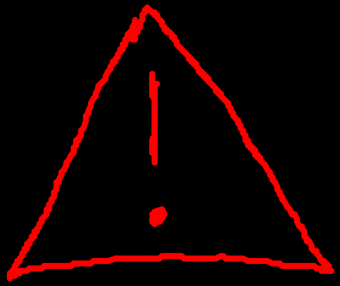
Condition-based Low-Degree Approximation of Real Polynomial Systems

Josué TONELLI-CUETO
UT San Antonio
Elias TSIGARIDAS
Inria Paris & IMJ-PRG

Slides at:
https://tonellicueto.xyz/pdf/JMM2023_slides.pdf



WARNING

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There will be one result on Eigenvalue Computations, the talk will focus on real polynomial systems

PROBLEM

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Given a real polynomial system

$$f_1(x) = \cdots = f_q(x) = 0$$

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in n variables,

1) what can we say about the conditioning of solving?
2) what does the condition say about the zero set?

Conditioning

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- The condition number depends on the metric

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  — how we measure errors —

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  — how we measure errors —
- The condition number depends on the encoding
  — how we write the problem —

Condition à la Demmel-Renegar

— conic framework

Condition à la Demmel-Renegar — conic framework

$\mathcal{I}$ input space

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$\Sigma \subseteq \mathcal{I}$ ill-posed inputs

Condition à la Demmel-Renegar — conic framework

$\mathcal{I}$ input space

$\Sigma \subseteq \mathcal{I}$ ill-posed inputs

$$C(i) := \frac{1}{d(i, \Sigma)}$$

WEYL

Setting

— projective

Set Up

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$$\mathcal{H}_d := \prod_{i=1}^{n} \mathbb{R}[X_0, \dots, X_n]_d .$$

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$$\Sigma = \{ f \in \mathcal{H}_d \mid f \text{ has a singular zero} \}$$

Weyl Norm

$$\|g\|_W := \sqrt{\sum_{i=1}^{n} \sum_{|\alpha|=d_i} \binom{d_i}{\alpha}^{-1} |g_{i,\alpha}|^2}$$

where

$$g_i = \sum g_{i,\alpha} X^\alpha \qquad \left(X^\alpha := X_0^{\alpha_0} \cdots X_n^{\alpha_n} \right)$$

$$\binom{d_i}{\alpha} = \frac{d_i!}{\alpha_0! \cdots \alpha_n!}$$

Weyl Condition Number

(Cucker, Krick, Malajovich, Wschebov)

$$\kappa_w(g) := \frac{\|g\|_w}{\mathrm{dist}_w(g, \Sigma)}$$

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$$\kappa_w(g) := \frac{\|g\|_w}{\mathrm{dist}_w(g, \Sigma)}$$

$$= \sup_{x \in S^n} \frac{\|g\|_w}{\sqrt{\|g(x)\|^2 + \|D_x g^{-1} \Delta^{1/2}\|^{-2}}}$$

where $\Delta = \mathrm{diag}(d_i)$ & $D_x g : T_x S^n \to \mathbb{R}^n$

Main Theorem (Simple form)

$$\#Z(\mathfrak{g}, \mathbb{P}^n) \leq D^{n/2}\,\mathrm{poly}(n, \log D)^n \log^n(2\kappa_w(\mathfrak{g}))$$

where $D = \max d_i$

MAIN THEOREM (COMPLEX FORM)

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There is a cover

$$\left\{ B\left(x, 1/c\sqrt{D}\right) \right\}_{x \in \mathcal{G}}$$ of size $O(D^{n/2})$

of $\mathbb{S}^n$

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s.t. $\# Z(g, T_x \mathbb{S}^n \cap B(x, \tfrac{1}{c\sqrt{D}})) \leq \# Z(\Phi_{x,g}, T_x \mathbb{S}^n)$.

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there is $\Phi_{x,\mathfrak{f}} \in \mathbb{R}[X_0,\dots,X_n]^n$ of degree

$$\leq \mathrm{poly}(n, \log D) \log(2\,\mathcal{K}_w(\mathfrak{f}))$$

s.t. $\#\mathcal{Z}(\mathfrak{f}, T_x\mathbb{S}^n \cap B(x, 1/c\sqrt{D})) \leq \#\mathcal{Z}(\Phi_{x,\mathfrak{f}}, T_x\mathbb{S}^n)$.

Moreover, for all $\zeta \in \mathcal{Z}(\mathfrak{f}, T_x\mathbb{S}^n \cap B(x, 1/c\sqrt{D}))$ there is $z \in \mathcal{Z}(\Phi_{x,\mathfrak{f}}, T_x\mathbb{S}^n)$ converging quadratically under Newton's method.

Corollary 1 of Main Result

- If $\#Z(\delta) \geq D^n$,

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$$\kappa_w(\delta) \geq 2^{D^{n/2}/\mathrm{poly}(n, \log D)^n}$$

Corollary 1 of Main Result

- If $\#Z(\mathcal{S}) \geq D^n$, then

$$\kappa_w(\mathcal{S}) \geq 2^{D^{n/2}/\mathrm{poly}(n,\log D)^n}$$

- Real systems with many zeros are badly-conditioned

COROLALLY 2 OF MAIN THEOREM

Let $f \in \mathcal{H}_d$ be random

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Let $f \in \mathcal{H}_d$ be random such that for all $x \in \mathbb{S}^n$, the $f_i(x)$ are independent, subgaussian and with anti-concentration.

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$$\left(\mathbb{E}\, \#Z(f, \mathbb{P}^n)^{\ell}\right)^{1/\ell} \leq D^{n/2} \operatorname{poly}(n, \log D)^n \ell^n$$

COROLALLY 2 OF MAIN THEOREM

Let $f \in \mathcal{H}_d$.

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Let $f \in \mathcal{H}_d$. Under very general random hypotheses,

$$\# Z(f, \mathbb{P}^n)^{1/n}$$

is subexponential with constant

$$D^{1/2} \operatorname{poly}(n, \log D)$$

I.e.

$$\mathbb{P}\left(\# Z(f, \mathbb{P}^n)^{1/n} \geq t\right) \leq e^{1 - \frac{t}{D^{1/2} \operatorname{poly}(n, \log D)}}$$

COROLALLY 2 OF MAIN THEOREM

A KSS random polynomial system

$$f \in \mathcal{H}_{(D,\dots,D)}$$

has its number of real zeros concentrated around

$$D^{n/2} = \mathbb{E}\,\#Z(f,\mathbb{P}^n)$$

COMPARISSON WITH LERARIO ET. AL.

(Lerario, Diatta; '22) (Breiding, Keneshlou, Lerario; '22)

COMPARISSON WITH LERARIO ET. AL.

OUR APPROACH | LERARIO'S APPROACH

(Lerario, Diatta; '22) (Breiding, Keneshlou, Lerario; '22)

COMPARISSON WITH LERARIO ET. AL.

OUR APPROACH

Taylor Approx.

LERARIO'S APPROACH

S. Harmonic Approx.

(Lerario, Diatta; '22) (Breiding, Keneshlou, Lerario; '22)

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Many local approx.

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A global approx.

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Comparisson with Lerario et. al.

Our Approach

Taylor Approx.

Many local approx.

Control the moments

Lerario's Approach

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Our Approach	Lerario's Approach
Taylor Approx.	S. Harmonic Approx.
Many local approx.	A global approx.
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Exploits analycity!	

(Lerario, Diatta; '22) (Breiding, Keneshlou, Lerario; '22)

1-Norm Setting

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$$I^n := [-1, 1]^n$$

$$\Sigma = \{ f \in \mathcal{P}_A \mid f \text{ has a singular zero in } I^n \}$$

1-Norm

$$\|g\|_1 := \max_{i} \sum_{\alpha \in A} |g_{i,\alpha}|$$

where

$$g_i = \sum g_{i,\alpha} X^\alpha \qquad \left(X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n} \right)$$

Cubic Condition Number

(TC, Tsigaridas)

$$C_1(f) := \sup_{x \in I^n} \frac{\|f\|_1}{\max\{\|f(x)\|_\infty, \|D_x f^{-1} \Delta\|_{\infty,\infty}^{-1}\}}$$

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$$\sim \frac{\|f\|_1}{\operatorname{dist}_1(f, \Sigma)}$$

where $\Delta = \operatorname{diag}(d_i)$ & $D_x f : \mathbb{R}^n \to \mathbb{R}^n$

Main Theorem (Simple form)

$$\#Z(g, I^n) \leq \mathrm{poly}(n)^n \log^{2n}(2D) \log(2C_1(g))$$

where $D = \max d_i$

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s.t. $\#Z(\delta,B)\leq\#Z(\Phi_{B,\delta})$.

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s.t. $\#Z(\mathfrak{f},B)\leq\#Z(\Phi_{B,\mathfrak{f}})$.

Moreover, for all $\zeta\in Z(\mathfrak{f},B)$, there is $z\in Z(\Phi_{B,\mathfrak{f}})$, converging quadratically under Newton's method.

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The Details

Main Tricks

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- Well-conditioned polynomials converge fast around zeros — as fast as a geometric series

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$$\gamma(g,x) := \max\left\{ 1,\ \sup_{k \geq 2} \left\| D_x g^{-1} \frac{1}{k!} D_x^k g \right\| \right\}$$

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Smales α-Theorem There is absolute $\alpha_* > 0$, s.t.

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Smales α-Theorem There is absolute $\alpha_* > 0$, s.t. if $\alpha(g,x) < \alpha_*$, then the Newton method at x converges quadratically. More concretely, $\mathrm{dist}(N_g^k(x), Z(g)) = O(2^{-2^k})$

Truncation Theorem (One version)

Let $g \in \mathbb{R}[X_1, \ldots, X_n]^n$, $\delta \in \mathbb{N}$, $x \in \mathbb{B}^n$

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Let $f \in \mathbb{R}[X_1, \dots, X_n]^n$, $\delta \in \mathbb{N}$, $x \in \mathbb{B}^n$ &

$$\tau(f, x; \delta) := \sup_{k \geq \delta+1} \left\| D_x f^{-1} \frac{2^k}{k!} D_0^k f \right\|$$

Truncation Theorem (One version)

Let $g \in \mathbb{R}[X_1, \dots, X_n]^n$, $\delta \in \mathbb{N}$, $x \in \mathbb{B}^n$ &

$$T(g, x; \delta) := \sup_{k \geq \delta+1} \left\| D_x g^{-1} \frac{2^k}{k!} D_0^k g \right\|$$

Consider

$$g_{|\delta}(X) := \sum_{k=0}^{\delta} \frac{1}{k!} D_0^k g(X, \dots, X)$$

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Then for

$$\delta - \log(\delta + 2) \geq \log T(f, x; \delta),$$

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Then for

$$\delta - \log(\delta + 2) \geq \log \tau(g, x; \delta),$$

$$\alpha(g_{|\delta}, x) \leq \frac{2\,\alpha(g, x) + 2^{1-\delta}\,\gamma(g, x)\,\tau(g, x; \delta)}{\left(1 - 2^{-\delta}(\delta+2)\,\tau(g, x; \delta)\right)^2}$$

I.e.

approximate zero of g à la Smale

$$\Downarrow$$

approximate zero of $g|\delta$ à la Smale

+ reverse & more ineqs.

Moroz's Lemma

W-Lemma: For $g \in \mathbb{R}[X_0, X_1]_d$ & $(x_0, x_1) \in S^1$,

$$\left| \frac{1}{k!} \frac{d^k}{dt^k}\Big|_{t=0} g\big((x_0, x_1) + t(x_1, -x_0)\big) \right| \leq \sqrt{\binom{d}{k}}\, \|g\|_W$$

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1-Lemma: For $g \in \mathbb{R}[X]_{\leq d}$, $a \in I$

and $\rho > 0$,

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1-Lemma: For $g \in \mathbb{R}[X]_{\leq d}$, $a \in I$ and $\rho > 0$, if either $2|a| < 1-\rho$ or $\rho < 1/2d$ then

$$\left| \frac{1}{k!} \frac{d^k}{dt^k}\Big|_{t=0} g(a+\rho t)\right| \leq 1/2^k\, \|g\|_1$$

Multivariate Moroz's 1-Lemma

Let $g \in \mathbb{R}[X_1, \ldots, X_n]^n_{\leq D}$, $a \in I^n$ & $\rho \in (0,1]^n$

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Consider

$$g_{a,\rho} := \left(g_i(a + PX) / \|g_i\|_1 \right)$$

where $P = \mathrm{diag}(\rho)$.

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If for all i,

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Consider

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If for all i,

either $2|a_i| \leq 1 - \rho_i$ or $\rho_i \leq 1/2D$

then for all e,

$$\left\| \frac{2^e}{e!} D_0^e \, g_{a,\rho} \right\|_{\infty,\infty} \leq \binom{D+n-1}{n-1} \leq \left(1 + \frac{\ln(n-1)}{n-1} D \right)^{n-1}$$

A Theorem about Eigenvalues

Triangle of competition

Triangle of competition

Eigenvalue
Problems

Triangle of competition

Eigenvalue Problems

$\searrow$

Eigenvalues

Triangle of competition

Eigenvalue Problems

Polynomial Systems

Eigenvalues / Zeros

Triangle of competition

Eigenvalue Problems $\xrightarrow{\text{char polynomials}}$ Polynomial Systems

Eigenvalues / Zeros

Triangle of competition

$$
\begin{tikzcd}
\text{Eigenvalue Problems} \arrow[rr, bend right, "\text{char polynomials}"'] \arrow[dr] & & \text{Polynomial Systems} \arrow[ll, bend right, "\text{Linearization}"'] \arrow[dl] \\
& \text{Eigenvalues/Zeros} &
\end{tikzcd}
$$

Numerical Analist's Rule

NEVER USE
CHARACTERISTIC
POLYNOMIALS
TO COMPUTE
EIGENVALUES

A formalization for Hermitian Matrices

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A formalization for Hermitian Matrices

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I.e. characteristic polynomials of Hermitian matrices are badly conditioned.

Future Work

- Can we make all this into fast algorithms?

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  - avoid condition estimation

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- Generalize it beyond zero-dim systems

- Can we make all this into fast algorithms?
  - avoid condition estimation
- Generalize it beyond zero-dim systems
  - volume & Betti numbers

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