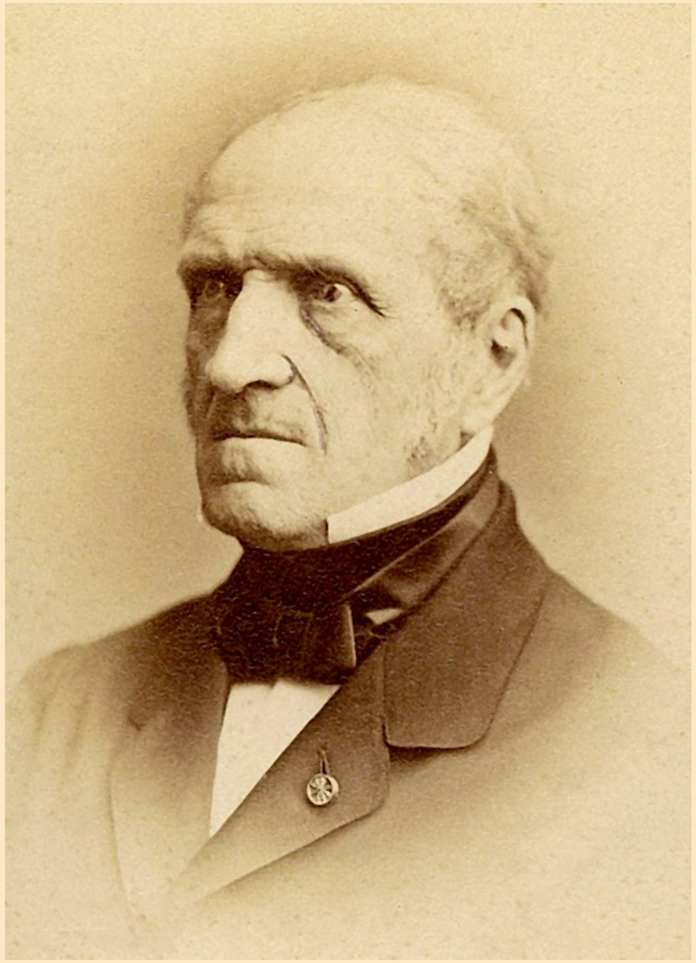
On this day...

the mathematian

Michel Chasles

would have

his 229th birthday



Famous for Chasles' identity:

$$\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac}$$

A whole career reduced to this!

Symbolic Computation Seminar
NCSU

Computing Numerically the Homology of a Semialgebraic Set

Josué Tonelli-Cueto (UT San Antonio)

Why do we care?

Central problem
in computational
semi alg. geometry

(with applications)

THE THEOREM:

There is an algorithm Hom that given a q-tuple $p \in \prod_{i=1}^{q} \mathbb{R}[X_1, \dots, X_n]_{\leq d_i}$ of polynomials and a semialgebraic formula Φ over p of size s, it computes the homology groups of

$$S_{\text{aff}}(p, \Phi) \subseteq \mathbb{R}^n$$

semialg. set defined by (p, Φ)

in time

$$\mathcal{O}\left(qs\left(128\, n\, D\, \overline{\kappa}_{\text{aff}}(p)\right)^{10n(n+2)}\right)$$

THE THEOREM: (Homogeneous Version)

There is an algorithm Hom that given a q-tuple $\mathfrak{f} \in \prod_{i=1}^{q} \mathbb{R}[X_0, \dots, X_n]_{d_i}$ of homogeneous polynomials and a semialgebraic formula Φ over $\mathfrak{f}$ of size s, it computes the homology groups of

$$S(\mathfrak{f}, \Phi) \subseteq \mathbb{S}^n$$

spherical semialg. set def. by $(\mathfrak{f}, \Phi)$

in time

$$\mathcal{O}\left(q s \left(128\, n D\, \overline{\kappa}(\mathfrak{f})\right)^{10 n(n+2)}\right)$$

THE THEOREM: (Homogeneous Version +)

There is an algorithm Hom_ℓ that given a q-tuple $\mathcal{F} \in \prod_{i=1}^{q} \mathbb{R}[X_0, \dots, X_n]_{d_i}$ of homogeneous polynomials and a semialgebraic formula Φ over $\mathcal{F}$ of size s, it computes the first ℓ homology groups of

$$S(\mathcal{F}, \Phi) \subseteq \mathbb{S}^n$$

spherical semialg. set def. by $(\mathcal{F}, \Phi)$

in time

$$\mathcal{O}\left(qs\left(128\, n\, D\, \overline{\kappa}(\mathcal{F})\right)^{10n(\ell+2)}\right)$$

THE PROB. THEOREM: (Homogeneous Ver)

Given a random KSS q-tuple f,

i.e. $f_i = \sum \binom{d_i}{\alpha}^{1/2} c_{i,\alpha} X^\alpha$ with the

the $c_{i,\alpha} \sim N(0,1)$ i.i.d., then HOM_ℓ

runs with

$$O\left(s\,(qD)^{O(n^2 \ell)}\, \mathfrak{a}^{10n(\ell+1)} \right)$$

with prob $\geq 1 - 1/\mathfrak{a}$

Also smoothed analysis
and more robust probabilistic models

Single exponential in n

with high probability!

Improves state of the art!

Algorithm at a glance

0. Condition number estimation
1. Reduction to Lax Case
2. Reduction to Basic Cases
3. Basic case

CONDITION NUMBER

$$\bar{\kappa}(g,x) := \max_{\substack{I \subseteq [q] \\ \#I \leq n+1}} \frac{\| g_I \|}{\sqrt{\| g_I(x) \|^2 + \sigma_{|I|}\left(\Delta_I^{-1/2} D_x g_I\right)}}$$

where $g_I := (g_i)_{i \in I}$ $\qquad D_x g_I : T_x \mathbb{S}^n \to \mathbb{R}^I$

tangent map

$\Delta = \operatorname{diag}(d_i)_{i \in I}$ $\qquad \sigma_{|I|}$ $|I|$-sing. val.

$$\bar{\kappa}(g) := \max_{x \in \mathbb{S}^n} \bar{\kappa}(g,x) \in [1, \infty)$$

Interpretation of $\bar{\kappa}(\gamma)$

$\bar{\kappa}(\gamma) < \infty$

$\Leftrightarrow$: $\forall i$, $Z_S(\gamma_i)$ smooth hypersur.

& $\forall I \subseteq [q]$,

$$\bigcap_{i \in I} Z_S(\gamma_i) \text{ transversal}$$

Note: If $|I| \geq n+1$, then $\bigcap_{i \in I} Z_S(\gamma_i)$ Transversal is empty

ESTIMATION OF $\bar{\kappa}(\gamma)$

Prop. $S^n \ni x \longmapsto 1/\bar{\kappa}(\gamma,x)$ is D-Lipschitz

Cor. If $G \subseteq S^n$ satisfies

$$d_H(G, S^n) < \varepsilon,$$

& $\max\{\bar{\kappa}(\gamma,x) \mid x \in G\}\, D\, \varepsilon < 1/2$, then

$$\bar{\kappa}(\gamma) \leq 2 \max\{\bar{\kappa}(\gamma,x) \mid x \in G\}$$

Reduction to Lax Case I

Gabrielov-Vorobjov Construnction:

$$\Gamma B_{\delta,\varepsilon}(g,\Phi) = S(g,\tilde{\Phi})$$

where in $\tilde{\Phi}$ $\begin{cases} g_i = 0 \rightsquigarrow |g_i| \leq \varepsilon \|g_i\|_w \\ g_i > 0 \rightsquigarrow g_i \geq \delta \|g_i\|_w \\ g_i < 0 \rightsquigarrow g_i \leq -\delta \|g_i\|_w \end{cases}$

$$\Gamma B_{\underline{\delta},\underline{\varepsilon}}(g,\Phi) = \bigcup_i \Gamma B_{\delta_i,\varepsilon_i}(g,\Phi)$$

Gabrielov-Vorobjov theorem:

If $0 << \varepsilon_1 << \delta_1 << \varepsilon_2 << \delta_2 << \ldots << \delta_m << 1$,

then $\exists\, H_k(\Gamma B_{\underline{\delta},\underline{\varepsilon}}(g,\Phi)) \to H_k(S(g,\Phi))$: iso for $k \leq m-1$

& surj. for $k = m$

Reduction to Lax Case II

Gabrielov-Vorobjov Construnction:

$$\Gamma B_{\delta,\varepsilon}(\gamma, \Phi) = S(\gamma, \tilde{\Phi})$$

where in $\tilde{\Phi}$ $\begin{cases} \gamma_i = 0 \rightsquigarrow |\gamma_i| \leq \varepsilon \|\gamma_i\|_w \\ \gamma_i > 0 \rightsquigarrow \gamma_i \geq \delta \|\gamma_i\|_w \\ \gamma_i < 0 \rightsquigarrow \gamma_i \leq -\delta \|\gamma_i\|_w \end{cases}$

$$\Gamma B_{\underline{\delta},\underline{\varepsilon}}(\gamma, \Phi) = \bigcup_i \Gamma B_{\delta_i,\varepsilon_i}(\gamma, \Phi)$$

Quantitative Gabrielov-Vorobjov theorem:

If $0 < \varepsilon_1 < \delta_1 < \varepsilon_2 < \dots < \delta_m < 1/\sqrt{2}\,\mathcal{K}(\gamma)$

then $\exists\, H_K(\Gamma B_{\underline{\delta},\underline{\varepsilon}}(\gamma, \Phi)) \to H_K(S(\gamma, \Phi))$ iso for $K \leq m-1$

& surj. for $K = m$

Reduction to basic case I

Main Idea:

- $\mathcal{X}_i^{\alpha}$ sample of $S(\delta_i\,\alpha\,0)$
- If for all basis $\phi = \bigwedge_{i\in I} (\delta_i\ \alpha_i\ 0)$ $\bigcap \mathcal{X}_i^{\alpha_i}$ good sample of $S(\delta, \phi)$, then for all semialg. formula Φ, $\Phi(\mathrm{Simp}(\mathcal{X}_i^{\alpha})) \simeq S(\delta, \Phi)$

Reduction to basic case II

Ingredients:

Explicit Functorial Nerve Theorem

Čech

$$\pi : \check{C}_\varepsilon(X) \longrightarrow B_\varepsilon(X) \quad \text{homology-equiv.}$$

$$\sum t_i [x_i] \longmapsto \sum t_i x_i$$

Homological Inclusion-Exclusion Transfer

$$\left.\begin{array}{l} f : X \to Y \text{ cont.} \\ X = \bigcup X_i,\ Y = \bigcup Y_i \\ f(X_i) \subseteq Y_i + \text{Tech.} \end{array}\right\} \quad \begin{array}{l} \forall I,\ f : \bigcap_{i \in I} X_i \to \bigcap_{i \in I} Y_i \\ \text{homology equiv.} \\ \\ \Rightarrow f : X \to Y \text{ hom. equiv.} \end{array}$$

Also hom. iso up to k and surj for $(k+1)$-homology

Basic Case TDA ingredients

Niyogi-Smale-Weinberger Thm

$$\exists\, d_H(\mathcal{X},X) < \varepsilon < \tfrac{1}{2}\tau(X) \Rightarrow B_\varepsilon(\mathcal{X}) \xrightarrow{\sim} X$$

Hausdorff distance

hom. equiv

reach

$$\tau(X) := \inf\{ r>0 \mid \exists\, p \in \mathbb{R}^n, x,\tilde{x} \in X \;\; d(p,X)=d(p,x)=d(p,\tilde{x})=r \;\&\; x \neq \tilde{x} \}$$

Attali-Lieutier-Salinas Thm

$$\exists\, d_H(\mathcal{X},X) < \varepsilon < \frac{1}{5}\tau(X) \Rightarrow VR_\varepsilon(\mathcal{X}) \xrightarrow{\sim} \check{C}_\varepsilon(\mathcal{X})$$

Vietoris-Rips

Čech

Basic Case Approx Results

Reach Bound

$$\tau\left(Y \cap \bigcap X_i\right) \geq \min_{I} \tau\left(Y \cap \bigcap_{i\in I} \partial X_i\right)$$

Reduction to reach of boundaries

ϕ basic semialg. form $\Rightarrow \exists\, D^{3/2}\,\bar{\kappa}(g)\,\tau(S(g,\phi)) > 1$

Condition-based bound of reach

Sampling Thm

$\mathcal{G} \subseteq \mathbb{S}^n$, $d_H(\mathcal{G}, \mathbb{S}^n) < r$

If $\sqrt{2}\, D^{1/2}\,\bar{\kappa}(g)\, r < 1$, then for all lax formulas

Φ, $d_H(\mathcal{G} \cap S_{D^{1/2}r}(g,\Phi), S(g,\Phi)) < \sqrt{2}\, D^{1/2}\,\bar{\kappa}(g)$

$\leftarrow$ Relaxation: $g \geq 0 \rightsquigarrow g \geq -D^{1/2} r, \ldots$

Improvements

- Using $\|g\|_\infty := \max_i \max_{x \in S^n} |g_i(x)|$ instead of $\|g\|_W$ reduces in one n the exponent of prob. run-time
- Can we implement it?

Eskerrik asko

bere arretagatik!