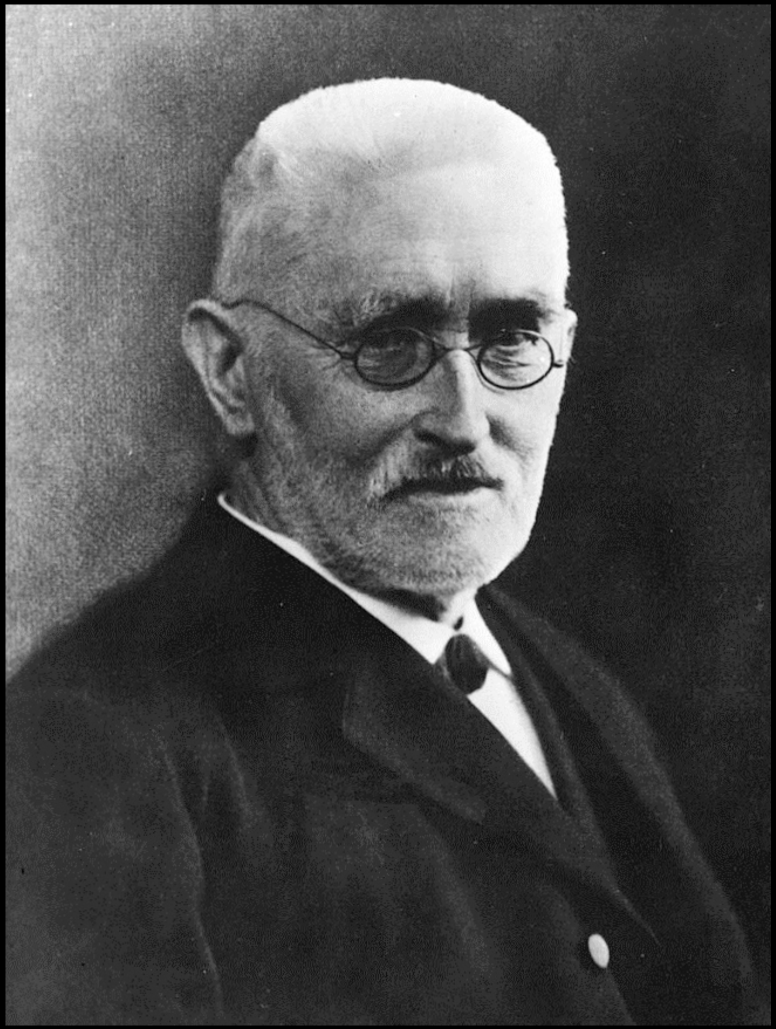
# FACT OF THE DAY



Today, October 6, Richard Dedekind would have turned 191 years old

8TH IBERIAN MATHEMATICAL MEETING
Sevilla, Spain

# Why does the DESCARTES Solver work?


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Photo while working on this project

# Real Root Isolation I: The Problem

INPUT:

$f \in \mathbb{Z}[X]$

OUTPUT:

Intervals $J_1, \ldots, J_k$ s.t.

0) $J_i = (a_i, b_i)$ with $a_i, b_i \in \mathbb{Q}$

1) $Z(f) \cap \mathbb{R} \subseteq \bigcup_{i=1}^{k} J_i$

2) $\forall i, \# Z(f) \cap J_i = 1$

INPUT SIZE PARAMETERS:

$d$ : degree of $f$

$\tau$ : bit-size of coefficients of $f$

MEASURE OF RUN-TIME

Bit complexity

# Real Root Isolation II: The State of the Art

| | |
|---|---|
| Sturm Solver | $\tilde{O}_B(d^4\tau^2)$ |
| Descartes Solver | $\tilde{O}_B(d^4\tau^2)$ |
| ANewDsc (Sagraloff & Mehlhorn; 2016) | $\tilde{O}_B(d^3+d^2\tau)$ |
| Pan's Algorithm (Pan; 2002) | $\tilde{O}_B(d^2\tau)$ |

Q: Can we beat the champion?

# Real Root Isolation III: What do we wish?

$$\tilde{O}_B(d\tau)$$

We wish to find real roots almost as fast as we read the polynomial!

# Descartes Solver I: Rule of Signs

$V(g) :=$ # sign variations of $g_0, g_1, \ldots$

Portrait by Frans Hal
Source: WikiMedia Commons

THM (Descartes' rule of signs)

$$\#Z(g, \mathbb{R}_>) \leq V(g)$$

Moreover,

$$V(g) \leq 1 \Rightarrow \text{Equality}$$

COR

$$\#Z(g,(a,b)) \leq V(g,(a,b)) := V\left((X+1)^d \, g\left(\frac{bX+a}{X+1}\right)\right)$$

$(0,\infty) \rightarrow (a,b)$
bijection

# Descartes Solver II: The Descartes' Oracle

1) Overcounting: $\#Z(g, J) \leq V(g, J)$

2) Exactness I: $V(g, J) \leq 1 \Rightarrow$ Equality

3) Exactness II:

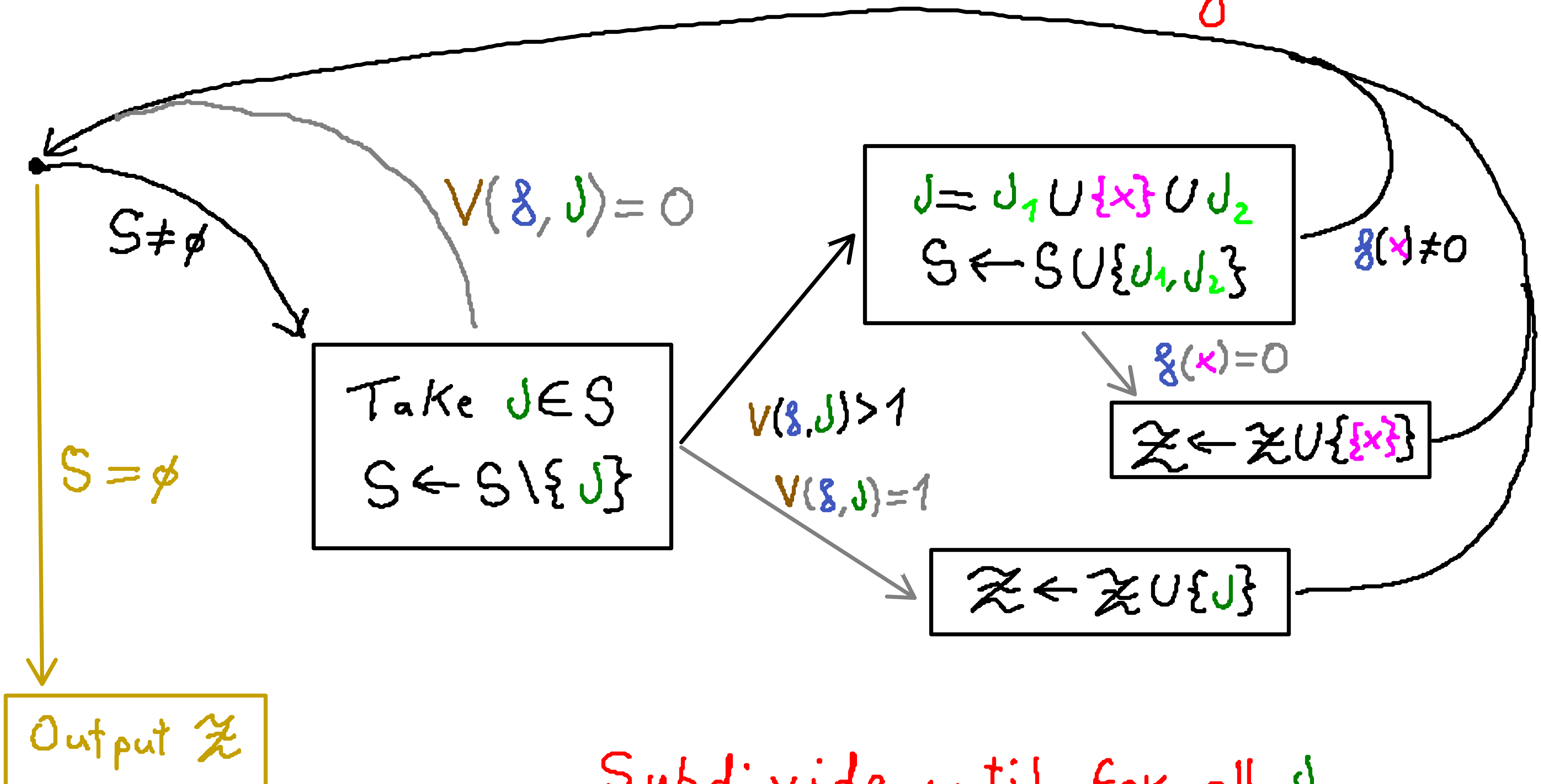
$$\#Z(g, D(m(J), cw(J))) \leq K \Rightarrow V(g, J) \leq K$$

Obreshkoff's Thm: Descartes sees the complex roots around!

4) Subadditivity:

$$\dot{\bigcup} J_i \subseteq J \Rightarrow \sum V(g, J_i) \leq V(g, J)$$

# Descartes Solver II: The Algorithm




Subdivide until for all $J$,

$V(g, J) \leq 1$!

# Real Root Isolation IV:

## Are we being pessimistic?

Worst-case complexity:

$$\max\{\mathrm{cost}(\text{SOLVER}, f) \mid \text{bit-size}(f) \leq \tau, \deg(f) \leq d\}$$

Pessimistic in practice

DESCARTES SOLVER seems to behave faster in practice!

¿Can we explain this?

# Real Root Isolation V: Beyond pessimism

## Worst-case complexity:

$$\max\{\mathrm{cost}(\mathrm{SOLVER}, g) \mid \text{bit-size}(g) \leq \tau, \deg(g) \leq d\}$$

(Goldstine & von Neumann, 1951)
(Demmel, 1988) (Smale; 1985, 1997)

(Roughgarden, 2021)

## Probabilistic complexity

$$\mathbb{E}\{\mathrm{cost}(\mathrm{SOLVER}, F)^{\ell} \mid \text{bit-size}(F) \leq \tau, \deg(F) \leq d\}$$

What's a 'good' random model for $F$?

Many choices of randomness 😱

# Beyond pessimism I: Uniform Random Bit Polynomials & a Simple Main Theorem

$$f = \sum_{k=0}^{d} f_k X^k$$

s.t. $f_k \sim \mathcal{U}([-2^\tau, 2^\tau] \cap \mathbb{Z})$ independent

SIMPLE MAIN THM (Ergür, T-C, Tsigaridas)

$$\mathbb{E}\, \mathrm{cost}(\text{DESCARTES}, f) = \widetilde{\mathcal{O}}_B(d^2 + d\tau)$$

On average, DESCARTES is almost optimal!

# Beyond pessimism II: Random Bit Polynomials

$$f = \sum_{k=0}^{d} f_k X^k \in \mathbb{Z}[X]$$

s.t. $f_k$ independent

bit-size of $f$:

$$\tau(f) := \min\{\tau \mid \forall k, \mathbb{P}(|f_k| \leq 2^\tau) = 1\}$$

weight of $f$:

No middle indexes!

$$w(f) := \max\{\mathbb{P}(f_k = c) \mid c \in \mathbb{R}, k \in \{0, 1, d-1, d\}\}$$

uniformity of $f$: $u(f) := \ln\left(w(f)(1 + 2^{\tau(f)+1})\right)$

# Beyond pessimism III:

## Main Theorem

Main Thm (Ergür, T-C, Tsigaridas)

$$\mathbb{E}\,\mathrm{cost}(\text{Descartes}, f) = \widetilde{\mathcal{O}}_B\left(d^2 + d\tau\right)\left(1 + u(f)\right)^4$$

Note: $f$ uniform $\Rightarrow$ $u(f) = 0$

Claim: For many cases, $u(f) = \mathcal{O}(1)$

If $\tau = \Omega(d)$, almost like reading!

On average, Descartes is almost optimal!

# Beyond pessimism IV: Examples of Random Bit Polynomials I

- Support control $\{0,1,d-1,d\} \subseteq A$

$$f = \sum_{k \in A} f_k X^k \quad \text{with } f_k \sim \mathcal{U}([-2^\tau, 2^\tau] \cap \mathbb{Z})$$

... then $u(f) = 0$

- Sign control $\sigma \in \{-1,+1\}^{\{0,\dots,d\}}$

$$f = \sum_{k=1}^{d} f_k X^k \quad \text{with } f_k \sim \mathcal{U}(\sigma_k([1, 2^\tau] \cap \mathbb{N}))$$

... then $u(f) \leq \ln 3$

# Beyond pessimism V:
# Examples of Random Bit Polynomials II

- Exact bitsize

$$F = \sum_{k=1}^{d} f_k X^k \quad \text{with} \quad f_k \sim \mathcal{U}(\{n \in \mathbb{Z} \mid \lfloor \log n \rfloor = \tau\})$$

... then $u(F) \leq \ln 3$

+ their combinations

Our random model is flexible!

# Beyond pessimism VI: Smoothed case included!

$F = \sum_{k=1}^{d} f_k X^k$ random bit polynomial

$g = \sum_{k=1}^{d} g_k X^k$ fix polynomial of entries of size $\tau$

$\sigma \in \mathbb{Z} \setminus \{0\}$

Then:

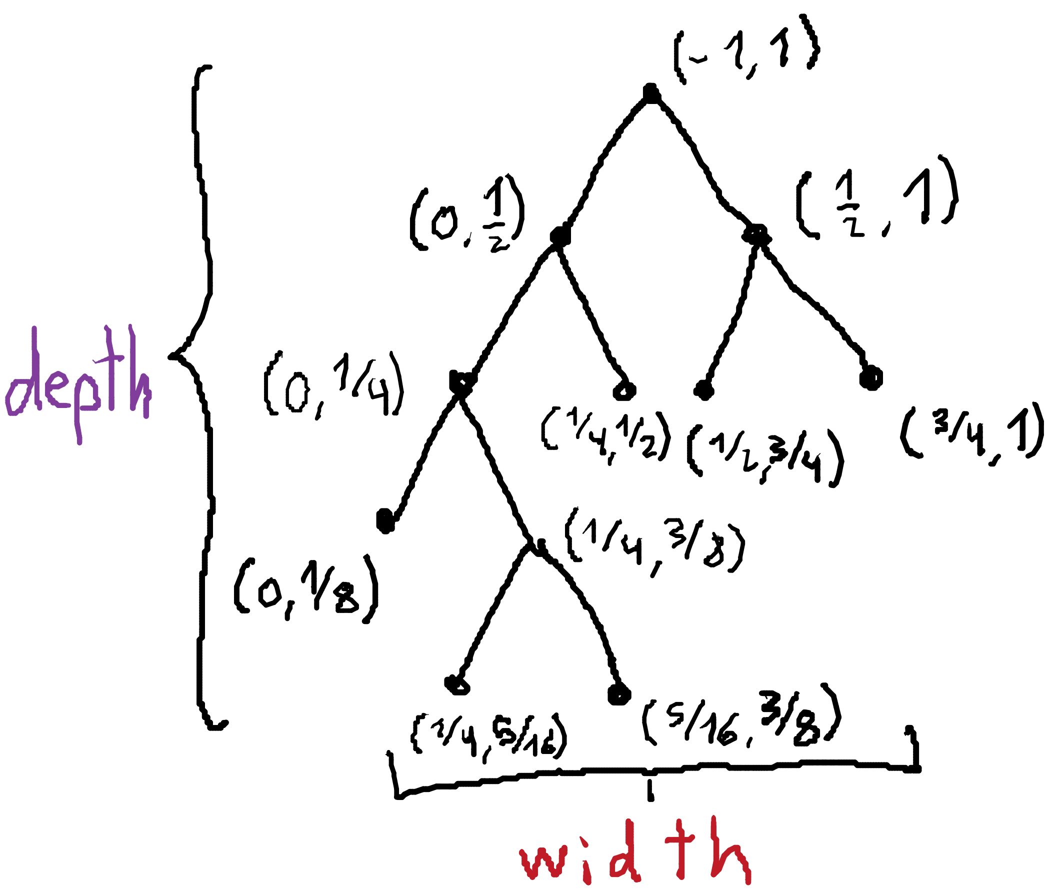
$F_\sigma = g + \sigma f$ random bit polynomial

& $u(F_\sigma) \leq 1 + u(F) + \max\{\tau - \tau(F), \tau(\sigma)\}$

LOTS OF DETAILS
WILL BE OMITTED

# How to bound the run-time of DESCARTES?

$\mathcal{T}(g) \leftarrow$ DESCARTES' Computation Tree




**Prop.**

$$\mathrm{cost}(\mathrm{DESCARTES}, g) \leq \mathcal{O}\left(d\tau\,\mathrm{width}\,\mathcal{T}(g)\,\mathrm{depth}\,\mathcal{T}(g) + d^2\,\mathrm{width}\,\mathcal{T}(g)\,\mathrm{depth}^2\,\mathcal{T}(g)\right)$$

I.e. size of $\mathcal{T}(g)$ bounds run-time of DESCARTES!

# The Ingredients of the Analysis I: Condition Number

$$C(g) := \max_{x\in[-1,1]} \frac{\sum_{k=0}^{d} |g_k|}{\max\{|g(x)|, |g'(x)|/d\}}$$

$C(g) = \infty \Leftrightarrow g$ has a singular root in $[-1,1]$

$$\frac{1}{C(g)} \sim \begin{cases} \text{How much I have to perturb} \\ \text{the coefficients of } g \text{ so that} \\ \tilde{g} \text{ has a singular root in } [-1,1] \end{cases}$$

# Bounding depth $T(g)$

**Prop.**

$$\text{depth } T(g) \leq 5 + \log d + \log C(g)$$

$$\text{sep. between complex roots of } g \text{ nearby } [-1,1] \geq \frac{1}{12\, d\, C(g)}$$

Here no → $\not\geq$

$$1/2^{\text{depth } T(g)}$$

Here it is essencial for roots being near [-1,1]!

# Bounding width $\tau(g)$

Prop

width $\tau(g)$

$\leq$ # complex roots of $g$ nearby $[-1, 1]$

Important for good bound of the RHS!

Secret tool to bound RHS:

Titchmarsh's thm.

# The Ingredients of the Analysis II: Cont. Probabilistic Toolbox

We can handle

$C(f)$ (& # complex roots of $f$ nearby $[-1,1]$)

for a wide class of random $f$

as long as continuous distribution

— using geometric functional analysis

... but we don't have a continuous dist.

# The Ingredients of the Analysis III: Ball's Smoothing Trick

$x \in \mathbb{Z}^N$ discrete random variable

$y \in \mathbb{R}^N$ s.t. $y_i \sim \mathcal{U}((-1/2, 1/2))$ i.i.d.

$\to$ $x + y$ continuous random var.

We can use our old cont. toolbox!

# SUMMING UP:

DESCARTES' SOLVER

IS ALMOST OPTIMAL ON AVERAGE!

... AND THAT'S WHY DESCARTES WORKS SO WELL

Muito Obrigado
pela Atenção!

¡Muchas Gracias
por su Atención!