## A  Analysis of Greedy Power Density

Denote $OPT$ as an optimal solution for the monotone submodular maximization problem under a knapsack constraint. Let $l$ be the size of the final solution set constructed by Greedy Power Density and $S_i$ be the partial solution set constructed by Greedy Power Density after selecting $i$ elements for $0 \le i \le l$. We define $l'$ as the largest index such that, up to the point when the partial solution set $S_{l'}$ is constructed, no element from the optimal set $OPT$ has been selected (i.e., chosen by maximizing power density) but not added to the solution set due to the budget constraint. Let $u_i := S_i \setminus S_{i-1}$. Hence, for all $1 \le i \le l'$, it holds that

$$\frac{f(u_i \mid S_{i-1})}{c(u_i)^\theta} \ge \max_{e \in OPT \setminus S_{i-1}} \frac{f(e \mid S_{i-1})}{c(e)^\theta}. \quad (1)$$

Let $C_i := \sum_{e \in OPT \setminus S_{i-1}} c(e)^\theta$ and $D := \sum_{i=1}^{l'} c(u_i)^\theta$.

LEMMA A.1. *For $i = 1, \cdots, l'$, it holds that*

$$f(S_i) - f(S_{i-1}) \ge \frac{c(u_i)^\theta}{C_i} [f(OPT) - f(S_{i-1})].$$

PROOF. Since $f$ is monotone and submodular, we have

$$f(OPT) \le f(OPT \cup S_{i-1})$$
$$\le f(S_{i-1}) + \sum_{e \in OPT \setminus S_{i-1}} f(e \mid S_{i-1})$$
$$= f(S_{i-1}) + \sum_{e \in OPT \setminus S_{i-1}} c(e)^\theta \cdot \frac{f(e \mid S_{i-1})}{c(e)^\theta}$$
$$\le f(S_{i-1}) + \frac{f(u_i \mid S_{i-1})}{c(u_i)^\theta} \cdot C_i$$
$$= f(S_{i-1}) + \frac{f(S_i) - f(S_{i-1})}{c(u_i)^\theta} \cdot C_i,$$

where the third inequality holds by Inequality (1). The lemma follows by rearranging the terms. $\square$

LEMMA A.2. *Let $\{a_i\}_{i=0}^m$ be a sequence with $a_0 = 0$, $\{b_i\}_{i=1}^m$ be a sequence with $0 \le b_i \le 1$ and $c > 0$ be a constant satisfying*

$$a_i - a_{i-1} \ge b_i(c - a_{i-1}). \quad (2)$$

*Then for every $i \in \{1, \dots, m\}$, it holds that*

$$a_i \ge \left[1 - \prod_{j=1}^i (1 - b_j)\right] c. \quad (3)$$

PROOF. We proceed by induction on $i$. From (2) with $i = 1$, it holds that $a_1 \ge b_1 c$, which matches (3) for $i = 1$. Assume (3) holds for $i - 1$. Then from (2), it holds that $a_i \ge (1 - b_i)a_{i-1} + b_i c$. Applying the induction hypothesis to $a_{i-1}$, it holds that $a_i \ge (1 - b_i)\left[1 - \prod_{j=1}^{i-1}(1 - b_j)\right]c + b_i c$. Factoring out $c$ and simplifying gives $a_i \ge \left[1 - \prod_{j=1}^i(1 - b_j)\right]c$, which completes the induction. $\square$

LEMMA A.3. *If $0 \le b_i \le 1$ for all $1 \le i \le n$, it holds that $1 - \prod_{i=1}^n(1 - b_i) \ge 1 - e^{-\sum_{i=1}^n b_i}$.*

PROOF. For $x \in [0, 1]$, it holds that $\ln(1 - x) \le -x$. Hence, it follows that

$$\prod_{i=1}^n (1 - b_i) = \exp\left(\sum_{i=1}^n \ln(1 - b_i)\right) \le \exp\left(-\sum_{i=1}^n -b_i\right),$$

which proves the lemma. $\square$

DEFINITION A.1 (SATISFIABLE BOUNDS). *We call a pair of bounds $(A, B)$ (possibly dependent on the cost ratio $r$ and budget $b$) is satisfiable if and only if*

(1) $A \ge C_i$ for all $1 \le i \le l'$;
(2) $c(u_i)^\theta / A \le 1$ for all $u_i$ ($1 \le i \le l'$);
(3) $B \le D$.

LEMMA A.4. *Given a pair of satisfiable bounds $(A, B)$, it holds that*

$$\frac{f(S_{l'})}{f(OPT)} \ge 1 - \exp\left(-\frac{B}{A}\right). \quad (4)$$

*In particular, if $\theta = 0$, it holds that*

$$\frac{f(S_{l'})}{f(OPT)} \ge 1 - \left(1 - \frac{1}{A}\right)^B. \quad (5)$$

PROOF. Since $C_i \le A$ for all $1 \le i \le l'$, it holds that

$$f(S_i) - f(S_{i-1}) \ge \frac{c(u_i)^\theta}{C_i}[f(OPT) - f(S_{i-1})]$$
$$\ge \frac{c(u_i)^\theta}{A}[f(OPT) - f(S_{i-1})]$$

for $i = 1, \cdots, l'$ by Lemma A.1.

Using Lemma A.2 by setting $a_i = f(S_i)$ with $a_0 = f(S_0) = 0$, $m = l'$, $b_i = \frac{c(u_i)^\theta}{A} \in [0, 1]$ and $c = f(OPT) > 0$, we could obtain that

$$f(S_i) \ge \left[1 - \prod_{j=1}^i \left(1 - \frac{c(u_j)^\theta}{A}\right)\right] f(OPT)$$

for $i = 1, \cdots, l'$. Consider the solution set $S_{l'}$, it holds that

$$f(S_{l'}) \ge \left[1 - \prod_{j=1}^{l'} \left(1 - \frac{c(u_j)^\theta}{A}\right)\right] f(OPT).$$

Using Lemma A.3 by setting $b_j := c(u_j)^\theta / A$, we have

$$1 - \prod_{j=1}^{l'} \left(1 - \frac{c(u_j)^\theta}{A}\right) \ge 1 - \exp\left(-\frac{D}{A}\right).$$

Furthermore, since $D \ge B$, it follows that $1 - \exp\left(-\frac{D}{A}\right) \ge 1 - \exp\left(-\frac{B}{A}\right)$. Hence, we have

$$\frac{f(S_{l'})}{f(OPT)} \ge 1 - \exp\left(-\frac{B}{A}\right).$$

In particular, if $\theta = 0$, it holds that $D = l' \ge B$ and

$$1 - \prod_{j=1}^{l'} \left(1 - \frac{c(u_j)^\theta}{A}\right) = 1 - \left(1 - \frac{1}{A}\right)^{l'} \ge 1 - \left(1 - \frac{1}{A}\right)^B.$$

$\square$

LEMMA A.5. *It holds that $C_i \le \min\{\lfloor b \rfloor r^\theta, \lfloor b \rfloor^{1-\theta} b^\theta\}$ for all $1 \le i \le l'$.*

PROOF. For each $e \in OPT$, it holds that $1 \leq c(e) \leq r$. Then it follows that $C_i \leq |OPT \setminus S_{i-1}| \cdot r^\theta \leq \lfloor b \rfloor r^\theta$. Since $\theta \in [0, 1]$ and $OPT$ is a feasible solution, by Hölder's inequality, it holds that

$$\sum_{e \in OPT \setminus S_{i-1}} c(e)^\theta \leq |OPT \setminus S_{i-1}|^{1-\theta} \left( \sum_{e \in OPT \setminus S_{i-1}} c(e) \right)^\theta$$

$$\leq \lfloor b \rfloor^{1-\theta} b^\theta.$$

Together with the earlier bound $C_i \leq \lfloor b \rfloor r^\theta$, it holds that $C_i \leq \min\{\lfloor b \rfloor r^\theta, \lfloor b \rfloor^{1-\theta} b^\theta\}$. □

Define

$$\alpha_1(r, b) := \begin{cases} b - r, & \text{if } b < r \lfloor b \rfloor, \\ \lfloor b \rfloor, & \text{otherwise.} \end{cases} \quad (6)$$

LEMMA A.6. For the solution set $S_{l'}$, it holds that $c(S_{l'}) \geq \alpha_1(r, b)$ where $\alpha_1(r, b)$ is defined as Equation (6).

PROOF. Assume $b \leq c(V)$ (otherwise, it is trivial that $OPT = V$).
Firstly, we always have $c(S_{l'}) \geq b - r$. If $l' < l$, for the solution set $S_{l'}$, there exists an element $e^* \in OPT \setminus S_{l'}$ such that $c(e^*) + c(S_{l'}) > b$, implying that $c(S_{l'}) > b - r$ since $c(e^*) \leq r$. If $l' = l$, we prove it by contradiction. If $c(S_l) < b - r$, there exists $e^* \in V \setminus S_l$ such that $c(e^*) + c(S_l) \leq r + c(S_l) \leq b$, which contradicts the fact that $S_l$ is the output of the Greedy algorithm. Hence, it holds that $c(S_l) \geq b - r$.
If $b \geq r \lfloor b \rfloor$, we assert that $|S_{l'}| \geq \lfloor b \rfloor$. In fact, it holds that $\lfloor b \rfloor \leq b/r \leq b$, implying that $\lfloor b/r \rfloor = \lfloor b \rfloor$. Hence, we have $c(S_{l'}) = \sum_{e \in S_{l'}} c(e) \geq |S_{l'}| \geq \lfloor b/r \rfloor = \lfloor b \rfloor$ because we can choose at least $\lfloor b/r \rfloor = \lfloor b \rfloor$ elements without discarding any element due to budget violation. □

LEMMA A.7. It holds that $l' \geq \lfloor b/r \rfloor$.

PROOF. By the definition of cost ratio $r$, for any element $e \in V$, it holds that $c(e) \leq r$. If $l' < \lfloor b/r \rfloor$, it holds that

$$c(S_{l'}) + c(e) \leq (\lfloor b/r \rfloor - 1)r + r \leq b \quad (7)$$

for any element $e \in V \setminus S_{l'}$.
By the definition of $l'$, there exists an element $e^* \in OPT \setminus S_{l'}$ such that $e^* = \arg\max_{e \in V \setminus S_{l'}} \frac{f(e|S_{l'})}{c(e)^\theta}$ and $c(S_{l'}) + c(e^*) > b$, which contradicts Inequality (7).
Hence, $l' \geq \lfloor b/r \rfloor$. □

LEMMA A.8. It holds that $D \geq \max\{\lfloor b/r \rfloor, \alpha_1(r, b)/r^{1-\theta}\}$.

PROOF. For any element $e \in V$ and $\theta \in [0, 1]$, it holds that $c(e)^\theta \geq 1$. Hence, it follows that $D = \sum_{i=1}^{l'} c(u_i)^\theta \geq l'$. Since $l' \geq \lfloor b/r \rfloor$ (Lemma A.7), it holds that $D \geq \lfloor b/r \rfloor$.
By the definition of cost ratio $r$, it holds that $c(e) \leq r$, implying that $c(e)^\theta \geq c(e)/r^{1-\theta}$. Hence, it follows that $D = \sum_{i=1}^{l'} c(u_i)^\theta \geq \frac{c(S_{l'})}{r^{1-\theta}}$. Since $c(S_{l'}) \geq \alpha_1(r, b)$ (Lemma A.6), it holds that $D \geq \frac{\alpha_1(r, b)}{r^{1-\theta}}$.
Combining the above two bounds gives

$$D \geq \max\{\lfloor b/r \rfloor, \alpha_1(r, b)/r^{1-\theta}\}.$$

□

In the following, we specify these lower and upper bounds to derive approximation guarantees.
Let $A_1 := \lfloor b \rfloor r^\theta$ and $A_2 := \lfloor b \rfloor^{1-\theta} b^\theta$. From Lemma A.5, it holds that $C_i \leq \min\{A_1, A_2\}$. Let $B_1 := \lfloor b/r \rfloor$, $B_2 := \frac{\alpha_1(r, b)}{r^{1-\theta}}$. From

Lemma A.8, it holds that $D \geq \max\{B_1, B_2\}$. For any $u_i$ ($1 \leq i \leq l'$), it holds that $c(u_i)^\theta / \min\{A_1, A_2\} \leq 1$, since $c(u_i) \leq r \leq b$ for any $1 \leq i \leq l'$. Hence, $(\min\{A_1, A_2\}, \max\{B_1, B_2\})$ is satisfiable. Hence, we could conclude the approximation ratio of Greedy Power Density as the following theorem from Lemma A.4:

THEOREM A.9. [Theorem 2.2 in the main text] For any $\theta \in [0, 1]$, Greedy Power Density achieves an approximation ratio of $1 - e^{-\gamma(\theta)}$ where $\gamma(\theta) := \frac{\max\{B_1, B_2\}}{\min\{A_1, A_2\}}$. In particular, it achieves $1 - \left(1 - \frac{1}{\min\{A_1, A_2\}}\right)^{\max\{B_1, B_2\}}$ -approximation if $\theta = 0$.

# B  Corollaries of Greedy Power Density

LEMMA B.1. When $\theta = 0$, Greedy Power Density achieves an approximation ratio of $1 - (1 - 1/\lfloor b \rfloor)^{\lfloor b/r \rfloor}$.

PROOF. When $\theta = 0$, $\min\{A_1, A_2\} = \lfloor b \rfloor$. Hence, the lemma follows by setting $\theta = 0$ in Theorem A.9 and $1 - (1 - 1/\lfloor b \rfloor)^{\max\{\lfloor b/r \rfloor, \alpha_1(r,b)/r\}} \geq 1 - (1 - 1/\lfloor b \rfloor)^{\lfloor b/r \rfloor}$. □

LEMMA B.2. When $r \leq b/\lfloor b \rfloor$, Greedy Power Density achieves a $(1 - e^{-1})$-approximation ratio when $\theta = 0$.

PROOF. If $r \leq b/\lfloor b \rfloor$, it holds that $b/r \geq \lfloor b \rfloor$, implying that $\lfloor b \rfloor = \lfloor b/r \rfloor$ since $1 \leq r \leq b$. Hence, we have $1 - (1 - 1/\lfloor b \rfloor)^{\lfloor b/r \rfloor} \geq 1 - e^{-1}$ by Lemma B.1. □

LEMMA B.3. When $r > b/\lfloor b \rfloor$ and $b/r \geq (1 - \ln(9/4))^{-1} \approx 5.2890$, Greedy Power Density achieves a $(5/9)$-approximation ratio when $\theta = 1$.

PROOF. When $r > b/\lfloor b \rfloor$ and $\theta = 1$, we have $\min\{A_1, A_2\} = b$ and $\max\{B_1, B_2\} = \max\{\lfloor b/r \rfloor, b - r\}$. By Theorem A.9, Greedy Power Density achieves an approximation ratio of $1 - e^{-\max\{\lfloor b/r \rfloor, b - r\}/b} \geq 1 - e^{-1 + r/b}$.
It is easy to verify that $1 - e^{-1+r/b} \geq \frac{5}{9}$ is equivalent to $b/r \geq (1 - \ln(9/4))^{-1}$, which proves the lemma. □

LEMMA B.4. When $2 \leq \lfloor b/r \rfloor \leq 5$ and $\lfloor b \rfloor \leq \lfloor b/r \rfloor + 1$, Greedy Power Density achieves a $(5/9)$-approximation ratio when $\theta = 0$.

PROOF. Let $k := \lfloor b/r \rfloor$ and $m = \lfloor b \rfloor$. The approximation ratio from Lemma B.1 could be written as $1 - (1 - 1/m)^k$. The condition $1 - (1 - 1/m)^k \geq 5/9$ is equivalent to $m \leq \frac{1}{1 - (9/4)^{-1/k}}$. Since $m$ is a positive integer, it holds that $m \leq \left\lfloor \frac{1}{1 - (9/4)^{-1/k}} \right\rfloor$.
Evaluating this inequality for $2 \leq k \leq 5$ gives the integer thresholds as

$$k = 2 \Rightarrow m \leq 3,$$
$$k = 3 \Rightarrow m \leq 4,$$
$$k = 4 \Rightarrow m \leq 5,$$
$$k = 5 \Rightarrow m \leq 6.$$

Combining these results, we could conclude that if $2 \leq \lfloor b/r \rfloor \leq 5$ and $\lfloor b \rfloor \leq \lfloor b/r \rfloor + 1$, it holds that $1 - (1 - 1/m)^k \geq 5/9$. (We focus on $2 \leq k \leq 5$ only, because by Lemma B.3, if $k \geq 6$, Greedy Power Density already achieves a $(5/9)$-approximation ratio when $\theta = 1$.) □

Summarizing Lemma B.2, Lemma B.4 and Lemma B.3, we have identified three regions in the $(r, b)$-plane:

$$A := \{(r, b) \in S : r \leq b/\lfloor b \rfloor\},$$

$$B := \{(r, b) \in S : b/r \geq (1 - \ln(9/4))^{-1}\},$$

$$C := \{(r, b) \in S : \lfloor b/r \rfloor \in \{2, 3, 4, 5\}, \ \lfloor b \rfloor \leq \lfloor b/r \rfloor + 1\}$$

where Greedy with Power Density could achieve a $(5/9)$-approximation ratio. This proves Lemma 2.4.