# Condition-based Low-Degree Approximation of Real Polynomial Systems

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**Slides at:**
**https://tonellicueto.xyz/pdf/JMM2023_slides.pdf**

# WARNING

There will be one result on Eigenvalue Computations, the talk will focus on real polynomial systems

# PROBLEM

Given a real polynomial system

$$f_1(x) = \cdots = f_q(x) = 0$$

in $n$ variables,

1) what can we say about the conditioning of solving?
2) what does the condition say about the zero set?

# Conditioning

- The condition number depends on the metric
  — how we measure errors —
- The condition number depends on the encoding
  — how we write the problem —

# Framework à la Renegar

— conic framework

$\mathcal{I}$ input space

$\Sigma \subseteq \mathcal{I}$ ill-posed inputs

$$C(i) := \frac{1}{d(i, \Sigma)}$$

# WEYL

# Setting

— projective

# Set Up

$$\mathcal{H}_d := \prod_{i=1}^{n} \mathbb{R}[X_0, \ldots, X_n]_d :$$

$\mathbb{P}^n$ Real Projective Space

$$\Sigma = \{ f \in \mathcal{H}_d \mid f \text{ has a singular zero} \}$$

# Weyl Norm

$$\|g\|_W := \sqrt{\sum_{i=1}^{n} \sum_{|\alpha|=d_i} \binom{d_i}{\alpha}^{-1} |g_{i,\alpha}|^2}$$

where

$$g_i = \sum g_{i,\alpha} X^\alpha \qquad \left( X^\alpha := X_0^{\alpha_0} \cdots X_n^{\alpha_n} \right)$$

$$\binom{d_i}{\alpha} = \frac{d_i!}{\alpha_0! \cdots \alpha_n!}$$

# Weyl Condition Number

(Cucker, Krick, Malajovich, Wschebor)

$$\kappa_W(g) := \frac{\|g\|_W}{\mathrm{dist}_W(g, \Sigma)}$$

$$= \sup_{x \in S^n} \frac{\|g\|_W}{\sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{1/2}\|^{-2}}}$$

where $\Delta = \mathrm{diag}(d_i)$ & $D_x g : T_x S^n \to \mathbb{R}^n$

# Main Theorem (Simple form)

$$\# Z(\mathfrak{g}, \mathbb{P}^n) \leq D^{n/2} \, \mathrm{poly}(n, \log D)^n \log^n(2 \kappa_w(\mathfrak{g}))$$

where $D = \max d_i$

# MAIN THEOREM (COMPLEX FORM)

There is a cover

$\{B(x, 1/c\sqrt{D})\}_{x\in\mathcal{G}}$ of size $\mathcal{O}(D^{n/2})$

of $\mathbb{S}^n$ s.t. for all $x \in \mathcal{G}$ & $\mathfrak{f} \in \mathcal{H}_d$,

there is $\Phi_{x,\mathfrak{f}} \in \mathbb{R}[X_0, \dots, X_n]^n$ of degree

$$\leq \mathrm{poly}(n, \log D) \log(2\, \mathcal{K}_w(\mathfrak{f}))$$

s.t. $\#\mathcal{Z}(\mathfrak{f}, T_x\mathbb{S}^n \cap B(x, 1/c\sqrt{D})) \leq \#\mathcal{Z}(\Phi_{x,\mathfrak{f}}, T_x\mathbb{S}^n)$.

Moreover, for all $\zeta \in \mathcal{Z}(\mathfrak{f}, T_x\mathbb{S}^n \cap B(x, 1/c\sqrt{D}))$ there is $z \in \mathcal{Z}(\Phi_{x,\mathfrak{f}}, T_x\mathbb{S}^n)$ converging quadratically under Newton's method.

# Corollary 1 of Main Result

- If $\#Z(\mathcal{F}) \geq D^n$, then
$$\kappa_w(\mathcal{F}) \geq 2^{D^{n/2}/\mathrm{poly}(n,\log D)^n}$$
- Real systems with many zeros are badly-conditioned

# COROLALLY 2 OF MAIN THEOREM

Let $f \in \mathcal{H}_d$ be random such that for all $x \in \mathbb{S}^n$, the $f_i(x)$ are independent, subgaussian and with anti-concentration. Then:

$$\left(\mathbb{E}\,\#Z(f,\mathbb{P}^n)^\ell\right)^{1/\ell} \leq D^{n/2}\,\mathrm{poly}(n,\log D)^n\,\ell^n$$

# COROLALLY 2 OF MAIN THEOREM

Let $f \in \mathcal{H}_d$. Under very general random hypotheses,

$$\# Z(f, \mathbb{P}^n)^{1/n}$$

is subexponential with constant

$$D^{1/2} \operatorname{poly}(n, \log D)$$

I.e.

$$\mathbb{P}\left(\# Z(f, \mathbb{P}^n)^{1/n} \geq t\right) \leq e^{1 - \frac{t}{D^{1/2} \operatorname{poly}(n, \log D)}}$$

# Corolally 2 of Main Theorem

A KSS random polynomial system

$$f \in \mathcal{H}_{(D,\dots,D)}$$

has its number of real zeros concentrated around

$$D^{n/2} = \mathbb{E}\,\#Z(f,\mathbb{P}^n)$$

# Comparisson with Lerario et. al.

## Our Approach

- Taylor Approx.
- Many local approx.
- Control the moments
- Robust
- Exploits analycity!

## Lerario's Approach

- S. Harmonic Approx.
- A global approx.
- Control only probability
- Only KSS

(Lerario, Diatta; '22) (Breiding, Keneshlou, Lerario; '22)

# 1-Norm Setting

# Set Up

$$\mathcal{P}_A := \{ f \in \mathbb{R}[X_0, \dots, X_n] \mid 0 \in \operatorname{supp} f \subseteq A \}^n$$

$$I^n := [-1, 1]^n$$

$$\Sigma = \{ f \in \mathcal{P}_A \mid f \text{ has a singular zero in } I^n \}$$

# Weyl Norm

$$\|g\|_1 := \max_i \sum_{\alpha \in A} |g_{i,\alpha}|$$

where

$$g_i = \sum g_{i,\alpha} X^\alpha \quad \left( X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n} \right)$$

# Cubic Condition Number

(TC, Tsigaridas)

$$C_1(f) := \sup_{x \in I^n} \frac{\|f\|_1}{\max\{\|f(x)\|_\infty, \|D_x f^{-1} \Delta\|_{\infty,\infty}^{-1}\}}$$

$$\sim \frac{\|f\|_1}{\mathrm{dist}_1(f, \Sigma)}$$

where $\Delta = \mathrm{diag}(d_i)$ & $D_x f : \mathbb{R}^n \to \mathbb{R}^n$

# Main Theorem (Simple form)

$$\# Z(g, I^n) \leq \operatorname{poly}(n)^n \log^{2n}(2D) \log(2C_1(g))$$

where $D = \max d_i$

# MAIN THEOREM (COMPLEX FORM)

There is a partition into boxes $\{B\}_{B\in\mathcal{B}}$ of size $\mathcal{O}(\log^n(2D))$ of $I^n$ s.t. for all $B\in\mathcal{B}$ & $g\in\mathcal{H}_d$, there is $\Phi_{B,g}\in\mathbb{R}[X_0,\dots,X_n]^n$ of degree

$$\leq \operatorname{poly}(n)\log D\log(2C_1(g))$$

s.t. $\#Z(g,B)\leq\#Z(\Phi_{B,g})$.

Moreover, for all $\zeta\in Z(g,B)$, there is $z\in Z(\Phi_{B,g})$, converging quadratically under Newton's method.

# Corollary 1 of Main Result

- If $\#Z(\mathfrak{F}, I^n) \geq D^K$, then

$$C_1(\mathfrak{F}) \geq 2^{D^K / \mathrm{poly}(n, \log D)^n}$$

- Real systems with many zeros are badly-conditioned

# COROLLARY 2 OF MAIN THEOREM

Let $f \in \mathcal{P}_A$ be random such that for all $x \in I^n$, the $f_i(x)$ are independent, subgaussian and with anti-concentration. Then:

$$\left(\mathbb{E}\, \# Z(f, I^n)^{\ell}\right)^{1/\ell} \leq \mathrm{poly}(n)^n \log^{2n}(2D)\, \ell^n$$

# Corolally 2 of Main Theorem

Let $f \in \mathcal{P}_A$. Under very general random hypotheses,

$$\#Z(f, I^n)^{1/n}$$

is subexponential with constant

$$\log^2(2D)\,\mathrm{poly}(n)$$

I.e.

$$\mathbb{P}\left(\#Z(f, \mathbb{P}^n)^{1/n} \geq t\right) \leq e^{1 - \frac{t}{\log^2(2D)\,\mathrm{poly}(n)}}$$

# The Details

# Main Tricks

- Smale's $\alpha$-theory is stable under analytic truncation
- Well-conditioned polynomials converge fast around zeros — as fast as a geometric series

# Smale's $\alpha$-Theory

$$\alpha(g,x) := \beta(g,x)\,\gamma(g,x)$$

$$\beta(g,x) := \| D_x g^{-1} g(x) \| = \| x - N_g(x) \|$$

$$\gamma(g,x) := \max\left\{ 1, \sup_{k\geq 2} \left\| D_x g^{-1} \frac{1}{k!} D_x^k g \right\| \right.$$

**Smales $\alpha$-Theorem** There is absolute $\alpha_* > 0$, s.t. if $\alpha(g,x) < \alpha_*$, then the Newton method at $x$ converges quadratically. More concretely, $\mathrm{dist}(N_g^k(x), Z(g)) = O(2^{-2^k})$

# Truncation Theorem (One version)

Let $g \in \mathbb{R}[X_1,\dots,X_n]^n$, $\delta \in \mathbb{N}$, $x \in \mathbb{B}^n$ &

$$T(g,x;\delta) := \sup_{k \geq \delta+1} \left\| D_x g^{-1} \frac{2^k}{k!} D_0^k g \right\|$$

Consider

$$g_{|\delta}(X) := \sum_{k=0}^{\delta} \frac{1}{k!} D_0^k g(X,\dots,X)$$

Then for

$$\delta - \log(\delta+2) \geq \log T(g,x;\delta),$$

$$\alpha(g_{|\delta}, x) \leq \frac{2\,\alpha(g,x) + 2^{1-\delta}\gamma(g,x)\,T(g,x;\delta)}{\left(1 - 2^{-\delta}(\delta+2)\,T(g,x;\delta)\right)^2}$$

I.e.

approximate zero of $g$ à la Smale

$$\Downarrow$$

approximate zero of $g|_\delta$ à la Smale

+ reverse & more ineqs.

# Moroz's Lemma

W-Lemma: For $g \in \mathbb{R}[X_0, X_1]_d$ & $(x_0, x_1) \in S^1$,

$$\left| \frac{1}{k!} \frac{d^k}{dt^k}\Big|_{t=0} g\big((x_0, x_1) + t(x_1, -x_0)\big) \right| \leq \sqrt{\binom{d}{k}}\, \|g\|_W$$

1-Lemma: For $g \in \mathbb{R}[X]_{\leq d}$, $a \in I$ and $\rho > 0$, if

either $2|a| < 1 - \rho$ or $\rho < 1/2d$

then $$\left| \frac{1}{k!} \frac{d^k}{dt^k}\Big|_{t=0} g(a + \rho t) \right| \leq 1/2^k\, \|g\|_1$$

# Multivariate Moroz's 1-Lemma

Let $f \in \mathbb{R}[X_1, \ldots, X_n]^n_{\leq D}$, $a \in I^n$ & $\rho \in (0,1]^n$

Consider

$$f_{a,\rho} := \left( f_i(a + PX) / \|f_i\|_1 \right)$$

where $P = \operatorname{diag}(\rho)$.

If for all $i$,

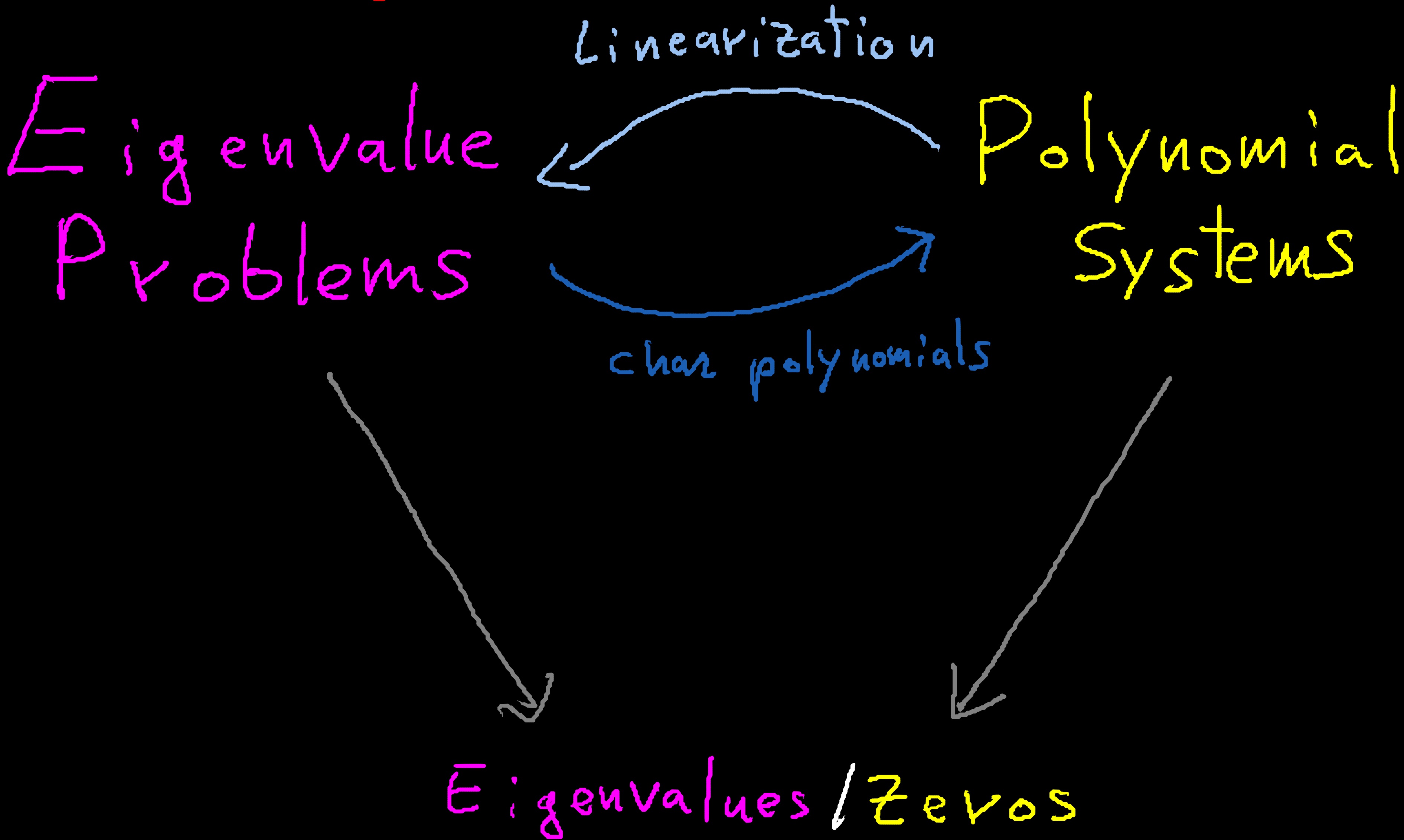
either $2|a_i| \leq 1 - \rho_i$ or $\rho_i \leq 1/2D$

then for all $e$,

$$\left\| \frac{2^e}{e!} D_0^e f_{a,\rho} \right\|_{\infty,\infty} \leq \binom{D+n-1}{n-1} \leq \left( 1 + \frac{\ln(n-1)}{n-1} D \right)^{n-1}$$

# A Theorem about Eigenvalues

# Triangle of competition



# Numerical Analist's Rule

NEVER USE CHARACTERISTIC POLYNOMIALS TO COMPUTE EIGENVALUES

# A formalization for Hermitian Matrices

**THM.** Let $A \in \mathrm{Herm}_d$. Then

$$\mathcal{K}_w(\chi_A^h) \geq 2^{\sqrt{d}/\mathrm{polylog}(d)}$$

$$\& \quad C_1(\chi_A) \geq 2^{d/\mathrm{polylog}(d)}$$

I.e. characteristic polynomials of Hermitian matrices are badly conditioned.

Future Work

- Can we make all this into fast algorithms?
  - avoid condition estimation
- Generalize it beyond zero-dim systems
  - volume & Betti numbers

# Thank You for your attention!

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