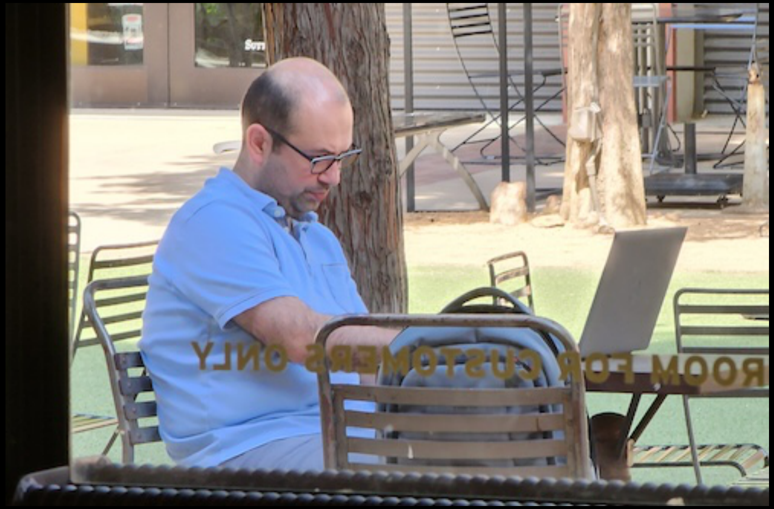
How many Real Zeros does a random sparse polynomial system have?

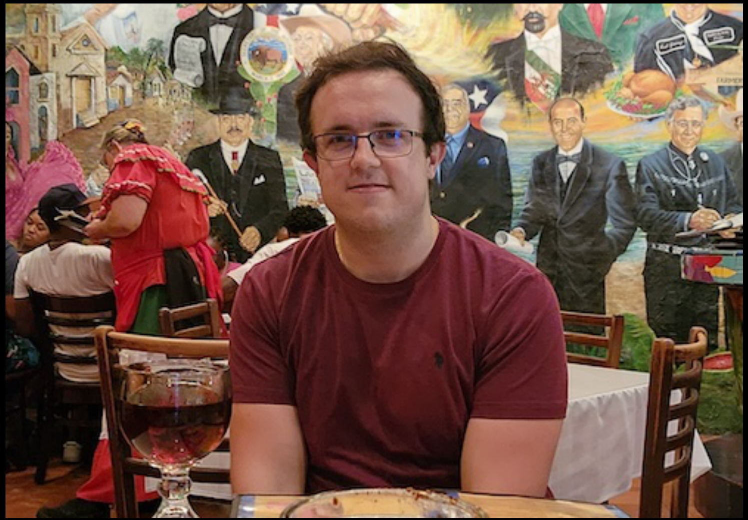
Josué Tonelli-Cueto
(Johns Hopkins University)

Algebra Seminar
UPV/EHU

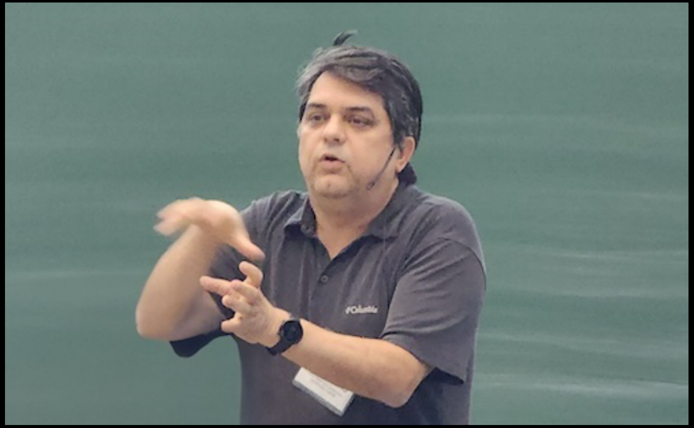
Joint work with...



Alperen A. Ergür



Maté L. Telek



Elias Tsigaridas

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

How many...

Complex Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

How many... $\mathbb{C}$omplex Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

d complex zeros (generically)

How many...

Real Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

How many... Real Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

≤ 2 real zeros (generically)

How many... Positive Zeros?

$$\begin{cases} a_1 + b_1 X + c_1 Y + d_1 XYZ^d = 0 \\ a_2 + b_2 X + c_2 Y + d_2 XYZ^d = 0 \\ a_3 + b_3 X + c_3 Y + d_3 XYZ^d = 0 \end{cases}$$

How many... Positive Zeros?

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0 or 1 positive zeros (generically)

General Phenomenon...

Many Complex Zeros,

but a lot fewer Real Zeros

Why do we care about Real Zeros?

[Applied answer]

Why do we care about Real Zeros?

- Chemical Reaction Networks

[Applied answer]

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- Chemical Reaction Networks
- Computer Vision

[Applied answer]

Why do we care about Real Zeros?

- Chemical Reaction Networks
- Computer Vision
- Phylogenetic Trees

[Applied answer]

Why do we care about $\mathbb{R}$eal Zeros?

In applications,
the $\mathbb{R}$eal & $\mathbb{P}$ositive Zeros
are the ones that matter

[Applied Answer]

Why do we care about Real Zeros?

[Pure answer]

Why do we care about $\mathbb{R}$eal Zeros?

The number of $\mathbb{C}$omplex zeros is a geometric quantity:

[Pure answer]

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is a geometric quantity:
degree, (mixed) volume of Newton polytopes

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The number of $\mathbb{R}$eal zeros
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Why do we care about $\mathbb{R}$eal Zeros?

The number of $\mathbb{C}$omplex zeros

is a geometric quantity:

degree, (mixed) volume of Newton polytopes

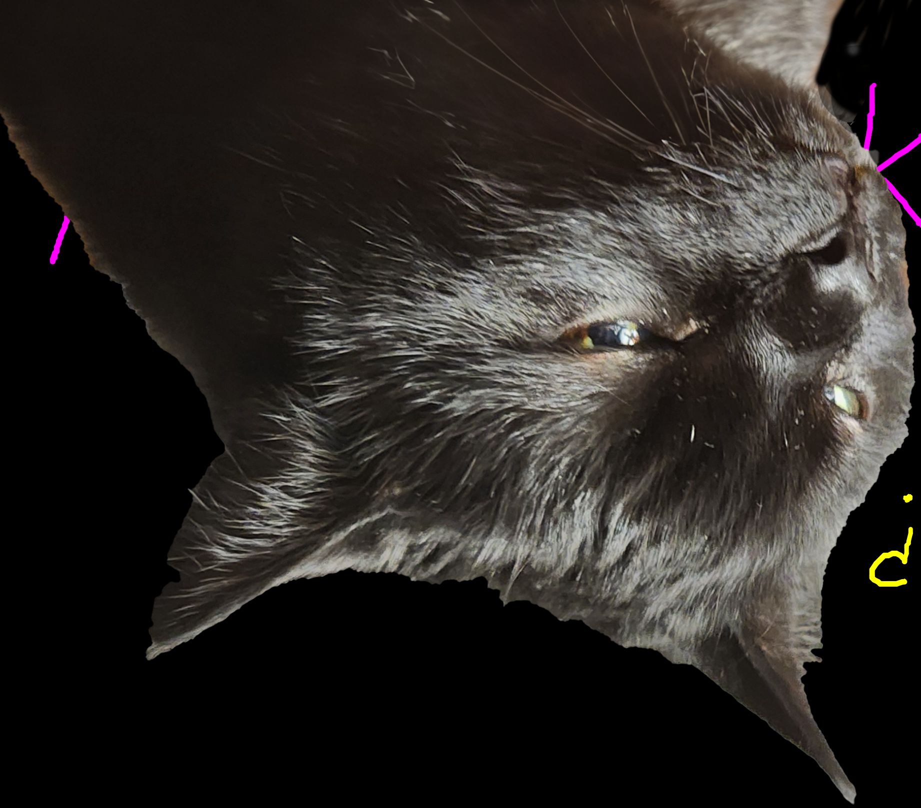
The number of $\mathbb{R}$eal zeros

is a more combinatorial quantity:

size of the support

[Pure answer]

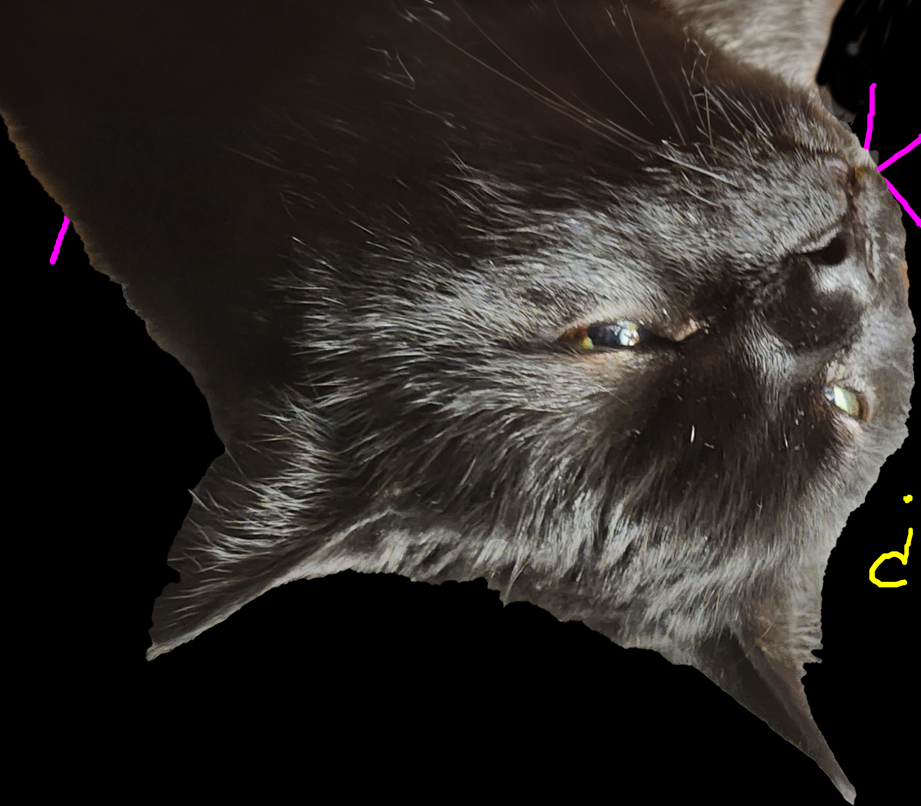
Khovanskii's Theorem on Fewnomial Systems



Kii's Theorem

Fewnomial Systems

d?

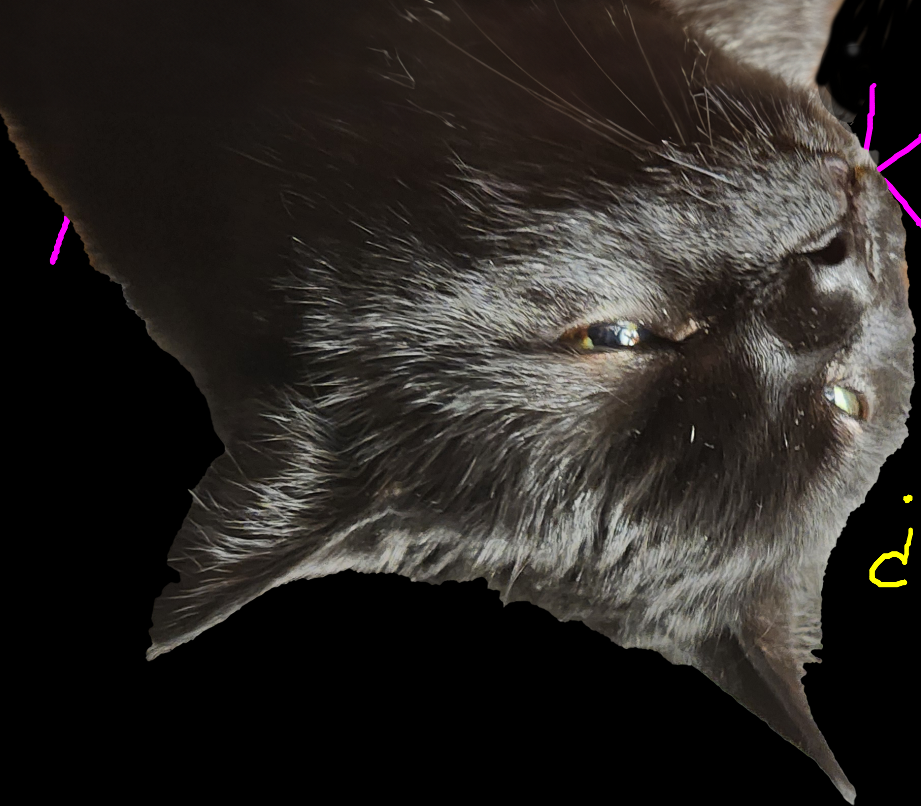


Kii's Theorem

Fewnomial Systems

¿?

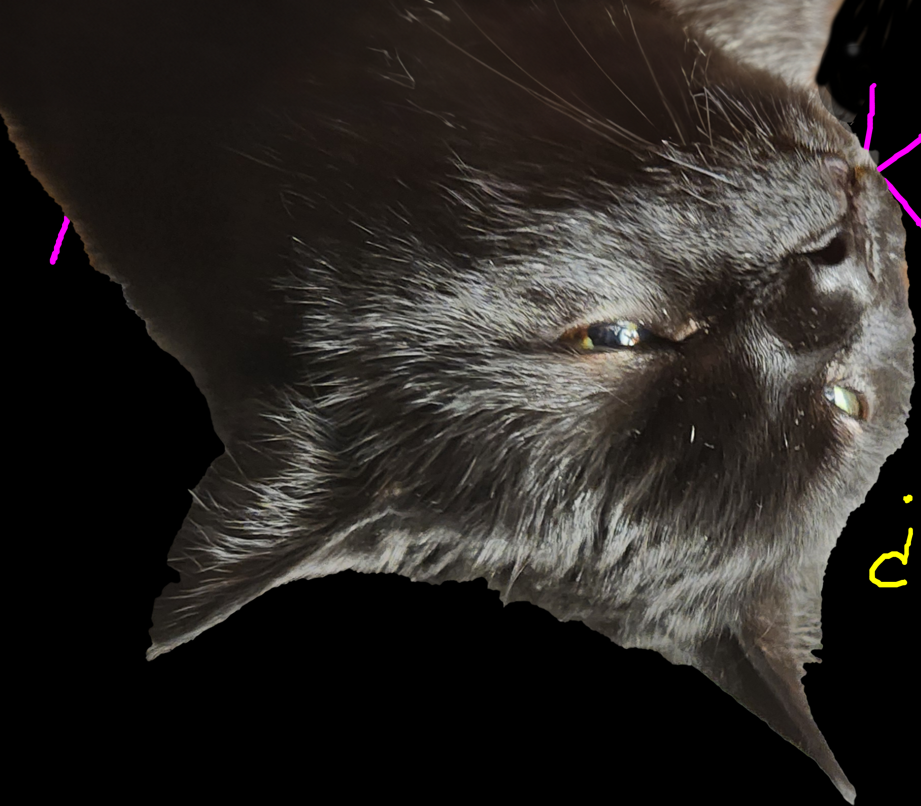
Polynomial



Kii's Theorem
Fewnomial Systems

¿?

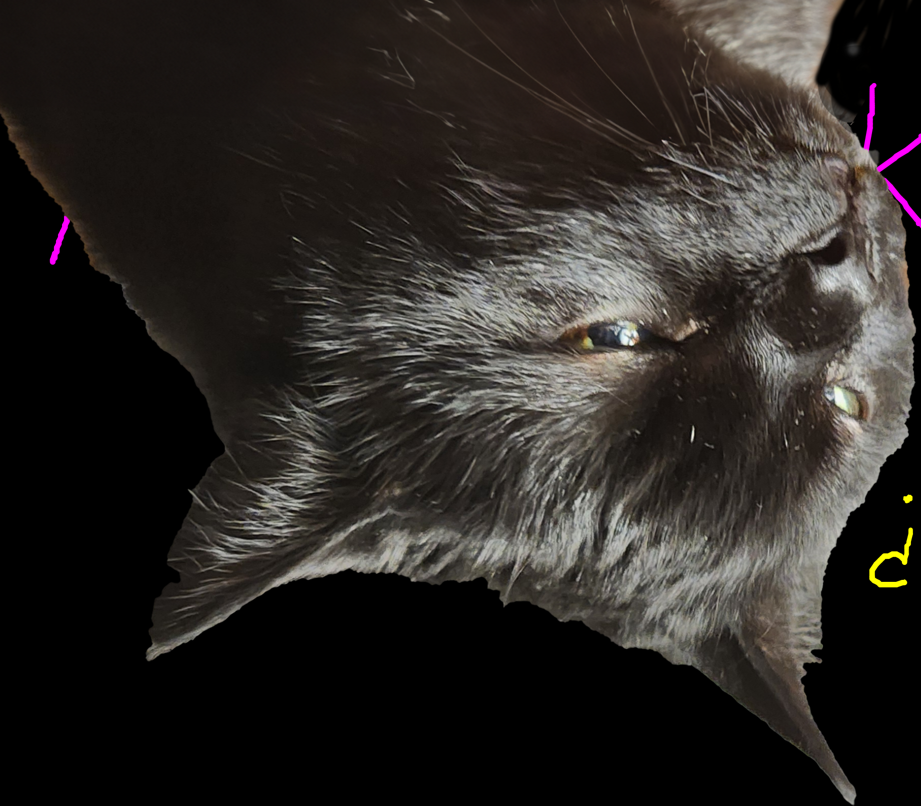
Polynomial → poly-nomial



Kii's Theorem
Fewnomial Systems

¿?

Polynomial → poly-nomial → много-член



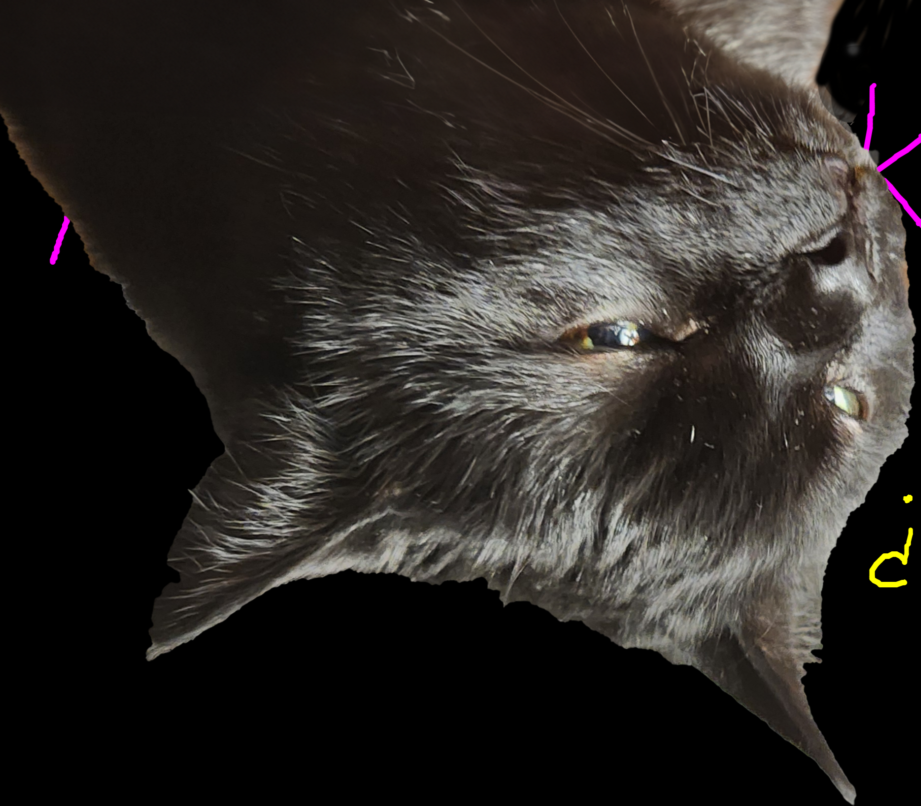
Kii's Theorem
Fewnomial Systems

¿?

Polynomial → poly-nomial → много-член

→ многочлен

"many monomial"



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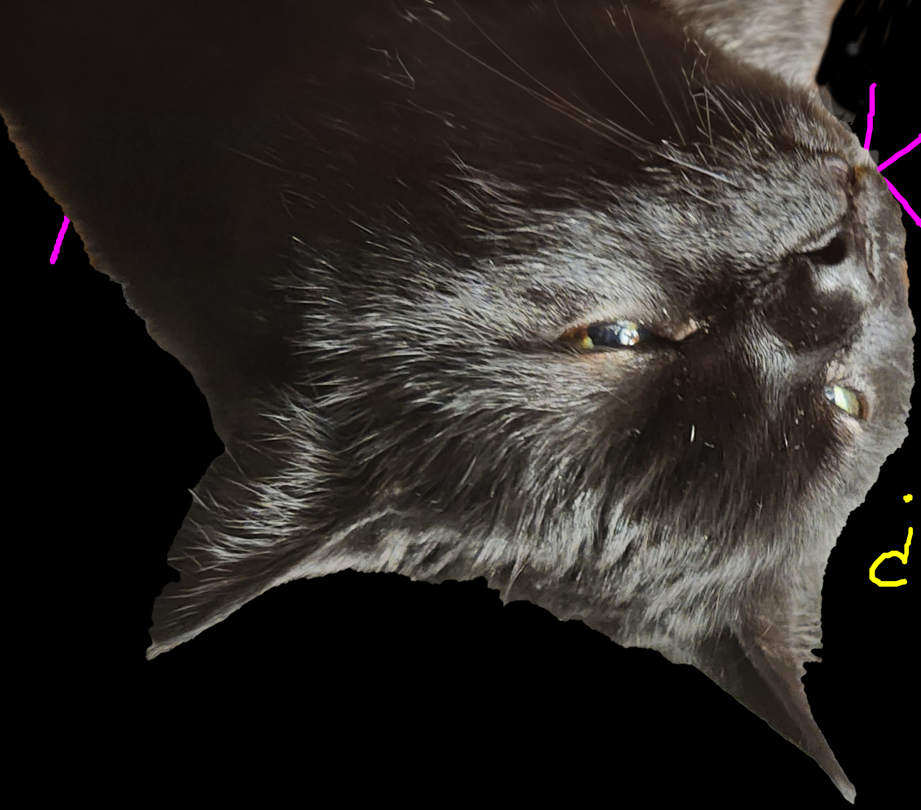
Polynomial → poly-nomial → много-член

→ многочлен → малочлен

"many monomial" "few monomials"



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Fewnomial Systems

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"many monomial" "few monomials"

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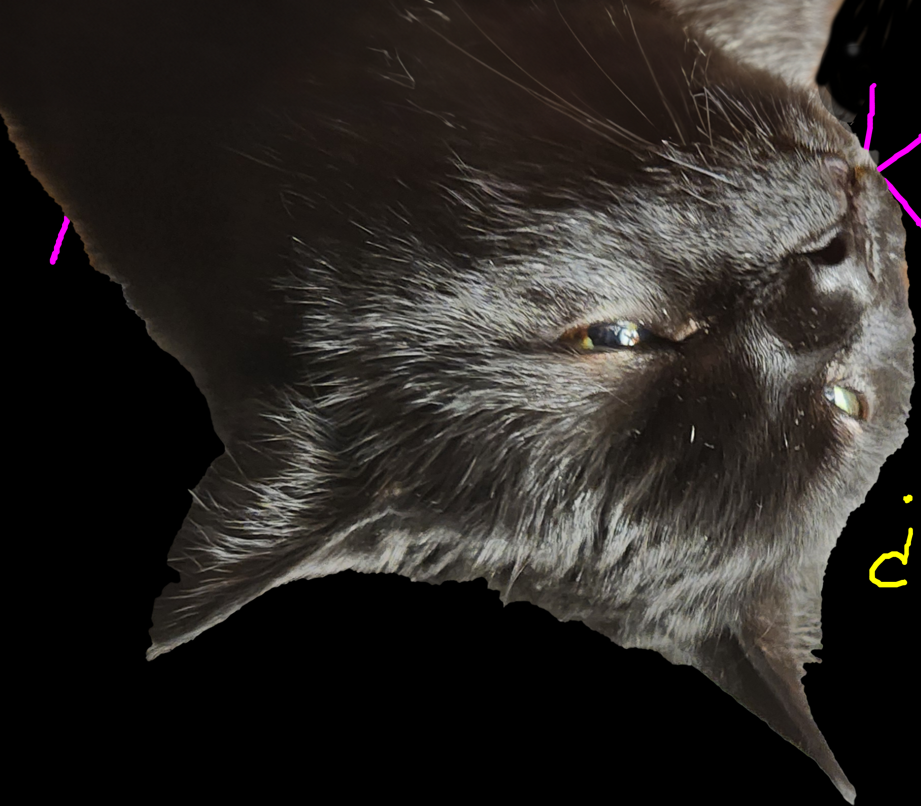


Kii's Theorem
Fewnomial Systems

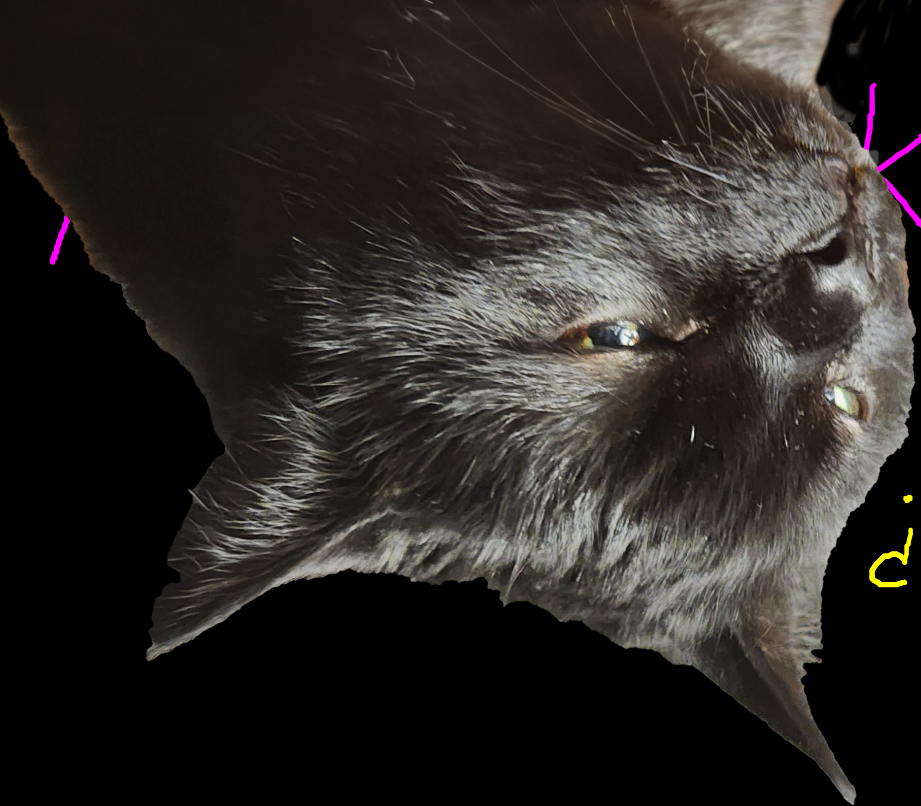
¿? →

Polynomial → poly-nomial → МНОГО-ЧЛЕН

→ МНОГОЧЛЕН → МАЛОЧЛЕН

"many monomial" "few monomials"

→ МАЛО-ЧЛЕН → few-nomial



Kii's Theorem
Fewnomial Systems

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Polynomial → poly-nomial → МНОГО-ЧЛеН

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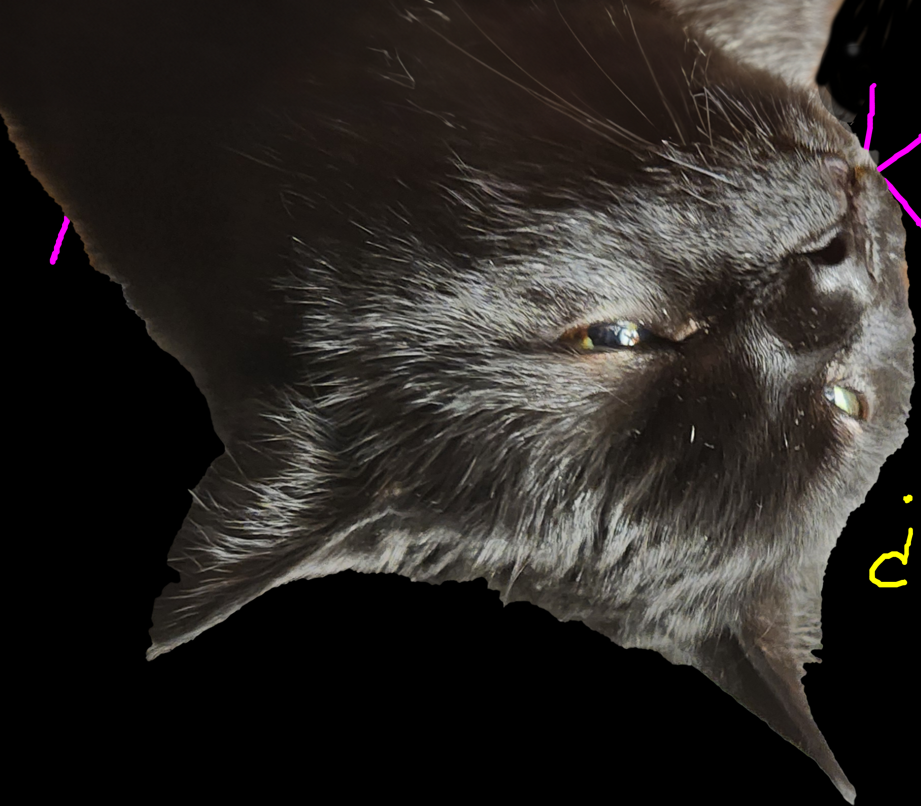
"many monomial" "few monomials"

→ МАЛО-ЧЛеН → few-nomial → Fewnomial



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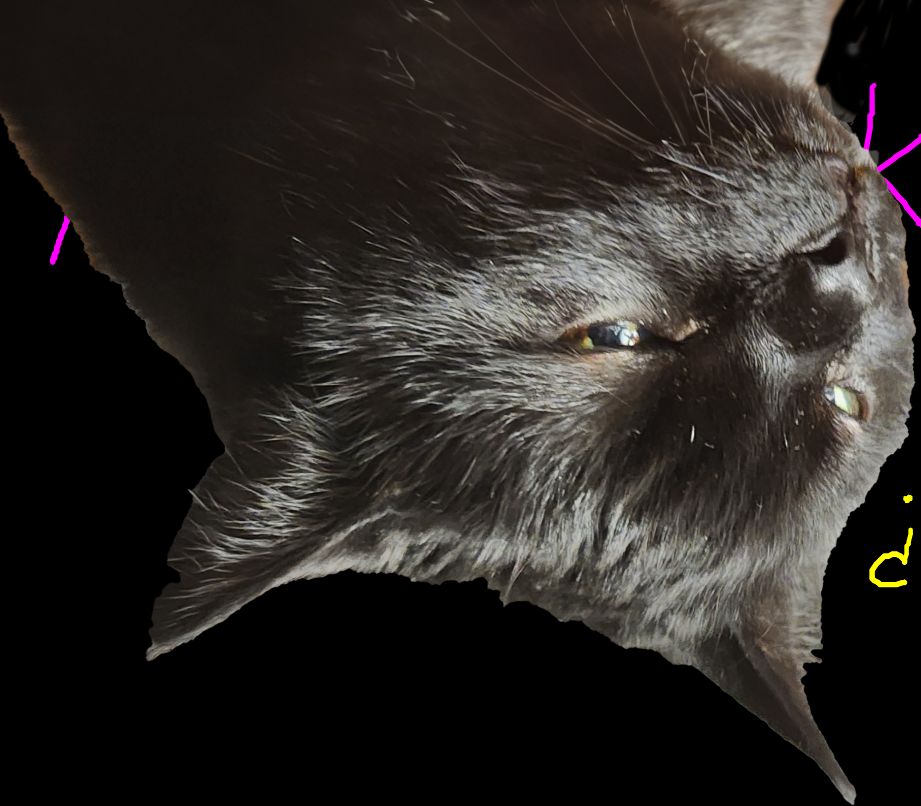
⇝ Gutxinomioa



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⇝ Gutxinomioa ⇝ Poconomio

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$$\# Z_r(g, \mathbb{R}_+^n) \leq 2^{\binom{t-1}{2}} (n+1)^{t-1}$$

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No degree!

$$\# \mathcal{Z}_r(g, \mathbb{R}^n_+) \leq 2^{\binom{t-1}{2}} (n+1)^{t-1}$$

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where g_i has t_i terms

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Is there universal $c > 0$ s.t.

$$\# Z_r(g, \mathbb{R}^n_+) \leq (t_1 \cdots t_n)^c \; ?$$

State of the Art (Zero dimensional case)

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(Sevastyanov, 1978; Koiran, Portier, Tavenas; 2015)

if $n=2$, $\# \mathcal{Z}_r(\mathcal{S}, \mathbb{R}^2_+) \leq \mathcal{O}(d_2^3 t_1 + d_2^2 t_1^3)$

State of the Art (Zero dimensional case)

Even the bivariate case ($n = 2$)

is widely open!

A Probabilistic Approach

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Randomize F,

A Probabilistic Approach

Randomize F,

what can we say about

$$\mathbb{E}\,\#Z_r(F, \mathbb{R}^n_+)?$$

Probabilistic State of the Art I

(Bügisser, Ergür, Tonelli-Cueto; 2018)

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$$\mathbb{E}\, \# \mathcal{Z}_r(f, \mathbb{R}^n_+) \leq \frac{1}{2^{n-1}} \binom{t}{n}$$

Probabilistic State of the Art II

(Bürgisser, ISSAC'2023)

$f_i = \sum_{\alpha \in A_i} f_{i,\alpha} X^{\alpha}$ with $f_{i,\alpha}$ i.i.d. standard Gaussian

$t_i := \# A_i$

$$\mathbb{E}\, \# Z_r(f, \mathbb{R}^n_+) \leq \frac{1}{(2\pi)^{\frac{n}{2}}} V\left(\sum_{i=1}^{n} P_i\right) t_1 \cdots t_n$$

where $P_i := \operatorname{conv}(A_i)$ & $V(P) := \#$ vertices of P

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all the same variance

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$$\mathbb{E}\,\#\mathcal{Z}_r(F, \mathbb{R}^n_+) \leq \frac{1}{4^n} t_1(t_1-1)\cdots t_n(t_n-1)$$

where t_i is # monomials of $\quad$;

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$t_i := \# A_i$ & $\Lambda_i : A_i \to \mathbb{R}$ s.t. $\Lambda_i(\alpha) := \log \mathrm{Var}(f_{i,\alpha})$

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$$\mathcal{L}(A, \Lambda) := \mathrm{conv}\left\{ \begin{pmatrix} \Lambda(\alpha) - s \\ \alpha \end{pmatrix} \,\middle|\, \alpha \in A \;\; s \geq 0 \right\}$$

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$\mathcal{L}(A, \Lambda) := \mathrm{conv}\left\{ \begin{pmatrix} \Lambda(\alpha) - s \\ \alpha \end{pmatrix} \,\middle|\, \alpha \in A \quad s \geq 0 \right\}$

$$\mathbb{E} \, \# Z_r(f, \mathbb{R}^n_+) \leq \frac{1}{4^n} V\left(\sum_{i=1}^n \mathcal{L}(A_i, \Lambda_i) \right) t_1 \cdots t_n$$

where $V(P) := \#$ vertices of P

Two Tools I: Rice's Formula

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$$\mathbb{E}\#Z(f, \mathbb{R}^n_+) = \int_{\mathbb{R}^n_+} \mathbb{E}\left(|\det D_x f| \,\middle|\, f(x) = 0\right) \rho_{f(x)}(0)\, dx$$

Two Tools II: Cauchy-Binet Formula

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A, B $m \times n$ matrices

$\det AB^T$

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A, B $m \times n$ matrices

$$\det AB^T = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I = m}} \det A_I \det B_I$$

where X_I is the submatrix of X

- whose columns are indexed by I

What about the variance?

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THM (Tsigaridas, Tonelli-Cueto; 2024+)

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$f \in \mathbb{R}[X_1, \dots, X_n]^n$ random fewnomial system with $\sum P_i$ simple, where $P_i = \operatorname{conv}(A_i)$ & A_i support of A_i

Under robust prob. assump.

$$\left(\mathbb{E}\, \# \mathcal{Z}_r(f, \mathbb{R}^n_+)^e\right)^{1/e} \leq V\left(\sum_{i=1}^n P_i\right) \mathcal{O}(\log^2 D + n \log D)^n e^n$$

where $V(P) := \#$ vertices of P & D = max degree

Future Work:

Can we answer deterministically Kashnirenko's question probabilistically?

Eskerrik

Asko!