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COMPLEXITY OF DEEP COMPUTATIONS VIA TOPOLOGY OF FUNCTION SPACES

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ABSTRACT. We use topological methods to study the complexity of deep computations and limit computations. We use topology of function spaces, specifically, the classification of Rosenthal compacta, to identify new complexity classes. We use the language of model theory, specifically, the concept of *independence* from Shelah’s classification theory, to translate between topology and computation. We use the theory of Rosenthal compacta to characterize approximability of deep computations, both deterministically and probabilistically.

7

INTRODUCTION

8 In this paper we study asymptotic behavior of computations, e.g., the depth of
9 a neural network tending to infinity, or the time interval between layers of a time-
10 series network tending toward zero. Recently, particular cases of this concept have
11 attracted considerable attention in deep learning research (e.g., Neural Ordinary
12 Differential Equations [CRBD], Physics-Informed Neural Networks [RPK19], and
13 deep equilibrium models [BKK]). The formal framework introduced here provides
14 a unified setting to study these limit phenomena from a foundational viewpoint.

15 Informed by model theory, to each computation in a given computation model,
16 we associate a continuous real-valued function, called the *type* of the computation,
17 that describes the logical properties of this computation with respect to the rest
18 of the model. This allows us to view computations in any given computational
19 model as elements of a space of real-valued functions, which is called the *space*
20 *of types* of the model. The idea of embedding models of theories into their type
21 spaces is central in model theory. In the context of this paper, the embedding of
22 computations into spaces of types allows us to utilize the vast theory of topology of
23 function spaces, known as C_p -theory, to obtain results about complexity of topolog-
24 ical limits of computations. As we shall indicate next, recent classification results
25 for spaces of functions provide an elegant and powerful machinery to classify com-
26 putations according to their levels of “tameness” or “wildness”, with the former
27 corresponding roughly to polynomial approximability and the latter to exponential
28 approximability. The viewpoint of spaces of types, which we have borrowed from

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model theory, thus becomes a “Rosetta stone” that allows us to interconnect various classification programs: In topology, the classification of Rosenthal compacta pioneered by Todorćević [Tod99]; in logic, the classification of theories developed by Shelah [She90]; and in statistical learning, the notions of PAC learning and VC dimension pioneered by Vapnik and Chervonenkis [VC74, VC71].

In a previous paper [ADIW24], the authors introduced the concept of limits of computations, which we called *ultracomputations* (given they arise as ultrafilter limits of standard computations) and *deep computations* (following usage in machine learning [BKK]). There is a technical difference between both designations, but in this paper, to simplify the nomenclature, we will ignore the difference and use only the term “deep computation”.

In [ADIW24], a new “tame vs wild” (i.e., polynomially approximable vs non-polynomially approximable) dichotomy for the complexity of deep computations was proved by invoking a classical result of Grothendieck from the 1950s [Gro52]. Under our model-theoretic Rosetta stone, the property of polynomial approximability of computations is identified with continuous extendibility in the sense of topology, and with the notions of *stability* and *type definability* in model theory.

Simplest among deep computations are those arising as pointwise limits of (continuous) computations. In topology, the *first Baire class*, or *Baire class 1* consists of functions (also called simply “*Baire-1*”) arising as pointwise limits of sequences of continuous functions. Intuitively, the Baire-1 class consists of functions with “controlled” discontinuities, and lies just one level of topological complexity above the Baire class 0 which (by definition) consists of continuous functions.

We prove a new “tame vs wild” Ramsey-theoretic dichotomy for complexity of general deep computations by invoking a famous paper by Bourgain, Fremlin and Talagrand from the late 70s [BFT78], and a new trichotomy for the class of “tame” deep computations by invoking an equally celebrated result of Todorćević, from the late 90s, for functions of the first Baire class [Tod99].

Todorćević’s trichotomy concerns *Rosenthal compacta*; these are special classes of topological spaces, defined as compact spaces that can be embedded (homeomorphically identified as a subset) within the space of Baire class 1 functions on some Polish (separable, completely metrizable) space, under the topology of pointwise convergence; that is, the subspace topology inherited from the product topology. Rosenthal compacta exhibit “topological tameness,” meaning that they behave in relatively controlled ways; since the late 70’s, they have played a crucial role in understanding the complexity of structures of functional analysis, especially Banach spaces. Todorćević’s trichotomy has been utilized to settle longstanding problems in topological dynamics and topological entropy [GM22]. It is noteworthy that Todorćević’s proof relies on sophisticated set-theoretic forcing and infinite Ramsey theory. At the time of writing this paper, decades after his original argument, no elementary proof has been found [Tod23, HT19].

Through our Rosetta stone, Rosenthal compacta in topology correspond to the important concept of “Non Independence Property” (known as “NIP”) in model theory, identified by Shelah [She71, She90], and to the concept of Probably Approximately Correct learning (known as “PAC learnability”) in statistical learning theory identified by Valiant [Val84].

Refining Todorćević’s trichotomy, we invoke a more recent heptachotomy for separable Rosenthal compacta obtained by Argyros, Dodos and Kanellopoulos [ADK08];

they identify seven fundamental “prototypes” of separable Rosenthal compacta, and show that any non-metrizable separable Rosenthal compactum must contain a “canonical” embedding of one of these prototypes.

We believe that the results presented in this paper show practitioners of computation, or topology, or descriptive set theory, or model theory, how classification invariants used in their field translate into classification invariants of other fields. In the interest of accessibility, we do not assume the reader to have previous familiarity with advanced topology, model theory, or computing. The only technical prerequisites to read this paper are undergraduate-level topology and measure theory. The necessary topological background beyond undergraduate topology is covered in section 1.

In section 1, we present basic topological and combinatorial preliminaries, and in section 2, we introduce the structural/model-theoretic viewpoint (no previous exposure to model theory is needed). Section 3 is devoted to the classification of deep computations.

Throughout the paper, our results pertain to classical models of computation (particularly computations involving real-valued quantities that are known and manipulated to a finite degree of precision). The final section, Section 4, introduces a probabilistic viewpoint, the development of which we intend to pursue in future research, extending the present framework to encompass non-deterministic and quantum computations.

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126 1. GENERAL TOPOLOGICAL PRELIMINARIES: FROM CONTINUITY TO BAIRE
127 CLASS 1

128 In this section we present some preliminaries from general topology and function
129 space theory. In the interest of completeness, we include some proofs that may be
130 safely skipped by readers familiar with these topics.

131 Recall that a subset of a topological space is F_σ if it is a countable union of closed
132 sets, and G_δ if it is a countable intersection of closed sets. A space is metrizable if
133 its topology agrees with the topology induced by some metric therein. Two such
134 metrics inducing the same topology may induce quite different properties in the
135 category of metric spaces. For example, the interval $(0, 1)$ with the usual metric (as
136 a subset) of the reals is not complete; however, $(0, 1)$ is homeomorphic to the real
137 line, which is complete with respect to the usual metric thereon. In a metrizable
138 space, every open set is F_σ ; equivalently, every closed set is G_δ .

139 A *Polish space* is a separable and completely metrizable topological space, i.e.,
140 admitting some complete metric inducing its topology. Although other (possibly
141 incomplete) metrics may induce the same topology, being Polish is a purely topolog-
142 ical property. One of the most important Polish spaces is the real line \mathbb{R} ; the others
143 include the Cantor space $2^\mathbb{N}$ and the Baire space $\mathbb{N}^\mathbb{N}$. The class of Polish spaces
144 is closed under countable topological products; in particular, the Cantor space $2^\mathbb{N}$
145 (the set of all infinite binary sequences, endowed with the product topology), the
146 Baire space $\mathbb{N}^\mathbb{N}$ (the set of all infinite sequences of naturals, also with the product
147 topology), and the space $\mathbb{R}^\mathbb{N}$ of sequences of real numbers are all Polish. Recall that
148 the product topology on these spaces is the topology of pointwise convergence: a
149 sequence converges in the space if and only if it converges at each coordinate index.

150 **Fact 1.1.** *A subset of a Polish space is itself Polish in the subspace topology if and*
151 *only if it is a G_δ set. In particular, closed subsets and open subsets of Polish spaces*
152 *are also Polish spaces.*

153 For a proof, see [Eng89, 4.3.24].

154 Given two topological spaces X and Y we denote by $C_p(X, Y)$ the set of all con-
155 tinuous functions $f : X \rightarrow Y$ endowed with the topology of pointwise convergence.
156 The space $C_p(X, \mathbb{R})$ of continuous real functions on X is denoted simply $C_p(X)$.
157 A natural question is, how do topological properties of X translate into $C_p(X)$
158 and vice versa? This general question, and the study of these spaces in general, is
159 the concern of C_p -theory, an active field of research in general topology which was
160 pioneered by A. V. Arhangel'skiĭ and his students in the 1970's and 1980's [Ark92].
161 This field has found many applications in model theory and functional analysis.
162 For a recent survey, see [Tka11].

163 A *Baire class 1* function between topological spaces is a function that can be
164 expressed as the pointwise limit of a sequence of continuous functions. In symbols,
165 $f : X \rightarrow Y$ is *Baire class 1* if there is a sequence of continuous functions $f_n : X \rightarrow Y$
166 such that for all $x \in X$, $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. If X and Y are topological spaces,

the space of Baire class 1 functions $f : X \rightarrow Y$ endowed with the topology of pointwise convergence is denoted $B_1(X, Y)$ (as above, $B_1(X, \mathbb{R})$ is denoted $B_1(X)$). Clearly, $C_p(X, Y) \subseteq B_1(X, Y) \subseteq Y^X$ and we give these the topology (called the *topology of pointwise convergence*) inherited from the product topology of Y^X . The Baire hierarchy of functions was introduced by René-Louis Baire in his 1899 doctoral thesis, *Sur les fonctions de variables réelles*. His work moved away from the 19th-century preoccupation with “pathological” functions toward a constructive classification based on pointwise limits. An elementary fact about Baire class 1 functions is that they are continuous except on a set of first category (also called a *meager* set, a set of first category is the countable union of sets whose closure has empty interior; intuitively, these sets are “topologically small”). Thus, Baire class 1 functions are continuous on a “topologically large” subset of their domain.

A topological space X is *perfectly normal* if it is normal and every closed subset of X is a G_δ (equivalently, every open subset of X is a F_σ). Every metrizable space (hence, every Polish space) is perfectly normal.

A topological space X is *Baire* if every countable intersection of dense open sets is dense. The Baire Category Theorem states that every compact Hausdorff or completely metrizable space (hence, every Polish space) is Baire.

The following fact was established by Baire in his 1899 thesis. A proof can be found in [Tod97, Section 10].

Fact 1.2 (Baire). *If X is perfectly normal, then the following conditions are equivalent for a function $f : X \rightarrow \mathbb{R}$:*

- (1) *f is a Baire class 1 function, that is, f is a pointwise limit of continuous functions.*
- (2) *$f^{-1}[U]$ is an F_σ subset of X whenever $U \subseteq \mathbb{R}$ is open.*

If, moreover, X is Baire, then (1) and (2) are equivalent to:

- (3) *For every closed $F \subseteq X$, the restriction $f|_F$ has a point of continuity.*

Moreover, if X is Polish and $f \notin B_1(X)$, then there exist countable $D_0, D_1 \subseteq X$ and reals $a < b$ such that

$$D_0 \subseteq f^{-1}(-\infty, a], \quad D_1 \subseteq f^{-1}[b, \infty), \quad \overline{D_0} = \overline{D_1}.$$

A subset L of a topological space X is *relatively compact* in X if the closure of L in X is compact. Relatively compact subsets of $B_1(X)$ (for X Polish) are of interest in analysis and topological dynamics. We begin with the following well-known result. Recall that a set $A \subseteq \mathbb{R}^X$ of real-valued functions is *pointwise bounded* if for every $x \in X$ there is $M_x \geq 0$ (a *pointwise bound at x*) such that $|f(x)| \leq M_x$ for all $f \in A$. We include a proof for the reader’s convenience:

Lemma 1.3. *Let X be a Polish space and $A \subseteq B_1(X)$ be pointwise bounded. The following are equivalent:*

- (i) *A is relatively compact in $B_1(X)$.*
- (ii) *A is relatively countably compact in $B_1(X)$, i.e., every countable subset of A has a limit point in $B_1(X)$.*
- (iii) *$\overline{A} \subseteq B_1(X)$, where \overline{A} denotes the closure in \mathbb{R}^X .*

Proof. (i) \Rightarrow (ii) Relatively compact subsets of any space are countably compact therein.

(ii) \Rightarrow (iii) Consider any $f \in \overline{A}$ and any countable subset $\{x_i\}_{i \in \mathbb{N}} \subseteq X$. We claim that there is a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ such that $\lim_{n \rightarrow \infty} f_n(x_i) = f(x_i)$ for all

212 $i \in \mathbb{N}$. Since A carries the relative product topology, for each $n \in \mathbb{N}$ there exists
 213 $f_n \in A$ such that $|f_n(x_i) - f(x_i)| < \frac{1}{n}$ for all $i \leq n$; the sequence $\{f_n\}$ is as claimed.
 214 Seeking a contradiction, assume that A is relatively countably compact in $B_1(X)$,
 215 but there exists some $f \in \overline{A} \setminus B_1(X)$. By Fact 1.2, there are countable $D_0, D_1 \subseteq X$
 216 with $\overline{D_0} = \overline{D_1}$, and $a < b$ such that $D_0 \subseteq f^{-1}(-\infty, a]$ and $D_1 \subseteq f^{-1}[b, \infty)$. Per
 217 the claim above, let $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ satisfy $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in D_0 \cup D_1$
 218 (the latter being a countable set). By relative countable compactness of A , there
 219 is a limit point $g \in B_1(X)$ of $\{f_n\}_{n \in \mathbb{N}}$; clearly, f and g agree on $D_0 \cup D_1$. Thus
 220 g takes values $g(x_i) = f(x_i) \leq a$ as well as values $g(x_j) = f(x_j) \geq b > a$ on any
 221 open subset of the closed set $\overline{D_0} = \overline{D_1}$, contradicting the implication (1) \Rightarrow (3) in
 222 Fact 1.2.

223 (iii) \Rightarrow (i) For each $x \in X$, let $M_x \geq 0$ be a pointwise bound for A . Since \overline{A}
 224 is a closed subset of the compact space $\prod_{x \in X} [-M_x, M_x] \subseteq \mathbb{R}^X$, it follows that \overline{A}
 225 is compact. By (iii), it is also the closure of A in $B_1(X)$. Thus, A is relatively
 226 compact in $B_1(X)$. \square

227 **1.1. From Rosenthal's dichotomy to the Bourgain-Fremelin-Talagrand di-**
 228 **chotomy to Shelah's NIP.** In metrizable spaces, points of the closure of some
 229 subset can always be approximated by points in the set proper, via a convergent
 230 sequence. For more complicated spaces, such as C_p -spaces, this fails in remarkable
 231 ways. The n -th coordinate map $p_n : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ on the Cantor space $X = 2^{\mathbb{N}}$
 232 ($= \{0, 1\}^{\mathbb{N}}$) is continuous for each $n \in \mathbb{N}$, and one can show (e.g., [Tod97, Chap-
 233 ter 1.1]) that $\{p_n\}_{n \in \mathbb{N}}$ has *no* convergent subsequences, in \mathbb{R}^X . In a sense, this
 234 example exhibits the worst failure of sequential convergence possible. The closure
 235 of $\{p_n\}$ in $\{0, 1\}^X$ (or in \mathbb{R}^X for that matter) is homeomorphic to the *Stone-Ćech*
 236 *compactification* of the discrete space of natural numbers, usually denoted $\beta\mathbb{N}$,
 237 which is an important object of study in general topology.

238 The following theorem, proved by Haskell Rosenthal in 1974, is fundamental in
 239 functional analysis and captures a sharp division in the behavior of sequences in a
 240 Banach space.

241 **Theorem 1.4** (Rosenthal's Dichotomy, [Ros74]). *If X is Polish and $\{f_n\} \subseteq C_p(X)$*
 242 *is pointwise bounded, then $\{f_n\}_{n \in \mathbb{N}}$ has a convergent subsequence, or a subsequence*
 243 *whose closure in \mathbb{R}^X is homeomorphic to $\beta\mathbb{N}$.*

244 Rosenthal's Dichotomy states that a pointwise bounded set of continuous func-
 245 tions contains either a convergent subsequence, or a subsequence whose closure is
 246 essentially the same as the example mentioned in the previous paragraphs (i.e.,
 247 "wildest" possible). The genesis of this theorem was Rosenthal's " ℓ_1 -Theorem",
 248 which states that a Banach space includes an isomorphic copy of ℓ_1 (the space of
 249 absolutely summable sequences), or else every bounded sequence therein is weakly
 250 Cauchy. The ℓ_1 -Theorem connects diverse areas: Banach space geometry, Ramsey
 251 theory, set theory, and topology of function spaces.

252 As we move from $C_p(X)$ to the larger space $B_1(X)$, a dichotomy paralleling the
 253 ℓ_1 -Theorem holds: Either every point of the closure of a set of functions is a Baire
 254 class 1 function, or there is a sequence in the set behaving in the wildest possible
 255 way. This result is usually not phrased as a dichotomy, but rather as an equivalence
 256 as in Theorem 1.5 below.

First, we introduce some useful notation. For any set $A \subseteq \mathbb{R}^X$ and any real a ,
define

$$X_{\leq a}^A := \bigcap_{f \in A} f^{-1}(-\infty, a] = \{x \in X : f(x) \leq a \text{ for all } f \in A\},$$

$$X_{\geq a}^A := \bigcap_{f \in A} f^{-1}[a, +\infty) = \{x \in X : f(x) \geq a \text{ for all } f \in A\}.$$

(In case $A = \emptyset$, we define $X_{\geq a}^\emptyset = X = X_{\leq a}^\emptyset$.) For any sequence $\{f_n\} \subseteq \mathbb{R}^X$ and $I \subseteq \mathbb{N}$, define $I^\complement := \mathbb{N} \setminus I$ and $f_I := \{f_i : i \in I\}$.

Theorem 1.5 (“The BFT Dichotomy”. Bourgain-Fremlin-Talagrand [BFT78]).
Let X be a Polish space and $A \subseteq C_p(X)$ be pointwise bounded. The following are equivalent:

- (i) A is relatively compact in $B_1(X)$.
- (ii) For every $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and every $a < b$ there is $I \subseteq \mathbb{N}$ such that $X_{\leq a}^{f_I} \cap X_{\geq b}^{f_{I^\complement}} = \emptyset$.

(As stated above, the BFT Dichotomy is a particular case of the equivalence (ii) \Leftrightarrow (v) in [BFT78, Corollary 4G].)

The sets $X_{\leq a}^{f_I}$ and $X_{\geq b}^{f_{I^\complement}}$ appearing in condition Theorem 1.5(ii) are defined, respectively, in terms of $|I|$ -many inequalities of the form $f_i(x) \leq a$, and $|I^\complement|$ -many of the form $f_j(x) \geq b$. Thus, at least one of $X_{\leq a}^{f_I}$ and $X_{\geq b}^{f_{I^\complement}}$ is defined by the satisfaction of infinitely (countably) many inequalities. For our purposes, it is more natural to consider only finitely many inequalities at a time, which motivates the definitions below.

Definition 1.6. We say that a function collection $A \subseteq \mathbb{R}^X$ has the *finitary No-Independence Property (NIP)* if, for all sequences $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and reals $a < b$, there exist finite disjoint sets $E, F \subseteq \mathbb{N}$ such that $X_{\leq a}^{f_E} \cap X_{\geq b}^{f_F} = \emptyset$. We say that such E, F witness *finitary NIP* for A , $\{f_n\}$ and a, b .

A set $A \subseteq \mathbb{R}^X$ has the *finitary Independence Property (IP)* if it does not have finitary NIP, i.e., if there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and reals $a < b$ such that for every pair of finite disjoint sets $E, F \subseteq \mathbb{N}$, we have $X_{\leq a}^{f_E} \cap X_{\geq b}^{f_F} \neq \emptyset$.

If the word “finite” is omitted in the above definitions, we obtain the definitions of *countable NIP* (weaker than finitary NIP) and *countable IP* (stronger than finitary IP), respectively.

If we insist on witnesses $E, F \subseteq \mathbb{N}$ such that $F = E^\complement$, we call the respective properties “BFT-NIP” (even weaker than countable NIP) and “BFT-IP” (even stronger than countable IP). Thus, Theorem 1.5 becomes that statement, for pointwise bounded function collections $A \subseteq C_p(X)$, that A is relatively compact in $B_1(X)$ if and only if A has BFT-NIP.

Unless otherwise unspecified, IP/NIP shall mean *finitary* IP/NIP henceforth.

Proposition 1.7. If X is compact and $A \subseteq C_p(X)$, then A has BFT-NIP if and only if it has finitary NIP.

(No pointwise boundedness is assumed of A .)

Proof. Trivially (as per the preceding discussion), finitary NIP implies BFT-NIP. Reciprocally, assume that X is compact and has finitary IP. Fix $A \subseteq C_p(X)$,

a sequence $\{f_n\} \subseteq A$ and reals $r < s$. For any $I, J \subseteq \mathbb{N}$ (almost disjoint in applications), write $X_{I,J}$ for $X_{\leq r}^{f_I} \cap X_{\geq s}^{f_J}$. For $I \subseteq I' \subseteq \mathbb{N}$ and $J \subseteq J' \subseteq \mathbb{N}$, we have $X_{I,J} \supseteq X_{I',J'}$; moreover, $X_{I,J} = \bigcap_{E \subseteq I, F \subseteq J} X_{E,F}$, where the index variables $E \subseteq I, F \subseteq J$ range over *finite* subsets of I, J , respectively. Clearly, $E, F \subseteq \mathbb{N}$ witness finitary NIP for $\{f_n\}$ if and only if $X^{E,F} = \emptyset$.

Fix $I \subseteq \mathbb{N}$. Since $\{f_n\} \subseteq A \subseteq C_p(X)$ is a sequence of continuous functions, and X is compact, the nested family $\{X_{E,F} : E \subseteq I, F \subseteq I^c\}$ consists of closed, thus compact, sets. Since A has finitary IP by hypothesis, the nested family consists of nonempty sets, hence its intersection $X_{I,I^c} \neq \emptyset$ by compactness. This holds for arbitrary $\{f_n\} \subseteq A$ and $r < s$, so A has BFT-IP. \square

Theorem 1.8. *Let X be a compact metrizable (hence Polish) space. For every pointwise bounded $A \subseteq C_p(X)$, the following properties are all equivalent:*

- (i) A is relatively compact in $B_1(X)$;
- (ii) A has BFT-NIP;
- (iii) A has countable NIP;
- (iv) A has finitary NIP.

(The equivalences (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) hold for arbitrary compact X .)

Proof. Corollary of Theorem 1.5 and Proposition 1.7. \square

Theorem 1.8 may be stated as the following dichotomy (under the assumptions): either A is relatively compact in $B_1(X)$, or A has IP (in either sense).

The Independence Property was first isolated by Saharon Shelah in model theory as a dividing line between theories whose models are “tame” (corresponding to NIP) and theories whose models are “wild” (corresponding to IP). See [She71, Definition 4.1], [She90]. We will discuss this dividing line in more detail in the next section.

1.2. NIP as a universal dividing line between polynomial and exponential complexity. The particular case of the BFT dichotomy (Theorem 1.5) when A consists of $\{0,1\}$ -valued (i.e., $\{\text{Yes}, \text{No}\}$ -valued) strings was discovered independently, around 1971-1972 in many foundational contexts related to polynomial (“tame”) vs exponential (“wild”) complexity: In model theory, by Saharon Shelah [She71], [She90], in combinatorics, by Norbert Sauer [Sau72], and Shelah [She72], [She90], and in statistical learning, by Vladimir Vapnik and Alexey Chervonenkis [VC71], [VC74].

In model theory: Shelah’s classification theory is a foundational program in mathematical logic devised to categorize first-order theories based on the complexity and structure of their models. A theory T is considered classifiable in Shelah’s sense if the number of non-isomorphic models of T of a given cardinality can be described by a bounded number of numerical invariants. In contrast, a theory T is unclassifiable if the number of models of T of a given cardinality is the maximum possible number. A key fact is that the number of models of T is directly impacted by the number of *types* over sets of parameters in models of T ; a controlled number of types is a characteristic of a classifiable theory.

In Shelah's classification program [She90], theories without the independence property (called NIP theories, or dependent theories) have a well-behaved, "tame" structure; the number of types over a set of parameters of size κ of such a theory is of polynomially or similar "slow" growth on κ .

In contrast, theories with the Independence Property (called IP theories) are considered "intractable" or "wild". A theory with the Independence Property produces the maximum possible number of types over a set of parameters; for a set of parameters of cardinality κ , the theory will have 2^{2^κ} -many distinct types.

In combinatorics: Sauer [Sau72] and Shelah [She72] proved the following independently: Let \mathcal{F} be a family of subsets of some set S . Either: for every $n \in \mathbb{N}$ there is a set $A \subseteq S$ with $|A| = n$ such that $|\{S_i \cap A : i \in \mathbb{N}\}| = 2^n$ (\mathcal{F} has "exponential complexity"); or: there exists $N \in \mathbb{N}$ such that for every $A \subseteq S$ with $|A| \geq N$, one has

$$|\{S_i \cap A : i \in \mathbb{N}\}| \leq \sum_{i=0}^{N-1} \binom{|A|}{i}.$$

(\mathcal{F} has "polynomial complexity"). Clearly, any family \mathcal{F} of subsets of a finite set S has polynomial complexity. The "polynomial" name is justified: indeed, for fixed $N > 0$, as a function of the size $|A| = m > 0$, we have

$$\sum_{i=0}^{N-1} \binom{m}{i} \leq \sum_{i=0}^{N-1} \frac{m^i}{i!} \leq \left(\sum_{i=0}^{N-1} \frac{1}{i!} \right) \cdot m^{N-1} < e \cdot m^{N-1} = O(m^N).$$

(More precisely, the order of magnitude is $O(m^{N-1})$: polynomial in m for N fixed.)

In machine learning: Readers familiar with statistical learning may recognize the Sauer-Shelah lemma as the dichotomy discovered and proved slightly earlier (1971) by Vapnik and Chervonenkis [VC71, VC74] to address uniform convergence in statistics. The least integer N given by the preceding paragraph, when it exists, is called the *VC-dimension* of \mathcal{F} ; it is a core concept in machine learning. If such an integer N does not exist, we say that the VC-dimension of \mathcal{F} is infinite. The lemma provides upper bounds on the number of data points (sample size) needed to learn a concept class of known VC dimension d up to a given admissible error in the statistical sense. The Fundamental Theorem of Statistical Learning states that a hypothesis class is PAC-learnable (PAC stands for "Probably Approximately Correct") if and only if its VC dimension is finite.

1.3. Rosenthal compacta. The universal classification implied by Theorem 1.5, as attested by the examples outlined in the preceding section, led to the following definition (by Gilles Godefroy [God80]):

Definition 1.9. A Rosenthal compactum is any topological space realized as a compact subset of the space $B_1(X) = B_1(X, \mathbb{R})$ (equipped with the topology of pointwise convergence) of all real functions of the first Baire class on some Polish space X .

A Rosenthal compactum K is necessarily Hausdorff since it is a topological subspace of the Hausdorff product space \mathbb{R}^X .

378 Rosenthal compacta possess significant topological and dynamical tameness prop-
 379 erties, and play an important role in functional analysis, measure theory, dynamical
 380 systems, descriptive set theory, and model theory. In this paper, we use them to
 381 study deep computations. For this, we shall first focus on countable languages,
 382 which is the theme of the next subsection.

383 **1.4. The special case $B_1(X, \mathbb{R}^{\mathcal{P}})$ with \mathcal{P} countable.** Fix an arbitrary (at most)
 384 countable set \mathcal{P} whose elements $P \in \mathcal{P}$ will be called *predicate symbols* or *for-*
 385 *mal predicates*. Our present goal is to characterize relatively compact subsets of
 386 $B_1(X, \mathbb{R}^{\mathcal{P}})$, where X is always assumed to be a perfectly normal space (often a
 387 Polish space).

388 The set \mathcal{P} shall be considered discrete whenever regarded as a topological space.
 389 Since $C_p(X, \mathbb{R}^{\mathcal{P}}) \subseteq B_1(X, \mathbb{R}^{\mathcal{P}}) \subseteq (\mathbb{R}^{\mathcal{P}})^X$, the “ambient” space $(\mathbb{R}^{\mathcal{P}})^X$ is quite rele-
 390 vant. The product $X \times \mathcal{P}$ will be regarded as either a pointset, or as a topological
 391 product depending on context. We have natural homeomorphic identifications

$$(\mathbb{R}^{\mathcal{P}})^X \cong \mathbb{R}^{X \times \mathcal{P}} \cong (\mathbb{R}^X)^{\mathcal{P}},$$

392 given by

$$\begin{aligned} \mathbb{R}^{X \times \mathcal{P}} &\rightarrow (\mathbb{R}^{\mathcal{P}})^X : \varphi \mapsto \hat{\varphi} \\ \mathbb{R}^{X \times \mathcal{P}} &\rightarrow (\mathbb{R}^X)^{\mathcal{P}} : \varphi \mapsto \varphi^\wedge, \end{aligned}$$

393 where

$$\hat{\varphi}(x) := \varphi(x, \cdot) \in \mathbb{R}^{\mathcal{P}}, \quad \varphi^\wedge(P) := \varphi(\cdot, P) \in \mathbb{R}^X.$$

394 Such identifications view X and \mathcal{P} as mere pointsets (the topology of X in particular
 395 plays no role).

396 For $x \in X$, define the “left projection” map

$$\lambda_x : \mathbb{R}^{X \times \mathcal{P}} \rightarrow \mathbb{R}^{\mathcal{P}} : \varphi \mapsto \lambda_x(\varphi) := \varphi(x, \cdot);$$

397 for $P \in \mathcal{P}$, the “right projection” map

$$\rho_P : \mathbb{R}^{X \times \mathcal{P}} \rightarrow \mathbb{R}^X : \varphi \mapsto \varphi(\cdot, P).$$

398 For fixed $x \in X$ and $P \in \mathcal{P}$, we also have canonical projection maps

$$\pi_x : \mathbb{R}^X \rightarrow \mathbb{R} : f \mapsto f(x), \quad \pi_P : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R} : f \mapsto f(P).$$

399 When clear from context, rather than using the specific symbols (“ λ ” for left, “ ρ ”
 400 for right) to denote projections, we may use the generic symbol “ π ”; thus, π_x may
 401 mean λ_x , and π_P may mean ρ_P .

402 The Proposition below reduces the study of $\mathbb{R}^{\mathcal{P}}$ -valued continuous or Baire-1
 403 functions on X to the special case of real-valued ones.

404 **Proposition 1.10.** *The identification $(\mathbb{R}^{\mathcal{P}})^X \cong \mathbb{R}^{X \times \mathcal{P}} \cong (\mathbb{R}^X)^{\mathcal{P}}$ induces identifi-*
 405 *cations*

$$C_p(X, \mathbb{R}^{\mathcal{P}}) \cong C_p(X \times \mathcal{P}) \cong C_p(X)^{\mathcal{P}}, \quad B_1(X, \mathbb{R}^{\mathcal{P}}) \cong B_1(X \times \mathcal{P}) \cong B_1(X)^{\mathcal{P}}.$$

406 (The cardinality of \mathcal{P} plays no role.)

407 *Proof.* The identification of C_p -spaces follows trivially from the definition of topo-
 408 logical product and the fact that \mathcal{P} is discrete: a continuous map $X \rightarrow \mathbb{R}^{\mathcal{P}}$ is
 409 precisely a \mathcal{P} -indexed family of continuous functions $X \rightarrow \mathbb{R}$, and these correspond
 410 to continuous functions $X \times \mathcal{P} \rightarrow \mathbb{R}$. The identification of Baire-1 spaces follows

immediately, since it is defined in terms of the purely topological notion of limit (in the ambient space) of sequences of continuous functions. \square

Given $A \subseteq Y^X$ and $K \subseteq X$ we write $A|_K := \{f|_K : f \in A\}$, i.e., the set of all restrictions of functions in A to K . The following Theorem is a slightly more general version of Theorem 1.5.

Theorem 1.11. *Assume that \mathcal{P} is countable, X is a Polish space, and $A \subseteq C_p(X, \mathbb{R}^{\mathcal{P}})$ is pointwise bounded in the sense that $\pi_P \circ A$ ($\subseteq C_p(X)$) is pointwise bounded for every $P \in \mathcal{P}$. The following are equivalent for every compact $K \subseteq X$:*

- (i) $A|_K$ is relatively compact in $B_1(K, \mathbb{R}^{\mathcal{P}})$;
- (ii) $\pi_P \circ A|_K$ has NIP for every $P \in \mathcal{P}$.

Proof. Compact subsets $K \subseteq X$ are closed, hence also Polish. Therefore, the asserted equivalence follows from Theorems 1.5 and 1.7. \square

Lastly, a simple but useful lemma that helps understand when we restrict a set of functions to a specific subspace of the domain space, we may always assume that the subspace is closed, as replacing the subspace by its closure has no effect on NIP.

Lemma 1.12. *Assume that X is Hausdorff and that $A \subseteq C_p(X)$. The following are equivalent for every $L \subseteq X$:*

- (i) $A|_L$ satisfies the NIP;
- (ii) $A|_{\overline{L}}$ satisfies the NIP.

Proof. It suffices to show that (i) \Rightarrow (ii). Suppose that (ii) does not hold, i.e., that there are $\{f_n\}_{n \in \mathbb{N}} \subseteq A$ and $a < b$ such that for all finite disjoint $E, F \subseteq \mathbb{N}$:

$$\overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in F} f_n^{-1}[b, \infty) \neq \emptyset.$$

Pick $a' < b'$ such that $a < a' < b' < b$. Then, for any finite disjoint $E, F \subseteq \mathbb{N}$ we can choose

$$x \in \overline{L} \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a') \cap \bigcap_{n \in F} f_n^{-1}(b', \infty)$$

By definition of closure:

$$L \cap \bigcap_{n \in E} f_n^{-1}(-\infty, a'] \cap \bigcap_{n \in F} f_n^{-1}[b', \infty) \neq \emptyset.$$

This contradicts (i). \square

2. COMPOSITIONAL COMPUTATION STRUCTURES: A STRUCTURAL APPROACH TO ARBITRARY-PRECISION ARITHMETIC

In this section, we connect function spaces with arbitrary-precision arithmetic computations. We start by summarizing some basic concepts from [ADIW24].

A *computation states structure* is a pair (L, \mathcal{P}) , where L is a set whose elements we call *states* and \mathcal{P} is a collection of real-valued functions on L that we call *predicates*. For a state $v \in L$, the *type* of a state v is the indexed family

$$\text{tp}(v) = (P(v))_{P \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}.$$

For each $P \in \mathcal{P}$, we call the value $P(v)$ the P -th *feature* of v . A *transition* of a computation states structure (L, \mathcal{P}) is a map $f : L \rightarrow L$.

Intuitively, L is the set of states of a computation, and the predicates $P \in \mathcal{P}$ are primitives that are given and accepted as computable. Each state $v \in L$ is uniquely characterized by its type $\text{tp}(v)$, so we may identify L with a subset of $\mathbb{R}^{\mathcal{P}}$. Important state spaces are $L = \mathbb{R}^{\mathbb{N}}$ and $L = \mathbb{R}^n$ for some positive integer n , endowed with predicate $P_i(v) = v_i$, one each for the i -th coordinate of v . We regard the space of types as a topological space, endowed with the topology of pointwise convergence induced by the product topology of $\mathbb{R}^{\mathcal{P}}$. Via the identification $v \mapsto \text{tp}(v)$, the states space L is correspondingly topologized; in particular, for each $P \in \mathcal{P}$, the projection map $v \mapsto P(v)$ is continuous.

Definition 2.1. Given a computation states structure (L, \mathcal{P}) , any element of $\mathbb{R}^{\mathcal{P}}$ in the image of L under the map $v \mapsto \text{tp}(v)$ will be called a *realized (state) type*. The topological closure of the set of realized types in $\mathbb{R}^{\mathcal{P}}$ (endowed with the pointwise convergence topology) will be called the *space of types* of (L, \mathcal{P}) , denoted \mathcal{L} ; elements $\xi \in \mathcal{L}$ are called *state types*. Elements of $\mathcal{L} \setminus L$ will be called *unrealized types*.

Intuitively, state types capture a notion of “limit state”.

As we combine ideas of model theory [Kei03] and topology [BFT78], we are interested in families of real-valued functions that are pointwise bounded. This leads us to the concepts of *sizer* and *shard* introduced first in [ADIW24]:

Definition 2.2. A *sizer* is a family $r_{\bullet} = (r_P)_{P \in \mathcal{P}}$ of positive real numbers, indexed by \mathcal{P} . Given a sizer r_{\bullet} , let $\mathbb{R}^{[r_{\bullet}]} = \prod_{P \in \mathcal{P}} [-r_P, r_P]$ (a compact space), and let the r_{\bullet} -*shard* of a states space L be

$$L[r_{\bullet}] = L \cap \mathbb{R}^{[r_{\bullet}]}.$$

For a sizer r_{\bullet} , the r_{\bullet} -*type shard* is defined as $\mathcal{L}[r_{\bullet}] = \overline{L[r_{\bullet}]}$ (a closed, hence compact subset of $\mathbb{R}^{[r_{\bullet}]}$).

Let also \mathcal{L}_{sh} be the union of all type-shards as the sizer r_{\bullet} varies.

In general, $\mathcal{L}_{\text{sh}} \subseteq \mathcal{L}$, and the inclusion may be proper. However, equality holds in the important special case when \mathcal{P} is countable (see [ADIW24]).

2.1. Compositional Computation Structures.

Definition 2.3. A *Compositional Computation Structure* (CCS) is a triple (L, \mathcal{P}, Γ) , where

- (L, \mathcal{P}) is a computation states structure, and
- $\Gamma \subseteq L^L$ is a semigroup under composition.

Elements of the semigroup Γ are called the *computations* of the structure (L, \mathcal{P}, Γ) . We assume that the identity map id on L is an element of Γ (which is thus not merely a semigroup but a monoid of transformations of L).

We topologize Γ as a subset of the topological product L^L , where the “exponent” L serves merely as an index set, but the “base” L is topologized by type; consequently, one may identify Γ with a subset of the topological product $(\mathbb{R}^{\mathcal{P}})^L$. More specifically, Γ is identified with a subset of \mathcal{L}^L , which is a closed subspace of $(\mathbb{R}^{\mathcal{P}})^L$. Therefore, we have an inclusion $\bar{\Gamma} \subseteq \mathcal{L}^L$. Elements $\xi \in \bar{\Gamma}$ are called (real-valued) *deep computations* or *ultracomputations*.

The reason why we require Γ to be a semigroup is because in many practical applications we want to perform an iterative process of computations (e.g., see subsection 2.3). In these scenarios we need the set of computations to be closed

under composition. This leads to other concepts that are not addressed in this paper but are rather discussed in [ADIW24, Section 5]. However, in other applications we do not need to work on a set of computations that is closed under composition (e.g., see subsection 2.4). Given a set $\Delta \subseteq L^L$ of computations (not necessarily a semigroup), we can always take the semigroup Γ generated by Δ , i.e., the smallest semigroup containing Δ .

A collection R of sizers is *exhaustive* if $L = \bigcup_{r_\bullet \in R} L[r_\bullet]$ (shards $L[r_\bullet]$ exhaust L). A transformation $\gamma \in \Gamma$ is *R-confined* if γ restricts to a map $\gamma|_{L[r_\bullet]} : L[r_\bullet] \rightarrow L[r_\bullet]$ (into $L[r_\bullet]$ itself) for every $r_\bullet \in R$. A subset $\Delta \subseteq \Gamma$ is *R-confined* if each $\gamma \in \Delta$ is.

Proposition 2.4. *If $\Delta \subseteq \Gamma$ is confined by an exhaustive sizer collection, then $\overline{\Delta}$ is a compact subset of $\mathcal{L}_{\text{sh}}^L$.*

Proof. Assume that R confines Δ . For each $v \in L$, let $r_\bullet^{(v)} \in R$ be a sizer such that $v \in L[r_\bullet^{(v)}]$. An arbitrary $\gamma \in \Delta$ restricts to a map $\gamma|_{L[r_\bullet^{(v)}]} : L[r_\bullet^{(v)}] \rightarrow L[r_\bullet^{(v)}]$, so $\Gamma \subseteq K := \prod_{v \in L} \mathcal{L}[r_\bullet^{(v)}]$. The space K is a product of compact spaces, hence compact, so $\overline{\Gamma}$ is a closed, hence compact subset thereof, and a subset of $\mathcal{L}_{\text{sh}}^L \supseteq K$ *a fortiori*. \square

For a CCS (L, \mathcal{P}, Γ) , we regard the elements of Γ as “standard” finitary computations, and the elements of $\overline{\Gamma}$, i.e., deep computations, as possibly infinitary limits of standard computations. The main goal of this paper is to study the computability, definability and computational complexity of deep computations. Since deep computations are defined through a combination of topological concepts (namely, topological closure) and structural and model-theoretic concepts (namely, models and types), we will import technology from both topology and model theory.

2.2. Computability and definability of deep computations and the Extendibility Axiom. Let $f : L \rightarrow \mathcal{L}$ be a function that maps each input state type $(P(v))_{P \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}$ to an output state type $(P \circ f(v))_{P \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}$.

(1) We will say that f is *definable* if for each $Q \in \mathcal{P}$, the output feature $Q \circ f : L \rightarrow \mathbb{R}$ is a definable predicate in the following sense: There is an *approximating function* $\varphi_{Q, K, \varepsilon} : L \rightarrow \mathbb{R}$ that can be built recursively out of a finite number of the (primitively computable) predicates in \mathcal{P} and by a finite number of iterations of the finitary lattice operations \wedge ($=\min$) and \vee ($=\max$), the operations of $\mathbb{R}^{\mathbb{R}}$ as a vector algebra (that is, vector addition and multiplication and scalar multiplication) and the operators \sup and \inf applied on individual variables from L , and such that

$$|\varphi_{Q, K, \varepsilon}(v) - Q \circ f(v)| \leq \varepsilon, \quad \text{for all } v \in K.$$

Remark: What we have defined above is a model-theoretic concept; it is a special case of the concept of *first-order definability* for real-valued predicates in the model theory of real-valued structures first introduced in [Iov94] for model theory of functional analysis and now standard in model theory (see [Kei03]). The \wedge ($=\min$) and \vee ($=\max$) operations correspond to the positive Boolean logical connectives “and” and “or”, and the \sup and \inf operators correspond to the first-order quantifiers, \forall and \exists .

(2) We will say that f is *computable* if it is definable in the sense defined above under (1), but without the use of the \sup/\inf operators; in other words, if for every choice of Q, K, ε , the approximation function $\varphi_{Q, K, \varepsilon} : L \rightarrow \mathcal{L}$ can

527 be constructed without any use of sup or inf operators. This is quantifier-
 528 free definability (i.e., definability as given by the preceding paragraph, but
 529 without use of quantifiers), which, from a logic viewpoint, corresponds to
 530 computability (the presence of the quantifiers \exists and \forall are the reason behind
 531 the undecidability of first-order logic).

532 It is shown in [ADIW24] that:

- 533 (1) For a definable $f : L \rightarrow \mathcal{L}$, the approximating functions $\varphi_{Q,K,\varepsilon}$ may be
 534 taken to be *polynomials* of the input features, and
- 535 (2) Definable transforms $f : L \rightarrow \mathcal{L}$ are precisely those that extend to contin-
 536 uous $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$.

537 To summarize, a function $f : L \rightarrow \mathcal{L}$ is computable if and only if it is definable
 538 if and only if it is polynomially approximable if and only if it can be extended to a
 539 continuous $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$. This is the reason for the following definition.

540 **Definition 2.5.** We say that a CCS (L, \mathcal{P}, Γ) satisfies the *Extendibility Axiom* if
 541 for all $\gamma \in \Gamma$, there is $\tilde{\gamma} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$ such that for every sizer r_\bullet there is a sizer s_\bullet
 542 such that $\tilde{\gamma}|_{\mathcal{L}[r_\bullet]} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[s_\bullet]$ is continuous. We refer to $\tilde{\gamma}$ as a *free* extension
 543 of γ .

544 For the rest of the paper, fix for each $\gamma \in \Gamma$ a free extension $\tilde{\gamma}$ of γ . For any
 545 $\Delta \subseteq \Gamma$, let $\tilde{\Delta}$ denote $\{\tilde{\gamma} : \gamma \in \Delta\}$.

546 For a more detailed discussion of the Extendibility Axiom, we refer the reader
 547 to [ADIW24].

548 **2.3. Newton's method as a CCS.** Let $p(z)$ be a non-constant polynomial with
 549 complex coefficients. We say that $(L_p, \mathcal{P}, \Gamma_p)$ is *Newton's method* for $p(z)$ if:

- L_p is the set of all $z \in \mathbb{C}$ such that there exists an open neighborhood U
 of z such that every sequence in $\{N_p^n : n \in \mathbb{N}\}$ has a subsequence that
 converges uniformly on compact subsets of U , where

$$N_p(z) = z - \frac{p(z)}{p'(z)}$$

550 and $N_p^n := N_p \circ N_p \circ \dots \circ N_p$ is the n th-iteration of N_p .

- $\mathcal{P} := \{P_1, P_2, P_3\}$ where

$$P_1(z) = \frac{2\text{Re}(z)}{|z|^2 + 1},$$

$$P_2(z) = \frac{2\text{Im}(z)}{|z|^2 + 1},$$

$$P_3(z) = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

- 551 • $\Gamma_p := \{N_p^n : n \in \mathbb{N}\}$.

552 *Remarks 2.6.*

- 553 (1) The set L_p is known as the *Fatou set* of N_p (see section 2 in [Bla84]). Its
 554 complement $\mathbb{C} \setminus L_p$, is known as the *Julia set* of N_p .
- 555 (2) The map $z \mapsto (P_1(z), P_2(z), P_3(z))$ is the stereographic projection into the
 556 Riemann Sphere S^2 .
- 557 (3) The set L_p is open and dense in \mathbb{C} ([Bla84], Corollary 4.6). Hence, its closure
 558 \mathcal{L} in \mathbb{R}^3 is the Riemann sphere S^2 (i.e., the extended complex plane).

- 559 (4) The set L_p is completely invariant under iterations of N_p , i.e., $N_p(L_p) = L_p$
 560 and $N_p^{-1}(L_p) = L_p$. This implies that all iterations $N_p^n : L_p \rightarrow L_p$ are
 561 transition maps.
 562 (5) Γ_p is the semigroup generated by $\{N_p\}$. Thus, $(L_p, \mathcal{P}, \Gamma_p)$ is a CCS.

Newton's method is an iterative method that is used to approximate a root of $p(z)$. The map $N_p(z)$ defined above is known as *Newton's map*. The method consists of taking an initial guess $z_0 \in \mathbb{C}$ and iterating the rational map N_p to obtain a sequence given by

$$z_{n+1} = N_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)}$$

563 For each root $r \in \mathbb{C}$ of $p(z)$, there exists an $\varepsilon > 0$ such that for any initial guess z_0
 564 in the ε -ball centered at r , Newton's iteration converges to r (provided $p'(r) \neq 0$)
 565 and the convergence is quadratic in that case, meaning the error at each step is
 566 roughly squared, causing the number of correct digits to double, leading to fast
 567 convergence.

Given a root r of $p(z)$, the set $B_r = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} N_p^n(z) = r\}$ is an open set called the *basin* of r . However, Newton's method can fail to converge to any root for some choices of z_0 . For example, consider the polynomial $p(z) = z^3 - 2z + 2$. The Newton map is given by

$$N_p(z) = z - \frac{z^3 - 2z + 2}{3z^2 - 2} = \frac{2z^3 - 2}{3z^2 - 2}$$

568 Notice that taking $z_0 = 0$ as an initial guess will yield the sequence $0, 1, 0, 1, 0, 1, \dots$
 569 that oscillates between 0 and 1 but none of them are roots of $p(z)$. Another more
 570 chaotic way Newton's method can fail to converge is when the sequence of iterations
 571 has no convergent subsequence. The set of such points, i.e. the *Julia set* associated
 572 to N_p , is typically a fractal. This can be visualized by adding a dash of color: let
 573 us give each complex number z_0 a color (R, G, B) where $R, G, B \in [0, 1]$ (so that
 574 $(1, 0, 0)$ is red, $(0, 1, 0)$ is green, $(0, 0, 1)$ is blue and $(0.5, 0, 0.5)$ is a light purple, for
 575 example). The values of R , G and B are determined by looking at the image of said
 576 number at each stage of the iteration, $N_p^n(z_0)$, and computing the current distance
 577 to each of the roots of $p(z)$; so $R = 1/d_r$ where d_r is the positive distance to the
 578 root which is colored red, and so on. In this way, the roots themselves are colored
 579 red, green, and blue, and every other point gets a mix of the three colors. As the
 580 number of iterations increases, each point gets a sharper color, as the sequence of
 581 images $\{N_p^n(z_0)\}_{n=1}^{\infty}$ converges to one of the three roots. At each stage, the complex
 582 plane looks as if out of focus because the coloring function is continuous. As the
 583 reader can see in Figure 1, the points at the boundary of each color class form the
 584 famous Newton's fractal (of which, interestingly, Newton was unaware).

585 Another example of a Newton's fractal is for $p(z) = z^3 - 1$. The roots of $p(z)$
 586 are the 3rd roots of unity and the Newton map is given by:

$$N_p(z) = z - \frac{z^3 - 1}{3z^2} = \frac{2z^3 + 1}{3z^2}$$

587 In this case, there are three basins of attraction (one for each root) and the
 588 complement of their union is the Julia set, i.e., the common boundary.

589 **Proposition 2.7.** *Let $p(z)$ be a non-constant polynomial. $(L_p, \mathcal{P}, \Gamma_p)$ satisfies the*
 590 *Extendibility Axiom.*



FIGURE 1. Newton's method approximating $p(z) = z^3 - 2z + 2$. Notice the regions of divergence.



FIGURE 2. Newton's method approximating $p(z) = z^3 - 1$.

591 *Proof.* $N_p : L_p \rightarrow L_p$ is a rational map. Rational maps can be continuously ex-
 592 tended to the extended complex plane, i.e., to \mathcal{L} . Composition of rational maps
 593 is a rational map, so by the same reasoning, computations $N_p^n : L_p \rightarrow L_p$ can be
 594 continuously extended to \mathcal{L} . \square

595 The set of deep computations $\bar{\Gamma}$ might behave different for various polynomials.
 596 Let us look at various examples:

Example 2.8. Computation of square roots. Let a be a positive real number and $p(x) = x^2 - a$. Let $L = \mathbb{R} \setminus \{0\}$. Let $\mathcal{P} = (P_1, P_2)$ where $x \mapsto (P_1(x), P_2(x))$ is the stereographic projection into $S^1 \subseteq \mathbb{R}^2$, i.e.,

$$P_1(x) = \frac{2x}{x^2 + 1},$$

$$P_2(x) = \frac{x^2 - 1}{x^2 + 1}.$$

Let $\Gamma = \{N_p^n : n \in \mathbb{N}\}$ where

$$N_p(x) = \frac{x^2 + a}{2x}.$$

As before, (L, \mathcal{P}, Γ) is a CCS. Note that $\mathcal{L} = S^1$ and that each iterate N_p^n can be continuously extended to the extended real line $\mathbb{R} \cup \{\infty\}$, i.e., \mathcal{L} . For example,

$$\tilde{N}_p(x) = \begin{cases} \frac{x^2+a}{2x}, & \text{if } x \in L; \\ \infty, & \text{if } x = 0, \infty. \end{cases}$$

For every initial guess $x \in L$, the limit $f(x) = \lim_{n \rightarrow \infty} N_p^n(x)$ converges pointwise to one of the roots. Moreover,

$$f(x) = \begin{cases} \sqrt{a}, & \text{if } x > 0; \\ -\sqrt{a}, & \text{if } x < 0. \end{cases}$$

Notice that f can be extended to \mathcal{L} by

$$\tilde{f}(x) = \begin{cases} \sqrt{a}, & \text{if } x > 0; \\ -\sqrt{a}, & \text{if } x < 0; \\ \infty, & \text{if } x = 0, \infty. \end{cases}$$

597 However, $\tilde{f} : \mathcal{L} \rightarrow \mathcal{L}$ is not continuous. The set $\bar{\Gamma}$ of deep computations is $\bar{\Gamma} \cup \{\tilde{f}\} \subseteq$
598 $B_1(\mathcal{L}, \mathcal{L})$.

Example 2.9. Newton's method for $p(z) = z^3 - 2z + 2$. Let r_1, r_2 and r_3 be the three roots of $p(z)$. Let B_1, B_2 and B_3 be their respective basins. Let B be the basin of the attractive cycle $0, 1, 0, 1, \dots$. Then, $L_p = B \cup \bigcup_{i=1}^3 B_i$. Notice that N_p^n does not converge pointwise. However, the subsequences N_p^{2n} and N_p^{2n+1} are pointwise convergent to functions f_1 and f_2 respectively. f_1 and f_2 are two distinct deep computations. Note that for $z \notin L_p$, no subsequence of $\tilde{N}_p^n(z)$ converges to a complex number. However, since $\mathcal{L} = S^2$ is compact there is a subsequence of $\tilde{N}_p^n(z)$ that converges to ∞ . We can extend $f_i : L_p \rightarrow \mathcal{L}$ to $\tilde{f}_i : \mathcal{L} \rightarrow \mathcal{L}$ by:

$$\tilde{f}_i(z) = \begin{cases} f_i(z), & \text{if } z \in L_p; \\ \infty, & \text{if } z \notin L_p. \end{cases}$$

599 Again, note that \tilde{f}_i for $i = 1, 2$ are not continuous and that $\tilde{f}_i \in \bar{\Gamma}$.

2.4. Finite precision threshold classifiers as a CCS. Let $L = 2^{\mathbb{N}}$, i.e., the set consisting of all infinite binary sequences with the topology of pointwise convergence. Let $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ equal the collection of projections, i.e., $P_n(x) = x(n)$ for each $x \in L$ and $n \in \mathbb{N}$. Notice that $L \subseteq \mathbb{R}^{\mathcal{P}}$ is closed. Therefore, $\mathcal{L} = L$. We denote by 0^∞ the infinite binary sequence consisting of 0s, and by 1^∞ the infinite binary sequence consisting of 1s. The set of finite binary strings is denoted by $2^{<\mathbb{N}}$. This set is naturally ordered by the lexicographic order \leq_{lex} . Given a finite binary string w , we consider the transition $\phi_w : L \rightarrow L$ given by the rule

$$\phi_w(x) = \begin{cases} 1^\infty, & \text{if } x|_{|w|} \leq_{\text{lex}} w; \\ 0^\infty, & \text{otherwise,} \end{cases}$$

600 where $|w|$ is the length of the string w and $x|_{|w|}$ is the prefix of x of length $|w|$.
601 That is, $\phi_w(x)$ is equal to the constant sequence of ones if $x|_{|w|}$ comes before or
602 is equal to w in the lexicographic order of strings, and it is equal to the constant
603 sequence of zeros otherwise. In words, ϕ_w checks if a number is less than or equal
604 to the scalar value of threshold w (the string w is finite, hence the classifier has
605 *finite precision*). Note that $P_n \circ \phi_w(x) = 1$ if and only if $x|_{|w|}$ comes before w .

606 **Proposition 2.10.** $\phi_w : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ is continuous for all $w \in 2^{<\mathbb{N}}$.

Proof. It suffices to prove that $P_n \circ \phi_w : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ is continuous for all $n \in \mathbb{N}$. For simplicity, let us call $f := P_n \circ \phi_w$, i.e.,

$$f(x) = \begin{cases} 1, & \text{if } x|_{|w|} \leq_{\text{lex}} w; \\ 0, & \text{otherwise.} \end{cases}$$

607 We first observe that $f^{-1}(1) = \{x \in 2^{\mathbb{N}} : x|_{|w|} \leq_{\text{lex}} w\}$ is an open set. Fix $x_0 \in$
 608 $f^{-1}(1)$. Let $t := x_0|_{|w|}$ and consider the basic open set $[t] = \{x \in 2^{\mathbb{N}} : x|_{|t|} = t\}$.
 609 Then it is not difficult to check that $x_0 \in [t] \subseteq f^{-1}(1)$. The same reasoning shows
 610 that $f^{-1}(0)$ is open. \square

611 Let $\Delta = \{\phi_w : w \in 2^{<\mathbb{N}}\} \cup \{\mathbf{0}^\infty, \mathbf{1}^\infty\}$, where $\mathbf{0}^\infty, \mathbf{1}^\infty : L \rightarrow L$ are the constant
 612 maps identical to 0^∞ and 1^∞ , respectively. Let Γ be the semigroup generated by
 613 Δ . The preceding proposition shows that Δ (and hence Γ) consists of continuous
 614 functions. In particular, the CCS (L, \mathcal{P}, Γ) satisfies the Extendibility Axiom. In
 615 contrast with Newton's method, the algebraic structure of Δ is quite simple: com-
 616 posing two classifiers results in something similar to a Boolean logic gate. The
 617 topological structure is far more interesting. Intuitively, the crucial difference be-
 618 tween Newton's method and threshold classifiers is that the complexity of the former
 619 comes from *depth*: the semigroup is generated by a single map but its iterates are
 620 highly complex. The complexity of threshold classification comes from *width*: the
 621 semigroup has infinitely many generators, but their compositions are simple.

622 Intuitively, the closure of Δ consists of the set of all possible threshold classifiers
 623 on the real line, but there are two sorts: the ones that classify strict inequalities
 624 and those that classify \leq . The members of Δ are finite-precision approximations
 625 of classifiers that check all bits of information. But here it gets interesting: what
 626 is the difference, in terms of arbitrary-precision arithmetic, between $x < 0.5$ and
 627 $x \leq 0.5$?

628 Suppose that f_a^+ represents the \leq classifier for a target $a \in L$. We identify
 629 the scalar truth values with constant sequences, formally $f_a^+ : L \rightarrow \{0^\infty, 1^\infty\}$ is
 630 defined by $f_a^+(x) = 1^\infty$ if $x \leq_{\text{lex}} a$ and $f_a^+(x) = 0^\infty$ otherwise. Note that if a is the
 631 constant 1^∞ , then $f_a^+ = \mathbf{1}^\infty$. Similarly, let f_a^- be the strict inequality $<$ classifier,
 632 i.e., $f_a^-(x) = 1^\infty$ if $x <_{\text{lex}} a$ and $f_a^-(x) = 0^\infty$ otherwise. Note that if a is the
 633 constant zero, then $f_a^- = \mathbf{0}^\infty$.

634 **Proposition 2.11.** $f_a^+, f_a^- \in \overline{\Delta}$ for all $a \in 2^{\mathbb{N}}$.

635 *Proof.* First, we show that $f_a^+ \in \overline{\Delta}$. If $a = 1^\infty$, then $f_a^+ = \mathbf{1}^\infty \in \Delta$. If a is
 636 not identically 1, we argue that the pointwise limit of the threshold classifiers on
 637 $w_n := a|_n \widehat{} 1$ (that is, the sequence obtained from appending a 1 to the first n
 638 bits of a) is precisely f_a^+ . Specifically, for every $x \in L$, we intend to prove that
 639 $\lim_{n \rightarrow \infty} \phi_{w_n}(x) = f_a^+(x)$. Assume that $x >_{\text{lex}} a$. Let m be the least index at which
 640 the two sequences differ. Then $a(m) = 0 < 1 = x(m)$, and for all $n \geq m$, w_n
 641 agrees with a up to m . Crucially, $w_n(m) = 0 < 1 = x(m)$, which implies that
 642 $w_n <_{\text{lex}} x|_{n+1}$, and hence $\phi_{w_n}(x) = 0^\infty = f_a^+(x)$ for large enough n . If $x \leq_{\text{lex}} a$,
 643 then $x|_{n+1} \leq_{\text{lex}} w_n$ for all $n \in \mathbb{N}$. Hence, $\phi_{w_n}(x) = 1^\infty = f_a^+(x)$ for all $n \in \mathbb{N}$.

644 Now, we prove that $f_a^- \in \overline{\Delta}$. If a is the constant zero, then $f_a^- = \mathbf{0}^\infty \in \Delta$.
 645 Suppose that a is not constantly zero; then we have two cases.

- 646 (1) If a is eventually zero (a is often called a *dyadic rational*), that is $a =$
 647 $u \frown 1 \frown 0^\infty$ (here \frown denotes concatenation) for some finite u . Let $w_n :=$
 648 $u \frown 0 \frown 1^n <_{\text{lex}} a$. We claim that $\lim_{n \rightarrow \infty} \phi_{w_n}(x) = f_a^-(x)$. Assume that
 649 $x <_{\text{lex}} a$. Then, $x|_{|w_n|} \leq_{\text{lex}} w_n$ for large enough n . Hence, $\phi_{w_n}(x) = 1^\infty =$
 650 $f_a^-(x)$ for large enough n . Now assume that $x \geq_{\text{lex}} a$. Then, $w_n <_{\text{lex}}$
 651 $a|_{|w_n|} \leq_{\text{lex}} x|_{|w_n|}$ so $\phi_{w_n}(x) = 0^\infty = f_a^-(x)$ for all $n \in \mathbb{N}$.
 652 (2) If a is not eventually zero, enumerate the indices of all positive bits in a ,
 653 $\{n \in \mathbb{N} : a(n) = 1\}$, strictly increasingly as $\{n_k : k \in \mathbb{N}\}$ (this is possible
 654 as the former set is infinite by assumption). Let $w_k := (a|_{n_k-1}) \frown 0$; that is,
 655 w_k is the result of flipping the k -th positive bit in a . Once again, observe
 656 that $w_k <_{\text{lex}} a$ for all k . The crucial step follows from the fact that for any
 657 $x <_{\text{lex}} a$, there is a large enough K such that $x <_{\text{lex}} w_k$ for all $k \geq K$.

658

□

659 The preceding proposition shows that the topological structure of deep compu-
 660 tations can be complicated. Indeed, $\overline{P_n \circ \Delta}$ contains the *Split Cantor* space for all
 661 $n \in \mathbb{N}$. (see Examples 3.3(3)).

2.5. **Finite precision prefix test.** In this subsection we present another example
 of a CCS with a more complicated set of deep computations. Let $L = 2^\mathbb{N}$ and $\mathcal{P} =$
 $\{P_n : n \in \mathbb{N}\}$ where $P_n(x) = x(n)$ are the projection maps so clearly $L \subseteq \mathbb{R}^\mathcal{P}$ and
 $\mathcal{L} = L$ (same computation states structure as subsection 2.4). For each $w \in 2^{<\mathbb{N}}$,
 let $\psi_w : L \rightarrow L$ be the transition given by:

$$\psi_w(x) = \begin{cases} 1^\infty, & \text{if } x|_{|w|} = w; \\ 0^\infty, & \text{otherwise.} \end{cases}$$

662 In other words, ψ_w determines whether the first $|w|$ bits of a binary sequence is
 663 exactly w . Let $\Delta = \{\psi_w : w \in 2^{<\mathbb{N}}\}$ and Γ be the semigroup generated by Δ . Since
 664 the sets $\{x \in 2^\mathbb{N} : x|_{|w|} = w\}$ are open and closed in $2^\mathbb{N}$, then the transitions ψ_w
 665 are all continuous. In particular, (L, \mathcal{P}, Γ) satisfies the Extendibility Axiom.

666 Let us analyze the set of deep computations of Δ . The idea of these finite
 667 precision prefix tests ψ_w is that they are approximating the equality relation on
 668 infinite binary sequences. For a given $a \in 2^\mathbb{N}$, let $\delta_a : L \rightarrow \{0^\infty, 1^\infty\}$ be the
 669 indicator function at a , i.e., $\delta_a(x) = 1^\infty$ if $x = a$ and $\delta_a(x) = 0^\infty$ otherwise.

670 **Proposition 2.12.** $\delta_a \in \overline{\Delta}$ for all $a \in 2^\mathbb{N}$.

671 *Proof.* Fix $a \in 2^\mathbb{N}$, and let $w_n := a|_n$ for each $n \in \mathbb{N}$. We claim that $\lim_{n \rightarrow \infty} \psi_{w_n}(x) =$
 672 $\delta_a(x)$ for all $x \in L$. If $x = a$, then $x|_{|w_n|} = w_n$ for all n and so $\psi_{w_n}(x) = 1^\infty = \delta_a(x)$
 673 for all n . If $x \neq a$, then $x|_{|w_n|} \neq w_n$ for large enough n . Hence, $\psi_{w_n}(x) = 0^\infty =$
 674 $\delta_a(x)$ for large enough n . □

675 These equality tests δ_a are not all the deep computations. The other deep
 676 computation we are missing is the constant map $\mathbf{0}^\infty$.

677 **Proposition 2.13.** $\mathbf{0}^\infty \in \overline{\Delta}$.

678 *Proof.* To show that $\mathbf{0}^\infty \in \overline{\Delta}$, for each $n \in \mathbb{N}$, consider, $w_n = 1^n \frown 0$, i.e., the string
 679 consisting of n consecutive 1s followed by a 0. If $x = 1^\infty$, then $x|_{|w_n|} \neq w_n$ for all
 680 $n \in \mathbb{N}$. Hence, $\psi_{w_n}(x) = 0^\infty$ for all $n \in \mathbb{N}$. If $x \neq 1^\infty$, let N be the smallest such
 681 that $x(N) = 0$. Then, $x|_{|w_n|} \neq w_n$ for all $n > N$. Hence, $\psi_{w_n}(x) = 0^\infty$ for large
 682 enough n . □

683 In fact, $\overline{\Delta} = \Delta \cup \{\delta_a : a \in 2^{\mathbb{N}}\} \cup \{\mathbf{0}^\infty\}$ and this space is known as the *Extended*
 684 *Alexandroff compactification of $2^{\mathbb{N}}$* (see Example 3.3(2)). One key topological prop-
 685 erty about this space is that $\mathbf{0}^\infty$ is not a G_δ point, i.e., $\{\mathbf{0}^\infty\}$ is not a countable
 686 intersection of open sets. Moreover, $\mathbf{0}^\infty$ is the only non- G_δ point. It is well-known
 687 that in a Hausdorff, first countable space every point is G_δ . This shows that our
 688 space of deep computations is not first countable. This space also contains a discrete
 689 subspace of size continuum, namely $\{\delta_a : a \in 2^{\mathbb{N}}\}$.

690

3. CLASSIFYING DEEP COMPUTATIONS

691 **3.1. NIP, Rosenthal compacta, and deep computations.** Under what condi-
 692 tions are deep computations Baire class 1, and thus well-behaved according to our
 693 framework, on type-shards? The following theorem says that, under the assump-
 694 tion that \mathcal{P} is countable, the space of deep computations is a Rosenthal compactum
 695 (when restricted to shards) if and only if the set of computations satisfies the NIP
 696 feature by feature. Hence, we can import the theory of Rosenthal compacta into
 697 this framework of deep computations.

698 **Theorem 3.1.** *Let (L, \mathcal{P}, Γ) be a compositional computational structure (Defini-*
 699 *tion 2.3) satisfying the Extendibility Axiom (Definition 2.5) with \mathcal{P} countable. Let*
 700 *R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. The following*
 701 *are equivalent.*

- 702 (i) $\overline{\Delta|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$;
 (ii) $\pi_P \circ \Delta|_{\mathcal{L}[r_\bullet]}$ satisfies the NIP for all $P \in \mathcal{P}$ and $r_\bullet \in R$; that is, for all $P \in \mathcal{P}$,
 $r_\bullet \in R$, $a < b$, $\{\gamma_n\}_{n \in \mathbb{N}} \subseteq \Delta$ there are finite disjoint $E, F \subseteq \mathbb{N}$ such that

$$L[r_\bullet] \cap \bigcap_{n \in E} (\pi_P \circ \gamma_n)^{-1}(-\infty, a] \cap \bigcap_{n \in F} (\pi_P \circ \gamma_n)^{-1}[b, \infty) = \emptyset.$$

703 Moreover, if any (hence all) of the preceding conditions hold, then every deep
 704 computation $f \in \overline{\Delta}$ can be extended to a Baire-1 function on shards, i.e., there is
 705 $\tilde{f} : \mathcal{L}_{sh} \rightarrow \mathcal{L}_{sh}$ such that $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$. In particular, on
 706 each shard every deep computation is the pointwise limit of a countable sequence of
 707 computations.

708 *Proof.* Since \mathcal{P} is countable, $\mathcal{L}[r_\bullet] \subseteq \mathbb{R}^{\mathcal{P}}$ is Polish. Also, the Extendibility Axiom
 709 implies that $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ is a pointwise bounded set of continuous functions for all
 710 $P \in \mathcal{P}$. Hence, Theorem 1.11 and Lemma 1.12 prove the equivalence of (i) and (ii).
 711 If (i) holds and $f \in \overline{\Delta}$, then write $f = \mathcal{U}\lim_i \gamma_i$ as an ultralimit. Define $\tilde{f} := \mathcal{U}\lim_i \tilde{\gamma}_i$.
 712 Hence, for all $r_\bullet \in R$ we have $\tilde{f}|_{\mathcal{L}[r_\bullet]} \in \overline{\tilde{\Delta}|_{\mathcal{L}[r_\bullet]}} \subseteq B_1(\mathcal{L}[r_\bullet], \mathcal{L}[r_\bullet])$. That every
 713 deep computation is a pointwise limit of a countable sequence of computations
 714 follows from the fact that for a Polish space X every compact subset of $B_1(X)$ is
 715 Fréchet-Urysohn (that is, a space where topological closures coincide with sequential
 716 closures, see Theorem 3F in [BFT78] or Theorem 4.1 in [Deb13]). \square

717 3.2. The Todorčević trichotomy, and levels of NIP and PAC learnability.

718 In this subsection we study the case when the set of deep computations is a separable
 719 Rosenthal compactum. We are interested in the separable case for two reasons:

- 720 (1) In practice, the set Δ of computations is countable. This implies that the
 721 set $\overline{\Delta}$ of deep computations is separable.

(2) The non-separable case lacks some tools and nice examples, which makes their study more complicated. In the separable case we have two important results, which are introduced in this subsection (Todorčević's Trichotomy) and the next subsection (Argyros-Dodos-Kanellopoulos heptachotomy). By introducing Todorčević's Trichotomy into this framework, we obtain a classification of the complexity of deep computations.

Given a countable set Δ of computations satisfying the NIP on features and shards (condition (ii) of Theorem 3.1), the set $\tilde{\Delta}_{\mathcal{L}[r_\bullet]}$ (for a fixed sizer r_\bullet) is a separable *Rosenthal compactum* (see Definition 1.9). Todorčević proved a remarkable trichotomy for Rosenthal compacta [Tod99] that was later refined through an heptachotomy proved by Argyros, Dodos, Kanellopoulos [ADK08]. In this section, inspired by the work of Glasner and Megrelishvili [GM22], we study ways in which this classification allows us to obtain different levels of PAC-learnability and NIP.

Recall that a topological space X is *hereditarily separable* if every subspace is separable, and that X is *first countable* if every point in X has a countable local basis. Every separable metrizable space is hereditarily separable, and R. Pol proved that every hereditarily separable Rosenthal compactum is first countable (see section 10 of [Deb13]). This suggests the following definition:

Definition 3.2. Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendibility Axiom and R be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be an R -confined countable set of computations satisfying the NIP on shards and features (condition (ii) in Theorem 3.1). We say that Δ is:

- (i) NIP₁ if $\tilde{\Delta}_{\mathcal{L}[r_\bullet]}$ is first countable for every $r_\bullet \in R$.
- (ii) NIP₂ if $\tilde{\Delta}_{\mathcal{L}[r_\bullet]}$ is hereditarily separable for every $r_\bullet \in R$.
- (iii) NIP₃ if $\tilde{\Delta}_{\mathcal{L}[r_\bullet]}$ is metrizable for every $r_\bullet \in R$.

Observe that $\text{NIP}_3 \Rightarrow \text{NIP}_2 \Rightarrow \text{NIP}_1 \Rightarrow \text{NIP}$. Todorčević, [Tod99], isolated three canonical examples of Rosenthal compacta that witness the failure of the converse implications above.

We now present some separable and non-separable examples of Rosenthal compacta. These show that the previously discussed classes NIP_i are not equal.

Examples 3.3.

- (1) *Alexandroff compactification of a discrete space of size continuum.* For each $a \in 2^{\mathbb{N}}$ consider the map $\delta_a : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ given by $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. Let $A(2^{\mathbb{N}}) = \{\delta_a : a \in 2^{\mathbb{N}}\} \cup \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero map. Notice that $A(2^{\mathbb{N}})$ is a compact subset of $B_1(2^{\mathbb{N}})$. In fact, $\{\delta_a : a \in 2^{\mathbb{N}}\}$ is an uncountable discrete subspace of $B_1(2^{\mathbb{N}})$, and its pointwise closure is precisely $A(2^{\mathbb{N}})$. Hence, this is a Rosenthal compactum which is not hereditarily separable (and therefore not first countable). In particular, this space does not satisfy separability, but it can be made separable by adding a countable set as the next example shows.
- (2) *Extended Alexandroff compactification.* For each finite binary sequence $s \in 2^{<\mathbb{N}}$, let $v_s : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by $v_s(x) = 1$ if x extends s and $v_s(x) = 0$ otherwise. Let $\hat{A}(2^{\mathbb{N}})$ be the pointwise closure of $\{v_s : s \in 2^{<\mathbb{N}}\}$, i.e., $\hat{A}(2^{\mathbb{N}}) = A(2^{\mathbb{N}}) \cup \{v_s : s \in 2^{<\mathbb{N}}\}$. Note that this space is a separable Rosenthal compactum which is not first countable. This is the example

discussed in Section 2.5. It is an example of a CCS that is NIP but not NIP₁.

- (3) *Split Cantor*. Let $<$ be the lexicographic order in the space of infinite binary sequences, i.e., $2^{\mathbb{N}}$. For each $a \in 2^{\mathbb{N}}$ let $f_a^- : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by $f_a^-(x) = 1$ if $x < a$ and $f_a^-(x) = 0$ otherwise. Let $f_a^+ : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by $f_a^+(x) = 1$ if $x \leq a$ and $f_a^+(x) = 0$ otherwise. The split Cantor is the space $S(2^{\mathbb{N}}) = \{f_a^- : a \in 2^{\mathbb{N}}\} \cup \{f_a^+ : a \in 2^{\mathbb{N}}\}$, which was obtained as the closure of the space discussed in Section 2.4, giving an example separating NIP₂ from NIP₃. This is a well known separable Rosenthal compactum. One example of a countable dense subset is the set of all f_a^+ and f_a^- where a is an infinite binary sequence that is eventually constant. Moreover, it is hereditarily separable, but it is not metrizable. It is homeomorphic to the space $2^{\mathbb{N}} \times \{0, 1\}$ with the lexicographic order topology via the identification $(a, 1) \mapsto f_a^+$ and $(a, 0) \mapsto f_a^-$.
- (4) *Alexandroff Duplicate*. Let K be any compact metric space and consider the Polish space $X = K \sqcup C(K)$, i.e., the disjoint union of $C(K)$ (with its supremum norm topology) and K . For each $a \in K$ define $g_a^0, g_a^1 : X \rightarrow \mathbb{R}$ as follows:

$$g_a^0(x) = \begin{cases} 0, & \text{if } x \in K \\ x(a), & \text{if } x \in C(K); \end{cases}$$

$$g_a^1(x) = \begin{cases} \delta_a(x), & \text{if } x \in K; \\ x(a), & \text{if } x \in C(K). \end{cases}$$

Let $D(K) = \{g_a^0 : a \in K\} \cup \{g_a^1 : a \in K\}$. Observe that all points g_a^1 are isolated and that open neighborhoods of $g_{a_0}^0$ are of the form $\{g_a^i : a \in U, i \in \{0, 1\}\} \setminus \{g_a^1 : a \in F\}$ where $U \subseteq K$ is an open neighborhood of a_0 and $F \subseteq K$ is a finite set. Another abstract way in which this space is presented is as the space $K \times \{0, 1\}$ whose basic open neighborhoods are given as before, identifying $(a, 0) \mapsto g_a^0$ and $(a, 1) \mapsto g_a^1$. We can also embed $D(K)$ into the product $A(K) \times K$ by identifying $(a, 0) \mapsto (\mathbf{0}, a)$ and $(a, 1) \mapsto (\delta_a, a)$. Notice that $D(K)$ is a first countable Rosenthal compactum. It is not separable if K is uncountable, thus we typically study the interesting case when $K = 2^{\mathbb{N}}$. As with the Alexandroff compactification $A(2^{\mathbb{N}})$, we can make the space $D(2^{\mathbb{N}})$ separable by adding a countable set. For example, the closure of the set $\{(v_s, s \smallfrown 0^\infty) : s \in 2^{<\mathbb{N}}\} \subseteq \hat{A}(2^{\mathbb{N}}) \times 2^{\mathbb{N}}$ is $\{(\mathbf{0}, a) : a \in 2^{\mathbb{N}}\} \cup \{(\delta_a, a) : a \in 2^{\mathbb{N}}\} \cup \{(v_s, s \smallfrown 0^\infty) : s \in 2^{<\mathbb{N}}\}$, where $\{(\mathbf{0}, a) : a \in 2^{\mathbb{N}}\} \cup \{(\delta_a, a) : a \in 2^{\mathbb{N}}\}$ is homeomorphic to $D(2^{\mathbb{N}})$.

- (5) *Extended Alexandroff Duplicate of the split Cantor*. For each finite binary sequence $t \in 2^{<\mathbb{N}}$ let $a_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 0's and let $b_t \in 2^{\mathbb{N}}$ be the sequence starting with t and ending with 1's. Define $h_t : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ by

$$h_t(x) = \begin{cases} 0, & \text{if } x < a_t; \\ 1/2, & \text{if } a_t \leq x \leq b_t; \\ 1, & \text{if } b_t < x. \end{cases}$$

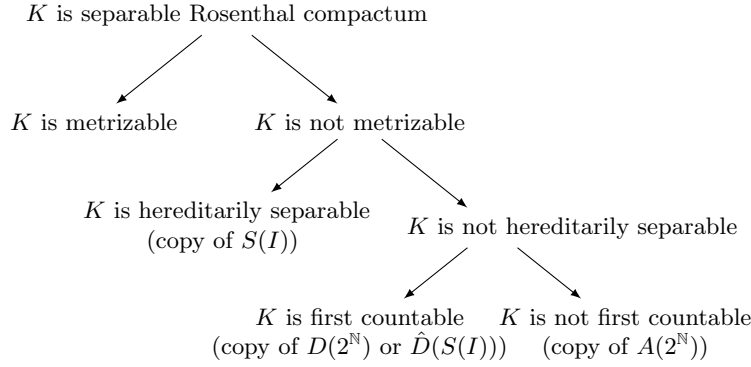
Let $\hat{D}(S(2^{\mathbb{N}}))$ be the pointwise closure of the set $\{h_t : t \in 2^{<\mathbb{N}}\}$. The identification $h_t \mapsto (v_t, f_{t \smallfrown 0^\infty}^+)$ lifts to a homeomorphism between $\hat{D}(S(2^{\mathbb{N}}))$

and the subspace of $\hat{A}(2^{\mathbb{N}}) \times S(2^{\mathbb{N}})$ consisting of $(\mathbf{0}, f_a^+)$, $(\mathbf{0}, f_a^-)$, (δ_a, f_a^+) and $(v_t, f_{t \smallfrown 0^\infty}^+)$ for $a \in 2^{\mathbb{N}}$ and $t \in 2^{<\mathbb{N}}$ (see 4.3.7 in [ADK08]). Hence, $\hat{D}(S(2^{\mathbb{N}}))$ is a separable first countable Rosenthal compactum which is not hereditarily separable. In fact, it contains an uncountable discrete subspace.

Theorem 3.4 (Todorćević's Trichotomy, [Tod99], Theorem 3 in [ADK08]). *Let K be a separable Rosenthal Compactum.*

- (i) *If K is hereditarily separable but non-metrizable, then $S(2^{\mathbb{N}})$ embeds into K .*
- (ii) *If K is first countable but not hereditarily separable, then either $D(2^{\mathbb{N}})$ or $\hat{D}(S(2^{\mathbb{N}}))$ embeds into K .*
- (iii) *If K is not first countable, then $A(2^{\mathbb{N}})$ embeds into K .*

We thus have the following classification:



808

Todorćević's Trichotomy suggests that in order to distinguish the classes NIP_i , the examples in 3.3 are essential. The following examples show that the levels NIP_i ($i = 1, 2, 3$) may be distinguished by the topological complexity of deep computations.

813 Examples 3.5.

- (1) Let (L, \mathcal{P}, Γ) be the computation of square root (example 2.8 with $\Delta = \Gamma$). We saw that $\bar{\Delta} = \tilde{\Delta} \cup \{\tilde{f}\} \subseteq B_1(\mathcal{L}, \mathcal{L})$. This corresponds to the Alexandroff compactification of a countable discrete set, which is metrizable. Hence, Δ is NIP_3 but it is not stable, in the sense that $\bar{\Delta} \not\subseteq C(\mathcal{L}, \mathcal{L})$.
- (2) Let (L, \mathcal{P}, Γ) be the finite precision threshold classifiers (Section 2.4) with $\Delta = \{\phi_w : w \in 2^{<\mathbb{N}}\} \cup \{\mathbf{0}^\infty, \mathbf{1}^\infty\}$. We saw that $\bar{\Delta}$ is homeomorphic to the Split Cantor space $S(2^{\mathbb{N}})$ (Example 3.3(3)), which is hereditarily separable but not metrizable. Hence, Δ is NIP_2 but not NIP_3 .
- (3) Let (L, \mathcal{P}, Γ) be the CCS given by $L = 2^{\mathbb{N}}$, $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ and Γ is the semigroup generated by $\Delta = \{\gamma_t : t \in 2^{<\mathbb{N}}\}$, where $P_n : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ is the projection map $P_n(x) = x(n)$ and $\gamma_t : L \rightarrow L$ is given by

$$\gamma_t(x) = \begin{cases} 0^\infty, & \text{if } x <_{\text{lex}} t \smallfrown 0^\infty; \\ (01)^\infty, & \text{if } t \smallfrown 0^\infty \leq_{\text{lex}} x \leq_{\text{lex}} t \smallfrown 1^\infty; \\ 1^\infty, & \text{if } t \smallfrown 1^\infty <_{\text{lex}} x. \end{cases}$$

where $(01)^\infty$ denotes the sequence of alternating bits: $010101\dots$. As in the other examples, it is not difficult to see that (L, \mathcal{P}, Γ) satisfies the Extendibility Axiom. For example, the condition $t \smallfrown 0^\infty \leq_{\text{lex}} x \leq_{\text{lex}} t \smallfrown 1^\infty$ is

822

823

824

- 825 equivalent to x extending t . Observe that the set of deep computations is
 826 homeomorphic to $\hat{D}(S(2^{\mathbb{N}}))$ (see Example 3.3(5)). This is an example of Δ
 827 which is NIP_1 but not NIP_2 .
- 828 (4) Let (L, \mathcal{P}, Γ) be the finite precision prefix test (Section 2.5) with $\Delta = \{\psi_w : w \in 2^{<\mathbb{N}}\}$. We saw that $\bar{\Delta}$ is homeomorphic to the Extended Alexandroff
 829 compactification $\hat{A}(2^{\mathbb{N}})$ (Example 3.3-(3)), which is separable but not first
 830 countable. Hence, Δ is NIP but not NIP_1 .
 831

832 The definitions provided here for NIP_i ($i = 1, 2, 3$) are topological. This raises
 833 the following question:

834 **Question 3.6.** Is there a non-topological characterization for NIP_i , $i = 1, 2, 3$?

835 **3.3. The Argyros-Dodos-Kanellopoulos heptachotomy, and approxima-**
 836 **bility of deep computation by minimal classes.** In the three separable cases
 837 given in 3.3, namely, $\hat{A}(2^{\mathbb{N}})$, $S(2^{\mathbb{N}})$ and $\hat{D}(S(2^{\mathbb{N}}))$, the countable dense subsets are
 838 indexed by the binary tree $2^{<\mathbb{N}}$. This choice of index is useful for two reasons:

- 839 (1) Our emphasis is computational. Real numbers can be represented as in-
 840 finite binary sequences, i.e., infinite branches of the binary tree $2^{<\mathbb{N}}$. We
 841 approximate real numbers or binary sequences with elements in $2^{<\mathbb{N}}$, i.e.,
 842 finite bitstrings. Indexing standard computations with finite bitstrings al-
 843 low us to better understand how deep computations arise and how they get
 844 approximated. Computationally, we are interested in the manner (and the
 845 efficiency) in which the approximations can occur.
- 846 (2) Infinite branches of the binary tree $2^{<\mathbb{N}}$ correspond to the Cantor space $2^{\mathbb{N}}$,
 847 the canonical perfect set (in the sense that any Polish space with no iso-
 848 lated points contains a copy of $2^{\mathbb{N}}$). The use of infinite dimensional Ramsey
 849 theory for trees (pioneered by the work of James D. Halpern, Hans Läuchli
 850 in [HL66] and also Keith Milliken in [Mil81], and Alain Louveau, Saharon
 851 Shelah, Boban Velickovic in [LSV93]) and perfect sets (Fred Galvin and An-
 852 dreas Blass in [Bla81], Arnold W. Miller in [Mil89], and Stevo Todorćević in
 853 [Tod99]) allowed S.A. Argyros, P. Dodos and V. Kanellopoulos in [ADK08]
 854 to obtain an improved version of Theorem 3.4. It is no surprise that Ram-
 855 sey Theory becomes relevant in the study of Rosenthal compacta as it was a
 856 key ingredient in Rosenthal's ℓ_1 Theorem. For this reason, the main results
 857 in [ADK08] (which we cite in this paper) are better explained by indexing
 858 Rosenthal compacta with the binary tree.

859 **Definition 3.7.** Let X be a Polish space.

- 860 (1) If I is countable and $\{f_i : i \in I\} \subseteq \mathbb{R}^X$, $\{g_i : i \in I\} \subseteq \mathbb{R}^X$ are two pointwise
 861 families by I , we say that $\{f_i : i \in I\}$ and $\{g_i : i \in I\}$ are *equivalent* if and
 862 only if the map $f_i \mapsto g_i$ is extended to a homeomorphism from $\overline{\{f_i : i \in I\}}$
 863 to $\overline{\{g_i : i \in I\}}$.
- 864 (2) If $\{f_t : t \in 2^{<\mathbb{N}}\}$ is a pointwise bounded family, we say that $\{f_t : t \in 2^{<\mathbb{N}}\}$
 865 is *minimal* if and only if for every dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$,
 866 $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to $\{f_t : t \in 2^{<\mathbb{N}}\}$.

867 One of the main results in [ADK08] is that, up to equivalence, there are seven
 868 minimal families of Rosenthal compacta and that for every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$ there is a dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$

is equivalent to one of the minimal families. We shall describe the seven minimal families next. We follow the same notation as in [ADK08]. For any node $t \in 2^{<\mathbb{N}}$, let us denote by $t \smallfrown 0^\infty$ ($t \smallfrown 1^\infty$) the infinite binary sequence starting with t and continuing with all 0's (respectively, all 1's). Fix a regular dyadic subtree $R = \{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ (i.e., a dyadic subtree such that every level of R is contained in a level of $2^{<\mathbb{N}}$) with the property that for all $s, s' \in R$, $s \smallfrown 0^\infty \neq s' \smallfrown 0^\infty$ and $s \smallfrown 1^\infty \neq s' \smallfrown 1^\infty$. Given $t \in 2^{<\mathbb{N}}$, let v_t be the characteristic function of the set $\{x \in 2^\mathbb{N} : x \text{ extends } t\}$. Let $<$ be the lexicographic order in $2^\mathbb{N}$. Given $a \in 2^\mathbb{N}$, let $f_a^+ : 2^\mathbb{N} \rightarrow \{0, 1\}$ be the characteristic function of $\{x \in 2^\mathbb{N} : a \leq x\}$ and let $f_a^- : 2^\mathbb{N} \rightarrow \{0, 1\}$ be the characteristic function of $\{x \in 2^\mathbb{N} : a < x\}$. Given two maps $f, g : 2^\mathbb{N} \rightarrow \mathbb{R}$ we denote by $(f, g) : 2^\mathbb{N} \sqcup 2^\mathbb{N} \rightarrow \mathbb{R}$ the function which is f on the first copy of $2^\mathbb{N}$ and g on the second copy of $2^\mathbb{N}$.

- (1) $D_1 = \{\frac{1}{|t|+1} v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_1} = A(2^\mathbb{N})$.
- (2) $D_2 = \{s_t \smallfrown 0^\infty : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_2} = 2^{\leq \mathbb{N}}$.
- (3) $D_3 = \{f_{s_t \smallfrown 0^\infty}^+ : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_3} = S(2^\mathbb{N})$.
- (4) $D_4 = \{f_{s_t \smallfrown 1^\infty}^- : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_4} = S(2^\mathbb{N})$.
- (5) $D_5 = \{v_t : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_5} = \hat{A}(2^\mathbb{N})$.
- (6) $D_6 = \{(v_{s_t}, s_t \smallfrown 0^\infty) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_6} = \hat{D}(2^\mathbb{N})$.
- (7) $D_7 = \{(v_{s_t}, x_{s_t \smallfrown 0^\infty}^+) : t \in 2^{<\mathbb{N}}\}$. This is discrete in $\overline{D_7} = \hat{D}(S(2^\mathbb{N}))$.

Theorem 3.8 (Heptachotomy of minimal families, Theorem 2 in [ADK08]). *Let X be Polish. For every relatively compact $\{f_t : t \in 2^{<\mathbb{N}}\} \subseteq B_1(X)$, there exists $i = 1, 2, \dots, 7$ and a regular dyadic subtree $\{s_t : t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to D_i . Moreover, all D_i are minimal and mutually non-equivalent.*

The implication of this result for deep computations is the following: for any countable set of computations Δ satisfying the NIP (for some CCS (L, \mathcal{P}, Γ)), we can always find a countable discrete set of deep computations that approximates all the other deep computations. For example: in the finite precision prefix test example (subsection 2.5), the prefix test computations (family D_5) approximate all other deep computations. However, note that this discrete set D_i may not consist of continuous functions (i.e., they will not be computable in general). For example, functions in D_3 are not continuous.

4. RANDOMIZED VERSIONS OF NIP AND MONTE CARLO COMPUTABILITY OF DEEP COMPUTATIONS

In this section, we replace deterministic computability by probabilistic ('Monte Carlo') computability. We do not assume that \mathcal{P} is countable. The main results of the section are Theorem 4.8 (connecting NIP and Monte Carlo computability) and 4.14 (connecting Talagrand stability and Monte Carlo computability).

Fundamental in this section is a measure-theoretic version of Theorem 1.11, namely, Theorem 4.5. For the proof of Theorem 1.11, we assumed countability of \mathcal{P} — this ensured that $\mathbb{R}^\mathcal{P}$ a Polish space. In this section, the countability assumption is not needed.

4.1. NIP and Monte Carlo computability of deep computations. The *raison d'être* of the Baire class-1 functions is to have with a class of functions that are obtained as equential limit points of continuous functions. By Fact 1.2, for perfectly

normal X , a function f is in $B_1(X, Y)$ if and only if $f^{-1}[U]$ is an F_σ subset of X for every open $U \subseteq Y$. In other words, we can view the class $B_1(X, Y)$ as a weaker notion of continuity. In this section we will study a more general class of functions: the space of universally measurable functions.

Definition 4.1. Given a Hausdorff space X and a measurable space (Y, Σ) , we say that $f : X \rightarrow Y$ is *universally measurable* (with respect to Σ) if $f^{-1}(E)$ is μ -measurable for every Radon measure μ on X and every $E \in \Sigma$. When $Y = \mathbb{R}$, we will always take $\Sigma = \mathcal{B}(\mathbb{R})$, the Borel σ -algebra of \mathbb{R} .

If X is a compact (Hausdorff) space, then every Radon measure μ on X is finite. Then, the measure given by $\nu(A) := \mu(A)/\mu(X)$ is a probability measure on X with the same null sets as μ . Hence, Radon measures on compact spaces are equivalent to (Radon) probability measures. We summarize this fact in the next remark:

Remark 4.2. If X is compact, then a function $f : X \rightarrow \mathbb{R}$ is universally measurable if and only if $f^{-1}(U)$ is μ -measurable for every Radon probability measure μ on X and every open set $U \subseteq \mathbb{R}$.

Intuitively, a function is universally measurable if it is “measurable no matter which reasonable way you try to measure things on its domain”. The concept of universal measurability emerged from work of Kallianpur and Sazonov, in the late 1950’s and 1960s — with later developments by Blackwell, Darst and others — building on earlier ideas of Gnedenko and Kolmogorov from the 1950s. See [Pap02, Chapters 1 and 2].

Notation 4.3. Following [BFT78], the collection of all universally measurable real-valued functions on X will be denoted by $M_r(X)$. Given a fixed Radon measure μ on X , the collection of all μ -measurable real-valued functions on X will be denoted by $\mathcal{M}^0(X, \mu)$.

In the context of deep computations, we are interested in transition maps of a state space $L \subseteq \mathbb{R}^{\mathcal{P}}$ into itself. There are two natural σ -algebras one can consider in the product space $\mathbb{R}^{\mathcal{P}}$: the Borel σ -algebra, i.e., the σ -algebra generated by open sets in $\mathbb{R}^{\mathcal{P}}$, and the cylinder σ -algebra (i.e., the σ -algebra generated by the sub-basic open sets in $\mathbb{R}^{\mathcal{P}}$, that is, the sets $\pi_P^{-1}(U)$ with $U \subseteq \mathbb{R}$ open and $P \in \mathcal{P}$). Note that when \mathcal{P} is countable, both σ -algebras coincide, but in general the cylinder σ -algebra is strictly smaller. We will use the cylinder σ -algebra to define universally measurable maps $f : \mathbb{R}^{\mathcal{P}} \rightarrow \mathbb{R}^{\mathcal{P}}$. The reason for this choice is the following characterization:

Proposition 4.4. Let X be a Hausdorff space and $Y = \prod_{i \in I} Y_i$ be any product of measurable spaces (Y_i, Σ_i) for $i \in I$. Let Σ_Y be the cylinder σ -algebra generated by the measurable spaces (Y_i, Σ_i) . The following are equivalent for $f : X \rightarrow Y$:

- (i) $f : X \rightarrow Y$ is universally measurable (with respect to Σ_Y).
- (ii) $\pi_i \circ f : X \rightarrow Y_i$ is universally measurable (with respect to Σ_i) for all $i \in I$.

Proof. (i) \Rightarrow (ii) is clear since the projection maps π_i are measurable and the composition of measurable functions is measurable. To prove (ii) \Rightarrow (i), suppose that $C = \prod_{i \in I} C_i$ is a measurable cylinder and let J be the finite subset of I such that $C_i \neq Y_i$ for $i \in J$. Then, $C = \bigcap_{i \in J} \pi_i^{-1}(C_i)$, so $f^{-1}(C) = \bigcap_{i \in J} (\pi_i \circ f)^{-1}(C_i)$ is universally measurable by assumption. \square

959 The preceding proposition says that a transition map is universally measurable
 960 if and only if it is universally measurable on all its features; in other words, we can
 961 check measurability of a transition just by checking measurability feature by feature.
 962 This is the same as in the Baire class-1 case (compare with Proposition 1.10).

963 The main result in section 3 is that, as long as we work with countably many
 964 features, PAC-learning (or NIP) corresponds to relative compactness in the space
 965 of Baire class-1 functions. The following result (which does not assume countability
 966 of the number of features) gives an analogous characterization of the NIP in terms
 967 of universal measurability:

968 **Theorem 4.5** (Bourgain-Fremlin-Talagrand, Theorem 2F in [BFT78]). *Let X be a*
 969 *Hausdorff space and $A \subseteq C(X)$ be pointwise bounded. The following are equivalent:*

- 970 (i) $\overline{A} \subseteq M_r(X)$.
- 971 (ii) For every compact $K \subseteq X$, $A|_K$ satisfies the NIP.
- 972 (iii) For every Radon measure μ on X , A is relatively countably compact in $\mathcal{M}^0(X, \mu)$,
 973 i.e., every countable subset of A has a limit point in $\mathcal{M}^0(X, \mu)$.

974 This result allows us to formalize the concept of a deep computation being *Monte*
 975 *Carlo computable*, which we define below (Definition 4.6). To motivate the defini-
 976 tion, let us first recall two facts:

- 977 (1) Littlewood's second principle states that every Lebesgue measurable func-
 978 tion is “nearly continuous”. The formal statement of this, which is Luzin's
 979 theorem, is that if (X, Σ, μ) a Radon measure space and Y is a second-
 980 countable topological space (e.g., $Y = \mathbb{R}^{\mathcal{P}}$ with \mathcal{P} countable) equipped with
 981 the Borel σ -algebra, then any given $f : X \rightarrow Y$ is measurable if and only if
 982 for every $E \in \Sigma$ and every $\varepsilon > 0$ there exists a closed $F \subseteq E$ such that the
 983 restriction $f|_F$ is continuous and $\mu(E \setminus F) < \varepsilon$.
- 984 (2) Computability of deep computations is characterized in terms of continuous
 985 extendibility of computations. This is at the core of [ADIW24].

986 These two facts motivate the following definition:

987 **Definition 4.6.** Let (L, \mathcal{P}, Γ) be a CCS. We say that a transition $f : L \rightarrow L$
 988 is *universally Monte Carlo computable* if and only if there exists $\tilde{f} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$
 989 extending f such that for every sizer r_{\bullet} there is a sizer s_{\bullet} such that the restriction
 990 $\tilde{f}|_{\mathcal{L}[r_{\bullet}]} : \mathcal{L}[r_{\bullet}] \rightarrow \mathcal{L}[s_{\bullet}]$ is universally measurable, i.e., $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_{\bullet}]} : \mathcal{L}[r_{\bullet}] \rightarrow [-s_P, s_P]$
 991 is μ -measurable for every Radon probability measure μ on $\mathcal{L}[r_{\bullet}]$ and $P \in \mathcal{P}$.

992 *Remark 4.7.* Condition (2) of Theorem 4.5 shows that to study measure-theoretic
 993 versions of NIP, we need only consider compact subsets of X . Now, every Radon
 994 measure on a compact space is finite; hence, by proper normalization, it can be
 995 treated as a probability measure. Therefore, in the context of Monte Carlo measur-
 996 ability, we focus on Radon probability measures rather than general Radon mea-
 997 sures.

998 **Theorem 4.8.** *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendibility Axiom. Let R*
 999 *be an exhaustive collection of sizers. Let $\Delta \subseteq \Gamma$ be R -confined. If $\pi_P \circ \Delta|_{\mathcal{L}[r_{\bullet}]}$*
 1000 *satisfies the NIP for all $P \in \mathcal{P}$ and all $r_{\bullet} \in R$, then every deep computation in Δ*
 1001 *is universally Monte Carlo computable.*

1002 *Proof.* Fix $P \in \mathcal{P}$ and $r_{\bullet} \in R$. By the Extendibility Axiom, $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_{\bullet}]}$ is a
 1003 set of pointwise bounded continuous functions on the compact set $\mathcal{L}[r_{\bullet}]$. Since

1004 $\pi_P \circ \tilde{\Delta}|_{L[r_\bullet]} = \pi_P \circ \Delta|_{L[r_\bullet]}$ has the NIP, so does $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]}$ by Lemma 1.12. By
 1005 Theorem 4.5, we have $\pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$ for all $r_\bullet \in R$ and $P \in \mathcal{P}$. Let
 1006 $f \in \overline{\Delta}$ be a deep computation. Write $f = \mathcal{U} \lim_i \gamma_i$ as an ultralimit of computations
 1007 in Δ . Define $\tilde{f} := \mathcal{U} \lim_i \tilde{\gamma}_i$. Then, $\tilde{f} : \mathcal{L}_{\text{sh}} \rightarrow \mathcal{L}_{\text{sh}}$ extends f . Since Δ is R -confined
 1008 we have that $f : L[r_\bullet] \rightarrow L[r_\bullet]$ and $\tilde{f} : \mathcal{L}[r_\bullet] \rightarrow \mathcal{L}[r_\bullet]$ for all $r_\bullet \in R$. Lastly, note that
 1009 for all $r_\bullet \in R$ and $P \in \mathcal{P}$ we have that $\pi_P \circ \tilde{f}|_{\mathcal{L}[r_\bullet]} \in \pi_P \circ \tilde{\Delta}|_{\mathcal{L}[r_\bullet]} \subseteq M_r(\mathcal{L}[r_\bullet])$. \square

1010 **Question 4.9.** Under the same assumptions of the preceding theorem, suppose
 1011 that every deep computation of Δ is universally Monte Carlo computable. Must
 1012 $\pi_P \circ \Delta|_{L[r_\bullet]}$ have the NIP for all $P \in \mathcal{P}$ and all $r_\bullet \in R$?

1013 **4.2. Talagrand stability and Monte Carlo computability of deep compu-**
 1014 **tations.** There is another notion closely related to NIP, introduced by Talagrand
 1015 in [Tal84] while studying Pettis integration. Suppose that X is a compact Haus-
 1016 dorff space and $A \subseteq \mathbb{R}^X$. Let μ be a Radon probability measure on X . Given a
 1017 μ -measurable set $E \subseteq X$, a positive integer k and real numbers $a < b$, we write:

$$D_k(A, E, a, b) = \bigcup_{f \in A} \{x \in E^{2k} : f(x_{2i}) \leq a, f(x_{2i+1}) \geq b \text{ for all } i < k\}.$$

1018 We say that A is *Talagrand μ -stable* if and only if for every μ -measurable set
 1019 $E \subseteq X$ of positive measure and for every $a < b$ there is a $k \geq 1$ such that

$$(\mu^{2k})^*(D_k(A, E, a, b)) < (\mu(E))^{2k},$$

1020 where μ^* denotes the outer measure (we need to work with outer measure since
 1021 the sets $D_k(A, E, a, b)$ need not be μ^{2k} -measurable). The inequality certainly holds
 1022 when A is a countable set of continuous (or μ -measurable) functions.

1023 The main result of this section is that deep computations, i.e., limit points
 1024 of a Talagrand stable set of computations are Monte Carlo computable; this is
 1025 Theorem 4.14 below. We now prove that limit points of a Talagrand μ -stable set
 1026 are μ -measurable. But first, let us state the following useful characterization of
 1027 measurable functions (compare with Fact 1.2):

1028 **Fact 4.10** (Lemma 413G in [Fre03]). *Suppose that (X, Σ, μ) is a measure space*
 1029 *and $\mathcal{K} \subseteq \Sigma$ is a collection of measurable sets satisfying the following conditions:*

- 1030 (1) (X, Σ, μ) is complete, i.e., for all $E \in \Sigma$ with $\mu(E) = 0$ and $F \subseteq E$ we have
 1031 $F \in \Sigma$.
- 1032 (2) (X, Σ, μ) is semi-finite, i.e., for all $E \in \Sigma$ with $\mu(E) = \infty$ there exists
 1033 $F \subseteq E$ such that $F \in \Sigma$ and $0 < \mu(F) < \infty$.
- 1034 (3) (X, Σ, μ) is saturated, i.e., $E \in \Sigma$ if and only if $E \cap F \in \Sigma$ for all $F \in \Sigma$
 1035 with $\mu(F) < \infty$.
- (4) (X, Σ, μ) is inner regular with respect to \mathcal{K} , i.e., for all $E \in \Sigma$

$$\mu(E) = \sup\{\mu(K) : K \in \mathcal{K} \text{ and } K \subseteq E\}.$$

1036 (In particular, if X is compact Hausdorff, μ is a Radon probability measure on X ,
 1037 Σ is the completion of the Borel σ -algebra by μ , and \mathcal{K} is the collection of compact
 1038 subsets of X , all these conditions hold). Then, $f : X \rightarrow \mathbb{R}$ is measurable if and
 1039 only if for every $K \in \mathcal{K}$ with $0 < \mu(K) < \infty$ and $a < b$, either $\mu^*(P) < \mu(K)$ or
 1040 $\mu^*(Q) < \mu(K)$ where $P = \{x \in K : f(x) \leq a\}$ and $Q = \{x \in K : f(x) \geq b\}$.

1041 The following technical lemma will be instrumental for proving Proposition 4.13,
1042 which, in turn, will yield the main result of the subsection, namely Theorem 4.14.

1043 **Lemma 4.11.** *If A is Talagrand μ -stable, then \bar{A} is also Talagrand μ -stable and*
1044 *$\bar{A} \subseteq \mathcal{M}^0(X, \mu)$.*

1045 *Proof.* First, we claim that a subset of a μ -stable set is μ -stable. To see this,
1046 suppose that $A \subseteq B$ and B is μ -stable. Fix any μ -measurable $E \subseteq X$ of positive
1047 measure and $a < b$. Let $k \geq 1$ such that

$$(\mu^{2k})^*(D_k(B, E, a, b)) < (\mu(E))^{2k}.$$

1048 Since $A \subseteq B$, we have $D_k(A, E, a, b) \subseteq D_k(B, E, a, b)$; therefore,

$$(\mu^{2k})^*(D_k(A, E, a, b)) \leq (\mu^{2k})^*(D_k(B, E, a, b)) < (\mu(E))^{2k}.$$

1049 We now show that \bar{A} is μ -stable. Fix $E \subseteq X$ measurable with positive measure
1050 and $a < b$. Let $a' < b'$ be such that $a < a' < b' < b$. Since A is μ -stable, let $k \geq 1$
1051 be such that

$$(\mu^{2k})^*(D_k(A, E, a', b')) < (\mu(E))^{2k}.$$

1052 If $x \in D_k(\bar{A}, E, a, b)$, then there is $f \in \bar{A}$ such that $f(x_{2i}) \leq a < a'$ and $f(x_{2i+1}) \geq$
1053 $b > b'$ for all $i < k$. By definition of pointwise convergence topology, there exists $g \in$
1054 A such that $g(x_{2i}) < a'$ and $g(x_{2i+1}) > b'$ for all $i < k$. Hence, $x \in D_k(A, E, a', b')$.
1055 We have shown that $D_k(\bar{A}, E, a, b) \subseteq D_k(A, E, a', b')$; hence,

$$(\mu^{2k})^*(D_k(\bar{A}, E, a, b)) \leq (\mu^{2k})^*(D_k(A, E, a', b')) < (\mu(E))^{2k}.$$

1056 It suffices to show that $\bar{A} \subseteq \mathcal{M}^0(X, \mu)$. Suppose that there exists $f \in \bar{A}$ such that
1057 $f \notin \mathcal{M}^0(X, \mu)$. By fact 4.10, there exists a μ -measurable set E of positive measure
1058 and $a < b$ such that $\mu^*(P) = \mu^*(Q) = \mu(E)$ where $P = \{x \in E : f(x) \leq a\}$ and
1059 $Q = \{x \in E : f(x) \geq b\}$. Then, for any $k \geq 1$: $(P \times Q)^k \subseteq D_k(\{f\}, E, a, b)$, so
1060 $(\mu^{2k})^*(D_k(\{f\}, E, a, b)) = (\mu^*(P)\mu^*(Q))^k = (\mu(E))^{2k}$. Thus, $\{f\}$ is not μ -stable.
1061 However, we argued above that a subset of a μ -stable set must be μ -stable, so we
1062 have a contradiction. \square

1063 **Definition 4.12.** We say that A is *universally Talagrand stable* if A is Talagrand
1064 μ -stable for every Radon probability measure μ on X .

1065 We first observe that universal Talagrand stability corresponds to a complexity
1066 class smaller than or equal to the NIP class:

1067 **Proposition 4.13.** *Let X be a compact Hausdorff space and $A \subseteq C(X)$ be point-*
1068 *wise bounded. If A is universally Talagrand stable, then A satisfies the NIP.*

1069 *Proof.* By Theorem 4.5, it suffices to show that A is relatively countably compact
1070 in $\mathcal{M}^0(X, \mu)$ for every Radon probability measure μ on X . Since A is Talagrand
1071 μ -stable for any such μ , we have $\bar{A} \subseteq \mathcal{M}^0(X, \mu)$ by Lemma 4.11. In particular, A
1072 is relatively countably compact in $\mathcal{M}^0(X, \mu)$. \square

1073 **Corollary 4.14.** *Let (L, \mathcal{P}, Γ) be a CCS satisfying the Extendibility Axiom. If*
1074 *$\pi_P \circ \Delta|_{L[r_\bullet]}$ is universally Talagrand stable for all $P \in \mathcal{P}$ and all sizes r_\bullet , then*
1075 *every deep computation is universally Monte Carlo computable.*

1076 *Proof.* This is a direct consequence of Proposition 4.13 and Theorem 4.8. \square

In the context of deep computations, we have identified two ways to obtain Monte Carlo computability, namely, NIP/PAC and Talagrand stability. It is natural to ask whether these two notions are equivalent. The following results show that, even in the simple case of countably many computations, this question is sensitive to the set-theoretic axioms. On the one hand, it is consistent (with respect to the standard ZFC axioms of set theory) that these two classes are the same:

Theorem 4.15 (Talagrand, Theorem 9-3-1(a) in [Tal84]). *Let X be a compact Hausdorff space and $A \subseteq M_r(X)$ be countable and pointwise bounded. Assume that $[0, 1]$ is not the union of $< \mathfrak{c}$ closed measure zero sets. If A satisfies the NIP, then A is universally Talagrand stable.*

(The assumption that $[0, 1]$ is not the union of $< \mathfrak{c}$ closed measure zero sets is a consequence of, for example, the Continuum Hypothesis.)

On the other hand, by fixing a particular well-known measure, namely the Lebesgue measure, we see that the other case is also consistent:

Theorem 4.16 (Fremlin, Shelah, [FS93]). *It is consistent with the usual axioms of set theory that there exists a countable pointwise bounded set of Lebesgue measurable functions with the NIP which is not Talagrand stable with respect to Lebesgue measure.*

Notice that the preceding two results apply to sets of measurable functions, a class of functions larger than the class of continuous functions. However, by the Extendibility Axiom, finitary computations are continuous, i.e., if A is a set of computations, then $A \subseteq C_p(X)$. The question of whether we can remove the set-theoretic assumption in Theorem 4.15 when $A \subseteq C_p(X)$ (instead of $A \subseteq M_r(X)$) remains open.

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