The University of Texas at San Antonio™

PROBLEM-SOLVING CLUB

(Spring 2023)

Date & Time: Tuesday, 5:00 pm – 7:00 pm

Location: MH 3.03.10

Organizers: Christopher Duffer & Dr. Josué Tonelli-Cueto

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1 Parity Principle

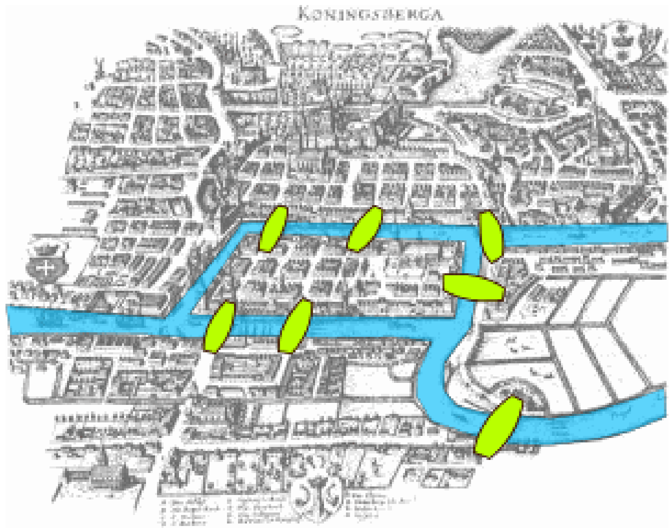
Problem 1.1. In a party, there are 2006 guest divided across four rooms. Prove that it is always possible to join the guest of the first room with the guests of another room in such a way that they can form couples for dancing without leaving anyone out.

Problem 1.2. A battlefield is divided into 64 sectors through a 8×8 grid. The objective of the attacking army is hidden in one of the sector and for trying to achieve it they throw their soldiers in parachute from the air. Assume that the sector in which a paratrooper falls is fully random and that a paratroopers cannot fall in a sector in which there is already a paratrooper. Once in land, the soldiers can move two sectors at a time but they cannot stop on the way to explore the adjacent sectors. Which is the minimum number of soldiers that the army should send to find the objective certainly and successfully? Does anything change if the army can choose where the soldiers land?

Problem 1.3. Is it possible to place 31 domino pieces forming a square in which the two opposite corners are missing? And if the corners don't have to be opposite to each other?

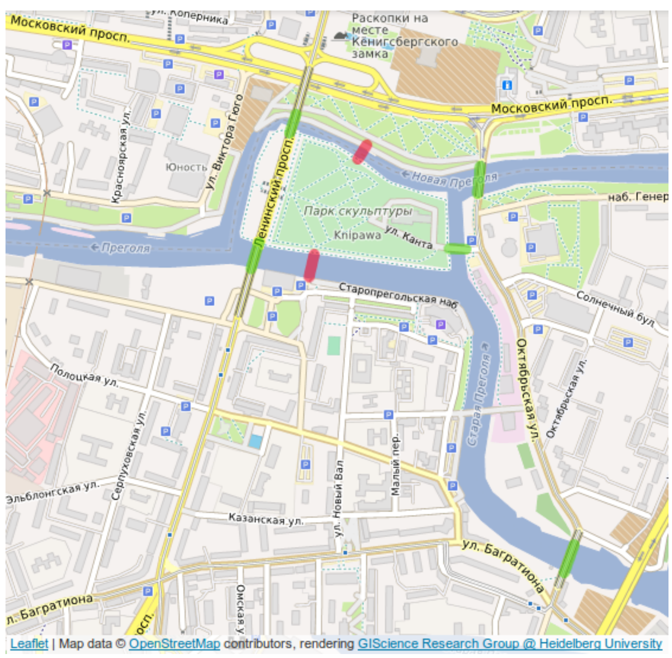
Problem 1.4. Can an ant walk through all edges of a cube in such a way that it does not pass twice through the same edge?

Problem 1.5. (Euler's Königsberg's Bridges' Problem) The following map of city of Königsberg emphasizes the river and bridges of the city.



A common pass-time at the time was trying to walk through the city passing through all the bridges once and only once. However, no one was able to achieve it. Can you explain why?

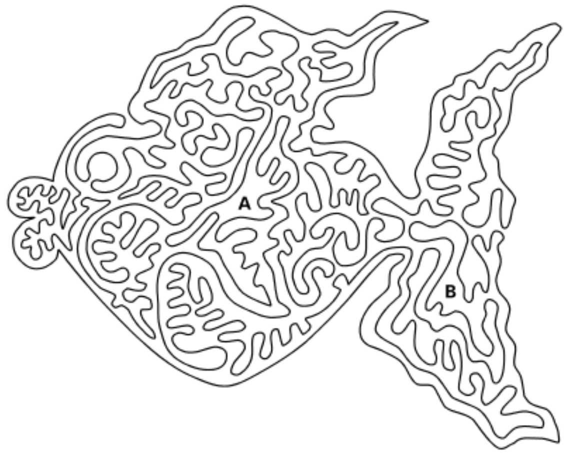
Nowadays, the map of of Königsberg (now called Kaliningrad) is the following one:



In this map, we can see in red the bridges that have been destroyed from Euler's times; in green, the ones that have survived; and in white, the new ones. Can we go through every bridge crossing each bridge only once now?

Problem 1.6. We put 13 points $P_1, P_2, \ldots, P_{13}$ in the plane and we draw the segments P_1P_2, P_2P_3, P_3P_4,. . .,$P_{12}P_{13}$, $P_{13}P_1$. Is it possible to draw a straight line in such a way that it cuts all segments in their interior (without touching the extremes)? And if instead of a straight line we draw a circle?

Problem 1.7. Is it possible to go from point A to point B in the drawing below without crossing the walls?



Problem 1.8. We put 40 cards face down forming 5 rows with 8 cards per row. Two players play a game as follows: In each turn, a player picks a row where not all the cards are face up and turns face up a card that is face down; then the player can turn any of the cards that follow in the row however they want—for each of the cards that follow, either E flips it or E leave leaves it as it is. The first player who cannot do any movement (because all cards are face up) loses the game. Prove that the first player can always win.

Problem 1.9. (Nim game) In the nim game we start with five piles of 1, 2, 3, 4 and 5 objects, respectively. Two players make alternatively the following move: they select a pile in which objects remain and remove from it any amount of objects they desire. The first player who cannot make any move loses, i.e., the player who leaves no objects into play at the end of their turn wins. Prove that the first player can always win. What happens if we increase the number of piles or vary the number of objects in each pile?

Problem 1.10. (Hats in a prison) In a prison, 100 prisoners are standing in a queue facing in one direction. Each prisoner is wearing either an orange hat or a blue hat. A prisoner can see the hats of all the prisoners in front of Em in the queue, but can see neither Eir own hat nor the hats of the prisoners standing behind. The warden of the prison is going to ask each prisioner for the color Eir hat starting from the last prisoner in queue. If a prisoner guesses the color of Eir hat, then E is liberated, otherwise E is executed. How many prisoners can be saved at most if the warden allows them to come up with an strategy before he starts asking?

2 General Problems

Problem 2.1. (UTSA Sandwich Problem) You eat sandwiches. You love sandwiches. No: you *love* sandwiches[1]. But *only* mini-sandwhiches that are exactly 2 in $\times$ 2in $\times$ 1in in size[2]. When visiting Mexico last summer, you realized that, delicious as the tacos were south of the border, you *missed* your sandwiches, so naturally you asked Eduardo Dueñez to ship you some sandwiches from the UTSA Problem-Solving Club. He packed full and sent you a cubic box with an edge-length of 20 in—i.e. a total of 2000 mini-sandwiches!

[1] Between 2018 and 2014, sandwiches—not pizza—were the snack choice at the UTSA Problem-Solving Club.

[2] Okay, truthfully, we like sandwiches of all shapes and sizes, but for the sake of this problem let's pretend we are very picky.

Given the urgency and sheer volume of the task, when it came to packing, all sandwiches ended up in random positions: some facing up or down, some facing either of the four sides of the box. When going through customs, an evil inspector took a very long needle and pierced right through the box (ouch!). Is there any chance that the needle may have missed every single sandwich (somehow finding its way right through their edges), or is it the case that necessarily some sandwich(es) must have been pierced by the needle?

Problem 2.2. **(Rainbow Hats Puzzle)** Seven prisoners are given the chance to be set free tomorrow. An executioner will put a hat on each prisoner's head. Each hat can be one of the seven colors of the rainbow and the hat colors are assigned completely at the executioner's discretion. Every prisoner can see the hat colors of the other six prisoners, but not his own. They cannot communicate with others in any form, or else they are immediately executed. Then each prisoner writes down his guess of his own hat color. If at least one prisoner correctly guesses the color of his hat, they all will be set free immediately; otherwise they will be executed. They are given the night to come up with a strategy. Is there a strategy that they can guarantee that they will be set free?

Problem 2.3. **(Hades vs. Sisyphus)** There are 1001 steps in a staircase. There are some rocks in some of the steps in such a way that there cannot be more than one rock per step. Sisyphus is fighting against Hades in this staircase by moving one rock alternatively. Each turn, Sisyphus has to take one rock and lift it, one or more steps, up to the first step that is empty. Then Hades takes one rock and brings it down to the step immediately inferior to it—which has to be empty, as no two rocks can be in the same step. In order to win, Sisyphus has to put a rock in the last step. Initially, there are 500 rocks in the initial 500 steps. If Sisyphus makes the first move, can Hades prevent Sisyphus to win?

Problem 2.4. **(Gas in a Circuit)** We have a circular circuit in which a car goes. The amount of gas needed for this car to go around the circuit is distributed in n deposits located in n fixed points of the circuit. At the beginning, there is no gas in the deposit of the car. Prove that no matter how the gas is distributed into the deposits and how the deposits are located we can always start at point in such a way that we can complete the circuit with the car. Assume that the consumption of gasoline is uniform and proportional to the distance moved. Moreover, assume that no gas is wasted when we stop to fill the deposit of the gas at a deposit and that the deposit of the car can hold all the gas needed to go around once.

References

[1] *Mathematics Stack Exchange*. `https://math.stackexchange.com/`.

[2] J. Sangroniz Gómez. *Los problemas de ingenio como recurso didáctico para las matemáticas en la enseñanza secundaria*. Programa Garatu 2006/2007. 2006.