List of Publications

The contents of this thesis have been published in the following articles:

- Section 1.5: Adapted from [5], where we solved a problem of Brian and Clontz.

- Chapter 2: Primarily based on my contributions to [2], specifically regarding the separation of learnability classes with computational structures, and the connections to other fields.

- Section 3.2: Focuses solely on my contributions to two main algorithms, their analysis, development and implementation detailed in [6].

- Section 3.3: Based on both the joint work and the sections and main results I contributed to [3] and [1].

- Section 3.4: Based on [4], specifically on my contributions to several conjectures and practical implementations.

Contents

Chapter 1

Descriptive Combinatorics

This is the descriptive combinatorics chapter of the thesis.

1.1 Background

This is the place to explain how to get from Cantor's classical results on Polish spaces, to the G_0 dichotomy and its applications, to more advanced topics inspired by these dichotomies.

1.2 A concise proof of the L_0 dichotomy

[Removed for pending review]

1.3 A Ramsey-type theorem for E_0 trees

I present a Ramsey-type theorem for E_0 trees, which is a key combinatorial ingredient in ongoing work.

1.4 Inexistence of a basis for digraphs of dichromatic number 3

I answer the open question of whether a basis exists for the class of digraphs with dichromatic number at least 3.

1.5 Uncountable sets and an infinite linear order game

This section addresses a question posed by Matthew Baker [Math. Mag. 80 (2007)] regarding the characterization of countable subsets of the reals via infinite games. While Brian and Clontz recently characterized this for countable payoff sets, the question remained open for general linear orders.

My Contribution: In collaboration with the author of [5], I provided the positive answer to the existence of winning strategies on uncountable sets. Furthermore, we constructed a dense linear order

of size κ for every infinite cardinal κ on which Player II has a winning strategy. This construction demonstrates the complexity of generalizing the Brian-Clontz characterization.

Chapter 2

Deep Computations

This is the deep computations chapter of the thesis.

This work was presented over the course of multiple sessions at the Fields Institute.

2.1   Introduction

2.2   Connections to learning theory

I formalize the connection between deep computations and learning theory. Here I include a series of examples connecting different variants of the NIP. A good explanation or analogy would be great in this section. Expanding on the initial concepts from [2], I explicitly map the relationships between the Non-Independence Property (NIP), VC-dimension, and PAC-learnability within this new context.

2.3   The Bourgain-Fremlin-Talagrand theorems

I synthesize the Bourgain-Fremlin-Talagrand theorems, organizing and refining the material presented during four seminars at the Fields Institute into a cohesive reference for this field. The first parts of the theory were presented on the date?? and then also on this other day.

2.4   The non-independence property

I analyze the model-theoretic aspects of deep computations, isolating the specific role NIP plays in the combinatorial properties of definable sets.

2.5   Compositional computation structures

This section details my primary contribution to [2]: the construction of concrete examples of computation structures that separate learnability classes. These examples provide the necessary counterpoints to establish the boundaries of the theory.

2.6 Quantum logics

Work in progress...

Chapter 3

Group Actions on Graphs

This is the group actions on graphs chapter of the thesis.

3.1   Introduction

3.2   Border tracing in arbitrary adjacency graphs

While border tracing is standard for finite planar graphs, I generalize these algorithms to arbitrary adjacency graphs.

My Contribution: I developed and analyzed two generalized algorithms for this setting.

3.3   Recursive constructions

I summarize my contributions to both [3] and [1] regarding recursive constructions of graphs with interesting combinatorial properties.

3.4   Amoeba trees

I present the properties of amoeba trees and recent advancements regarding their conjectures.

My Contribution: I independently conceptualized several key properties discussed in this section. Additionally, I authored abstractions and object-oriented frameworks that had a significant impact on the results presented in [4]. Apart from my immediate contributions, I focus on unpublished ideas that helped shaped the research.

3.5   Infinitary amoeba graphs

I synthesize ideas on infinite graphs, focusing on the interaction between automorphism groups, ends, and amenable groups, specifically placing my own results and ideas concerning amoeba graphs within this broader theoretical context.

Definition 3.1 (Local Amoeba). A graph G is called a *local amoeba* if $\overline{\mathrm{Fer}(G)} = S_X$.

Assume that G is a graph defined on an infinite countable set X. In this case, X^X is homeomorphic with the Baire space ω^ω, and so we are dealing with separable and completely metrizable spaces (i.e. Polish spaces). One can show that S_X, being a G_δ subspace of the Baire space, is also Polish and thus its topology is induced by a complete metric. Indeed, observe that

$$S_X = \bigcap_{\substack{x,y \in X \\ x \neq y}} \{f \in X^X : f(n) \neq f(m)\} \cap \bigcap_{y \in X} \bigcup_{x \in X} \{f \in X^X : f(x) = y\}$$

is a countable (whenever X is) intersection of open sets. Giving an explicit complete metric is done in the example at the end of the section.

Group actions on graphs may be quite different in the infinite case. For instance, consider the automorphism of a random graph. Most finite random graphs are asymmetric but almost all countably infinite random graphs are highly symmetric.

Many properties of finite amoebas translate to our more general definition, such as the fact that $\mathrm{Fer}(G) = \mathrm{Fer}(\overline{G})$ for any graph G. Also, any complete graph on any set X is a local amoeba, and thus so is the discrete graph (the graph with no edges). As a first non-trivial example, the bi-infinite path $P_\mathbb{Z}$, i.e. the graph defined on $X = \mathbb{Z}$ where the edges are given by consecutive pairs, is interestingly not a local amoeba. In fact in this context, bi-infinite paths play a similar role as finite cycles. Indeed, it straightforward to verify that $\mathrm{Fer}(P_\mathbb{Z}) = \mathrm{Aut}(P_\mathbb{Z})$. Notice that the basic open set $[(0\ 1)(2)]$ is disjoint from $\mathrm{Fer}(G)$ since the only automorphism interchanging 0 and 1 is a reflection (which doesn't fix the element 2), and thus $(0\ 1) \notin \overline{\mathrm{Fer}(G)}$.

Lemma 3.2. *Let $\Gamma \leq S_X$ and $\varphi \in S_X$. Then $\varphi \in \overline{\Gamma}$ if and only if for every finite $F \subseteq X$, there is $\sigma \in \Gamma$ such that $\varphi \restriction F = \sigma \restriction F$.*

Proof. For the forward direction, take φ and F as above. Clearly, $[\varphi \restriction F]$ is a basic open neighborhood of φ and hence must intersect Γ. Any σ witnessing this has the sought property.

Conversely, take an arbitrary $t \in \omega^{<\omega}$ extended by φ. Pick $\sigma \in \Gamma$ with $\varphi \restriction \mathrm{dom}\, t = \sigma \restriction \mathrm{dom}\, t$. Clearly, $\sigma \in \Gamma \cap [t]$, proving that $\varphi \in \overline{\Gamma}$. $\square$

A group action on X is called *n-transitive* if $|X| \geq n$ and for any two pairwise distinct n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, there is a group element g such that $gx_i = y_i$ for all $1 \leq i \leq n$. Notice that 1-transitive is just the usual definition of *transitive*, i.e. that there is a single orbit. If for all $n \geq 1$, $\mathrm{Fer}(G)$ acts n-transitively on X, we say $\mathrm{Fer}(G)$ is *highly transitive*.

Proposition 3.3. *The following are equivalent.*

1. *The graph G is a local amoeba.*

2. *Every finite $F \subseteq X$ and every $\varphi \in S_X$, there is $\sigma \in \mathrm{Fer}(G)$ such that $\varphi \restriction F = \sigma \restriction F$.*

3. *$\mathrm{Fer}(G)$ is highly transitive.*

Proof. The implication $[1 \implies 2]$ follows from the previous lemma. Assume (2) and let $n \geq 1$ and $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two pairwise distinct. Let $F := \{x_i : 1 \leq i \leq n\}$, then fix a bijection $\omega \setminus F \to \omega \setminus \{y_i : 1 \leq i \leq n\}$ and define φ such that it agrees with said bijection and in addition satisfies $\varphi(x_i) = y_i$. By (2), some $\sigma \in \mathrm{Fer}(G)$ has the desired property.

Finally, assume (3), let φ be an arbitrary permutation on X and $F = \{x_1, \ldots, x_n\} \subseteq X$. Apply (3) to the tuples $(x_1, \ldots, x_n)$ and $(\varphi(x_1), \ldots, \varphi(x_n))$ to get $\sigma \in \mathrm{Fer}(G)$. It follows that σ agrees with φ on F. Since F was arbitrary, by the previous lemma, G must be a local amoeba. $\qquad\square$

I point out that in the example of the bi-infinite path, $\mathrm{Fer}(P_{\mathbb{Z}})$ acts transitively on $P_{\mathbb{Z}}$ but not 2-transitively, as the same aforementioned example clearly shows, namely the pairs $(0, 2)$ and $(1, 2)$. In such cases, one could use the largest n for which the group acts n-transitively on G to compare how close G is to being a local amoeba.

Denote by P_{n+1} the path on the vertex set $n+1$ with edges given by consecutive pairs of numbers. For a label $i \in X$, denote by $\mathrm{Fer}^i(G)$ the subgroup of $\mathrm{Fer}(G)$ generated by those generators of $\mathrm{Fer}(G)$ that fix i. We say a graph G is *stem-symmetric with respect to i* (or to the vertex labeled by i) if $\mathrm{Fer}^i(G)$ is the symmetric group on $X \setminus \{i\}$.

The next result provides an example of a graph where the gap between the automorphism group and the Fer group is as large as possible. Concretely, the countably infinite path on $X = \omega$, denoted P_ω where the edges are given by consecutive pairs of naturals, has a trivial automorphism group, but $\mathrm{Fer}(P_\omega)$ is dense in S_ω.

We need the following combinatorial lemma to prove that infinite paths are local amoebas. The only need the fact that finite paths are stem-symmetric with respect to an endpoint.

Lemma 3.4. *The graph P_{n+1} is stem-symmetric with respect to k if and only if $k \leq n$ and, if $n \geq 5$ is odd, $k \neq \frac{n-1}{2}$.*

Proof. We prove the case $k = n$ as the others are similar. Let $1 \leq \ell < n$ and notice that the graph $P_{n+1} - (\ell, \ell+1) + (0, \ell+1)$ is a path on $n+1$ vertices and thus isomorphic to P_{n+1}. We can find an explicit isomorphism that fixes n and decompose it into transpositions. In symbols,

$$\prod_{k=0}^{\ell} (k \quad \ell - k) : (\ell \; (\ell+1) \to 0 \; (\ell+1)).$$

The second step is realizing that these permutations indeed generate the symmetric group on (the set) n. This is straightforward and can be done in many ways. For example, every transposition of the form $(0 \; \ell)$, for $1 \leq \ell < n$, is in $\mathrm{Fer}^n(P_{n+1})$ by an inductive argument on ℓ, taking conjugates and noticing that the transposition in question is the left-most factor in the ℓ-th permutation above. $\qquad\square$

Proposition 3.5. *P_ω is a local amoeba.*

Proof. Let $F \subseteq \omega$ be finite and $\varphi \in S_\omega$. Define $n := \max(F \cup \varphi[F]) + 1$ and notice that $t := \varphi \upharpoonright F$ is a bijection between finite subsets of n. In particular, there exists a bijection $s : n \setminus F \to n \setminus t[F]$. Now define $\sigma : X \to X$ by

$$\sigma(x) := \begin{cases} s(x) & x \in n \setminus F \\ t(x) & x \in F \\ x & x \geq n \end{cases}$$

Routine arguments show that $\sigma \in S_X$. Now consider the induced subgraph P_{n+1} and the permutation $\sigma \upharpoonright (n+1)$. Observe that this permutation fixes the endpoint of the path, n, by definition. Thus by the previous lemma, there is a sequence $f_0, \ldots, f_m$ of generators of $\mathrm{Fer}(P_{n+1})$, all of which fix n, such that $\sigma \upharpoonright (n+1) = f_0 \cdots f_m$.

For each $i < m$, we can extend f_i to a permutation $\overline{f_i}$ on X by fixing every label $x > n$. Clearly, each $\overline{f_i}$ is a member of $\mathrm{Fer}(P_\omega)$ and thus $\sigma = \overline{f_0} \cdots \overline{f_m} \in \mathrm{Fer}(P_\omega)$. Finally, the choice of σ makes it evident that $\sigma \upharpoonright F = t = \varphi \upharpoonright F$, and thus P_ω is a local amoeba by Proposition 3.3. $\qquad\square$

Bibliography

[1] Y. Caro, I. González, A. Hansberg, M. Jácome, T. Matos-Wiederhold, and A. Montejano. Graphs with constant balancing number. *Procedia Computer Science*, 223:325–333, 2023. XII Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS 2023). URL: `https://www.sciencedirect.com/science/article/pii/S1877050923010177`, `doi:10.1016/j.procs.2023.08.244`.

[2] Eduardo Dueñez, José Iovino, Tonatiuh Matos-Wiederhold, Luciano Salvetti, and Franklin D. Tall. Complexity of deep computations via topology of function spaces, 2026. Preprint. `arXiv:2601.00528`.

[3] L. Eslava, A. Hansberg, T. Matos-Wiederhold, and D. Ventura. New recursive constructions of amoebas and their balancing number. *Aequationes Mathematicae*, 99:1265–1299, 2025. Preprint available at `https://https://arxiv.org/abs/2311.17182`.

[4] S. Gómez, I. González, A. Hansberg, and T. Matos-Wiederhold. Amoeba trees. Preprint.

[5] Tonatiuh Matos-Wiederhold and Luciano Salvetti. Uncountable sets and an infinite linear order game. *Topology Proceedings*, 66:177–184, 2025. E-published on July 22, 2025. Preprint available at `https://arxiv.org/abs/2408.14624`.

[6] Petra Wiederhold and Tonatiuh Matos-Wiederhold. Border tracing in oriented adjacency graphs of polygonal tilings. *Theoretical Computer Science*, 1065:115729, 2026. URL: `https://www.sciencedirect.com/science/article/pii/S0304397525006668`, `doi:10.1016/j.tcs.2025.115729`.