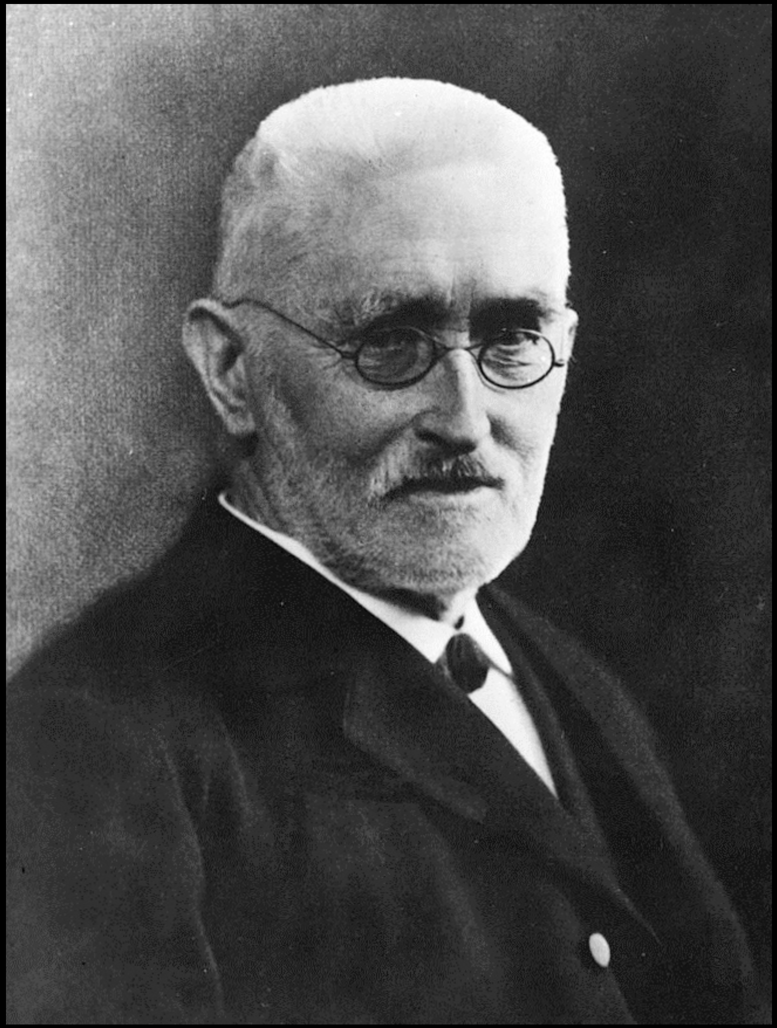
Fact of the Day



Today, October 6, Richard Dedekind would have turned 191 years old

Slides available at:
tonellicueto.xyz/pdf/8IMM_slides.pdf

8TH IBERIAN MATHEMATICAL MEETING
Sevilla, Spain

Why does the DESCARTES Solver work?

Alperen A. ERGÜR

Josué TONELLI-CUETO

Elias TSIGARIDAS

The University of Texas at San Antonio™

Inria

IMJ-PRG

Photo while working on this project

Real Root Isolation I: The Problem

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INPUT:

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$f \in \mathbb{Z}[X]$

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Intervals $J_1, \ldots, J_k$ s.t.

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Bit complexity

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STURM SOLVER $\qquad \tilde{O}_B(d^4\tau^2)$

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Real Root Isolation II: The State of the Art

STURM SOLVER $\tilde{O}_B(d^4\tau^2)$

DESCARTES SOLVER $\tilde{O}_B(d^4\tau^2)$

ANEWDSC $\tilde{O}_B(d^3+d^2\tau)$

(Sagraloff & Mehlhorn; 2016)

Real Root Isolation II: The State of the Art

Sturm Solver	$\tilde{O}_B(d^4\tau^2)$
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Pan's Algorithm (Pan; 2002)	$\tilde{O}_B(d^2\tau)$

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Pan's Algorithm $\tilde{O}_B(d^2\tau)$

(Pan; 2002)

Q: Can we beat the champion?

Real Root Isolation III: What do we wish?

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$$\tilde{O}_B(d\tau)$$

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What do we wish?

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We wish to find real roots almost as fast as we read the polynomial!

Descartes Solver I:
Rule of Signs

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$V(g) :=$ # sign variations of $g_0, g_1, \ldots$

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Portrait by Frans Hal
Source: WikiMedia Commons

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$(0,\infty) \to (a,b)$
bijection

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The Descartes' Oracle

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Obreshkoff's Thm: Descartes sees the complex roots around!

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4) Subadditivity:

$$\dot{\bigcup} J_i \subseteq J \Rightarrow \sum V(g, J_i) \leq V(g, J)$$

Descartes Solver III:

The Algorithm

-

Descartes Solver III:

The Algorithm

$S \neq \emptyset$

Descartes Solver III: The Algorithm

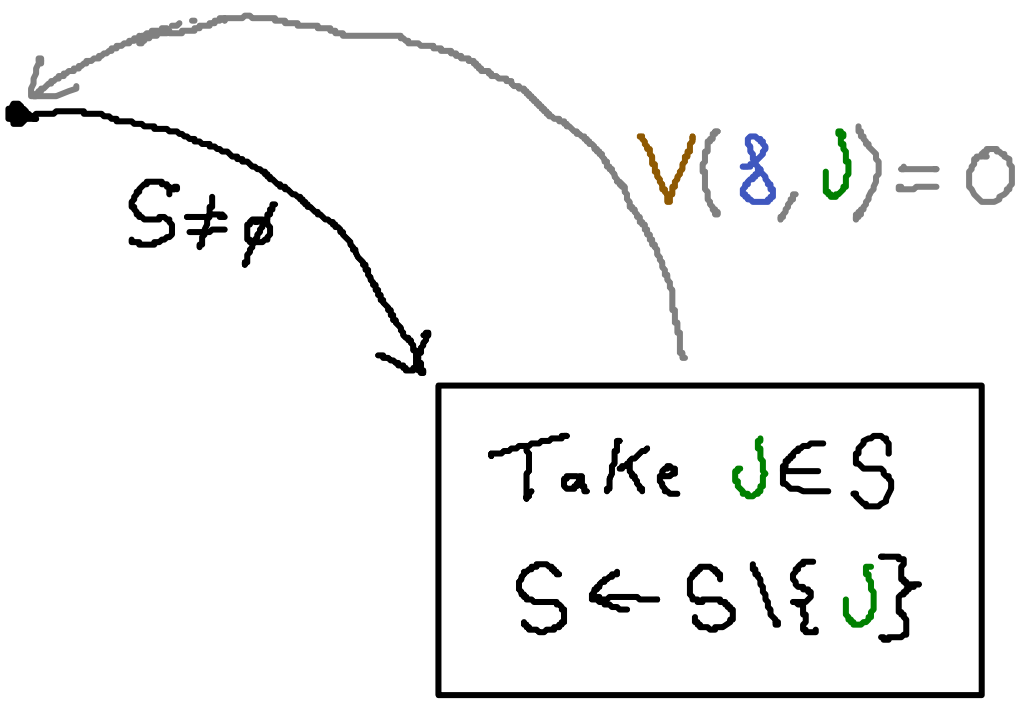
$S \neq \emptyset$

Take $J \in S$

$S \leftarrow S \setminus \{J\}$

DESCARTES SOLVER III:

The Algorithm



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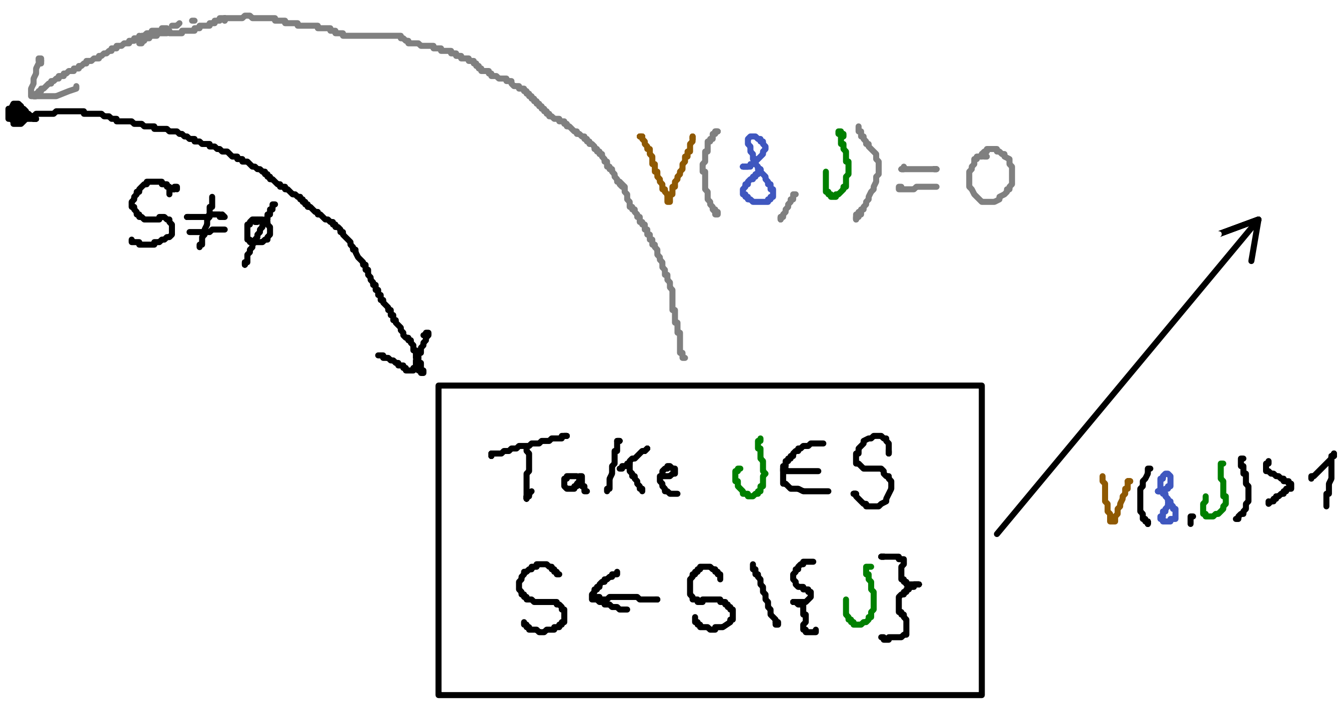

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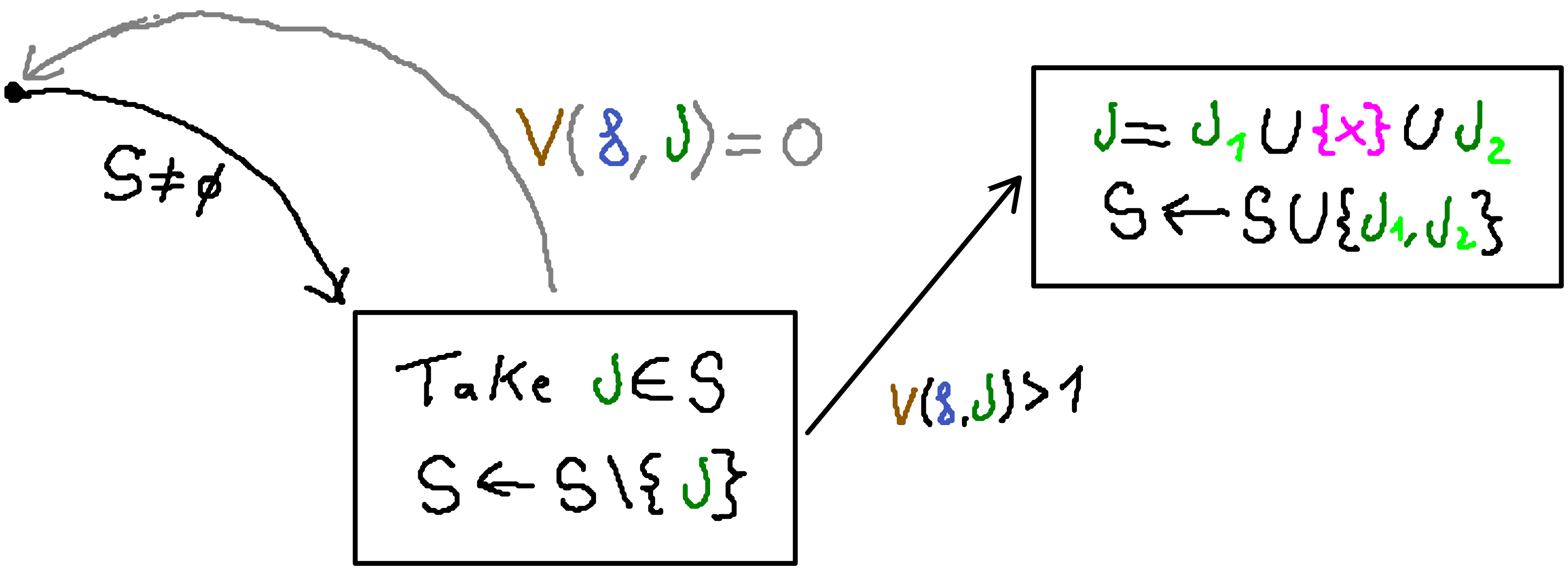

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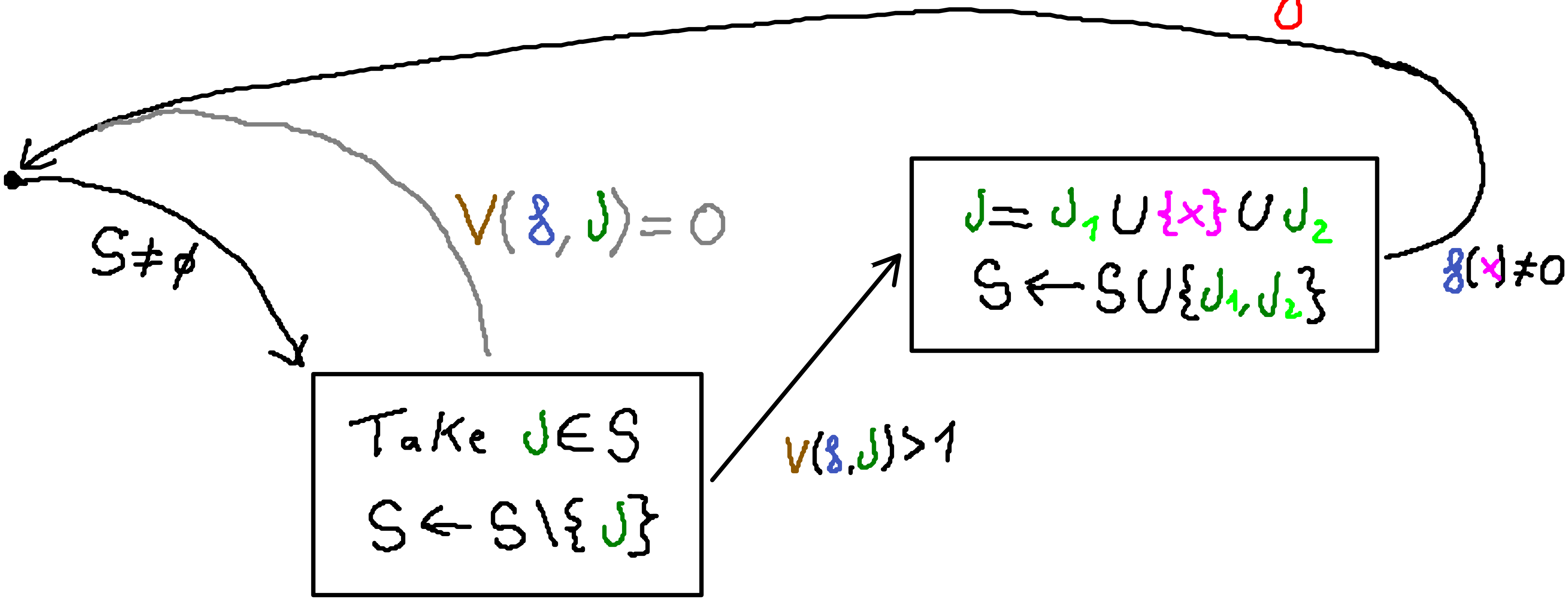

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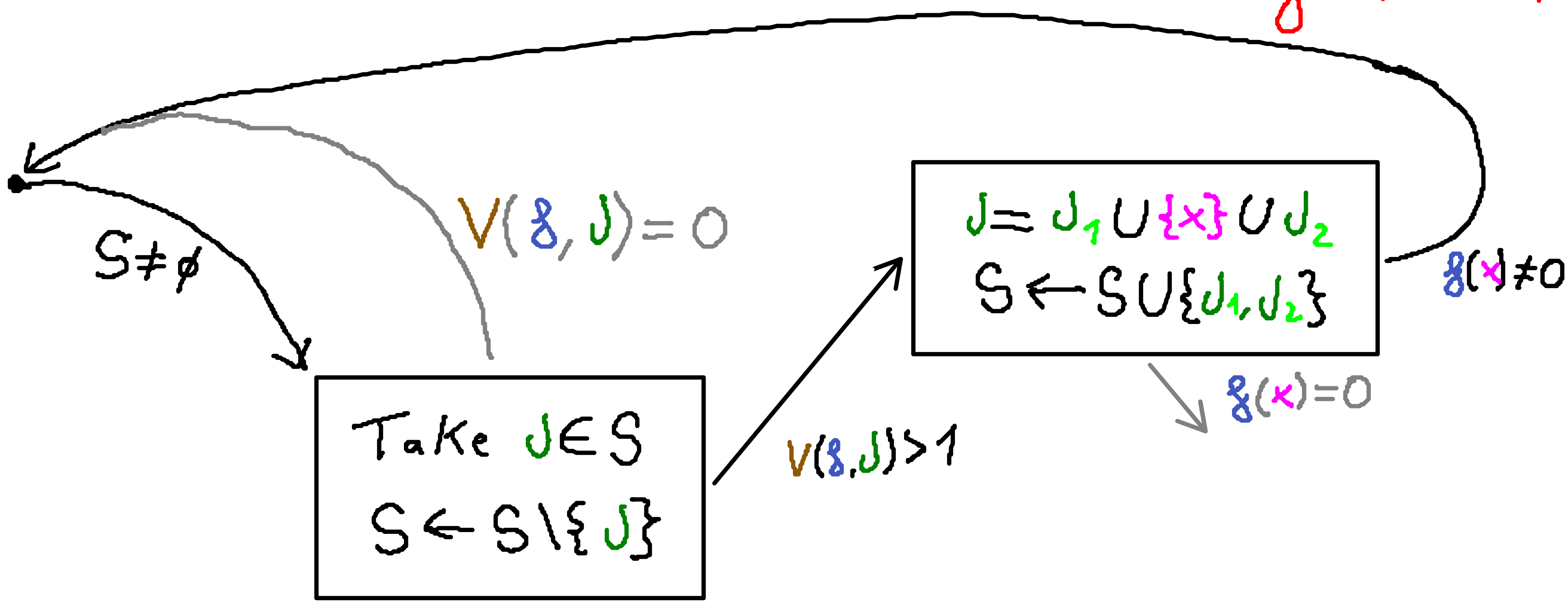

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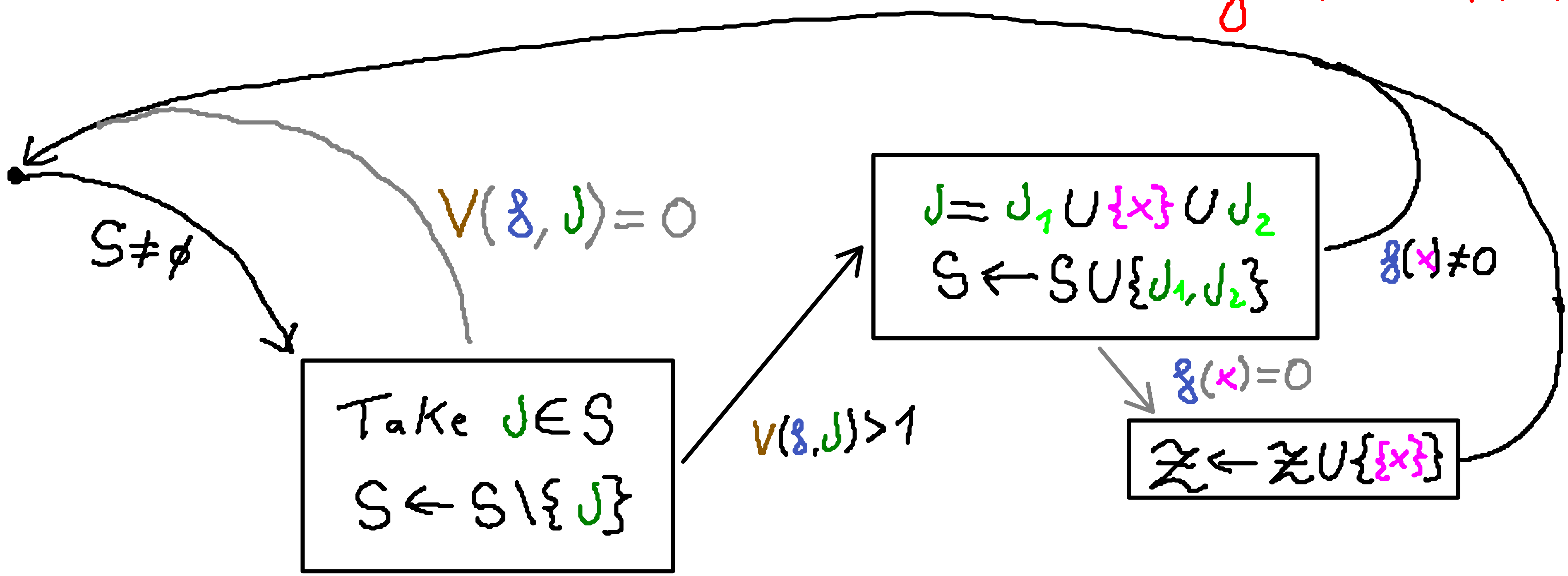

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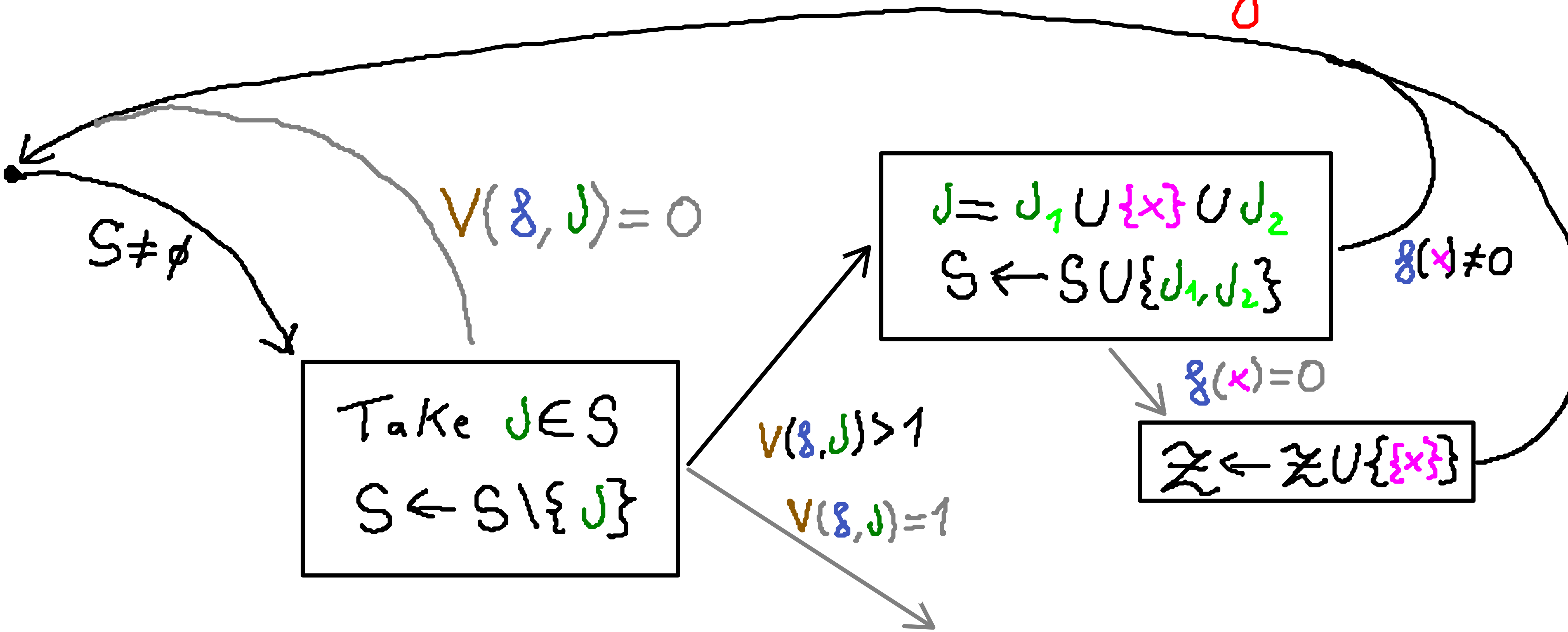

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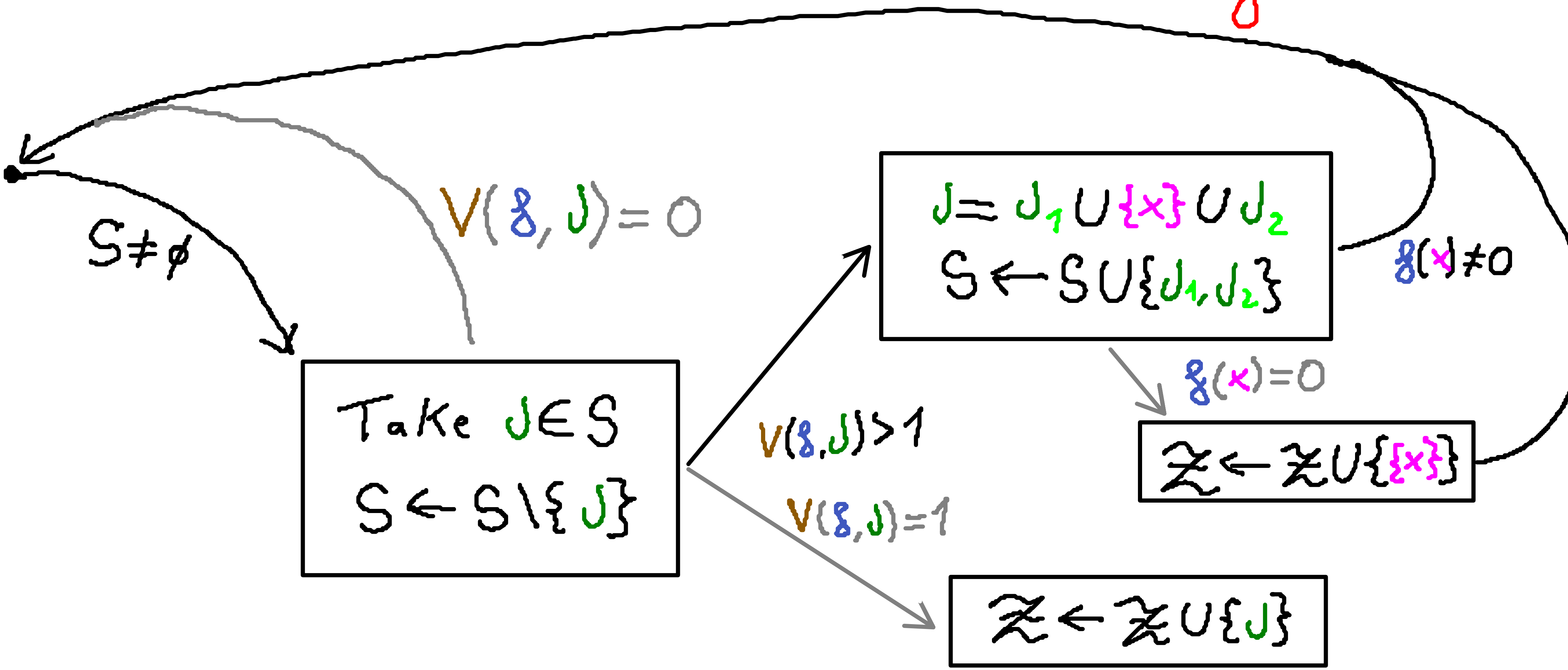

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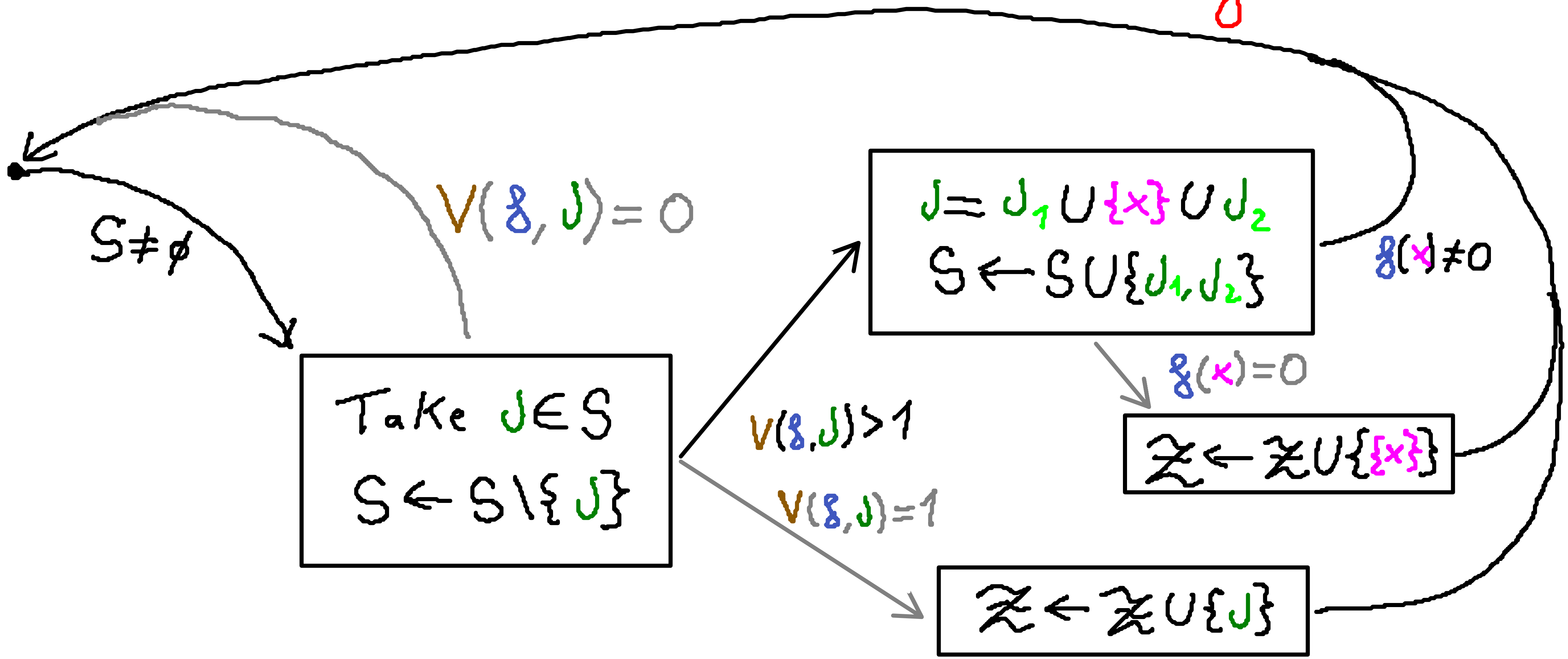

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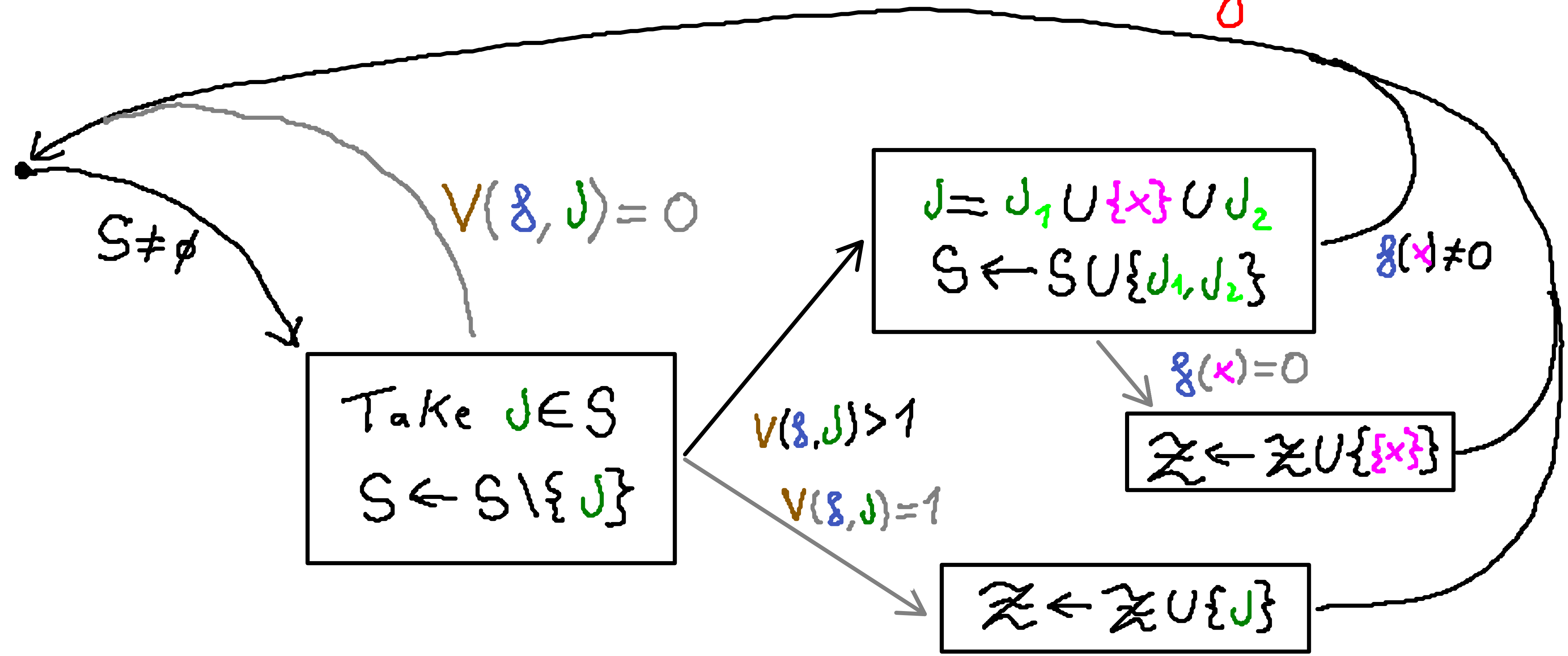

Subdivide until for all J,

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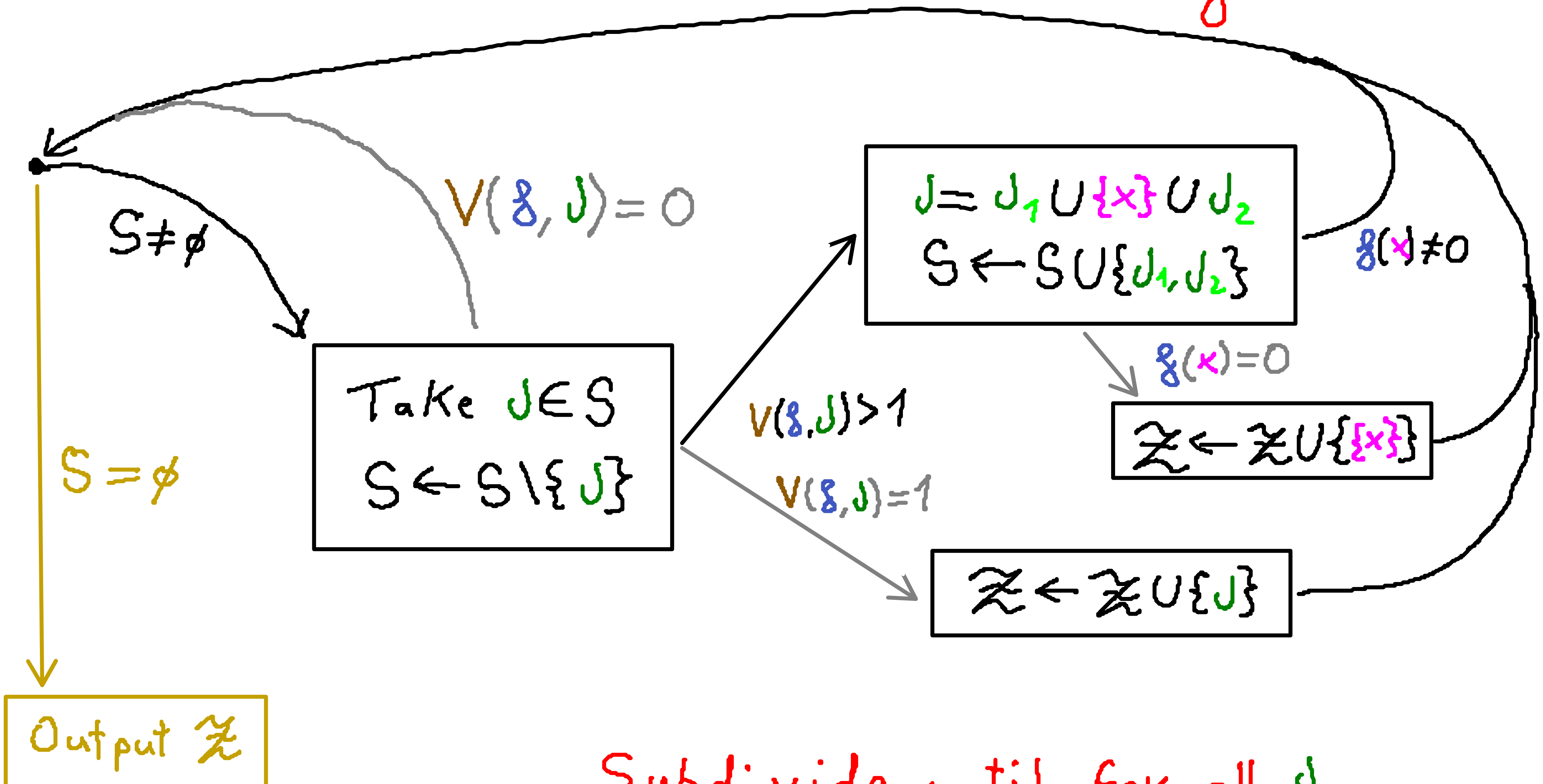

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Are we being pessimistic?

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↑

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DESCARTES SOLVER seems to behave faster in practice!

¿Can we explain this?

Real Root Isolation V:

Beyond pessimism

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Many choices of randomness 😱

Beyond pessimism I:
Uniform Random Bit Polynomials
& A Simple Main Theorem

Beyond pessimism I: Uniform Random Bit Polynomials & a Simple Main Theorem

$$F = \sum_{k=0}^{d} f_k X^k$$

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On average, DESCARTES is almost optimal!

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uniformity of f: $u(f):=\ln\left(w(f)\left(1+2^{\tau(f)+1}\right)\right)$

Beyond pessimism III:

Main Theorem

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Main Thm (Ergür, T-C, Tsigaridas)

$$\mathbb{E}\,\mathrm{cost}(\mathrm{Descartes}, f) = \tilde{\mathcal{O}}_B\left(d^2 + d\tau\right)\left(1 + u(f)\right)^4$$

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Claim: For many cases, $u(f) = \mathcal{O}(1)$

Beyond pessimism III: MAIN THEOREM

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IF $\tau = \Omega(d)$, almost like reading!

Beyond pessimism III: Main Theorem

Main Thm (Ergür, T-C, Tsigaridas)

$$\mathbb{E}\,\mathrm{cost}(\textsc{Descartes}, f) = \widetilde{\mathcal{O}}_B\big(d^2 + d\tau\big)\big(1 + u(f)\big)^4$$

Note: f uniform $\Rightarrow u(f) = 0$

Claim: For many cases, $u(f) = \mathcal{O}(1)$

IF $\tau = \Omega(d)$, almost like reading!

On average, Descartes is almost optimal!

Beyond pessimism IV:

Examples of Random Bit Polynomials I

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Beyond pessimism V:

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Our random model is flexible!

Lots of Details
will be Omited

How to bound
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$\tau(g) \leftarrow$ DESCARTES' Computation Tree

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$\bullet\ (-1,1)$

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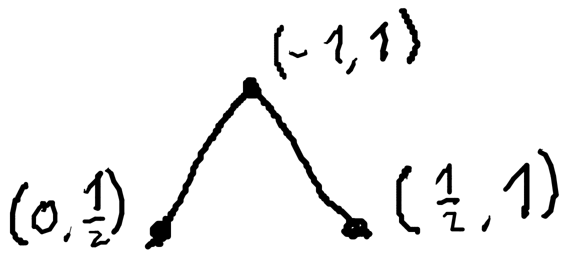
$\mathcal{T}(g) \leftarrow$ Descartes' Computation Tree



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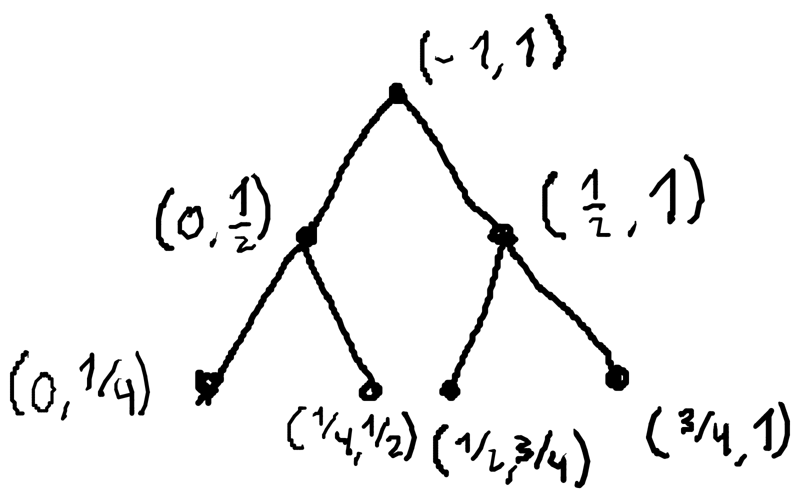

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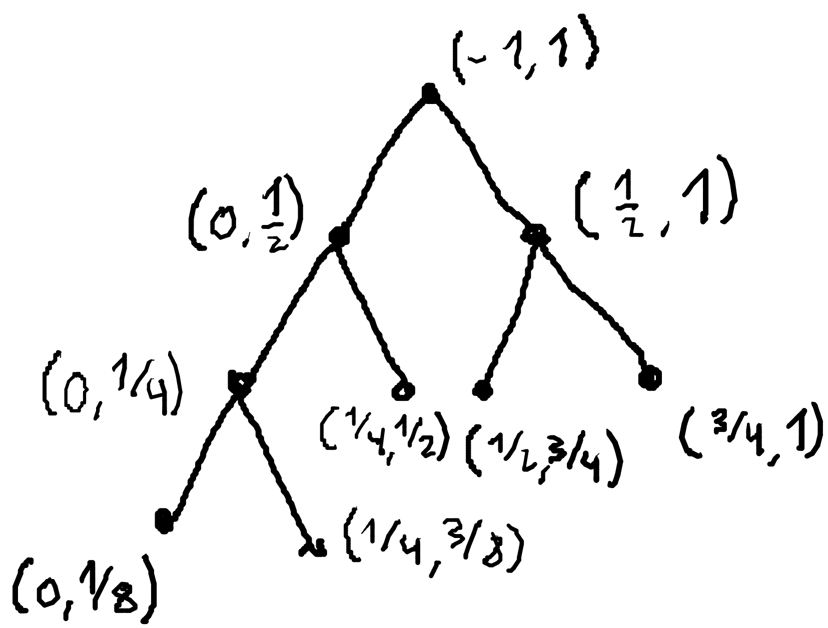

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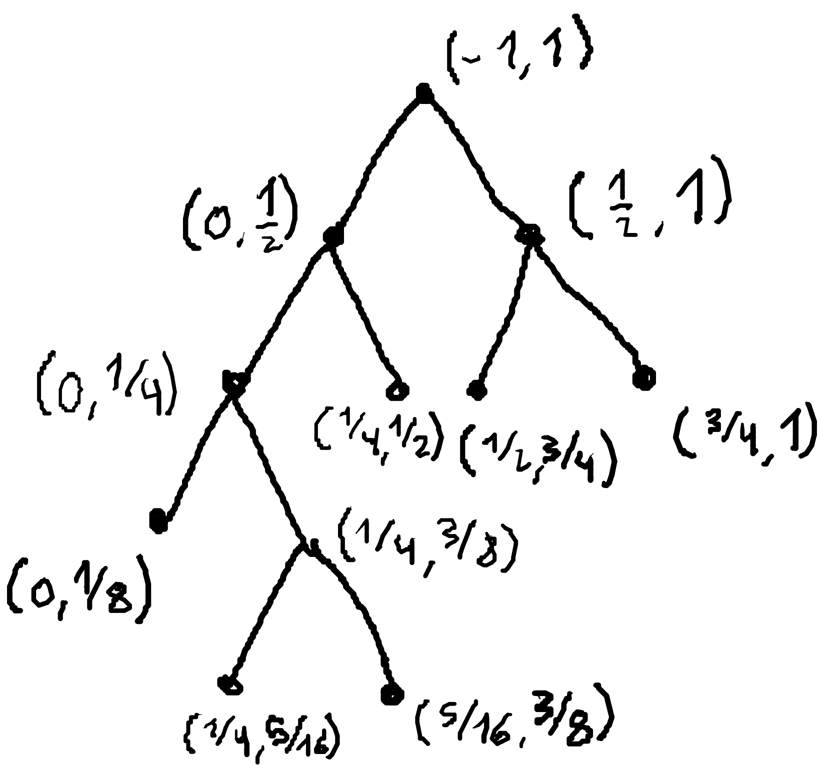

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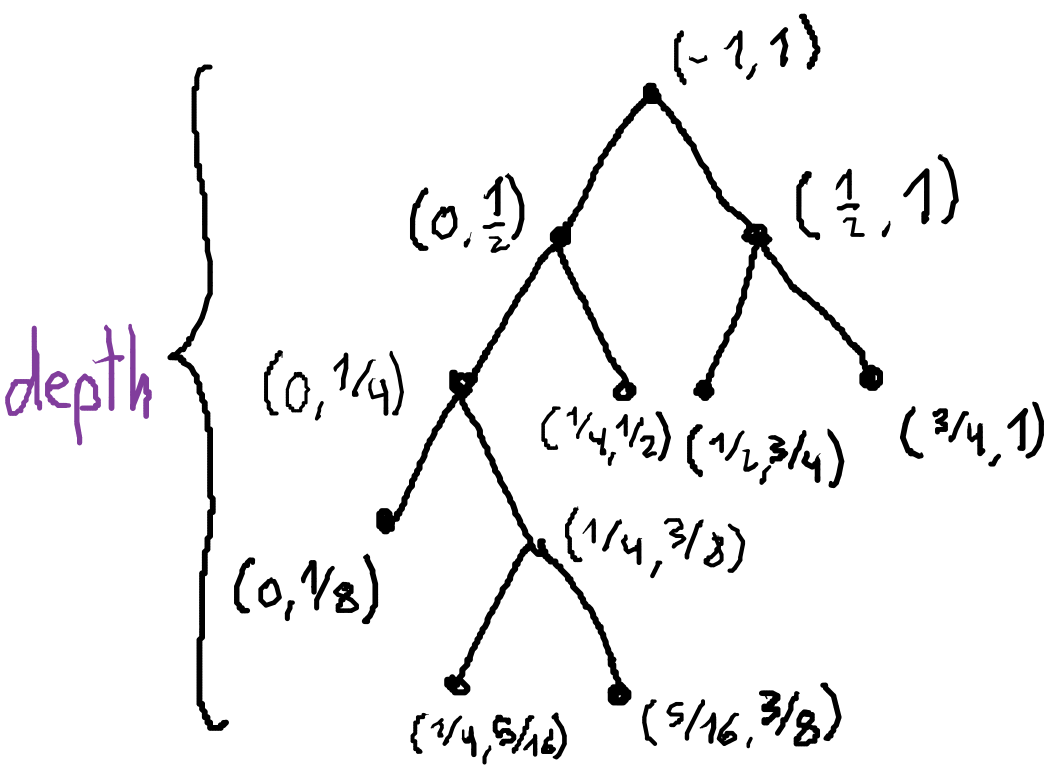

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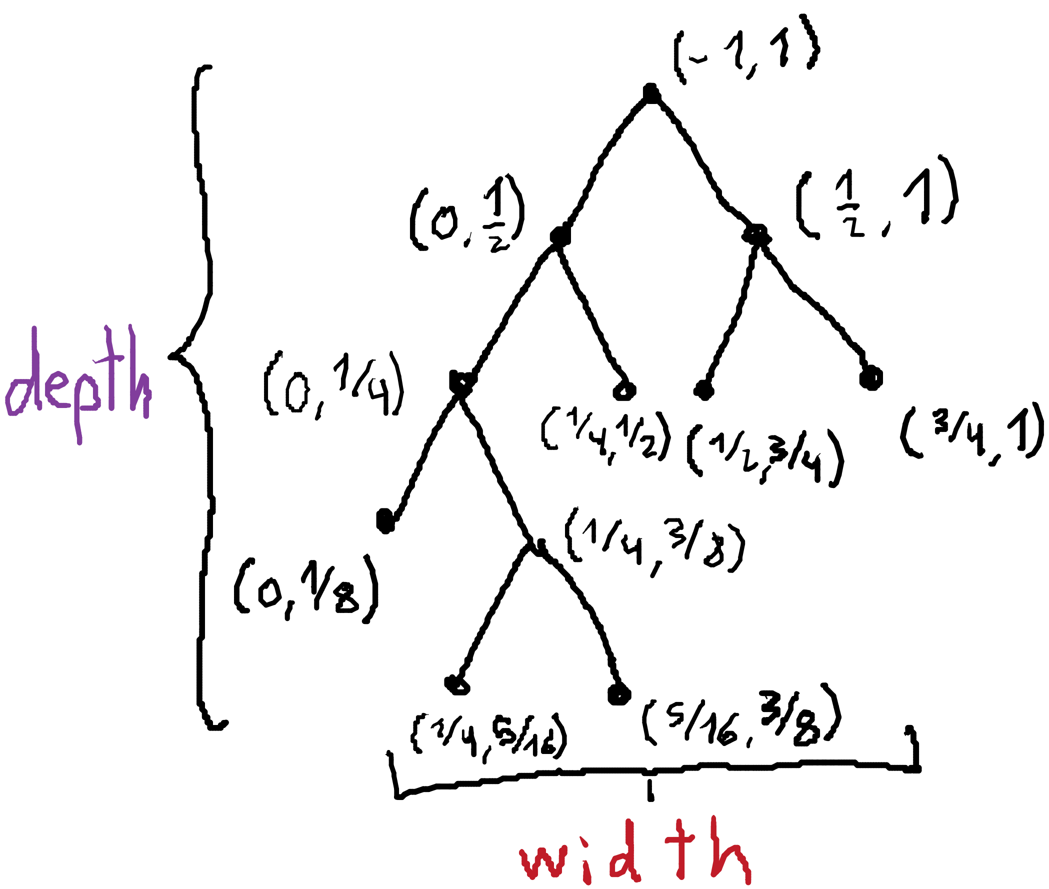

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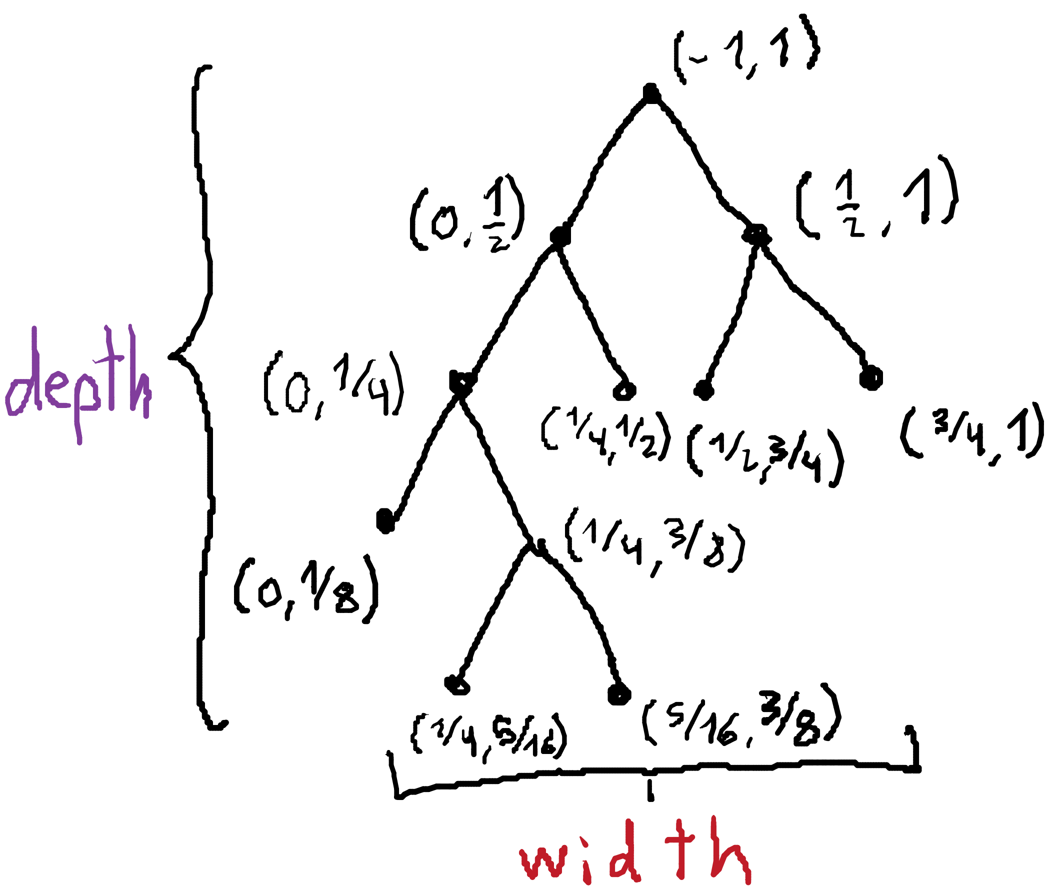
Prop.

$\mathrm{cost}(\text{Descartes}, g)$



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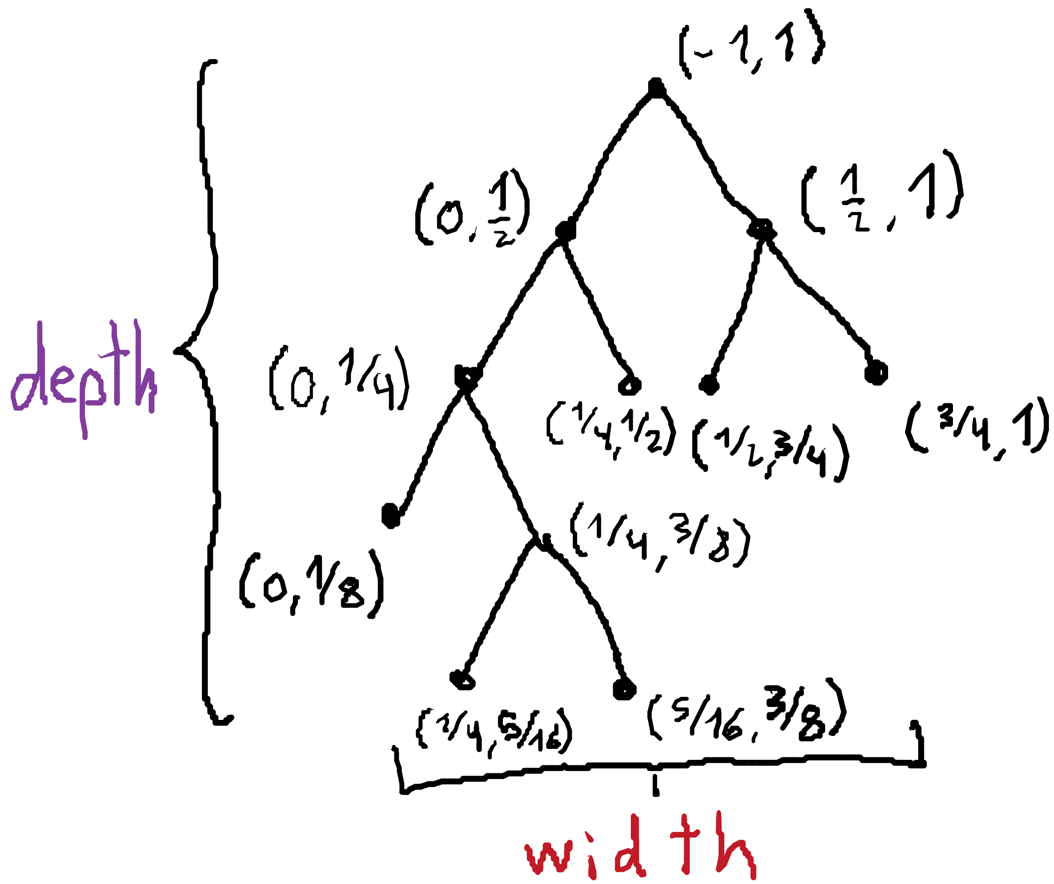

Prop.

$$\mathrm{cost}(\text{Descartes}, g) \leq \mathcal{O}\big(d\tau \,\mathrm{width}\,\mathcal{T}(g)\,\mathrm{depth}\,\mathcal{T}(g) + d^2\,\mathrm{width}\,\mathcal{T}(g)\,\mathrm{depth}^2\,\mathcal{T}(g)\big)$$

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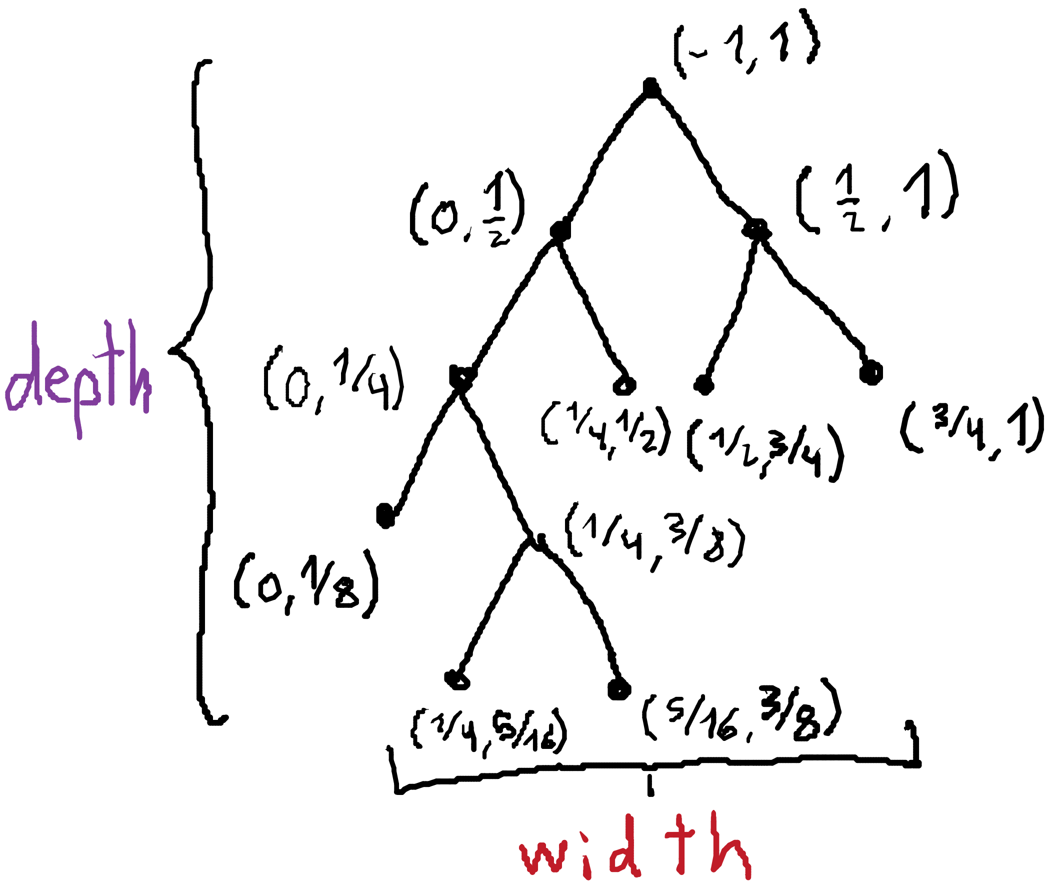

Prop.

$$\mathrm{cost}(\text{DESCARTES}, f) \leq O\big(d\tau\,\mathrm{width}\,\mathcal{T}(f)\,\mathrm{depth}\,\mathcal{T}(f) + d^2\,\mathrm{width}\,\mathcal{T}(f)\,\mathrm{depth}^2\,\mathcal{T}(f)\big)$$

I.e. size of $\mathcal{T}(f)$ bounds run-time of DESCARTES!

The Ingredients of the Analysis I: Condition Number

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$$C(g) := \max_{x \in [-1,1]} \frac{\sum_{k=0}^{d} |g_k|}{\max\{|g(x)|, |g'(x)|/d\}}$$

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$C(g) = \infty \iff g$ has a singular root in $[-1,1]$

$$\frac{1}{C(g)} \sim \left\{ \begin{array}{l} \text{How much I have to perturb} \\ \quad \text{the coefficients of } g \text{ so that} \\ \quad\quad \tilde{g} \text{ has a singular root in } [-1,1] \end{array} \right.$$

Bounding depth $\Upsilon(g)$

Bounding depth $\Upsilon(\delta)$

PROP.

$$\operatorname{depth} \Upsilon(\delta) \leq 5 + \log d + \log C(\delta)$$

Bounding depth $\Upsilon(g)$

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sep. between complex roots of g nearby $[-1, 1]$

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sep. between complex roots of g nearby $[-1,1]$ $\geq \dfrac{1}{12\, d\, C(g)}$

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Here it is essencial for roots being near [-1, 1]!

Bounding depth $\gamma(g)$

Prop.

$$\text{depth}\,\gamma(g) \leq 5 + \log d + \log C(g)$$

$$\text{sep. between complex roots of } g \text{ nearby } [-1,1] \geq \frac{1}{12\, d\, C(g)}$$

$$\text{sep. between complex roots of } g \text{ nearby } [-1,1] \gg 1/2^{\text{depth}\,\gamma(g)}$$

Here no

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Bounding width $\Upsilon(8)$

Bounding width $\tau(8)$

PROP

width $\tau(8)$

Bounding width $\mathcal{T}(g)$

Prop

width $\mathcal{T}(g)$

$$\leq \#\text{complex roots of } g \text{ nearby } [-1,1]$$

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Prop

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Important for good bound of the RHS!

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Secret tool to bound RHS:

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Titchmarsh's thm.

The Ingredients of the Analysis II: Cont. Probabilistic Toolbox

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... but we don't have a continuous dist.

The Ingredients of the Analysis III: Ball's Smoothing Trick

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$y \in \mathbb{R}^N$ s.t. $y_i \sim \mathcal{U}((-1/2, 1/2))$ i.i.d.

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$\longrightarrow x + y$ continuous random var.

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$\longrightarrow$ $x + y$ continuous random var.

We can use our old cont. toolbox!

Summing Up:

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Descartes' Solver

is almost optimal on average!

SUMMING UP:

DESCARTES' SOLVER

IS ALMOST OPTIMAL ON AVERAGE!

... AND THAT'S WHY DESCARTES WORKS SO WELL

Muito Obrigado

pela Atenção!

¡Muchas Gracias

por su Atención!