Finding real zeros a lot faster through an adaptive grid

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Internal Story of the Problem

THE PROBLEM

$\mathcal{H}_d[n] \ni f$

DETERMINISTIC
+
NUMERICALLY
STABLE
+
'GOOD' PROBABILISTIC
RUN-TIME (for random f)

$Z_{\mathbb{P}}(f)$

↑
The ALGORITHM
we want!

Real Homogeneous
Polynomial System
in $X_0, \ldots, X_n$
& with $\deg f_i = d_i$

projective
zeros of f

What do we mean by 'numerically stable'?

1) The algorithm can run in finite precision

More precisely: **forward stability**

If the used precision is 'small enough', then the algorithm's output is correct

This might depend on the input

2) Stability **à la Smale**

The obtained approximations can be refined through an iterative scheme

What do we mean by "'good' probabilistic run-time"?

Fact. Numerical algorithms can have very different run-times on inputs of the same size

↳ Why? Run-times might depend on the condition number

Solution (von Neumann, Goldstine, Demmel, Smale)

average framework:

See what is the run-time for a random input.

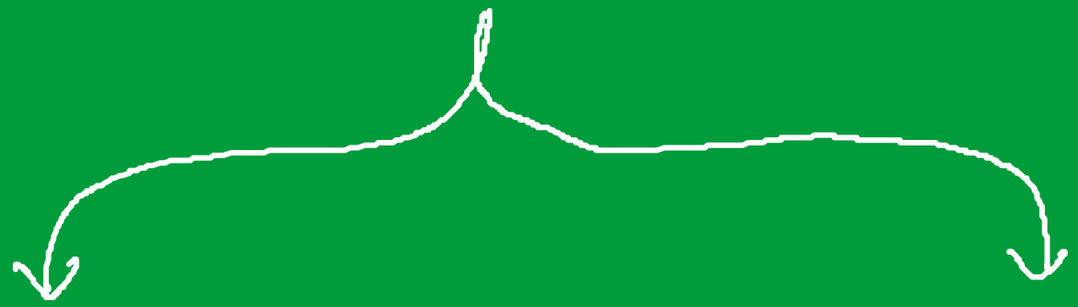
↑ which distribution?

(Spielman, Teng) smoothed framework:

random perturbation of an arbitrary input

What do we mean by "'good' probabilistic run-time"?

Answer: The run-time is reasonably small in the probabilistic sense



Expectation is small

$\mathbb{E}$ run-time $\leq T$

expected complexity

small with high probability

$\mathbb{P}(\text{run-time} \geq T) \leq \varepsilon$

weak complexity

(Ameluxen, Lotz)

Excludes 'black swans'

The Original Trilogy (by Cucker, Krick, Malajovich, Wschebor)

Journal of Complexity 24 (2008) 582–605

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Journal of Complexity

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A numerical algorithm for zero counting, I: Complexity and accuracy

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1. Introduction

In recent years considerable attention has been paid to the complexity of counting problems over the reals. The counting complexity class $\#\mathrm{P}_{\mathbb{R}}$ was introduced [20] and completeness results for $\#\mathrm{P}_{\mathbb{R}}$ were established [3] for natural geometric problems notably, for the computation of the Euler characteristic of semialgebraic sets. As one could expect, the "basic" $\#\mathrm{P}_{\mathbb{R}}$-complete problem consists of counting the real zeros of a system of polynomial equations.

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Journal of Fixed Point Theory and Applications

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In this paper we replace $\kappa(f)$—which was originally defined using the computationally friendly infinity norm—by a version $\widetilde{\kappa}(f)$ (defined in Section 2 below) which uses instead Euclidean norms. This difference is of little consequence in complexity estimates since one has (cf. Proposition 3.3 below)

$$\frac{\widetilde{\kappa}(f)}{\sqrt{n}} \leq \kappa(f) \leq \sqrt{2n}\,\widetilde{\kappa}(f). \tag{1}$$

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Part 1 Algorithm & condition-based complexity

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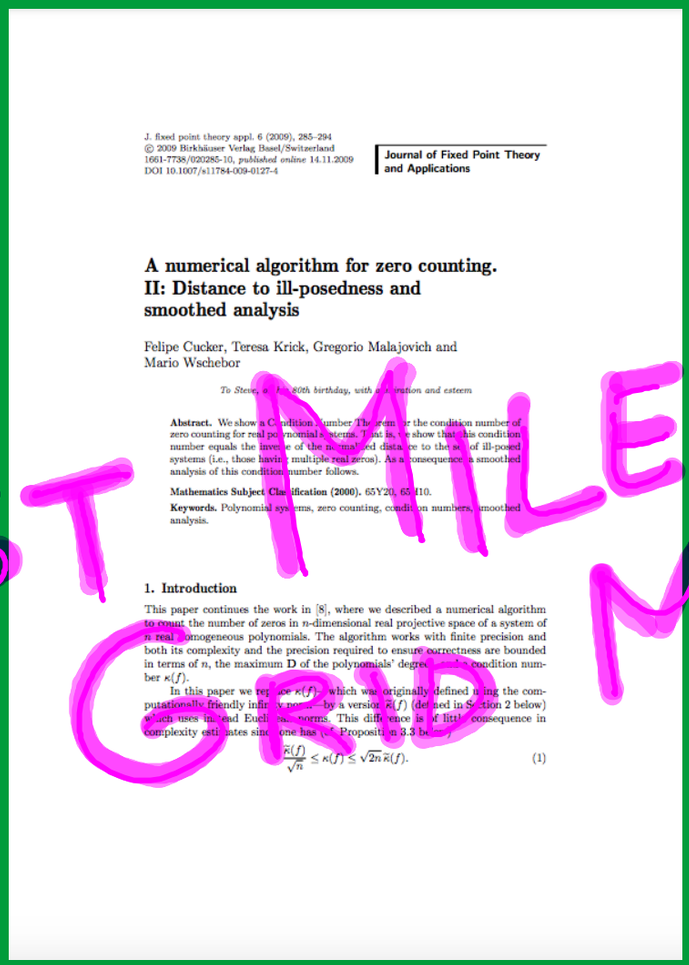

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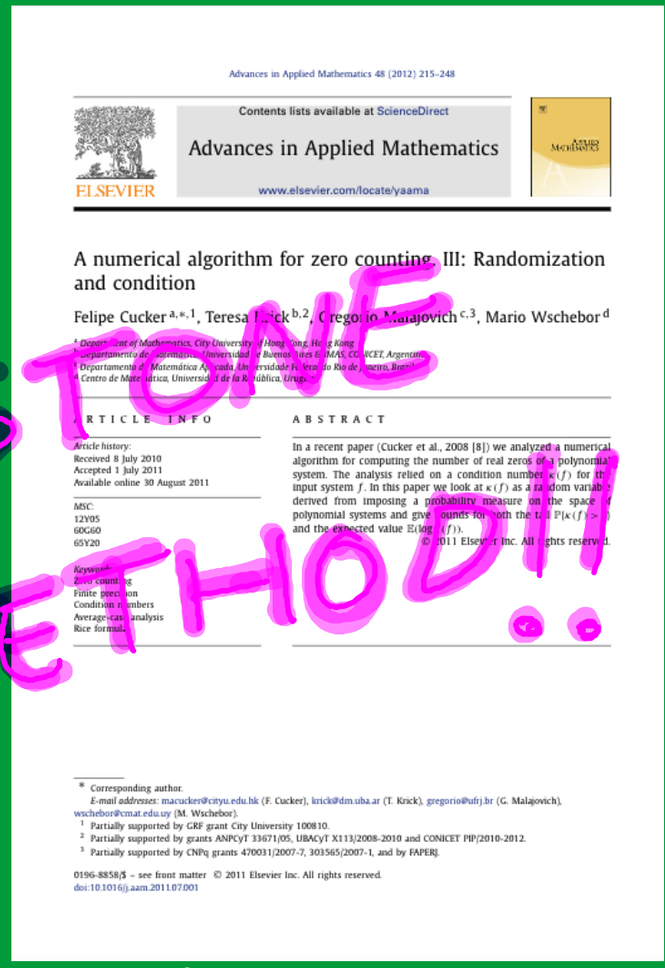

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FIRST MILESTONE OF THE CUCKER METHOD!!

Part 1 ↑ Algorithm & condition-based complexity

↑ Part 2 Condition Number Theorem & probabilistic complexity

↱ Part 3 Probabilistic analysis without integral geometry

The CKMW algorithm

$\mathcal{H}_d[n] \ni g$ → CKMW → $Z_{\mathbb{P}}(g)$

- DETERMINISTIC
- NUMERICALLY STABLE

'GOOD' PROBABILISTIC RUN-TIME

With 'high probability', run-time(CKMW, g) $\leq D^{O(n^2)}$

for g KSS (average / smoothed)

KSS = Kostlan-Shub-Smale Gaussian

$D := \max d_i$

Probabilistic Model

$g \in \mathcal{H}_d[n]$ KSS random system if

$$g_i = \sum_\alpha \sqrt{\binom{d_i}{\alpha}}\, c_{i,\alpha} X^\alpha$$

with $c_{i,\alpha}$ independent, normal random variables of mean 0 and variance 1.

We also have a smoothed version!

Why this? Invariant under orthogonal changes of variables

$f \circ \|\cdot\|_w g$

The Spin-offs (by Ergür, Paouris, Rojas)

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FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
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Probabilistic Condition Number Estimates for Real Polynomial Systems I: A Broader Family of Distributions

Alperen A. Ergür[1] · **Grigoris Paouris**[2] · **J. Maurice Rojas**[2]

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Keywords Condition number · Epsilon net · Probabilistic bound · Kappa · Real-solving · Overdetermined · Subgaussian

Communicated by Felipe Cucker.

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Springer

1st non-gaussian average complexity in Numerical Alg. Geom!

SMOOTHED ANALYSIS FOR THE CONDITION NUMBER OF STRUCTURED REAL POLYNOMIAL SYSTEMS

ALPEREN A. ERGÜR, GRIGORIS PAOURIS, AND J. MAURICE ROJAS

Abstract. We consider the sensitivity of real zeros of structured polynomial systems to perturbations of their coefficients. In particular, we provide explicit estimates for condition numbers of structured random real polynomial systems, and extend these estimates to smoothed analysis setting.

1. Introduction

Efficiently finding real roots of real polynomial systems is one of the main objectives of computational algebraic geometry. There are numerous algorithms for this task, but the core steps of these algorithms are easy to outline: They are some combination of algebraic manipulation, a discrete/polyhedral computation, and a numerical iterative scheme.

From a computational complexity point of view, the cost of numerical iteration is much less transparent than the cost of algebraic or discrete computation. This paper constitutes a step toward understanding the complexity of numerically solving structured real polynomial systems. Our main results are Theorems 1.14, 1.16, and 1.18 below, but we will first need to give some context for our results.

1.1. **How to control accuracy and complexity of numerics in real algebraic geometry?** In the numerical linear algebra tradition, going back to von Neumann and Turing, condition numbers play a central role in the control of accuracy and speed of algorithms (see, e.g., [3, 6] for further background). Shub and Smale initiated the use of condition numbers for polynomial system solving over the field of complex numbers [36, 37]. Subsequently, condition numbers played a central role in the solution of Smale's 17th problem [2, 5, 25].

The numerics of solving polynomial systems over the real numbers is more subtle than complex case: small perturbations can cause the solution set to change cardinality. One can even go from having no real zero to many real zeros by an arbitrarily small change in the coefficients. This behaviour doesn't appear over the complex numbers as one has theorems (such as the Fundemantel Theorem of Algebra) proving that root counts are "generically" constant. Luckily, a condition number theory that captures these subtleties was developed by Cucker [11]. Now we set up the notation and present Cucker's definition.

Definition 1.1 (Bombieri-Weyl Norm). *We set* $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *where* $\alpha := (\alpha_1, \ldots, \alpha_n)$, *and let* $P = (p_1, \ldots, p_{n-1})$ *be a system of homogenous polynomials with degree pattern* $d_1, \ldots, d_{n-1}$. *Let* $c_{i,\alpha}$ *denote the coefficient of* x^α *in a* p_i. *We define the* Weyl-Bombieri norms *of* p_i *and* P *to be, respectively,*

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1st non-gaussian smoothed complexity in NAG! (+ structured systems)

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ROBUST PROBABILISTIC TOOLBOX

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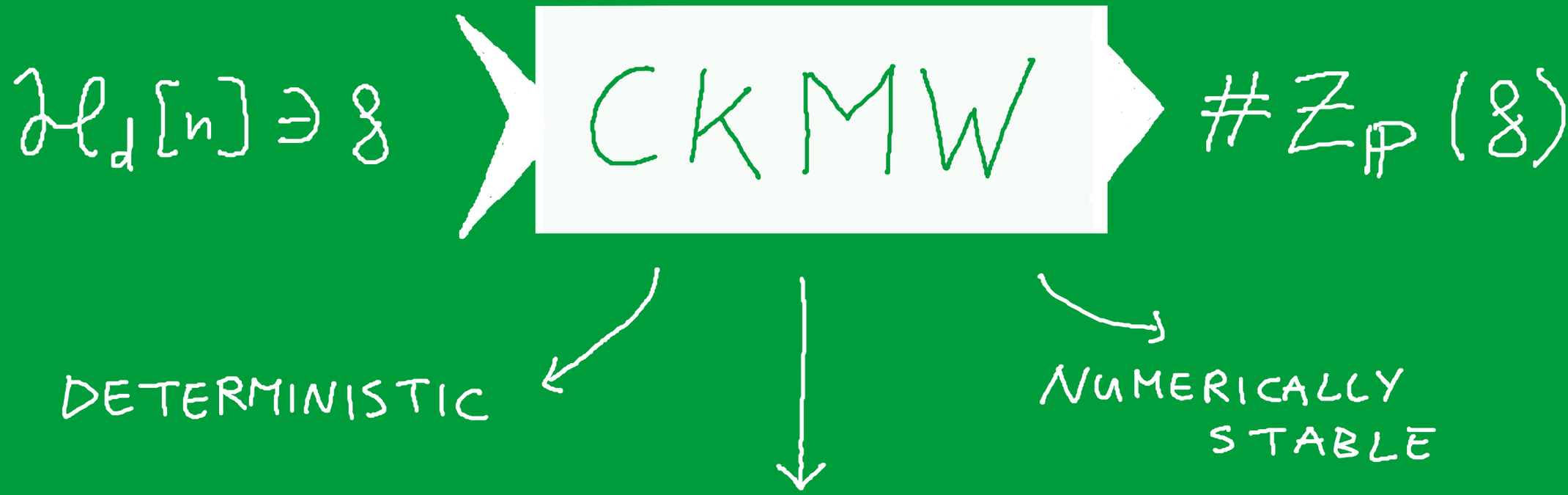
1st non-gaussian average complexity in Numerical Alg. Geom!

1st non-gaussian smoothed complexity in NAG! (+ structured systems)

The CKMW algorithm (after the spin-offs)



With 'high probability', run-time(CKMW, $\mathcal{S}$) $\leq D^{O(n^2)}$

for a wide class of random systems

$D := \max d_i$

Probabilistic Model II

$g \in \mathcal{H}_d[n]$ dobro random system

$$g_i = \sum_\alpha \sqrt{\binom{d_i}{\alpha}}\, c_{i,\alpha} X^\alpha$$

with $c_{i,\alpha}$ independent,

Also with smoothed version!

centered
i.e. $\mathbb{E}\, c_{i,\alpha} = 0$

subgaussian with cte. $\leq K_i$;
i.e. $\mathbb{E}\,|c_{i,\alpha}|^q \leq K_i^q\, q^{q/2}$ for $q \geq 1$

anticoncentration cte. $\leq \rho_i$;
i.e. $\mathbb{P}(|c_{i,\alpha} - t| \leq \varepsilon) \leq 2\rho_i \varepsilon$ for $t \in \mathbb{R}$

Important properties of gaussians!

For random $g \in \mathcal{L}_d[n]$ as before,

$\mathbb{E}_g$ run-time$(CKMW, g) < \infty$?

NO!

Why? run-time$(CKMW, g) \leq O^{O(n)}\, \kappa(g)^n$

& $\mathbb{E}_g\, \kappa(g)^n = \infty$

$\|g\|_W = \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} g_{i,\alpha}^2}$ (Weyl norm)

$\kappa(g) = \max_{x \in S^n} \kappa(g,x)$ $\quad \kappa(g,x) = \|g\|_W / \sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{1/2}\|^{-2}}$ $\quad \Delta = \mathrm{diag}(d_i)$

Idea!

Make CKMW adaptive,

then complexity should depend on

$$\mathbb{E}_{x \in S^n} \mathcal{K}(f,x)^n$$

which has <u>finite expectation</u>

for a random f!

$\|f\|_w = \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} f_{i,\alpha}^2}$ (Weyl norm) $\mathcal{K}(f,x) = \|f\|_w / \sqrt{\|f(x)\|^2 + \|D_x f \Delta^{-1}\|^{-2}}$ $\Delta = \mathrm{diag}(d_i)$

Inspiration: (Cucker, Ergür, T.-C.; 2019) while studying PV algorithm

Naive adaptive version fails!

(Eckhardt, 2020) (Han, 2018)

Run-time bound in terms of

$$\mathop{\mathbb{E}}_{x \in S^n} \mathcal{K}(g,x)^{2n}$$

which has <u>infinite expectation</u>

for a random g!

$\|g\|_W = \sqrt{\sum_{i,\alpha} \left(\frac{d_i}{\alpha}\right)^{-1} g_{i,\alpha}^2}$ (Weyl norm) $\mathcal{K}(g,x) = \|g\|_W / \sqrt{\|g(x)\|^2 + \|D_x g^{-T} \Delta^g\|^{-2}}$ $\Delta = \mathrm{diag}(d_i)$

What goes wrong?

The criterion to select zeros!

The CKMW algorithm

1) Refine grid $G \subseteq \mathbb{S}^n$ until $d_{\mathbb{S}}(G, \mathbb{S}^n)$ 'small'

2 {
- Exclude points $x \in G$ s.t. $\|g(x)\|/\|g\|_w$ 'big'
- Include points $x \in G$ s.t. $\|g(x)\|/\|g\|_w$ 'small'
}

3 Post-process the selected points to get approximation of $Z_{ip}(g)$

How do we exclude points?

Exclusion Lemma. Let $g \in \mathcal{H}_d[n]$, the map

$$S^n \ni x \mapsto g(x)/\|g\|_w$$

is $\sqrt{D}$-Lipschitz. In particular, if

$$r \geq \frac{\|g(x)\|}{\sqrt{D}\,\|g\|_w},$$

the $B_S(x,r) \cap \mathcal{Z}_S(g) = \emptyset$.

$\|g\|_w = \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} g_{i,\alpha}^2}$ (Weyl norm) $\qquad D := \max d_i$

How do we include points?

Spherical Newton operator: $N_g(x) := \dfrac{x - D_x g^{-1} g(x)}{\|x - D_x g^{-1} g(x)\|}$

$$N_g^{n+1}(x) = N_g(N_g^n(x))$$

Smale's α-criterion:

$$\alpha(g,x) := \beta(g,x)\gamma(g,x) \leq \alpha_*$$

$$\hookrightarrow \#B_S(x, 1.5\beta(g,x)) \cap Z_S(g) = 1$$

$$\& \quad N_g^n(x) \xrightarrow[\text{quadratically}]{} \text{zero of } g$$

where $\beta(g,x) := \|D_x g^{-1} g(x)\|$ & $\gamma(g,x) := \sup_{k\geq 2} \|D_x g^{-1} \frac{1}{k!} D_x^k g\|$

Assume $\sqrt{2}\,\kappa(g,x)\,\|g(x)\|/\|g\|_w < 1 \ldots$

- Higher Derivative Estimate: $\gamma(g,x) \leq \frac{1}{2} D^{3/2} \kappa(g,x)$
- An estimate for β: $\beta(g,x) \leq \kappa(g,x)\,\|g(x)\|/\|g\|_w$

The CKMW algorithm

1) Refine grid $G \subseteq \mathbb{S}^n$ until $\underbrace{d_{\mathbb{S}}(G, \mathbb{S}^n)}_{=:\delta} \leq \frac{1}{cD^2\kappa(g)^2}$

2 {
- Exclude points $x \in G$ s.t. $\|g(x)\|/\|g\|_w \geq \sqrt{D}\,\delta$
- Include points $x \in G$ s.t. $\|g(x)\|/\|g\|_w \leq \frac{1}{\tilde{c}D^2\kappa(g)^2}$

3 Post-process the selected points to get approximation of $Z_{ip}(g)$

Note quadratic condition in the inclusion criterion!

$\kappa(g) := \max_{x \in \mathbb{S}^n} \|g\|_w \,/\, \sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{1/2}\|^{-2}}$ condition number

The adaptive CKMW algorithm

NAIVE EDITION

1) Refine adaptively $G \subseteq \mathbb{S}^n \times (0,\infty)$ so that
   1) $\mathbb{S}^n \subseteq \bigcup \{B_g(x,r) \mid (x,r) \in G\}$
   & 2) $\forall (x,r) \in G,\ r \leq 1/cD\kappa(g,x)^2$
2) $\begin{cases} \text{Exclude } (x,r) \in G & \text{if } \|g(x)\|/\|g\|_w \geq \sqrt{D}\, r \\ \text{Include } (x,r) \in G & \text{if } \|g(x)\|/\|g\|_w \leq 1/\hat{c}D^2\kappa(g,x)^2 \end{cases}$
3) Post-process the selected points to get approximation of $Z_{lp}(g)$

Still quadratic inclusion criterion!

$\kappa(g,x) := \|g\|_w / \sqrt{\|g(x)\|^2 + \|D_x g^{-1}\Delta^{1/2}\|^{-2}}$ local condition number

Where does the square come from?

$$\beta(\xi, x) \leq \mathcal{K}(\xi, x) \frac{\|\xi(x)\|}{\|\xi\|_W}$$

is a very bad estimate!

The $\mathcal{K}(\xi, x)$ in the upper bound causes the square

We should use β directly!

Converse Smale's α-theorem:

$$\gamma(f,x)\,\mathrm{dist}_S(x, Z_S(f)) < 1$$

$$\Leftrightarrow \quad \alpha(f,x) \le \frac{\gamma(f,x)\,\mathrm{dist}_S(x, Z_S(f))}{1-\gamma(f,x)\,\mathrm{dist}_S(x, Z_S(f))}$$

'If x is sufficiently near $Z_S(f)$,
then Smale's α-criterion at x holds'

Corollary. If $\sqrt{2}\,K(f,x)\,\|f(x)\|/\|f\|_W < 1$,

then $\alpha(f,x) < \alpha_*$

or $B_S\left(x, c/D^2K(f,x)\right) \cap Z_S(f) = \emptyset$

The adaptive CKMW algorithm

NON NAIVE EDITION!!!

1) Refine adaptively $G \subseteq S^n \times (0,\infty)$ so that
   1) $S^n \subseteq \bigcup \{B_S(x,r) \mid (x,r) \in G\}$
   & 2) $\forall (x,r) \in G,\ r \leq 1/c D K(g,x)$
2) Exclude $(x,r) \in G$ if $\|g(x)\|/\|g\|_w \geq \sqrt{D}\, r$
   Include $(x,r) \in G$ if $\beta(g,x) \leq 1/c D^2 K(g,x)$
3) Post-process the selected points to get approximation of $Z_{IP}(g)$

Using β gives the desired $\mathbb{E}_{x \in S^n} K(g,x)^n$ bound!

$K(g,x) := \|g\|_w / \sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{1/2}\|^{-2}}$ local condition number

Some extra tricks

$$\chi_2(g,x) := \frac{\|g\|_w}{\sqrt{\|g(x)\|^2 + \|D_x g^{-1}\Delta^{1/2}\|^{-2}}}$$

Change of norm

$$\hookrightarrow C(g,x) := \frac{\|g\|_\infty}{\max\{\|\Delta^{-1}g(x)\|_\infty, \|D_x g^{-1}\Delta^2\|_{\infty,2}^{-1}\}}$$

where $\Delta := \mathrm{diag}(d_i)$

Extra normalization

Row-normalization

$$\hat{g} := (g_i/\|g_i\|_\infty)_i$$

Interlude: $\|\cdot\|_w$ vs. $\|\cdot\|_\infty$

Definition

$g \in \mathcal{H}_d[n]$

$$\|g\|_w := \sqrt{\sum_{i,\alpha} \binom{d_i}{\alpha}^{-1} g_{i,\alpha}^2} \qquad \|g\|_\infty := \max_{i, x \in S^n} |g_i(x)|$$

Similar inequalities $\Rightarrow$ Similar condition-based theory

$$\|D_x g(v)\|_w \le D \|g\|_w \qquad \|D_x g(v)\|_\infty \le D \|g\|_\infty$$

where $D_x g(v) = \sum_j \frac{\partial g}{\partial x_j} v_j \in \mathcal{H}_{d-1}[n]$ ← Kellogg's inequality

Different probabilistic behaviours

$$\mathbb{E}_g \|g\|_w \sim \sqrt{N} \sim \min\{n^{D/2}, D^{n/2}\} \qquad \mathbb{E}_g \|g\|_\infty \sim \sqrt{n \log D}$$

There is a catch! $\|g\|_\infty$ harder to compute

Main Result

MAIN THEOREM

There is a DETERMINISTIC, NUMERICALLY STABLE algorithm a CKMW that given $f \in \mathbb{Z}_q[u]$ computes $\#Z_p(f)$ and such that

$$\mathbb{E}_f \text{ run-time}(\text{a CKMW}, f) \leq 2^{\mathcal{O}(u \log u)} D^u N + 2^{\mathcal{O}(u \log u)} \mathcal{D}^{2.5} (N + \mathcal{D})$$

(for f dobro random (with bounded parameters)

'GOOD' PROBABILISTIC RUN-TIME

where $D := \max d_i$ $\mathcal{D} := \prod d_i$

$N := \sum \binom{u+d_i}{u} = \#$ of zero & nu-zero coeff. of f

+ PARALLELIZABLE

Future Work

We can produce correct adaptive samples!

Homology computation of semialgebraic sets

Post-processing step has to be improved!

Can we have a Monte-Carlo version without computing $\|\delta\|_\infty$?

Obrigado
pela atenção!