Average minus Weak Complexity equals Adaptive Algorithms

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based in discussions and work with
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August 23, 2019

Complexity of Numerical Computation
A conference in honor of Felipe Cucker

Slides at `https://tonellicueto.xyz/pdf/felipefest2019_slides.pdf`

Collaborators and me

This talk is based on discussions and work with

- Peter Bürgisser (Technische Universität Berlin),
- Felipe Cucker (CityU Hong Kong), and
- Alperen Ergür (Technische Universität Berlin).

To be more developed and written out in

- future work, and
- chapter 5 of my PhD thesis
  "Condition and Homology in Semialgebraic Geometry"

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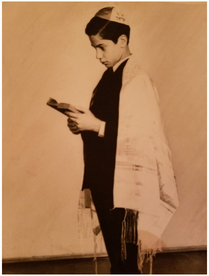
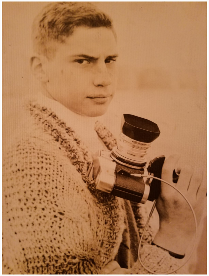
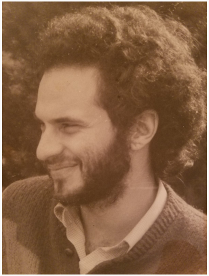
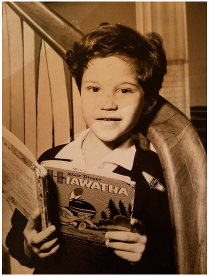
Young Felipe









Thanks to Antonia!

Philosophy of the talk

符号算法、
数值算法，
只要能解决未解决的问题就
是好算法。

Table of contents

Felipe and me somewhere in Lantau Island

Grid method until now

Grid method

Sketch of the grid method:

- Put a grid of points approaching the space
- Check a condition in all points of the grid
- Refine if the condition is not satisfied at some of the points
    - Non-adaptive: Uniform refinement of the whole grid
    - Adaptive: Refinement only around the points where something goes wrong

Philosophy of the method:

- *divide et impera*
- Smaller operations lead to smaller errors and errors cannot sequentially accumulate

Grid method in real algebraic geometry I

(CS98) (Cucker & Smale, 1998)
Grid method is introduced for testing feasibility of a semialgebraic set
Condition-based analysis, but no probabilistic estimate

(C99) (Cucker, 1999)
Improvements over CS99 are done
Condition number $\kappa(f)$ is introduced

(CKMW1) (Cucker, Krick, Malajovich & Wschebor, 2008)
Counting zeros of real algebraic set: Condition-based complexity

(CKMW2) (Cucker, Krick, Malajovich & Wschebor, 2009)
Counting zeros of real algebraic set: Condition number theorem and probabilistic analysis

(CKMW3) (Cucker, Krick, Malajovich & Wschebor, 2012)
Counting zeros of real algebraic set: Alternative probabilistic analysis (without using conic structure)

Grid method in real algebraic geometry II

write the "bible of condition"
Grid method appears in Chapter 19

Grid method in real algebraic geometry III

(CKS16) (Cucker, Krick & Shub, 2016)
Numerical computation of homology of real projective sets
[using the reach together with the approximation theorem of (Niyogi, Smale & Weinberger, 2008)]

(BCL17) (Bürgisser, Cucker & Lairez, 2017)
...of basic semialgebraic sets

(BCTC1) (Bürgisser, Cucker & T.-C., 2019)
...of closed semialgebraic sets

(BCTC2) (Bürgisser, Cucker & T.-C., 2019)
...of arbitrary semialgebraic sets
[using construction of (Grabrielov & Vorobjov, 2009)]

Successes and defeats

Successes:

- Algorithms work in single exponential time, i.e., $\mathrm{poly}(q, D)^{\mathrm{poly}(n)}$, with high probability (weak complexity)
- Highly parallelizable algorithm

Defeats:

- Algorithm does not work for all inputs...
  (inevitable in numerical algorithms)
  We can deal with it with an hybrid approach: Under a condition threshold go numerical, and above the threshold go symbolic.
- Infinite expectation (unlike the complex case)
- Why are adaptive algorithms not better?

Felipe making pamplonas

Plantinga-Vegter algorithm enters the game

Setting

What do we have?

- An implicit curve C inside $[-a, a]^2$ given by a C^1 function $f : [-a, a]^2 \to \mathbb{R}$
- Interval approximations $\Box f$ of f and $\Box\partial f$ of ∂f

What do we want?

- Piecewise-linear approximation L of C in $[-a, a]^2$such that $([-a, a]^2, C)$ and $([-a, a]^2, L)$ are isotopic

Any assumptions?

- C smooth
- C Intersects the boundary of $[-a, a]^2$ transversely

Plantinga-Vegter algorithm for curves I

Algorithm: PV Algorithm for curves (Plantinga, Vegter; 2004)

Input: $a \in (0, \infty)$ and $f : \mathbb{R}^2 \to \mathbb{R}$
with interval approximations $\square[f]$ and $\langle \square[\partial f], \square[\partial f] \rangle$

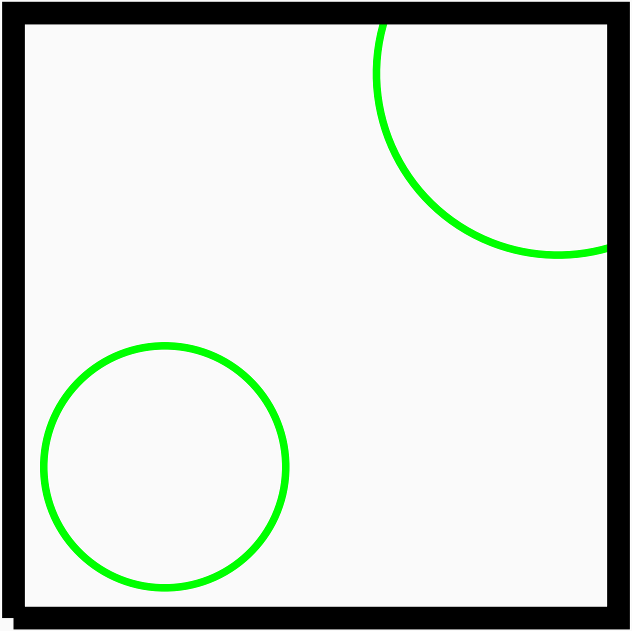
SUBDIVISION:
Starting with the trivial subdivision $\mathcal{S} := \{[-a, a]^n\}$, repeatedly subdivide each $J \in \mathcal{S}$ into 4 squares until for all $J \in \mathcal{S}$,

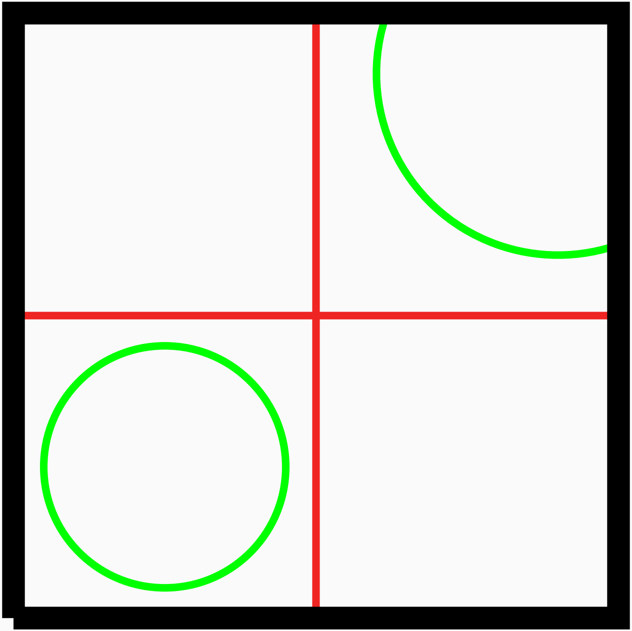
$$0 \notin \square f(J) \text{ or } 0 \notin \langle \square \partial f(J), \square \partial f(J) \rangle$$

CONSTRUCTION:
Construct piecewise-linear curve L
joining the midpoints of “small” edges of each $J \in \mathcal{S}$ with oposite f-signs at their vertices

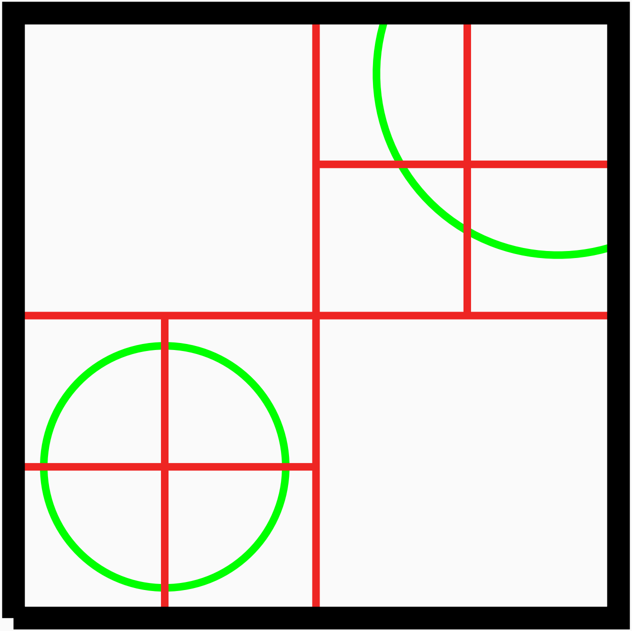
Output: Piecewise-linear approximation L of $C = f^{-1}(0) \cap [-a, a]^2$ isotopic to it

Plantinga-Vegter algorithm for curves II: Example

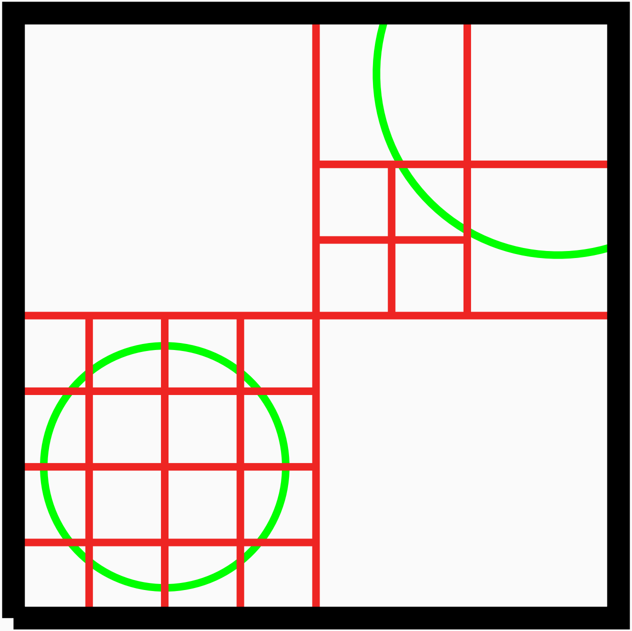




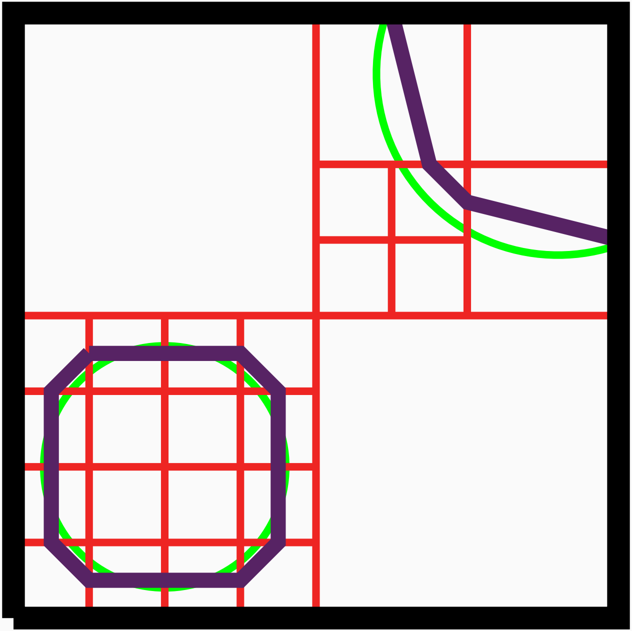
Plantinga-Vegter algorithm for curves II: Example



Plantinga-Vegter algorithm for curves II: Example



Plantinga-Vegter algorithm for curves II: Example



Plantinga-Vegter algorithm in higher dimensions

1. (Plantinga, Vegter; 2004) introduced the algorithm for curves and surfaces
   Very efficient in practice
2. (Burr, Gao & Tsigaridas, ISSAC2017) generalized the subdivision method to higher dimensions
   but no construction of the piecewise-linear approximation...

Note that...

$$\text{cost(subdivision algorithm)} \sim \#\,\text{(subdivisions)} \cdot \text{cost(evaluations)}$$

Subdivision in Plantinga-Vegter algorithm

Algorithm: Subdivision of PV Algorithm (Burr, Gao & Tsigaridas, ISSAC2017)

Input: $a \in (0, \infty)$, $f : \mathbb{R}^n \to \mathbb{R}$
with interval approximations $\square[hf]$ and $\square[h'\partial f]$
for some functions $h, h' : \mathbb{R}^n \to (0, \infty)$

Starting with the trivial subdivision $\mathcal{S} := \{[-a, a]^n\}$, repeatedly subdivide each $J \in \mathcal{S}$ into 2^n cubes until the condition

$$C_f(J) : 0 \notin \square[hf](J) \text{ or } 0 \notin \langle \square[h'\partial f], \square[h'\partial f] \rangle$$

holds for all $J \in \mathcal{S}$

Output: Subdivision $\mathcal{S} \subseteq \mathcal{I}_n$ of $[-a, a]^n$
such that for all $J \in \mathcal{S}$, $C_f(J)$ is true

h, h' depend on the setting and the interval arithmetic one uses

Local size bound

Idea: Give a bound to the size of the smallest box containing a point not satisfying the condition.

Definition (Burr, Gao & Tsigaridas, ISSAC2017)

A *local size bound* for f is a function $b_f : \mathbb{R}^n \to [0, \infty)$ such that for all $x \in \mathbb{R}^n$,

$$b_f(x) \leq \inf\{\mathrm{vol}\,(J) \mid x \in J \in \mathcal{I}_n \text{ and } C_f(J) \text{ false}\},$$

where $C_f(J)$: $0 \notin \square[hf](J)$ or $0 \notin \langle \square[h'\partial f], \square[h'\partial f] \rangle$

1st bound in terms of the local size bound

Idea: Boxes cover the cube. The volume of the boxes should add to that of the cube. Volume of the boxes is at least number of boxes times volume of smallest box.

Proposition (Burr, Gao & Tsigaridas, ISSAC2017)

The number of n-cubes of the final subdivision of the subdivision of the PV *algoirhtm on input* (f, a), *regardless of how the subdivision step is done, is at most*

$$(2a)^n / \inf\{b_f(x) \mid x \in [-a, a]^n\} \qquad \square$$

where b_f *is a local size bound for* f*.*

(Burr, Gao & Tsigaridas, ISSAC2017) construct a local size bound for an integer polynomial $f \in \mathcal{P}_{n,d}$ and obtain the bound

$$2^{\mathcal{O}(nd^{n+1}(n\tau+nd\log(nd)+9n+d)\log a)}$$

where τ is the bit-size of the coefficients.

2nd bound in terms of the local size bound

Refinement of the idea: Not all boxes have the same size. "$\frac{\mathrm{d}x}{b_f(x)}$ is, up to constant, the infinitesimal number of boxes needed to cover x."
Formalized by (Burr, 2016) using the technique known as *continuous amortization* introduced in (Burr, Krahmer & Yap, 2009)

Theorem (Burr, Gao & Tsigaridas, ISSAC2017)

The number of n-cubes of the final subdivision of the PV *algorithm on input (f, a) is at most*

$$\max\left\{1, \int_{[-a,a]^n} \frac{2^n}{b_f(x)}\,\mathrm{d}x\right\}$$

where b_f is a local size bound for f. Moreover, the bound is finite if and only if the algorithm terminates. □

Techniques of (Burr, Gao & Tsigaridas, ISSAC2017) cannot exploit it!

Can we exploit it?

(Burr, Gao & Tsigaridas, ISSAC2017) said...

> *Even though our bounds are optimal, in practice, these are quite pessimistic [...]*

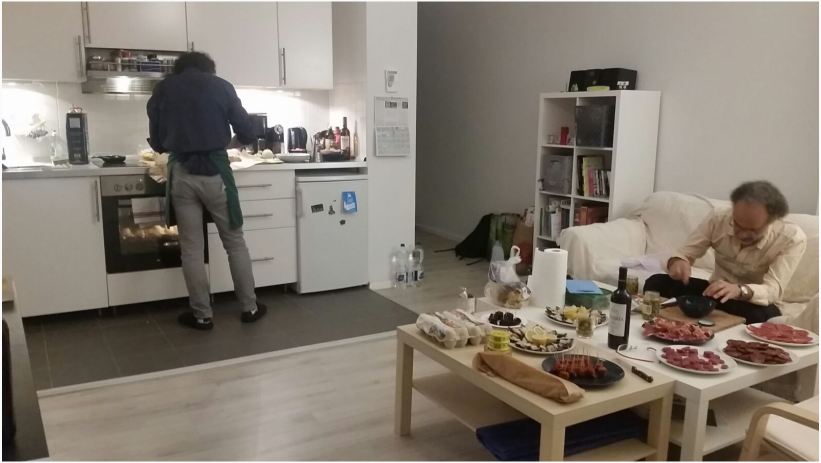
and

> *Since the complexity of the algorithm can be exponential in the inputs [size], the integral must be described in terms of additional geometric and intrinsic parameters.*

What can be a solution to these issues?

CONDITION NUMBERS

Felipe peeling garlics while I cook



A condition-based analysis of Plantinga-Vegter

Local condition number

Definition (Cucker)

Let $f \in \mathcal{H}_{n,d}[q]$, the *local condition number* of f at $y \in \mathbb{S}^n$ is

$$\kappa(f, y) := \frac{\|f\|}{\sqrt{f(y)^2 + \sigma_q(\Delta^{-1}\mathrm{D}_y f)^2}}$$

where $\|f\|$ is the Weyl norm of f, σ_q the qth singular value and $\mathrm{D}_y f$ the tangent map with respect the sphere.
Given $f \in \mathcal{P}_{n,d}[q]$, the *local affine condition number* of f at $x \in \mathbb{R}^n$ is

$$\kappa_{\mathrm{aff}}(f, x) := \kappa(f^{\mathrm{h}}, \phi(x))$$

where f^{h} is the homogeneization of f and $\phi : x \mapsto \frac{1}{\sqrt{1+\|x\|^2}} \begin{pmatrix} 1 \\ x \end{pmatrix}$.

Unfortunately the theory is developed for the homogenous setting.
Too many translations!

Observation

$\kappa(f,x) = \infty$ iff $V_{\mathbb{S}}(f)$ has a singularity at x

Ergo $\kappa(f,x)$ says how near $V_{\mathbb{S}}(f)$ is of having a singularity at x.
Can we be more concrete? YES!

Theorem (Condition Number Theorem)

Let $x \in \mathbb{S}^n$ and

$$\Sigma_x := \{g \in \mathcal{H}_{n,d}[q] \mid g(x) = 0,\ \mathrm{rank}\mathrm{D}_x g < q\},$$

i.e., Σ_x is the set of f having x as a singular point. Then for every $f \in \mathcal{H}_{n,d}[q]$,

$$\frac{\|f\|}{\kappa(f,x)} = \mathrm{dist}(f, \Sigma_x)$$

where $\|\cdot\|$ is the Weyl norm of $\mathcal{H}_{n,d}[q]$ and the distance is the induced by the Weyl norm of $\mathcal{H}_{n,d}[q]$. □

The wanted "additional geometric and intrinsic parameter"

Condition-based local size bounds

Theorem (Cucker, Ergür & T.-C., ISSAC2019)
Let $f \in \mathcal{P}_{n,d}[1]$. Then

$$x \mapsto 1/\left(2^{5/2} d n \kappa_{\mathrm{aff}}(f,x)\right)^n$$

is a local size bound for f with the interval approximation of (Cucker, Ergür & T.-C., ISSAC2019), and

$$x \mapsto 1/\left(2^{3n} d^2 \kappa_{\mathrm{aff}}(f,x)\right)^n$$

with the interval approximation of Remark 2.2. of (Burr, Tsigaridas, Yap; ISSAC2017) □

Condition-based cost (Using 1st bound)

Theorem (Cucker, Ergür & T.-C., ISSAC2019)

The number of n-cubes in the final subdivision of the subdision of the PV *algorithm on input* (f, a) *is at most*

$$d^n \max\{1, a^n\} 2^{n \log n + 9n/2} \max_{x \in [-a,a]^n} \left(\kappa_{\mathrm{aff}}(f, x)^n\right)$$

if the interval approximation is that of (Cucker, Ergür & T.-C., ISSAC2019), and at most

$$d^{2n} \max\{1, a^n\} 2^{3n^2+2n} \max_{x \in [-a,a]^n} \left(\kappa_{\mathrm{aff}}(f, x)^n\right)$$

if the interval approximation is that of (Burr, Tsigaridas, Yap; ISSAC2017). □

Note that

$$\max_{x \in [-a,a]^n} \left(\kappa_{\mathrm{aff}}(f, x)^n\right) \leq \kappa(f^{\mathrm{h}}) := \max_{x \in \mathbb{S}^n} \kappa(f^{\mathrm{h}}, x).$$

Condition-based cost (Using 2nd bound)

Theorem (Cucker, Ergür & T.-C., ISSAC2019)
The number of n-cubes in the final subdivision of the subdision of the PV *algorithm on input* (f, a) *is at most*

$$d^n \max\{1, a^n\} 2^{n \log n + 9n/2} \, \mathbb{E}_{x \in [-a,a]^n} \left(\kappa_{\mathrm{aff}}(f, x)^n\right)$$

if the interval approximation is that of (Cucker, Ergür & T.-C., ISSAC2019), and at most

$$d^{2n} \max\{1, a^n\} 2^{3n^2 + 2n} \, \mathbb{E}_{x \in [-a,a]^n} \left(\kappa_{\mathrm{aff}}(f, x)^n\right)$$

if the interval approximation is that of (Burr, Tsigaridas, Yap; ISSAC2017). □

Note that

$$\mathbb{E}_{x \in [-a,a]^n} \left(\kappa_{\mathrm{aff}}(f, x)^n\right) \leq \mathbb{E}_{x \in \mathbb{S}^n} \left(\kappa(f^{\mathrm{h}}, x)^n\right)$$

What is this expectation?

Can

$$\mathbb{E}_{x \in \mathbb{S}^n} \left(\kappa(f, x)^n \right)$$

be better than

$$\kappa(f) := \max_{x \in \mathbb{S}^n} \kappa(f, x)?$$

Yes!

Dobro random polynomials I

Definition (Cucker, Ergür, T.C.; ISSAC2019)
A *dobro random polynomial* $f \in \mathcal{H}_{n,d}[1]$ with parameters K and ρ is a polynomial

$$f := \sum_{|\alpha|=d} \binom{d}{\alpha}^{1/2} c_\alpha X^\alpha$$

such that the c_α are independent random variables such that

- P1 $\mathbb{E}c_\alpha = 0$ (centered),
- P2 $(\mathbb{E}|c_\alpha|^p)^{\frac{1}{p}} \leq K\sqrt{p}$ for $p \geq 1$ (subgaussian with Ψ_2-norm $\leq K$), and
- P3 $\max_{u\in\mathbb{R}}\{\mathbb{P}(|c_\alpha - u| \leq \varepsilon)\}$ (anti-concentration with constant ρ).

A *dobro random polynomial* $f \in \mathcal{P}_{n,d}[1]$ is a polynomial f such that its homogenization f^{h} is so.

Dobro random polynomials II

Examples of dobro random polynomials:

N *KSS random polynomial*: (KSS=Kostlan-Smale-Shub)

- c_α is Gaussian with unit variance
- $K\rho = 1/\sqrt{2\pi}$

U *Weyl random polynomial*:

- c_α is uniform distribution in $[-1, 1]$
- $K\rho \leq 1$

E A *p-random polynomial*:

- c_α has density function $\delta_p e^{-|t|^p}$ where δ_p being the appropriate constant and $p \geq 2$

Note: Dobro (добро) is a Russian adjective derived from добрый (dóbryj) which means kind, good, genial, gentle, soft, etc.

An unexpected bound

Theorem (Cucker, Ergür, T.C.; ISSAC2019)

Let $f \in \mathcal{H}_{n,d}[1]$ *be a dobro random polynomial with parameters* K *and* ρ. *Then*

$$\mathbb{E}_f \mathbb{E}_{x \in \mathbb{S}^n} \left(\kappa(f^{\mathrm{h}}, x)^n \right) \leq d^{\frac{n^2+n}{2}} 2^{\frac{n^2+3\log n+9}{2}} (c_1 c_2 K \rho)^{n+1}$$

where c_1, c_2 *are universal constants.* □

Corollary

Plantinga-Vegter algorithm has average polynomial time in the degree.

With an improvement of our condition-based techniques (i.e. changing to the max norm) we can eliminate the n^2 of the exponent!

Also available in smoothed form!

$$\mathcal{O}\left(d^5\right) \text{ and } \mathcal{O}\left(d^6\right)$$

with the interval arithmetic of, respectively, (Cucker, Ergür & T.-C., ISSAC2019) and (Burr, Gao, & Tsigaridas, ISSAC2017)

Interlude: Back to curves

$$\mathcal{O}\left(d^{5/2}\log^{5/2} d\right) \text{ and } \mathcal{O}\left(d^{3}\log^{3} d\right)$$

with the interval arithmetic of, respectively, (Cucker, Ergür & T.-C., ISSAC2019) and (Burr, Gao & Tsigaridas, ISSAC2017) With the new techniques! Compare to bit-complexity

$$\tilde{\mathcal{O}}(d^5\tau + d^6)$$

of the deterministic algorithm of (Kobel & Sagraloff, 2015) and (Diatta, Diatta, Rouillier, Roy & Sagraloff; 2018)

Some comments:

- Difference should be careful, complexity measured in different ways! But still, why so efficient?!
- PV algorithm does not work with singular curves, although there is work in this direction (Burr, Choi, Galehouse & Yap, 2012)
- Can we develop an hybrid approach working on all inputs with the same worst case, but faster on average?

Working with Felipe

A condition for adaptive grids?

A new hope

Proposition (T.-C., Thesis)

Let $f \in \mathcal{H}_{n,d}[q]$*. Then*

$$\kappa(f) < \infty \text{ iff } \mathbb{E}_{x \in \mathbb{S}^n}\left(\kappa(f,x)^n\right) < \infty.$$

$\square$

Theorem (T.-C., Thesis)

Let $f \in \mathcal{H}_{n,d}[q]$ *be KSS and* $\alpha < 1 + 1/n$*. Then*

$$\mathbb{E}_f\left(\mathbb{E}_{x \in \mathbb{S}^n}\left(\kappa(f,x)^n\right)\right)^{\alpha} = \mathcal{O}\left(\frac{qD}{\frac{n+1}{n} - \alpha}\right)^{\mathcal{O}(n^2)}$$

$\square$

Dependence on α cannot be improved. This can be shown for hypersurfaces...

Using the max norm, we can get rid of the square in the exponent

Fast computation of minimum of Lipschitz functions

Theorem (Han, 2019)

Let $\Lambda : \mathbb{S}^n \to (0,1]$ *be a L-Lipschitz map. Then there is an adaptive grid algorithm computing a grid* $\mathcal{G} \subseteq \mathbb{S}^n$ *and* $r \in \mathbb{R}_{>}^{\mathcal{G}}$ *such that*

(1) $\{B(x, r_x) \mid x \in \mathcal{G}\}$ *covers* $\mathbb{S}^n$

(2) *For all* $x \in \mathcal{G}$, $r_x \leq \Lambda(x)$

and whose complexity is

$$\mathcal{O}\,(2+L)^n\,\mathbb{E}_{x\in\mathbb{S}^n}\left(\Lambda(x)^{-n}\right).$$

In particular, the $\min_{x\in\mathbb{S}^n} \Lambda(x)$ *can be computed in the above time.*

Note that

1. It's easy to see that the above bound is optimal
2. Note similarity with adaptive homotopy continuation

Fast computation of $\kappa(f)$

Easy observation: $x \mapsto \kappa(f, x)^{-1}$ is D-Lipschitz

Theorem (T.-C., Thesis)

There is an adaptive grid algorithm that computes $\kappa(f)$ in average single exponential time, when $f \in \mathcal{H}_{n,d}[q]$ be KSS.

Observe that funnily $\kappa(f)$ does not have finite expectation.

Corollary (T.-C., Thesis)

There is an adaptive grid algorithm that computes a lower bound of the reach of an spherical algebraic set $\mathcal{Z}(f)$ in average single exponential time, when $f \in \mathcal{H}_{n,d}[q]$ be KSS.

What about homology?

The condition used is quadratic on $\kappa(f, x)$, because of the use of Smale's α-theory. Assuming all TDA techniques work, this gives a condition-based bound with

$$\mathbb{E}_{x \in \mathbb{S}^n} \left(\kappa(f, x)^{2n} \right)$$

for basic semialgebraic sets. This does not give finite expectation!

For general semialgebraic sets, the above cannot even be obtained yet, due to even more technical problems to solve.

More work to do!

Is there a numerical algorithm computing homology of semialgebraic sets in average single exponential time?

Can we develop a hybrid symbolic-numerical algorithm computing homology of semialgebraic sets in average single exponential time and worst-case doubly exponential time?

Guztiagatik eskerrik asko!