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Is there a convenient category of spectra?

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Abstract

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The construction of the smash product of two spectra is one of the most unsatisfactory aspects of every available treatment of the stable category. Increased interest in enriched ring and module spectra has made the misbehavior of smash products a source of growing frustration. This paper conveys the unhappy message that this frustration is unavoidable. Five simple, obviously desirable axioms for a good category of spectra with a well-behaved smash product are listed. Then it is shown that no category can satisfy all five of these minimal axioms.

Introduction

Recently, there has been significant interest in spectra with various enriched ring structures, in module spectra over such ring spectra, and in complex constructions for spectra like the bar construction. The study of these spectra and constructions necessarily draws attention away from the stable category $h\mathscr{S}$ to a category $\mathscr{S}$ of spectra and actual maps of spectra, from which $h\mathscr{S}$ can be derived by inverting some class of weak equivalences. There are several possible choices for such a category of spectra and actual maps [1–5, 9, 11, 12, 18, 21, 23–25]. Each choice has its advantages, but also serious disadvantages. Many of the disadvantages are tied to the behavior of the smash product of spectra. In this article, it is shown that some disadvantages associated with the smash product are intrinsic. This is done by listing a rather minimal set of axioms for a convenient category of spectra and then showing that these axioms are inconsistent. The axioms and the inconsistency theorem are stated in Section 1. Section 2 contains a few comments on symmetric monoidal categories which are necessary for the proof of Theorem 1.1. This theorem is derived there from an observation about

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strict ring spectra. The final section discusses the relation of the axioms to several other properties desirable in a category of spectra.

1. An ideal category of spectra?

Throughout this article, $\mathcal{T}$ is the category of based, compactly generated, weak Hausdorff spaces [17, 26] and $\mathcal{S}$ is a putative convenient category of spectra. The category obtained from $\mathcal{T}$ by inverting all of the weak equivalences in $\mathcal{T}$ is denoted $h\mathcal{T}$; the stable category is denoted $h\mathcal{S}$. For any based space X,

$$QX = \operatorname*{colim}_n \Omega^n \Sigma^n X$$

is the usual free infinite loop space generated by X and

$$\iota : X \to QX$$

is the standard inclusion.

Certainly, any category $\mathcal{S}$ of spectra should come equipped with a functor

$$\Omega^\infty : \mathcal{S} \to \mathcal{T}$$

that assigns to each spectrum E its associated infinite loop space $\Omega^\infty E$. Similarly, there should be a functor

$$\Sigma^\infty : \mathcal{T} \to \mathcal{S}$$

assigning to each space X its associated suspension spectrum $\Sigma^\infty X$. The smash product $D \wedge E$ of two spectra D and E should also be defined in $\mathcal{S}$.

The five axioms listed below impose conditions on the functors Q, Σ^∞, and Ω^∞ and on the smash products in both $\mathcal{T}$ and $\mathcal{S}$. We use $\wedge$ to denote the smash product in both $\mathcal{T}$ and $\mathcal{S}$, relying on the context to indicate which is intended.

(A1) The category $\mathcal{S}$ is a symmetric monoidal category with respect to the smash product.

(A2) The functor Σ^∞ is left adjoint to the functor Ω^∞.

(A3) The unit for the smash product in $\mathcal{S}$ is the suspension spectrum $\Sigma^\infty S^0$.

(A4) Either there is a natural transformation

$$\phi : (\Omega^\infty D) \wedge (\Omega^\infty E) \to \Omega^\infty (D \wedge E),$$

for D and E in $\mathcal{S}$, or there is a natural transformation

$$\gamma : \Sigma^\infty (X \wedge Y) \to (\Sigma^\infty X) \wedge (\Sigma^\infty Y),$$

for X and Y in $\mathcal{T}$, which commutes with the unity, commutativity, and associativity isomorphisms of the categories $\mathcal{T}$ and $\mathcal{S}$.

(A5) There is a natural weak equivalence

$$\theta : \Omega^\infty \Sigma^\infty X \to QX ,$$

for X in $\mathcal{T}$, making the diagram

$$\begin{tikzcd} & \Omega^\infty \Sigma^\infty X \arrow[d, "\theta"] \\ X \arrow[ur, "\eta"] \arrow[r, "\iota"'] & QX \end{tikzcd}$$

commute, where η is the unit of the $(\Sigma^\infty, \Omega^\infty)$-adjunction.

The exact sense in which the maps ϕ or γ of Axiom (A4) are expected to commute with the various isomorphisms of $\mathcal{T}$ and $\mathcal{S}$ is described in Section 2. Moreover, there it is shown that, given Axioms (A1), (A2), and (A3), the existence of a natural transformation ϕ making the appropriate diagrams commute is equivalent to the existence of a natural transformation γ making the appropriate diagrams commute. Hence the 'or' in Axiom (A4).

The first thing to be said in defense of these axioms is that, if $\mathcal{S}$ and $\mathcal{T}$ are replaced by $h\mathcal{S}$ and $h\mathcal{T}$ respectively, then the analogous axioms are satisfied. The desirability of Axioms (A1) through (A3) should be obvious. Infinite loop space theorists would certainly insist on Axiom (A5). If E is a ring spectrum, then its infinite loop space $\Omega^\infty E$ should inherit some sort of ring structure. The natural transformation ϕ of Axiom (A4) provides this structure. In practice, one expects the natural transformation γ of Axiom (A4) to be a natural isomorphism. Such an isomorphism is used to show that the smash product of an m-sphere spectrum and an n-sphere spectrum is an $(m + n)$-sphere spectrum and is essential for the proof that the smash product of two CW-spectra is a CW-spectrum, Moreover, the natural transformation γ^{-1} is needed to ensure that, if M is a topological monoid, then $\Sigma^\infty M^+$ is a ring spectrum. Nevertheless, we assume only the existence of the natural transformation γ since this suffices for our nonexistence theorem. Besides, the nature of the functors Σ^∞, Ω^∞, and $\wedge$ ensures that, in almost any category of spectra, the natural transformation γ is the easiest to define. The more frequently used transformations ϕ and γ^{-1} are then constructed from γ in the obvious ways.

The following disappointing conclusion is proved in Section 2:

Theorem 1.1. *There is no category $\mathcal{S}$ satisfying Axioms* (A1) *through* (A5).

Since Elmendorf's category of spectra [9] is the only published symmetric monoidal category of spectra, it is instructive to see how it behaves with respect to the axioms above. Let $\mathcal{S}_E$ be Elmendorf's category of spectra and let $\mathcal{J}$ be his category of universes. Recall from [9] that the objects of $\mathcal{J}$ are all real inner product spaces of finite or countably infinite dimension. Elmendorf's category comes equipped with an augmentation functor

$$\varepsilon : \mathscr{S}_{\mathrm{E}} \to \mathscr{I} .$$

For any inner product space U in $\mathscr{I}$, let $\varepsilon^{-1}(U)$ be the subcategory of $\mathscr{S}_{\mathrm{E}}$ whose objects are those sent to U by ε and whose morphisms are those sent by ε to the identity map $1_U : U \to U$. Let 0 denote the unique real inner product space of dimension 0; $\varepsilon^{-1}(0)$ is a full subcategory of $\mathscr{S}_{\mathrm{E}}$.

There is an obvious functor

$$\Omega_{\mathrm{E}}^{\infty} : \mathscr{S}_{\mathrm{E}} \to \mathscr{T} .$$

Typically, this functor associates an infinite loop space $\Omega_{\mathrm{E}}^{\infty} D$ to an object D of $\mathscr{S}_{\mathrm{E}}$. However, the restriction of $\Omega_{\mathrm{E}}^{\infty}$ to $\varepsilon^{-1}(0)$ is an isomorphism between $\varepsilon^{-1}(0)$ and $\mathscr{T}$. This isomorphism of categories identifies the smash product of spaces with the restriction to $\varepsilon^{-1}(0)$ of the smash product on $\mathscr{S}_{\mathrm{E}}$. Moreover, the inverse image of S^0 under this isomorphism is the unit for the smash product on $\mathscr{S}_{\mathrm{E}}$.

Let $\Sigma_0^{\infty} : \mathscr{T} \to \mathscr{S}_{\mathrm{E}}$ be the inverse of the isomorphism of $\varepsilon^{-1}(0)$ with $\mathscr{T}$, composed with the inclusion of $\varepsilon^{-1}(0)$ in $\mathscr{S}_{\mathrm{E}}$. With the choice of the pair Σ_0^{∞} and $\Omega_{\mathrm{E}}^{\infty}$ for Σ^{∞} and Ω^{∞}, Elmendorf's category satisfies Axioms (A1) through (A4). Since the composite $\Omega_{\mathrm{E}}^{\infty}\Sigma_0^{\infty}$ is, by definition, the identity functor, Elmendorf's category does not satisfy Axiom (A5). Adjoints are unique up to natural isomorphism, so any other choice for Σ^{∞} would not satisfy Axiom (A2).

Essentially, Elmendorf has produced a symmetric monoidal category of spectra by including a copy of the category $\mathscr{T}$ of spaces as a full subcategory of his category $\mathscr{S}_{\mathrm{E}}$ of spectra. He uses S^0 in this full subcategory as the unit for his smash product of spectra. It appears that the real import of Theorem 1.1 may be that any symmetric monoidal category of spectra $\mathscr{S}$ satisfying Axioms (A1) through (A4) must be formed in this fashion; that is, the functor Σ^{∞} must be the inclusion of $\mathscr{T}$ as a subcategory of $\mathscr{S}$.

Infinite loop space theorists will be comforted to know that there are other functors from $\mathscr{T}$ to $\mathscr{S}_{\mathrm{E}}$ which satisfy a suitable variant of Axiom (A5). For any countably infinite dimensional U in $\mathscr{I}$, there exist: a functor

$$\Sigma_U^{\infty} : \mathscr{T} \to \mathscr{S}_{\mathrm{E}} ;$$

a natural transformation

$$\eta_U : X \to \Omega_{\mathrm{E}}^{\infty}\Sigma_U^{\infty} X ;$$

and a natural homeomorphism

$$\theta_U : \Omega_{\mathrm{E}}^{\infty}\Sigma_U^{\infty} X \to QX$$

making the diagram

$$\begin{tikzcd} & \Omega^\infty_{\mathrm{E}} \Sigma^\infty_U X \arrow[d, "\theta_U"] \\ X \arrow[ur, "\eta_U"] \arrow[r, "\iota"'] & QX \end{tikzcd}$$

commute. The functor Σ^∞_U factors through $\varepsilon^{-1}(U)$. Regarded as a functor into $\varepsilon^{-1}(U)$, Σ^∞_U is left adjoint to the restriction of $\Omega^\infty_{\mathrm{E}}$ to $\varepsilon^{-1}(U)$. The map η_U is the unit of this adjunction. These observations may be interpreted as saying that, if the pair Σ^∞_U and $\Omega^\infty_{\mathrm{E}}$ are used for Σ^∞ and Ω^∞, then $\mathscr{S}_{\mathrm{E}}$ satisfies variants of Axioms (A2) and (A5). With these choices, $\mathscr{S}_{\mathrm{E}}$ also satisfies Axiom (A1) and a variant of Axiom (A4), but cannot satisfy any variant of (A3).

2. Monoidal functors and the category of spectra

From a categorical point of view, Axioms (A3) and (A4) assert that the functor Ω^∞ is a (lax) monoidal functor and the functor Σ^∞ is a (lax) comonoidal functor. The definition and a few basic properties of such functors are reviewed here (see also [8, 10, 14, 16]). Then Theorem 1.1 is proved.

Throughout this section, $\mathscr{A}_1$ and $\mathscr{A}_2$ are to be symmetric monoidal categories whose products are denoted $\wedge_1$ and $\wedge_2$, respectively. The units for these products are denoted e_1 and e_2, respectively. The associativity, commutativity, and unity isomorphisms of these categories are, for $i = 1$ or 2,

$$\boldsymbol{a}_i : (x \wedge_i y) \wedge_i z \xrightarrow{\cong} x \wedge_i (y \wedge_i z)\,, \qquad \boldsymbol{c}_i : x \wedge_i y \xrightarrow{\cong} y \wedge_i x\,,$$
$$\boldsymbol{l}_i : e_i \wedge_i x \xrightarrow{\cong} x\,, \qquad \boldsymbol{r}_i : x \wedge_i e_i \xrightarrow{\cong} x\,.$$

Definition 2.1. (i) A (lax) monoidal functor $(\boldsymbol{\Phi}, \phi_0, \phi)$ from $\mathscr{A}_1$ to $\mathscr{A}_2$ consists of a functor

$$\boldsymbol{\Phi} : \mathscr{A}_1 \to \mathscr{A}_2\,,$$

a map

$$\phi_0 : e_2 \to \Phi e_1\,,$$

and a natural transformation

$$\phi : (\boldsymbol{\Phi} x) \wedge_2 (\boldsymbol{\Phi} y) \to \boldsymbol{\Phi}(x \wedge_1 y)$$

such that, for all x, y, and z in $\mathscr{A}_1$, the following diagrams commute:

$$\begin{CD} e_2 \wedge_2 \Phi x @>{l_2}>> \Phi x \\ @V{\phi_0 \wedge_2 1}VV @AA{\Phi l_1}A \\ (\Phi e_1) \wedge_2 (\Phi x) @>{\phi}>> \Phi(e_1 \wedge_1 x) \end{CD}$$

$$\begin{CD} (\Phi x) \wedge_2 e_2 @>{r_2}>> \Phi x \\ @V{1 \wedge_2 \phi_0}VV @AA{\Phi r_1}A \\ (\Phi x) \wedge_2 (\Phi e_1) @>{\phi}>> \Phi(x \wedge_1 e_1) \end{CD}$$

$$\begin{CD} (\Phi x) \wedge_2 (\Phi y) @>{c_2}>> (\Phi y) \wedge_2 (\Phi x) \\ @V{\phi}VV @VV{\phi}V \\ \Phi(x \wedge_1 y) @>{\Phi c_1}>> \Phi(y \wedge_1 x) \end{CD}$$

$$\begin{CD} ((\Phi x) \wedge_2 (\Phi y)) \wedge_2 (\Phi z) @>{a_2}>> (\Phi x) \wedge_2 ((\Phi y) \wedge_2 (\Phi z)) \\ @V{\phi \wedge_2 1}VV @VV{1 \wedge_2 \phi}V \\ \Phi(x \wedge_1 y) \wedge_2 (\Phi z) @. (\Phi x) \wedge_2 \Phi(y \wedge_1 z) \\ @V{\phi}VV @VV{\phi}V \\ \Phi((x \wedge_1 y) \wedge_1 z) @>{\Phi a_1}>> \Phi(x \wedge_1 (y \wedge_1 z)) \end{CD}$$

(ii) A (lax) comonoidal functor $(\Gamma, \gamma_0, \gamma)$ from $\mathscr{A}_2$ to $\mathscr{A}_1$ consists of a functor

$$\Gamma : \mathscr{A}_2 \to \mathscr{A}_1 ,$$

a map

$$\gamma_0 : \Gamma e_2 \to e_1 ,$$

and a natural transformation

$$\gamma : \Gamma(u \wedge_2 v) \to (\Gamma u) \wedge_1 (\Gamma v)$$

making obvious diagrams analogous to the four above commute.

Axiom (A4) can now be stated precisely.

(A4′) Either there is a natural transformation

$$\phi : (\Omega^\infty D) \wedge (\Omega^\infty E) \to \Omega^\infty(D \wedge E) ,$$

for D and E in $\mathscr{S}$, which, together with the unit

$$\eta : S^0 \to \Omega^\infty \Sigma^\infty S^0$$

of the $(\Sigma^\infty, \Omega^\infty)$-adjunction, makes Ω^∞ into a monoidal functor, or there is a natural transformation

$$\gamma : \Sigma^\infty(X \wedge Y) \to (\Sigma^\infty X) \wedge (\Sigma^\infty Y) ,$$

for X and Y in $\mathcal{T}$, which, together with the identify map $1 : \Sigma^{\infty} S^0 \to \Sigma^{\infty} S^0$, makes Σ^{∞} into a comonoidal functor.

The following result, due to Kelly [14, Theorem 1.2], justifies the 'or' in this axiom by showing that, given Axioms (A1), (A2), and (A3), the two conditions in the axiom are equivalent. Enough of the proof of this lemma is given to clarify the connections between the transformations ϕ and γ and between the maps $\eta : S^0 \to \Omega^{\infty}\Sigma^{\infty} S^0$ and $1 : \Sigma^{\infty} S^0 \to \Sigma^{\infty} S^0$ in the axiom.

Lemma 2.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be symmetric monoidal categories and let $\Gamma : \mathcal{A}_2 \to \mathcal{A}_1$ be left adjoint to $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$. Then Φ is a monoidal functor if and only if Γ is a comonoidal functor.*

Proof. The adjunction provides an obvious connection between the maps

$$\phi_0 : e_2 \to \Phi e_1 \quad \text{and} \quad \gamma_0 : \Gamma e_2 \to e_1$$

needed to make Φ monoidal and Γ comonoidal. Given a natural transformation

$$\phi : (\Phi x) \wedge_2 (\Phi y) \to \Phi(x \wedge_1 y)\,,$$

one obtains a natural transformation

$$\gamma : \Gamma(u \wedge_2 v) \to (\Gamma u) \wedge_1 (\Gamma v)$$

as the adjoint of the composite

$$u \wedge_2 v \xrightarrow{\eta \wedge_2 \eta} (\Phi\Gamma u) \wedge_2 (\Phi\Gamma v) \xrightarrow{\phi} \Phi((\Gamma u) \wedge_1 (\Gamma v))\,,$$

where η is the unit of the (Γ, Φ)-adjunction. Similarly, given γ, one obtains ϕ as the adjoint of the composite

$$\Gamma((\Phi x) \wedge_2 (\Phi y)) \xrightarrow{\gamma} (\Gamma\Phi x) \wedge_1 (\Gamma\Phi y) \xrightarrow{\varepsilon \wedge_1 \varepsilon} x \wedge_1 y\,,$$

where ε is the counit of the (Γ, Φ)-adjunction. It is easy to see that, if the pairs (ϕ_0, ϕ) and (γ_0, γ) are related in this fashion, then the pair (ϕ_0, ϕ) makes Φ into a monoidal functor if and only if the pair (γ_0, γ) makes Γ into a comonoidal functor. □

If $\mathcal{S}$ is a symmetric monoidal category of spectra, then the usual homotopy-theoretic notion of a ring spectrum in $h\mathcal{S}$ can be lifted to the notion of a strict ring spectrum in $\mathcal{S}$.

Definition 2.3. (i) Let $\mathcal{S}$ be a category of spectra satisfying Axioms (A1) and (A3). A strict ring spectrum is a spectrum E in $\mathcal{S}$ together with a unit map $u : \Sigma^\infty S^0 \to E$ and a multiplication $\mu : E \wedge E \to E$ making the usual associativity, commutativity, and unity diagrams commute in $\mathcal{S}$; that is, E is a commutative monoid [19, p. 166] in the category $\mathcal{S}$.

(ii) Assume that $\mathcal{S}$ satisfies Axioms (A1) through (A4) and that E is a strict ring spectrum in $\mathcal{S}$. Let

$$\tilde{u} : S^0 \to \Omega^\infty E$$

be the adjoint of the unit of E and let $\tilde{\mu}$ be the composite

$$\tilde{\mu} : \Omega^\infty E \times \Omega^\infty E \to \Omega^\infty E \wedge \Omega^\infty E \xrightarrow{\phi} \Omega^\infty(E \wedge E) \xrightarrow{\Omega^\infty \mu} \Omega^\infty E .$$

The space $\Omega^\infty E$ is a commutative topological monoid whose multiplication is $\tilde{\mu}$ and whose unit is the image under $\tilde{u}$ of the non-basepoint of S^0. Let $\Omega_1^\infty E$ be the path component of the unit in $\Omega^\infty E$. Then $\tilde{\mu}$ restricts to a multiplication

$$\hat{\mu} : \Omega_1^\infty E \times \Omega_1^\infty E \to \Omega_1^\infty E ,$$

under which $\Omega_1^\infty E$ is also a commutative topological monoid.

The following immediate consequence of [22] or [20, Proposition 3.6] indicates that there are not many interesting strict ring spectra.

Lemma 2.4. *Let $\mathcal{S}$ be a category of spectra satisfying Axioms* (A1) *through* (A4) *and let E be a strict ring spectrum in $\mathcal{S}$. Then $\Omega_1^\infty E$ is a product of Eilenberg–Mac Lane spaces.* □

Theorem 1.1 follows directly since the unit for the smash product must be a strict ring spectrum:

Proof of Theorem 1.1. Assume that $\mathcal{S}$ is a category of spectra satisfying Axioms (A1) through (A5). Since $\Sigma^\infty S^0$ is the unit for the smash product on $\mathcal{S}$, $\Omega_1^\infty \Sigma^\infty S^0$ must be a product of Eilenberg–Mac Lane spaces. Axiom (A5) would then imply that the path component of the identity map in QS^0 is a product of Eilenberg–Mac Lane spaces, but this is false. □

3. Remarks on Axiom (A4′)

Axiom (A4′) is the only one of the axioms whose motivation may not be obvious. Perhaps its best justification is the improbability of the existence of a category of spectra that fails to satisfy it and yet is still sufficiently well-behaved to be useful. As evidence for this improbability, we show here that this axiom follows from obvious continuity conditions on the functors Σ^∞, Ω^∞, and $? \wedge E$ together with a universal property for either the smash product of two spectra or the smash product of a space and a spectrum. After the introduction of some notation, the appropriate continuity and universality properties are codified in a series of obviously desirable additional axioms for a category of spectra. We then show that various combinations of these axioms imply Axiom (A4′).

To impose continuity conditions on the functors Σ^∞, Ω^∞, and $? \wedge E$, we must assume that our category $\mathcal{S}$ of spectra is a topological category. In other words, we assume that, for each E and F in $\mathcal{S}$, the morphism set $\mathcal{S}(E, F)$ has a topology and that, for every D, E, and F in $\mathcal{S}$, the composition map

$$\mathcal{S}(E, F) \times \mathcal{S}(D, E) \to \mathcal{S}(D, F)$$

is continuous. Both May's [18] and Elmendorf's [9] categories of spectra are topological. Denote the space of morphisms in $\mathcal{S}$ from E to F by $\mathcal{S}[E, F]$ so that this space may be distinguished from its underlying set $\mathcal{S}(E, F)$. Similarly, denote the space of continuous, based maps from X to Y by $\mathcal{T}[X, Y]$. We also assume that there is a unique trivial map between any two spectra E and F in $\mathcal{S}$ and that the composite of any map with a trivial map is trivial. By taking the trivial map as the basepoint in each morphism set $\mathcal{S}[E, F]$ of $\mathcal{S}$, we may regard $\mathcal{S}$ as a based topological category; that is, a category enriched over $\mathcal{T}$. May's category of spectra is an example of such a category. Unfortunately, Elmendorf's category $\mathcal{S}_{\mathrm{E}}$ is not a based topological category. For any space X, countably infinite-dimensional inner product space U, and spectrum E in the subcategory $\varepsilon^{-1}(U)$ of $\mathcal{S}_{\mathrm{E}}$, there are no maps (not even a trivial map) in $\mathcal{S}_{\mathrm{E}}$ from E to $\Sigma^\infty X$.

In addition to having the space of functions between any two spectra, we would also like to have a spectrum of functions between any two spectra; that is, we would like a functor assigning to each pair E and F in $\mathcal{S}$, a spectrum $\mathcal{S}^*[E, F]$. The spectra $\mathcal{S}^*[E, F]$ should be function spectra in the sense that, for each E in $\mathcal{S}$, the functor $\mathcal{S}^*[E, ?]$ should be right adjoint to the functor $? \wedge E$. The existence of such function spectra would make $\mathcal{S}$ a symmetric monoidal closed category [15]. So far, no one has constructed such a category of spectra. Nevertheless, the existence of such a closed structure is a highly desirable property for $\mathcal{S}$ and one closely related to Axiom (A4′).

Given topologies on the morphism sets of $\mathcal{S}$, Axioms (A1) and (A2) can be strengthened to impose obvious continuity conditions on Σ^∞, Ω^∞, and the smash product construction. Axiom (A1) may be further strengthened to require the existence of well-behaved function spectra. If there is a functor $\wedge: \mathcal{T} \times \mathcal{S} \to \mathcal{S}$ providing the smash product $X \wedge D$ of a space X and a spectrum D, then this mixed smash product may also be required to have an adjoint. The appropriate axioms are:

(A1′) The category $\mathscr{S}$ is a symmetric monoidal category with respect to smash product. Moreover, for every D, E, and F in $\mathscr{S}$, the operation of taking the smash product with E induces a continuous map

$$\mathscr{S}[D, F] \to \mathscr{S}[D \wedge E, F \wedge E].$$

(A1″) The category $\mathscr{S}$ is a symmetric monoidal closed category. Moreover, for every D, E, and F in $\mathscr{S}$, the adjunction isomorphism between $? \wedge E$ and $\mathscr{S}^*[E, ?]$ is actually a homeomorphism

$$\mathscr{S}[D \wedge E, F] \cong \mathscr{S}[D, \mathscr{S}^*[E, F]].$$

(A2′) The functor Σ^∞ is left adjoint to the functor Ω^∞. Moreover, for each space X and spectrum F, the $(\Sigma^\infty, \Omega^\infty)$-adjunction isomorphism is actually a homeomorphism

$$\mathscr{S}[\Sigma^\infty X, F] \cong \mathscr{T}[X, \Omega^\infty F].$$

(A6) There is a functor $\wedge : \mathscr{T} \times \mathscr{S} \to \mathscr{S}$ providing $\mathscr{S}$ with smash products of spaces and spectra. Moreover, for each D in $\mathscr{S}$, the functor $? \wedge D : \mathscr{T} \to \mathscr{S}$ is left adjoint to the functor $\mathscr{S}[D, ?] : \mathscr{S} \to \mathscr{T}$.

Remarks 3.1. (i) These axioms are not implausible. Axiom (A1′) is satisfied by Elmendorf's category. May's category of spectra is not symmetric monoidal because his smash product is neither associative, commutative, nor unital before passage to the stable category. However, his smash product does satisfy the continuity condition in Axiom (A1′). As we have already noted, no one has yet constructed a category of spectra satisfying Axiom (A1″); however, May's category partially satisfies this axiom in the sense that it has function spectra providing adjoints to the smash product and the associated adjunction isomorphisms are homeomorphisms [18]. Axiom (A2′) is satisfied by both May's [18] and Elmendorf's [9] categories.

(ii) Any category of spectra $\mathscr{S}$ has an obvious candidate for the mixed smash product functor $\wedge : \mathscr{T} \times \mathscr{S} \to \mathscr{S}$ of Axiom (A6)—namely, the functor sending a space X and a spectrum D to the spectrum $\Sigma^\infty X \wedge D$. Unfortunately, this functor need not satisfy the adjunction condition imposed in Axiom (A6). For example, Elmendorf's category $\mathscr{S}_E$ cannot satisfy Axiom (A6) with this or any other choice of a functor from $\mathscr{T} \times \mathscr{S}_E$ to $\mathscr{S}_E$ because, if the spectrum D is indexed on an infinite universe, then the functor $\mathscr{S}[D, ?]$ takes values, not in $\mathscr{T}$, but in the category of unpointed spaces. It is therefore not the right adjoint of any functor whose domain is $\mathscr{T}$. May's category satisfies Axiom (A6), but only with respect to a different functor $\wedge : \mathscr{T} \times \mathscr{S} \to \mathscr{S}$. The functors $X \wedge D$ and $(\Sigma^\infty X) \wedge D$ in May's category are related only by a natural weak equivalence

$$X \wedge D \to \Sigma^\infty X \wedge D.$$

Lemma 3.4 below gives a condition under which the functor $(\Sigma^\infty ?) \wedge ?$ satisfies the adjunction condition of Axiom (A6).

(iii) Axiom (A1″) is stronger than Axiom (A1′) since the map

$$\mathcal{S}[D, F] \rightarrow \mathcal{S}[D \wedge E, F \wedge E]$$

can be written as the composite

$$\begin{aligned} \mathcal{S}[D, F] &\xrightarrow{\eta_*} \mathcal{S}[D, \mathcal{S}^*[E, F \wedge E]] \\ &\cong \mathcal{S}[D \wedge E, F \wedge E], \end{aligned}$$

where $\eta : F \rightarrow \mathcal{S}^*[E, F \wedge E]$ is the unit of the $(? \wedge E, \mathcal{S}^*[E, ?])$-adjunction.

(iv) Axiom (A6) asserts that the smash product of a space and a spectrum is an example of what is called a tensor in the theory of enriched categories [7, 10, 13, 15].

The following two results relate our new axioms to Axiom (A4′).

Proposition 3.2. *Let $\mathcal{S}$ be a based topological category of spectra with functors Σ^∞ and Ω^∞. If $\mathcal{S}$ satisfies Axioms* (A1′), (A2′), (A3), and (A6), *then $\mathcal{S}$ satisfies Axiom* (A4′).

Proof. Let F be in $\mathcal{S}$ and let X and Y be in $\mathcal{T}$. Applying the Yoneda Lemma to the chain of isomorphisms

$$\begin{aligned} \mathcal{S}(X \wedge \Sigma^\infty Y, F) &\cong \mathcal{T}(X, \mathcal{S}[\Sigma^\infty Y, F]) \\ &\cong \mathcal{T}(X, \mathcal{T}[Y, \Omega^\infty F]) \\ &\cong \mathcal{T}(X \wedge Y, \Omega^\infty F) \\ &\cong \mathcal{S}(\Sigma^\infty(X \wedge Y), F) \end{aligned}$$

produces a natural isomorphism

$$\tau : X \wedge \Sigma^\infty Y \rightarrow \Sigma^\infty(X \wedge Y) .$$

A natural map

$$\lambda : X \wedge (D \wedge E) \rightarrow (X \wedge D) \wedge E ,$$

defined for every X in $\mathcal{T}$ and every pair D and E in $\mathcal{S}$, may be obtained by considering the image of the identity map in $\mathcal{S}(X \wedge D, X \wedge D)$ under the chain of maps

$$\begin{aligned}\mathcal{S}(X \wedge D, X \wedge D) &\cong \mathcal{T}(X, \mathcal{S}[D, X \wedge D]) \\ &\xrightarrow{(? \wedge E)_*} \mathcal{T}(X, \mathcal{S}[D \wedge E, (X \wedge D) \wedge E]) \\ &\cong \mathcal{S}(X \wedge (D \wedge E), (X \wedge D) \wedge E)\,.\end{aligned}$$

Both τ and λ are examples of standard maps describing the behavior of tensors with respect to continuous functors [7, 10, 13, 15]. For any spaces X and Y, let

$$\gamma : \Sigma^\infty(X \wedge Y) \to (\Sigma^\infty X) \wedge (\Sigma^\infty Y)$$

be the composite

$$\begin{aligned}\Sigma^\infty(X \wedge Y) &\xrightarrow{\tau^{-1}} X \wedge \Sigma^\infty Y \\ &\xrightarrow{1 \wedge l^{-1}} X \wedge ((\Sigma^\infty S^0) \wedge (\Sigma^\infty Y)) \\ &\xrightarrow{\lambda} (X \wedge \Sigma^\infty S^0) \wedge (\Sigma^\infty Y) \\ &\xrightarrow{\tau \wedge 1} (\Sigma^\infty(X \wedge S^0)) \wedge (\Sigma^\infty Y) \\ &\cong (\Sigma^\infty X) \wedge (\Sigma^\infty Y)\,,\end{aligned}$$

in which l is the left unity isomorphism of $\mathcal{S}$. Proving that γ and the identity map $1 : \Sigma^\infty S^0 \to \Sigma^\infty S^0$ make Σ^∞ into a comonoidal functor is just an exercise in the coherence properties of enriched categories. □

Proposition 3.3. *Let $\mathcal{S}$ be a based topological category of spectra with functors Σ^∞ and Ω^∞ satisfying Axioms* (A1″), (A2′), *and* (A3). *Then $\mathcal{S}$ satisfies Axiom* (A4′).

We have already observed in Remark 3.1(iii) that Axiom (A1″) implies Axiom (A1′). Thus, Proposition 3.3 follows immediately from Proposition 3.2 and the following result:

Lemma 3.4. *Let $\mathcal{S}$ be a based topological category of spectra with functors Σ^∞ and Ω^∞ satisfying Axioms* (A1″), (A2′), *and* (A3). *Then $\mathcal{S}$ satisfies Axiom* (A6).

Proof. Assume that $\mathcal{S}$ is a topological category of spectra satisfying Axioms (A1″), (A2′), and (A3). We show that the functor from $\mathcal{T} \times \mathcal{S}$ to $\mathcal{S}$ which sends a space X and a spectrum D to the spectrum $(\Sigma^\infty X) \wedge D$ provides $\mathcal{S}$ with a smash product of spaces and spectra satisfying the appropriate adjunction. For any D and F in $\mathcal{S}$, define the natural homeomorphism

$$\beta : \mathcal{S}[D, F] \to \Omega^\infty \mathcal{S}^*[D, F]$$

to be the composite

$$\begin{aligned}\mathcal{S}[D, F] &\xrightarrow{l^*} \mathcal{S}[(\Sigma^\infty S^0) \wedge D, F] \\ &\cong \mathcal{S}[\Sigma^\infty S^0, \mathcal{S}^*[D, F]] \\ &\cong \mathcal{T}[S^0, \Omega^\infty \mathcal{S}^*[D, F]] \\ &\cong \Omega^\infty \mathcal{S}^*[D, F],\end{aligned}$$

where l is the left unit isomorphism of $\mathcal{S}$. The chain of natural isomorphisms

$$\begin{aligned}\mathcal{S}((\Sigma^\infty X) \wedge D, F) &\cong \mathcal{S}(\Sigma^\infty X, \mathcal{S}^*[D, F]) \\ &\cong \mathcal{T}(X, \Omega^\infty \mathcal{S}^*[D, F]) \\ &\xrightarrow{(\beta^{-1})_*} \mathcal{T}(X, \mathcal{S}[D, F]),\end{aligned}$$

for D and F in $\mathcal{S}$ and X in $\mathcal{T}$, provides the adjunction isomorphism required by Axiom (A6). □

Note added in proof. I learned of the following closely related paper of Hastings after my paper was accepted for publication:

H.M. Hastings, Stabilizing tensor products, Proc. Amer. Math. Soc. 49 (1975) 1–7.

Theorem 1 of this paper is especially noteworthy because of its consequences for symmetric monoidal structures on any category of spectra.

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