

Tensors

A *tensor* is a multi-indexed list of numbers, i.e., a map

$$\{1, \dots, n_1\} \times \dots \times \{1, \dots, n_d\} \ni (j_1, \dots, j_d) \mapsto t_{j_1, \dots, j_d}$$

We denote the space of these tensors by $\mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}$.

So...

A *vector* is a list of numbers
A *matrix* is a table of numbers
A *tensor* is a multidimensional box of numbers
—a 1-tensor is a vector and a 2-tensor a matrix

Rank-One Tensors

A *rank-one tensor* is a tensor $\lambda \mathbf{x}^1 \otimes \dots \otimes \mathbf{x}^d$ of the form

$$\lambda \mathbf{x}^1 \otimes \dots \otimes \mathbf{x}^d := \left(\lambda x_{j_1}^1 \dots x_{j_d}^d \right)$$

where λ is a scalar and the \mathbf{x}^i are vectors.

Observation. If $\lambda \mathbf{x}^1 \otimes \dots \otimes \mathbf{x}^d$ is a real tensor, then we can assume without loss of generality that $\lambda, \mathbf{x}^1, \dots, \mathbf{x}^d$ are real.

So...

Every 1-tensor (vector) is rank-one
A rank-one 2-tensor is just a rank-one matrix

Frobenius norm for tensors

Given a tensor $T = (t_{j_1, \dots, j_d})$, its *Frobenius norm* is

$$\|T\| := \sqrt{\sum_{j_1, \dots, j_d} |t_{j_1, \dots, j_d}|^2}.$$

This norm induces a Hermitian inner product that we denote by $\langle \cdot, \cdot \rangle$.

So...

In the case of 2-tensors (matrices), this agrees with the usual definition

Best rank-one approximation

Given a tensor $T = (t_{j_1, \dots, j_d}) \in \mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}$, a *best rank-one approximation* of T is a rank-one tensor $\alpha \mathbf{z}^1 \otimes \dots \otimes \mathbf{z}^d \in \mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}$ such that for every rank-one tensor $\lambda \mathbf{x}^1 \otimes \dots \otimes \mathbf{x}^d \in \mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}$,

$$\|T - \alpha \mathbf{z}^1 \otimes \dots \otimes \mathbf{z}^d\| \leq \|T - \lambda \mathbf{x}^1 \otimes \dots \otimes \mathbf{x}^d\|.$$

Motivating question.

HOW BAD CAN A RANK-ONE APPROXIMATION BE?

Note...

When working with real tensors,
we limit ourselves to real best rank-one approximations

Best Rank-One Approximation Ratio

Qi & Lu (2017) showed that for a tensor $T \in \mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}$, any of its best rank-one approximations $\alpha \mathbf{z}^1 \otimes \dots \otimes \mathbf{z}^d \in \mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}$ satisfies

$$\frac{\|T - \alpha \mathbf{z}^1 \otimes \dots \otimes \mathbf{z}^d\|^2}{\|T\|^2} = 1 - \left(\max_{\mathbf{x}^i \in \mathbb{K}^{n_i}} \left\{ \frac{|\langle \mathbf{x}^1 \otimes \dots \otimes \mathbf{x}^d, T \rangle|}{\|\mathbf{x}^1\| \dots \|\mathbf{x}^d\| \|T\|} \right\} \right)^2.$$

Moreover, the maximum on the right-hand side is achieved at $\mathbf{z}^1 \otimes \dots \otimes \mathbf{z}^d$.

Best Rank-One Approximation Ratio for X

Given a linear subspace $X \subseteq \mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}$, the *rank-one approximation ratio* for X is

$$\mathcal{A}(X) := \min_{T \in X} \max_{\mathbf{x}^i \in \mathbb{K}^{n_i}} \frac{|\langle \mathbf{x}^1 \otimes \dots \otimes \mathbf{x}^d, T \rangle|}{\|\mathbf{x}^1\| \dots \|\mathbf{x}^d\| \|T\|} \in (0, 1].$$

What does $\mathcal{A}(X)$ measure? The quality of the worst-approximating best-rank approximation of tensors in X

Bounds for $\mathcal{A}(\mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d})$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$)

The result was known, we provided an explicit upper bound...

Theorem

$$\frac{1}{\sqrt{\min_j \prod_{i \neq j} n_i}} \leq \mathcal{A}(\mathbb{K}^{n_1} \otimes \dots \otimes \mathbb{K}^{n_d}) \leq \frac{10\sqrt{d \ln d}}{\sqrt{\min_j \prod_{i \neq j} n_i}}$$

Note the bound does not care if \mathbb{K} is either \mathbb{R} or \mathbb{C} !

Proof techniques...

GEOMETRIC FUNCTIONAL ANALYSIS, INTEGRAL IDENTITIES & PROBABILITY!

All the details available at...
ARXIV:2201.02191



What about symmetric tensors?

A *symmetric tensor* $T = (t_{j_1, \dots, j_d}) \in (\mathbb{K}^n)^{\otimes d} := \mathbb{K}^n \otimes \dots \otimes \mathbb{K}^n$ is a tensor such that for every permutation $\sigma \in \Sigma_d$ and all (j_1, \dots, j_d) ,

$$t_{j_1, \dots, j_d} = t_{j_{\sigma(1)}, \dots, j_{\sigma(d)}}.$$

We denote by $\text{Sym}^d(\mathbb{K}^n) \subseteq (\mathbb{K}^n)^{\otimes d}$ the subspace of symmetric tensors.

Bounds for $\mathcal{A}(\text{Sym}^d(\mathbb{C}^n))$ & $\mathcal{A}(\text{Sym}^d(\mathbb{R}^n))$

COMPLEX THEOREM (K. & T.-C., '22 +)

For any $d \geq 3$ and $n \geq 2$,

$$\max \left\{ \left(\frac{d+n-1}{d} \right)^{-\frac{1}{2}}, \frac{1}{n^{\frac{d-1}{2}}} \right\} \leq \mathcal{A}(\text{Sym}^d(\mathbb{C}^n)) \leq 10\sqrt{n \ln d} \left(\frac{d+n-1}{d} \right)^{-\frac{1}{2}}.$$

In particular,

$$\frac{1}{n^{\frac{d-1}{2}}} \leq \mathcal{A}(\text{Sym}^d(\mathbb{C}^n)) \leq 6 \left(1 + \frac{1}{\ln d} \right) \sqrt{d! \ln d} \frac{1}{n^{\frac{d-1}{2}}},$$

and, for $d \geq n^2/4$,

$$\sqrt{\frac{(n-1)!}{d^{n-1}}} \left(1 - \frac{n^2}{4d} \right) \leq \mathcal{A}(\text{Sym}^d(\mathbb{C}^n)) \leq 10\sqrt{\frac{n! \ln d}{d^{n-1}}}.$$

REAL THEOREM (K. & T.-C., '22 +)

For any $d \geq 3$ and $n \geq 2$,

$$\max \left\{ \frac{1}{2^{\frac{d}{2}}} \left(\frac{d+n-1}{d} \right)^{-\frac{1}{2}}, \frac{1}{n^{\frac{d-1}{2}}} \right\} \leq \mathcal{A}(\text{Sym}^d(\mathbb{R}^n)) \leq \frac{6\sqrt{n \ln d}}{2^{\frac{d}{2}}} \left(\frac{d+\frac{n}{2}-1}{d} \right)^{-\frac{1}{2}}.$$

In particular,

$$\frac{1}{n^{\frac{d-1}{2}}} \leq \mathcal{A}(\text{Sym}^d(\mathbb{R}^n)) \leq 6 \left(1 + \frac{1}{\ln d} \right) \sqrt{d! \ln d} \frac{1}{n^{\frac{d-1}{2}}},$$

and, for $d \geq n^2/4$,

$$\sqrt{\frac{(n-1)!}{2^d d^{n-1}}} \left(1 - \frac{n^2}{4d} \right) \leq \mathcal{A}(\text{Sym}^d(\mathbb{R}^n)) \leq 9\sqrt{\frac{(\frac{n}{2})! \ln d}{2^d d^{\frac{n}{2}-1}}} \left(1 + \frac{1}{4d} \right).$$

COROLLARY (K. & T.-C., '22 +)

For a fixed $d \geq 3$, there is a constant $C_d > 0$ (depending on d) such that

$$\mathcal{A}((\mathbb{K}^n)^{\otimes d}) \leq \mathcal{A}(\text{Sym}^d(\mathbb{R}^n)), \mathcal{A}(\text{Sym}^d(\mathbb{C}^n)) \leq C_d \mathcal{A}((\mathbb{K}^n)^{\otimes d}).$$

COROLLARY (K. & T.-C., '22 +)

For a fixed $n \geq 3$,

$$\lim_{d \rightarrow \infty} \frac{\mathcal{A}(\text{Sym}^d(\mathbb{R}^n))}{\mathcal{A}(\text{Sym}^d(\mathbb{C}^n))} = \lim_{d \rightarrow \infty} \frac{\mathcal{A}((\mathbb{K}^n)^{\otimes d})}{\mathcal{A}(\text{Sym}^d(\mathbb{R}^n))} = 0.$$

Also for partially symmetric tensors!