

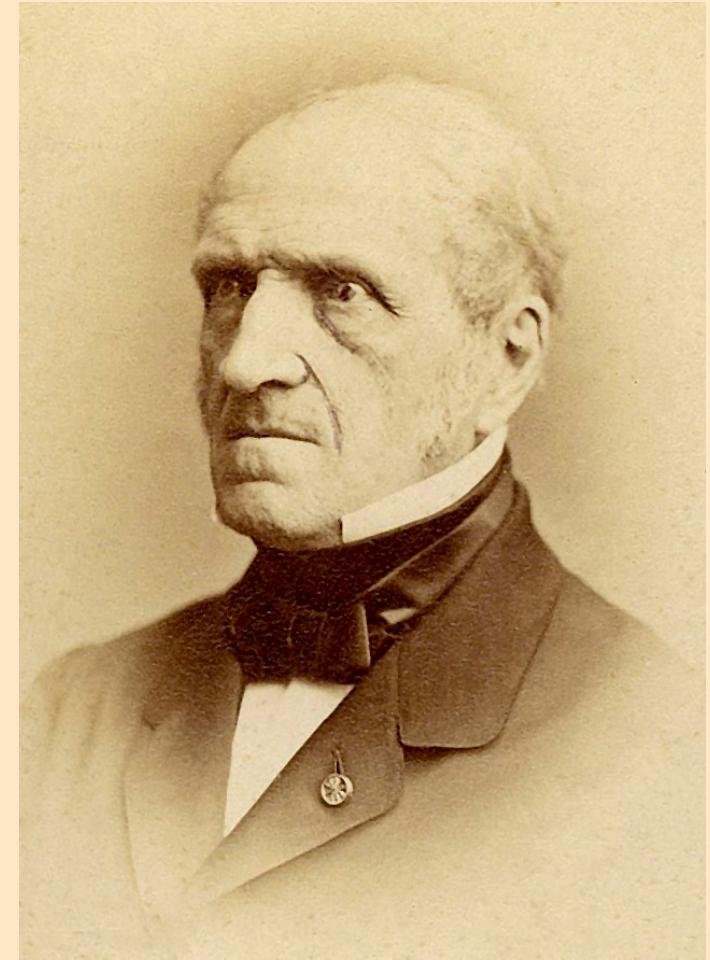
On this day...

the mathematician

Michel Chasles

would have

his 229<sup>th</sup> birthday



Famous for Chasles' identity: A whole career  
reduced to this!

$$\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac}$$



Symbolic Computation Seminar  
NCSU

# Computing Numerically the Homology of a Semialgebraic Set

Josué TONELLI-CUETO (UT San Antonio)

Joint work with:

Peter Bürgisser  
& Felipe Cucker



References.

Computing the Homology of Semialgebraic Sets. I: Lax Formulas  
Foundations of Computational Mathematics, 20(1):71–119, 2020.  
DOI 10.1007/s10208-019-09418-y

Computing the Homology of Semialgebraic Sets. II: General Formulas  
Foundations of Computational Mathematics, 21(5):1279–1316, 2021.  
DOI 10.1007/s10208-020-09483-8

Condition and Homology in Semialgebraic Geometry.  
PhD thesis, Technische Universität Berlin, DepositOnce Repository, December 2019.  
DOI 10.14279/depositonce-9453

Why do we care?

Central problem

in computational

Semialg. geometry

(with applications)

# THE THEOREM:

There is an algorithm  $\text{Hom}$  that given  
a  $q$ -tuple  $p \in \prod_{i=1}^q \mathbb{R}[x_1, \dots, x_n]_{\leq d_i}$  of  
polynomials and a semialgebraic formula

$\Phi$  over  $P$  of size  $s$ , it computes the  
homology groups of  
 $S_{\text{aff}}(P, \Phi) \subseteq \mathbb{R}^n \leftarrow$  semialg. set  
defined by  $(P, \Phi)$

in time

$$O(qs(128nD\bar{\kappa}_{\text{aff}}(P))^{10n(n+2)})$$

# THE THEOREM: (Homogeneous Version)

There is an algorithm  $\text{Hom}$  that given  
a  $q$ -tuple  $\gamma \in \overline{\bigcap_{i=1}^q \mathbb{R}[X_0, \dots, X_n]}_{d_i}$  of homogeneous  
polynomials and a semialgebraic formula  
 $\Phi$  over  $\gamma$  of size  $s$ , it computes the  
homology groups of  
 $S(\gamma, \Phi) \subseteq \mathbb{S}^n$  ← spherical  
semialg. set  
def. by  $(\gamma, \Phi)$   
in time

$$O(qs(128nD\bar{K}(\gamma))^{10n(n+2)})$$

# THE THEOREM: (Homogeneous Version +)

There is an algorithm  $\text{Hom}_\ell$  that given  
a  $q$ -tuple  $\gamma \in \prod_{i=1}^q \mathbb{R}[X_0, \dots, X_n]_{d_i}$  of homogeneous  
polynomials and a semialgebraic formula  
 $\Phi$  over  $\gamma$  of size  $s$ , it computes the  
first  $\ell$  homology groups of spherical  
 $S(\gamma, \Phi) \subseteq \mathbb{S}^n$  ← semialg. set  
def. by  $(\gamma, \Phi)$   
in time

$$O(qs(128nD\bar{K}(\gamma))^{10n(e+2)})$$

# THE PRDB. THEOREM: (Homogeneous Ver)

Given a random KSS q-tuple  $f$ ,

i.e.  $f_i = \sum \binom{d_i}{\alpha}^{1/2} c_{i,\alpha} X^\alpha$  with the

the  $c_{i,\alpha} \sim N(0,1)$  i.i.d., then  $\text{Hom}_e$   
runs with

$$O(s(qD)^{O(n^2e)} \lambda^{10n(e+1)})$$

with prob  $\geq 1 - 1/\lambda$

Also smoothed analysis

and more robust probabilistic models

Single exponential in  $n$   
with high probability!

Improves state of the art!

# Algorithm at a glance

0. Condition number estimation

1. Reduction to Lax Case

2. Reduction to Basic Cases

3. Basic case

# Condition Number

$$\bar{\kappa}(\delta, x) := \max_{\substack{I \subseteq [q] \\ \#I \leq n+1}} \frac{\|\delta_I\|_w}{\sqrt{\|\delta_I(x)\|^2 + \sigma_{|I|}(\Delta_I^{-1/2} D_x \delta_I)}}$$

where  $\delta_I := (\delta_i)_{i \in I}$   $D_x \delta_I : T_x S^n \rightarrow \mathbb{R}^{|I|}$   
 $\Delta = \text{diag}(d_i)_{i \in I}$  tangent map  
 $\sigma_{|I|}$   $|I|$ -sing. val.

$$\bar{\kappa}(\delta) := \max_{x \in S^n} \bar{\kappa}(\delta, x) \in [1, \infty)$$

# Interpretation of $\bar{\chi}(g)$

$$\bar{\chi}(g) < \infty$$

$\Leftrightarrow: \forall i, \chi_g(g_i)$  smooth hypersur.

&  $\forall I \subseteq [q],$

$$\bigcap_{i \in I} \chi_g(g_i) \text{ transversal}$$

Note: If  $|I| \geq n+1$ , then  $\bigcap_{i \in I} \chi_g(g_i)$  Transversal is empty

## ESTIMATION OF $\bar{\kappa}(g)$

Prop.  $\mathbb{S}^n \ni x \mapsto \frac{1}{\bar{\kappa}(g, x)}$  is  $D$ -Lipschitz

Cor. If  $G \subseteq \mathbb{S}^n$  satisfies

$$d_H(G, \mathbb{S}^n) < \varepsilon,$$

&  $\max \{ \bar{\kappa}(g, x) \mid x \in G \} D \varepsilon < 1/2$ , then

$$\bar{\kappa}(g) \leq 2 \max \{ \bar{\kappa}(g, x) \mid x \in G \}$$

# REDUCTION TO LAX CASE I

Gabrielov-Vorobjov Construction:

$$\Gamma B_{S, \varepsilon}(\underline{s}, \bar{\Phi}) = S(\underline{s}, \tilde{\Phi})$$

where in  $\tilde{\Phi}$

$$\left\{ \begin{array}{l} s_i = 0 \rightsquigarrow |s_i| \leq \varepsilon \|s_i\|_w \\ s_i > 0 \rightsquigarrow s_i \geq \delta \|s_i\|_w \\ s_i < 0 \rightsquigarrow s_i \leq -\delta \|s_i\|_w \end{array} \right.$$

$$\Gamma B_{S, \varepsilon}(\underline{s}, \bar{\Phi}) = \bigcup_i \Gamma B_{S_i, \varepsilon_i}(\underline{s}, \bar{\Phi})$$

Gabrielov-Vorobjov theorem:

If  $0 \ll \varepsilon_1 \ll \delta_1 \ll \varepsilon_2 \ll \delta_2 \ll \dots \ll \delta_m \ll 1$ ,

then  $\exists H_K(\Gamma B_{S, \varepsilon}(\underline{s}, \bar{\Phi})) \rightarrow H_k(S(\underline{s}, \bar{\Phi}))$ : iso for  $K \leq m-1$   
 & surj. for  $K = m$

# REDUCTION TO LAX CASE II

Gabrielov-Vorobjov Construction:

$$\Gamma B_{S,\varepsilon}(g, \bar{\Phi}) = S(g, \tilde{\Phi})$$

where in  $\tilde{\Phi}$

$$\left\{ \begin{array}{l} g_i = 0 \rightsquigarrow |g_i| \leq \varepsilon \|g_i\|_w \\ g_i > 0 \rightsquigarrow g_i \geq \delta \|g_i\|_w \\ g_i < 0 \rightsquigarrow g_i \leq -\delta \|g_i\|_w \end{array} \right.$$

$$\Gamma B_{S,\varepsilon}(g, \bar{\Phi}) = \bigcup_i \Gamma B_{S_i, \varepsilon_i}(g, \bar{\Phi})$$

Quantitative Gabrielov-Vorobjov theorem:

If  $0 < \varepsilon_1 < \delta_1 < \varepsilon_2 < \dots < \delta_m < \sqrt{\varepsilon} R(g)$

then  $\exists H_K(\Gamma B_{S,\varepsilon}(g, \bar{\Phi})) \rightarrow H_k(S(g, \bar{\Phi}))$ : iso for  $K \leq m-1$   
 & surj. for  $K = m$

# Reduction to basic case I

Main Idea:

- $x_i^\alpha$ : sample of  $S(\mathcal{S}; \alpha; 0)$
- If for all basis  $\phi = \bigwedge_{i \in I} (\mathcal{S}; \alpha_i; 0)$

$\bigwedge x_i^\alpha$ : good sample of  $S(\mathcal{S}, \phi)$ ,

then for all semialg. Formula  $\Phi$ ,

$$\Phi(Simp(x_i^\alpha)) \simeq S(\mathcal{S}, \Phi)$$

# Reduction to basic case II

Ingredients:

Explicit Functorial Nerve Theorem

$$\begin{array}{c} \pi: \check{C}_\varepsilon(x) \rightarrow B_\varepsilon(x) \text{ homology-equiv.} \\ \text{Cech} \quad \nearrow \quad \sum t_i[x_i] \mapsto \sum t_i x_i \end{array}$$

Homological Inclusion-Exclusion Transfer

$$g: X \rightarrow Y \text{ cont.}$$

$$X = \bigcup X_i, Y = \bigcup Y_i$$

$$g(X_i) \subseteq Y_i + \text{Tech.}$$

$$\forall I, g: \bigcap_{i \in I} X_i \rightarrow \bigcap_{i \in I} Y_i$$

homology equiv.

$$\Rightarrow g: X \rightarrow Y \text{ hom. equiv.}$$

Also hom. iso up to  $k$  and surv; for  $(k+1)$ -homology

$$\exists d_H(x, X) < \varepsilon < \frac{1}{2}\gamma(X) \Rightarrow B_\varepsilon(x) \overset{\sim}{\hookrightarrow} X$$

Hausdorff  
distance

hom. equiv

$$\gamma(X) := \inf\{r > 0 \mid \exists p \in \mathbb{R}^n, x, \tilde{x} \in X \text{ such that } d(p, X) = d(p, x) = d(p, \tilde{x}) = r \text{ and } x \neq \tilde{x}\}$$

$$\exists d_H(x, X) < \varepsilon < \frac{1}{5} \gamma(X) \Rightarrow VR_\varepsilon(x) \xleftarrow{\text{Vietoris-Rips}} \check{C}_\varepsilon(x) \xrightarrow{\text{Čech}}$$

# Basic Case Approx Results

Reach Bound

$$\gamma(Y \cap \bigcap X_i) \geq \min_I \gamma(Y \cap \bigcap_{i \in I} \partial X_i)$$

Reduction to reach of boundaries

$$\phi \text{ basic semialg. form} \Rightarrow \exists D^{3/2} \bar{\kappa}(g) \gamma(S(g, \phi)) > 1$$

Condition-based bound of reach

Sampling Thm  $g \subseteq S^n, d_H(g, S^n) < r$

If  $\sqrt{D^{1/2}} \bar{\kappa}(g) r < 1$ , then for all lax formulas

$$\Phi, d_H(g \cap S_{D^{1/2}r}(g, \Phi), S(g, \Phi)) < \sqrt{D^{1/2}} \bar{\kappa}(g) r$$

Relaxation:  $g \geq 0 \rightsquigarrow g \geq -D^{1/2}r, \dots$

# Improvements

- Using  $\|\varphi\|_\infty := \max_i \max_{x \in S^n} |\varphi_i(x)|$  instead of  $\|\varphi\|_W$  reduces in one  $n$  the exponent of prob. run-time
- Can we implement it?

Eskerrik asko  
bere arretagatik!  
.