Functional Norms, Condition Numbers, and Numerical Algebraic Geometry

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The Idea

Numerical linear algebra:

- various matrix norms
- the selection of a norm in algorithms' design/analysis is often done to minimize complexity

Numerical polynomial algebra:

- a single norm (Weyl, 1932) dominates the literature
- it is easy to compute and unitarily/orthogonally invariant

A Tale of Two Norms

The Weyl norm

$$f \in \mathcal{H}_d^{\mathbb{F}}[1] \qquad f = \sum_{|\alpha|=d} f_\alpha X^\alpha$$

where $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$.

$$\|f\|_W := \sqrt{\sum_{|\alpha|=d} \binom{d}{\alpha}^{-1} |f_\alpha|^2}$$

where $\binom{d}{\alpha}$ is the multinomial coefficient $\frac{d!}{\alpha_0! \ldots \alpha_n!}$.

For $f = (f_1, \ldots, f_q) \in \mathcal{H}_{\mathrm{d}}[q]$ the Weyl norm extends as

$$\|f\|_W := \sqrt{\|f_1\|_W^2 + \cdots + \|f_q\|_W^2}$$

The ∞ norm

$$\|f\|_\infty^{\mathbb{F}} := \begin{cases} \max_{x\in\mathbb{S}^n} \|f(x)\|_\infty = \max_{x\in\mathbb{S}^n} \max_i |f_i(x)| & \text{if } \mathbb{F} = \mathbb{R} \\ \max_{z\in\mathbb{P}^n} \|f(z)\|_\infty = \max_{z\in\mathbb{P}^n} \max_i |f_i(z)| & \text{if } \mathbb{F} = \mathbb{C} \end{cases}$$

Why bother to choose $\|f\|_\infty^{\mathbb{F}}$ over $\|f\|_W$?

Why bother?

Reason 1:
There is a huge gain for random data!

In the worst-case,

$$\|f\|_{\infty}^{\mathbb{F}} \leq \|f\|_{W}$$

In the random case,

Theorem

For random $\mathfrak{f} \in \mathcal{H}_{\mathrm{d}}^{\mathbb{F}}[q]$,

$$\mathop{\mathbb{E}}_{\mathfrak{f}} \frac{\|\mathfrak{f}\|_{\infty}^{\mathbb{F}}}{\|\mathfrak{f}\|_{W}} \leq \mathcal{O}\left(\sqrt{\frac{n\ln(e\mathrm{D})}{N}}\right) \sim \mathcal{O}\left(\sqrt{\frac{\ln(e\mathrm{D})}{\mathrm{D}^{n}}}\right) \text{ (}\textit{for large}\text{ D)}$$

Huge gain for ‘typical’ input

Why bother?

Reason 2:

The ∞-norm can still control the derivatives!

Theorem

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $f \in \mathcal{H}_{\mathrm{d}}^{\mathbb{F}}[1]$, $x \in \mathbb{F}^{n+1}$ *and* $v \in \mathbb{F}^{n+1}$, *then*

$$\left|\overline{\mathrm{D}}_x f\, v\right| \leq d^{\frac{1}{2}} \|f\|_W \|x\|_2^{d-1} \|v\|_2.$$

Theorem (Kellogg's Inequality)

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $f \in \mathcal{H}_{\mathrm{d}}^{\mathbb{F}}[1]$, $x \in \mathbb{F}^{n+1}$ *and* $v \in \mathbb{F}^{n+1}$, *then*

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Similar complexity analyses. . .

. . . with similar condition numbers

Complex setting:

$$\mu_{\mathrm{norm}}(f,\zeta) := \|f\|_W \left\| \mathrm{D}_\zeta f^\dagger \Delta^{1/2} \right\|_{2,2}.$$

$$\downarrow$$

$$\mathsf{M}(f,\zeta) = \sqrt{q}\|f\|_\infty^{\mathbb{C}} \left\| \mathrm{D}_\zeta f^\dagger \Delta \right\|_{2,2}.$$

. . . with similar condition numbers

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Real setting:

$$\kappa(f) := \sup_{x\in\mathbb{S}^n} \frac{\|f\|_W}{\sqrt{\|f(x)\|_2^2 + \left\|\mathrm{D}_x f^\dagger \Delta^{1/2}\right\|_{2,2}^{-2}}}.$$

$$\downarrow$$

$$\mathsf{K}(f) := \sup_{x\in\mathbb{S}^n} \frac{\sqrt{q}\|f\|_\infty^{\mathbb{R}}}{\max\left\{\|f(x)\|, \|\mathrm{D}_x f^\dagger \Delta\|_{2,2}^{-1}\right\}}.$$

Any problems?

$\|\ \|_\infty$ is not cheap to estimate

Proposition

Given $(f, k) \in \mathcal{H}_{\mathrm{d}}^{\mathbb{F}}[q] \times \mathbb{N}$ *we can compute* T *such that*

$$(1 - 2^{-k})\mathsf{T} \leq \|f\|_\infty \leq \mathsf{T}$$

with cost

$$\mathcal{O}\left(2^{n \log n} \mathsf{D}^n 2^{\frac{(k+1)n}{2}} N\right).$$

Gains are big enough to compensate for this

THREE Applications

1st Application: Computing the Betti numbers of (Semi-)Algebraic Sets

State of the art

Theorem

There is a numerical algorithm BETTI *that, given $f \in \mathcal{H}_{\mathbf{d}}[q]$, returns the Betti numbers of its zero set $Z(f) \subset \mathbb{S}^n$. The cost of* BETTI *on input f is bounded as*

$$\mathtt{cost}(f) \le 2^{\mathcal{O}(n^2 \log n)} D^{\mathcal{O}(n^2)} \kappa(f)^{\mathcal{O}(n^2)}.$$

Furthermore, it satisfies

$$\mathtt{cost}(p) \le q^{\mathcal{O}(n)} (nD)^{\mathcal{O}(n^3)}$$

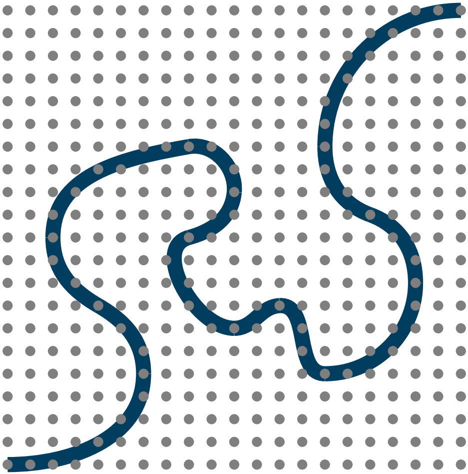
with probability at least $1 - (nqD)^{-n}$.

The result holds for a class of distributions extending the Gaussian

Outside a set of vanishingly small measure
this yields an exponential acceleration over all previous algorithms

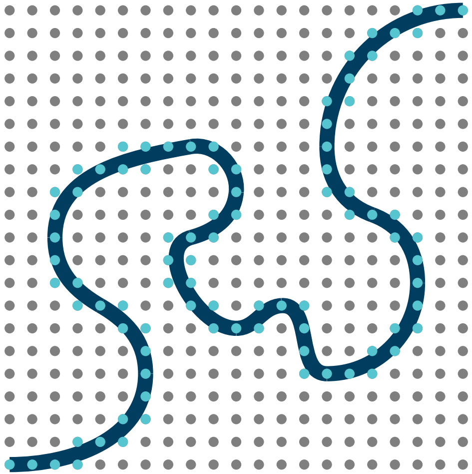
The Algorithm

The Algorithm



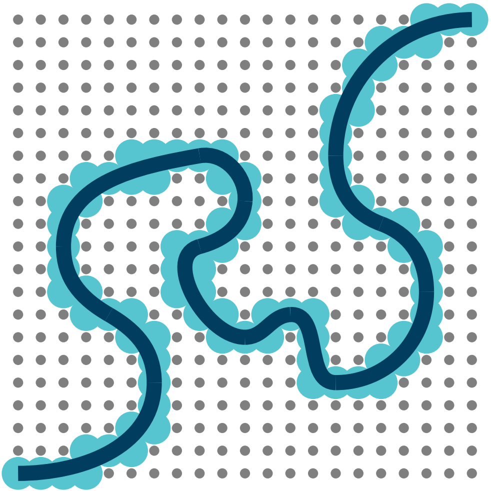
$\kappa(f)$ controls the mesh of the grid!

The Algorithm



$\kappa(f)$ is in the criterion to determine which points are near!

The Algorithm



$\kappa(f)$ determines how big we should take the balls!
(Through the Niyogi-Smale-Weinberger Theorem
and a bound on the reach!)

The Algorithm

Union of Balls

$$\downarrow$$

$\boxed{\text{some TDA (e.g. Nerve Lemma)}}$

$$\downarrow$$

Betti numbers of zero set

(Even torsion coefficients!)

Replacing $\| \ \|_W$ with $\| \ \|_\infty$

(1) The same scheme can be applied using K instead of κ

(2) $\dfrac{\mathtt{cost}(\textsc{Betti}_\infty, f)}{\mathtt{cost}(\textsc{Betti}_W, f)} \leq \left(\dfrac{\mathsf{K}(f)}{\kappa(f)}\right)^{10n}$

(3) For random $\mathfrak{f}$

$$\frac{\mathtt{cost}(\textsc{Betti}_\infty, \mathfrak{f})}{\mathtt{cost}(\textsc{Betti}_W, \mathfrak{f})} \leq \left(\frac{Cn\sqrt{qD\ln(eD)}}{\sqrt{N-20n}}\right)^{10n}$$

with probability at least $1 - \frac{1}{N}$

For fixed n and large D, the ratio in the right-hand side is of the order of

$$\left(\frac{C\sqrt{\ln(e\mathrm{D})}}{D^{\frac{n-1}{2}}}\right)^{10n}.$$

2nd Application:
The Plantinga-Vegter Algorithm

- Given a real polynomial f, the PV algorithm meshes the real zero set.
- Mostly used for two and three variables by computer graphics

community, reported to be efficient, and quite popular

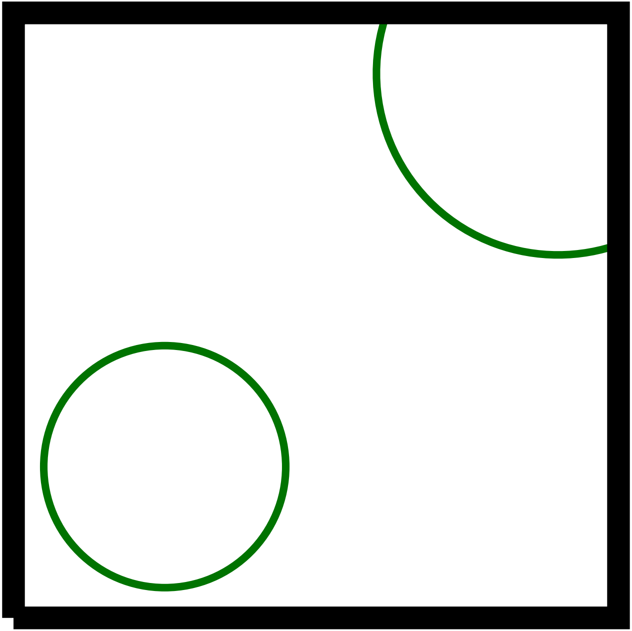
- Concretely speaking:

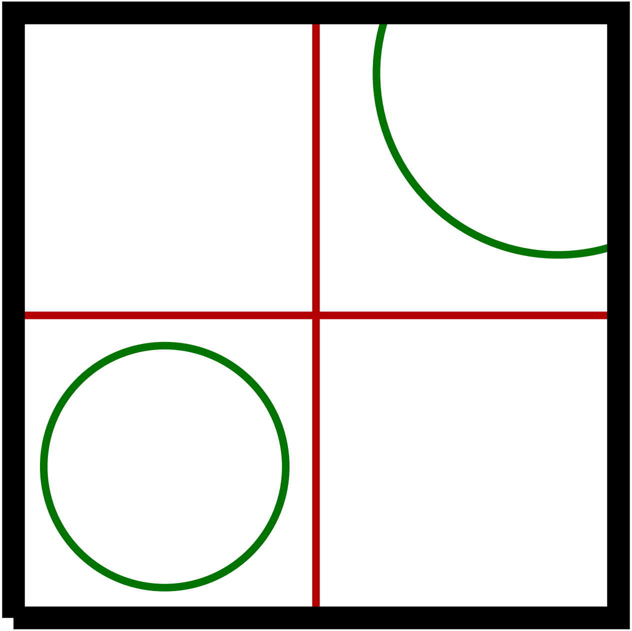
Given a polynomial $f \in \mathbb{R}[X, Y]$ (or $f \in \mathbb{R}[X, Y, Z]$) with degree d it computes an isotopic piecewise linear approximation of the zero set of f within a given square in $\mathbb{R}^2$ (cube in $\mathbb{R}^3$, respectively).

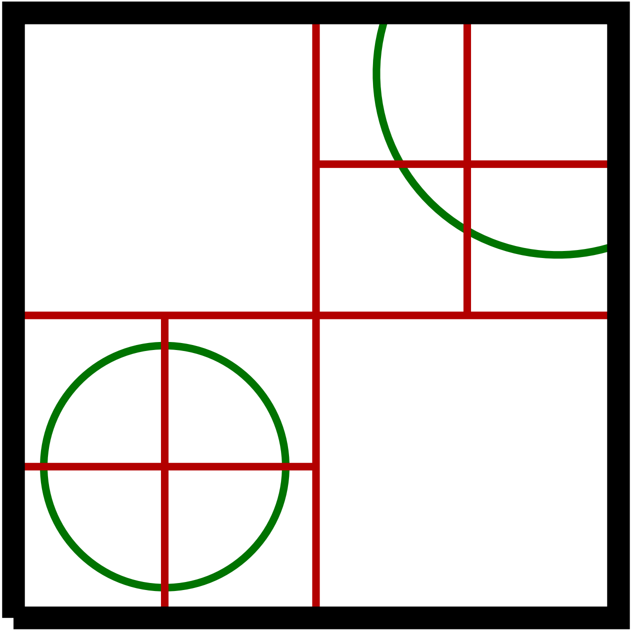
- Ambiguous for precision control

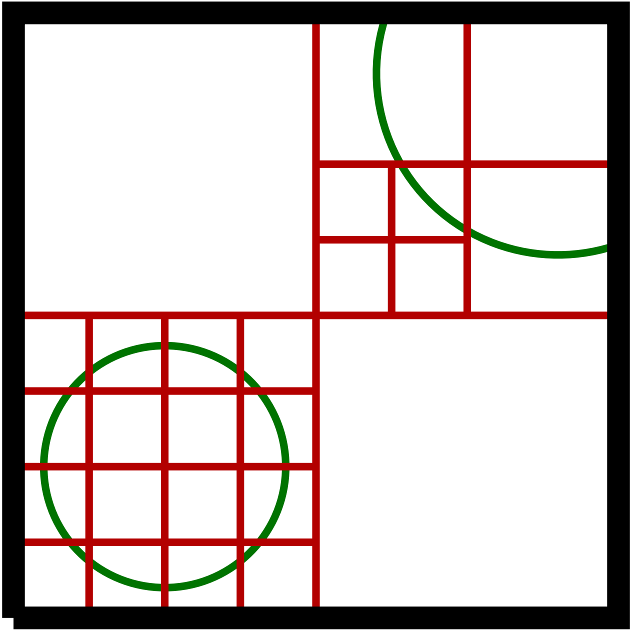
- Worst-case complexity analysis by Burr, Gao, Tsigaridas came after 14 years and predicted exponential running time

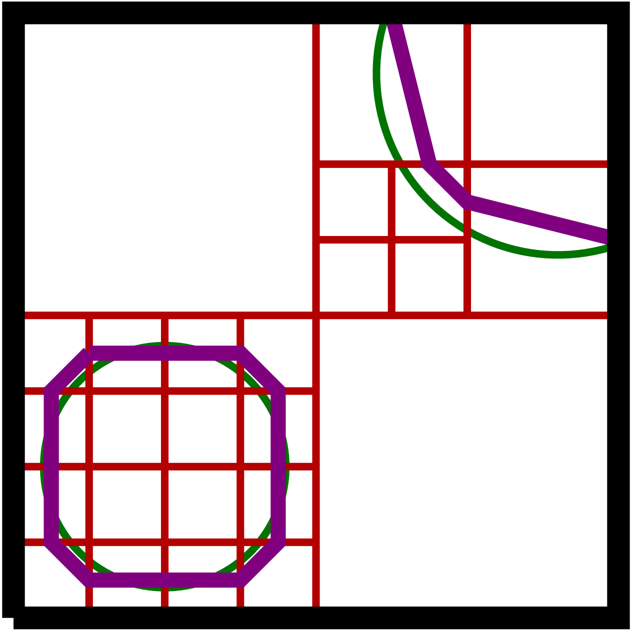
- We use condition numbers for precision control and beyond-worst-case complexity analysis











Smoothed Analysis of Algorithms

- Perturb a deterministic input g with a random input $\mathfrak{h}$:

$$g + \sigma \|g\| \mathfrak{h}$$

where $\sigma \in (0, \infty)$ controls the "variance"

- For the algorithm of interest, we bound the quantity

$$\sup_{g} \mathbb{E}_{\mathfrak{h}} \operatorname{cost}(g + \sigma \|g\| \mathfrak{h})$$

  - $\sigma = 0$ gives the worst-case complexity analysis
  - $\sigma \to \infty$ gives the average case complexity analysis
  - $\sigma \in (0, \infty)$ gives the smoothed complexity analysis

- Smoothed analysis explains run-time in practice!

- Note that we need to choose a probability distribution for $\mathfrak{h}$

In our case, $\mathfrak{h}$ is a dobro random polynomial, i.e., subgaussian coefficients with bounded continuous density

Worst-case case complexity of the PV algorithm

$$2^{\mathcal{O}(d^n)}$$

Smoothed complexity of the PV algorithm

With the Weyl norm,

$$d^{\mathcal{O}(n^2)}$$

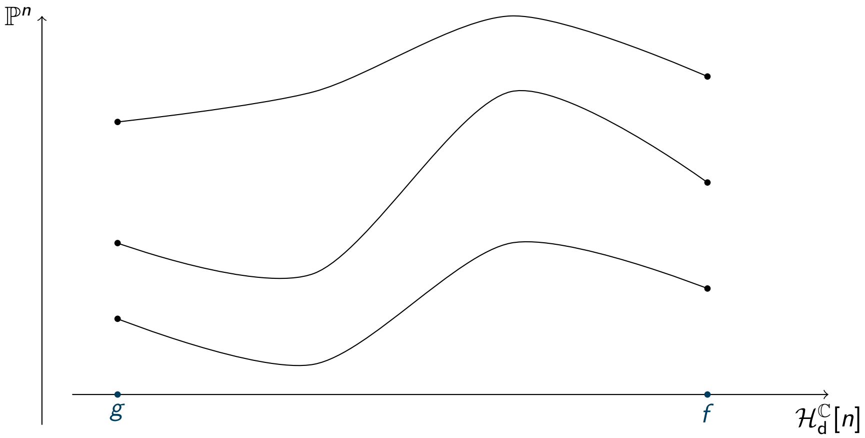
With the ∞-norm,

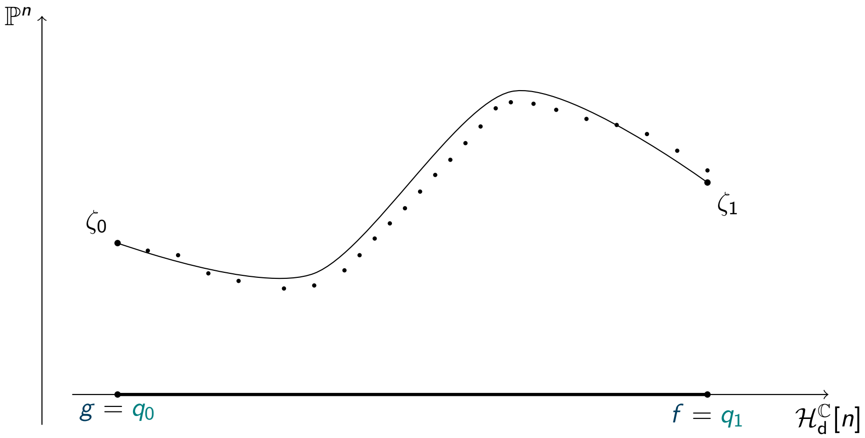
$$(d \log d)^{\mathcal{O}(n)}$$

Smoothed complexity of the PV algorithm for low dimensions

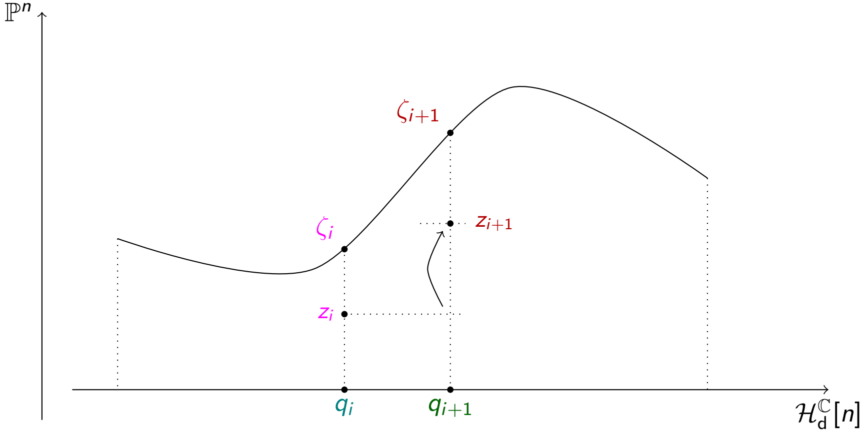
	$n = 2$	$n = 3$
PV_W	$\mathcal{O}\left(d^8\right)$	$\mathcal{O}\left(d^{13}\right)$
PV_∞	$\mathcal{O}\left(d^7 \log^{1.5}(d)\right)$	$\mathcal{O}\left(d^{10} \log^2(d)\right)$

3rd Application:
Systems
of
complex quadratic equations

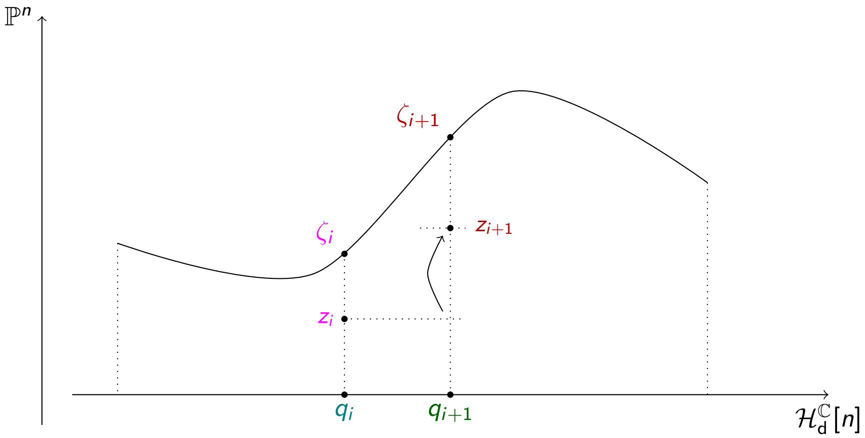

$\mathbb{P}^n$
g
f
$\mathcal{H}_d^{\mathbb{C}}[n]$



$$q_t := tf + (1-t)g$$



$$d_{\mathbb{S}}(q_i, q_{i+1}) := \frac{0.008535284}{\mathrm{dist}_{\mathbb{S}}(f, g)\mathrm{D}^{3/2}\mu_{\mathrm{norm}}(q_i, z_i)^2} \qquad z_{i+1} := N_{q_{i+1}}(z_i).$$



$$d_{\mathbb{S}}(q_i, q_{i+1}) := \frac{0.03}{\frac{\|f-g\|_\infty^{\mathbb{C}}}{\|q_i\|_\infty^{\mathbb{C}}} \mathrm{DM}(q_i, z_i)^2} \qquad z_{i+1} := N_{q_{i+1}}(z_i).$$

	EXPECTED # STEPS	COST OF STEP	TOTAL COST
W	$\mathcal{O}\left(n\mathrm{D}^{3/2}N\right)$	$\mathcal{O}(N)$	$\mathcal{O}\left(n\mathrm{D}^{3/2}N^2\right)$
∞	$\mathcal{O}(n^3\mathrm{D}\log(e\mathrm{D}))$	Large	Large

The case of quadratic equations: $\mathrm{D} = 2$ **(**$N = \mathcal{O}(n^3)$**)**

	EXPECTED # STEPS	COST OF STEP	TOTAL COST
W	$\mathcal{O}\left(n^4\right)$	$\mathcal{O}(n^3)$	$\mathcal{O}\left(n^7\right)$
∞	$\mathcal{O}(n^3)$	$\mathcal{O}(n^{1.5+\omega})$	$\mathcal{O}(n^{4.5+\omega})$

Note that $\omega < 2.375$!

Conclusion

As in the case of numerical linear algebra,

a careful choice of norms can improve algorithm efficiency

¡Muchas Gracias!

Teşekkürler!

Eskerrik asko!