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27/9 Locally compact (Hausdorff) abelian groups.
We compute RHom between LCA. LCA don't form an abelian cat., by RHom we mean RHom for Cond(Ab). (Pointset topology exercise: Locally compact $\Rightarrow$ compactly generated.)

[**Def.** $X \in Top$ is locally compact : X is Hausdorff and any $x \in X$ has open U, compact K s.t. $x \in K \subseteq U$.]

1. Recall & Complement on Cond(Ab).

① $\otimes$ is sheafification of $S \mapsto M(S) \otimes N(S)$ for $M, N \in Cond(Ab)$, $S \in CHaus$.

- $\forall T_i \in Cond$, $\mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2] = \mathbb{Z}[T_1 \times T_2] \in Cond(Ab)$
(check : ~~LHS(S) =~~, on presheaf level,
$LHS(S) = \mathbb{Z}[Map(S, T_1)] \otimes \mathbb{Z}[Map(S, T_2)] = \mathbb{Z}[Map(S, T_1) \times Map(S, T_2)] = \mathbb{Z}[Map(S, T_1 \times T_2)] = RHS(S)$
where Map means map in Top.)

- $\forall T \in Cond$, $\mathbb{Z}[T]$ is flat.
(because, on presheaf level,
$A \otimes \mathbb{Z}[T] : S \mapsto A(S) \otimes \mathbb{Z}[Map(S, T)]$ and this is exact since $\mathbb{Z}[Map(S,T)] \in Ab$ is free)

② For $M, N \in Cond(Ab)$, $Hom(M, N) \in Ab$.
There is a natural way to enrich it to a Cond(Ab), i.e. defining an internal $\underline{Hom}$:
$\forall P \in Cond(Ab)$, $Hom(P, \underline{Hom}(M,N)) \cong Hom(P \otimes M, N)$.

$$\left(\begin{tikzcd} Cond(Ab) \arrow[r, shift left=2, "\otimes M"] \arrow[r, phantom, "\perp"] & Cond(Ab) \arrow[l, shift left=2, "\underline{Hom}(M,-)"] \end{tikzcd}\right)$$

- Take $P = \mathbb{Z}[S]$, $S \in CHaus$, get
$$Hom(\mathbb{Z}[S], \underline{Hom}(M,N)) \cong Hom(\mathbb{Z}[S] \otimes M, N)$$
$$= Hom(S, \underline{Hom}(M,N))$$
$$= \underline{Hom}(M,N)(S)$$

for top. gp. A, B, endow $\mathrm{Hom}(A,B)$ with compact-open topology. (PSTop exercise: $\forall S \in \mathrm{Top}$. Map. $S \to \mathrm{Hom}(A,B)$ continuous $\iff S \times A \to B$ continuous.

Question: Is $\underline{\mathrm{Hom}(M,N)} \cong \underline{\mathrm{Hom}(M,N)}$? where RHS is ~~discrete~~

Partial answer: NO. $\underline{\mathrm{Hom}}$ remembers "topology" of M, N:

Prop A, B ... Hausdorff top. ab. gp.,
A compactly generated (as top. sp.)
Then $\exists$ natural iso.

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \xrightarrow{\sim} \underline{\mathrm{Hom}(A,B)} \quad \left(\begin{array}{l}\mathrm{Hom}(A,B) \text{ is with} \\ \text{comp-open top.}\end{array}\right)$$

proof: $\forall S \in \mathrm{cHaus}$, $\underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) = \mathrm{Hom}(\underline{A} \otimes \mathbb{Z}[S], \underline{B})$.
~~go further:~~ $=$ $\underline{\mathrm{Hom}(A,B)}(S) = \mathrm{Map}(S, \mathrm{Hom}(A,B)) = \mathrm{Map}(A \times S, B)$.

Notice surjection $\mathbb{Z}[\underline{A}] \to \underline{A} \to 0$ (this means
$[a] \mapsto a$ .... 'universal' for AG.

$\forall S \in$ ~~cHaus~~ ExtDis, $\mathbb{Z}[\underline{A}](S) \to \underline{A}(S) \to 0$

$$\begin{array}{ccc} \mathbb{Z}[\underline{A}](S) & \to & \underline{A}(S) \\ \| & & \| \\ \mathbb{Z}[\mathrm{Map}(S,A)] & & \mathrm{Map}(S,A) \\ {[a]} & \mapsto & a \end{array}$$
, here $a \in \mathrm{Map}(S, A)$.

So $\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[S] \to \underline{A} \otimes \mathbb{Z}[S] \to 0$ ($\mathbb{Z}[S]$ flat)

$$\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[S] = \mathbb{Z}[\underline{A} \times S]$$

So any element in $\mathrm{Hom}(\underline{A} \otimes \mathbb{Z}[S], \underline{B})$ induces an element in $\mathrm{Hom}(\mathbb{Z}[\underline{A} \times S], \underline{B})$ which by adjunction $=$ ~~$\mathrm{Map}(\underline{A} \times S$~~, $\mathrm{Hom}(\underline{A} \times S, \underline{B})$ again by adjunction $= \mathrm{Map}(A \times S, B)$ in Top.

This is clearly injective.

To show surjectivity, need to show: given map $A \times S \to B$, the induced map $\mathbb{Z}[\underline{A} \times S] \to \underline{B}$ factors through $\underline{A} \otimes \mathbb{Z}[S]$, i.e. $\ker(\mathbb{Z}[\underline{A}] \to \underline{A}) \otimes \mathbb{Z}[S]$ is "mapped to 0", i.e. $\mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[S] \to \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[S] \to \underline{B}$ composition is 0, where

$$\begin{array}{ccccccc} \mathbb{Z}[\underline{A} \times \underline{A}] & \to & \mathbb{Z}[\underline{A}] & \to & \underline{A} & \to & 0 \\ {[(a,b)]} & \mapsto & [a+b]-[a]-[b] & & & & \end{array}$$

But $\mathbb{Z}[\underline{A}\times\underline{A}\times \underline{S}] \longrightarrow \mathbb{Z}[\underline{A}\times \underline{S}] \longrightarrow \underline{B}$

$\leftrightsquigarrow$ $A\times A\times S \longrightarrow A\times S \longrightarrow B$

$(a,b,s) \longmapsto (ab-a-b, s)$

$(\overset{\shortparallel}{0}, s) \longmapsto 0$ (since $(A\times S\to B) \sim (S\to \mathrm{Hom}(A,B))$)

$\square$

③ Form $D(\mathrm{Cond}(Ab))$. Use projective resolutions to define $R\mathrm{Hom}$, $\otimes^L$.

Define internal $R\underline{\mathrm{Hom}}$ via: $\forall P\in D^-(\mathrm{Cond}(Ab))$,

$$\mathrm{Hom}(P, R\underline{\mathrm{Hom}}(M,N)) \cong \mathrm{Hom}(P\otimes^L M, N).$$

~~[illegible]~~

Our goal is to compute $R\underline{\mathrm{Hom}}(LCA, LCA)$.

④ "Recall" def. of condensed cohomology:

$T\in \mathrm{Cond}$, $M\in \mathrm{Cond}(Ab)$.

- $\Gamma(T,M) := \mathrm{Hom}_{\mathrm{Cond}}(T,M) \underset{\text{adj}}{=} \mathrm{Hom}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[T], M) \in Ab$

- $R\Gamma(T,M) := R\mathrm{Hom}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[T], M) \in D(Ab)$, where we use proj. res. w.r.t. $\mathbb{Z}[T]$.

$$H^i_{\mathrm{Cond}}(T,M) := H^i(R\mathrm{Hom}(\mathbb{Z}[T], M))$$

Previous talk showed: for $T\in$ locally compact Hausdorff, M discrete,

$\cancel{H^i_{\mathrm{Cond}}(T,M) \cong H^i_{\mathrm{Sheaf}}(T,M)}$

$R\Gamma(\underline{T},M) \cong R\Gamma_{\mathrm{sheaf}}(T,M)$

Q:
- $R\underline{\mathrm{Hom}}(\mathbb{Z}[T], M)$ as a refinement of $R\Gamma(T,M)$?
- the "topology" of $\mathbb{Z}[\underline{T}]\in\mathrm{Cond}$, for T locally compact Hausdorff?

2. Now compute $\underline{RHom}(LCA, LCA)$.

"Recall":

$\underline{\text{Thm}}$ i) Any LCA is an extension of $\mathbb{R}^n$, discrete, compact.

ii) Pontryagin $\mathbb{D}: A \mapsto \operatorname{Hom}(A, \mathbb{T})$ $(\mathbb{T} = \mathbb{R}/\mathbb{Z})$ well-defined on LCA. $\mathbb{D}^2$ is iso., switches dis. – comp.

So we can make various reductions:

- by i), reduce to $\mathbb{R}$, dis., comp.
- any discrete M has $0 \to \mathbb{Z}^{\oplus J} \to \mathbb{Z}^{\oplus I} \to M \to 0$, where $\mathbb{Z}^{\oplus J} = \bigoplus_J \mathbb{Z}$

  so reduce to $\mathbb{Z}^{\oplus I}$

  $\underline{RHom}(\mathbb{Z}^{\oplus I}, M) = \prod_I M$.

- any compact M has $0 \to \mathbb{Z}^{\oplus J} \to \mathbb{Z}^{\oplus I} \to \operatorname{Hom}(M, \mathbb{T}) \to 0$

  apply $\mathbb{D}$ get $0 \to M \to \mathbb{T}^I \to \mathbb{T}^J \to 0$, where $\mathbb{T}^J = \prod_J \mathbb{T}$

  So reduce to $\mathbb{T}^I$, arbitrary I.

- $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$ relates $\mathbb{Z}, \mathbb{R}, \mathbb{T}$.

In essence there are two (more) things to compute:

$$(*)\quad \begin{cases} \underline{RHom}(\mathbb{T}^I, M) & M \text{ dis.} \\ \underline{RHom}(\mathbb{T}^I, \mathbb{R}) & \end{cases}$$

($\underline{RHom}$ commutes with limit in second variable)

Idea: recall $\underline{RHom}(\mathbb{Z}[\overset{\text{Cond}}{T}], \overset{\text{dis. Cond(Ab)}}{M})(\overset{\text{CHaus}}{S}) = RHom(\mathbb{Z}[T \times S], M)$

$= R\Gamma(T \times S, M)$, which we know from sheaf coho.

We will relate $(*)$ to these, using

$\underline{\text{FACT}}$ In Ab, $\exists$ $\underline{\text{functorial}}$ res.

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

<u>Remark</u>. • [Xena, Jan. 2021] mentions the res. can be explicitly constructed.
• Functoriality implies similar res. for Cond(Ab).
But Note this <u>won't</u> be a projective res. in general unless all $A^{r_{i,j}}$ are ExtDis. (for Cond(Ab))

① $R\underline{Hom}(\mathbb{T}^I, M) = \bigoplus_I M[-1]$, M dis.
• I finite: reduces to $I = \{*\}$. We show $R\underline{Hom}(\mathbb{R}, M) = 0$ and use $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$.

Apply <u>FACT</u> ~~to~~ to $\mathbb{R} \to 0$:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathbb{R}^{r_{i,j}}] \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \bigoplus \mathbb{Z}[0^{r_{i,j}}] \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

If these were proj. res. then we may apply $R\underline{Hom}(-, M)$ and we are done. But upper row is not, so we should take proj. res. then $R\underline{Hom}(-, M)$.
You might ask: why not take proj. res. of $\mathbb{R}$ directly.
This is because:

Aside.
[015G] <u>FACT'</u> (Cartan-Eilenberg resolution)
[Weibel §5] $F : \mathcal{A} \to \mathcal{B}$ left exact ~~between~~ contravariant between abelian cat.
$K_{\bullet \geq 0}$ a complex in $\mathcal{A}$.
Then $\exists$ a double complex $P_{\bullet,\bullet}$ <u>functorial</u> w.r.t. K. and
a) each term is projective. nonzero only in first quadrant. (i.e. $\bullet \geq 0$)
b) $\mathrm{Tot}(P_{\bullet,\bullet}) \to K$ is a projective res. of K.
c) the spectral seq. associated to $F(P_{\bullet,\bullet})$ satisfies

$$ {}^{I}E_1^{p,q} = R^q F(K_p^{\bullet}) \Rightarrow H^*(F(\mathrm{Tot}(P_{\cdot,\cdot}))) = R^*F(K_\cdot). $$

("I" means filtration by columns)

Apply this to

$R\mathrm{Hom}(-, M) : \mathrm{Cond}(Ab) \to Ab$

$K_\cdot = (0 \leftarrow A \leftarrow \mathbb{Z}[A] \leftarrow \dots \leftarrow \bigoplus \mathbb{Z}[A^{r_{ij}}] \leftarrow \dots) \otimes \mathbb{Z}[S]$

we get a spectral seq. $\left(\begin{array}{l} A \in \mathrm{Cond}(Ab) \\ S \in \mathrm{CHaus} \\ M \in \mathrm{Cond}(Ab) \end{array} \right)$

$$ E_1^{p,q} = R^q \mathrm{Hom}\left(\bigoplus_{j=1}^{n_p} \mathbb{Z}[A^{r_{p,j}} \times S], M\right) $$

$$ = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \Rightarrow R^* \mathrm{Hom}(K_\cdot, M) $$

$$ = R^*\underline{\mathrm{Hom}}(A, M)(S). $$

[note i) $K_\cdot$ is q.i.s to $A[0]$.
ii) $\underline{\mathrm{Hom}}(A,M)(S) = \mathrm{Hom}(A \otimes \mathbb{Z}[S], M)$
$\Rightarrow R\underline{\mathrm{Hom}}(A, M)(S) = R\mathrm{Hom}(A \otimes \mathbb{Z}[S], M)$]

In our situation $\mathbb{R} \to 0$, we get spectral seq. $E_{1,\mathbb{R}}^{p,q}$ and $E_{1,0}^{p,q}$ converging to $R^*\underline{\mathrm{Hom}}(\mathbb{R}, M)(S)$ and 0 respectively.

We ~~show~~ note the induced $E_{1,\mathbb{R}}^{p,q} \leftarrow E_{1,0}^{p,q}$ is ~~a q.i.s~~ an iso. because

$$ H^q(\mathbb{R}^r \times S, M) \xleftarrow{\sim} H^q(S, M) \quad \text{as sheaf coho.} $$

- I infinite: suffices to show

$$ \operatorname*{colim}_{\substack{J \subseteq I \\ \text{finite}}} R\underline{\mathrm{Hom}}(\mathbb{T}^J, M) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{T}^I, M) $$

because $\mathrm{LHS} \overset{\text{above}}{\cong} \operatorname{colim} \bigoplus_J M[-1] \cong \bigoplus_I M[-1]$.

• Apply FACT to. $\mathbb{T}^J \xleftarrow{\text{proj}} \mathbb{T}^I$. As above, everything reduces to

$$\operatorname*{colim}_{\substack{J\subseteq I\\ \text{finite}}} H^q(\mathbb{T}^J\times S, M) \xrightarrow{\sim} H^q(\mathbb{T}^I\times S, M)$$

which is true for sheaf coho. □

② $\underline{R\mathrm{Hom}}(\mathbb{T}^I, \mathbb{R}) = 0$. Omit.

IRRELEVANT

③ $\underline{R\mathrm{Hom}}(\mathbb{Z}^I, M) = \bigoplus_I M$, M drs.

Use same proof as ① we can show $\underline{R\mathrm{Hom}}(\mathbb{R}^I, M) = 0$.
Then ~~apply~~ use $0 \to \mathbb{Z}^I \to \mathbb{R}^I \to \mathbb{T}^I \to 0$. (Note: this is exact for arbitrary I because AB4* holds in Cond(Ab))
Apply $\underline{R\mathrm{Hom}}(-, M)$:

$$\underline{R\mathrm{Hom}}(\mathbb{Z}^I, M) = \mathrm{cone}\Big(\bigoplus_I M[-1] \to 0\Big) = \bigoplus_I M$$

~~④ $\underline{R\mathrm{Hom}}(\mathbb{Z}^I, \mathbb{R}) = \bigoplus_I \mathbb{R}$~~

Consequences: $\underline{R\mathrm{Hom}}(A, B)$

a) (drs., drs.): $0 \to \mathbb{Z}^{\oplus I} \to \mathbb{Z}^{\oplus J} \to A \to 0$
$\rightsquigarrow \underline{R\mathrm{Hom}}(A,B) = \mathrm{cone}(B^J \to B^I)$

b) (comp., drs.): $0 \to A \to \mathbb{T}^I \to \mathbb{T}^J \to 0$
$\rightsquigarrow \underline{R\mathrm{Hom}}(A,B) = \mathrm{cocone}\Big(\bigoplus_I B[-1] \to \bigoplus_J B[-1]\Big)$

c) $(\mathbb{R}$, drs.) 0

d) (drs., comp.) as a)

e) (comp. comp.) as b) $\rightsquigarrow = \mathrm{cocone}(\underline{R\mathrm{Hom}}(\mathbb{T}^J, B) \to \underline{R\mathrm{Hom}}(\mathbb{T}^I, B))$

$\underline{R\mathrm{Hom}}(\mathbb{T}^I, B) = \mathrm{cocone}(\underline{R\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{I'}) \to \underline{R\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^{J'}))$

$\underline{R\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}^J) = \underline{R\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T})^J$.

$\underline{R\mathrm{Hom}}(\mathbb{T}^I, \mathbb{T}) = \mathrm{cone}\Big(\bigoplus_I \mathbb{Z}[-1] \to 0\Big) = \bigoplus_I \mathbb{Z}$

f) ($\mathbb{R}$, comp.) as c)

g) (dis., $\mathbb{R}$) as a)

h) (comp. $\mathbb{R}$) 0

i) ($\mathbb{R}$, $\mathbb{R}$) use $0 \to \mathbb{Z} \to \mathbb{R} \twoheadrightarrow \mathbb{T} \to 0$ $\rightsquigarrow$ $= \mathbb{R}$.

<u>Cor</u> $\underline{\mathrm{Ext}}^{\geq 2}(A, B) = 0$ for A, B LCA.

Added:

0->Z->R->T->0 is exact as abelian groups. They are still exact as condensed abelian groups. To see this, it suffices to check by evaluating on extremally disconnected groups, which in turn is an exercise in point set topology. [For details, see, e.g. arxiv: 2109.07816, Prop. 2.18]