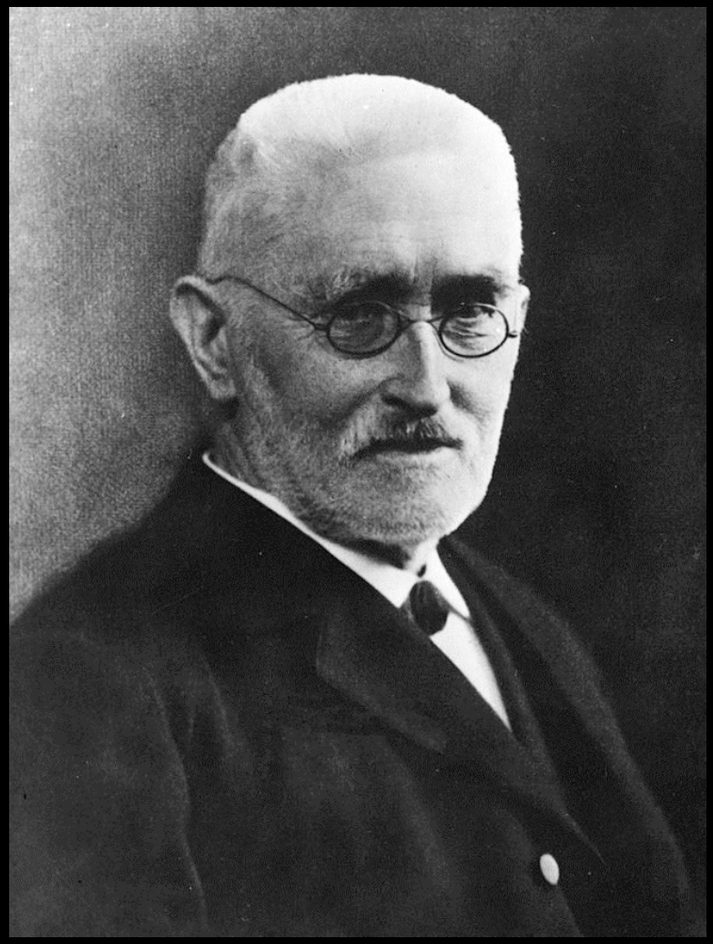
Fact of the Day



Today, October 6, Richard Dedekind would have turned 191 years old

Slides available at:
tonellicueto.xyz/pdf/8IMM_slides.pdf

8TH IBERIAN MATHEMATICAL MEETING
Sevilla, Spain

Why does the DESCARTES Solver work?

Alperen A. ERGÜR
Josué TONELLI-CUETO
Elias TSIGARIDAS

UTSA The University of Texas at San Antonio™

Inria

IMJ-PRG

Photo while working on this project

Real Root Isolation I: The Problem

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Q: Can we beat the champion?

Real Root Isolation III:

What do we wish?

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$$\tilde{\sigma}_B(d\tau)$$

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We wish to find real roots
almost as fast as we read the polynomial!

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Rule of Signs

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Portrait by Frans Hal
Source: WikiMedia Commons

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$(0,\infty) \to (a,b)$ bijection

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Descartes Solver III:

The Algorithm

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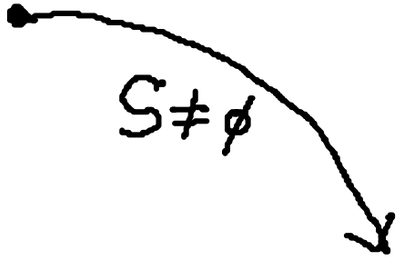

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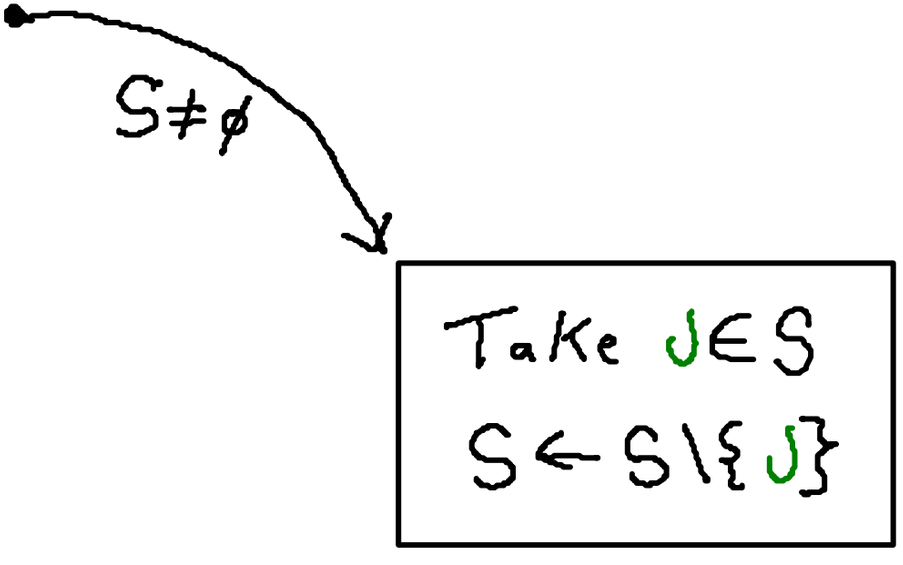

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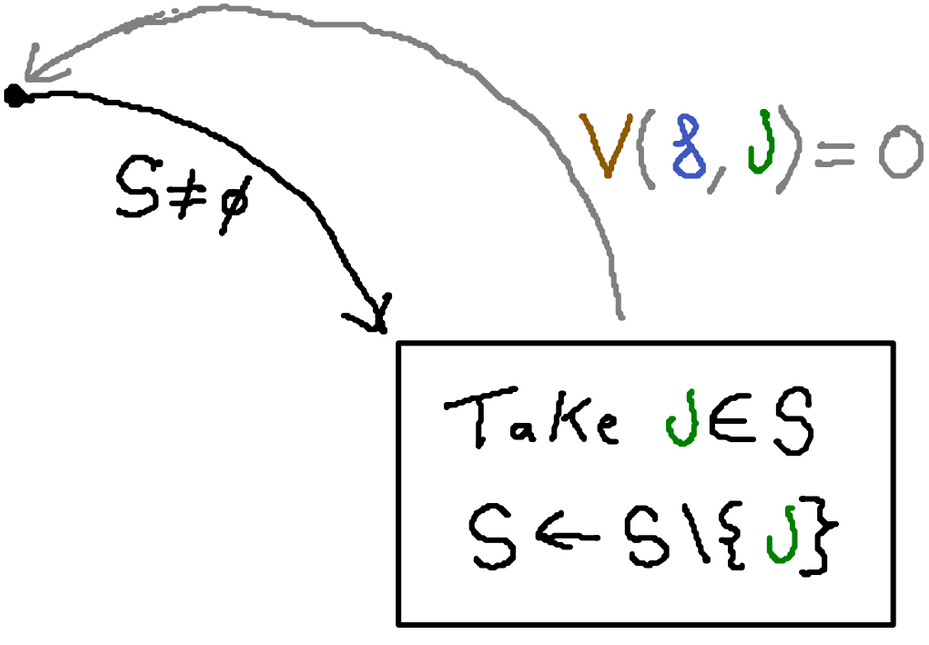

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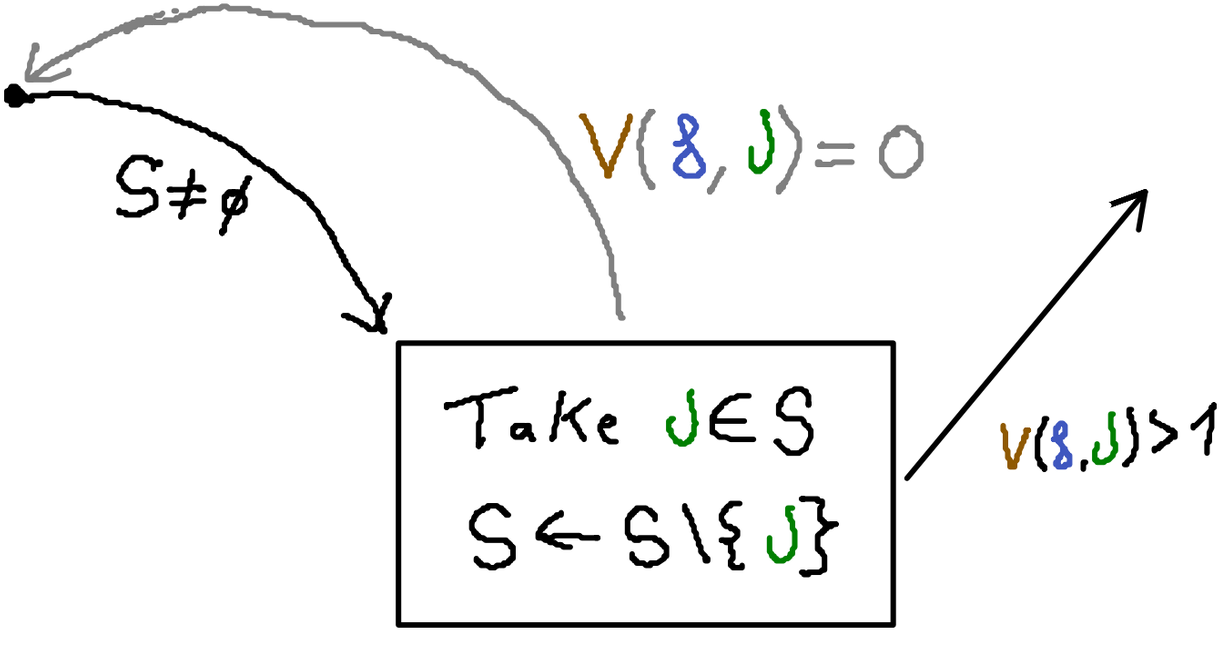

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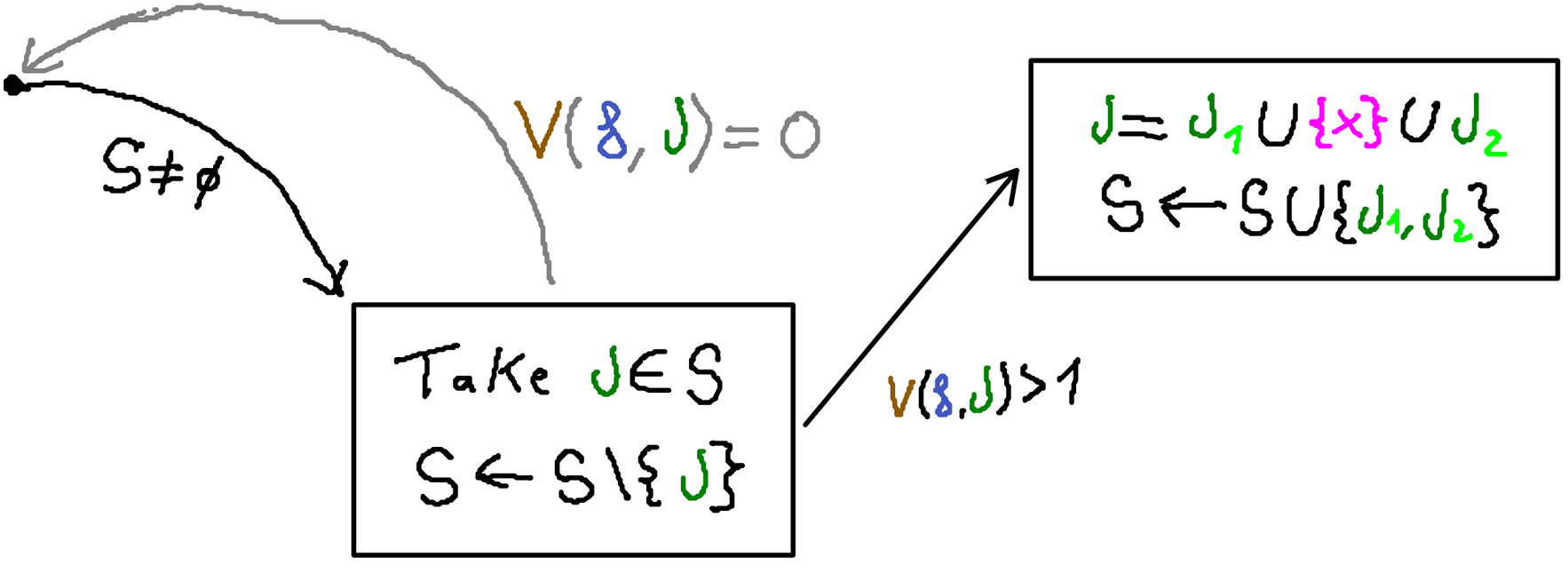

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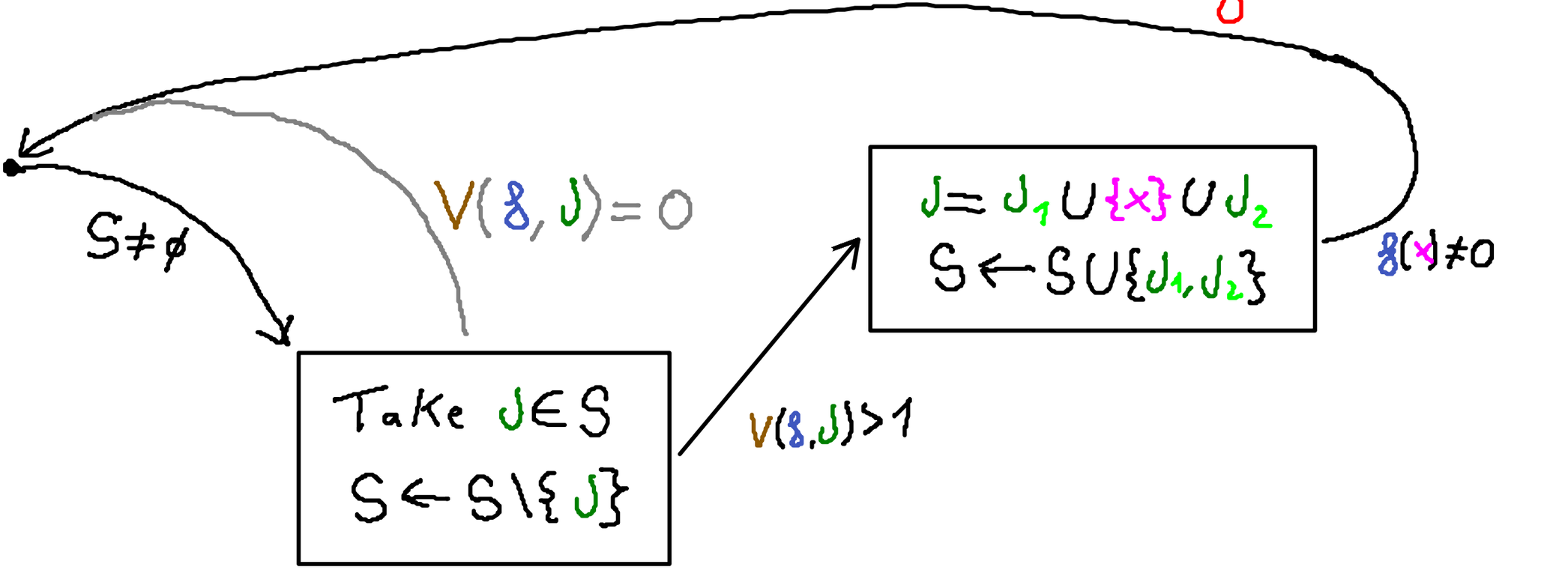

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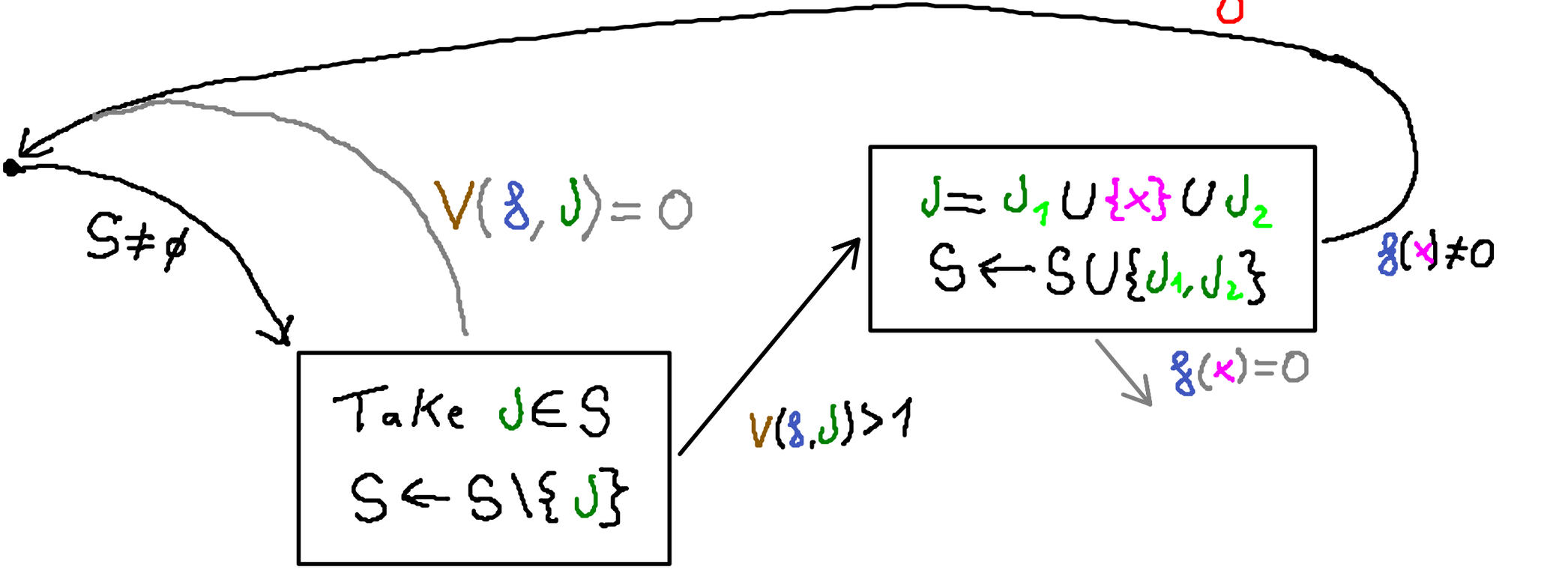

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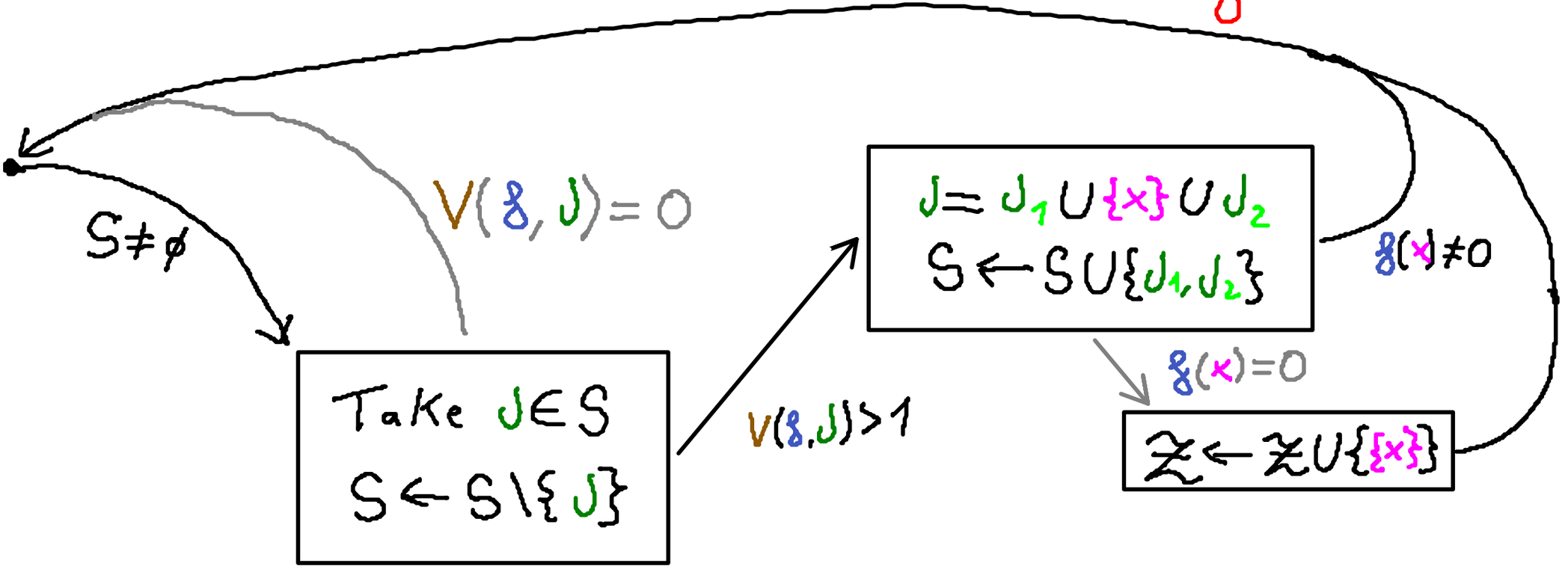

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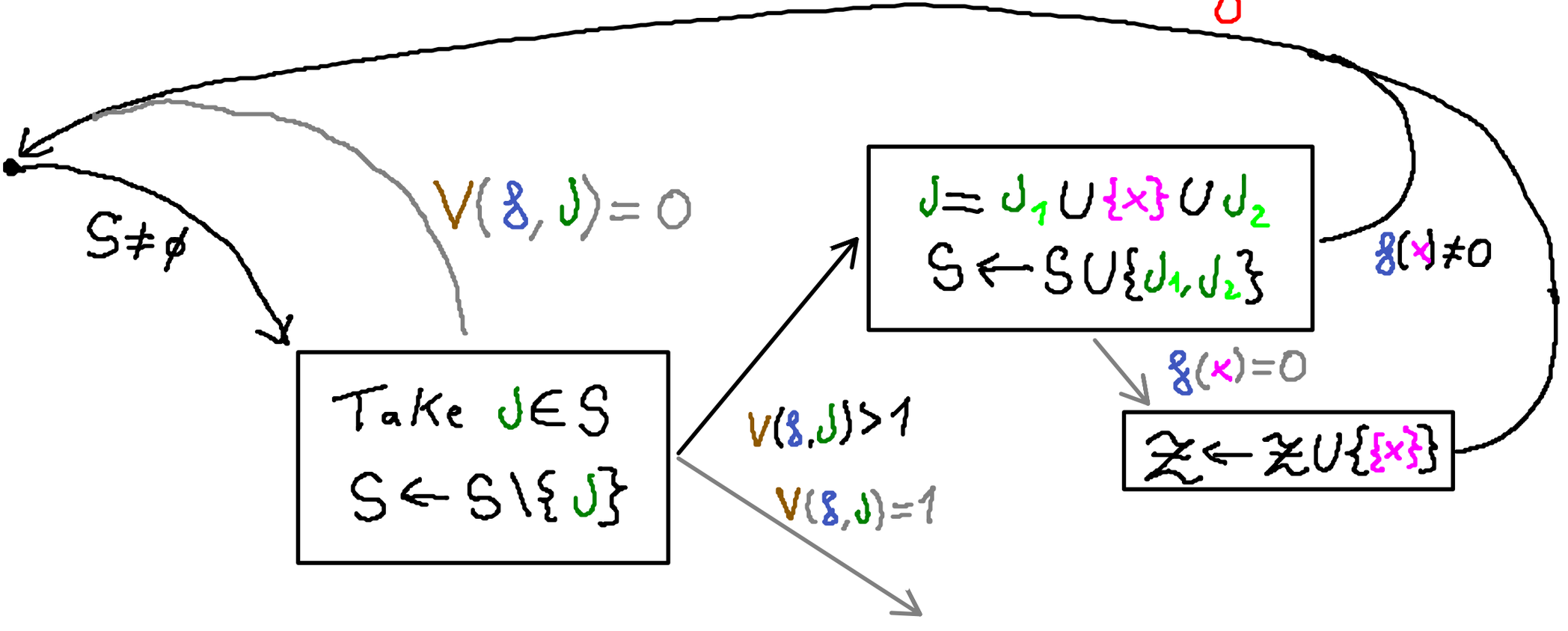

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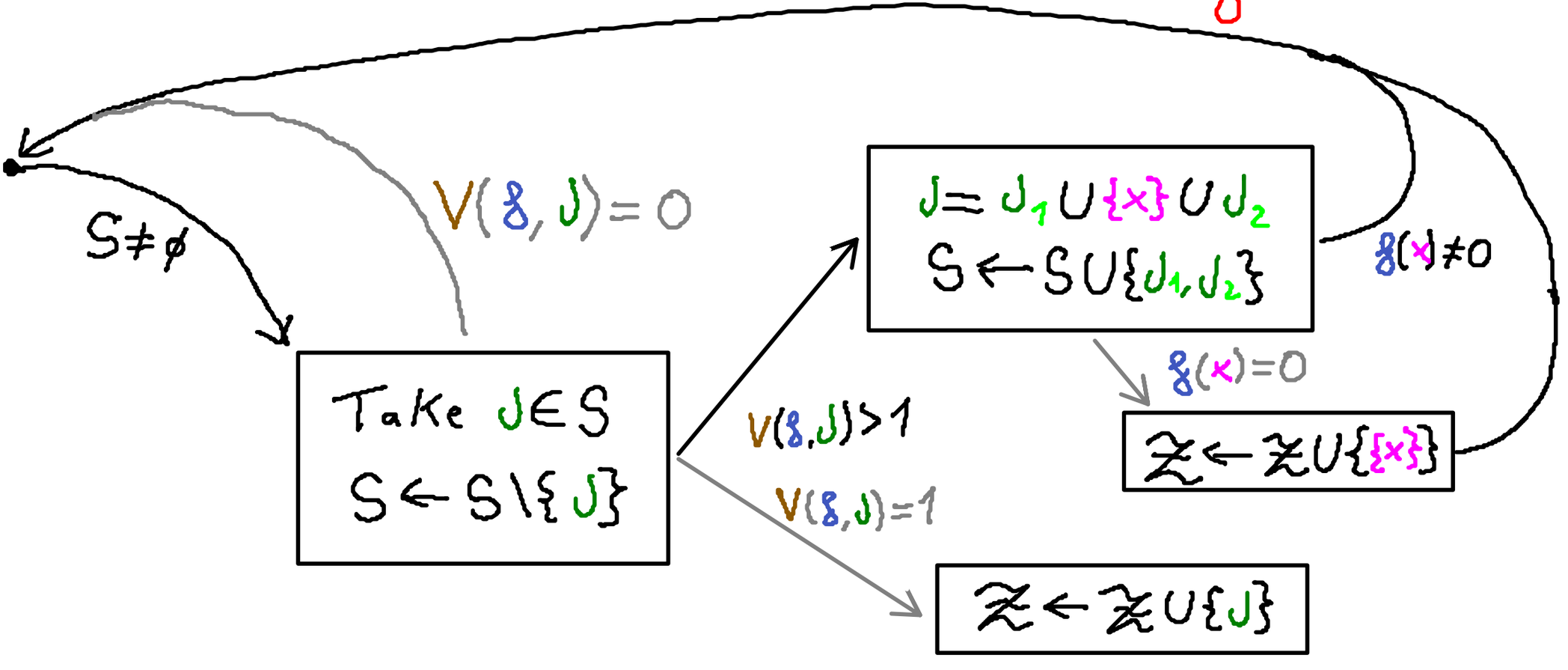

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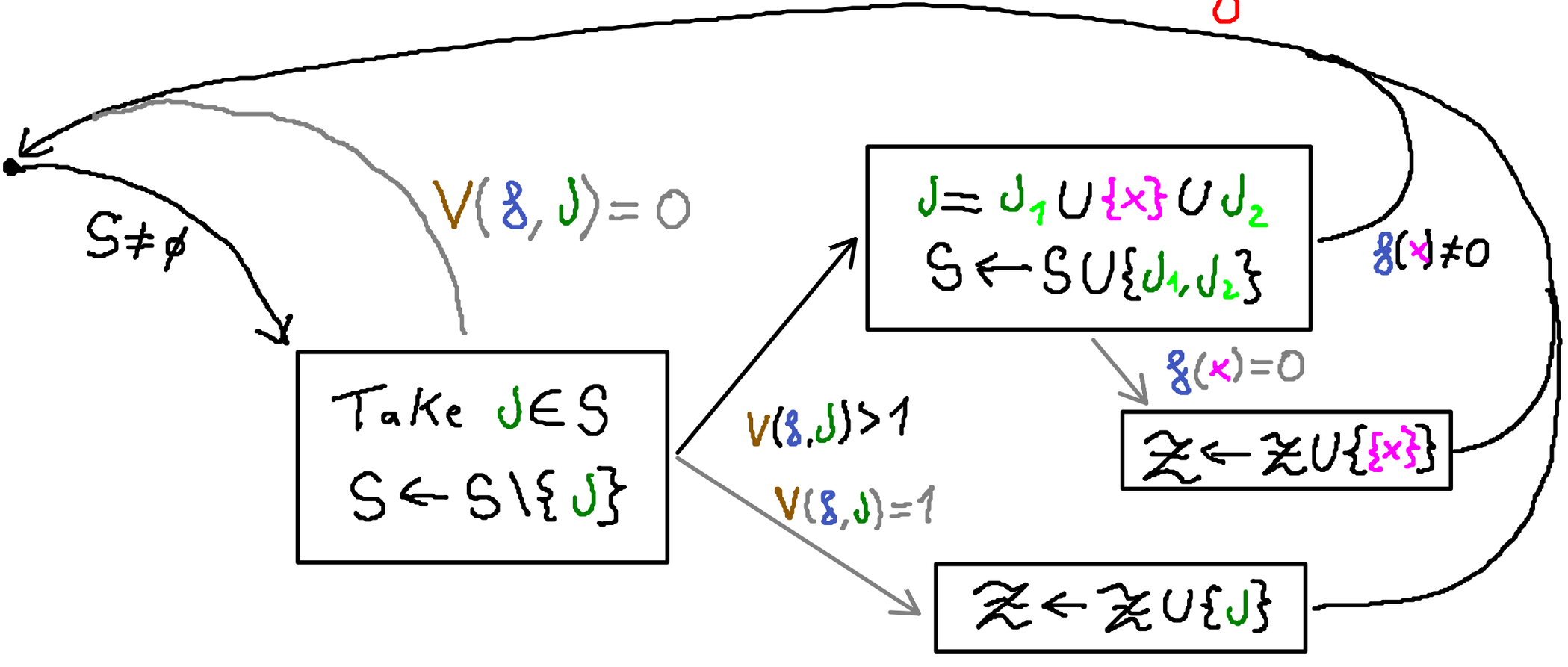

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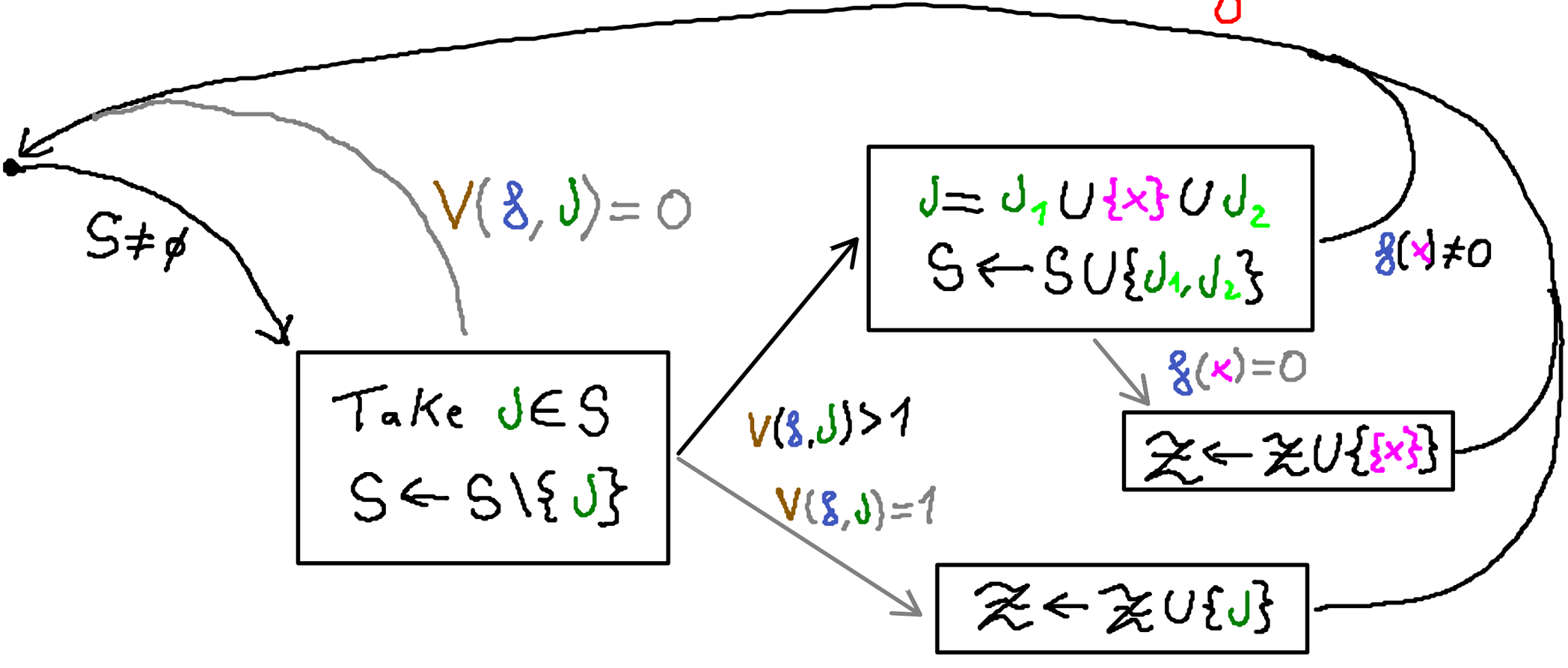

Subdivide until for all J,

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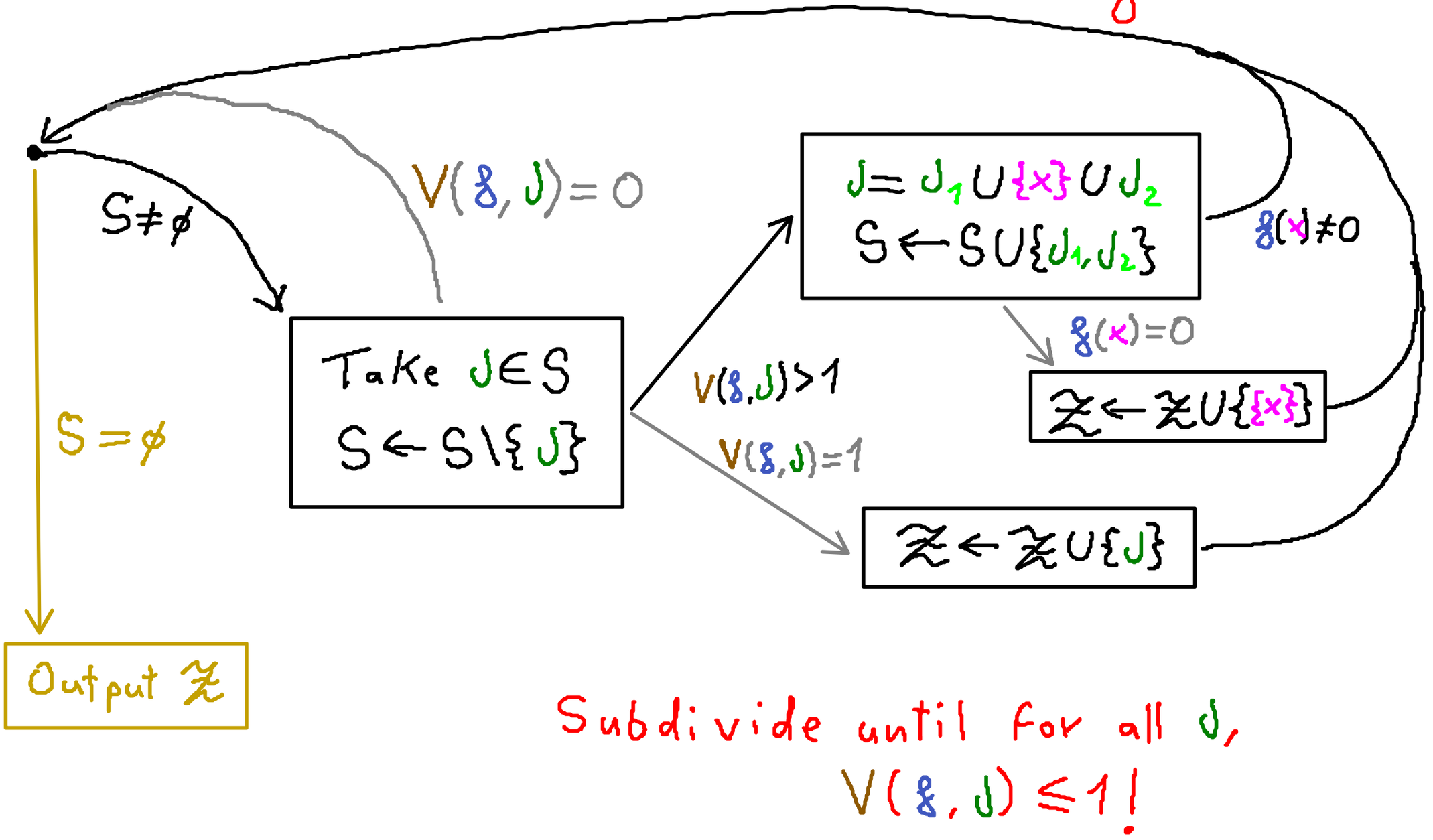

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Are we being pessimistic?

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$$\max\{\text{cost}(\text{SOLVER}, f) \mid \text{bit-size}(f) \leq \tau, \deg(f) \leq d\}$$

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DESCARTES SOLVER seems to behave faster in practice!

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¿Can we explain this?

Real Root Isolation V:

Beyond pessimism

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What's a 'good' random model for f?

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Many choices of randomness 😱

Beyond pessimism I:
Uniform Random Bit Polynomials
& A Simple Main Theorem

Beyond pessimism I: Uniform Random Bit Polynomials & A Simple Main Theorem

$$F = \sum_{k=0}^{d} F_k X^k$$

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s.t. $f_k \sim \mathcal{U}([-2^r, 2^r] \cap \mathbb{Z})$ independent

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SIMPLE MAIN THM (Ergür, T-C, Tsigaridas)

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SIMPLE MAIN THM (Ergür, T-C, Tsigaridas)

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On average, DESCARTES is almost optimal!

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$$w(F) := \max\{\mathbb{P}(f_k = c) \mid c \in \mathbb{R}, k \in \{0, 1, d-1, d\}\}$$

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No middle indexes!

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On average, Descartes is almost optimal!

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Our random model is flexible!

LOTS OF DETAILS
WILL BE OMITED

How to bound the run-time of DESCARTES?

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$\Upsilon(\mathcal{S})$ ← DESCARTES' Computation Tree

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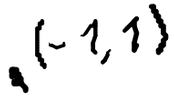

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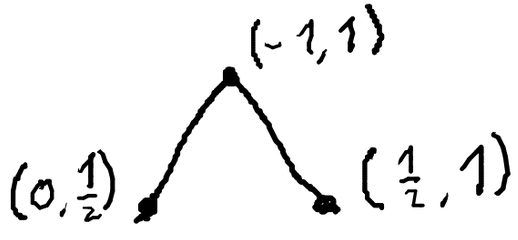

How to bound the run-time of DESCARTES?

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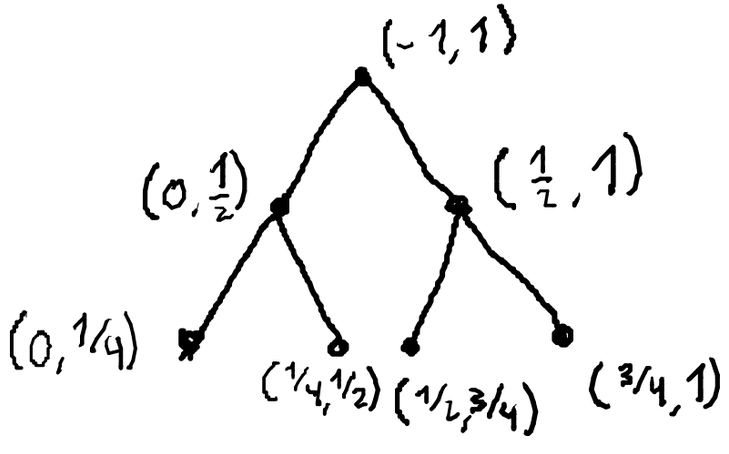

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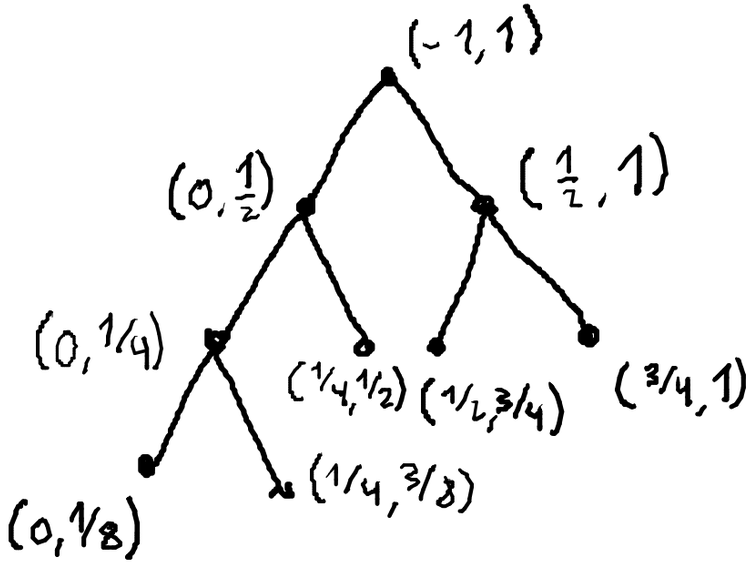
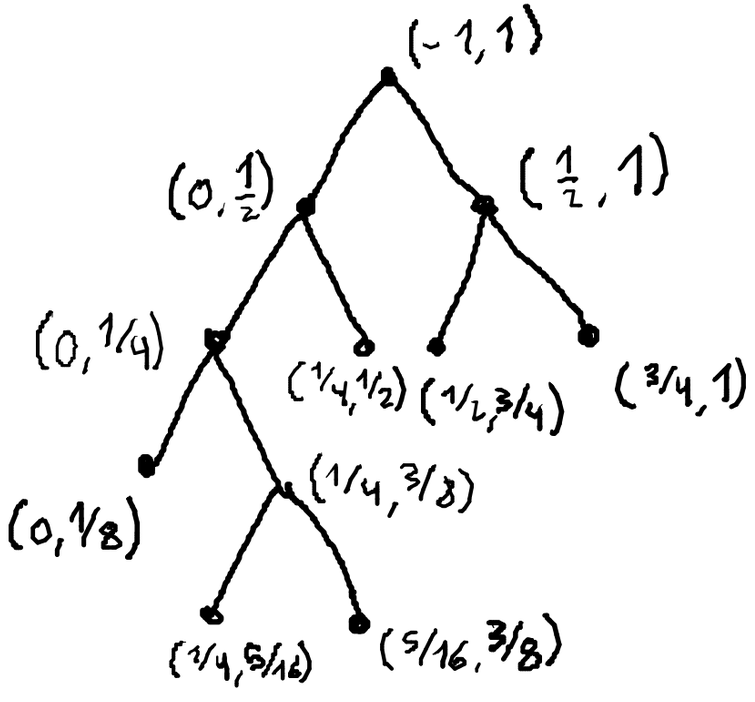


How to bound the run-time of DESCARTES?

$T(8) \leftarrow$ DESCARTES' Computation Tree



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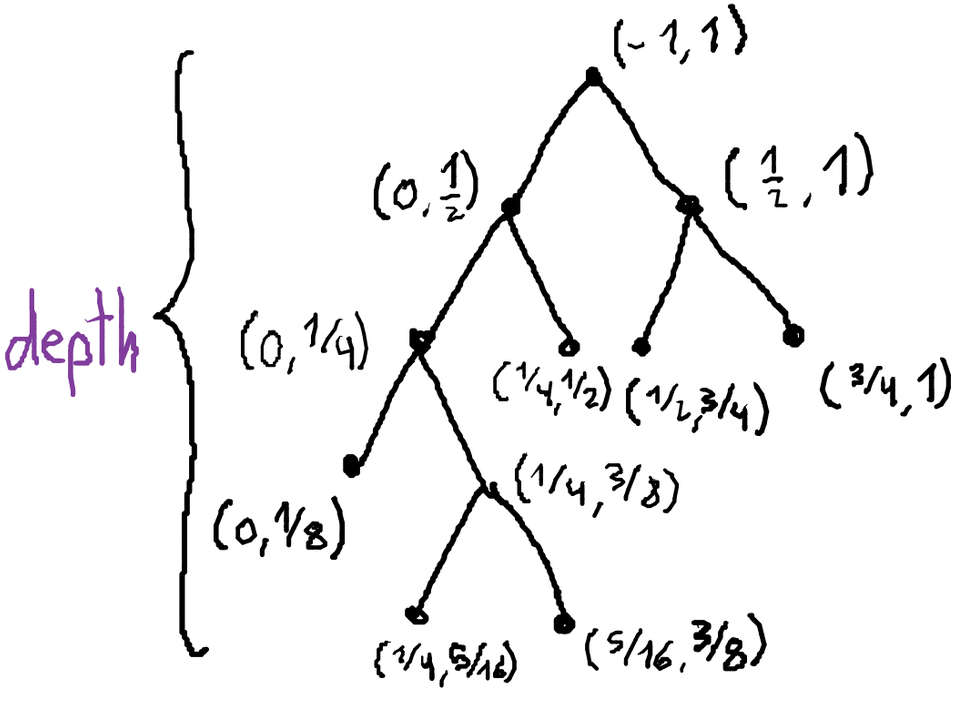
$\Upsilon(\delta) \leftarrow$ DESCARTES' Computation Tree



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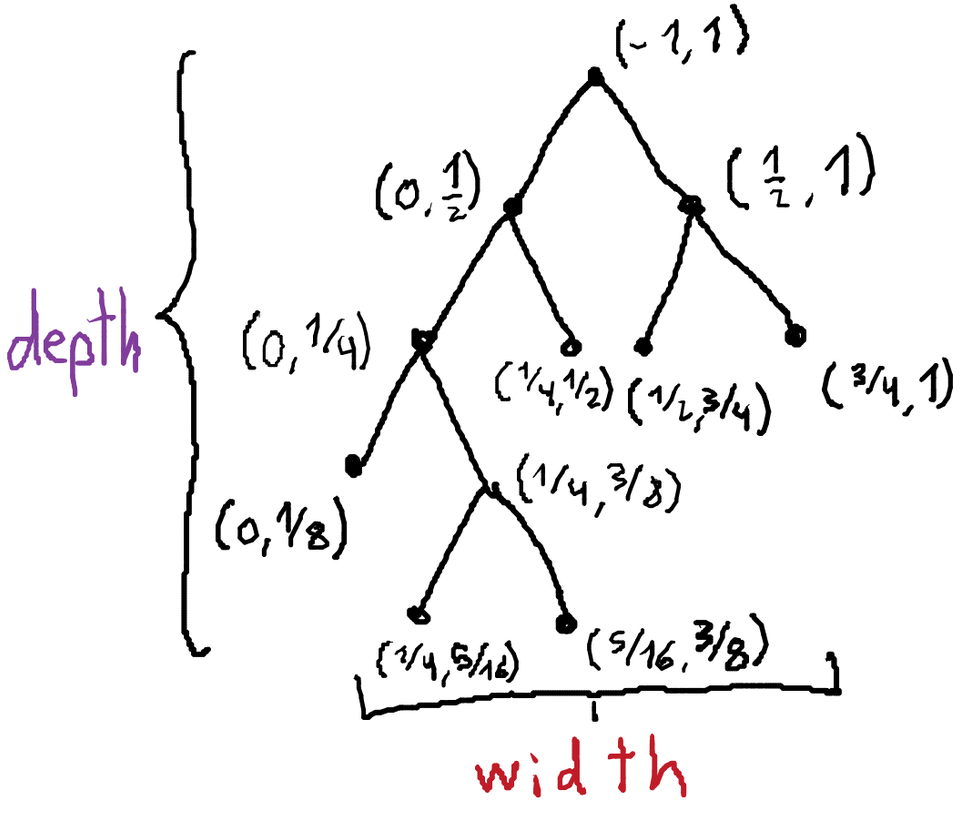

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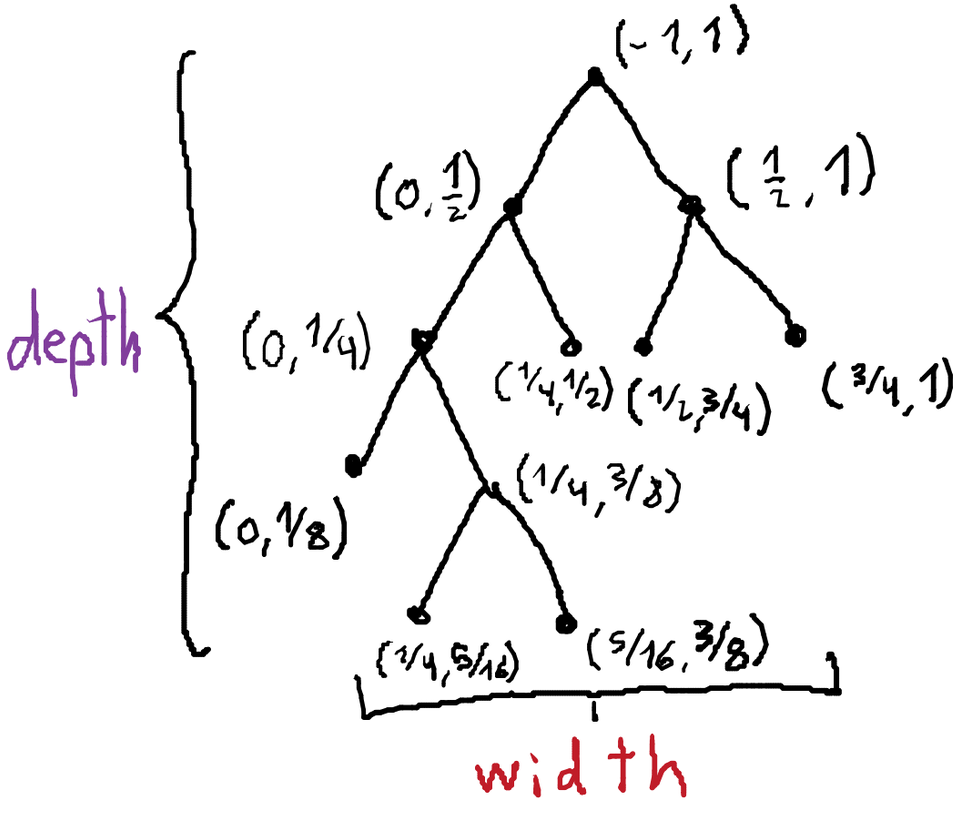
$\Upsilon(\delta) \leftarrow$ DESCARTES' Computation Tree

PROP.

cost(DESCARTES, δ)



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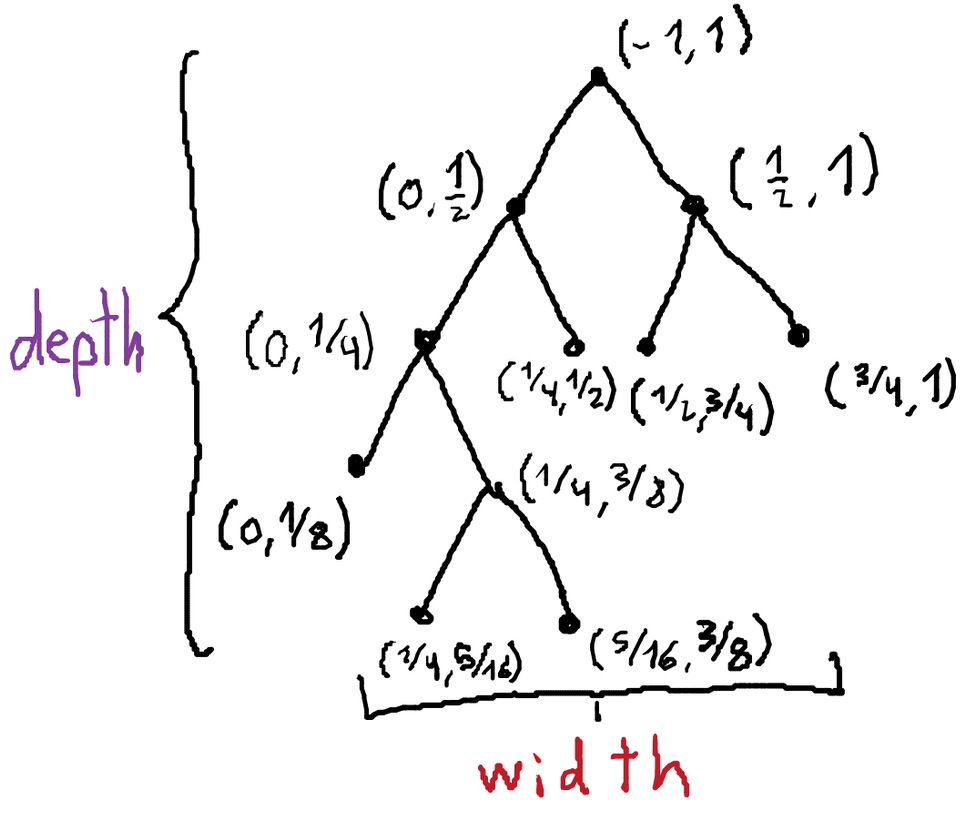
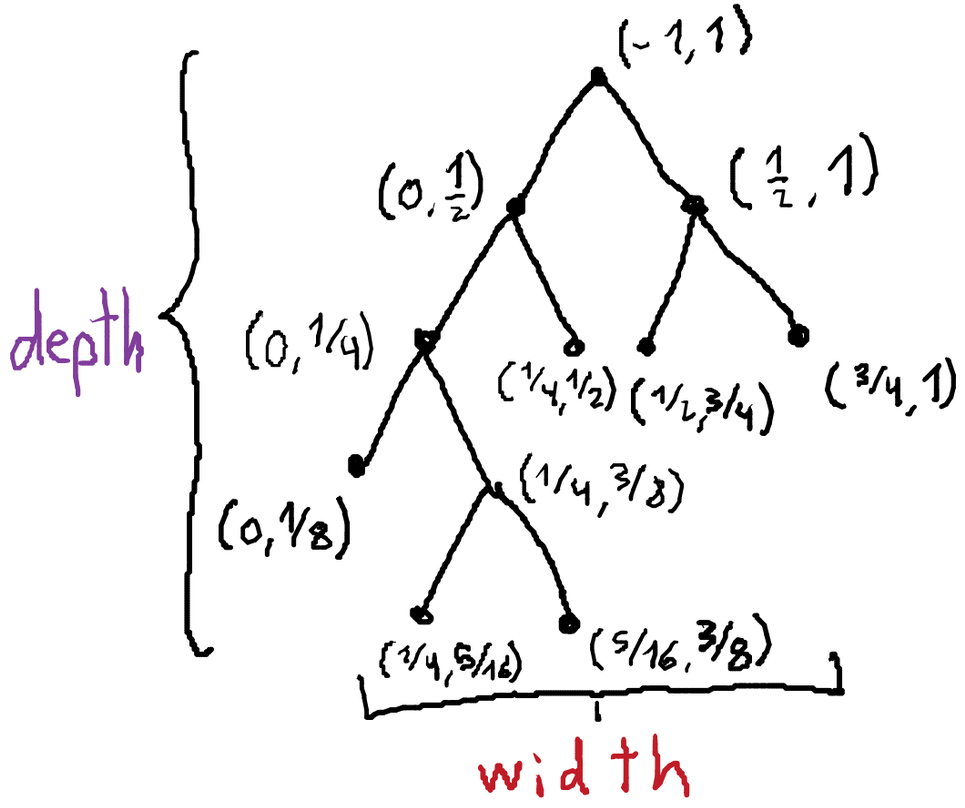


PROP.

$$\text{cost}(\text{DESCARTES}, \mathcal{S}) \;/\!/\; O(d\,\mathrm{width}\,\mathcal{T}(\mathcal{S})\,\mathrm{depth}\,\mathcal{T}(\mathcal{S}) + d^2\,\mathrm{width}\,\mathcal{T}(\mathcal{S})\,\mathrm{depth}^2\,\mathcal{T}(\mathcal{S}))$$

How to bound the run-time of DESCARTES?

$\mathcal{T}(g)$ ← DESCARTES' Computation Tree



PROP.

$$\text{cost}(\text{DESCARTES}, g) \in O(d\,\mathcal{T}\text{width}\,\mathcal{T}(g)\,\text{depth}\,\mathcal{T}(g) + d^2\,\text{width}\,\mathcal{T}(g)\,\text{depth}^2\,\mathcal{T}(g))$$

I.e. size of $\mathcal{T}(g)$ bounds run-time of DESCARTES!

The Ingredients of the Analysis I: Condition Number

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$$C(g) := \max_{x \in [-1,1]} \frac{\sum_{k=0}^{d} |g_k|}{\max\{|g(x)|, |g'(x)|/d\}}$$

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$\frac{1}{C(g)} \sim$ How much I have to perturb the coefficients of g so that $\tilde{g}$ has a singular root in $[-1,1]$

Bounding depth $\gamma(g)$

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Prop.

$$\text{depth}\ \gamma(g) \leq 5 + \log d + \log C(g)$$

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sep. between complex roots of g near by $[-1, 1]$

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$\wedge\!\wedge$

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Here no $\rightarrow$ $1/2^{\text{depth } \gamma(g)}$

Bounding width $\gamma(8)$

Bounding width $\gamma(\delta)$

PROP

width $\gamma(\delta)$

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PROP

width $\gamma(\delta)$

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Secret tool to bound RHS:

Bounding width $\gamma(\delta)$

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Secret tool to bound RHS:

Titchmarsh's thm.

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... but we don't have a continuous dist.

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Ball's Smoothing Trick

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We can use our old cont. toolbox!

Summing Up:

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Descartes' Solver

is almost optimal on average!

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... and that's why Descartes works so well

Muito Obrigado
pela Atenção!

¡Muchas Gracias
por su Atención!