

# MCA 2021

Congreso Matemático de las Américas

**12-23 Julio**

Mathematical Congress of the Americas

**12-23 July**

**Buenos Aires** Argentina

## Symbolic vs Numerical: Allies or Enemies?

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& Josué Tonelli-Cueto (Inria Paris/IMJ-PRG)

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**Sesión especial / Special Session**

*Symbolic and Numerical Computation  
with Polynomials*

Miércoles / Wednesday 14, 13:00-16:00

Viernes / Friday 16, 11:00-16:00

Miércoles / Wednesday 21, 16:00-21:00

# POLYNOMIALS EVERYWHERE I

Polynomials are used for modelling many phenomena:

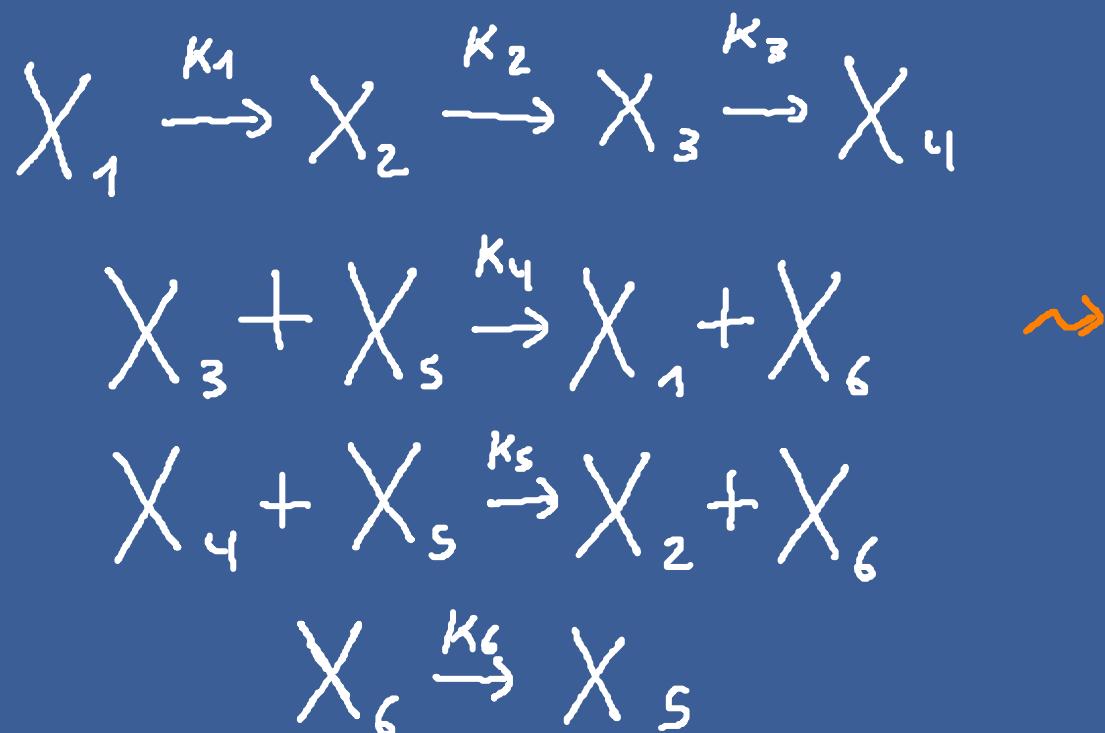
applied

&

theoretical

# EXAMPLE 1: CHEMICAL REACTION NETWORKS

Two-component system  
with hybrid histidine kinase

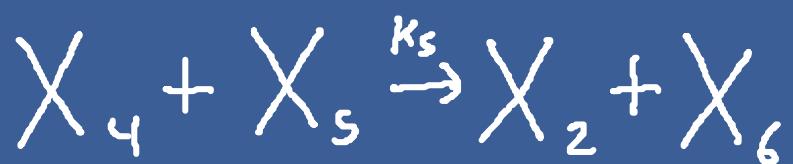
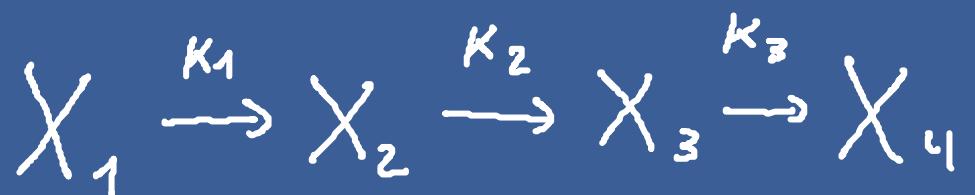


ODE of concentrations

$$\left\{
 \begin{aligned}
 \dot{X}_1 &= -K_1 X_1 + K_4 X_3 X_5 \\
 \dot{X}_2 &= K_1 X_1 - K_2 X_2 + K_5 X_4 X_5 \\
 \dot{X}_3 &= K_2 X_2 - K_3 X_3 - K_4 X_3 X_5 \\
 \dot{X}_4 &= K_3 X_3 - K_5 X_4 X_5 \\
 \dot{X}_5 &= -K_4 X_3 X_5 - K_5 X_4 X_5 + K_6 X_6 \\
 \dot{X}_6 &= K_4 X_3 X_5 + K_5 X_4 X_5 - K_6 X_6
 \end{aligned}
 \right.$$

# EXAMPLE 1: CHEMICAL REACTION NETWORKS

Two-component system  
with hybrid histidine kinase



ODE of concentrations

$$\left\{ \begin{array}{l} \dot{0} = -K_1 X_1 + K_4 X_3 X_5 \\ \dot{0} = K_1 X_1 - K_2 X_2 + K_5 X_4 X_5 \\ \dot{0} = K_2 X_2 - K_3 X_3 - K_4 X_3 X_5 \\ \dot{0} = K_3 X_3 - K_5 X_4 X_5 \\ \dot{0} = -K_4 X_3 X_5 - K_5 X_4 X_5 + K_6 X_6 \\ \dot{0} = K_4 X_3 X_5 + K_5 X_4 X_5 - K_6 X_6 \end{array} \right.$$

For which  $K_i$  we have more than one equilibrium (i.e. positive zero)?

More in: Bihau, Dickenstein & Giaroli. Lower bounds for positive roots and regions of multistationarity in chemical reaction networks.

## EXAMPLE 2. ALGEBRAIC VISION



$$(x_1, y_1) \in \mathbb{RP}^2 \times \mathbb{RP}^2$$
$$(x_2, y_2) \in \mathbb{RP}^2 \times \mathbb{RP}^2$$
$$(x_3, y_3) \in \mathbb{RP}^2 \times \mathbb{RP}^2$$
$$(x_4, y_4) \in \mathbb{RP}^2 \times \mathbb{RP}^2$$
$$(x_5, y_5) \in \mathbb{RP}^2 \times \mathbb{RP}^2$$

X

Y

Essential matrix:  $E \in \mathbb{R}^{3 \times 3}$  (gives relative orientations)

$$2EE^T E = \text{tr}(EE^T)E \quad \det E = 0 \quad y_i^T E x_i = 0$$

More in: Nister. An efficient solution to the five-point relative pose problem.

# POLYNOMIALS EVERYWHERE II

(Theoretical)  
Questions

Applications

Problems with polynomials

How  
to understand  
them?

How  
to solve them?

GEOMETRY OF POLYNOMIALS

Commutative algebra

Algebraic geometry

Differential geometry ...

# A CASE STUDY FOR COMPUTATION: POLYNOMIAL SYSTEM SOLVING

How is  
THE SYSTEM  
GIVEN?

coefficients

$\mathbb{C}$     $\mathbb{Q}$     $\mathbb{Q}_{\text{cp}}$   
 $\mathbb{R}$    ...  
 $\mathbb{F}_p$

exact  
or approximate?

Cost of  
operation  
bit arithmetic

structured  
dense   sparse

WHAT MEANS TO FIND A ZERO?

Zeros in  
 $\mathbb{C}^n$     $\mathbb{R}^n$   
 $\mathbb{R}_+^n$     $\mathbb{Q}^n$   
...

HOW MUCH  
DOES COMPUTATION

COST?

Complexity framework

worst-case   condition-based

probabilistic

how to represent it?

approximation

RUR

a la Smale   Floating  
isolating point  
boxes

algebraic  
extensions  
(Gröbner  
bases)

WARNING:  
NOT EVERYTHING IS FINDING ZEROS!

Existence  
of solutions

Computing  
topological  
invariants

Elimination

Implicitization

Counting  
solutions

Computing  
degree/dimension

# SYMBOLIC vs. NUMERICAL

exact  
vs.  
approximate

input operations  
output operations

implicit assumptions  
of computation

cost  
bit vs. arithmetic

Complexity framework

worst-case

condition-based

probabilistic

# A ROADMAP TO THE SESSION

Existence  
of Solutions

Dickenstein  
Sombra

Elimination/  
Implicitization

D'Andrea Krick  
Tsigaridas

Computing  
over p-adics

Rojas

Solving Structured  
Polynomial Systems

Buse' Regan

Homotopy Continuation

Cucker Rodriguez

Malajovich Walker

Computing  
over  $\mathbb{R}$

Szanto Yap

Numerical  
Complexity Theory

Burr Carrasco

Positive dimensional  
systems

Herrero Sottile

Ergür

Spheres; Etayo

Applications

Bürgisser Garrote Jeronimo



And for the Europe-based researchers,  
one advice for the session...

