Finding real zeros a lot faster through an adaptive grid

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the

Problem

THE PROBLEM

$\mathcal{H}_d[n] \ni f$

DETERMINISTIC
+
NUMERICALLY STABLE
+
'GOOD' PROBABILISTIC RUN-TIME (for random f)

$Z_{\mathbb{P}}(f)$

The ALGORITHM we want!

Real Homogeneous Polynomial System in $X_0, \dots, X_n$ & with $\deg f_i = d_i$

projective zeros of f

What do we mean by
"'good' probabilistic run-time"?

Fact. Numerical algorithms can have very different run-times on inputs of the same size

Why? Run-times migth depend on the condition number

Solution (von Neumann, Goldstine, Demmel, Smale)

average framework:

See what is the run-time for a random input.

which distribution?

(Spielman, Teng) smoothed framework:

random perturbation of an arbitrary input

What do we mean by 'numerically stable'?

1) The algorithm can run in finite precision

More precisely: forward stability

If the used precision is 'small enough',
then the algorithm's output is correct

This might
depend on the input

2) Stability à la Smale

The obtained approximations can be
refined through an iterative scheme

What do we mean by

"'good' probabilistic run-time"?

Answer: The run-time is reasonably small in the probabilistic sense

Expectation is small

$\mathbb{E}\,\text{run-time} \leq T$

expected complexity

small with high probability

$\mathbb{P}(\text{run-time} \geq T) \leq \varepsilon$

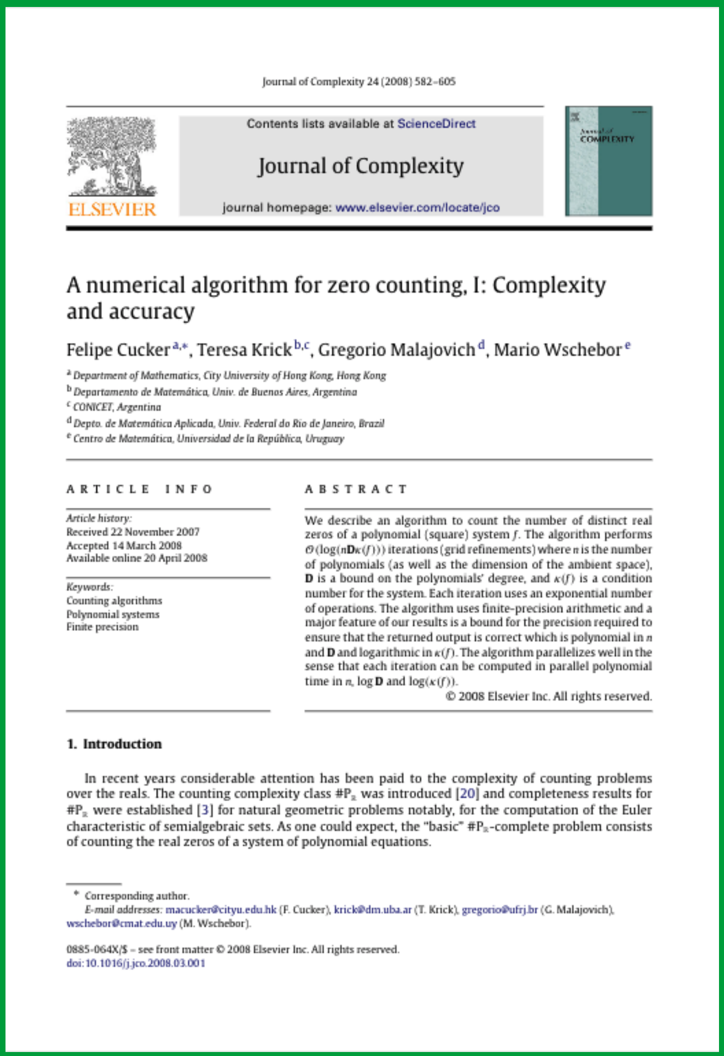
weak complexity

(Amelunxen, Lotz)

Excludes 'black swans'

A 'weak' solution to the Problem

The Original Trilogy
(by Cucker, Krick, Malajovich, Wschebor)



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A numerical algorithm for zero counting, I: Complexity and accuracy

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ABSTRACT

We describe an algorithm to count the number of distinct real zeros of a polynomial (square) system f. The algorithm performs $\mathcal{O}(\log(n\mathbf{D}\kappa(f)))$ iterations (grid refinements) where n is the number of polynomials (as well as the dimension of the ambient space), $\mathbf{D}$ is a bound on the polynomials' degree, and $\kappa(f)$ is a condition number for the system. Each iteration uses an exponential number of operations. The algorithm uses finite-precision arithmetic and a major feature of our results is a bound for the precision required to ensure that the returned output is correct which is polynomial in n and $\mathbf{D}$ and logarithmic in $\kappa(f)$. The algorithm parallelizes well in the sense that each iteration can be computed in parallel polynomial time in n, $\log \mathbf{D}$ and $\log(\kappa(f))$.

1. Introduction

In recent years considerable attention has been paid to the complexity of counting problems over the reals. The counting complexity class $\#P_{\mathbb{R}}$ was introduced [20] and completeness results for $\#P_{\mathbb{R}}$ were established [3] for natural geometric problems notably, for the computation of the Euler characteristic of semialgebraic sets. As one could expect, the "basic" $\#P_{\mathbb{R}}$-complete problem consists of counting the real zeros of a system of polynomial equations.

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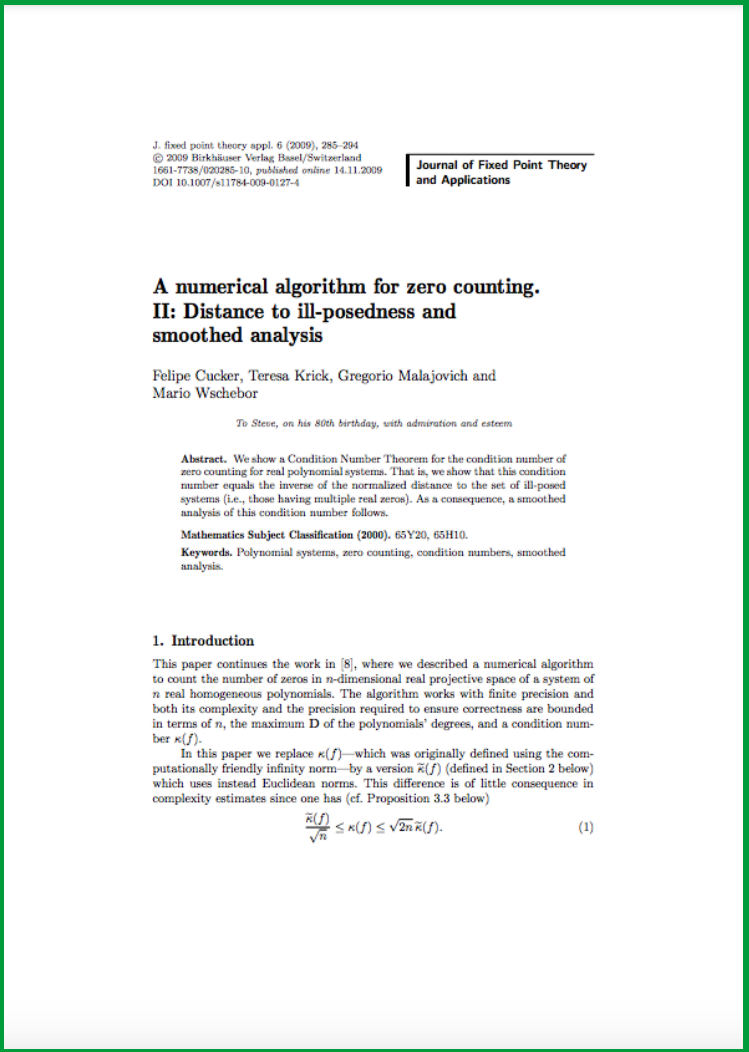

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Journal of Fixed Point Theory and Applications

A numerical algorithm for zero counting. II: Distance to ill-posedness and smoothed analysis

Felipe Cucker, Teresa Krick, Gregorio Malajovich and Mario Wschebor

To Steve, on his 80th birthday, with admiration and esteem

Abstract. We show a Condition Number Theorem for the condition number of zero counting for real polynomial systems. That is, we show that this condition number equals the inverse of the normalized distance to the set of ill-posed systems (i.e., those having multiple real zeros). As a consequence, a smoothed analysis of this condition number follows.

Mathematics Subject Classification (2000). 65Y20, 65H10.

Keywords. Polynomial systems, zero counting, condition numbers, smoothed analysis.

1. Introduction

This paper continues the work in [8], where we described a numerical algorithm to count the number of zeros in n-dimensional real projective space of a system of n real homogeneous polynomials. The algorithm works with finite precision and both its complexity and the precision required to ensure correctness are bounded in terms of n, the maximum $\mathbf{D}$ of the polynomials' degrees, and a condition number $\kappa(f)$.

In this paper we replace $\kappa(f)$—which was originally defined using the computationally friendly infinity norm—by a version $\widetilde{\kappa}(f)$ (defined in Section 2 below) which uses instead Euclidean norms. This difference is of little consequence in complexity estimates since one has (cf. Proposition 3.3 below)

$$\frac{\widetilde{\kappa}(f)}{\sqrt{n}} \le \kappa(f) \le \sqrt{2n}\,\widetilde{\kappa}(f). \tag{1}$$

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A numerical algorithm for zero counting. III: Randomization and condition

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ABSTRACT

In a recent paper (Cucker et al., 2008 [8]) we analyzed a numerical algorithm for computing the number of real zeros of a polynomial system. The analysis relied on a condition number $\kappa(f)$ for the input system f. In this paper we look at $\kappa(f)$ as a random variable derived from imposing a probability measure on the space of polynomial systems and give bounds for both the tail $\mathbb{P}(\kappa(f) > a)$ and the expected value $\mathbb{E}(\log \kappa(f))$.

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Part 1 Algorithm & condition-based complexity

Part 2 Condition Number Theorem & probabilistic complexity

Part 3 Probabilistic analysis without integral geometry

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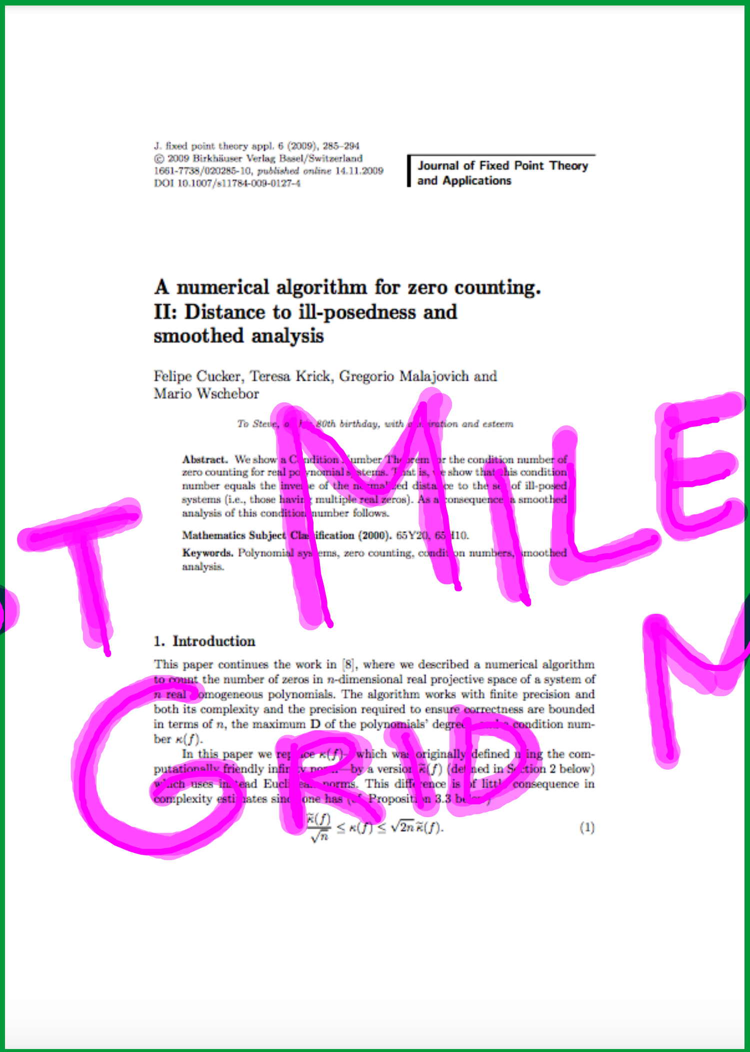
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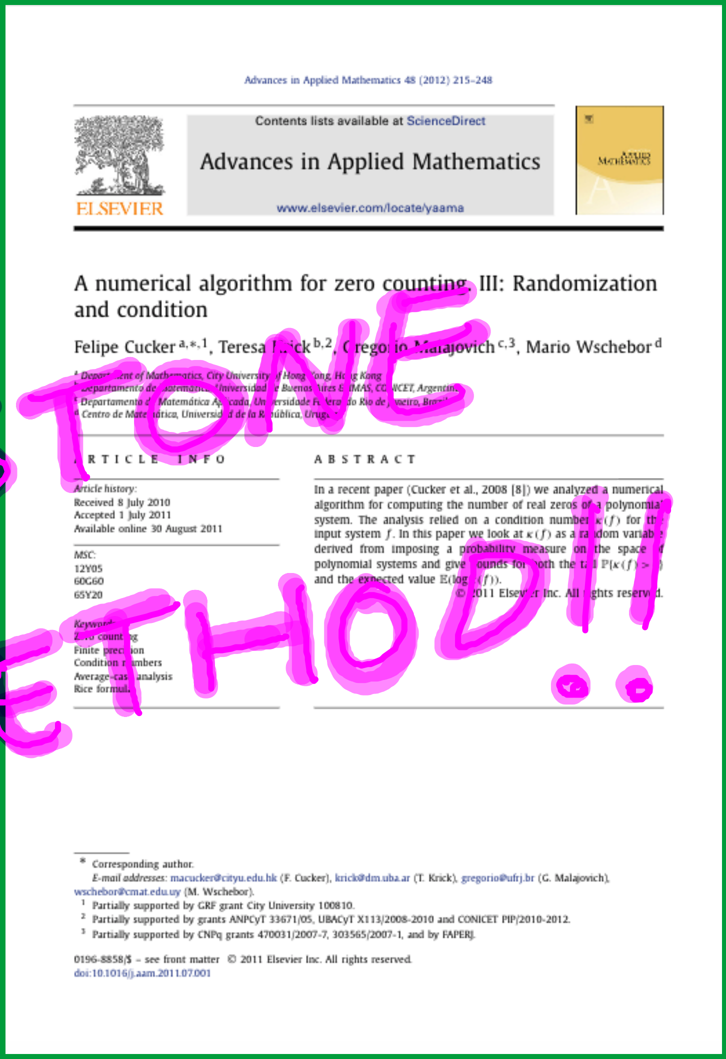
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A numerical algorithm for zero counting. III: Randomization and condition

FIRST MILESTONE OF THE GRID METHOD!!

Part 1 Algorithm & condition-based complexity

Part 2 Condition Number Theorem & probabilistic complexity

Part 3 Probabilistic analysis without integral geometry

The CKMW algorithm

$$\mathcal{H}_d[n] \ni f \longrightarrow \boxed{\text{CKMW}} \longrightarrow Z_{\mathbb{P}}(f)$$

DETERMINISTIC

'GOOD' PROBABILISTIC RUN-TIME

NUMERICALLY STABLE

With 'high probability', $\text{run-time}(\text{CKMW}, f) \leq D^{\mathcal{O}(n^2)}$

for f KSS (average / smoothed)

KSS = Kostlan-Shub-Smale Gaussian

$D := \max d_i$

PROBABILISTIC MODEL

$g \in \mathcal{H}_d[n]$ KSS random system if

$$g_i = \sum_\alpha \sqrt{\binom{d_i}{\alpha}}\, c_{i,\alpha} X^\alpha$$

with $c_{i,\alpha}$ independent, normal random variables of mean 0 and variance 1.

We also have a smoothed version!

$f + \sigma \|g\|_W g$

Why this? Invariant under orthogonal changes of variables

THE SPIN-OFFS
(by Ergür, Paouris, Rojas)

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Probabilistic Condition Number Estimates for Real Polynomial Systems I: A Broader Family of Distributions

Alperen A. Ergür[1] · Grigoris Paouris[2] · J. Maurice Rojas[2]

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Abstract We consider the sensitivity of real roots of polynomial systems with respect to perturbations of the coefficients. In particular—for a version of the condition number defined by Cucker and used later by Cucker, Krick, Malajovich, and Wschebor—we establish new probabilistic estimates that allow a much broader family of measures than considered earlier. We also generalize further by allowing overdetermined systems. In Part II, we study smoothed complexity and how sparsity (in the sense of restricting which terms can appear) can help further improve earlier condition number estimates.

Keywords Condition number · Epsilon net · Probabilistic bound · Kappa · Real-solving · Overdetermined · Subgaussian

Communicated by Felipe Cucker.

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SMOOTHED ANALYSIS FOR THE CONDITION NUMBER OF STRUCTURED REAL POLYNOMIAL SYSTEMS

ALPEREN A. ERGÜR, GRIGORIS PAOURIS, AND J. MAURICE ROJAS

ABSTRACT. We consider the sensitivity of real zeros of structured polynomial systems to perturbations of their coefficients. In particular, we provide explicit estimates for condition numbers of structured random real polynomial systems, and extend these estimates to smoothed analysis setting.

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1. INTRODUCTION

Efficiently finding real roots of real polynomial systems is one of the main objectives of computational algebraic geometry. There are numerous algorithms for this task, but the core steps of these algorithms are easy to outline: They are some combination of algebraic manipulation, a discrete/polyhedral computation, and a numerical iterative scheme.

From a computational complexity point of view, the cost of numerical iteration is much less transparent than the cost of algebraic or discrete computation. This paper constitutes a step toward understanding the complexity of numerically solving structured real polynomial systems. Our main results are Theorems 1.14, 1.16, and 1.18 below, but we will first need to give some context for our results.

1.1. **How to control accuracy and complexity of numerics in real algebraic geometry?** In the numerical linear algebra tradition, going back to von Neumann and Turing, condition numbers play a central role in the control of accuracy and speed of algorithms (see, e.g., [3, 6] for further background). Shub and Smale initiated the use of condition numbers for polynomial system solving over the field of complex numbers [36, 37]. Subsequently, condition numbers played a central role in the solution of Smale's 17th problem [2, 5, 25].

The numerics of solving polynomial systems over the real numbers is more subtle than complex case: small perturbations can cause the solution set to change cardinality. One can even go from having no real zero to many real zeros by an arbitrarily small change in the coefficients. This behaviour doesn't appear over the complex numbers as one has theorems (such as the Fundemantel Theorem of Algebra) proving that root counts are "generically" constant. Luckily, a condition number theory that captures these subtleties was developed by Cucker [11]. Now we set up the notation and present Cucker's definition.

Definition 1.1 (Bombieri-Weyl Norm). *We set* $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *where* $\alpha := (\alpha_1, \ldots, \alpha_n)$, *and let* $P = (p_1, \ldots, p_{n-1})$ *be a system of homogenous polynomials with degree pattern* $d_1, \ldots, d_{n-1}$. *Let* $c_{i,\alpha}$ *denote the coefficient of* x^α *in a* p_i. *We define the* Weyl-Bombieri norms *of* p_i *and* P *to be, respectively,*

$$\|p_i\|_W := \sqrt{\sum_{\alpha_1+\cdots+\alpha_n=d_i} \frac{|c_{i,\alpha}|^2}{\binom{d_i}{\alpha}}}$$

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1

1st non-gaussian average complexity in Numerical Alg. Geom!

1st non-gaussian smoothed complexity in NAG! (+ structured systems)

THE SPIN-OFFS
(by Ergür, Paouris, Rojas)

ROBUST PROBABILISTIC TOOLBOX!!

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1st non-gaussian average complexity in Numerical Alg. Geom!

1st non-gaussian smoothed complexity in NAG! (+ structured systems)

The CKMW algorithm (after the spin-offs)

$$\mathcal{H}_d[n] \ni f \longrightarrow \boxed{\text{CKMW}} \longrightarrow \# Z_{\mathbb{P}}(f)$$

DETERMINISTIC

'GOOD' PROBABILISTIC RUN-TIME

NUMERICALLY STABLE

With 'high probability', run-time(CKMW, f) $\leq D^{\mathcal{O}(n^2)}$

for f wide class of random systems

$D := \max d_i$

PROBABILISTIC MODEL II

$g \in \mathcal{H}_d[n]$ dobro random system

$$g_i = \sum_{\alpha} \sqrt{\binom{d_i}{\alpha}}\, c_{i,\alpha} X^{\alpha}$$

with $c_{i,\alpha}$ independent,

centered

i.e. $\mathbb{E}\, c_{i,\alpha} = 0$

subgaussian with cte. $\leq K$;

i.e. $\mathbb{E}\, |c_{i,\alpha}|^{\ell} \leq K_i^{\ell}\, \ell^{\ell/2}$ for $\ell \geq 1$

anticoncentration cte. $\leq \rho$;

i.e. $\mathbb{P}(|c_{i,\alpha} - t| \leq \varepsilon) \leq 2\rho_i \varepsilon$ for $t \in \mathbb{R}$

Also with smoothed version!

Important properties of gaussians!

For random $f \in \mathcal{H}_d[n]$ as before,

$$\mathbb{E}_f \text{ run-time}(\text{CKMW}, f) < \infty \,?$$

NO!

Why? $\text{run-time}(\text{CKMW}, f) \leq D^{O(n)} \mathcal{K}(f)^n$

$$\& \quad \mathbb{E}_f \mathcal{K}(f)^n = \infty$$

$\|f\|_W = \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} f_{i,\alpha}^2}$ (Weyl norm)

$\mathcal{K}(f) = \max_{x \in \mathbb{S}^n} \mathcal{K}(f, x)$ $\quad$ $\mathcal{K}(f,x) = \|f\|_W / \sqrt{\|f(x)\|^2 + \|D_x f^{\dagger} \Delta^{1/2}\|^{-2}}$ $\quad$ $\Delta = \mathrm{diag}(d_i)$

An 'expected' solution

to the Problem

Ideal

Make CKMW adaptive,

then complexity should depend on

$$\mathbb{E}_{x \in \mathbb{S}^n} \mathcal{K}(g,x)^n$$

which has finite expectation

for a random g!

$\|g\|_W = \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} g_{i,\alpha}^2}$ (Weyl norm) $\mathcal{K}(g,x) = \|g\|_W / \sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{1/2}\|^{-2}}$ $\Delta = \mathrm{diag}(d_i)$

Inspiration: (Cucker, Ergür, T.-C.; 2019) while studying PV algorithm

Naive adaptive version fails!

(Eckhardt, 2020) (Han, 2018)

Run-time bound in terms of

$$\mathbb{E}_{x \in S^n} \mathcal{K}(g,x)^{2n}$$

which has infinite expectation

for a random g!

$\|g\|_W = \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} g_{i,\alpha}^2}$ (Weyl norm) $\mathcal{K}(g,x) = \|g\|_W / \sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{\frac{1}{2}}\|^{-2}}$ $\Delta = \mathrm{diag}(d_i)$

What goes wrong?

The criterion to select zeros!

The CKMW algorithm

1) Refine grid $\mathcal{G} \subseteq \mathbb{S}^n$ until $d_{\mathbb{S}}(\mathcal{G}, \mathbb{S}^n)$ 'small'

2 Exclude points $x \in \mathcal{G}$ s.t. $\|f(x)\| / \|f\|_W$ 'big'
Include points $x \in \mathcal{G}$ s.t. $\|f(x)\| / \|f\|_W$ 'small'

3 Post-process the selected points to get approximation of $Z_{\mathbb{P}}(f)$

How do we exclude points?

Exclusion Lemma. Let $f \in \mathcal{H}_d[n]$, the map

$$\mathbb{S}^n \ni x \mapsto f(x)/\|f\|_W$$

is $\sqrt{D}$-Lipschitz. In particular, if

$$r \leq \frac{\|f(x)\|}{\sqrt{D}\,\|f\|_W},$$

the $B_{\mathbb{S}}(x,r) \cap Z_{\mathbb{S}}(f) = \emptyset$.

$\|f\|_W = \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} f_{i,\alpha}^2}$ (Weyl norm) $\qquad D := \max d_i$

How do we include points?

Spherical Newton operator: $N_g(x) := \dfrac{x - D_x g^{-1} g(x)}{\| x - D_x g^{-1} g(x) \|}$

$$N_g^{n+1}(x) = N_g(N_g^n(x))$$

Smale's α-criterion:

$$\alpha(g, x) := \beta(g, x)\gamma(g, x) \leq \alpha_*$$

$$\hookrightarrow \# B_S(x, 1.5\beta(g,x)) \cap Z_S(g) = 1$$

$$\& \quad N_g^n(x) \xrightarrow[\text{quadratically}]{} \text{zero of } g$$

where $\beta(g, x) := \| D_x g^{-1} g(x) \|$ & $\gamma(g, x) := \sup_{k \geq 2} \| D_x g^{-1} \frac{1}{k!} D_x^k g \|$

Assume $\sqrt{2}\, \mathcal{K}(g, x) \, {\| g(x) \|}/{\| g \|_W} < 1 \ldots$

- Higher Derivative Estimate: $\gamma(g, x) \leq \frac{1}{2} D^{3/2} \mathcal{K}(g, x)$
- An estimate for β: $\beta(g, x) \leq \mathcal{K}(g, x) \, {\| g(x) \|}/{\| g \|_W}$

The CKMW algorithm

1) Refine grid $\mathcal{G} \subseteq \mathbb{S}^n$ until $\underbrace{d_{\mathbb{S}}(\mathcal{G},\mathbb{S}^n)}_{=:\delta} \leq \frac{1}{cD^2\kappa(f)^2}$

2 $\begin{cases} \text{Exclude points } x \in \mathcal{G} \text{ s.t. } \|f(x)\|/\|f\|_W > \sqrt{D}\,\delta \\ \text{Include points } x \in \mathcal{G} \text{ s.t. } \|f(x)\|/\|f\|_W \leq \frac{1}{\tilde{c}D^2\kappa(f)^2} \end{cases}$

3 Post-process the selected points to get approximation of $Z_{\mathbb{P}}(f)$

Note quadratic condition in the inclusion criterion!

$\kappa(f) := \max_{x \in \mathbb{S}^n} \|f\|_W \Big/ \sqrt{\|f(x)\|^2 + \|D_x f^{-1} \Delta^{1/2}\|^{-2}}$ condition number

The adaptive CKMW algorithm

NAIVE EDITION

1) Refine adaptively $\mathcal{G} \subseteq \mathbb{S}^n \times (0,\infty)$ so that
   1) $\mathbb{S}^n \subseteq \bigcup \{B_{\mathbb{S}}(x,r) \mid (x,r) \in \mathcal{G}\}$
   & 2) $\forall (x,r) \in \mathcal{G},\ r \leq 1/cD\kappa(f,x)^2$

2 $\begin{cases} \text{Exclude } (x,r) \in \mathcal{G} \text{ if } \|f(x)\|/\|f\|_W \geq \sqrt{D}\, r \\ \text{Include } (x,r) \in \mathcal{G} \text{ if } \|f(x)\|/\|f\|_W \leq 1/\tilde{c}D^2\kappa(f,x)^2 \end{cases}$

3 Post-process the selected points to get approximation of $Z_{\mathbb{P}}(f)$

Still quadratic inclusion criterion!

$\kappa(f,x) := \|f\|_W \Big/ \sqrt{\|f(x)\|^2 + \|D_x f^{-1} \Delta^{1/2}\|^{-2}}$ local condition number

Where does the square come from?

$$\beta(g,x) \leq \kappa(g,x)\,\frac{\|g(x)\|}{\|g\|_W}$$

is a very bad estimate!

The $\kappa(g,x)$ in the upper bound causes the square

We should use β directly!

Converse Smale's α-theorem:

$$\gamma(g,x)\,\mathrm{dist}_S(x, Z_S(g)) < 1$$

$$\Longrightarrow \quad \alpha(g,x) \leq \frac{\gamma(g,x)\,\mathrm{dist}_S(x,Z_S(g))}{1-\gamma(g,x)\,\mathrm{dist}_S(x,Z_S(g))}$$

'If x is sufficiently near $Z_S(g)$,
then Smale's α-criterion at x holds'

Corollary. If $\sqrt{2}\,\mathcal{K}(g,x)\,\|g(x)\|/\|g\|_W < 1$,

then $\alpha(g,x) < \alpha_*$

or $B_S(x, c/D^2\mathcal{K}(g,x)) \cap Z_S(g) = \emptyset$

The adaptive CKMW algorithm

NON NAIVE EDITION!!!

1) Refine adaptively $\mathcal{G} \subseteq \mathbb{S}^n \times (0,\infty)$ so that
   1) $\mathbb{S}^n \subseteq \bigcup \{B_{\mathbb{S}}(x,r) \mid (x,r) \in \mathcal{G}\}$
   & 2) $\forall (x,r) \in \mathcal{G},\ r \leq 1/cD\kappa(\mathfrak{f},x)$

2
- Exclude $(x,r) \in \mathcal{G}$ if $\|\mathfrak{f}(x)\|/\|\mathfrak{f}\|_W \geq \sqrt{D}\, r$
- Include $(x,r) \in \mathcal{G}$ if $\beta(\mathfrak{f},x) \leq 1/\tilde{c}D^2\kappa(\mathfrak{f},x)$

3 Post-process the selected points to get approximation of $Z_{\mathbb{P}}(\mathfrak{f})$

Using β gives the desired $\mathbb{E}_{x\in\mathbb{S}^n}\kappa(\mathfrak{f},x)^n$ bound!

$\kappa(\mathfrak{f},x) := \|\mathfrak{f}\|_W / \sqrt{\|\mathfrak{f}(x)\|^2 + \|D_x\mathfrak{f}^{-1}\Delta^{1/2}\|^{-2}}$ local condition number

Some extra tricks

$$\mathcal{K}(g,x) := \frac{\|g\|_W}{\sqrt{\|g(x)\|^2 + \|D_x g^{-1}\Delta^{1/2}\|^{-2}}}$$

Change of norm

$$C(g,x) := \frac{\|g\|_\infty}{\max\left\{\|\Delta^{-1} g(x)\|_\infty, \|D_x g^{-1}\Delta^{2}\|_{\infty,2}^{-1}\right\}}$$

Extra normalization

where $\Delta := \mathrm{diag}(d_i)$

Row-normalization

$$\hat{g} := \left(g_i / \|g_i\|_\infty\right)_i$$

Interlude: $\|\cdot\|_W$ vs. $\|\cdot\|_\infty$

Definition

$f \in \mathcal{H}_d[n]$

$$\|f\|_W := \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} f_{i,\alpha}^2} \qquad \|f\|_\infty := \max_{i,x\in\mathbb{S}^n} |f_i(x)|$$

Similar inequalities $\Rightarrow$ Similar condition-based theory

$$\|D_x f(v)\|_W \leq D\|f\|_W \qquad \|D_x f(v)\|_\infty \leq D\|f\|_\infty$$

Kellogg's inequality

where $D_x f(v) = \sum_j \frac{\partial f}{\partial x_j} v_j \in \mathcal{H}_{d-1}[n]$

Different probabilistic behaviours

$$\mathbb{E}_f \|f\|_W \sim \sqrt{N} \sim \min\{n^{D/2}, D^{n/2}\} \qquad \mathbb{E}_f \|f\|_\infty \sim \sqrt{n \log D}$$

There is a catch! $\|f\|_\infty$ harder to compute

Main Result

MAIN THEOREM

There is a DETERMINISTIC, NUMERICALLY STABLE algorithm aCKMW that given $f \in \mathcal{H}_d[n]$ computes $\#Z_{\mathbb{P}}(f)$ and such that

$$\mathbb{E}_f \text{ run-time}(\text{aCKMW}, f) \leq 2^{\mathcal{O}(n \log n)} D^n N + 2^{\mathcal{O}(n \log n)} \mathcal{D}^{2.5} (N + \mathcal{D})$$

for f dobro random (with bounded parameters)

'GOOD' PROBABILISTIC RUN-TIME

where $D := \max_i d_i$ $\quad \mathcal{D} := \prod_i d_i$

$$N := \sum_i \binom{n+d_i}{n} = \#\text{ of zero \& non-zero coeff. of } f$$

+ PARALLELIZABLE

Future Work

Homology computation of semialgebraic sets

We can produce correct adaptive samples!

Post-processing step has to be improved!

Can we have a Monte-Carlo version without computing $\|\delta\|_\infty$?

Obrigado
pela atenção!