arXiv:2306.06784

On the Number of Real Zeros of Random Sparse Polynomial Systems

or...

how few real zeros does a random fewnomial system have?

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Josué Tonelli-Cueto
(Johns Hopkins University)

Joint work with...

Both pictures taken in San Antonio, Texas, USA

Joint work with...

Alperen A. Ergür

Both pictures taken in San Antonio, Texas, USA

Joint work with...

Alperen A. Ergür

Maté L. Telek

Both pictures taken in San Antonio, Texas, USA

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Is there a bound of the form

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Propagandistic version

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poconomio (es)
gutxinomio (eu)

Khovanskii's Answer

& Refinements

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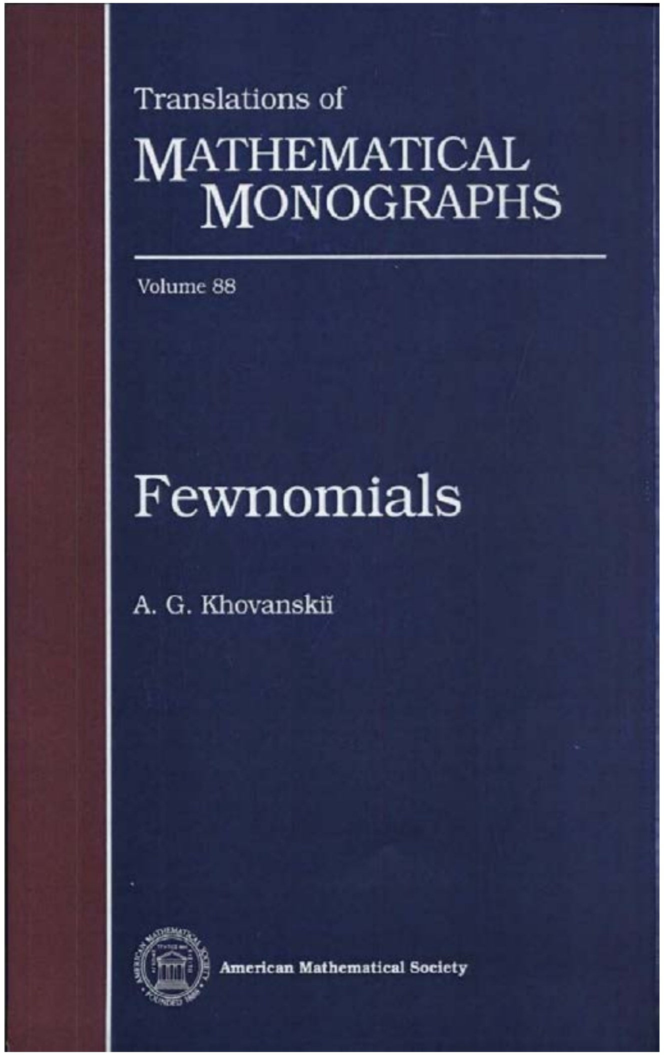

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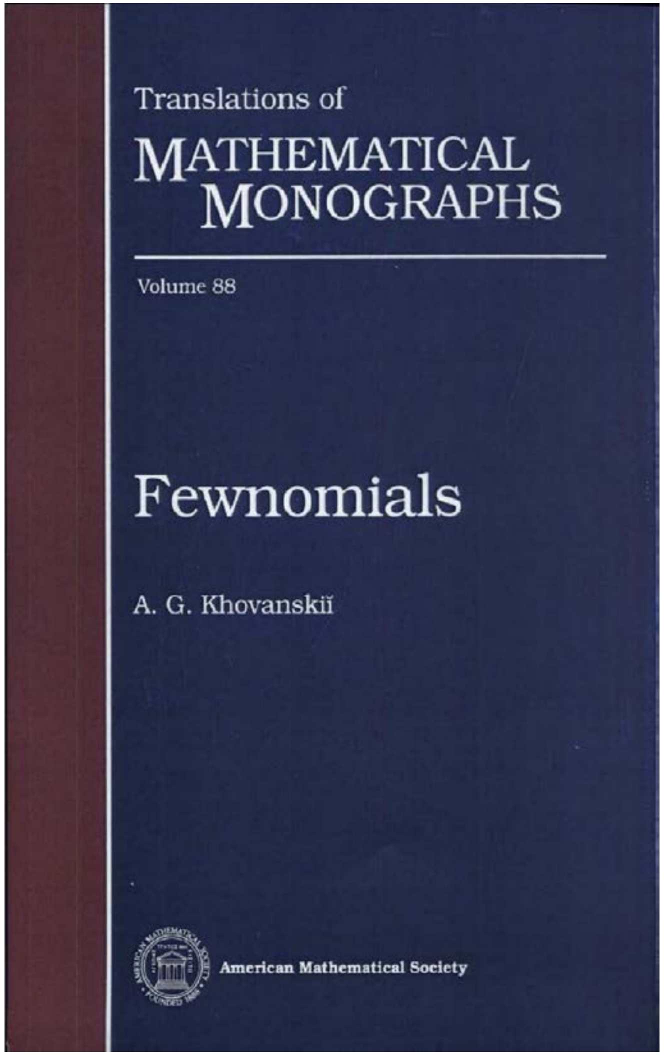

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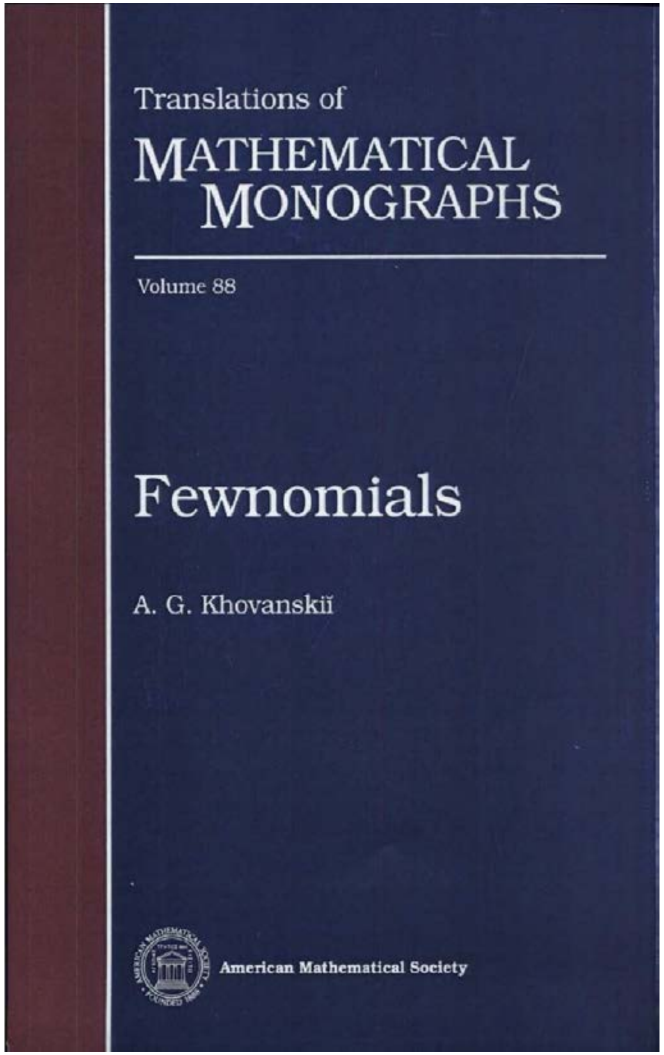

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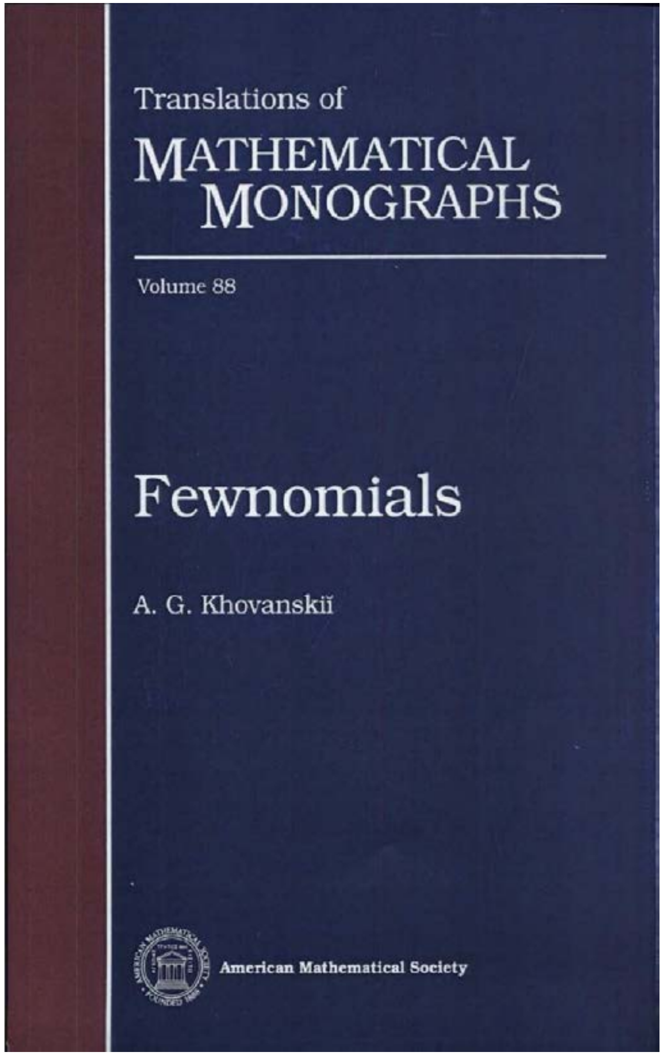

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(Khovanskii, 1991)

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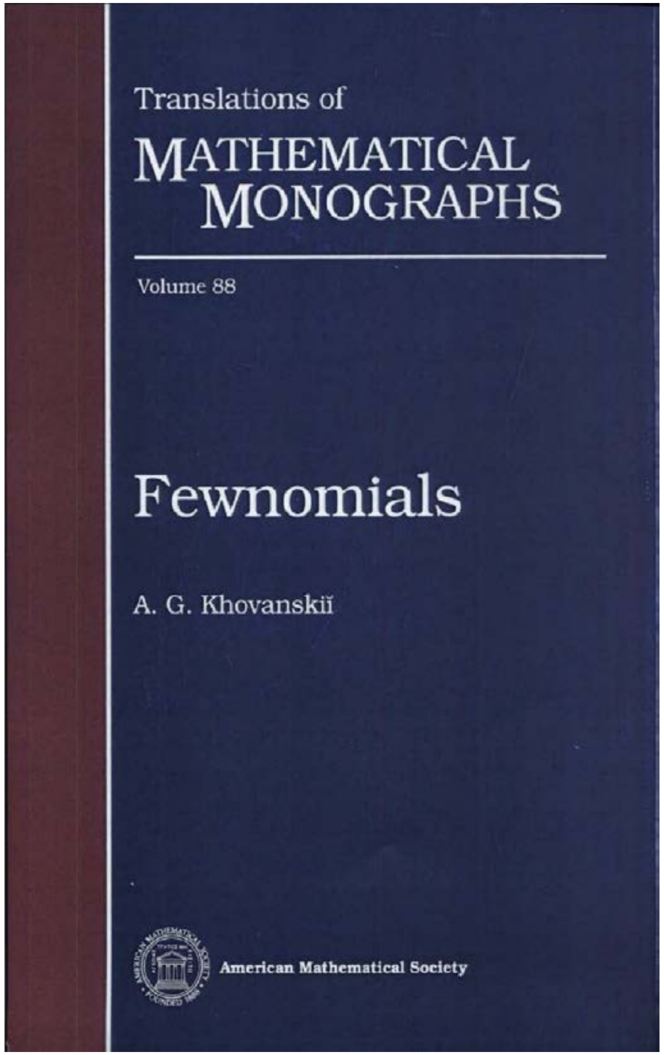

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(Bihan & Sottile, 2007)

$$\#Z_r(f, \mathbb{R}^n_+) \le \frac{e^2+3}{4} 2^{\binom{t-n-1}{2}} n^{t-n-1}$$

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STILL OPEN FOR $n=2$!!!

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Can we say

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spoiler: yes!

Reasons to care...

about random fewnomial systems

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Better understanding of typical behaviour

— not the worst case, but the average typical case

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Robust probabilistic results might be a pathway to deterministic ones

$$\sup_{g} \# Z_r(g, \mathbb{R}^n_+) = \sup_{e \geq 1} \left(\mathbb{E}_F \# Z_r(F, \mathbb{R}^n_+)^e \right)^{1/e}$$

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(Bürgisser, Ergür & TC; 2019) & (Bürgisser, ISSAC'23) are particular cases of our model

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presented at MEGA2019!

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THEN

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Minkowski sum

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IF $\begin{cases} \text{(VM1) for all } i \text{ and } \alpha \in A_i,\ \mathbb{V}(f_{i,\alpha}) \leq 1 \\ \text{(VM2) for all } i \text{ and } \alpha \in A_i \text{ vertex of } P_i := \operatorname{conv}(A_i), \\ \qquad \mathbb{V}(f_{i,\alpha}) = 1 \end{cases}$

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#vertices

Minkowski sum

Improves (Bürgisser, ISSAC'23)

Our Result: The Unmixed Case

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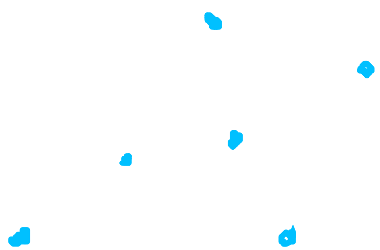

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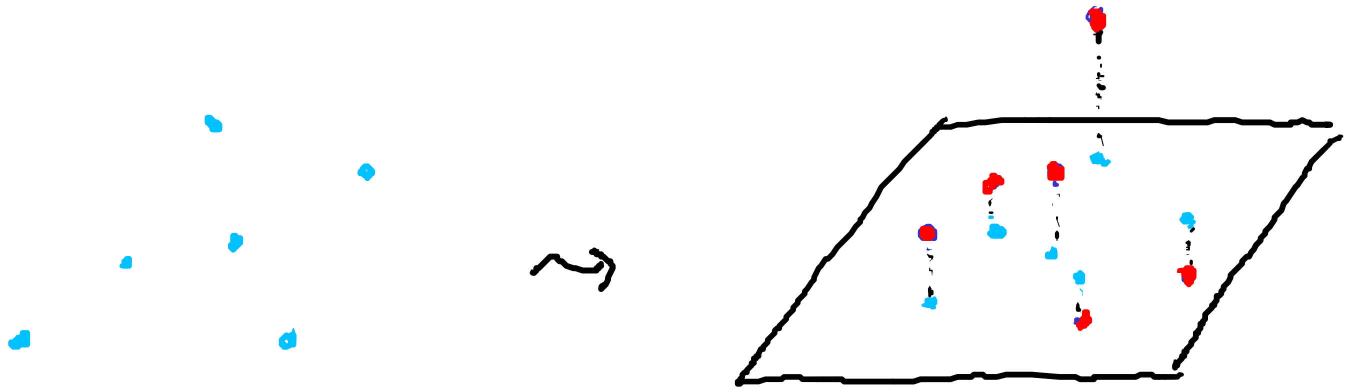

Regular (Mixed) Subdivisions?

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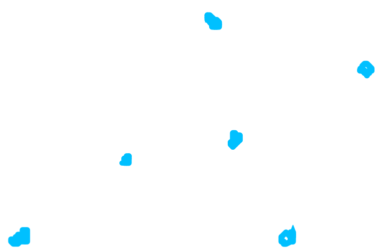
upper envelope of A wr π

$$\mathcal{L}(A,\pi) := \operatorname{conv}\left\{ \begin{pmatrix} \pi(\alpha)-s \\ \alpha \end{pmatrix} \,\middle|\, \alpha \in A,\ s \geq 0 \right\}$$



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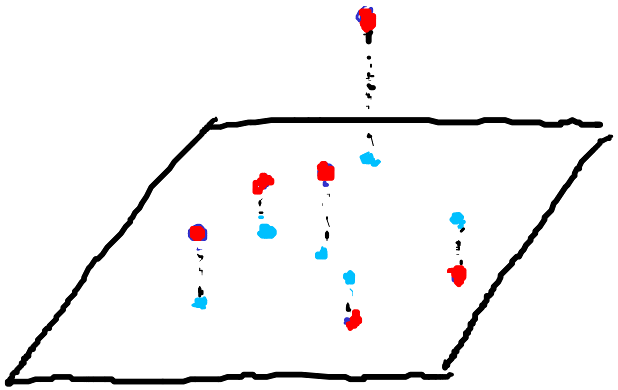

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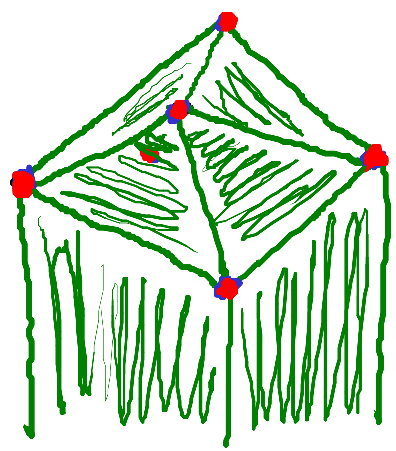

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⚠ #vertices of regular mixed subdivision induced by variance on supports

Three Tools

that made this possible

Tool I: Kac-Rice Formula

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under some technical assumptions...

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$$\mathbb{E}_{f} \, Z(f, \Omega)$$

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$$\mathbb{E}_{f} Z(f,\Omega) = \int_{\Omega} \mathbb{E}\left(|\det D_x f| \,\middle|\, f(x)=0 \right) \delta_{f(x)}(0)\, dx$$

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conditional expectation

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conditional expectation

density of $f(x)$

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$\det(AB^T)$

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Tool II: Cauchy-Binet Formula

(aka the Fundamental Lemma of Fewnomial Theory)

$A, B \in \mathbb{F}^{m \times n}$

$$\det(AB^T) = \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ \# J = m}} \det(A_J)\det(B_J)$$

where $A_J = (A_{i,j})_{\substack{i \in \{1,\ldots,m\} \\ j \in J}}$, $B_J := (B_{i,j})_{\substack{i \in \{1,\ldots,m\} \\ j \in J}}$

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Tool III: Normal Fan

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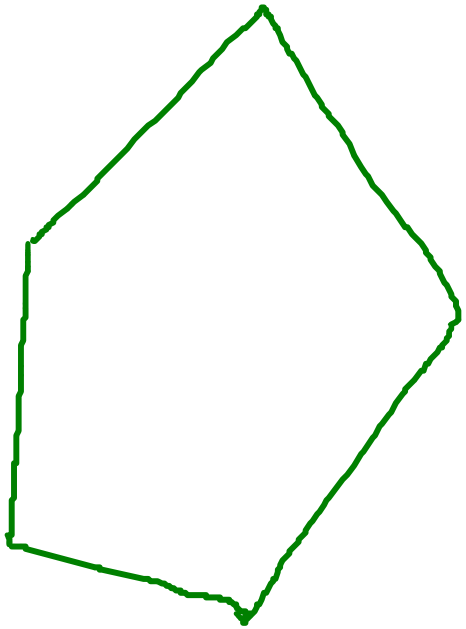

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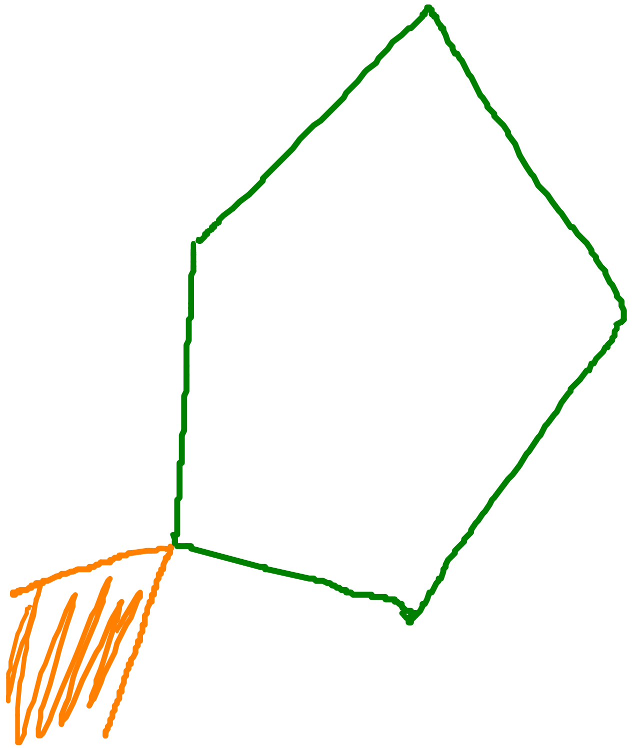

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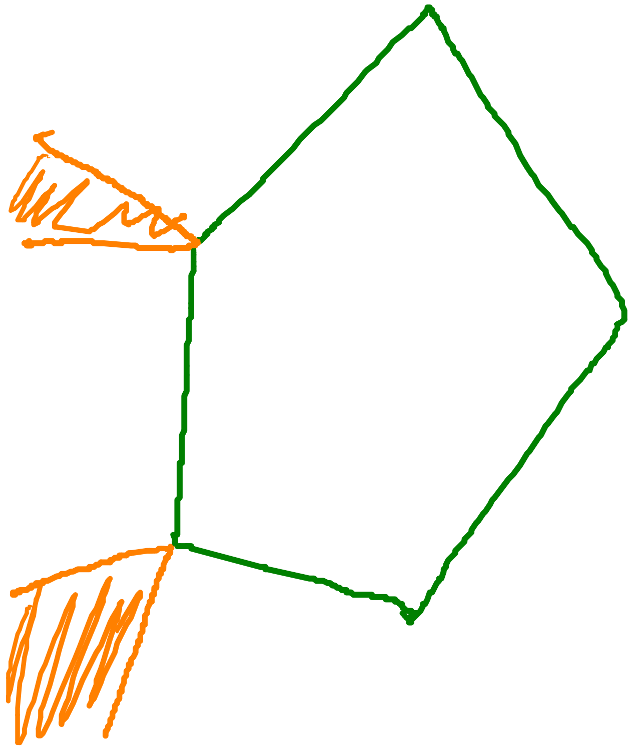

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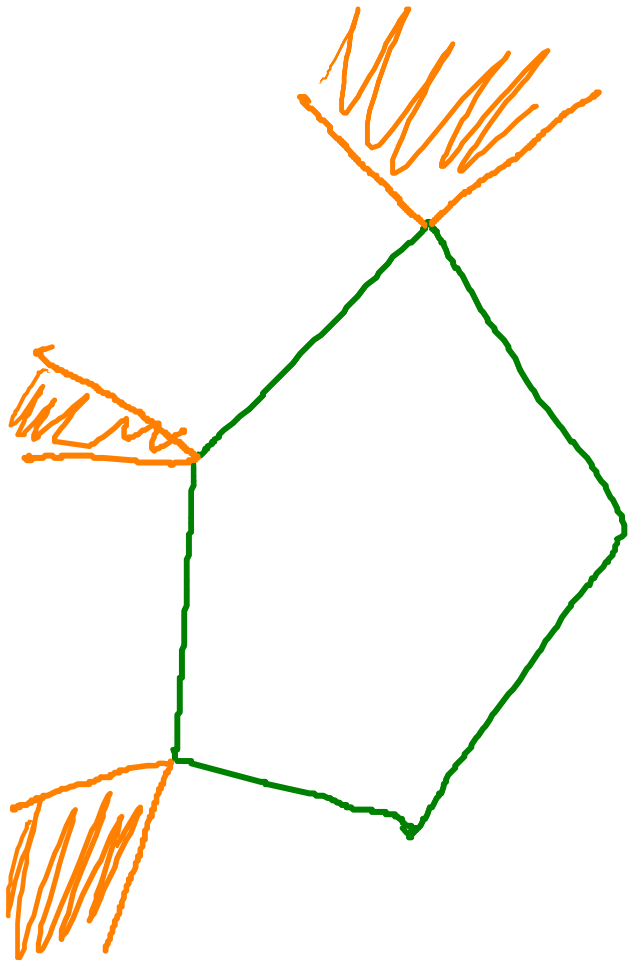

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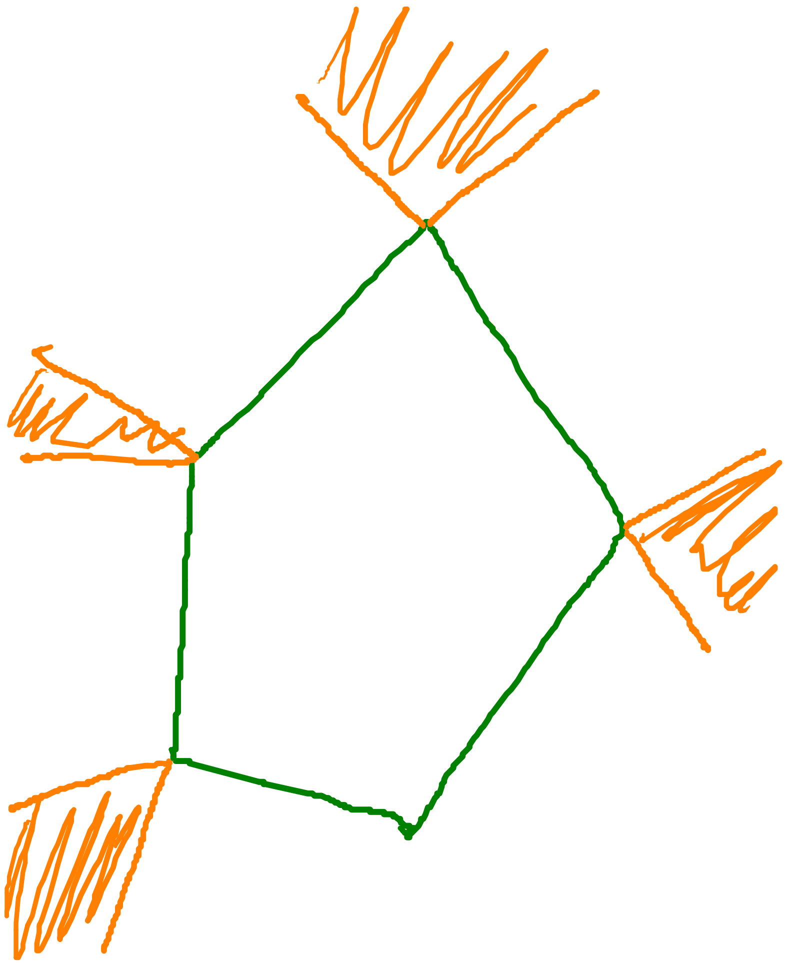

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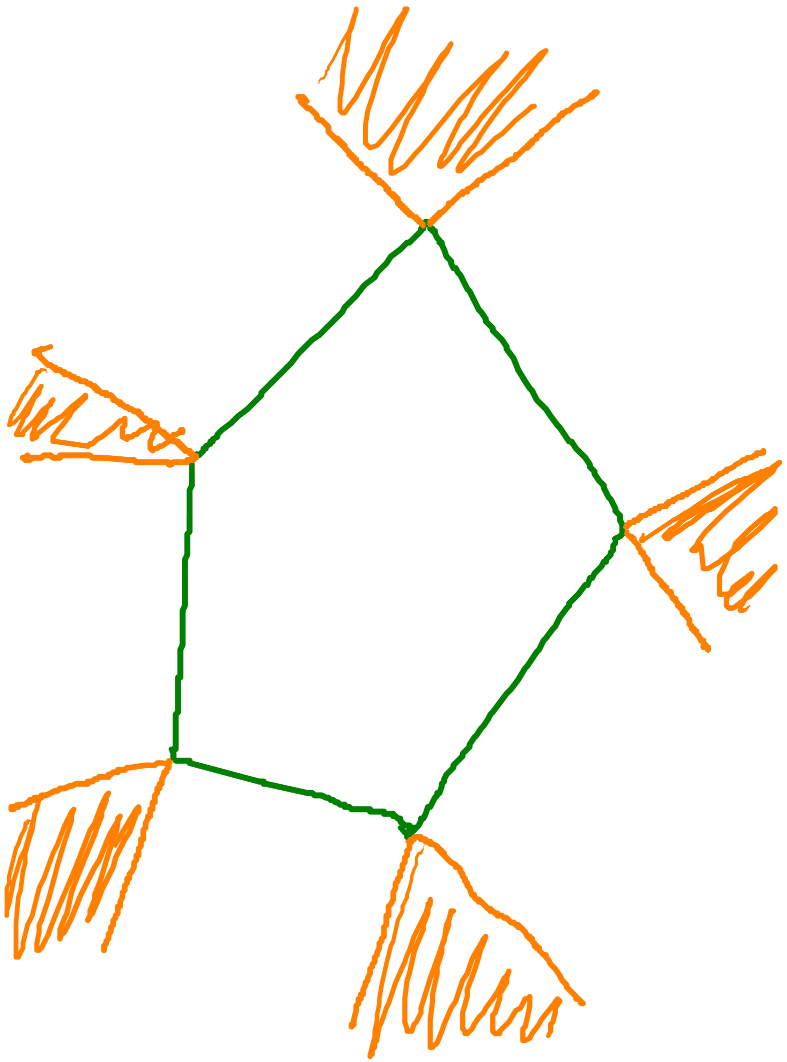

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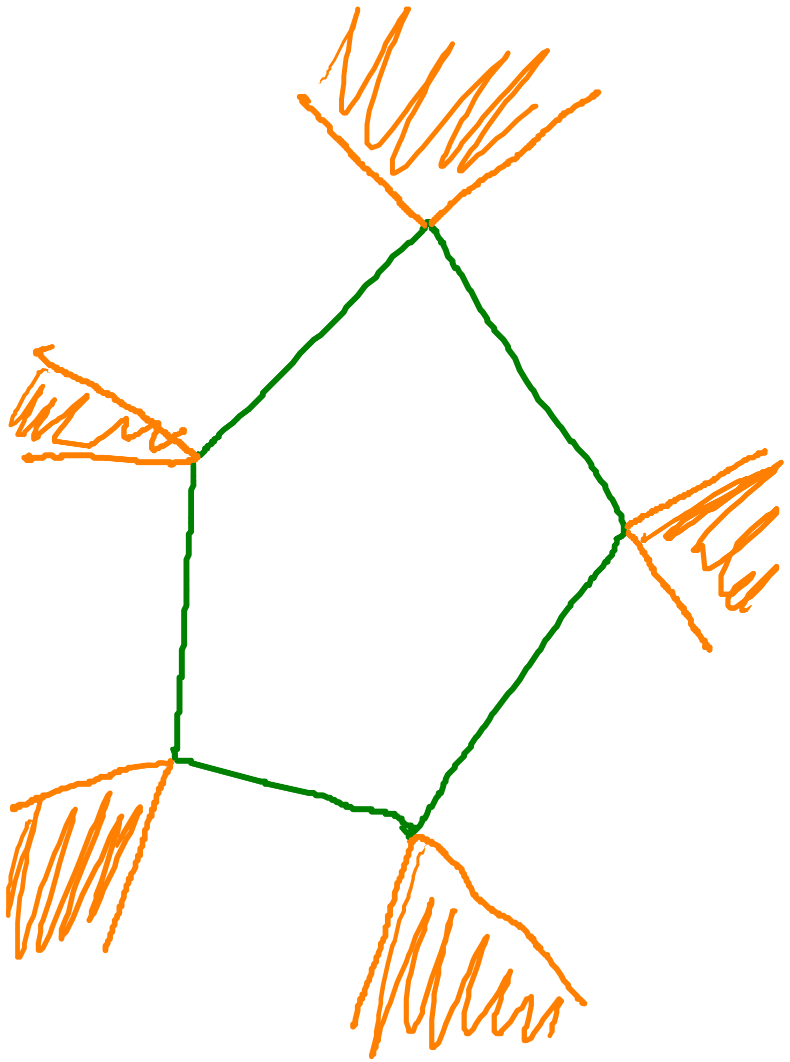

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In particular, what about

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$$\bigvee_{F} Z_{\mathbb{R}}(F, \mathbb{R}^n_+) \ ?$$

What about more general Gaussian fewnomial systems?

Vielen Dank

für

Aufmerksamkeit!