Probability Revision

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Overview

Introduction

Conditional Probability

Random Variable

Sample Space, Event

- A sample space S is a set whose elements are called elementary events.
- An event is a subset of the sample space S.
- For an event A, the event $\overline{A} = S - A$.
- Example: For throwing a dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$.
  The event of throwing an even number is $A = \{2, 4, 6\}$.

Probability Distribution

- A probability distribution Pr() is a mapping from events to real numbers such that
- $Pr(A) \geq 0$, for all events A
- $Pr(S) = 1$
- $Pr(A \cup B) = Pr(A) + Pr(B)$ for mutually exclusive (disjoint) events A and B.
- For dice, a uniform distribution is a reasonable model.
  $Pr(i) = 1/6$, for $i = 1, 2, 3, 4, 5, 6$.
  $Pr(\{2, 4, 6\}) = 1/6 + 1/6 + 1/6 = 3/6 = 1/2$.

- In general: $Pr(A) \leq Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) \leq Pr(A) + Pr(B)$.
- If $A = \{2,4,6\}$, $B = \{1,2\}$. Then $Pr(A \cup B) = Pr(A) \cup Pr(B) - Pr(A \cap B) = 1/2 + 1/3 - 1/6 = 2/3$.
- Two events A and B are called independent if $Pr(A \cap B) = Pr(A) \cdot Pr(B)$.
- Example: If we flip two fair coins. Let A be the event that the first coin is heads. Let B be the event that second coin is heads.
  Then $Pr(A \cap B) = Pr(A) \cdot Pr(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Conditional probability

- The conditional probability of an event A given another event B (where $Pr(B) \neq 0$) is defined as

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

- Bayes' Theorem:

$$Pr(A|B) = \frac{Pr(A)Pr(B|A)}{Pr(B)}$$

$$= \frac{Pr(A) \cdot Pr(B|A)}{Pr(A) \cdot Pr(B|A) + Pr(\overline{A}) \cdot Pr(B|\overline{A})}$$

Bayes' Theorem: example

Suppose we have one fair coin and one biased coin that always gives heads. We choose one coin out of these two coins, uniformly at random. Suppose when we toss the chosen coin twice, we get heads both times. What is the probability that the coin chosen was biased?

Let A be the event of choosing the biased coin.

Let B be the event that both coin tosses with the chosen coin are heads.

$Pr(A) = \frac{1}{2}$, $Pr(B|A) = 1$, $Pr(\overline{A}) = \frac{1}{2}$, $Pr(B|\overline{A}) = \frac{1}{4}$.

$$Pr(A|B) = \frac{Pr(A) \cdot Pr(B|A)}{Pr(A) \cdot Pr(B|A) + Pr(\overline{A}) \cdot Pr(B|\overline{A})}$$

$$= \frac{(1/2) \cdot 1}{(1/2) \cdot 1 + (1/2) \cdot (1/4)} = \frac{4}{5}$$

Random Variable

A random variable X is a function that maps the sample space S to real numbers.

The function $f(x) = Pr(X = x)$ is the probability density function of X.

Example: In a roll of a pair of dice, let X be the max of the two values shown on the dice.

Then $Pr(X = 3) = 5/36$, because the elementary events which give $X = 3$ are $(1, 3), (2, 3), (3, 3), (3, 1), (3, 2)$, and there are a total of 36 elementary events in the sample space (each of which is equally likely)

Expectation

- The expectation or mean of a random variable X is

$$E(X) = \sum_x x \cdot Pr(X = x)$$

- Example: Suppose X is the outcome of a dice.
  $E(X) = \sum_i i \cdot Pr(X = i) = \sum_i (i/6) = 3.5$

Linearity of Expectations

- For any two events X, Y and a constant a

$$E(X + Y) = E(X) + E(Y)$$

$$E(aX) = aE(X)$$

- If X and Y are independent then

$$E(XY) = E(X) \cdot E(Y)$$

Bernoulli Trial

- An instance of a Bernoulli trial has probability p of success and probability $q = 1 - p$ of failure.
- Suppose we have a sequence of independent Bernoulli trials, each with probability p of success. Let X be the number of trials needed to obtain a success for the first time.
  Then,

$$Pr(X = k) = q^{k-1}p$$

$$E(X) = \frac{1}{p}$$

- Suppose X is the number of successes in n Bernoulli trials. Then

$$Pr(X = k) = \binom{n}{k} p^k q^{n-k}$$

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