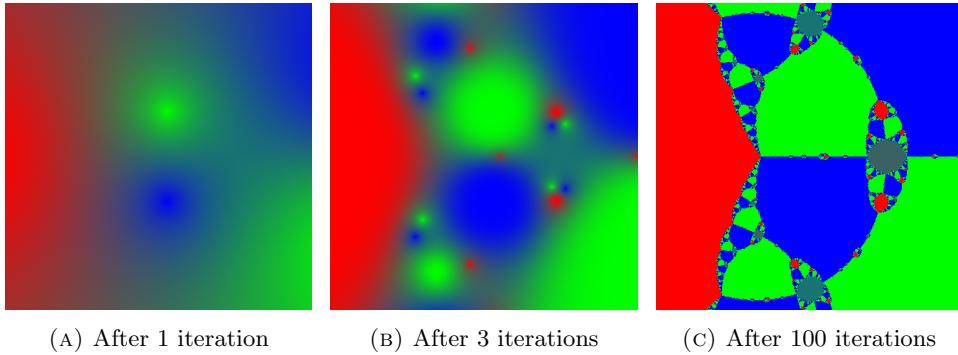


FIGURE 1. Newton's method approximating  $p(z) = z^3 - 2z + 2$ .  
Notice the regions of divergence.

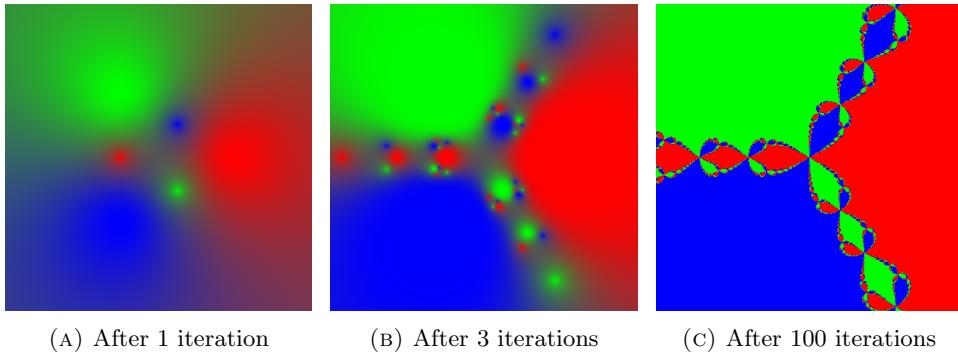


FIGURE 2. Newton's method approximating  $p(z) = z^3 - 1$ .

591 *Proof.*  $N_p : L_p \rightarrow L_p$  is a rational map. Rational maps can be continuously ex-  
592 tended to the extended complex plane, i.e., to  $\mathcal{L}$ . Composition of rational maps  
593 is a rational map, so by the same reasoning, computations  $N_p^n : L_p \rightarrow L_p$  can be  
594 continuously extended to  $\mathcal{L}$ .  $\square$

595 The set of deep computations  $\bar{\Gamma}$  might behave different for various polynomials.  
596 Let us look at various examples:

**Example 2.8. Computation of square roots.** Let  $a$  be a positive real number and  $p(x) = x^2 - a$ . Let  $L = \mathbb{R} \setminus \{0\}$ . Let  $\mathcal{P} = (P_1, P_2)$  where  $x \mapsto (P_1(x), P_2(x))$  is the stereographic projection into  $S^1 \subseteq \mathbb{R}^2$ , i.e.,

$$P_1(x) = \frac{2x}{x^2 + 1},$$

$$P_2(x) = \frac{x^2 - 1}{x^2 + 1}.$$

Let  $\Gamma = \{N_p^n : n \in \mathbb{N}\}$  where

$$N_p(x) = \frac{x^2 + a}{2x}.$$

As before,  $(L, \mathcal{P}, \Gamma)$  is a CCS. Note that  $\mathcal{L} = S^1$  and that each iterate  $N_p^n$  can be continuously extended to the extended real line  $\mathbb{R} \cup \{\infty\}$ , i.e.,  $\mathcal{L}$ . For example,

$$\tilde{N}_p(x) = \begin{cases} \frac{x^2+a}{2x}, & \text{if } x \in L; \\ \infty, & \text{if } x = 0, \infty. \end{cases}$$

For every initial guess  $x \in L$ , the limit  $f(x) = \lim_{n \rightarrow \infty} N_p^n(x)$  converges pointwise to one of the roots. Moreover,

$$f(x) = \begin{cases} \sqrt{a}, & \text{if } x > 0; \\ -\sqrt{a}, & \text{if } x < 0. \end{cases}$$

Notice that  $f$  can be extended to  $\mathcal{L}$  by

$$\tilde{f}(x) = \begin{cases} \sqrt{a}, & \text{if } x > 0; \\ -\sqrt{a}, & \text{if } x < 0; \\ \infty, & \text{if } x = 0, \infty. \end{cases}$$

597 However,  $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$  is not continuous. The set  $\bar{\Gamma}$  of deep computations is  $\bar{\Gamma} \cup \{\tilde{f}\} \subseteq$   
598  $B_1(\mathcal{L}, \mathcal{L})$ .

**Example 2.9. Newton's method for**  $p(z) = z^3 - 2z + 2$ . Let  $r_1, r_2$  and  $r_3$  be the three roots of  $p(z)$ . Let  $B_1, B_2$  and  $B_3$  be their respective basins. Let  $B$  be the basin of the attractive cycle  $0, 1, 0, 1, \dots$ . Then,  $L_p = B \cup \bigcup_{i=1}^3 B_i$ . Notice that  $N_p^n$  does not converge pointwise. However, the subsequences  $N_p^{2n}$  and  $N_p^{2n+1}$  are pointwise convergent to functions  $f_1$  and  $f_2$  respectively.  $f_1$  and  $f_2$  are two distinct deep computations. Note that for  $z \notin L_p$ , no subsequence of  $\tilde{N}_p^n(z)$  converges to a complex number. However, since  $\mathcal{L} = S^2$  is compact there is a subsequence of  $\tilde{N}_p^n(z)$  that converges to  $\infty$ . We can extend  $f_i : L_p \rightarrow \mathcal{L}$  to  $\tilde{f}_i : \mathcal{L} \rightarrow \mathcal{L}$  by:

$$\tilde{f}_i(z) = \begin{cases} f_i(z), & \text{if } z \in L_p; \\ \infty, & \text{if } z \notin L_p. \end{cases}$$

599 Again, note that  $\tilde{f}_i$  for  $i = 1, 2$  are not continuous and that  $\tilde{f}_i \in \bar{\Gamma}$ .

**2.4. Finite precision threshold classifiers as a CCS.** Let  $L = 2^{\mathbb{N}}$ , i.e., the set consisting of all infinite binary sequences with the topology of pointwise convergence. Let  $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$  equal the collection of projections, i.e.,  $P_n(x) = x(n)$  for each  $x \in L$  and  $n \in \mathbb{N}$ . Notice that  $L \subseteq \mathbb{R}^{\mathcal{P}}$  is closed. Therefore,  $\mathcal{L} = L$ . We denote by  $0^\infty$  the infinite binary sequence consisting of 0s, and by  $1^\infty$  the infinite binary sequence consisting of 1s. The set of finite binary strings is denoted by  $2^{<\mathbb{N}}$ . This set is naturally ordered by the lexicographic order  $\leq_{\text{lex}}$ . Given a finite binary string  $w$ , we consider the transition  $\phi_w : L \rightarrow L$  given by the rule

$$\phi_w(x) = \begin{cases} 1^\infty, & \text{if } x|_{|w|} \leq_{\text{lex}} w; \\ 0^\infty, & \text{otherwise,} \end{cases}$$

600 where  $|w|$  is the length of the string  $w$  and  $x|_{|w|}$  is the prefix of  $x$  of length  $|w|$ .  
601 That is,  $\phi_w(x)$  is equal to the constant sequence of ones if  $x|_{|w|}$  comes before or  
602 is equal to  $w$  in the lexicographic order of strings, and it is equal to the constant  
603 sequence of zeros otherwise. In words,  $\phi_w$  checks if a number is less than or equal  
604 to the scalar value of threshold  $w$  (the string  $w$  is finite, hence the classifier has  
605 *finite precision*). Note that  $P_n \circ \phi_w(x) = 1$  if and only if  $x|_{|w|}$  comes before  $w$ .

606 **Proposition 2.10.**  $\phi_w : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$  is continuous for all  $w \in 2^{<\mathbb{N}}$ .

*Proof.* It suffices to prove that  $P_n \circ \phi_w : 2^{\mathbb{N}} \rightarrow \{0, 1\}$  is continuous for all  $n \in \mathbb{N}$ . For simplicity, let us call  $f := P_n \circ \phi_w$ , i.e.,

$$f(x) = \begin{cases} 1, & \text{if } x|_{|w|} \leq_{\text{lex}} w; \\ 0, & \text{otherwise.} \end{cases}$$

607 We first observe that  $f^{-1}(1) = \{x \in 2^{\mathbb{N}} : x|_{|w|} \leq_{\text{lex}} w\}$  is an open set. Fix  $x_0 \in$   
 608  $f^{-1}(1)$ . Let  $t := x_0|_{|w|}$  and consider the basic open set  $[t] = \{x \in 2^{\mathbb{N}} : x|_{|t|} = t\}$ .  
 609 Then it is not difficult to check that  $x_0 \in [t] \subseteq f^{-1}(1)$ . The same reasoning shows  
 610 that  $f^{-1}(0)$  is open.  $\square$

611 Let  $\Delta = \{\phi_w : w \in 2^{<\mathbb{N}}\} \cup \{\mathbf{0}^\infty, \mathbf{1}^\infty\}$ , where  $\mathbf{0}^\infty, \mathbf{1}^\infty : L \rightarrow L$  are the constant  
 612 maps identical to  $0^\infty$  and  $1^\infty$ , respectively. Let  $\Gamma$  be the semigroup generated by  
 613  $\Delta$ . The preceding proposition shows that  $\Delta$  (and hence  $\Gamma$ ) consists of continuous  
 614 functions. In particular, the CCS  $(L, \mathcal{P}, \Gamma)$  satisfies the Extendibility Axiom. In  
 615 contrast with Newton's method, the algebraic structure of  $\Delta$  is quite simple: com-  
 616 posing two classifiers results in something similar to a Boolean logic gate. The  
 617 topological structure is far more interesting. Intuitively, the crucial difference be-  
 618 tween Newton's method and threshold classifiers is that the complexity of the former  
 619 comes from *depth*: the semigroup is generated by a single map but its iterates are  
 620 highly complex. The complexity of threshold classification comes from *width*: the  
 621 semigroup has infinitely many generators, but their compositions are simple.

622 Intuitively, the closure of  $\Delta$  consists of the set of all possible threshold classifiers  
 623 on the real line, but there are two sorts: the ones that classify strict inequalities  
 624 and those that classify  $\leq$ . The members of  $\Delta$  are finite-precision approximations  
 625 of classifiers that check all bits of information. But here it gets interesting: what  
 626 is the difference, in terms of arbitrary-precision arithmetic, between  $x < 0.5$  and  
 627  $x \leq 0.5$ ?

628 Suppose that  $f_a^+$  represents the  $\leq$  classifier for a target  $a \in L$ . We identify  
 629 the scalar truth values with constant sequences, formally  $f_a^+ : L \rightarrow \{0^\infty, 1^\infty\}$  is  
 630 defined by  $f_a^+(x) = 1^\infty$  if  $x \leq_{\text{lex}} a$  and  $f_a^+(x) = 0^\infty$  otherwise. Note that if  $a$  is the  
 631 constant  $1^\infty$ , then  $f_a^+ = \mathbf{1}^\infty$ . Similarly, let  $f_a^-$  be the strict inequality  $<$  classifier,  
 632 i.e.,  $f_a^-(x) = 1^\infty$  if  $x <_{\text{lex}} a$  and  $f_a^-(x) = 0^\infty$  otherwise. Note that if  $a$  is the  
 633 constant zero, then  $f_a^- = \mathbf{0}^\infty$ .

634 **Proposition 2.11.**  $f_a^+, f_a^- \in \overline{\Delta}$  for all  $a \in 2^{\mathbb{N}}$ .

635 *Proof.* First, we show that  $f_a^+ \in \overline{\Delta}$ . If  $a = 1^\infty$ , then  $f_a^+ = \mathbf{1}^\infty \in \Delta$ . If  $a$  is  
 636 not identically 1, we argue that the pointwise limit of the threshold classifiers on  
 637  $w_n := a|_n^\infty 1$  (that is, the sequence obtained from appending a 1 to the first  $n$   
 638 bits of  $a$ ) is precisely  $f_a^+$ . Specifically, for every  $x \in L$ , we intend to prove that  
 639  $\lim_{n \rightarrow \infty} \phi_{w_n}(x) = f_a^+(x)$ . Assume that  $x >_{\text{lex}} a$ . Let  $m$  be the least index at which  
 640 the two sequences differ. Then  $a(m) = 0 < 1 = x(m)$ , and for all  $n \geq m$ ,  $w_n$   
 641 agrees with  $a$  up to  $m$ . Crucially,  $w_n(m) = 0 < 1 = x(m)$ , which implies that  
 642  $w_n <_{\text{lex}} x|_{n+1}$ , and hence  $\phi_{w_n}(x) = 0^\infty = f_a^+(x)$  for large enough  $n$ . If  $x \leq_{\text{lex}} a$ ,  
 643 then  $x|_{n+1} \leq_{\text{lex}} w_n$  for all  $n \in \mathbb{N}$ . Hence,  $\phi_{w_n}(x) = 1^\infty = f_a^+(x)$  for all  $n \in \mathbb{N}$ .

644 Now, we prove that  $f_a^- \in \overline{\Delta}$ . If  $a$  is the constant zero, then  $f_a^- = \mathbf{0}^\infty \in \Delta$ .  
 645 Suppose that  $a$  is not constantly zero; then we have two cases.

- 646 (1) If  $a$  is eventually zero ( $a$  is often called a *dyadic rational*), that is  $a =$   
 647  $u \cap 1 \cap 0^\infty$  (here  $\cap$  denotes concatenation) for some finite  $u$ . Let  $w_n :=$   
 648  $u \cap 0 \cap 1^n <_{\text{lex}} a$ . We claim that  $\lim_{n \rightarrow \infty} \phi_{w_n}(x) = f_a^-(x)$ . Assume that  
 649  $x <_{\text{lex}} a$ . Then,  $x|_{|w_n|} \leq_{\text{lex}} w_n$  for large enough  $n$ . Hence,  $\phi_{w_n}(x) = 1^\infty =$   
 650  $f_a^-(x)$  for large enough  $n$ . Now assume that  $x \geq_{\text{lex}} a$ . Then,  $w_n <_{\text{lex}}$   
 651  $a|_{|w_n|} \leq_{\text{lex}} x|_{|w_n|}$  so  $\phi_{w_n}(x) = 0^\infty = f_a^-(x)$  for all  $n \in \mathbb{N}$ .
- 652 (2) If  $a$  is not eventually zero, enumerate the indices of all positive bits in  $a$ ,  
 653  $\{n \in \mathbb{N} : a(n) = 1\}$ , strictly increasingly as  $\{n_k : k \in \mathbb{N}\}$  (this is possible  
 654 as the former set is infinite by assumption). Let  $w_k := (a|_{n_{k-1}}) \cap 0$ ; that is,  
 655  $w_k$  is the result of flipping the  $k$ -th positive bit in  $a$ . Once again, observe  
 656 that  $w_k <_{\text{lex}} a$  for all  $k$ . The crucial step follows from the fact that for any  
 657  $x <_{\text{lex}} a$ , there is a large enough  $K$  such that  $x <_{\text{lex}} w_k$  for all  $k \geq K$ .

□

659 The preceding proposition shows that the topological structure of deep computa-  
 660 tions can be complicated. Indeed,  $\overline{P_n \circ \Delta}$  contains the *Split Cantor* space for all  
 661  $n \in \mathbb{N}$ . (see Examples 3.3(3)).

**2.5. Finite precision prefix test.** In this subsection we present another example of a CCS with a more complicated set of deep computations. Let  $L = 2^\mathbb{N}$  and  $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$  where  $P_n(x) = x(n)$  are the projection maps so clearly  $L \subseteq \mathbb{R}^\mathcal{P}$  and  $\mathcal{L} = L$  (same computation states structure as subsection 2.4). For each  $w \in 2^{<\mathbb{N}}$ , let  $\psi_w : L \rightarrow L$  be the transition given by:

$$\psi_w(x) = \begin{cases} 1^\infty, & \text{if } x|_{|w|} = w; \\ 0^\infty, & \text{otherwise.} \end{cases}$$

662 In other words,  $\psi_w$  determines whether the first  $|w|$  bits of a binary sequence is  
 663 exactly  $w$ . Let  $\Delta = \{\psi_w : w \in 2^{<\mathbb{N}}\}$  and  $\Gamma$  be the semigroup generated by  $\Delta$ . Since  
 664 the sets  $\{x \in 2^\mathbb{N} : x|_{|w|} = w\}$  are open and closed in  $2^\mathbb{N}$ , then the transitions  $\psi_w$   
 665 are all continuous. In particular,  $(L, \mathcal{P}, \Gamma)$  satisfies the Extendibility Axiom.

666 Let us analyze the set of deep computations of  $\Delta$ . The idea of these finite  
 667 precision prefix tests  $\psi_w$  is that they are approximating the equality relation on  
 668 infinite binary sequences. For a given  $a \in 2^\mathbb{N}$ , let  $\delta_a : L \rightarrow \{0^\infty, 1^\infty\}$  be the  
 669 indicator function at  $a$ , i.e.,  $\delta_a(x) = 1^\infty$  if  $x = a$  and  $\delta_a(x) = 0^\infty$  otherwise.

670 **Proposition 2.12.**  $\delta_a \in \overline{\Delta}$  for all  $a \in 2^\mathbb{N}$ .

671 *Proof.* Fix  $a \in 2^\mathbb{N}$ , and let  $w_n := a|_n$  for each  $n \in \mathbb{N}$ . We claim that  $\lim_{n \rightarrow \infty} \psi_{w_n}(x) =$   
 672  $\delta_a(x)$  for all  $x \in L$ . If  $x = a$ , then  $x|_{|w_n|} = w_n$  for all  $n$  and so  $\psi_{w_n}(x) = 1^\infty = \delta_a(x)$   
 673 for all  $n$ . If  $x \neq a$ , then  $x|_{|w_n|} \neq w_n$  for large enough  $n$ . Hence,  $\psi_{w_n}(x) = 0^\infty =$   
 674  $\delta_a(x)$  for large enough  $n$ . □

675 These equality tests  $\delta_a$  are not all the deep computations. The other deep  
 676 computation we are missing is the constant map  $\mathbf{0}^\infty$ .

677 **Proposition 2.13.**  $\mathbf{0}^\infty \in \overline{\Delta}$ .

678 *Proof.* To show that  $\mathbf{0}^\infty \in \overline{\Delta}$ , for each  $n \in \mathbb{N}$ , consider,  $w_n = 1^n \cap 0$ , i.e., the string  
 679 consisting of  $n$  consecutive 1s followed by a 0. If  $x = 1^\infty$ , then  $x|_{|w_n|} \neq w_n$  for all  
 680  $n \in \mathbb{N}$ . Hence,  $\psi_{w_n}(x) = 0^\infty$  for all  $n \in \mathbb{N}$ . If  $x \neq 1^\infty$ , let  $N$  be the smallest such  
 681 that  $x(N) = 0$ . Then,  $x|_{|w_n|} \neq w_n$  for all  $n > N$ . Hence,  $\psi_{w_n}(x) = 0^\infty$  for large  
 682 enough  $n$ . □

683 In fact,  $\overline{\Delta} = \Delta \cup \{\delta_a : a \in 2^{\mathbb{N}}\} \cup \{\mathbf{0}^\infty\}$  and this space is known as the *Extended*  
 684 *Alexandroff compactification* of  $2^{\mathbb{N}}$  (see Example 3.3(2)). One key topological prop-  
 685 erty about this space is that  $\mathbf{0}^\infty$  is not a  $G_\delta$  point, i.e.,  $\{\mathbf{0}^\infty\}$  is not a countable  
 686 intersection of open sets. Moreover,  $\mathbf{0}^\infty$  is the only non- $G_\delta$  point. It is well-known  
 687 that in a Hausdorff, first countable space every point is  $G_\delta$ . This shows that our  
 688 space of deep computations is not first countable. This space also contains a discrete  
 689 subspace of size continuum, namely  $\{\delta_a : a \in 2^{\mathbb{N}}\}$ .

### 690 3. CLASSIFYING DEEP COMPUTATIONS

691 **3.1. NIP, Rosenthal compacta, and deep computations.** Under what condi-  
 692 tions are deep computations Baire class 1, and thus well-behaved according to our  
 693 framework, on type-shards? The following theorem says that, under the assump-  
 694 tion that  $\mathcal{P}$  is countable, the space of deep computations is a Rosenthal compactum  
 695 (when restricted to shards) if and only if the set of computations satisfies the NIP  
 696 feature by feature. Hence, we can import the theory of Rosenthal compacta into  
 697 this framework of deep computations.

698 **Theorem 3.1.** *Let  $(L, \mathcal{P}, \Gamma)$  be a compositional computational structure (Defini-  
 699 tion 2.3) satisfying the Extendibility Axiom (Definition 2.5) with  $\mathcal{P}$  countable. Let  
 700  $R$  be an exhaustive collection of sizers. Let  $\Delta \subseteq \Gamma$  be  $R$ -confined. The following  
 701 are equivalent.*

- 702 (i)  $\overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$  for all  $r_\bullet \in R$ ;  
 703 (ii)  $\pi_P \circ \tilde{\Delta}|_{L[r_\bullet]}$  satisfies the NIP for all  $P \in \mathcal{P}$  and  $r_\bullet \in R$ ; that is, for all  $P \in \mathcal{P}$ ,  
 704  $r_\bullet \in R$ ,  $a < b$ ,  $\{\gamma_n\}_{n \in \mathbb{N}} \subseteq \Delta$  there are finite disjoint  $E, F \subseteq \mathbb{N}$  such that

$$L[r_\bullet] \cap \bigcap_{n \in E} (\pi_P \circ \gamma_n)^{-1}(-\infty, a] \cap \bigcap_{n \in F} (\pi_P \circ \gamma_n)^{-1}[b, \infty) = \emptyset.$$

705 Moreover, if any (hence all) of the preceding conditions hold, then every deep  
 706 computation  $f \in \overline{\Delta}$  can be extended to a Baire-1 function on shards, i.e., there is  
 707  $\tilde{f} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$  such that  $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$  for all  $r_\bullet \in R$ . In particular, on  
 708 each shard every deep computation is the pointwise limit of a countable sequence of  
 709 computations.

710 *Proof.* Since  $\mathcal{P}$  is countable,  $\mathcal{L}[r_\bullet] \subseteq \mathbb{R}^{\mathcal{P}}$  is Polish. Also, the Extendibility Axiom  
 711 implies that  $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$  is a pointwise bounded set of continuous functions for all  
 712  $P \in \mathcal{P}$ . Hence, Theorem 1.11 and Lemma 1.12 prove the equivalence of (i) and (ii).  
 713 If (i) holds and  $f \in \overline{\Delta}$ , then write  $f = \text{Ulim}_i \gamma_i$  as an ultralimit. Define  $\tilde{f} := \text{Ulim}_i \tilde{\gamma}_i$ .  
 714 Hence, for all  $r_\bullet \in R$  we have  $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in \overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ . That every  
 715 deep computation is a pointwise limit of a countable sequence of computations  
 716 follows from the fact that for a Polish space  $X$  every compact subset of  $B_1(X)$  is  
 717 Fréchet-Urysohn (that is, a space where topological closures coincide with sequential  
 718 closures, see Theorem 3F in [BFT78] or Theorem 4.1 in [Deb13]).  $\square$

719 **3.2. The Todorčević trichotomy, and levels of NIP and PAC learnability.**  
 720 In this subsection we study the case when the set of deep computations is a separable  
 721 Rosenthal compactum. We are interested in the separable case for two reasons:

- 722 (1) In practice, the set  $\Delta$  of computations is countable. This implies that the  
 723 set  $\overline{\Delta}$  of deep computations is separable.

722 (2) The non-separable case lacks some tools and nice examples, which makes  
 723 their study more complicated. In the separable case we have two important  
 724 results, which are introduced in this subsection (Todorčević's Trichotomy)  
 725 and the next subsection (Argyros-Dodos-Kanellopoulos heptachotomy). By  
 726 introducing Todorčević's Trichotomy into this framework, we obtain a clas-  
 727 sification of the complexity of deep computations.

728 Given a countable set  $\Delta$  of computations satisfying the NIP on features and  
 729 shards (condition (ii) of Theorem 3.1), the set  $\tilde{\Delta}_{\mathcal{L}[r_\bullet]}$  (for a fixed sizer  $r_\bullet$ ) is a  
 730 separable *Rosenthal compactum* (see Definition 1.9). Todorčević proved a remark-  
 731 able trichotomy for Rosenthal compacta [Tod99] that was later refined through an  
 732 heptachotomy proved by Argyros, Dodos, Kanellopoulos [ADK08]. In this section,  
 733 inspired by the work of Glasner and Megrelishvili [GM22], we study ways in which  
 734 this classification allows us to obtain different levels of PAC-learnability and NIP.

735 Recall that a topological space  $X$  is *hereditarily separable* if every subspace is  
 736 separable, and that  $X$  is *first countable* if every point in  $X$  has a countable lo-  
 737 cal basis. Every separable metrizable space is hereditarily separable, and R. Pol  
 738 proved that every hereditarily separable Rosenthal compactum is first countable  
 739 (see section 10 of [Deb13]). This suggests the following definition:

740 **Definition 3.2.** Let  $(L, \mathcal{P}, \Gamma)$  be a CCS satisfying the Extendibility Axiom and  $R$   
 741 be an exhaustive collection of sizers. Let  $\Delta \subseteq \Gamma$  be an  $R$ -confined countable set of  
 742 computations satisfying the NIP on shards and features (condition (ii) in Theorem  
 743 3.1). We say that  $\Delta$  is:

- 744 (i)  $\text{NIP}_1$  if  $\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$  is first countable for every  $r_\bullet \in R$ .
- 745 (ii)  $\text{NIP}_2$  if  $\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$  is hereditarily separable for every  $r_\bullet \in R$ .
- 746 (iii)  $\text{NIP}_3$  if  $\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$  is metrizable for every  $r_\bullet \in R$ .

747 Observe that  $\text{NIP}_3 \Rightarrow \text{NIP}_2 \Rightarrow \text{NIP}_1 \Rightarrow \text{NIP}$ . Todorčević, [Tod99], isolated three  
 748 canonical examples of Rosenthal compacta that witness the failure of the converse  
 749 implications above.

750 We now present some separable and non-separable examples of Rosenthal com-  
 751 pacta. These show that the previously discussed classes  $\text{NIP}_i$  are not equal.

### 752 Examples 3.3.

- 753 (1) *Alexandroff compactification of a discrete space of size continuum.* For  
 754 each  $a \in 2^{\mathbb{N}}$  consider the map  $\delta_a : 2^{\mathbb{N}} \rightarrow \mathbb{R}$  given by  $\delta_a(x) = 1$  if  $x = a$   
 755 and  $\delta_a(x) = 0$  otherwise. Let  $A(2^{\mathbb{N}}) = \{\delta_a : a \in 2^{\mathbb{N}}\} \cup \{\mathbf{0}\}$ , where  $\mathbf{0}$   
 756 is the zero map. Notice that  $A(2^{\mathbb{N}})$  is a compact subset of  $B_1(2^{\mathbb{N}})$ . In  
 757 fact,  $\{\delta_a : a \in 2^{\mathbb{N}}\}$  is an uncountable discrete subspace of  $B_1(2^{\mathbb{N}})$ , and its  
 758 pointwise closure is precisely  $A(2^{\mathbb{N}})$ . Hence, this is a Rosenthal compactum  
 759 which is not hereditarily separable (and therefore not first countable). In  
 760 particular, this space is does not satisfy separability, but it can be made  
 761 separable by adding a countable set as the next example shows.
- 762 (2) *Extended Alexandroff compactification.* For each finite binary sequence  $s \in$   
 763  $2^{<\mathbb{N}}$ , let  $v_s : 2^{\mathbb{N}} \rightarrow \mathbb{R}$  be given by  $v_s(x) = 1$  if  $x$  extends  $s$  and  $v_s(x) = 0$   
 764 otherwise. Let  $\hat{A}(2^{\mathbb{N}})$  be the pointwise closure of  $\{v_s : s \in 2^{<\mathbb{N}}\}$ , i.e.,  
 765  $\hat{A}(2^{\mathbb{N}}) = A(2^{\mathbb{N}}) \cup \{v_s : s \in 2^{<\mathbb{N}}\}$ . Note that this space is a separable  
 766 Rosenthal compactum which is not first countable. This is the example

discussed in Section 2.5. It is an example of a CCS that is NIP but not  $\text{NIP}_1$ .

- (3) *Split Cantor.* Let  $<$  be the lexicographic order in the space of infinite binary sequences, i.e.,  $2^{\mathbb{N}}$ . For each  $a \in 2^{\mathbb{N}}$  let  $f_a^- : 2^{\mathbb{N}} \rightarrow \mathbb{R}$  be given by  $f_a^-(x) = 1$  if  $x < a$  and  $f_a^-(x) = 0$  otherwise. Let  $f_a^+ : 2^{\mathbb{N}} \rightarrow \mathbb{R}$  be given by  $f_a^+(x) = 1$  if  $x \leq a$  and  $f_a^+(x) = 0$  otherwise. The split Cantor is the space  $S(2^{\mathbb{N}}) = \{f_a^- : a \in 2^{\mathbb{N}}\} \cup \{f_a^+ : a \in 2^{\mathbb{N}}\}$ , which was obtained as the closure of the space discussed in Section 2.4, giving an example separating  $\text{NIP}_2$  from  $\text{NIP}_3$ . This is a well known separable Rosenthal compactum. One example of a countable dense subset is the set of all  $f_a^+$  and  $f_a^-$  where  $a$  is an infinite binary sequence that is eventually constant. Moreover, it is hereditarily separable, but it is not metrizable. It is homeomorphic to the space  $2^{\mathbb{N}} \times \{0, 1\}$  with the lexicographic order topology via the identification  $(a, 1) \mapsto f_a^+$  and  $(a, 0) \mapsto f_a^-$ .
- (4) *Alexandroff Duplicate.* Let  $K$  be any compact metric space and consider the Polish space  $X = K \sqcup C(K)$ , i.e., the disjoint union of  $C(K)$  (with its supremum norm topology) and  $K$ . For each  $a \in K$  define  $g_a^0, g_a^1 : X \rightarrow \mathbb{R}$  as follows:

$$g_a^0(x) = \begin{cases} 0, & \text{if } x \in K \\ x(a), & \text{if } x \in C(K); \end{cases}$$

$$g_a^1(x) = \begin{cases} \delta_a(x), & \text{if } x \in K; \\ x(a), & \text{if } x \in C(K). \end{cases}$$

Let  $D(K) = \{g_a^0 : a \in K\} \cup \{g_a^1 : a \in K\}$ . Observe that all points  $g_a^1$  are isolated and that open neighborhoods of  $g_{a_0}^0$  are of the form  $\{g_a^0 : a \in U, i \in \{0, 1\}\} \setminus \{g_a^1 : a \in F\}$  where  $U \subseteq K$  is an open neighborhood of  $a_0$  and  $F \subseteq K$  is a finite set. Another abstract way in which this space is presented is as the space  $K \times \{0, 1\}$  whose basic open neighborhoods are given as before, identifying  $(a, 0) \mapsto g_a^0$  and  $(a, 1) \mapsto g_a^1$ . We can also embed  $D(K)$  into the product  $A(K) \times K$  by identifying  $(a, 0) \mapsto (\mathbf{0}, a)$  and  $(a, 1) \mapsto (\delta_a, a)$ . Notice that  $D(K)$  is a first countable Rosenthal compactum. It is not separable if  $K$  is uncountable, thus we typically study the interesting case when  $K = 2^{\mathbb{N}}$ . As with the Alexandroff compactification  $A(2^{\mathbb{N}})$ , we can make the space  $D(2^{\mathbb{N}})$  separable by adding a countable set. For example, the closure of the set  $\{(v_s, s^\frown 0^\infty) : s \in 2^{<\mathbb{N}}\} \subseteq \hat{A}(2^{\mathbb{N}}) \times 2^{\mathbb{N}}$  is  $\{(\mathbf{0}, a) : a \in 2^{\mathbb{N}}\} \cup \{(\delta_a, a) : a \in 2^{\mathbb{N}}\} \cup \{(v_s, s^\frown 0^\infty) : s \in 2^{<\mathbb{N}}\}$ , where  $\{(\mathbf{0}, a) : a \in 2^{\mathbb{N}}\} \cup \{(\delta_a, a) : a \in 2^{\mathbb{N}}\}$  is homeomorphic to  $D(2^{\mathbb{N}})$ .

- (5) *Extended Alexandroff Duplicate of the split Cantor.* For each finite binary sequence  $t \in 2^{<\mathbb{N}}$  let  $a_t \in 2^{\mathbb{N}}$  be the sequence starting with  $t$  and ending with 0's and let  $b_t \in 2^{\mathbb{N}}$  be the sequence starting with  $t$  and ending with 1's. Define  $h_t : 2^{\mathbb{N}} \rightarrow \mathbb{R}$  by

$$h_t(x) = \begin{cases} 0, & \text{if } x < a_t; \\ 1/2, & \text{if } a_t \leq x \leq b_t; \\ 1, & \text{if } b_t < x. \end{cases}$$

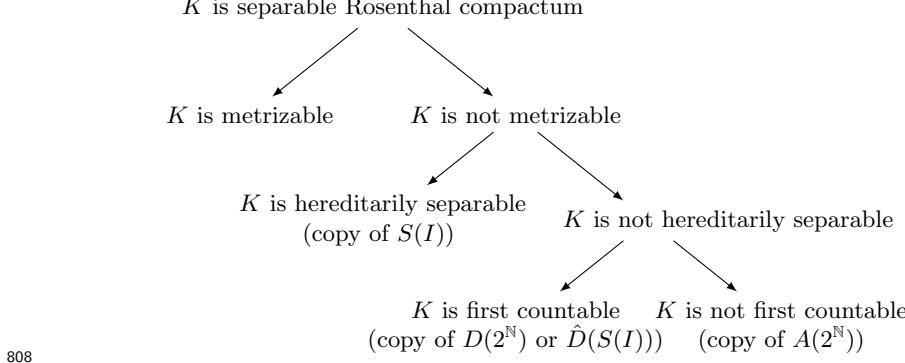
Let  $\hat{D}(S(2^{\mathbb{N}}))$  be the pointwise closure of the set  $\{h_t : t \in 2^{<\mathbb{N}}\}$ . The identification  $h_t \mapsto (v_t, f_{t^\frown 0^\infty}^+)$  lifts to a homeomorphism between  $\hat{D}(S(2^{\mathbb{N}}))$

and the subspace of  $\hat{A}(2^{\mathbb{N}}) \times S(2^{\mathbb{N}})$  consisting of  $(\mathbf{0}, f_a^+), (\mathbf{0}, f_a^-), (\delta_a, f_a^+)$  and  $(v_t, f_{t^\frown 0^\infty}^+)$  for  $a \in 2^{\mathbb{N}}$  and  $t \in 2^{<\mathbb{N}}$  (see 4.3.7 in [ADK08]). Hence,  $\hat{D}(S(2^{\mathbb{N}}))$  is a separable first countable Rosenthal compactum which is not hereditarily separable. In fact, it contains an uncountable discrete subspace.

**Theorem 3.4** (Todorčević's Trichotomy, [Tod99], Theorem 3 in [ADK08]). *Let  $K$  be a separable Rosenthal Compactum.*

- (i) *If  $K$  is hereditarily separable but non-metrizable, then  $S(2^{\mathbb{N}})$  embeds into  $K$ .*
- (ii) *If  $K$  is first countable but not hereditarily separable, then either  $D(2^{\mathbb{N}})$  or  $\hat{D}(S(2^{\mathbb{N}}))$  embeds into  $K$ .*
- (iii) *If  $K$  is not first countable, then  $A(2^{\mathbb{N}})$  embeds into  $K$ .*

We thus have the following classification:



Todorčević's Trichotomy suggests that in order to distinguish the classes  $\text{NIP}_i$ , the examples in 3.3 are essential. The following examples show that the levels  $\text{NIP}_i$  ( $i = 1, 2, 3$ ) may be distinguished by the topological complexity of deep computations.

### Examples 3.5.

- (1) Let  $(L, \mathcal{P}, \Gamma)$  be the computation of square root (example 2.8 with  $\Delta = \Gamma$ ). We saw that  $\bar{\Delta} = \tilde{\Delta} \cup \{\tilde{f}\} \subseteq B_1(\mathcal{L}, \mathcal{L})$ . This corresponds to the Alexandroff compactification of a countable discrete set, which is metrizable. Hence,  $\Delta$  is  $\text{NIP}_3$  but it is not stable, in the sense that  $\bar{\Delta} \not\subseteq C(\mathcal{L}, \mathcal{L})$ .
- (2) Let  $(L, \mathcal{P}, \Gamma)$  be the finite precision threshold classifiers (Section 2.4) with  $\Delta = \{\phi_w : w \in 2^{<\mathbb{N}}\} \cup \{\mathbf{0}^\infty, \mathbf{1}^\infty\}$ . We saw that  $\bar{\Delta}$  is homeomorphic to the Split Cantor space  $S(2^{\mathbb{N}})$  (Example 3.3(3)), which is hereditarily separable but not metrizable. Hence,  $\Delta$  is  $\text{NIP}_2$  but not  $\text{NIP}_3$ .
- (3) Let  $(L, \mathcal{P}, \Gamma)$  be the CCS given by  $L = 2^{\mathbb{N}}$ ,  $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$  and  $\Gamma$  is the semigroup generated by  $\Delta = \{\gamma_t : t \in 2^{<\mathbb{N}}\}$ , where  $P_n : 2^{\mathbb{N}} \rightarrow \{0, 1\}$  is the projection map  $P_n(x) = x(n)$  and  $\gamma_t : L \rightarrow L$  is given by

$$\gamma_t(x) = \begin{cases} 0^\infty, & \text{if } x <_{\text{lex}} t^\frown 0^\infty; \\ (01)^\infty, & \text{if } t^\frown 0^\infty \leq_{\text{lex}} x \leq_{\text{lex}} t^\frown 1^\infty; \\ 1^\infty, & \text{if } t^\frown 1^\infty <_{\text{lex}} x. \end{cases}$$

where  $(01)^\infty$  denotes the sequence of alternating bits:  $010101\dots$ . As in the other examples, it is not difficult to see that  $(L, \mathcal{P}, \Gamma)$  satisfies the Extendibility Axiom. For example, the condition  $t^\frown 0^\infty \leq_{\text{lex}} x \leq_{\text{lex}} t^\frown 1^\infty$  is

equivalent to  $x$  extending  $t$ . Observe that the set of deep computations is homeomorphic to  $\hat{D}(S(2^{\mathbb{N}}))$  (see Example 3.3(5)). This is an example of  $\Delta$  which is NIP<sub>1</sub> but not NIP<sub>2</sub>.

- (4) Let  $(L, \mathcal{P}, \Gamma)$  be the finite precision prefix test (Section 2.5) with  $\Delta = \{\psi_w : w \in 2^{<\mathbb{N}}\}$ . We saw that  $\overline{\Delta}$  is homeomorphic to the Extended Alexandroff compactification  $\hat{A}(2^{\mathbb{N}})$  (Example 3.3-(3)), which is separable but not first countable. Hence,  $\Delta$  is NIP but not NIP<sub>1</sub>.

The definitions provided here for NIP<sub>i</sub> ( $i = 1, 2, 3$ ) are topological. This raises the following question:

**Question 3.6.** Is there a non-topological characterization for NIP<sub>i</sub>,  $i = 1, 2, 3$ ?

**3.3. The Argyros-Dodos-Kanellopoulos heptachotomy, and approximability of deep computation by minimal classes.** In the three separable cases given in 3.3, namely,  $\hat{A}(2^{\mathbb{N}})$ ,  $S(2^{\mathbb{N}})$  and  $\hat{D}(S(2^{\mathbb{N}}))$ , the countable dense subsets are indexed by the binary tree  $2^{<\mathbb{N}}$ . This choice of index is useful for two reasons:

- (1) Our emphasis is computational. Real numbers can be represented as infinite binary sequences, i.e., infinite branches of the binary tree  $2^{<\mathbb{N}}$ . We approximate real numbers or binary sequences with elements in  $2^{<\mathbb{N}}$ , i.e., finite bitstrings. Indexing standard computations with finite bitstrings allow us to better understand how deep computations arise and how they get approximated. Computationally, we are interested in the manner (and the efficiency) in which the approximations can occur.
- (2) Infinite branches of the binary tree  $2^{<\mathbb{N}}$  correspond to the Cantor space  $2^{\mathbb{N}}$ , the canonical perfect set (in the sense that any Polish space with no isolated points contains a copy of  $2^{\mathbb{N}}$ ). The use of infinite dimensional Ramsey theory for trees (pioneered by the work of James D. Halpern, Hans Läuchli in [HL66] and also Keith Milliken in [Mil81], and Alain Louveau, Saharon Shelah, Boban Velickovic in [LSV93]) and perfect sets (Fred Galvin and Andreas Blass in [Bla81], Arnold W. Miller in [Mil89], and Stevo Todorcević in [Tod99]) allowed S.A. Argyros, P. Dodos and V. Kanellopoulos in [ADK08] to obtain an improved version of Theorem 3.4. It is no surprise that Ramsey Theory becomes relevant in the study of Rosenthal compacta as it was a key ingredient in Rosenthal's  $\ell_1$  Theorem. For this reason, the main results in [ADK08] (which we cite in this paper) are better explained by indexing Rosenthal compacta with the binary tree.

**Definition 3.7.** Let  $X$  be a Polish space.

- (1) If  $I$  is countable and  $\{f_i : i \in I\} \subseteq \mathbb{R}^X$ ,  $\{g_i : i \in I\} \subseteq \mathbb{R}^X$  are two pointwise families by  $I$ , we say that  $\{f_i : i \in I\}$  and  $\{g_i : i \in I\}$  are *equivalent* if and only if the map  $f_i \mapsto g_i$  is extended to a homeomorphism from  $\overline{\{f_i : i \in I\}}$  to  $\overline{\{g_i : i \in I\}}$ .
- (2) If  $\{f_t : t \in 2^{<\mathbb{N}}\}$  is a pointwise bounded family, we say that  $\{f_t : t \in 2^{<\mathbb{N}}\}$  is *minimal* if and only if for every dyadic subtree  $\{s_t : t \in 2^{<\mathbb{N}}\}$  of  $2^{<\mathbb{N}}$ ,  $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$  is equivalent to  $\{f_t : t \in 2^{<\mathbb{N}}\}$ .

One of the main results in [ADK08] is that, up to equivalence, there are seven minimal families of Rosenthal compacta and that for every relatively compact  $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$  there is a dyadic subtree  $\{s_t : t \in 2^{<\mathbb{N}}\}$  such that  $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$

is equivalent to one of the minimal families. We shall describe the seven minimal families next. We follow the same notation as in [ADK08]. For any node  $t \in 2^{<\mathbb{N}}$ , let us denote by  $t^\frown 0^\infty$  ( $t^\frown 1^\infty$ ) the infinite binary sequence starting with  $t$  and continuing with all 0's (respectively, all 1's). Fix a regular dyadic subtree  $R = \{s_t : t \in 2^{<\mathbb{N}}\}$  of  $2^{<\mathbb{N}}$  (i.e., a dyadic subtree such that every level of  $R$  is contained in a level of  $2^{<\mathbb{N}}$ ) with the property that for all  $s, s' \in R$ ,  $s^\frown 0^\infty \neq s' \frown 0^\infty$  and  $s^\frown 1^\infty \neq s' \frown 1^\infty$ . Given  $t \in 2^{<\mathbb{N}}$ , let  $v_t$  be the characteristic function of the set  $\{x \in 2^\mathbb{N} : x \text{ extends } t\}$ . Let  $<$  be the lexicographic order in  $2^\mathbb{N}$ . Given  $a \in 2^\mathbb{N}$ , let  $f_a^+ : 2^\mathbb{N} \rightarrow \{0, 1\}$  be the characteristic function of  $\{x \in 2^\mathbb{N} : a \leq x\}$  and let  $f_a^- : 2^\mathbb{N} \rightarrow \{0, 1\}$  be the characteristic function of  $\{x \in 2^\mathbb{N} : a < x\}$ . Given two maps  $f, g : 2^\mathbb{N} \rightarrow \mathbb{R}$  we denote by  $(f, g) : 2^\mathbb{N} \sqcup 2^\mathbb{N} \rightarrow \mathbb{R}$  the function which is  $f$  on the first copy of  $2^\mathbb{N}$  and  $g$  on the second copy of  $2^\mathbb{N}$ .

- (1)  $D_1 = \{\frac{1}{|t|+1} v_t : t \in 2^{<\mathbb{N}}\}$ . This is discrete in  $\overline{D_1} = A(2^\mathbb{N})$ .
- (2)  $D_2 = \{s_t^\frown 0^\infty : t \in 2^{<\mathbb{N}}\}$ . This is discrete in  $\overline{D_2} = 2^{\leq N}$ .
- (3)  $D_3 = \{f_{s_t^\frown 0^\infty}^+ : t \in 2^{<\mathbb{N}}\}$ . This is discrete in  $\overline{D_3} = S(2^\mathbb{N})$ .
- (4)  $D_4 = \{f_{s_t^\frown 1^\infty}^- : t \in 2^{<\mathbb{N}}\}$ . This is discrete in  $\overline{D_4} = S(2^\mathbb{N})$ .
- (5)  $D_5 = \{v_t : t \in 2^{<\mathbb{N}}\}$ . This is discrete in  $\overline{D_5} = \hat{A}(2^\mathbb{N})$ .
- (6)  $D_6 = \{(v_{s_t}, s_t^\frown 0^\infty) : t \in 2^{<\mathbb{N}}\}$ . This is discrete in  $\overline{D_6} = \hat{D}(2^\mathbb{N})$ .
- (7)  $D_7 = \{(v_{s_t}, x_{s_t^\frown 0^\infty}^+) : t \in 2^{<\mathbb{N}}\}$ . This is discrete in  $\overline{D_7} = \hat{D}(S(2^\mathbb{N}))$ .

**Theorem 3.8** (Heptachotomy of minimal families, Theorem 2 in [ADK08]). *Let  $X$  be Polish. For every relatively compact  $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$ , there exists  $i = 1, 2, \dots, 7$  and a regular dyadic subtree  $\{s_t : t \in 2^{<\mathbb{N}}\}$  of  $2^{<\mathbb{N}}$  such that  $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$  is equivalent to  $D_i$ . Moreover, all  $D_i$  are minimal and mutually non-equivalent.*

The implication of this result for deep computations is the following: for any countable set of computations  $\Delta$  satisfying the NIP (for some CCS  $(L, \mathcal{P}, \Gamma)$ ), we can always find a countable discrete set of deep computations that approximates all the other deep computations. For example: in the finite precision prefix test example (subsection 2.5), the prefix test computations (family  $D_5$ ) approximate all other deep computations. However, note that this discrete set  $D_i$  may not consist of continuous functions (i.e., they will not be computable in general). For example, functions in  $D_3$  are not continuous.

#### 4. RANDOMIZED VERSIONS OF NIP AND MONTE CARLO COMPUTABILITY OF DEEP COMPUTATIONS

In this section, we replace deterministic computability by probabilistic ('Monte Carlo') computability. We do not assume that  $\mathcal{P}$  is countable. The main results of the section are Theorem 4.8 (connecting NIP and Monte Carlo computability) and 4.14 (connecting Talagrand stability and Monte Carlo computability).

Fundamental in this section is a measure-theoretic version of Theorem 1.11, namely, Theorem 4.5. For the proof of Theorem 1.11, we assumed countability of  $\mathcal{P}$  — this ensured that  $\mathbb{R}^\mathcal{P}$  a Polish space. In this section, the countability assumption is not needed.

**4.1. NIP and Monte Carlo computability of deep computations.** The *raison d'être* of the Baire class-1 functions is to work with a class of functions that are obtained as limit points of continuous functions (deep computations) but are

not too far from being continuous. By Fact 1.2, for perfectly normal  $X$ , a function  $f$  is in  $B_1(X, Y)$  if and only if  $f^{-1}[U]$  is an  $F_\sigma$  subset of  $X$  for every open  $U \subseteq Y$ . Recall that a function  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}[U]$  is open in  $X$  for every open  $U \subseteq Y$ . In other words, we can view the class  $B_1(X, Y)$  as a weaker notion of continuity. In this section we will study a different (and larger) class of functions: the space of universally measurable functions.

**Definition 4.1.** Given a Hausdorff space  $X$  and a measurable space  $(Y, \Sigma)$ , we say that  $f : X \rightarrow Y$  is *universally measurable* (with respect to  $\Sigma$ ) if  $f^{-1}(E)$  is  $\mu$ -measurable for every Radon measure  $\mu$  on  $X$  and every  $E \in \Sigma$ . When  $Y = \mathbb{R}$ , we will always take  $\Sigma = \mathcal{B}(\mathbb{R})$ , the Borel  $\sigma$ -algebra of  $\mathbb{R}$ .

If  $X$  is a compact (Hausdorff) space, then every Radon measure  $\mu$  on  $X$  is finite. Then, the measure given by  $\nu(A) := \mu(A)/\mu(X)$  is a probability measure on  $X$  with the same null sets as  $\mu$ . Hence, Radon measures on compact spaces are equivalent to (Radon) probability measures. We summarize this fact in the next remark:

**Remark 4.2.** If  $X$  is compact, then a function  $f : X \rightarrow \mathbb{R}$  is universally measurable if and only if  $f^{-1}(U)$  is  $\mu$ -measurable for every Radon probability measure  $\mu$  on  $X$  and every open set  $U \subseteq \mathbb{R}$ .

Intuitively, a function is universally measurable if it is “measurable no matter which reasonable way you try to measure things on its domain”. The concept of universal measurability emerged from work of Kallianpur and Sazonov, in the late 1950’s and 1960s — with later developments by Blackwell, Darst and others — building on earlier ideas of Gnedenko and Kolmogorov from the 1950s. See [Pap02, Chapters 1 and 2].

**Notation 4.3.** Following [BFT78], the collection of all universally measurable real-valued functions on  $X$  will be denoted by  $M_r(X)$ . Given a fixed Radon measure  $\mu$  on  $X$ , the collection of all  $\mu$ -measurable real-valued functions on  $X$  will be denoted by  $\mathcal{M}^0(X, \mu)$ .

In the context of deep computations, we are interested in transition maps of a state space  $L \subseteq \mathbb{R}^\mathcal{P}$  into itself. There are two natural  $\sigma$ -algebras one can consider in the product space  $\mathbb{R}^\mathcal{P}$ : the Borel  $\sigma$ -algebra, i.e., the  $\sigma$ -algebra generated by open sets in  $\mathbb{R}^\mathcal{P}$ , and the cylinder  $\sigma$ -algebra (i.e., the  $\sigma$ -algebra generated by the sub-basic open sets in  $\mathbb{R}^\mathcal{P}$ , that is, the sets  $\pi_P^{-1}(U)$  with  $U \subseteq \mathbb{R}$  open and  $P \in \mathcal{P}$ ). Note that when  $\mathcal{P}$  is countable, both  $\sigma$ -algebras coincide, but in general the cylinder  $\sigma$ -algebra is strictly smaller. We will use the cylinder  $\sigma$ -algebra to define universally measurable maps  $f : \mathbb{R}^\mathcal{P} \rightarrow \mathbb{R}^\mathcal{P}$ . The reason for this choice is the following characterization:

**Proposition 4.4.** Let  $X$  be a Hausdorff space and  $Y = \prod_{i \in I} Y_i$  be any product of measurable spaces  $(Y_i, \Sigma_i)$  for  $i \in I$ . Let  $\Sigma_Y$  be the cylinder  $\sigma$ -algebra generated by the measurable spaces  $(Y_i, \Sigma_i)$ . The following are equivalent for  $f : X \rightarrow Y$ :

- (i)  $f : X \rightarrow Y$  is universally measurable (with respect to  $\Sigma_Y$ ).
- (ii)  $\pi_i \circ f : X \rightarrow Y_i$  is universally measurable (with respect to  $\Sigma_i$ ) for all  $i \in I$ .

**Proof.** (i) $\Rightarrow$ (ii) is clear since the projection maps  $\pi_i$  are measurable and the composition of measurable functions is measurable. To prove (ii) $\Rightarrow$ (i), suppose that  $C = \prod_{i \in I} C_i$  is a measurable cylinder and let  $J$  be the finite subset of  $I$  such that

959  $C_i \neq Y_i$  for  $i \in J$ . Then,  $C = \bigcap_{i \in J} \pi_i^{-1}(C_i)$ , so  $f^{-1}(C) = \bigcap_{i \in J} (\pi_i \circ f)^{-1}(C_i)$  is  
 960 universally measurable by assumption.  $\square$

961 The preceding proposition says that a transition map is universally measurable  
 962 if and only if it is universally measurable on all its features; in other words, we can  
 963 check measurability of a transition just by checking measurability feature by feature.  
 964 This is the same as in the Baire class-1 case (compare with Proposition 1.10).

965 The main result in section 3 is that, as long as we work with countably many  
 966 features, PAC-learning (or NIP) corresponds to relative compactness in the space  
 967 of Baire class-1 functions. The following result (which does not assume countability  
 968 of the number of features) gives an analogous characterization of the NIP in terms  
 969 of universal measurability:

970 **Theorem 4.5** (Bourgain-Fremlin-Talagrand, Theorem 2F in [BFT78]). *Let  $X$  be a  
 971 Hausdorff space and  $A \subseteq C(X)$  be pointwise bounded. The following are equivalent:*

- 972 (i)  $\overline{A} \subseteq M_r(X)$ .
- 973 (ii) For every compact  $K \subseteq X$ ,  $A|_K$  satisfies the NIP.
- 974 (iii) For every Radon measure  $\mu$  on  $X$ ,  $A$  is relatively countably compact in  $\mathcal{M}^0(X, \mu)$ ,  
 975 i.e., every countable subset of  $A$  has a limit point in  $\mathcal{M}^0(X, \mu)$ .

976 This result allows us to formalize the concept of a deep computation being *Monte  
 977 Carlo computable*, which we define below (Definition 4.6). To motivate the defini-  
 978 tion, let us first recall two facts:

- 979 (1) Littlewoood's second principle states that every Lebesgue measurable func-  
 980 tion is "nearly continuous". The formal statement of this, which is Luzin's  
 981 theorem, is that if  $(X, \Sigma, \mu)$  a Radon measure space and  $Y$  is a second-  
 982 countable topological space (e.g.,  $Y = \mathbb{R}^{\mathcal{P}}$  with  $\mathcal{P}$  countable) equipped with  
 983 the Borel  $\sigma$ -algebra, then any given  $f : X \rightarrow Y$  is measurable if and only if  
 984 for every  $E \in \Sigma$  and every  $\varepsilon > 0$  there exists a closed  $F \subseteq E$  such that the  
 985 restriction  $f|_F$  is continuous and  $\mu(E \setminus F) < \varepsilon$ .
- 986 (2) Computability of deep computations is characterized in terms of continuous  
 987 extendibility of computations. This is at the core of [ADIW24].

988 These two facts motivate the following definition:

989 **Definition 4.6.** Let  $(L, \mathcal{P}, \Gamma)$  be a CCS. We say that a transition  $f : L \rightarrow L$   
 990 is *universally Monte Carlo computable* if and only if there exists  $\tilde{f} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$   
 991 extending  $f$  such that for every sizer  $r_\bullet$  there is a sizer  $s_\bullet$  such that the restriction  
 992  $\tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$  is universally measurable, i.e.,  $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow [-s_P, s_P]$   
 993 is  $\mu$ -measurable for every Radon probability measure  $\mu$  on  $\mathcal{L}[r_\bullet]$  and  $P \in \mathcal{P}$ .

994 *Remark 4.7.* Condition (2) of Theorem 4.5 shows that to study measure-theoretic  
 995 versions of NIP, we need only consider compact subsets of  $X$ . Now, every Radon  
 996 measure on a compact space is finite; hence, by proper normalization, it can be  
 997 treated as a probability measure. Therefore, in the context of Monte Carlo measur-  
 998 ability, we focus on Radon probability measures rather than general Radon mea-  
 999 sures.

1000 **Theorem 4.8.** *Let  $(L, \mathcal{P}, \Gamma)$  be a CCS satisfying the Extendibility Axiom. Let  $R$   
 1001 be an exhaustive collection of sizers. Let  $\Delta \subseteq \Gamma$  be  $R$ -confined. If  $\pi_P \circ \Delta|_{L[r_\bullet]}$   
 1002 satisfies the NIP for all  $P \in \mathcal{P}$  and all  $r_\bullet \in R$ , then every deep computation in  $\Delta$   
 1003 is universally Monte Carlo computable.*

1004 *Proof.* Fix  $P \in \mathcal{P}$  and  $r_\bullet \in R$ . By the Extendibility Axiom,  $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$  is a  
1005 set of pointwise bounded continuous functions on the compact set  $\mathcal{L}[r_\bullet]$ . Since  
1006  $\pi_P \circ \tilde{\Delta}|_{L[r_\bullet]} = \pi_P \circ \Delta|_{L[r_\bullet]}$  has the NIP, so does  $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$  by Lemma 1.12. By  
1007 Theorem 4.5, we have  $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$  for all  $r_\bullet \in R$  and  $P \in \mathcal{P}$ . Let  
1008  $f \in \overline{\Delta}$  be a deep computation. Write  $f = \mathcal{U} \lim_i \gamma_i$  as an ultralimit of computations  
1009 in  $\Delta$ . Define  $\tilde{f} := \mathcal{U} \lim_i \tilde{\gamma}_i$ . Then,  $\tilde{f} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$  extends  $f$ . Since  $\Delta$  is  $R$ -confined  
1010 we have that  $f : L[r_\bullet] \rightarrow L[r_\bullet]$  and  $\tilde{f} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[r_\bullet]$  for all  $r_\bullet \in R$ . Lastly, note that  
1011 for all  $r_\bullet \in R$  and  $P \in \mathcal{P}$  we have that  $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_\bullet]} \in \pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$ .  $\square$

1012 **Question 4.9.** Under the same assumptions of the preceding theorem, suppose  
1013 that every deep computation of  $\Delta$  is universally Monte Carlo computable. Must  
1014  $\pi_P \circ \Delta|_{L[r_\bullet]}$  have the NIP for all  $P \in \mathcal{P}$  and all  $r_\bullet \in R$ ?

1015 **4.2. Talagrand stability and Monte Carlo computability of deep computations.** There is another notion closely related to NIP, introduced by Talagrand  
1016 in [Tal84] while studying Pettis integration. Suppose that  $X$  is a compact Haus-  
1017 dorff space and  $A \subseteq \mathbb{R}^X$ . Let  $\mu$  be a Radon probability measure on  $X$ . Given a  
1018  $\mu$ -measurable set  $E \subseteq X$ , a positive integer  $k$  and real numbers  $a < b$ , we write:

$$D_k(A, E, a, b) = \bigcup_{f \in A} \{x \in E^{2k} : f(x_{2i}) \leq a, f(x_{2i+1}) \geq b \text{ for all } i < k\}.$$

1020 We say that  $A$  is *Talagrand  $\mu$ -stable* if and only if for every  $\mu$ -measurable set  
1021  $E \subseteq X$  of positive measure and for every  $a < b$  there is a  $k \geq 1$  such that

$$(\mu^{2k})^*(D_k(A, E, a, b)) < (\mu(E))^{2k},$$

1022 where  $\mu^*$  denotes the outer measure (we need to work with outer measure since  
1023 the sets  $D_k(A, E, a, b)$  need not be  $\mu^{2k}$ -measurable). The inequality certainly holds  
1024 when  $A$  is a countable set of continuous (or  $\mu$ -measurable) functions.

1025 The main result of this section is that deep computations, i.e., limit points  
1026 of a Talagrand stable set of computations are Monte Carlo computable; this is  
1027 Theorem 4.14 below. We now prove that limit points of a Talagrand  $\mu$ -stable set  
1028 are  $\mu$ -measurable. But first, let us state the following useful characterization of  
1029 measurable functions (compare with Fact 1.2):

1030 **Fact 4.10** (Lemma 413G in [Fre03]). *Suppose that  $(X, \Sigma, \mu)$  is a measure space  
1031 and  $\mathcal{K} \subseteq \Sigma$  is a collection of measurable sets satisfying the following conditions:*

- 1032 (1)  *$(X, \Sigma, \mu)$  is complete, i.e., for all  $E \in \Sigma$  with  $\mu(E) = 0$  and  $F \subseteq E$  we have  
1033  $F \in \Sigma$ .*
- 1034 (2)  *$(X, \Sigma, \mu)$  is semi-finite, i.e., for all  $E \in \Sigma$  with  $\mu(E) = \infty$  there exists  
1035  $F \subseteq E$  such that  $F \in \Sigma$  and  $0 < \mu(F) < \infty$ .*
- 1036 (3)  *$(X, \Sigma, \mu)$  is saturated, i.e.,  $E \in \Sigma$  if and only if  $E \cap F \in \Sigma$  for all  $F \in \Sigma$   
1037 with  $\mu(F) < \infty$ .*
- 1038 (4)  *$(X, \Sigma, \mu)$  is inner regular with respect to  $\mathcal{K}$ , i.e., for all  $E \in \Sigma$*

$$\mu(E) = \sup\{\mu(K) : K \in \mathcal{K} \text{ and } K \subseteq E\}.$$

1038 *(In particular, if  $X$  is compact Hausdorff,  $\mu$  is a Radon probability measure on  $X$ ,  
1039  $\Sigma$  is the completion of the Borel  $\sigma$ -algebra by  $\mu$ , and  $\mathcal{K}$  is the collection of compact  
1040 subsets of  $X$ , all these conditions hold). Then,  $f : X \rightarrow \mathbb{R}$  is measurable if and*

1041 only if for every  $K \in \mathcal{K}$  with  $0 < \mu(K) < \infty$  and  $a < b$ , either  $\mu^*(P) < \mu(K)$  or  
 1042  $\mu^*(Q) < \mu(K)$  where  $P = \{x \in K : f(x) \leq a\}$  and  $Q = \{x \in K : f(x) \geq b\}$ .

1043 The following technical lemma will be instrumental for proving Proposition 4.13,  
 1044 which, in turn, will yield the main result of the subsection, namely Theorem 4.14.

1045 **Lemma 4.11.** *If  $A$  is Talagrand  $\mu$ -stable, then  $\overline{A}$  is also Talagrand  $\mu$ -stable and  
 1046  $\overline{A} \subseteq \mathcal{M}^0(X, \mu)$ .*

1047 *Proof.* First, we claim that a subset of a  $\mu$ -stable set is  $\mu$ -stable. To see this,  
 1048 suppose that  $A \subseteq B$  and  $B$  is  $\mu$ -stable. Fix any  $\mu$ -measurable  $E \subseteq X$  of positive  
 1049 measure and  $a < b$ . Let  $k \geq 1$  such that

$$(\mu^{2k})^*(D_k(B, E, a, b)) < (\mu(E))^{2k}.$$

1050 Since  $A \subseteq B$ , we have  $D_k(A, E, a, b) \subseteq D_k(B, E, a, b)$ ; therefore,

$$(\mu^{2k})^*(D_k(A, E, a, b)) \leq (\mu^{2k})^*(D_k(B, E, a, b)) < (\mu(E))^{2k}.$$

1051 We now show that  $\overline{A}$  is  $\mu$ -stable. Fix  $E \subseteq X$  measurable with positive measure  
 1052 and  $a < b$ . Let  $a' < b'$  be such that  $a < a' < b' < b$ . Since  $A$  is  $\mu$ -stable, let  $k \geq 1$   
 1053 be such that

$$(\mu^{2k})^*(D_k(A, E, a', b')) < (\mu(E))^{2k}.$$

1054 If  $x \in D_k(\overline{A}, E, a, b)$ , then there is  $f \in \overline{A}$  such that  $f(x_{2i}) \leq a < a'$  and  $f(x_{2i+1}) \geq b > b'$  for all  $i < k$ . By definition of pointwise convergence topology, there exists  $g \in E$   
 1055 such that  $g(x_{2i}) < a'$  and  $g(x_{2i+1}) > b'$  for all  $i < k$ . Hence,  $x \in D_k(A, E, a', b')$ .  
 1056 We have shown that  $D_k(\overline{A}, E, a, b) \subseteq D_k(A, E, a', b')$ ; hence,

$$(\mu^{2k})^*(D_k(\overline{A}, E, a, b)) \leq (\mu^{2k})^*(D_k(A, E, a', b')) < (\mu(E))^{2k}.$$

1057 It suffices to show that  $\overline{A} \subseteq \mathcal{M}^0(X, \mu)$ . Suppose that there exists  $f \in \overline{A}$  such that  
 1058  $f \notin \mathcal{M}^0(X, \mu)$ . By fact 4.10, there exists a  $\mu$ -measurable set  $E$  of positive measure  
 1059 and  $a < b$  such that  $\mu^*(P) = \mu^*(Q) = \mu(E)$  where  $P = \{x \in E : f(x) \leq a\}$  and  
 1060  $Q = \{x \in E : f(x) \geq b\}$ . Then, for any  $k \geq 1$ :  $(P \times Q)^k \subseteq D_k(\{f\}, E, a, b)$ , so  
 1061  $(\mu^{2k})^*(D_k(\{f\}, E, a, b)) = (\mu^*(P)\mu^*(Q))^k = (\mu(E))^{2k}$ . Thus,  $\{f\}$  is not  $\mu$ -stable.  
 1062 However, we argued above that a subset of a  $\mu$ -stable set must be  $\mu$ -stable, so we  
 1063 have a contradiction.  $\square$

1064 **Definition 4.12.** We say that  $A$  is *universally Talagrand stable* if  $A$  is Talagrand  
 1065  $\mu$ -stable for every Radon probability measure  $\mu$  on  $X$ .

1066 We first observe that universal Talagrand stability corresponds to a complexity  
 1067 class smaller than or equal to the NIP class:

1068 **Proposition 4.13.** *Let  $X$  be a compact Hausdorff space and  $A \subseteq C(X)$  be point-  
 1069 wise bounded. If  $A$  is universally Talagrand stable, then  $A$  satisfies the NIP.*

1070 *Proof.* By Theorem 4.5, it suffices to show that  $A$  is relatively countably compact  
 1071 in  $\mathcal{M}^0(X, \mu)$  for every Radon probability measure  $\mu$  on  $X$ . Since  $A$  is Talagrand  
 1072  $\mu$ -stable for any such  $\mu$ , we have  $\overline{A} \subseteq \mathcal{M}^0(X, \mu)$  by Lemma 4.11. In particular,  $A$   
 1073 is relatively countably compact in  $\mathcal{M}^0(X, \mu)$ .  $\square$

1074 **Corollary 4.14.** *Let  $(L, \mathcal{P}, \Gamma)$  be a CCS satisfying the Extendibility Axiom. If  
 1075  $\pi_P \circ \Delta|_{L[r_\bullet]}$  is universally Talagrand stable for all  $P \in \mathcal{P}$  and all sizers  $r_\bullet$ , then  
 1076 every deep computation is universally Monte Carlo computable.*

1077 *Proof.* This is a direct consequence of Proposition 4.13 and Theorem 4.8.  $\square$

1079 In the context of deep computations, we have identified two ways to obtain Monte  
 1080 Carlo computability, namely, NIP/PAC and Talagrand stability. It is natural to  
 1081 ask whether these two notions are equivalent. The following results show that,  
 1082 even in the simple case of countably many computations, this question is sensitive  
 1083 to the set-theoretic axioms. On the one hand, it is consistent (with respect to the  
 1084 standard ZFC axioms of set theory) that these two classes are the same:

1085 **Theorem 4.15** (Talagrand, Theorem 9-3-1(a) in [Tal84]). *Let  $X$  be a compact  
 1086 Hausdorff space and  $A \subseteq M_r(X)$  be countable and pointwise bounded. Assume that  
 1087  $[0, 1]$  is not the union of  $< \mathfrak{c}$  closed measure zero sets. If  $A$  satisfies the NIP, then  
 1088  $A$  is universally Talagrand stable.*

1089 (The assumption that  $[0, 1]$  is not the union of  $< \mathfrak{c}$  closed measure zero sets is a  
 1090 consequence of, for example, the Continuum Hypothesis.)

1091 On the other hand, by fixing a particular well-known measure, namely the  
 1092 Lebesgue measure, we see that the other case is also consistent:

1093 **Theorem 4.16** (Fremlin, Shelah, [FS93]). *It is consistent with the usual axioms of  
 1094 set theory that there exists a countable pointwise bounded set of Lebesgue measur-  
 1095 able functions with the NIP which is not Talagrand stable with respect to Lebesgue  
 1096 measure.*

1097 Notice that the preceding two results apply to sets of measurable functions, a  
 1098 class of functions larger than the class of continuous functions. However, by the  
 1099 Extendibility Axiom, finitary computations are continuous, i.e., if  $A$  is a set of  
 1100 computations, then  $A \subseteq C_p(X)$ . The question of whether we can remove the set-  
 1101 theoretic assumption in Theorem 4.15 when  $A \subseteq C_p(X)$  (instead of  $A \subseteq M_r(X)$ )  
 1102 remains open.

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