assignment01

Wednesday, January 21, 2026 7:46 PM

1. (a) Simplify: $9^{\log_3 n} = n^2$

$\log_{10} 2^{n^3} = n^3 \cdot \log_{10} 2$ (Assume without proof)

$n^3 - n^2\log n \in \Theta(n^3)$

$\log_{10} 2 \cdot n^3 \in \Theta(n^3)$

Polynomials: $\sqrt{n} < n^{1.5} < 9^{\log_3 n} < n^3 - n^2\log n < (\log_{10} 2) \cdot n^3$

Compare $n!$, 4^{2n}, by stirling's approximation $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for big n

$$\lim_{n\to\infty} \frac{n!}{4^{2n}} = \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{16e}\right)^n = \infty \Rightarrow n! \in \omega(4^{2n}) \Rightarrow 4^{2n} < n!$$

Compare $n!$, n^n

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{e^n} = \lim_{n\to\infty} \frac{\frac{\sqrt{2\pi}}{2}\cdot\frac{1}{\sqrt{n}}}{e^n} = 0 \Rightarrow n! \in o(n^n) \Rightarrow n! < n^n$$

Since polynomial < exponential in terms of order of growth

$\log_{10} 2^{(n^3)} < 4^{2n}$

So overall: $\sqrt{n} < n^{1.5} < 9^{\log_3 n} < n^3 - n^2\log n < \log_{10} 2^{n^3} < 4^{2n} < n! < n^n$

(b) let $n = 3^x$ where $x \in \mathbb{Z}_+$

$$\begin{aligned}
g(n) = 3g\left(\tfrac{n}{3}\right) + n = G(x) = g(3^x) &= 3g(3^{x-1}) + 3^x \\
&= 3G(x-1) + 3^x \\
&= 3(3G(x-2) + 3^{x-1}) + 3^x \\
&= 3^2 G(x-2) + 2\cdot 3^x \\
&\;\;\vdots \quad g(3)=3 \\
&= 3^{x-1}\underbrace{G(1)}_{} + (x-1)3^x
\end{aligned}$$

$G(x) = 3^x + (x-1)3^x$

$g(n) = n + n\cdot(\log_3 n - 1) = n\log_3 n$

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2. Hypothesis: $\forall m \in \mathbb{Z}_{\geq 0}$, m can be expressed as a sum of finite set of Fib numbers, s.t. no two are the same and consequtive

Base Case: $m = 0 = Fib(0)$ ✓

$m = 1 = Fib(1)$ ✓

Inductive step: Assume $\forall x < m$, the hypothesis is true.

$\exists k \in \mathbb{Z}_+$, s.t. $F_k \leq m < F_{k+1}$, $F_{k+1} = F_k + F_{k-1}$

$\Leftrightarrow m < F_k + F_{k-1} \Leftrightarrow F_k + F_{k-1}$ cannot be summed to m

let $m = F_k + m^*$, then $m^* < m$, by assumption

* m^* is sum of non-duplicate, non-consequtive Fib numbers.

and m^* doesn't include F_{k-1} since $F_{k-1} + F_k > m$

therefore m is a sum of non-duplicate/non-consequtive Fib numbers.

By the principle of MI, the hypothesis is true.

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3. let $f(n) = n$, $g(n) = n^2$, $n \in o(n^2)$, let $0 \leq \alpha, \beta < 1$ s.t. $\log n + \alpha = \lceil \log n \rceil$

$\log n^2 + \beta = \lceil \log n^2 \rceil$

$$\lim_{n\to\infty} \frac{\lceil \log n \rceil}{\lceil \log n^2 \rceil} = \lim_{n\to\infty} \frac{\log n + \alpha}{\log n^2 + \beta} = \lim_{n\to\infty} \frac{\log n}{\log n^2} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2}{n}} = \frac{1}{2} \neq 0$$

therefor $\lceil \log n \rceil \notin o\lceil \log n^2 \rceil$ (L'Hopital's Rule)

$\Rightarrow$ not true