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FOR SCIENCE
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Bilbao
M8
EMAKUMEEN NAZIOARTEKO EGUNA
DÍA INTERNACIONAL DE LAS MUJERES
GUZTION AHOTSA
PARA NOMBRAR, SER, ESTAR

arXiv:2306.06784

How few Real Zeros does a random fewnomial system have?

Josué Tonelli-Cueto
(Johns Hopkins University)

Joint work with...

Both pictures taken in San Antonio, Texas, USA

Joint work with...

Alperen A. Ergür

Both pictures taken in San Antonio, Texas, USA

Joint work with...

Alperen A. Ergür

Maté L. Telek

Both pictures taken in San Antonio, Texas, USA

How many zeros does it have?

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$$\begin{cases} a_1 + b_1 X + \gamma_1 Y + c_1 XYZ^d = 0 \\ a_2 + b_2 X + \gamma_2 Y + c_2 XYZ^d = 0 \\ a_3 + b_3 X + \gamma_3 Y + c_3 XYZ^d = 0 \end{cases}$$

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≤ 1 in $\mathbb{R}^3_+$

Kushnirenko's Question:

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What do we call such a system with few terms?

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Is there a bound of the form

$$
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Propagandistic version

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$\wr$ many nomials $\quad$ $\wr$ few nomials

poconomio (es)

gutxinomio (eu)

Khovanskii's Answer

& Refinements

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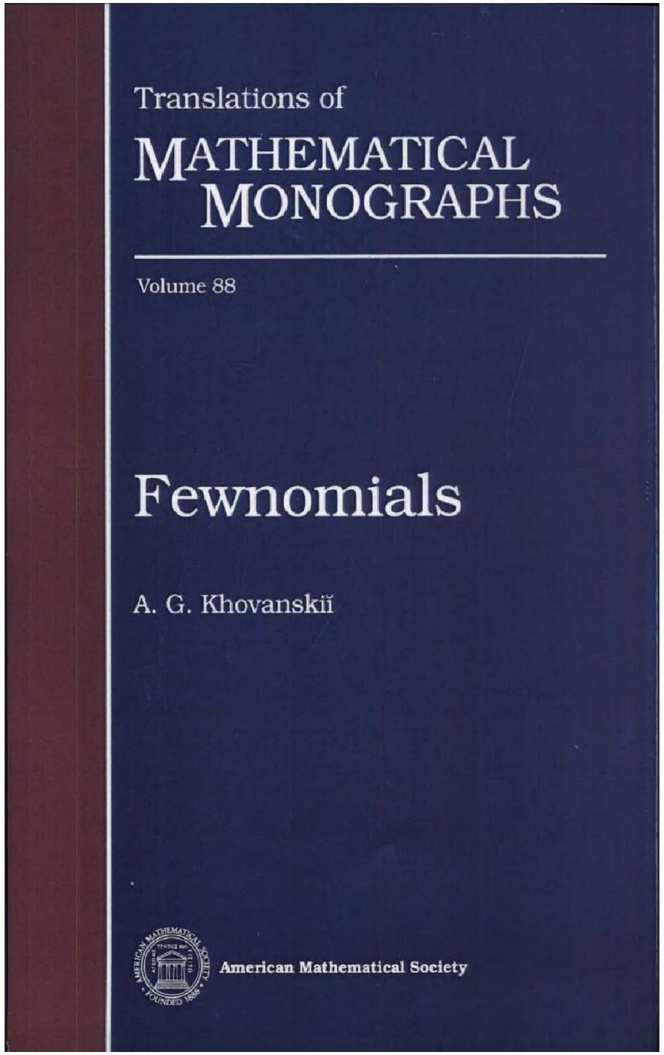

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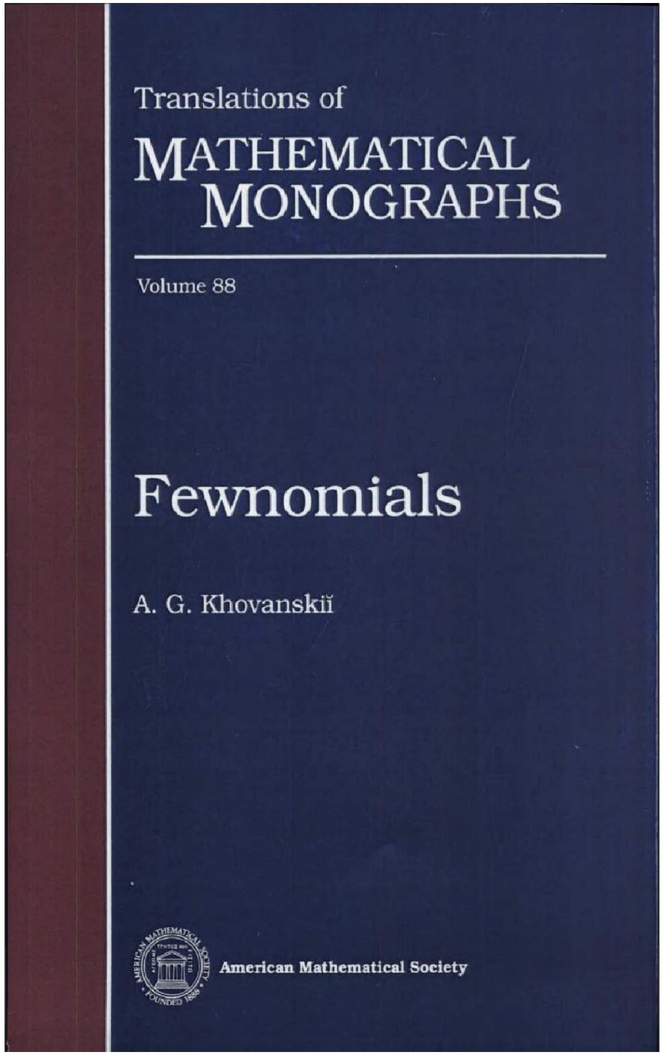

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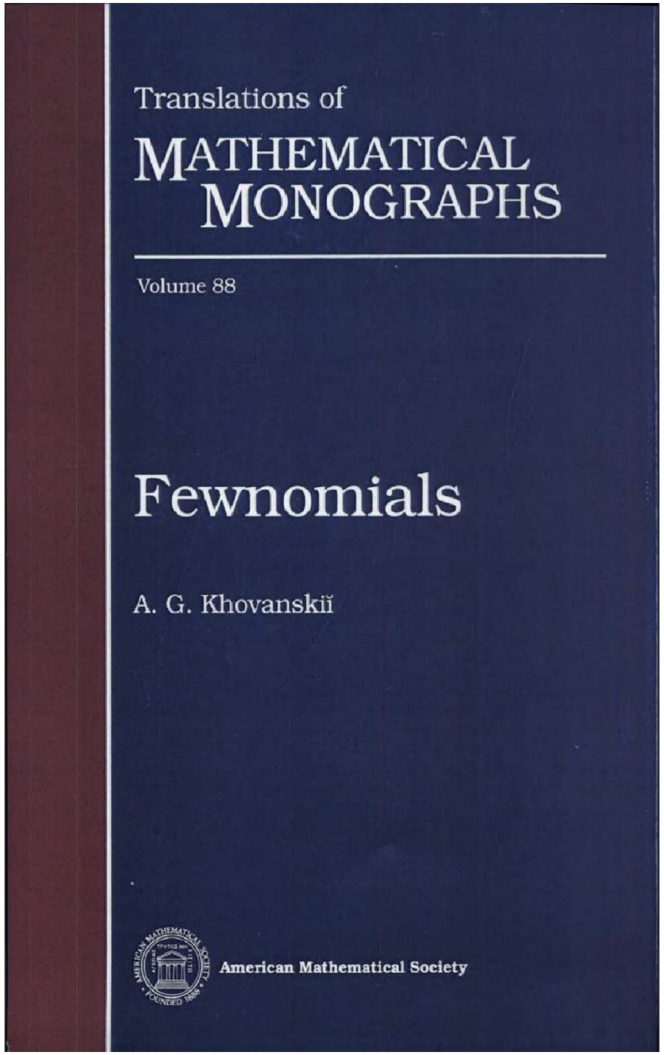

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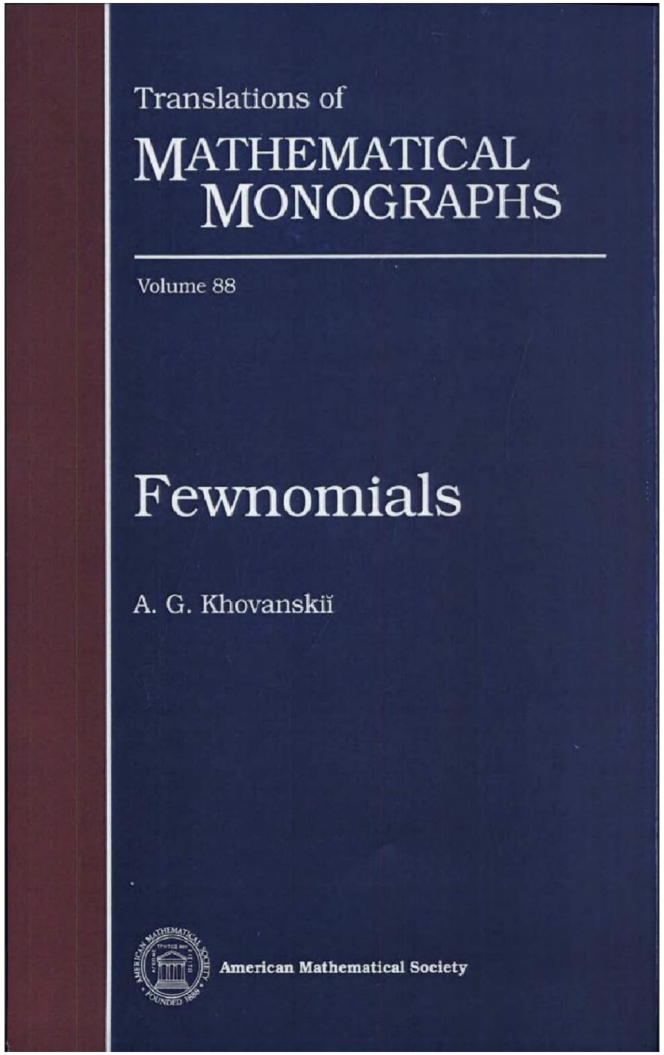

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(Khovanskii, 1991)

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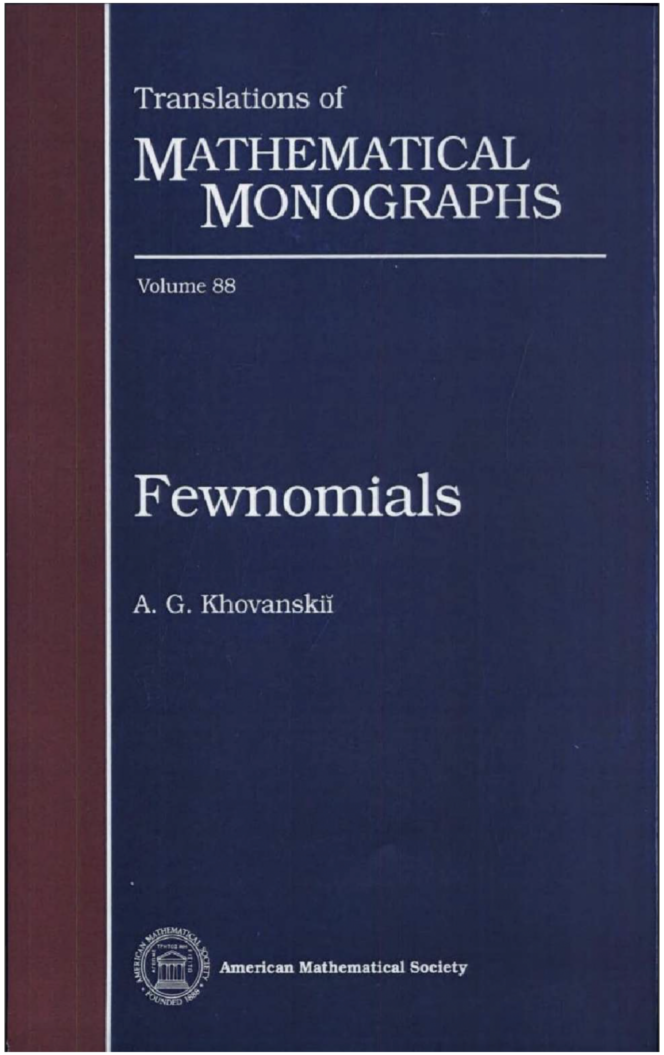

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(Bihan & Sottile; 2007)

$$\#Z_r(F, \mathbb{R}^n_+) \le \frac{e^2+3}{4} 2^{\binom{t-n-1}{2}} n^{t-n-1}$$

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STILL OPEN FOR $n=2$!!!

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Can we say

SOMETHING

for a random fewnomial system?

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spoiler: yes!

Reasons to care...

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Better understanding

of typical behaviour

— not the worst case, but the ~~average~~ typical case

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Robust probabilistic results

might be a pathway to deterministic ones

$$\sup_{g} \# Z_r(g, \mathbb{R}^n_+) = \sup_{e \geq 1} \left(\mathbb{E}_F \# Z_r(F, \mathbb{R}^n_+)^e \right)^{1/e}$$

Our Random Model

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(Bürgisser, Ergür & TC; 2019) & (Bürgisser, ISSAC'23) are particular cases of our model

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presented at MEGA2019!

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Our Result: Unit Variance Case

Improves (Bürgisser, ISSAC'23)

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THEN

$$\mathbb{E}_F \, Z_r(F, \mathbb{R}^n_+) \leq \frac{1}{4^n} V\left(\sum_{i=1}^n P_i\right) \prod_{i=1}^n (t_i - 1)$$

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Minkowski sum

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$F := \{ f_i := \sum_{\alpha \in A_i} f_{i,\alpha} X^\alpha \quad (i = 1, \ldots, n)$

s.t. the $f_{i,\alpha}$ are independent, centered & Gaussian

IF (VM1) for all i and $\alpha \in A_i$, $\mathbb{V}(f_{i,\alpha}) \leq 1$

(VM2) For all i and $\alpha \in A_i$ vertex of $P_i := \mathrm{conv}(A_i)$,

$$\mathbb{V}(f_{i,\alpha}) = 1$$

THEN

$$\mathbb{E}_F Z_r(F, \mathbb{R}_+^n) \leq \frac{1}{4^n} V\left(\sum_{i=1}^n P_i\right) \prod_{i=1}^n (t_i - 1)$$

#vertices

Minkowski sum

Improves (Bürgisser, ISSAC'23)

Our Result: The Unmixed Case

Improves (Bürgisser, Ergür & TC: 2019)

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$$\mathbb{E}_F\, Z_r(F, \mathbb{R}^n_+) \leq \frac{n+1}{4^n} \binom{t}{n+1}$$

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Regular (Mixed) Subdivisions?

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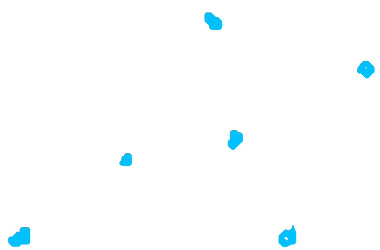

Regular (Mixed) Subdivisions?

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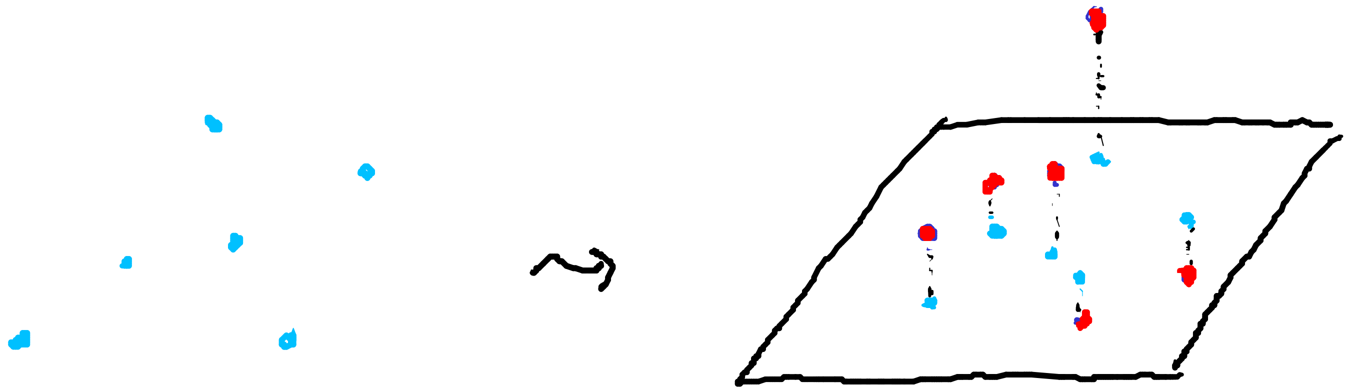
$\pi : A \to \mathbb{R}$ lifting



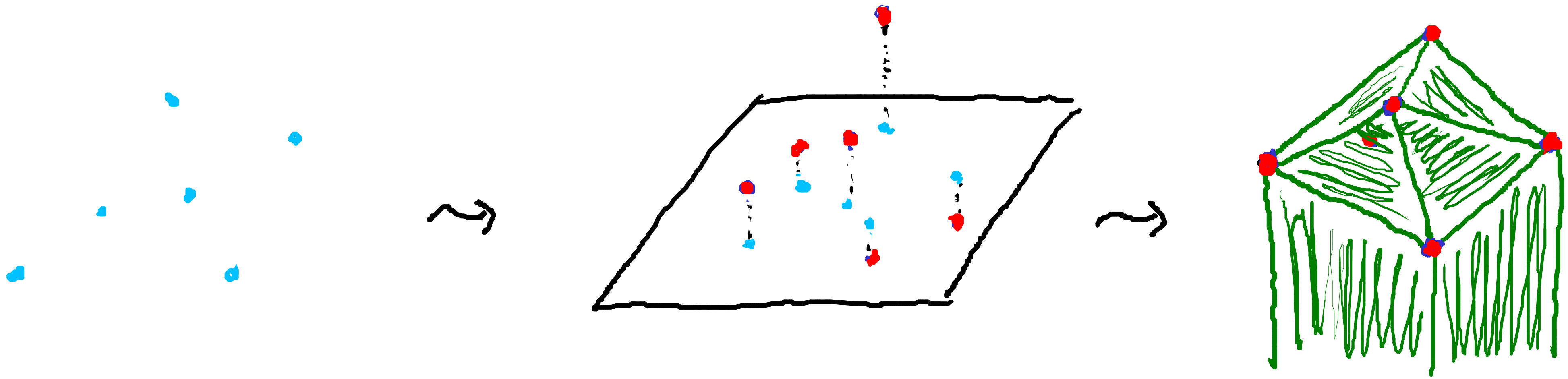
Regular (Mixed) Subdivisions?

$A \subseteq \mathbb{R}^n$

$\pi : A \to \mathbb{R}$ lifting

upper envelope of A wr π

$$\mathcal{L}(A,\pi) := \operatorname{conv}\left\{ \begin{pmatrix} \pi(\alpha) - s \\ \alpha \end{pmatrix} \,\middle|\, \alpha \in A,\ s \geq 0 \right\}$$



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\# vertices \# Minkowski sum

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#vertices #Minkowski sum

#vertices of regular mixed subdivision induced by variance on supports

Three Tools

that made this possible

Tool I: Kac-Rice Formula

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under some technical assumptions...

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$$\mathbb{E}_{f}\, \mathcal{Z}(f, \Omega)$$

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$$\mathbb{E}_{f}\, Z(f, \Omega) = \int_{\Omega} \mathbb{E}\left(|\det D_x f| \;\middle|\; f(x) = 0 \right) \delta_{f(x)}(0)\, dx$$

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conditional expectation

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conditional expectation

density of $f(x)$

Tool II: Cauchy-Binet Formula

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used also in the work of Bihan & Sottile

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$\det(AB^T)$

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Tool II: Cauchy-Binet Formula

(aka the Fundamental Lemma of Fewnomial Theory)

$A, B \in \mathbb{F}^{m \times n}$

$$\det(AB^T) = \sum_{\substack{J \subseteq \{1,\dots,n\} \\ \# J = m}} \det(A_J)\det(B_J)$$

where $A_J = (A_{i,j})_{\substack{i \in \{1,\dots,m\} \\ j \in J}}$, $B_J := (B_{i,j})_{\substack{i \in \{1,\dots,m\} \\ j \in J}}$

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Tool III: Normal Fan

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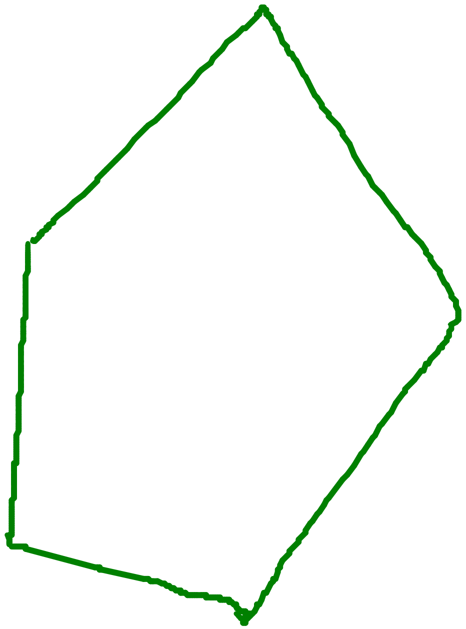

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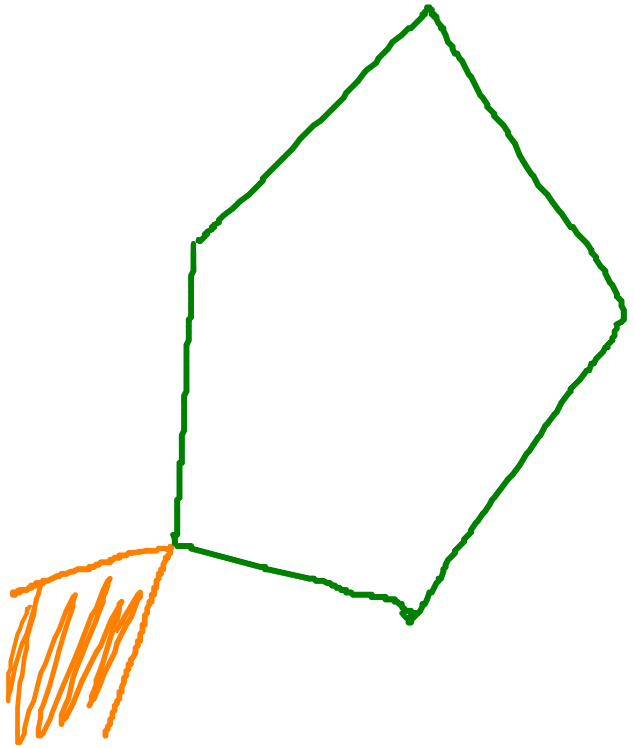

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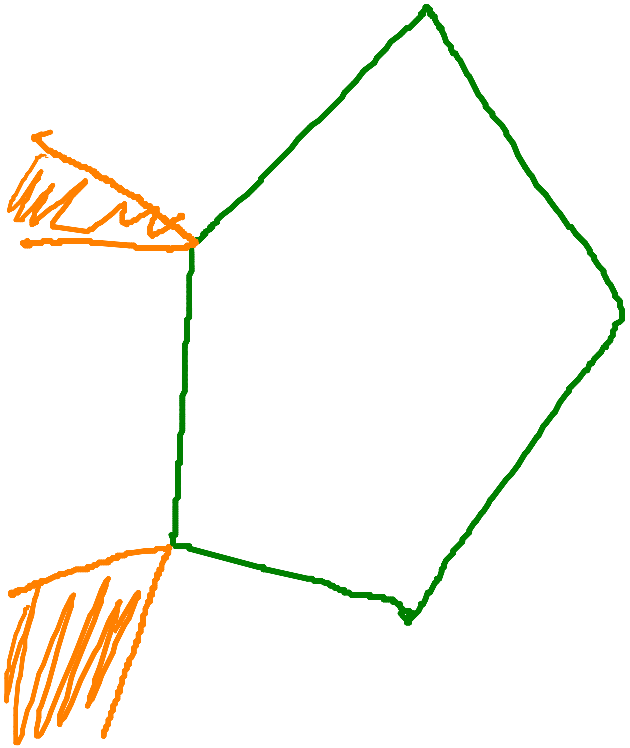

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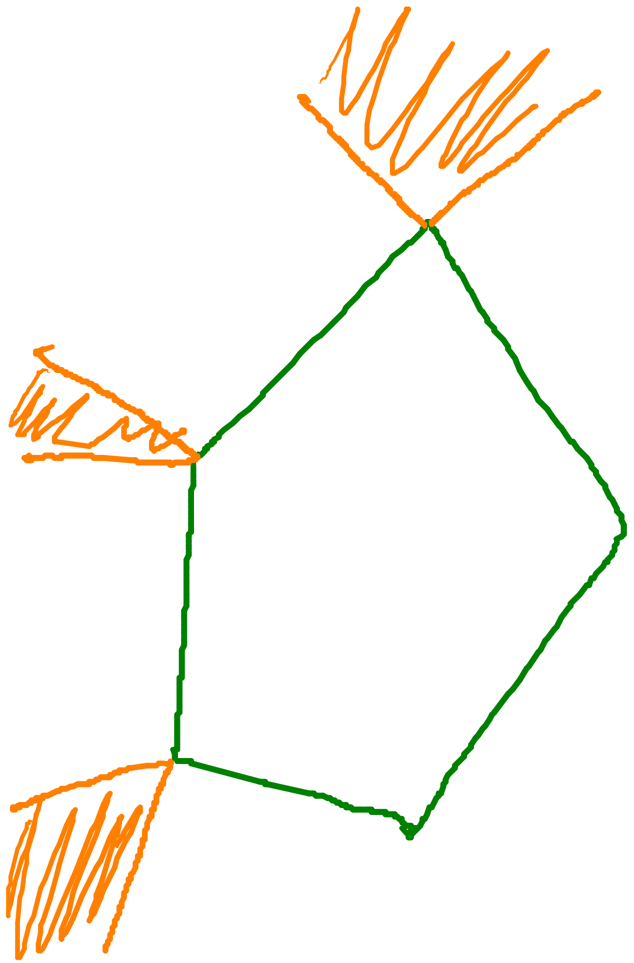

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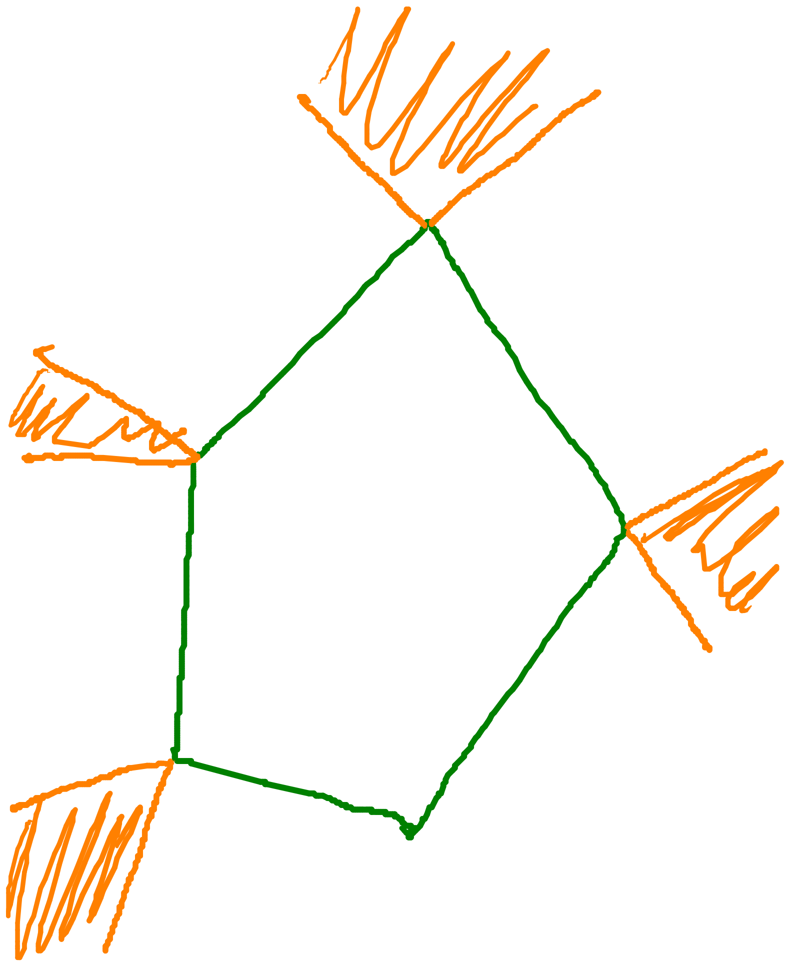

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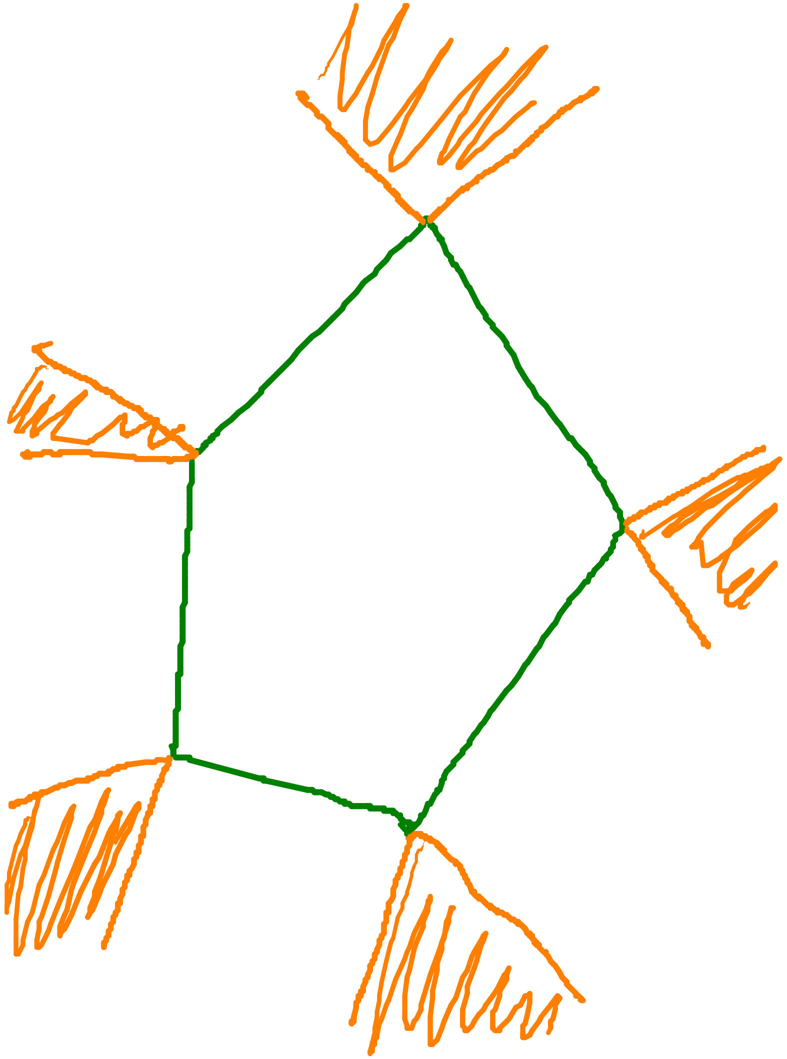

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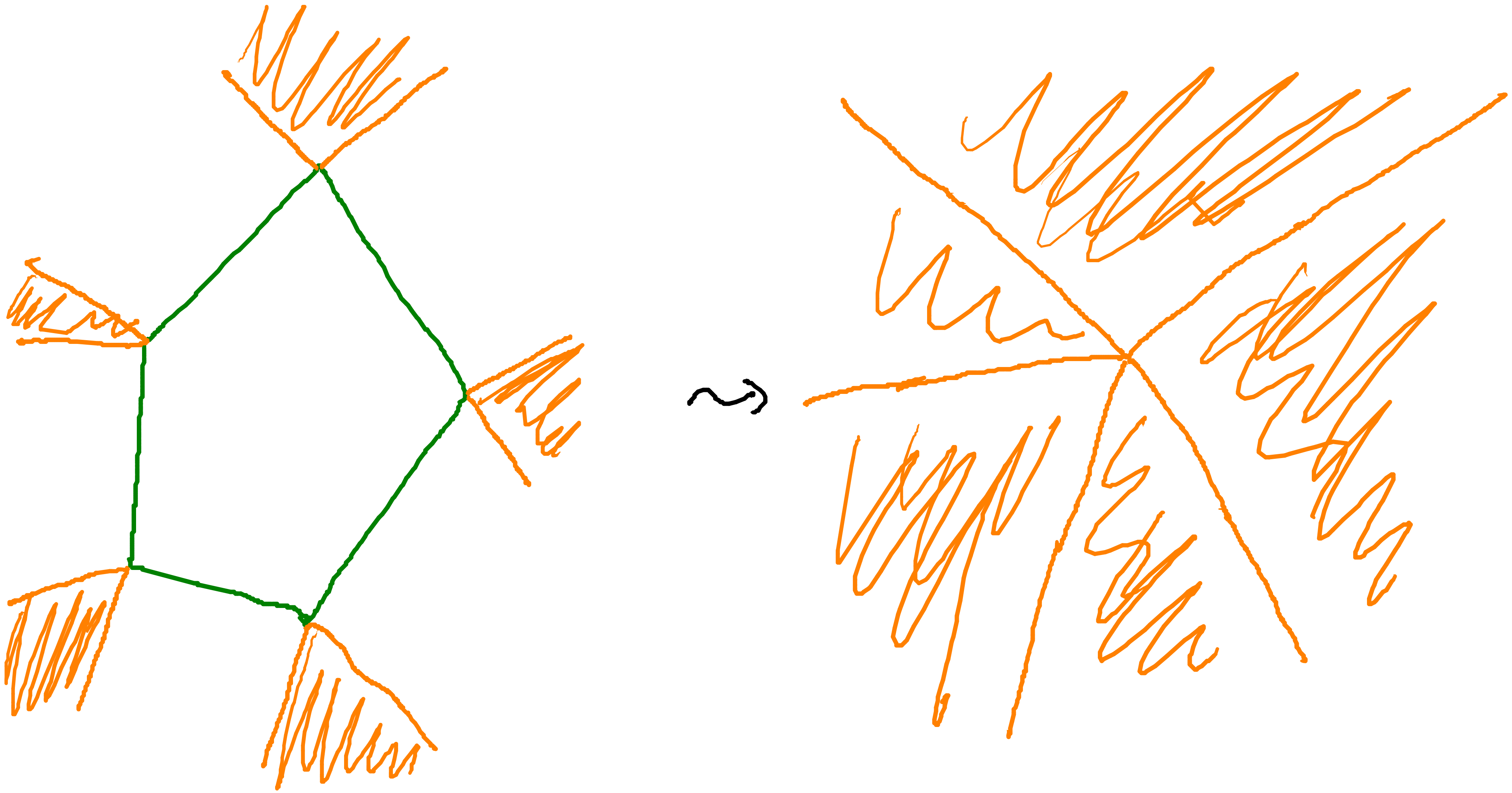

Open Questions

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What about higher moments?

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In particular, what about

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In particular, what about

$$\bigvee_{F} Z_r(F, \mathbb{R}_+^n) \; ?$$

What about more general Gaussian fewnomial systems?

Vielen Dank

für

Aufmerksamkeit!