Kushnirenko's Fewnomials, the Number of $\mathbb{R}$eal Zeros & Condition Numbers

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What is a fewnomial?

A *fewnomial* is a polynomial with few monomials.

The polynomial
$$1 - T + 7T^5 + T^{1978}$$
has degree 1978, but, despite this, it has very **few** monomials, so it is an example of a fewnomial.

Why don't we say oligonomial?

polynomial → poly-nomial → много-члены →
много члены ⟶ мало члены
(many monomials) (few monomials)
→ мало-члены → few-nomial → fewnomial

Kushnirenko Hypotheses

Kushnirenko Hypothesis I

Topological complexity (e.g., Betti numbers) of the zero set of a real polynomial system can be controlled by the complexity of such a system (e.g., number of monomials) rather than by the degree (or the Newton polygon)

Kushnirenko Hypothesis II

The number of real zeros of a general real polynomial system
$$f_1(X_1,\dots,X_n) = \cdots = f_n(X_1,\dots,X_n) = 0$$
can be bounded from above by the total number of nonzero terms of $f_1,\dots,f_n$.

Kushnirenko Hypothesis III (corrected)

The number of nondegenerate real zeros of a real polynomial system
$$f_1(X_1,\dots,X_n) = \cdots = f_n(X_1,\dots,X_n) = 0$$
can be bounded from above by
$$O\left(\prod_{i=1}^{n} \#\mathrm{supp}(f_i)\right).$$

Kushnirenko Hypothesis IV

The number of nondegenerate real zeros of a real polynomial system
$$f_1(X_1,X_2) = f_2(X_1,X_2) = 0$$
can be bounded by $Z(\deg f_1, \#\mathrm{supp}\, f_2)$, where Z is some universal function.

Why hypotheses?

In some Slavic languages, "hypothesis" is also used as "conjecture"

Examples

Descartes' rule of signs: A real univariate polynomial f has at most $1 + 2\#\,\mathrm{supp}\, f$ real roots.

For a generic choice of coefficients, the system
$$\begin{cases} \alpha_1 + \beta_1 X + \gamma_1 Y + \delta_1 XYZ^d = 0 \\ \alpha_2 + \beta_2 X + \gamma_2 Y + \delta_2 XYZ^d = 0 \\ \alpha_3 + \beta_3 X + \gamma_3 Y + \delta_3 XYZ^d = 0 \end{cases}$$
has always d complex solutions, but at most 2 real solutions.

More on fewnomial history...

Kushnirenko's Letter to Prof. Sottile, and Appendix F of J. Tonelli-Cueto (2019), *Condition and Homology in Semialgebraic Geometry*, PhD thesis, Technische Universität Berlin.

Some Milestones

Sevastyanov (1978): Kushnirenko hypothesis IV is true, but original version of hypothesis III is false.

Khovanskii (1991): Kushnirenko hypotheses I and II are true.

Khovanskii's Theorem (1991)

The number of nondegenerate positive zeros of a fewnomial system $f_1 = \cdots = f_n = 0$ in n variables is at most
$$2^{\binom{t-1}{2}}(n+1)^{t-1}$$
where $t = \#\left(\cup_{i=1}^{n} \mathrm{supp}\, f_i\right)$.

Bihan, **Sottile** (2007): Improvement of Khovanskii's bound.
Bihan, **Sottile**, **Rojas** (2008): Improvement of Khovanskii's bound for number of connected components
Bihan, **Sottile** (2009): Improvement of Khovanskii's bound for sum of Betti numbers

Bihan, Sottile (2007)

The number of nondegenerate positive zeros of a fewnomial system $f_1 = \cdots = f_n = 0$ in n variables is at most
$$3 \cdot 2^{\binom{t-n-1}{2}} n^{t-n-1}$$
where $t = \#\left(\cup_{i=1}^{n} \mathrm{supp}\, f_i\right)$.

Koiran, **Portier**, **Tavenas** (2015): Sevastyanov's lost theorem is reproven
Bürgisser, **Ergür**, **Tonelli-Cueto** (2018): A probabilistic Kushnirenko hypothesis III is proven

Bürgisser, Ergür, Tonelli-Cueto (2018)

Let $A \subset \mathbb{N}^n$ of size t and $\mathfrak{f}_i = \sum_{\alpha\in A} \mathfrak{f}_{i,\alpha} X^\alpha$ random polynomials in n variables such that the $\mathfrak{f}_{i,\alpha}$ are independent centered Gaussian whose variance only depends on $\alpha \in A$. Then
$$\mathbb{E}_{\mathfrak{f}} \#\mathcal{Z}_r(\mathfrak{f}, \mathbb{R}^n_+) \le \frac{1}{2^{n-1}}\binom{t}{n}$$
where $\mathcal{Z}_r(\mathfrak{f}, \mathbb{R}^n_+)$ is the set of nondegenerate zeros of $\mathfrak{f}_1 = \cdots = \mathfrak{f}_n = 0$ in $\mathbb{R}^n_+$.

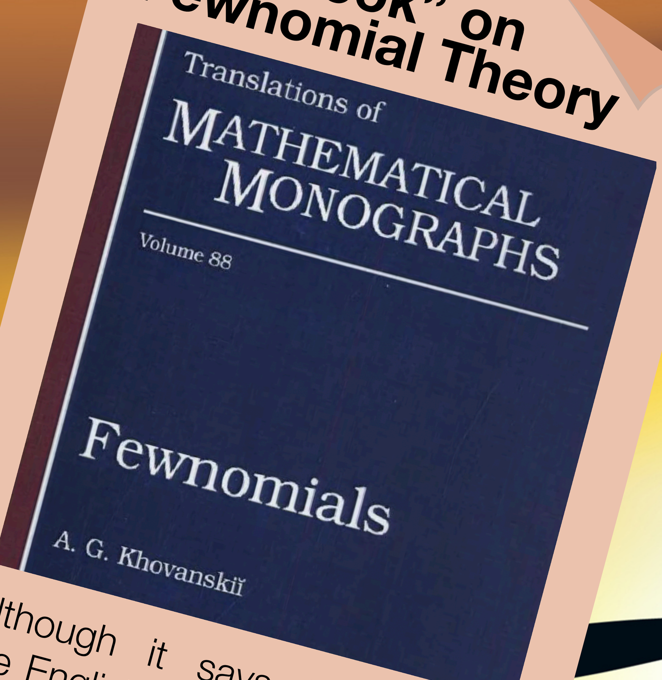
Probabilistic Approach

Probabilistic Reformulation

Let $\mathfrak{f}_i = \sum_{\alpha\in A} \mathfrak{f}_{i,\alpha} X^\alpha$ be random polynomials in n variables with some absolutely continuous distribution whose density does not vanish. Then
$$\sup_{\ell} \left(\mathbb{E}_{\mathfrak{f}} \#\mathcal{Z}_r(\mathfrak{f}, \mathbb{R}^n)^\ell\right)^{\frac{1}{\ell}} = \max_{f} \#\mathcal{Z}_r(f, \mathbb{R}^n)$$
where $\mathcal{Z}_r(f)$ is the set of nondegenerate zeros of $f_1 = \cdots = f_n = 0$ in $\mathbb{R}^n$.

Proof idea: A fewnomial system with maximum number of nondegenerate zeros is stable.

The "book" on Fewnomial Theory



Although it says "translation", the English edition precedes the Russian one.

Condition Number

Let $f = (f_1,\dots,f_n)$ be a real polynomial system in n variables with f_i of degree at most d_i, its *condition number* is
$$\mathrm{C}(f) := \sup_{x\in[-1,1]^n} \frac{\|f\|}{\max\{\|f(x)\|, \|D_x f^{-1}\Delta\|^{-1}\}}$$
where $\Delta := \mathrm{diag}(d_1,\dots,d_n)$

How to think about the condition number?

Roughly, $\mathrm{C}(f)$ is the inverse of the distance to the discriminant variety.

A new bound!

Main Theorem (T.-C., Ts.; '22 +)

Let $f = (f_1,\dots,f_n)$ be a real polynomial system in n variables. Then
$$\#\mathcal{Z}(f, [-1,1]^n) \le O\left(\log \mathbf{D} \max\{n \log \mathbf{D}, \log \mathrm{C}(f)\}\right)^n$$
where $\mathbf{D}$ is the maximum degree.

Corollary:
Well-posed Real Polynomial Systems
have Few Real Zeros

Probabilistic Consequences

Prob. Theorem (T.-C., Ts.; '22 +)

Let $\mathfrak{f} = (\mathfrak{f}_1,\dots,\mathfrak{f}_n)$ be a random real fewnomial system in n variables whose coefficients are independent and uniformly distributed in $[-1,1]$. Then
$$\mathbb{E}_{\mathfrak{f}} \#\mathcal{Z}_r(\mathfrak{f}, \mathbb{R}^n)^\ell \le O\left(n\ell \log^2 \mathbf{D}\right)^{n\ell}$$
where $\mathcal{Z}_r(\mathfrak{f}, \mathbb{R}^n)$ is the set of nondegenerate real zeros of $\mathfrak{f}_1 = \cdots = \mathfrak{f}_n = 0$, and $\mathbf{D}$ is the maximum degree.

Corollary:
Fewnomial systems with many zeros
are very improbable

More general...

We can cover a wide range of probabilistic assumptions

Algorithmic Consequences

Proof is fully constructible!
Issue: Computing $\mathrm{C}(f)$ is expensive

Alg. Theorem (T.-C., Ts.; '22 +)

There is a explicit partition $\mathcal{B}$ of $[-1,1]^n$ into $O(\log \mathbf{D})^n$ boxes such that for all real polynomial system $f = (f_1,\dots,f_n)$ in n variables of degree at most $\mathbf{D}$ and all $B \in \mathcal{B}$, there is a polynomial
$$\phi_{f,B}$$
of degree $O(\max\{n \log \mathbf{D}, \log \mathrm{C}(f)\})$ such that
$$\#\mathcal{Z}(f, B) \le \#\mathcal{Z}(\phi_{f,B}, \mathbb{R}^n).$$
Moreover, every real zero of f in B has a zero of $\phi_{f,B}$ that converges quadratically to it under Newton's method.

Proof idea: Well-conditioned polynomials are fast converging Taylor series

On the sphere...

We can cover also random polynomial system with the Weyl scaling. However, in this case, we are unable to cover fewnomials, and we get probabilistic bounds of the form $O\left(\sqrt{\mathbf{D}} \log \mathbf{D}\right)^n$ appears in the bound.