Farewell to Weyl: Condition-based analysis with a Banach norm in numerical algebraic geometry

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Table of contents

Motivation

$A \in \mathbb{C}^{m\times n}$ a matrix and $m \leq n$.
Two norms:

1. **Spectral norm**.

$$\|A\| := \max_{x\in\mathbb{S}(\mathbb{C}^n)} \|Ax\|$$

2. **Fröbenius norm**.

$$\|A\|_F := \sqrt{\sum_{i,j} \left|A^i_j\right|^2}$$

$A \in \mathbb{C}^{m\times n}$ a matrix and $m \leq n$

$$\Sigma := \{B \in \mathbb{C}^{m\times n} \mid \operatorname{rank} B < m\}$$

...and two conic condition numbers:

1. $\kappa(A) := \frac{\|A\|}{\operatorname{dist}(A,\Sigma)} = \|A\|\|A^\dagger\|$
2. $\kappa_F(A) := \frac{\|A\|_F}{\operatorname{dist}_F(A,\Sigma)}$

Curiously,

$$\frac{\|A\|}{\kappa(A)} = \operatorname{dist}(A,\Sigma) = \operatorname{dist}_F(A,\Sigma) = \frac{\|A\|_F}{\kappa(A)_F}$$

In general,

$$\frac{1}{m}\|A\|_F \leq \|A\| \leq \|A\|_F$$

but for random A,

$$\mathbb{E}_A \frac{\|A\|}{\|A\|_F} = \mathcal{O}\left(\frac{1}{\sqrt{m}}\right)$$

Also,

$$\frac{\kappa(A)}{\kappa_F(A)} = \frac{\|A\|}{\|A\|_F}$$

So...

changing the norm improves the condition of large matrices!

Norms on polynomials

Notation

- $X_0, X_1, \ldots, X_n$ variables
- $n+1 :=$ number of variables
- $q :=$ number of distinct polynomials
- $\boldsymbol{d} = (d_1, \ldots, d_q)$ tuple of degrees
- $D := \max\{d_1, \ldots, d_q\}$
- $\mathcal{H}_{\boldsymbol{d}}[q]$ space of q-tuples f, where f_i is homogeneous polynomial of degree d_i in the $n+1$ variables $X_0, X_1, \ldots, X_n$
- $N := \sum_{i=1}^{q} \binom{n+d_i}{n} = q \min\left\{\mathcal{O}(D^n), \mathcal{O}(n^D)\right\} = \dim \mathcal{H}_{\boldsymbol{d}}[q]$
- $\Delta := \mathrm{diag}(\sqrt{\boldsymbol{d}})$
- $\mathrm{D}_x f$ tangent map $\mathrm{T}_x\mathbb{S}^n \to \mathbb{R}^q$ or $\mathrm{T}_{[x]}\mathbb{P}^n \to \mathbb{C}^q$

Weyl norm

$$\|f\|_W := \sqrt{\sum_{i=1}^{q} \|f_i\|_W^2}$$

where

$$\|f_i\|_W = \sqrt{\sum_{\alpha} \binom{d_i}{\alpha}^{-1} |f_{i,\alpha}|^2} \quad \text{and} \quad f_i = \sum_{\alpha} f_{i,\alpha} X^{\alpha}$$

Some properties:

1. Invariant under orthogonal/unitary transformations
2. It controls evaluation: $\|f(x)\| \leq \|f\|_W$
3. It controls the norm of the derivative: $\|\partial f\|_W \leq D\|f\|_W$
4. It comes from an inner product

Max norm

$$\|f\|_\infty := \max_{x\in\mathbb{S}^n} \|f(x)\|$$

and

$$\|f\|_{\mathfrak{m}} := \max_{x\in\mathbb{S}^n} \sqrt{\|f(x)\|^2 + \|\Delta^{-1}\mathrm{D}_x f\|^2}$$

Some properties:

1. Invariant under orthogonal/unitary transformations
2. It controls evaluation: $\|f(x)\| \leq \|f\|_\infty \leq \|f\|_{\mathfrak{m}}$
3. It controls the norm of the derivative: $\|\partial f\|_\infty \leq \sqrt{2}D\|f\|_\infty$ (Kellogs' Theorem)
4. $\|f\|_\infty$ better for computation and polynomial inequalities and $\|f\|_{\mathfrak{m}}$ better for condition inequalities, but they are computationally equivalent

$$\|f\|_\infty \leq \|f\|_{\mathfrak{m}} \leq \sqrt{2}\min\{D, \sqrt{qD}\}\|f\|_\infty$$

Considerations I

Example

$f \in \mathcal{H}_{\mathbb{1}}[q]$, i.e., f linear map given by $A \in \mathbb{C}^n$

$$\|f\|_\infty = \|A\|.$$

$$\|f\|_{\mathfrak{m}} = \sqrt{\|A\|^2 + \sigma_2(A)^2}$$

Proposition

Let $f \in \mathcal{H}_{\boldsymbol{d}}[q]$. *Then*

$$\|f\|_\infty \leq \|f\|_{\mathfrak{m}} \leq \|f\|_W \leq \sqrt{qN}\|f\|_\infty^{\mathbb{C}}.$$

Considerations II

Theorem

Let $f \in \mathcal{H}_{\mathbf{d}}[q]$ be a KSS random polynomial tuple and c_0 an absolute constant. Then

$$\mathbb{P}\left(\|f\|_W \geq c_0 N t\right) \leq \exp(1 - N t^2),$$

and

$$\mathbb{P}\left(\|f\|_\infty \geq c_0 \sqrt{n} \log(D) t\right) \leq \exp(1 - n \log(D) t^2)$$

Remark

We can also make this for dobro random polynomials...

Condition numbers

Old condition number

$$\mu(f,x) := \frac{\|f\|_W}{\sigma_q(\Delta^{-1}\mathrm{D}_x f)}$$

$$\kappa(f,x) := \frac{\|f\|_W}{\sqrt{\|f(x)\|^2 + \sigma_q(\Delta^{-1}\mathrm{D}_x f)^2}}$$

Theorem (Condition Number Theorem)

$$\kappa(f,x) = \|f\|_W / \mathrm{dist}_W(f, \Sigma_x)$$

where

$$\Sigma_x := \{g \mid g \textit{ singular at } x\}.$$

Why does it work?

1. Higher Derivative Estimate: It controls Smale's Gamma,

$$\gamma(f,x) := \sup_{k\geq 2}\left\|\frac{1}{k!}\mathrm{D}_x f^{\dagger} D_x^k f\right\|^{\frac{1}{k-1}} \leq \frac{1}{2}D^{3/2}\mu(f,x)$$

2. It's inverse is Lipschitz with respect to f,

$$\left|\frac{\|f\|_W}{\mu(f,x)} - \frac{\|g\|_W}{\mu(g,x)}\right| \leq \|f-g\|_W \text{ and } \left|\frac{\|f\|_W}{\kappa(f,x)} - \frac{\|g\|_W}{\kappa(g,x)}\right| \leq \|f-g\|_W;$$

3. and with respect to x,

$$\left|\frac{\|f\|_W}{\mu(f,x)} - \frac{\|f\|_W}{\mu(f,y)}\right| \leq D\|x-y\| \text{ and } \left|\frac{\|f\|_W}{\kappa(f,x)} - \frac{\|g\|_W}{\kappa(g,x)}\right| \leq D\|x-y\|.$$

These are what makes everything work!

New condition numbers?

$$\mathsf{M}(f,x) := \frac{\|f\|_{\mathfrak{m}}}{\sigma_q(\Delta^{-1}\mathrm{D}_x f)}$$

$$\mathsf{K}(f,x) := \frac{\|f\|_{\mathfrak{m}}}{\sqrt{\|f(x)\|^2 + \sigma_q(\Delta^{-1}\mathrm{D}_x f)^2}}$$

Theorem (Condition Number Theorem)

$$\mathsf{K}(f,x) = \|f\|_{\mathfrak{m}} / \mathrm{dist}_{\mathfrak{m}}(f, \Sigma_x)$$

where

$$\Sigma_x := \{g \mid g \textit{ singular at } x\}.$$

Do they still work?

1. Higher Derivative Estimate: It controls Smale's Gamma,

$$\gamma(f,x) := \sup_{k\geq 2}\left\|\frac{1}{k!}\mathrm{D}_x f^\dagger D_x^k f\right\|^{\frac{1}{k-1}} \leq \min\{\sqrt{q},\sqrt{D}\}D^{3/2}\mathrm{M}(f,x)$$

2. It's inverse is Lipschitz with respect to f,

$$\left|\frac{\|f\|_{\mathfrak{m}}}{\mathrm{M}(f,x)}-\frac{\|g\|_{\mathfrak{m}}}{\mathrm{M}(g,x)}\right| \leq \|f-g\|_{\mathfrak{m}} \text{ and } \left|\frac{\|f\|_{\mathfrak{m}}}{\mathrm{K}(f,x)}-\frac{\|g\|_{\mathfrak{m}}}{\mathrm{K}(g,x)}\right| \leq \|f-g\|_{\mathfrak{m}};$$

3. and with respect to x,

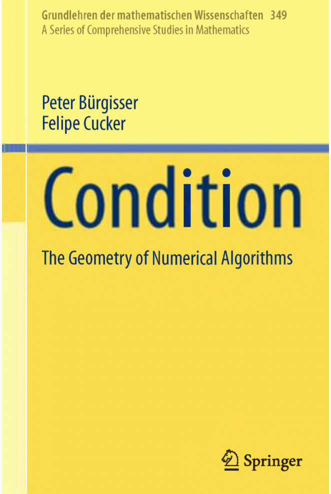
$$\left|\frac{\|f\|_{\mathfrak{m}}}{\mathrm{M}(f,x)}-\frac{\|f\|_{\mathfrak{m}}}{\mathrm{M}(f,y)}\right| \leq D\|x-y\| \text{ and } \left|\frac{\|f\|_{\mathfrak{m}}}{\mathrm{K}(f,x)}-\frac{\|g\|_{\mathfrak{m}}}{\mathrm{K}(g,x)}\right| \leq \sqrt{2}D\|x-y\|.$$

This means that...

We can carry,
up to parameters and constants,
the same condition-based
complexity analysis!

How?

Just follow the book!



...and some other papers!

(Proof-analysis of all it)

Case of linear homotopy

State-of-the-art

	Expected number of iterations	
Beltrán, Pardo; 2011	$\mathcal{O}\left(D^{3/2} n N\right)$	
Armentano, Beltrán, Bürgisser, Cucker, Shub; 2016	$\mathcal{O}\left(D^{3/2} n N^{1/2}\right)$	
Lairez; 2017	$\mathcal{O}\left(D^2 n^5\right)$	Not for linear homotopy!
Cucker, Ergür, T-C; $\leq$ 2020	$\mathcal{O}\left(D^{5/2} \log(D)^2 n^{5/2}\right)$	Some work to do...

Work to do

1. Can we compute $\|f\|_\infty$ up to a $\mathrm{poly}(D, n)$-factor in $\mathcal{O}(N)$-time?
    - To make the complexity bound effective, we need to be able to approximate the max norm fast
    - It can be with $\mathcal{O}(D)^n$ parallel evaluations and $\mathcal{O}(n \log(D))$ comparisons (Non-adaptive grid)
2. More general distributions
3. More general functions?

Case of grid and subdivision methods

Grid and subdivision methods

Based on a simple idea:

1. Subdivide region (or refine grid),
2. evaluate, and
3. compare.

Two types of subdivisions:

- Uniform subdivisions $\rightarrow$ effective (weak complexity)
  - Zero location (Cucker, Krick, Malajovich, Wschebor; 2008-12)
  - Homology computation of semialgebraic sets (Cucker, Krick, Shub; 2017), (Bürgisser, Cucker, Lairez; 2018) and (Bürgisser, Cucker, T.-C.; 2018&19)
- Adaptive subdivisions $\rightarrow$ efficient (average complexity) – recent!
  - Plantinga-Vegter algorithm (Next slide. . .)
  - Real condition estimation (Jiadong, Lairez; 2018)

Moreover, we can compute max norms on the way!

Plantinga-Vegter algorithm I

1. (Plantinga, Vegter; 2004)
    - Determination of isotopy type of smooth implicit curves inside a square and smooth implicit surfaces inside a box
    - Certification via interval arithmetic
    - No complexity analysis
2. (Burr, Gao, Tsigaridas; 2017)
    - Generalization of subdivision to arbitrary dimensions
    - Local size bound and continuous amortization
    - Worst-case bound for integer polynomials of degree D
3. (Cucker, Ergür, T.-C.; 2019)
    - Condition-based analysis (using Weyl norm) of the local size bound
    - Average and smoothed analysis for dobro polynomials, obtaining
      $$\tilde{\mathcal{O}}\left(D^{\frac{n^2+3n}{2}}\right)$$
      subdivisions on average
    - More at ISSAC19 next week in Beijing!

Plantinga-Vegter algorithm II

With the new norm…

$$\tilde{\mathcal{O}}\left(D^{\frac{n^2+3n}{2}})\right) \to \tilde{\mathcal{O}}\left(D^{\frac{3n}{2}} \log^{n+1} D\right)$$

So for curves…

$$\mathcal{O}\left(D^3 \log^3 D\right),$$

i.e., a lot better on average that many symbolic algorithms ($\mathcal{O}(D^{16} \log^5 D)$ c.f. (González-Vega, El Kahou; 1996))

Bere arretagatik eskerrik asko!

Galderak?