

A Numerical Algorithm for Zero Counting

IV

An Adaptive Speedup

Josué Tonelli-Cueto

Inria Paris / IMJ-PRG



Norway | Tromsø, June 7-11, 2021

THE PROBLEM

$$\mathcal{H}_d[n] \ni g$$



Real Homogeneous
Polynomial System
in x_0, \dots, x_n
& with $\deg g_i = d_i$

DETERMINISTIC
+
NUMERICALLY
STABLE
+
'GOOD' PROBABILISTIC
RUN-TIME (for random g)

↑
The ALGORITHM
we want!

$$\#Z_P(g)$$



projective
zeros of g

The ORIGINAL TRILOGY

(by Cucker, Krick, Malajovich, Wschebor)



A numerical algorithm for zero counting, I: Complexity and accuracy

Felipe Cucker^{a,*}, Teresa Krick^{b,c}, Gregorio Malajovich^d, Mario Wschebor^e

^a Department of Mathematics, City University of Hong Kong, Hong Kong

^b Departamento de Matemática, Univ. de Buenos Aires, Argentina

^c CONICET, Argentina

^d Depto. de Matemática Aplicada, Univ. Federal do Rio de Janeiro, Brazil

^e Centro de Matemática, Universidad de la República, Uruguay

ARTICLE INFO

Article history:
Received 22 November 2007
Accepted 14 March 2008
Available online 20 April 2008

Keywords:
Counting algorithms
Polynomial systems
Finite precision

ABSTRACT

We describe an algorithm to count the number of distinct real zeros of a polynomial (square) system f . The algorithm performs $\mathcal{O}(\log(nD\kappa(f)))$ iterations (grid refinements) where n is the number of polynomials (as well as the dimension of the ambient space), D is a bound on the polynomials' degree, and $\kappa(f)$ is a condition number for the system. Each iteration uses an exponential number of operations. The algorithm uses finite-precision arithmetic and a major feature of our results is a bound for the precision required to ensure that the returned output is correct which is polynomial in n and D and logarithmic in $\kappa(f)$. The algorithm parallelizes well in the sense that each iteration can be computed in parallel polynomial time in n , $\log D$ and $\log(\kappa(f))$.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

In recent years considerable attention has been paid to the complexity of counting problems over the reals. The counting complexity class $\#P_R$ was introduced [20] and completeness results for $\#P_R$ were established [3] for natural geometric problems notably, for the computation of the Euler characteristic of semialgebraic sets. As one could expect, the “basic” $\#P_R$ -complete problem consists of counting the real zeros of a system of polynomial equations.

* Corresponding author.
E-mail addresses: macucker@cityu.edu.hk (F. Cucker), krick@dm.uba.ar (T. Krick), gregorio@ufrj.br (G. Malajovich), wschebor@mat.uba.ar (M. Wschebor).

J. Fixed point theory appl. 6 (2008), 285–294
© 2009 Birkhäuser Verlag Basel/Switzerland
1661-7738/020285-10, published online 14.11.2009
DOI 10.1007/s11784-009-0127-4

Journal of Fixed Point Theory and Applications

A numerical algorithm for zero counting. II: Distance to ill-posedness and smoothed analysis

Felipe Cucker, Teresa Krick, Gregorio Malajovich and Mario Wschebor

To Steve, on his 80th birthday, with admiration and esteem

Abstract. We show a Condition Number Theorem for the condition number of zero counting for real polynomial systems. That is, we show that this condition number equals the inverse of the normalized distance to the set of ill-posed systems (i.e., those having multiple real zeros). As a consequence, a smoothed analysis of this condition number follows.

Mathematics Subject Classification (2000). 65Y20, 65H10.

Keywords. Polynomial systems, zero counting, condition numbers, smoothed analysis.

1. Introduction

This paper continues the work in [8], where we described a numerical algorithm to count the number of zeros in n -dimensional real projective space of a system of n real homogeneous polynomials. The algorithm works with finite precision and both its complexity and the precision required to ensure correctness are bounded in terms of n , the maximum D of the polynomials' degrees, and a condition number $\kappa(f)$.

In this paper we replace $\kappa(f)$ —which was originally defined using the computationally friendly infinity norm—by a version $\tilde{\kappa}(f)$ (defined in Section 2 below) which uses instead Euclidean norms. This difference is of little consequence in complexity estimates since one has (cf. Proposition 3.3 below)

$$\frac{\tilde{\kappa}(f)}{\sqrt{n}} \leq \kappa(f) \leq \sqrt{2n} \tilde{\kappa}(f). \quad (1)$$



A numerical algorithm for zero counting. III: Randomization and condition

Felipe Cucker^{a,*}, Teresa Krick^{b,2}, Gregorio Malajovich^{c,3}, Mario Wschebor^d

^a Department of Mathematics, City University of Hong Kong, Hong Kong

^b Departamento de Matemática, Universidad de Buenos Aires & IMAS, CONICET, Argentina

^c Departamento de Matemática Aplicada, Universidade Federal do Rio de Janeiro, Brazil

^d Centro de Matemática, Universidad de la República, Uruguay

ARTICLE INFO

Article history:
Received 8 July 2010
Accepted 1 July 2011
Available online 30 August 2011

MSC:
12Y05
60G60
65Y20

Keywords:
Zero counting
Finite precision
Condition numbers
Average-case analysis
Rice formula

ABSTRACT

In a recent paper (Cucker et al., 2008 [8]) we analyzed a numerical algorithm for computing the number of real zeros of a polynomial system. The analysis relied on a condition number $\kappa(f)$ for the input system f . In this paper we look at $\kappa(f)$ as a random variable derived from imposing a probability measure on the space of polynomial systems and give bounds for both the tail $\mathbb{P}(\kappa(f) > a)$ and the expected value $\mathbb{E}(\log \kappa(f))$.
© 2011 Elsevier Inc. All rights reserved.

* Corresponding author.
E-mail addresses: macucker@cityu.edu.hk (F. Cucker), krick@dm.uba.ar (T. Krick), gregorio@ufrj.br (G. Malajovich), wschebor@mat.uba.ar (M. Wschebor).

¹ Partially supported by GRF grant City University 100810.

² Partially supported by grants ANPCYT 33671/05, UBACYT X113/2008-2010 and CONICET PIP/2010-2012.

³ Partially supported by CNPq grants 470031/2007-7, 303565/2007-1, and by FAPERJ.

Part 1 Algorithm
& condition-based complexity

Part 2 Condition Number Theorem
& probabilistic complexity

Part 3 Probabilistic analysis
without integral geometry

The ORIGINAL TRILOGY

(by Cucker, Krick, Malajovich, Wschebor)



A numerical algorithm for zero counting, I: Complexity and accuracy

Felipe Cucker^{a,*}, Teresa Krick^{b,c}, Gregorio Malajovich^d, Mario Wschebor^e

^a Department of Mathematics, City University of Hong Kong, Hong Kong
^b Departamento de Matemática, Univ. de Buenos Aires, Argentina
^c CONICET, Argentina
^d Depto. de Matemática Aplicada, Univ. Federal do Rio de Janeiro, Brazil
^e Centro de Matemática, Universidad de la República, Uruguay

ARTICLE INFO

Article history:
 Received 22 November 2007
 Accepted 14 March 2008
 Available online 20 April 2008

Keywords:
 Counting algorithms
 Polynomial systems
 Finite precision

ABSTRACT

We describe an algorithm to count the number of distinct real zeros of a polynomial (square) system f . The algorithm performs $\mathcal{O}(\log(nD\kappa(f)))$ iterations (grid refinements) where n is the number of polynomials (as well as the dimension of the ambient space), D is a bound on the polynomials' degree, and $\kappa(f)$ is a condition number for the system. Each iteration uses an exponential number of operations. The algorithm uses finite-precision arithmetic as a major feature of our results. A bound for the precision required to ensure that the returned output is correct which is polynomial in n and D and logarithmic in $\kappa(f)$. The algorithm parallelizes well in the sense that each iteration can be computed in parallel polynomial time in n , $\log D$ and $\log(\kappa(f))$.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

In recent years considerable attention has been paid to the complexity of counting problems over the reals. The counting complexity class $\#P_R$ was introduced [20] and completeness results for $\#P_R$ were established [3] for natural geometric problems notably, for the computation of the Euler characteristic of semialgebraic sets. As one could expect, the "basic" $\#P_R$ -complete problem consists of counting the real zeros of a system of polynomial equations.

* Corresponding author.
 E-mail addresses: macucker@cityu.edu.hk (F. Cucker), krick@dm.uba.ar (T. Krick), gregorio@ufrj.br (G. Malajovich), wschebor@cmat.edu.uy (M. Wschebor).

J. Fixed point theory appl. 6 (2008), 285–294
 © 2009 Birkhäuser Verlag Basel/Switzerland
 1661-7738/08/020285-10, published online 14.11.2009
 DOI 10.1007/s11784-009-0127-4

Journal of Fixed Point Theory and Applications

A numerical algorithm for zero counting. II: Distance to ill-posedness and smoothed analysis

Felipe Cucker, Teresa Krick, Gregorio Malajovich and Mario Wschebor

To Steve, on his 80th birthday, with affection and esteem

Abstract. We show a Condition Number Theorem for the condition number of zero counting for real polynomial systems. That is, we show that this condition number equals the inverse of the normalized distance to the set of ill-posed systems (i.e., those having multiple real zeros). As a consequence a smoothed analysis of this condition number follows.

Mathematics Subject Classification (2000). 65Y20, 65H10.

Keywords. Polynomial systems, zero counting, condition numbers, smoothed analysis.

1. Introduction

This paper continues the work in [8], where we described a numerical algorithm to count the number of zeros in n -dimensional real projective space of a system of n real homogeneous polynomials. The algorithm works with finite precision and both its complexity and the precision required to ensure correctness are bounded in terms of n , the maximum D of the polynomials' degree, and the condition number $\kappa(f)$.

In this paper we replace $\kappa(f)$, which was originally defined using the computationally friendly inf-norm, by a version $\tilde{\kappa}(f)$ (defined in Section 2 below) which uses instead Euclidean norms. This difference is of little consequence in complexity estimates since one has (cf. Proposition 3.3 below)

$$\frac{\tilde{\kappa}(f)}{\sqrt{n}} \leq \kappa(f) \leq \sqrt{2n} \tilde{\kappa}(f). \quad (1)$$

Advances in Applied Mathematics 48 (2012) 215–248



Contents lists available at ScienceDirect
 Advances in Applied Mathematics
 www.elsevier.com/locate/yaama



A numerical algorithm for zero counting. III: Randomization and condition

Felipe Cucker^{a,*}, Teresa Krick^{b,2}, Gregorio Malajovich^{c,3}, Mario Wschebor^d

^a Department of Mathematics, City University of Hong Kong, Hong Kong
^b Departamento de Matemática, Universidad de Buenos Aires, Argentina
^c Departamento de Matemática Aplicada, Universidade Federal do Rio de Janeiro, Brazil
^d Centro de Matemática, Universidad de la República, Uruguay

ARTICLE INFO

Article history:
 Received 8 July 2010
 Accepted 1 July 2011
 Available online 30 August 2011

MSC:
 12Y05
 65G10
 65Y20

Keywords:
 Zero counting
 Finite precision
 Condition numbers
 Average-case analysis
 Rice formula

ABSTRACT

In a recent paper (Cucker et al., 2008 [8]) we analyzed a numerical algorithm for computing the number of real zeros of a polynomial system. The analysis relied on a condition number $\kappa(f)$ for the input system f . In this paper we look at $\kappa(f)$ as a random variable derived from imposing a probability measure on the space of polynomial systems and give bounds for both the tail $\mathbb{P}(\kappa(f) > t)$ and the expected value $\mathbb{E}(\log \kappa(f))$.

© 2011 Elsevier Inc. All rights reserved.

* Corresponding author.
 E-mail addresses: macucker@cityu.edu.hk (F. Cucker), krick@dm.uba.ar (T. Krick), gregorio@ufrj.br (G. Malajovich), wschebor@cmat.edu.uy (M. Wschebor).

¹ Partially supported by GRF grant City University 100810.

² Partially supported by grants ANPCYT 33671/05, UBACYT X113/2008–2010 and CONICET PIP/2010–2012.

³ Partially supported by CNPq grants 470031/2007-7, 303565/2007-1, and by FAPERJ.

Part 1 Algorithm & condition-based complexity

Part 2 Condition Number Theorem & probabilistic complexity

Part 3 Probabilistic analysis without integral geometry

The CKMW algorithm



DETERMINISTIC

NUMERICALLY
STABLE

'GOOD' PROBABILISTIC RUN-TIME

With 'high probability', $\text{run-time}(\text{CKMW}, \mathcal{S}) \leq D^{O(n^2)}$
for \mathcal{S} KSS (average / smoothed)

KSS = Kostlan-Shub-Smale
Gaussian

$D := \max d_i$

THE SPIN-OFFS (by Ergür, Paouris, Rojas)

Found Comput Math (2019) 19:131–157
https://doi.org/10.1007/s10208-018-9380-5

FOUNDATIONS OF
COMPUTATIONAL
MATHEMATICS

The Journal of the Society for the Foundations of Computational Mathematics



Probabilistic Condition Number Estimates for Real Polynomial Systems I: A Broader Family of Distributions

Alperen A. Ergür¹ · Grigoris Paouris² ·
J. Maurice Rojas²

Received: 8 December 2016 / Revised: 6 June 2017 / Accepted: 30 October 2017 /
Published online: 31 January 2018
© SFOCM 2018

Abstract We consider the sensitivity of real roots of polynomial systems with respect to perturbations of the coefficients. In particular—for a version of the condition number defined by Cucker and used later by Cucker, Krick, Malajovich, and Wschebor—we establish new probabilistic estimates that allow a much broader family of measures than considered earlier. We also generalize further by allowing overdetermined systems. In Part II, we study smoothed complexity and how sparsity (in the sense of restricting which terms can appear) can help further improve earlier condition number estimates.

Keywords Condition number · Epsilon net · Probabilistic bound · Kappa · Real-solving · Overdetermined · Subgaussian

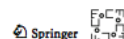
Communicated by Felipe Cucker.

Alperen A. Ergür was partially supported by NSF Grant CCF-1409020, NSF CAREER Grant DMS-1151711 and Einstein Foundation, Berlin. Grigoris Paouris was partially supported by BSF Grant 2010288 and NSF CAREER Grant DMS-1151711. J. Maurice Rojas was partially supported by NSF Grants CCF-1409020 and DMS-1460766.

✉ Alperen A. Ergür
erguer@math.m-bberlin.de
Grigoris Paouris
grigoris@math.tamu.edu
J. Maurice Rojas
rojas@math.tamu.edu

¹ Institut für Mathematik, Technische Universität Berlin, Sekretariat MA 3-2, Strasse des 17. Juni 136, 10623 Berlin, Germany

² Department of Mathematics, Texas A&M University TAMU 3368, College Station, TX 77843-3368, USA



SMOOTHED ANALYSIS FOR THE CONDITION NUMBER OF STRUCTURED REAL POLYNOMIAL SYSTEMS

ALPEREN A. ERGÜR, GRIGORIS PAOURIS, AND J. MAURICE ROJAS

ABSTRACT. We consider the sensitivity of real zeros of structured polynomial systems to perturbations of their coefficients. In particular, we provide explicit estimates for condition numbers of structured random real polynomial systems, and extend these estimates to smoothed analysis setting.

1. INTRODUCTION

Efficiently finding real roots of real polynomial systems is one of the main objectives of computational algebraic geometry. There are numerous algorithms for this task, but the core steps of these algorithms are easy to outline: They are some combination of algebraic manipulation, a discrete/polyhedral computation, and a numerical iterative scheme.

From a computational complexity point of view, the cost of numerical iteration is much less transparent than the cost of algebraic or discrete computation. This paper constitutes a step toward understanding the complexity of numerically solving structured real polynomial systems. Our main results are Theorems 1.14, 1.16, and 1.18 below, but we will first need to give some context for our results.

1.1. How to control accuracy and complexity of numerics in real algebraic geometry? In the numerical linear algebra tradition, going back to von Neumann and Turing, condition numbers play a central role in the control of accuracy and speed of algorithms (see, e.g., [3, 6] for further background). Shub and Smale initiated the use of condition numbers for polynomial system solving over the field of complex numbers [36, 37]. Subsequently, condition numbers played a central role in the solution of Smale's 17th problem [2, 5, 25].

The numerics of solving polynomial systems over the real numbers is more subtle than complex case: small perturbations can cause the solution set to change cardinality. One can even go from having no real zero to many real zeros by an arbitrarily small change in the coefficients. This behaviour doesn't appear over the complex numbers as one has theorems (such as the Fundamental Theorem of Algebra) proving that root counts are "generically" constant. Luckily, a condition number theory that captures these subtleties was developed by Cucker [11]. Now we set up the notation and present Cucker's definition.

Definition 1.1 (Bombieri-Weyl Norm). We set $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha := (\alpha_1, \dots, \alpha_n)$, and let $P = (p_1, \dots, p_{n-1})$ be a system of homogenous polynomials with degree pattern d_1, \dots, d_{n-1} . Let $c_{i,\alpha}$ denote the coefficient of x^α in p_i . We define the Weyl-Bombieri norms of p_i and P to be, respectively,

$$\|p_i\|_W := \sqrt{\sum_{\alpha_1 + \dots + \alpha_n = d_i} \frac{|c_{i,\alpha}|^2}{\binom{d_i}{\alpha}}}$$

A.E. was partially supported by Einstein Foundation, Berlin and by the Pravesh Kothari of CMU. G.P. was partially supported by Simons Foundation Collaboration grant 527498 and NSF grant DMS-1812240. J.M.R. was partially supported by NSF grants CCF-1409020, DMS-1460766, and CCF-1900881.

arXiv:1809.03626v2 [math.AG] 17 Feb 2020

1st non-gaussian
average complexity
in Numerical Alg. Geom!

1st non-gaussian
smoothed complexity in NAG!
(+ structured systems)

THE SPIN-OFFS (by Ergür, Paouris, Rojas)

Found Comput Math (2019) 19:131–157
https://doi.org/10.1007/s10208-018-9380-5

FOUNDATIONS OF
COMPUTATIONAL
MATHEMATICS

The Journal of the Society for the Foundations of Computational Mathematics



Probabilistic Condition Number Estimates for Real Polynomial Systems I: A Broader Family of Distributions

Alperen A. Ergür¹ · Grigoris Paouris² ·
J. Maurice Rojas²

Received: 8 December 2016 / Revised: 6 June 2017 / Accepted: 30 October 2017 /
Published online: 21 January 2018
© Society for the Foundations of Computational Mathematics 2018

Abstract We consider the sensitivity of real roots of polynomial systems with respect to perturbations of the coefficients. In particular—for a version of the condition number defined by Cucker and used later by Cucker, Krick, Mačajevič, and Wochebor—we establish new probabilistic estimates that allow a much broader family of measures than considered earlier. We also generalize further, by allowing overdetermined systems. In Part II, we study smoothed complexity and how sparsity (in the sense of restricting which terms can appear) can help further improve earlier condition number estimates.

Keywords Condition number · Epsilon net · Probabilistic bound · Kappa · Real-solving · Overdetermined · Subgaussian

Communicated by Felipe Cucker.

Alperen A. Ergür was partially supported by NSF Grant CCF-1409020, NSF CAREER Grant DMS-1151711 and Einstein Foundation, Berlin. Grigoris Paouris was partially supported by BSF Grant 2010288 and NSF CAREER Grant DMS-1151711. J. Maurice Rojas was partially supported by NSF Grants CCF-1409020 and DMS-1460766.

✉ Alperen A. Ergür
erguer@math.m-b-berlin.de
Grigoris Paouris
grigoris@math.tamu.edu
J. Maurice Rojas
rojas@math.tamu.edu

¹ Institut für Mathematik, Technische Universität Berlin, Sekretariat MA 3-2, Strasse des 17.
Juni 136, 10623 Berlin, Germany

² Department of Mathematics, Texas A&M University TAMU 3368, College Station,
TX 77843-3368, USA



SMOOTHED ANALYSIS FOR THE CONDITION NUMBER OF STRUCTURED REAL POLYNOMIAL SYSTEMS

ALPEREN A. ERGÜR, GRIGORIS PAOURIS, AND J. MAURICE ROJAS

ABSTRACT. We consider the sensitivity of real zeros of structured polynomial systems to perturbations of their coefficients. In particular, we provide explicit estimates for condition numbers of structured random real polynomial systems, and extend these estimates to smoothed analysis setting.

1. INTRODUCTION

Efficiently finding real roots of real polynomial systems is one of the main objectives of computational algebraic geometry. There are numerous algorithms for this task, but the core steps of these algorithms are easy to outline: They are some combination of algebraic manipulation, a discrete/polyhedral computation, and a numerical iterative scheme.

From a computational complexity point of view, the cost of numerical iteration is much less transparent than the cost of algebraic or discrete computation. This paper constitutes a step toward understanding the complexity of numerically solving structured real polynomial systems. Our main results are Theorems 1.14, 1.16, and 1.18 below, but we will first need to give some context for our results.

1.1. How to control accuracy and complexity of numerics in real algebraic geometry? In the numerical linear algebra tradition, going back to von Neumann and Turing, condition numbers play a central role in the control of accuracy and speed of algorithms (see, e.g., [3, 6] for further background). Shub and Smale initiated the use of condition numbers for polynomial system solving over the field of complex numbers [36, 37]. Subsequently, condition numbers played a central role in the solution of Smale's 17th problem [2, 5, 25].

The numerics of solving polynomial systems over the real numbers is more subtle than complex case: small perturbations can cause the solution set to change cardinality. One can even go from having no real zero to many real zeros by an arbitrarily small change in the coefficients. This behaviour doesn't appear over the complex numbers as one has theorems (such as the Fundamental Theorem of Algebra) proving that root counts are “generically” constant. Luckily, a condition number theory that captures these subtleties was developed by Cucker [11]. Now we set up the notation and present Cucker's definition.

Definition 1.1 (Bombieri-Weyl Norm). We set $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha := (\alpha_1, \dots, \alpha_n)$, and let $P = (p_1, \dots, p_{n-1})$ be a system of homogenous polynomials with degree pattern d_1, \dots, d_{n-1} . Let $c_{i,\alpha}$ denote the coefficient of x^α in a p_i . We define the Weyl-Bombieri norms of p_i and P to be, respectively,

$$\|p_i\|_W := \sqrt{\sum_{\alpha_1 + \dots + \alpha_n = d_i} \frac{|c_{i,\alpha}|^2}{\binom{d_i}{\alpha}}}$$

A.E. was partially supported by Einstein Foundation, Berlin and by the Pravesh Kothari of CMU. G.P. was partially supported by Simons Foundation Collaboration grant 527498 and NSF grant DMS-1812240. J.M.R. was partially supported by NSF grants CCF-1409020, DMS-1460766, and CCF-1900881.

arXiv:1809.03626v2 [math.AG] 17 Feb 2020

1st non-gaussian
average complexity
in Numerical Alg. Geom!

1st non-gaussian
smoothed complexity in NAG!
(+ structured systems)

The CKMW algorithm (after the spinoffs)



DETERMINISTIC

NUMERICALLY
STABLE

'GOOD' PROBABILISTIC RUN-TIME

With 'high probability', $\text{run-time}(\text{CKMW}, 8) \leq D^{O(n^2)}$
for 8 wide class of random systems

$D := \max d_i$

For random $g \in \mathcal{H}_d[n]$ as before,

$\mathbb{E}_g \text{run-time}(\text{CKMW}, g) < \infty?$

NO!

Idea!

Make CKMW adaptive,

then complexity should depend on

$$\mathbb{E}_{x \in S^n} \mathcal{K}(\xi, x)^n$$

which has finite expectation

for a random ξ !

$$\mathcal{K}(\xi, x) = \|\xi\|_W / \sqrt{\|\xi(x)\|^2 + \|D_x \xi^T \Delta^{-1/2} \eta\|^2} \quad \Delta = \text{diag}(d_i)$$

Inspiration: (Cucker, Ergür, T.-C.; 2019) while studying PV algorithm

Naive adaptive version fails!

(Eckhardt, 2020) (Han, 2018)

Run-time bound in terms of

$$\mathbb{E}_{x \in S^n} \mathcal{K}(\mathcal{g}, x)^{2n}$$

which has infinite expectation
for a random \mathcal{g} !

$$\mathcal{K}(\mathcal{g}, x) = \|\mathcal{g}\|_w / \sqrt{\|\mathcal{g}(x)\|^2 + \|D_x \mathcal{g}^{-1} \Delta^{\frac{1}{2}}\|^2} \quad \Delta = \text{diag}(d_i)$$

What goes wrong?

The criterion to select zeros!

The CKMW algorithm

- 1) Refine grid $G \subseteq S^n$ until $d_S(G, S^n)$ 'small'
- 2 $\left\{ \begin{array}{l} \text{Exclude points } x \in G \text{ s.t. } \|g(x)\|/\|g\|_w \text{ 'big'} \\ \text{Include points } x \in G \text{ s.t. } \|g(x)\|/\|g\|_w \text{ 'small'} \end{array} \right.$
- 3 Post-process the selected points to get
 $\#Z_{IP}(g)$

The CKMW algorithm

- 1) Refine grid $G \subseteq S^n$ until $\underbrace{d_S(G, S^n)}_{=: \delta} \leq \frac{1}{c D^2 \kappa(\delta)^2}$
- 2 $\left\{ \begin{array}{l} \text{Exclude points } x \in G \text{ s.t. } \|\delta(x)\| / \|\delta\|_w \geq \sqrt{D} \delta \\ \text{Include points } x \in G \text{ s.t. } \|\delta(x)\| / \|\delta\|_w \leq \frac{1}{\tilde{c} D^2 \kappa(\delta)^2} \end{array} \right.$
- 3 Post-process the selected points to get
 $\# Z_{IP}(\delta)$

Note quadratic condition in the inclusion criterion!

$$\kappa(\delta) := \max_{x \in S^n} \|\delta\|_w / \sqrt{\|\delta(x)\|^2 + \|D_x \delta^{-1} \Delta^{1/2}\|^{-2}} \quad \text{condition number}$$

The adaptive CKMW algorithm

NAIVE EDITION

- 1) Refine adaptively $G \subseteq \mathbb{S}^n \times (0, \infty)$ so that
 - 1) $\mathbb{S}^n \subseteq \bigcup \{B_G(x, r) \mid (x, r) \in G\}$
 - & 2) $\forall (x, r) \in G, r \leq 1/c D \kappa(\mathcal{F}, x)^2$
- 2 $\begin{cases} \text{Exclude } (x, r) \in G & \text{if } \|\mathcal{F}(x)\| / \|\mathcal{F}\|_w \geq \sqrt{D} \\ \text{Include } (x, r) \in G & \text{if } \|\mathcal{F}(x)\| / \|\mathcal{F}\|_w \leq 1/\tilde{c} D^2 \kappa(\mathcal{F}, x)^2 \end{cases}$
- 3 Post-process the selected points to get
 $\#Z_{IP}(\mathcal{F})$

Still quadratic inclusion criterion!

$$\kappa(\mathcal{F}, x) := \|\mathcal{F}\|_w / \sqrt{\|\mathcal{F}(x)\|^2 + \|D_x \mathcal{F}^{-1} \Delta^{1/2}\|^{-2}} \quad \text{local condition number}$$

Where does the square come from?

Smale's α -criterion:

$$\alpha(g, x) := \beta(g, x) \gamma(g, x) \leq \alpha_*$$

$$\Leftrightarrow \# B_S(x, 1.5 \beta(g, x)) \cap Z_S(g) = 1$$

$$\& \quad N_g^u(x) \xrightarrow{\text{quadratically}} \text{zero of } g$$

$$\text{where } \beta(g, x) := \|D_x g^{-1} g(x)\| \& \gamma(g, x) := \sup_{k \geq 2} \|D_x g^{-1} \frac{1}{k!} D_x^k g\|$$

Assume $\sqrt{2} \kappa(g, x) \|g(x)\| / \|g\|_W < 1 \dots$

• Higher Derivative Estimate: $\gamma(g, x) \leq \frac{1}{2} D^{3/2} \kappa(g, x)$

• A bad bound for β : $\beta(g, x) \leq \kappa(g, x) \|g(x)\| / \|g\|_W$

$$N_g(x) := \frac{x - D_x g^{-1} g(x)}{\|x - D_x g^{-1} g(x)\|}, \quad N_g^{h+1}(x) = N_g(N_g^h(x))$$

Where does the square come from?

Smale's α -criterion:

$$\alpha(g, x) := \beta(g, x) \gamma(g, x) \leq \alpha_*$$

$$\Leftrightarrow \# B_S(x, 1.5 \beta(g, x)) \cap Z_S(g) = 1$$

$$\& \quad N_g^u(x) \xrightarrow{\text{quadratically}} \text{zero of } g$$

$$\text{where } \beta(g, x) := \|D_x g^{-1} g(x)\| \& \gamma(g, x) := \sup_{k \geq 2} \|D_x g^{-1} \frac{1}{k!} D_x^k g\|$$

Assume $\sqrt{2} \mathcal{K}(g, x) \|g(x)\| / \|g\|_W < 1 \dots$

• Higher Derivative Estimate: $\gamma(g, x) \leq \frac{1}{2} D^{3/2} \mathcal{K}(g, x)$

• A bad bound for β : $\beta(g, x) \leq \mathcal{K}(g, x) \|g(x)\| / \|g\|_W$

↑ This creates the square!

$$N_g(x) := \frac{x - D_x g^{-1} g(x)}{\|x - D_x g^{-1} g(x)\|}, \quad N_g^{h+1}(x) = N_g(N_g^h(x))$$

We should use β directly!

Converse Smale's α -theorem:

$$\gamma(f, x) \operatorname{dist}_S(x, Z_S(f)) < 1$$

$$\Leftrightarrow \alpha(f, x) \leq \frac{\gamma(f, x) \operatorname{dist}_S(x, Z_S(f))}{1 - \gamma(f, x) \operatorname{dist}_S(x, Z_S(f))}$$

'If x is sufficiently near $Z_S(f)$,
then Smale's α -criterion at x holds'

Corollary. If $\sqrt{2} \kappa(f, x) \|f(x)\| / \|f\|_W < 1$,

then $\alpha(f, x) < d_*$

or $B_S(x, c/D^2 \kappa(f, x)) \cap Z_S(f) = \emptyset$

The adaptive CKMW algorithm

NON NAIVE EDITION!!!

- 1) Refine adaptively $G \subseteq \mathbb{S}^n \times (0, \infty)$ so that
 - 1) $\mathbb{S}^n \subseteq \bigcup \{B_G(x, r) \mid (x, r) \in G\}$
 - & 2) $\forall (x, r) \in G, \quad r \leq 1/c D \kappa(\mathcal{F}, x)$
- 2 $\begin{cases} \text{Exclude } (x, r) \in G & \text{if } \|\mathcal{F}(x)\| / \|\mathcal{F}\|_w \geq \sqrt{D} \vee \\ \text{Include } (x, r) \in G & \text{if } \beta(\mathcal{F}, x) \leq 1/c^2 D^2 \kappa(\mathcal{F}, x) \end{cases}$
- 3 Post-process the selected points to get
 $\#Z_{IP}(\mathcal{F})$

Using β gives the desired $\mathbb{E}_{x \in \mathbb{S}^n} \kappa(\mathcal{F}, x)^4$ bound!

$\kappa(\mathcal{F}, x) := \|\mathcal{F}\|_w / \sqrt{\|\mathcal{F}(x)\|^2 + \|D_x \mathcal{F}^{-1} \Delta^{1/2}\|^{-2}}$ local condition number

Some extra tricks

$$\chi(g, x) := \frac{\|g\|_w}{\sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{1/2}\|^{-2}}}$$

Change of norm

$$\hookrightarrow C(g, x) := \frac{\|g\|_\infty}{\max\{\|\Delta^{-1}g(x)\|_\infty, \|D_x g^{-1} \Delta^2\|_{\infty, 2}^{-1}\}}$$

where $\Delta := \text{diag}(d_i)$

Extra normalization

Row-normalization

$$\hat{g} := (g_i / \|g_i\|_\infty)_i$$

PROBABILISTIC MODEL

$g \in \mathcal{H}_d[n]$ dobro random system

$$g_i = \sum_{\alpha} \sqrt{\binom{d_i}{\alpha}} c_{i,\alpha} X^{\alpha}$$

with $c_{i,\alpha}$ independent,
centered

i.e. $\mathbb{E} c_{i,\alpha} = 0$

subgaussian with $\text{cte.} \leq K$;

i.e. $\mathbb{E} |c_{i,\alpha}|^e \leq K^e e^{e/2}$ for $e \geq 1$

anticoucentration $\text{cte.} \leq \rho$;

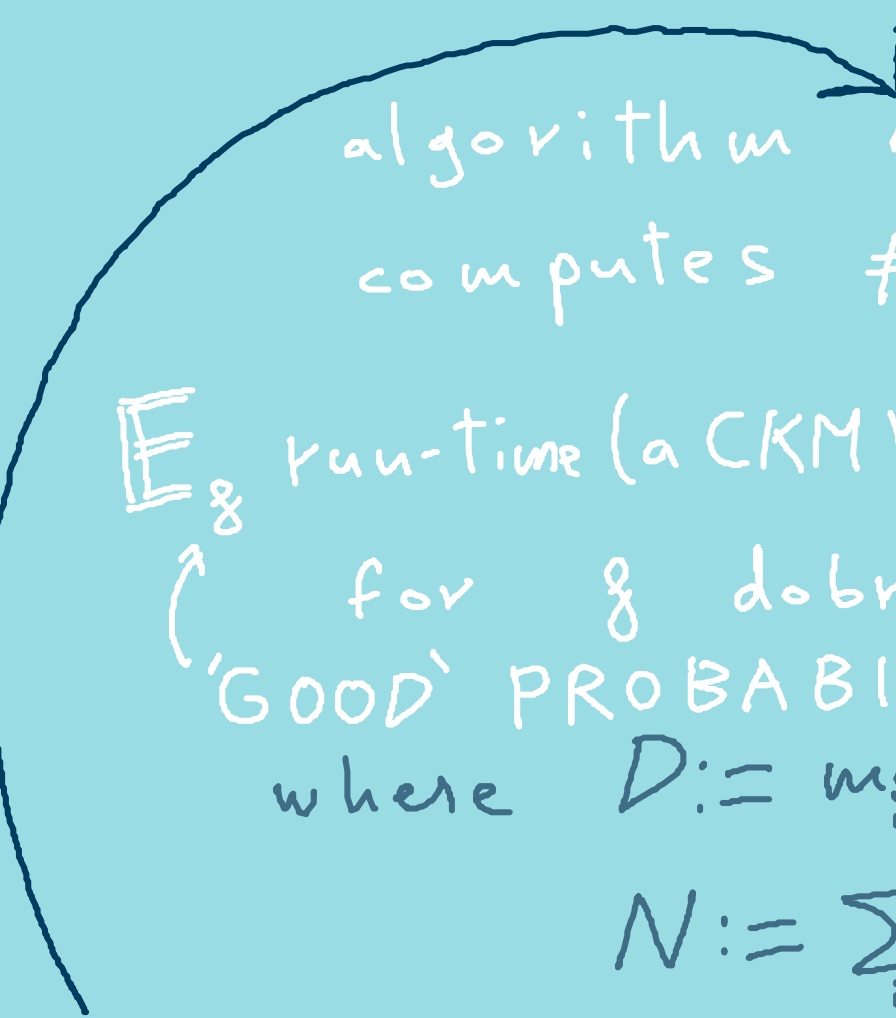
i.e. $\mathbb{P}(|c_{i,\alpha} - t| \leq \varepsilon) \leq 2\rho\varepsilon$ for $t \in \mathbb{R}$

We also
have a smoothed
version!

It generalizes KSS random systems where $c_{i,\alpha} \sim N(0,1)$

MAIN THEOREM

There is a DETERMINISTIC,
NUMERICALLY STABLE

algorithm  a CKMW that given $g \in \mathcal{H}_d[n]$
computes $\#Z_P(g)$ and such that

$$\mathbb{E}_g \text{run-time}(\text{a CKMW}, g) \leq 2^{O(n \log n)} D^N + 2^{O(n \log n)^{2.5}} \mathcal{D}^{(N+\mathcal{D})}$$

for g dobro random (with bounded parameters)

'GOOD' PROBABILISTIC RUN-TIME

where $D := \max_i d_i$ $\mathcal{D} := \prod_i d_i$

$N := \sum_i \binom{n+d_i}{n} = \# \text{ of zero \& non-zero coeff. of } g$

+ PARALLELIZABLE

FUTURE WORK

Homology computation

of semialgebraic sets

We can produce
correct adaptive
samples!

Post-processing
step has to be improved!

TUSEN TAKK!



E
M 21 A
G

Norway | Tromsø, June 7-11, 2021