1. (a) $a = 95$ $b = 10$ $\log_b a < 2$

$f(n) = 223n^2 - 16$
$= \Theta(n^2)$

since $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$,

it fillows that

$\forall \varepsilon > 0, \cdots$ 不需要取值

$\exists \varepsilon_0 > 0, \cdots$

(regularity) $95 \cdot f(\frac{x}{10}) = 95 \cdot \frac{223n^2}{100} - 16$

v.s. $c \cdot f(x) = 223n^2 - 16$

to specify?

let $c = 1$, it holds

By Master Theorem case 3, we have $T(n) = \Theta(n^2)$

$\square$

(b) $a = 36$ $b = 6$ $\log_b a = 2$

$f(n) = \Theta(n^2) = \Theta(n^{\log_b a})$ $k = 0$

$\Theta(n^2 \log^k n)$

By Master Theorem case 2, we have $T(n) = \Theta(n^2 \log^{k+1} n)$

(c) Can not directly apply Master Theorem.

$T(n) \geq 2n \geq n$, $T(n) \leq T(\frac{n}{2}) + T(\frac{n}{2}) + 2n$

? $T(\sqrt{n}) \leq T(\frac{n}{2})$

$T(x) = x$

$\sqrt{n} > \frac{n}{2}$ $\quad n=1$

$\lim_{n \to \infty} \sqrt{n}$ $\quad \lim \frac{n}{2}$ $\to \lim_{n \to \infty} \frac{\frac{n}{2}}{\sqrt{n}} \to \infty$

T( ) 计算过程复杂度

$n \geq n_0$ 行为决定 T 的大小

$\Theta(n \log n)$ ✗ $\quad$ by Master Theorem $O(n \log n)$

$RHS = 2T(\frac{n}{2}) + 2n$ $\qquad RHS = \Theta(n \log n)$

$T(n) \leq RHS \implies T(n) = O(n \log n)$

$\cancel{= O(n^2)}$

$T(n) \geq 2n \implies T(n) = \Omega(n)$

? $T(n) = O(n)$ 同构、 $\iff T(n) \leq cn$ for some constant c. $\quad \forall n \geq n_0$

怎么找c?

Made with Goodnotes

assume $\forall m < n$, it holds that $T(m) \leq cm$.

$\frac{n}{2} < n$, $\sqrt{n} < n$,

$$T(n) = T(\tfrac{n}{2}) + T(\sqrt{n}) + 2n$$

$$\leq \frac{c}{2}n + c\sqrt{n} + 2n$$

① $\sqrt{n} \leq n$

② $\sqrt{n} \leq \frac{n}{2}$ $\forall n \geq 4$

③ $\sqrt{n} \leq \frac{n}{4}$ $\forall n \geq 16$

选① × $>$

$$\leq (\tfrac{3}{2}c + 2)n \ !! \leq cn$$

选③

$$\leq (\tfrac{3}{4}c + 2)n \leq cn$$

$$\iff \tfrac{3}{4}c + 2 \leq c$$

$$\iff c \geq 8$$

$\exists\ c = 8$, $n_0 = 16$, $\forall n \geq 16$, it holds that

$T(n) \leq 8n$.

Hence, $T(n) = \Theta(n)$.

$$\frac{c}{2}n + c\sqrt{n} + 2n = \frac{c}{2}n + c\,o(n) + 2n$$

$$\leq \frac{c}{2}n + c\varepsilon n + 2n \quad \ldots \quad \leq cn$$

2. $H(n) = H(n-1) + 2$

$= H(n-2) + 2 + 2$

$\vdots$

$= H(1) + \overbrace{2 + \cdots + 2}^{n-1}$

$= 2 + 2(n-1)$

$= 2n$

$T(n) = T(n-1) + 1 + 2(n-1)$

$= T(n-2) + 1$

(Telescoping) $T(n+1) - T(n) = 2n+1$

$$T(N) - T(1) = \sum_{i=1}^{N-1} T(i+1) - T(i)$$

$$= \sum_{i=1}^{N-1} 2i+1$$

$$= (N-1) + 2\frac{(1+N-1)(N-1)}{2}$$

$$= N^2 - 1$$

$T(N) = N^2$

$\square$

3. $m \le n$ 猜 $O(n)$

$$T(n) = \begin{cases} 1 \\ T(m) \end{cases}$$

例子: 令 $m=6$, $n=9$

iteration	m	n
0	6	9
1	3	6
2	0	3

terminate

return $n=3$

3 is (6, 9)'s GCD.

令 $m=5$, $n=7$

0	5	7
1	2	5
2	1	2
3	0	1

$O(1)$ divide

$$T(m,n) = T(n \bmod m, m) + O(1)$$

$$\vdots$$

$$= T(0, m) + \overbrace{O(1) \cdots + O(1)}^{?}$$

$$= O(1)$$

$(a, b) \to (b \bmod a, a)$

① if $b \geq 2a$, then $b \bmod a < a \leq \frac{b}{2}$;

$T(n) \to T(\frac{n}{2})$

② otherwise $b < 2a$, then $a \leq$

$b \bmod a = b - a < b - \frac{b}{2} = \frac{b}{2}$

$(b-a, a)$

$T(n) \leq T(\frac{n}{2}) + O(1)$ $\quad n \to 1$

$= O(\log n)$ $\quad \frac{n}{2^k} = 1$

$k = \log_2 n$

不是特别严谨，explanation.

Fibonacci Series