4. Dibuja las elipsoides sólidas $\{x \mid (x-\bar{x})' S^{-1} (x-\bar{x}) \leq 1\}$ para las sig matrices y determina los valores de los ejes mayores y menores

a) $S = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

$$S^{-1} = \frac{1}{25-16}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix}$$

$$(x-\bar{x}, y-\bar{y})\begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix}\begin{bmatrix} x-\bar{x} \\ y-\bar{y} \end{bmatrix} = \left[\frac{5}{9}(x-\bar{x}) - \frac{4}{9}(y-\bar{y}),\ -\frac{4}{9}(x-\bar{x}) + \frac{5}{9}(y-\bar{y})\right]\begin{bmatrix} x-\bar{x} \\ y-\bar{y} \end{bmatrix}$$

$$= \frac{5}{9}(x-\bar{x})^2 - \frac{4}{9}(y-\bar{y})(x-\bar{x}) - \frac{4}{9}(y-\bar{y})(x-\bar{x}) + \frac{5}{9}(y-\bar{y})^2$$

$$= \frac{5}{9}(x-\bar{x})^2 - \frac{8}{9}(y-\bar{y})(x-\bar{x}) + \frac{5}{9}(y-\bar{y})^2$$

$$\rightarrow \text{elipse}_1 = \frac{5}{9}(x-\bar{x})^2 - \frac{8}{9}(y-\bar{y})(x-\bar{x}) + \frac{5}{9}(y-\bar{y})^2 \leq 1 = c^2$$

Sabemos que el término cruzado corresponde a la rotación de la elipse

Para los ejes, debemos sacar los eigenvectores de S^{-1} (los mismos que de S)

$$S = \begin{pmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - 16 = 25 - 10\lambda + \lambda^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda-9)(\lambda-1)$$

$$\rightarrow \lambda_1 = 1 \qquad \lambda_2 = 9$$

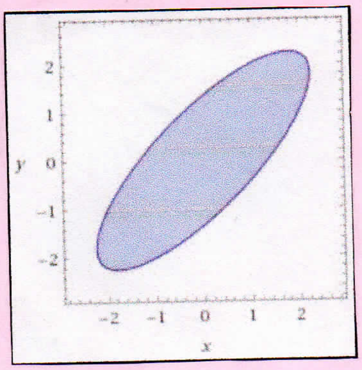
v_1: $S = \begin{pmatrix} 5-1 & 4 \\ 4 & 5-1 \end{pmatrix} = \left(\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow x+y=0 \leftrightarrow \begin{array}{l} x=x \\ y=-x \end{array} \quad \therefore v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

v_2: $S = \begin{pmatrix} 5-9 & 4 \\ 4 & 5-9 \end{pmatrix} = \left(\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow -x+y=0 \leftrightarrow \begin{array}{l} x=x \\ y=y \end{array} \quad \therefore v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Tenemos que los ejes están dados por $c^2\sqrt{\lambda_i}\, v_i$

$\rightarrow$ el eje mayor es $\frac{3}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

el eje menor es $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ //



b) $S = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$

$$S^{-1} = \frac{1}{25-16}\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \begin{bmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{bmatrix}$$

para la elipse

$$(x-\bar{x}, y-\bar{y})\begin{bmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{bmatrix}\begin{bmatrix} x-\bar{x} \\ y-\bar{y} \end{bmatrix} = \left[\frac{5}{9}(x-\bar{x}) + \frac{4}{9}(y-\bar{y}),\ \frac{4}{9}(x-\bar{x}) + \frac{5}{9}(y-\bar{y})\right]\begin{bmatrix} x-\bar{x} \\ y-\bar{y} \end{bmatrix}$$

$$= \frac{5}{9}(x-\bar{x})^2 + \frac{4}{9}(x-\bar{x})(y-\bar{y}) + \frac{4}{9}(x-\bar{x})(y-\bar{y}) + \frac{5}{9}(y-\bar{y})^2$$

$$= \frac{5}{9}(x-\bar{x})^2 + \frac{8}{9}(x-\bar{x})(y-\bar{y}) + \frac{5}{9}(y-\bar{y})^2$$

$$\therefore\ \frac{5}{9}(x-\bar{x})^2 + \frac{8}{9}(x-\bar{x})(y-\bar{y}) + \frac{5}{9}(y-\bar{y})^2 \leq 1$$

Para los ejes sacamos eigenpares

$$\begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = (\lambda-1)(\lambda-9)$$

¡Son los mismos eigenpares que el ejercicio anterior!

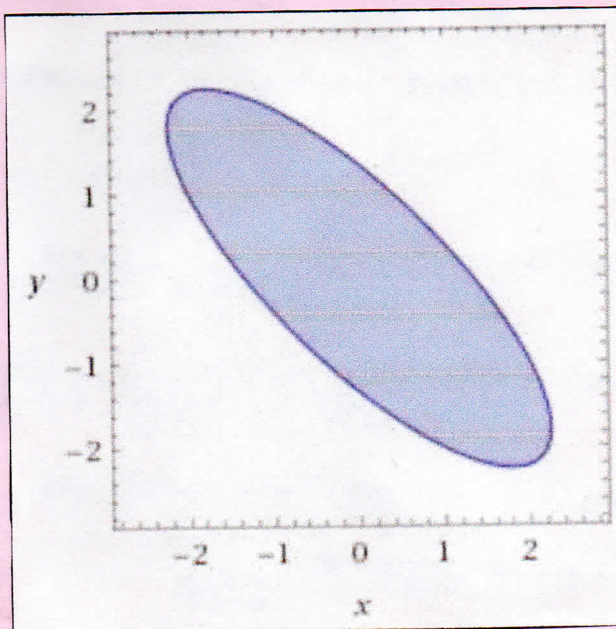
(λ_1, v_1): $\lambda_1 = 1$ $\quad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(λ_2, v_2): $\lambda_2 = 9$ $\quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Los ejes están dados por $c^2\sqrt{\lambda_i}\,v_i$

$\therefore$ el eje mayor $= \frac{3}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

el eje menor $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ //



c) $S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$S^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$(x-\bar{x}, y-\bar{y})\, S^{-1} \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} = (x-\bar{x}, y-\bar{y}) \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} = \left(\frac{1}{3}(x-\bar{x}), \frac{1}{3}(y-\bar{y})\right) \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}$$

$$= \frac{1}{3}(x-\bar{x})^2 + \frac{1}{3}(y-\bar{y})^2$$

$\therefore \; \frac{1}{3}(x-\bar{x})^2 + \frac{1}{3}(y-\bar{y})^2 \leq 1$ (es un circulo)

Para los ejes, necesitamos los eigenpares

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 \rightarrow \lambda = 3$$

Arbitrariamente elegimos los eigenvectores $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ y $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$c^2\sqrt{\lambda}$

$\therefore$ Los ejes son $3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ y $3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Como es un circulo, eje mayor = eje menor //