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PRODUCTS OF COCYCLES AND EXTENSIONS OF MAPPINGS

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1. Introduction

The Brouwer concept of *degree* led to the algebraic enumeration of the classes of homotopically equivalent maps of an n-sphere S^n on itself. Two such maps are homotopic if and only if they have the same degree. Any degree can be realized by a suitable map.

H. Hopf [8] extended this result to the maps of an n-dimensional complex K^n in S^n. The result, as reformulated by Whitney [17], reads: Two maps f, g of K^n in S^n are homotopic ($f \cong g$) if and only if $f^*s^n \sim g^*s^n$ in K^n (where s^n is the generating n-cocycle on S^n and f^*, g^* are the cochain homomorphisms induced by f, g). Any n-cocycle in K^n is an f^*s^n for a suitable f. Thus the homotopy classes can be put in 1–1 correspondence, in a natural way, with the n^{th} cohomology group $H^n(K^n)$.

The problem of enumerating the homotopy classes of maps of S^3 on S^2 was solved by Hopf [7, 9]. The classes form the elements of an infinite cyclic group—the Hurewicz [10] homotopy group $\pi_3(S^2)$. Freudenthal [5] and Pontrjagin [13] extended this result by showing that $\pi_{n+1}(S^n)$ is cyclic of order 2 for $n > 2$.[1]

Pontrjagin [14] then obtained an enumeration of the homotopy classes of maps of a 3-complex K^3 on S^2. If $f \cong g$, by the Hopf results $f^*s^2 \sim g^*s^2$. Assuming this necessary condition, then $g \cong g'$ where g' and f coincide on the 2-dimensional section K^2 of K^3. For any 3-cell σ, the two maps f, g' define an element $d^3(f, g', \sigma) \in \pi_3(S^2)$. Then $f \cong g'$ (and therefore to g) if and only if there exists a 1-cocycle e^1 in K^3 such that

$$d^3(f, \; g') \;\sim\; 2e^1 \;\smile\; f^*s^2.$$

(Here $\smile$ is the Alexander-Čech-Whitney [1, 3, 18] cup product of cocycles.) Furthermore any pair consisting of a 2 and a 3-cocycle on K^3 can be realized as f^*s^2 and $d^3(f, g)$ by suitable f, g.

This result provides an algebraic enumeration of the homotopy classes as follows. All classes divide into disjoint collections of classes according to the cohomology class z^2 of f^*s^2. The classes within the collection corresponding to a z^2 can be put in 1–1 correspondence with the elements of the factor group of $H^3(K^3)$ by the subgroup of products of $H^1(K^3)$ with $2z^2$.

The present paper solves the homotopy classification problem for maps of an $(n + 1)$-complex K in S^n. In order to achieve this, the $\smile$ product is generalized as follows. For each integer $i \geqq 0$ and cochains u, v of dimensions p, q a product $u \smile_i v$ is defined having dimension $p + q - i$. In particular, $u \smile_0 v = u \smile v$. In general, a product of cocycles need not be a cocycle if

[1] Since $\pi_i(S^1) = 0$ for $i > 1$, a complete classification of maps of an n-complex in S^1 is provided by the Hopf theorem.

$i > 0$ (see the coboundary formula (5.1)). However, if u is a cocycle, $u \smile_i u$ is a cocycle (cocycle mod 2) if $p - i$ is odd (even). These self-products, or squares, lead to invariant homomorphisms:

$$\mathrm{Sq}_i\colon H^p(K) \to H^{2p-i}(K).$$

(The group on the right has mod 2 coefficients if $p - i$ is even.) The i^{th} square of any element is always of order 2. Part I of the paper is devoted to establishing the basic properties of these new products.[2]

The Pontrjagin formula for $n > 2$ becomes

$$d^{n+1}(f, g) \sim e^{n-1} \smile_{n-3} e^{n-1}.$$

Thus the homotopy classes within the collection corresponding to a particular $z^n \in H^n(K)$ can be put in 1–1 correspondence with the finite group $H^{n+1}(K)/\mathrm{Sq}_{n-3} H^{n-1}(K)$.[3]

Much of the proof is based on a paper of Eilenberg [4] in which f^*s^n, $d^{n+1}(f, g)$ and related cocycles are defined under general circumstances and their basic properties established. As emphasized by Eilenberg, f^*s^n and e^{n-1} have coefficients in $\pi_n(S^n)$ and d^{n+1} has coefficients in $\pi_{n+1}(S^n)$. In order that the cup products should be defined, products of elements of $\pi_n(S^n)$ with values in $\pi_{n+1}(S^n)$ must be defined. This is done so that the square of a generator of the former group is a generator of the latter.

The homotopy classification theorem is obtained as a corollary of an extension theorem. In its non-relative form, it states that a map f of the n-section K^n of an $(n + 2)$-complex K in S^n can be extended to a map of K in S^n if and only if f^*s^n is a cocycle in K and its square of order $n - 2$ is ~ 0. Attention should be called to the crucial role played in the proof by the complex projective plane M^4. The theorem is first verified in the special case $n = 2$ and $K = M^4$. The general case is deduced from the special by a series of constructions.

The relative form of the extension theorem (see 24.1) requires a new type of invariant operation. If f, g are two maps of K in K' which coincide on a subcomplex L of K, then there are induced homomorphisms $(f - g)^*$ of the *absolute* cohomology groups of K' into the *relative* cohomology groups of K mod L which are a measure of the extent in which f and g differ on $K - L$. As is natural, these are dual to homomorphisms of the relative homology groups of K mod L into the absolute homology groups of K'.

Part I. Products

2. Definitions

The symbol K will denote a finite simplicial complex, and G a commutative group, written additively. The group of p-dimensional cochains with co-

[2] The $\smile_1$ product reduced mod 2 coincides with a product used by Pontrjagin in a brief note [15].

[3] This result contradicts results announced by Pontrjagin [12] in 1938. To the author's knowledge, no proofs of these results have yet appeared. It also contradicts results announced by Freudenthal [6] in 1939. An error in these results has been pointed out by G. Whitehead [16].

efficients in G is denoted by $C^p(K, G)$ (the symbol G is omitted if G is the group of integers). Let

$$\delta: \ C^p(K, G) \ \rightarrow \ C^{p+1}(K, G)$$

be the coboundary operator for cochains. Let $B^p(K, G)$, $Z^p(K, G)$ be the subgroups of $C^p(K, G)$ of coboundaries and cocycles; and let

$$H^p(K, G) = Z^p(K, G)/B^p(K, G)$$

denote the p^{th} cohomology group of K. If u is a p-cocycle $\{u\} \in H^p(K, G)$ denotes its cohomology class. And $\{u\} = \{v\}$ is abbreviated $u \sim v$; this means, of course, that $u - v$ is a coboundary. The relative cohomology groups of K modulo a subcomplex L are denoted by $H^p(K, L, G)$.

An order in K is a partial order of is vertices such that the vertices of any simplex are simply ordered. A fixed order α in K will be assumed until further notice. A symbol such as σ, τ or ζ will denote, ambiguously, either (1) a simplex of K, or (2) the array of vertices of the simplex ordered as in α, or (3) the orientation of the simplex determined by this order, or (4) the elementary cochain which attaches $+1$ to this oriented simplex and 0 to all others. The ambiguity can usually be resolved by examining the context in which the symbol is used.

Let σ, τ have dimensions p, q, and let $i \geqq 0$ be an integer. The *ordered* pair (σ, τ) is *i-regular* in the order α if the following conditions are satisfied.

(-1). The vertices of σ, τ together span a $(p + q - i)$-simplex ζ. In this case, σ, τ have $i + 1$ vertices in common, denoted by $V^0, V^1, \cdots, V^i$ in the order α.

(0). V^0 is the first vertex of τ.

(1). V^0, V^1 are adjacent vertices in σ.

(2). V^1, V^2 are adjacent vertices in τ.

. . . .

$(j + 1)$. V^j, V^{j+1} are adjacent vertices in σ (τ) if j is even (odd).

. . . .

$(i + 1)$. V^i is the last vertex of σ (τ) if i is even (odd).

Note that 0-regularity requires the last vertex of σ to be the first vertex of τ.

If (σ, τ) is i-regular, let σ_0 be the face of σ spanned by its vertices $\leqq V^0$, let σ_{2j} $(0 < 2j \leqq i)$ be the face of σ spanned by its vertices $\geqq V^{2j-1}$ and $\leqq V^{2j}$, and, if i is odd, let σ_{i+1} be the face of σ spanned by its vertices $\geqq V^i$. Similarly, let τ_{2j+1} $(1 \leqq 2j + 1 < i + 1)$ be the face of τ spanned by its vertices $\geqq V^{2j}$ and $\leqq V^{2j+1}$, and, if i is even, let τ_{i+1} be the face of τ spanned by its vertices $\geqq V^i$. By the i-regularity condition, σ, τ can be written as joins of subsimplexes:

$$\sigma = \sigma_0 \cdot \sigma_2 \cdot \cdots \cdot \sigma_{2k}, \quad \tau = \tau_1 \cdot \tau_3 \cdot \cdots \cdot \tau_{2k+(-1)^i}$$

where $2k = i$ if i is even, and $2k = i + 1$ if i is odd. Let τ'_{2j+1} be the face of τ_{2j+1} obtained by deleting the vertices V^{2j} and V^{2j+1}, and, if i is even, let τ'_{i+1}

be the face of τ_{i+1} obtained by deleting V^i. Then the simplex ζ spanned by the vertices of σ and τ is the join

$$\zeta = \sigma_0 \cdot \tau_1' \cdot \sigma_2 \cdot \tau_3' \cdot \cdots \cdot \begin{cases} \tau_{i+1}' & i \text{ odd}, \\ \sigma_{i+1} & i \text{ even}. \end{cases}$$

Define $\sigma \smile_i \tau = 0$ in the group of $(p + q - i)$-cochains if (σ, τ) is not i-regular, and define $\sigma \smile_i \tau = \pm\zeta$ if (σ, τ) is i-regular. The sign is $+$ if $i = 0$. In general, it is the sign of the permutation required to bring the ordered array of vertices

$$\sigma_0, \; \sigma_2, \; \cdots, \; \sigma_{2k}, \; \tau_1', \; \tau_3', \; \cdots, \; \tau_{2k+(-1)^i}'$$

into the order α:

$$\sigma_0, \; \tau_1', \; \sigma_2, \; \tau_3', \; \cdots, \; \sigma_j, \; \tau_{j+1}', \; \cdots.$$

It is to be noted that $\sigma \smile_0 \tau$ coincides with the Alexander-Čech-Whitney [1, 3, 18] product $\sigma \smile \tau$.

Let G, G' be abelian groups *paired* to a third abelian group G''. This means that there is a bilinear product $g \cdot g' \in G''$ defined for $g \in G, g' \in G'$. Let $u^p \in C^p(K, G)$, $v^q \in C^q(K, G')$, and let $u^p = \sum g_j \sigma_j^p$, $v^q = \sum g_k' \sigma_k^q$ be their unique representations in terms of the distinct p and q-simplexes of K oriented by the order α. Define

$$u^p \smile_i v^q = \textstyle\sum_{j,k} (g_j g_k') \sigma_j^p \smile_i \sigma_k^q.$$

Since the right side is a linear form in oriented $(p + q - i)$-simplexes, $u^p \smile_i v^q \in C^{p+q-i}(K, G'')$.

It is easily verified that the product $\smile_i$ is bilinear, so that $C^p(K, G)$, $C^q(K, G')$ are paired to $C^{p+q-i}(K, G'')$.

Theorem 2.1. *$u^p \smile_i v^q = 0$ if $i > p$ or q.*

This follows from the fact that the common face of σ_j^p, σ_k^q has dimension $\leqq$ Min (p, q). Then (σ_j^p, σ_k^q) is not i-regular.

Remark. In the functional notation for cochains, a p-cochain is a function $u^p(A^0, \cdots, A^p)$ with values in G and is defined on each ordered set of $p + 1$ vertices whose union spans a simplex. It is alternating in the order of the vertices and is zero if the vertices do not span a p-simplex. In this notation, if $A^0, \cdots, A^{p+q}$ span a $(p + q)$-simplex and $A^0 < \cdots < A^{p+q}$, then

$$u^p \smile_0 v^q(A^0, \cdots, A^{p+q}) = u^p(A^0, \cdots, A^p) \cdot v^q(A^p, \cdots, A^{p+q}).$$

Under similar conditions,

$$u^p \smile_1 v^q(A^0, \cdots, A^{p+q-1})$$
$$= \sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} u^p(A^0, \cdots, A^j, A^{j+q}, \cdots, A^{p+q-1}) \cdot v^q(A^j, \cdots, A^{j+q}).$$

In general, if ζ is a $(p + q - i)$-simplex, any i-face of ζ determines a splitting into a product: $\sigma \smile_i \tau = \pm\zeta$. Then $u^p \smile_i v^q(\zeta) = \sum \pm u^p(\sigma)\cdot v^q(\tau)$, the sum is taken over those i-faces of ζ such that $\dim \sigma = p$, $\dim \tau = q$. These formulas are awkward to handle for the following reason: Both sides of each equation are defined for any set of vertices on a simplex in any order; however they are equal only if the order is α.

3. Simplicial maps

Let K, K' be simplicial complexes, and $f: K' \to K$ be a simplicial map. For any coefficient group G, f induces a homomorphism

$$f^*: C^p(K, G) \to C^p(K', G)$$

defined as follows. If $u^p \in C^p(K, G)$, and $\sigma' = A^0 \cdots A^p$ is a p-simplex of K', then

$$f^*u^p(A^0 \;\cdots\; A^p) = u^p(f(A^0) \;\cdots\; f(A^p)).$$

Explicitly, if $f(\sigma')$ is degenerate, f^*u^p has the value 0 on σ', otherwise $f^*u^p(\sigma') = u^p(f(\sigma'))$. Let δ, δ' be the coboundary operators in K and K'. As is well known, $\delta' f^* = f^*\delta$ so that f^* maps cocycles into cocycles, coboundaries into coboundaries and thereby induces a homomorphism

$$f^*: H^p(K, G) \to H^p(K', G).$$

In case K' is a subcomplex of K and f is the identity map $f(x) = x$ for $x \in K'$, then the cochain mapping f^* has a very simple form. If $u^p = \sum g_j\sigma_j^p$, f^*u^p is obtained by replacing g_j by zero for each σ_j^p not in K'.

If α, α' are orders in K, K', then f is said to be *order preserving* if $A' \leqq B'$ in α' implies $f(A') \leqq f(B')$ in α. If f and α are given, there exist orders α' such that f is order preserving. For each vertex A of K, introduce a simple order among the vertices of $f^{-1}(A)$ and then order these blocks of vertices as their images are ordered in K.

THEOREM 3.1. *If f: $K' \to K$ is an order preserving simplicial map, then $f^*(u \smile_i v) = f^*u \smile_i f^*v$.*

This is proved most readily using the functional notation for cochains. Suppose ζ' is a $(p + q - i)$-simplex of K', and $\zeta' = \sigma' \smile_i \tau'$ in the order α' where $\dim \sigma' = p$, $\dim \tau' = q$. If $f(\zeta')$ is degenerate, then either $f(\sigma')$ or $f(\tau')$ is degenerate or they have more than an i-face in common. In this case both sides have the value 0 on ζ'. If $f(\zeta')$ is non-degenerate, then

$$f^*(u \smile_i v)(\zeta') = u \smile_i v(f(\zeta')).$$

Since $f \mid \zeta'$ is a 1–1 and order preserving map of ζ' on $f(\zeta')$, it follows that $f(\zeta') = \pm f(\sigma') \smile_i f(\tau')$ and any splitting $f(\zeta') = \pm\sigma \smile_i \tau$ can be obtained in this manner. Therefore

$$\begin{aligned} u \smile_i v(f(\zeta')) &= \sum \pm\, u(f(\sigma')) \cdot v(f(\tau')) \\ &= \sum \pm\, f^*u(\sigma') \cdot f^*v(\tau') = (f^*u \smile_i f^*v)(\zeta'). \end{aligned}$$

4. Join formulas

If σ is an ordered p-simplex and A is a vertex, then σA is defined as follows. It is the $0 \in C^{p+1}(K)$ if A is a vertex of σ or if the vertices of σ together with A do not span a simplex of K. Otherwise it is the ordered $(p + 1)$-simplex consisting of the ordered vertices of σ followed by A. If $u^p = \sum g_j \sigma_j^p$, define $u^p A = \sum g_j(\sigma_j^p A) \in C^{p+1}(K, G)$. It lies in the star of A.

THEOREM. *If the vertex A follows all vertices of σ^p and τ^q in the order α, then*

$$\sigma^p \smile_i (\tau^q A) = \begin{cases} (\sigma^p \smile_i \tau^q)A & i \text{ even}, \\ 0 & i \text{ odd}, \end{cases} \tag{4.1}$$

$$(\sigma^p A) \smile_i \tau^q = \begin{cases} 0 & i \text{ even}, \\ (-1)^{q+1}(\sigma^p \smile_i \tau^q)A & i \text{ odd}, \end{cases} \tag{4.2}$$

$$(\sigma^p A) \smile_i (\tau^q A) = (-1)^{q+i+1}(\sigma^p \smile_{i-1} \tau^q)A. \tag{4.3}$$

REMARK. These formulas may be taken as the basis of an inductive definition of $\smile_i$, the induction is one on the number of vertices involved. If A is the last vertex of σ, and B the last of τ in the order α, apply (4.1) if $A < B$, (4.2) if $A > B$, and (4.3) if $A = B$. To start the induction, $\sigma \smile_i \tau$ must be defined directly if either σ or τ is a single vertex or if $i = 0$. From the strictly logical point of view, it would be better to take these formulas as definitions since all subsequent work could be based on them.

PROOF 4.1. If i is odd, since A is not in σ, the last vertex common to σ and τA is not A. Thus condition $(i + 1)$ for regularity of $(\sigma, \tau A)$ fails. Let i be even. If the vertices of σ, τ, A together do not span a $(p + q - i + 1)$-simplex, both sides vanish. Suppose they do. Then $\tau A \neq 0$ and condition (-1) for i-regularity holds for both (σ, τ) and $(\sigma, \tau A)$. If $(\sigma, \tau A)$ is not i-regular, whichever regularity condition fails for $(\sigma, \tau A)$ will also fail for (σ, τ). Again, both sides vanish. Suppose $(\sigma, \tau A)$ is i-regular. Then (σ, τ) is i-regular, because the common i-face is the same in the two cases, and the presence or absence of A does not disturb the order relations. The number of permutations in calculating $\sigma \smile_i (\tau A)$ is the same as the number in calculating $(\sigma \smile_i \tau)A$ because the last block $\tau'_{2k+1}A$ of vertices in τA is the last block in ζA since i is even.

PROOF 4.2. It is similar to that of 4.1, except in the calculation of the sign of the permutation. In this case the last vertex A of σA must be permuted with $\tau'_1, \tau'_3, \cdots, \tau'_i$ making a total of $q + 1 - (i + 1)$ more interchanges for the right side than the left. Since i is odd, $q + 1 - (i + 1) \equiv q + 1 \bmod 2$.

PROOF 4.3. If the vertices of σ, τ, A do not together span a $(p + q + 2 - i)$-simplex both sides vanish. If they do, condition (-1) for regularity of (σ, τ) and $(\sigma A, \tau A)$ is satisfied. In addition $V^i = A$ so that condition $(i + 1)$ for the i-regularity of $(\sigma A, \tau A)$ is satisfied whether i is even or odd. Condition (i) for i-regularity becomes: $V^{i-1}A$ must be adjacent in σA (τA) if i is odd (even). This is equivalent to: V^{i-1} is the last vertex of σ (τ) if $i - 1$ is even (odd). This is precisely condition (i) for the $(i - 1)$-regularity of (σ, τ). Thus the two conditions hold or fail to hold together. Similarly conditions (j) $(j = 0, 1, \cdots, i-1)$ for the i-regularity of $(\sigma A, \tau A)$ and $(i-1)$-regularity of (σ, τ) hold or fail to hold together. Suppose they all hold. In calculating the sign of the permutation of the left side, the vertex A in σA must be permuted with the τ''s which have $q + 1 - i$ vertices in all. This accounts for the sign in (4.3).

5. The coboundary formula

THEOREM 5.1. *If u, v are cohains of dimensions p, q, then* $\delta(u \smile_i v) = (-1)^{p+q-i}u \smile_{i-1} v + (-1)^{pq+p+q}v \smile_{i-1} u + \delta u \smile_i v + (-1)^p u \smile_i \delta v$.

If $f: K' \to K$ is an order preserving map, and u, v satisfy (5.1) in K, it follows immediately that f^*u, f^*v satisfy (5.1) in K'. It is only necessary to apply the facts that f^* and δ commute, f^* is linear, and f^* preserves products (3.1). In particular, let K' be a subcomplex of K, and let f be the identity map, then any cochain u' of K' is f^*u for some cochain u in K. (Define $u(\sigma)$ to be $u'(\sigma)$ if σ is in K', otherwise $u(\sigma) = 0$). It follows from this that, if (5.1) holds in K, it holds in any subcomplex of K.

Any complex can be realized as a subcomplex of a simplex, namely, the simplex spanned by all the vertices of the complex. It follows that it suffices to prove (5.1) for a complex which consists of all the faces of simplex.

The proof will proceed by induction on the number of vertices of the simplex. To start the induction, let the simplex consist of the vertex A alone. Then the only non-zero product of simplexes is $A \smile_0 A = A$. Thus all terms of (5.1) vanish except possibly in the two cases $i = 0, 1$. In the case $i = 0$, every term vanishes since $A \smile_{-1} A = 0$ and $\delta A = 0$. In the case $i = 1$, all terms involving δ vanish, the remaining two terms become $(-1)A \smile_0 A + (-1)^0 A \smile_0 A = 0$.

Suppose now that (5.1) holds in a simplex $s' = A^0 \cdots A^{n-1}$. Let A be a vertex distinct from $A^0, \cdots, A^{n-1}$ and let $s = A^0 \cdots A^{n-1}A$ with the vertices ordered as indicated. Let δ, δ' be the coboundary operators in s, s'. Then, by the definition of coboundary, for any p-simplex σ in s':

$$\delta\sigma = \delta'\sigma + (-1)^{p+1}\sigma A. \tag{5.2}$$

Since all terms of (5.1) are bilinear in u and v, it suffices to prove the formula in the case $u =$ an oriented p-simplex σ and $v =$ an oriented q-simplex τ. Four cases arise.

CASE 1: *σ and τ are in s'.* Apply (5.2) to $\sigma \smile_i \tau$, and then the assumption that (5.1) holds in s'.

$$\begin{aligned} \delta(\sigma \smile_i \tau) &= \delta'(\sigma \smile_i \tau) + (-1)^{p+q-i+1}(\sigma \smile_i \tau)A = (-1)^{p+q-i}\sigma \smile_{i-1} \tau \\ &+ (-1)^{pq+p+q}\tau \smile_{i-1} \sigma + \delta'\sigma \smile_i \tau + (-1)^p \sigma \smile_i \delta\tau + (-1)^{p+q-i+1}(\sigma \smile_i \tau)A. \end{aligned} \tag{5.3}$$

The fact that $\smile$ in s' equals $\smile$ in s is used here (this is a special case of (3.1) with $f: s' \to s$ the identity map). Thus the first two terms of (5.3) are as desired. Apply (5.2) to the first factor of $\delta\sigma \smile_i \tau$ and to the second of $\sigma \smile_i \delta\tau$, add and obtain

$$\begin{aligned} \delta\sigma \smile_i \tau + (-1)^p \sigma \smile_i \delta\tau &= \delta'\sigma \smile_i \tau + (-1)^p \sigma \smile_i \delta'\tau \\ &+ (-1)^{p+1}\sigma A \smile_i \tau + (-1)^{p+q+1}\sigma \smile_i \tau A. \end{aligned} \tag{5.4}$$

If i is even, by (4.1) and (4.2), the third term on the right of (5.4) is zero and the fourth term equals the last term of (5.3). If i is odd, by (4.1) and (4.2), the fourth term on the right of (5.4) is zero and the third term equals the last term of (5.3). Thus, in either case, the last three terms of (5.3) coincide with the left side of (5.4).

CASE 2: *σ not in s', τ in s'.* Suppose first $\sigma = A$. Then $A \smile_i \tau = A \smile_{i-1} \tau = \tau \smile_{i-1} A = 0$ since A, τ have no common vertex. Also $A \smile_i \delta\tau = A \smile_i \delta'\tau + (-1)^{q+1} A \smile_i \tau A$ by (5.2). Each term is zero since regularity condition (-1) or (0) fails to hold. Finally $\delta A \smile_i \tau = 0$; for $(A'A, \tau)$ is not i-regular since condition (-1) fails if $i > 0$, and condition (1) fails if $i = 0$. Thus fall terms of (5.1) vanish. Suppose then that $\sigma = \sigma' A$ where σ' is in s'. There are two subcases.

CASE 2′: *i is even.* By definition of δ,

$$\delta(\sigma' A) = (\delta'\sigma')A. \tag{5.5}$$

Now calculate each term of (5.1):

$$\begin{aligned}
\delta(\sigma \smile_i \tau) &= \delta(\sigma' A \smile_i \tau) = \delta(0) = 0 && \text{by (4.2),}\\
\sigma \smile_i \tau &= \sigma' A \smile_{i-1} \tau = (-1)^{q+1}(\sigma' \smile_{i-1} \tau)A && \text{by (4.2),}\\
\tau \smile_{i-1} \sigma &= \tau \smile_{i-1} \sigma' A = 0 && \text{by (4.1),}\\
\delta\sigma \smile_i \tau &= \delta(\sigma' A) \smile_i \tau = (\delta'\sigma')A \smile_i \tau = 0 && \text{by (5.5), (4.2),}\\
\sigma \smile_i \delta\tau &= \sigma' A \smile_i (\delta'\tau + (-1)^{q+1} \tau A) && \text{by (5.2)}\\
&= 0 + (\sigma' \smile_{i-1} \tau)A && \text{by (4.2), (4.3)}
\end{aligned}$$

Thus there are just two non-zero terms in (5.1) and these cancel.

CASE 2″: *i is odd.* Calculate each term of (5.1). For the first term apply (5.5) with $\sigma' \smile_i \tau$ in place of σ', and then apply (5.1) in s'.

$$\begin{aligned}
\delta(\sigma \smile_i \tau) &= \delta(\sigma' A \smile_i \tau) = (-1)^{q+1}\delta[(\sigma' \smile_i \tau)A] = (-1)^{q+1}[\delta'(\sigma' \smile_i \tau)]A\\
&= (-1)^{q+1}[(-1)^{p-1+q-i}\sigma' \smile_{i-1} \tau + (-1)^{pq+p-1} \tau \smile_{i-1} \sigma'\\
&\qquad + \delta'\sigma' \smile_i \tau + (-1)^{p-1}\sigma' \smile_i \delta'\tau]A\\
&= (-1)^{p+1}(\sigma' \smile_{i-1} \tau)A + (-1)^{pq+p+q}(\tau \smile_{i-1} \sigma')A + (-1)^{p+1}\\
&\qquad (\delta'\sigma' \smile_i \tau)A + (-1)^{p+q}(\sigma' \smile_i \delta'\tau)A,
\end{aligned}$$

$$\begin{aligned}
\sigma \smile_{i-1} \tau &= \sigma' A \smile_{i-1} \tau = 0 && \text{by (4.2),}\\
\tau \smile_{i-1} \sigma &= \tau \smile_{i-1} \sigma' A = (\tau \smile_{i-1} \sigma')A && \text{by (4.1),}\\
\delta\sigma \smile_i \tau &= (\delta'\sigma')A \smile_i \tau = (-1)^{q+1}(\delta'\sigma' \smile_i \tau)A && \text{by (4.2),}\\
\sigma \smile_i \delta\tau &= \sigma' A \smile_i (\delta'\tau + (-1)^{q+1}\tau A) && \text{by (5.2),}\\
&= (-1)^{q+2}(\sigma' \smile_i \delta'\tau)A - (\sigma' \smile_{i-1} \tau)A && \text{by (4.2), (4.3).}
\end{aligned}$$

Substitute in (5.1) the computed value of each term. The two sides of the equation are then identical in form.

CASE 3: *σ in s', τ not in s'.* Suppose first $\tau = A$. Then $\sigma \smile_i A = \sigma \smile_{i-1} A = A \smile_{i-1} \sigma = 0$ since σ, A have no common vertex. Also $\delta\sigma \smile_i A = \delta'\sigma \smile_i A + (-1)^{p+1}(\sigma A) \smile_i A$, and the first term is zero for the same reason. The second is also zero if $i > 0$. Otherwise $\delta\sigma \smile_0 A = (-1)^{p+1}\sigma A$. Since σ and $A'A$ have at most A' in common, $\sigma \smile_i \delta A = 0$ if $i > 0$. If $i = 0$, the only non-zero term of $\sigma \smile_0 \delta A$ is $\sigma \smile_0 A'A = \sigma A$ where A' is the last vertex of σ. Thus if $i > 0$, all terms of (5.1) vanish; if $i = 0$, there are just two non-zero terms and they cancel. Suppose then $\tau = \tau' A$ where τ' is in s'.

CASE 3': *i is odd.* Computing each term of (5.1) and substituting shows that both sides of (5.1) vanish:

$$\begin{aligned}
\delta(\sigma \smile_i \tau) &= \delta(\sigma \smile_i \tau'A) = \delta(0) = 0 && \text{by (4.1)},\\
\sigma \smile_{i-1} \tau &= \sigma \smile_{i-1} \tau'A = (\sigma \smile_{i-1} \tau')A && \text{by (4.1)},\\
\tau \smile_{i-1} \sigma &= \tau'A \smile_{i-1} \sigma = 0 && \text{by (4.2)},\\
\delta\sigma \smile_i \tau &= (\delta'\sigma + (-1)^{p+1}\sigma A) \smile_i \tau'A && \text{by (5.2)},\\
&= 0 + (-1)^{p+q}(\sigma \smile_{i-1} \tau')A && \text{by (4.1), (4.3)},\\
\sigma \smile_i \delta\tau &= \sigma \smile_i (\delta'\tau')A = 0 && \text{by (5.5), (4.1)}
\end{aligned}$$

CASE 3'': *i is even.* Applying (4.1), (5.5), and (5.1) in s' gives

$$\begin{aligned}
\delta(\sigma \smile_i \tau) &= \delta(\sigma \smile_i \tau'A) = \delta[(\sigma \smile_i \tau')A] = [\delta'(\sigma \smile_i \tau')]A\\
&= (-1)^{p+q-1}(\sigma \smile_{i-1} \tau')A + (-1)^{pq+q-1}(\tau' \smile_{i-1} \sigma)A\\
&\qquad + (\delta'\sigma \smile_i \tau')A + (-1)^p(\sigma \smile_i \delta'\tau')A,\\
\sigma \smile_{i-1} \tau &= \sigma \smile_{i-1} \tau'A = 0 && \text{by (4.1)},\\
\tau \smile_{i-1} \sigma &= \tau'A \smile_{i-1} \sigma = (-1)^{p+1}(\tau' \smile_{i-1} \sigma)A && \text{by (4.2)},\\
\delta\sigma \smile_i \tau &= (\delta'\sigma + (-1)^{p+1}\sigma A) \smile_i \tau'A && \text{by (5.2)},\\
&= (\delta'\sigma \smile_i \tau')A + (-1)^{p+q+1}(\sigma \smile_{i-1} \tau')A && \text{by (4.1), (4.3)},\\
\sigma \smile_i \delta\tau &= \sigma \smile_i (\delta'\tau')A = (\sigma \smile_i \delta'\tau')A && \text{by (5.5), (4.1)}.
\end{aligned}$$

Substitute the computed values in (5.1) and the two sides become identical in form.

CASE 4: *neither σ nor τ in s'.* There are four subcases.

CASE 4.1: $\sigma = \tau = A$. All terms of (5.1) vanish unless $i = 0$ or 1. If $i = 0$, then $\delta(A \smile_0 A) = \delta A$, $A \smile_{-1} A = 0$, $\delta A \smile_0 A = \delta A$, an $A \smile_0 \delta A = 0$. If $i = 1$, all terms vanish except the first two on the right which reduce to $-A + A$.

CASE 4.2: $\sigma = A$, $\tau = \tau'A$. All terms of (5.1) vanish unless $i = 1$. In this case, all terms vanish except $\tau \smile_0 \sigma = \tau'A \smile_0 A = \tau'A$ and $\delta\sigma \smile_1 \tau = (-1)^{q+1}\tau'A$.

CASE 4.3: $\sigma = \sigma'A$, $\tau = A$. All terms of (5.1) vanish unless $i = 0$ or 1.

In either case, $\tau \smile_{i-1} \sigma = 0$. If $i = 0$, the only non-zero terms are $\delta(\sigma \smile_0 \tau) = \delta(\sigma' A \smile_0 A) = \delta(\sigma)$ and $\delta\sigma \smile_0 \tau = \delta(\sigma' A) \smile_0 A = (\delta'\sigma')A \smile_0 A = (\delta'\sigma')A = \delta\sigma$. If $i = 1$, the only non-zero terms are $\sigma \smile_0 \tau = \sigma' A \smile_0 A = \sigma' A$ and $\sigma \smile_1 \delta\tau = \sigma$.

CASE 4.4: $\sigma = \sigma' A$, $\tau = \tau' A$. Apply (4.3), (5.5), and (5.1) in s' to obtain:

$$
\begin{aligned}
\delta(\sigma \smile_i \tau) &= \delta(\sigma' A \smile_i \tau' A) = (-1)^{q+i}\delta[(\sigma' \smile_{i-1} \tau')A] = (-1)^{q+i}[\delta'(\sigma' \smile_{i-1} \tau')]A \\
&= (-1)^{q+i}[(-1)^{p+q+i+1}(\sigma' \smile_{i-2} \tau')A + (-1)^{pq+1}(\tau' \smile_{i-2} \sigma')A \\
&\qquad + (\delta'\sigma' \smile_{i-1} \tau')A + (-1)^{p-1}(\sigma' \smile_{i-1} \delta'\tau')A], \\
\sigma \smile_{i-1} \tau &= (-1)^{q+i+1}(\sigma' \smile_{i-2} \tau')A && \text{by (4.3)}, \\
\tau \smile_{i-1} \sigma &= (-1)^{p+i+1}(\tau' \smile_{i-2} \sigma')A && \text{by (4.3)}, \\
\delta\sigma \smile_i \tau &= (\delta'\sigma')A \smile_i \tau' A = (-1)^{q+i}(\delta'\sigma \smile_{i-1} \tau')A && \text{by (5.5), (4.3)}, \\
\sigma \smile_i \delta\tau &= \sigma' A \smile_i (\delta'\tau')A = (-1)^{q+i+1}(\sigma' \smile_{i-1} \delta'\tau')A && \text{by (5.5), (4.3)},
\end{aligned}
$$

Substitute the computed values in (5.1) and the two sides become identical in form. This completes the proof of (5.1).

6. The squaring operations

If u, v are cocycles ($\delta u = \delta v = 0$), then the last two terms of the formula for $\delta(u \smile_i v)$ vanish. However the first two terms are not necessarily zero unless $i = 0$. Thus, products of cocycles need not be cocycles except in the case $i = 0$. Since the fact that $u \smile_0 v$ is a cocycle has been thoroughly exploited, the main interest lies in what can be accomplished in the general case.

Suppose now and henceforth that G is paired with itself to G'. If $u, v \in Z^p(K, G)$, and $w \in C^{p-1}(K, G)$, direct substitution in the coboundary formula (5.1) gives

$$\delta(u \smile_{i+1} v) = (-1)^{i+1}u \smile_i v + (-1)^p v \smile_i u, \tag{6.1}$$

$$\delta(u \smile_i u) = [(-1)^i + (-1)^p]u \smile_{i-1} u, \tag{6.2}$$

$$\delta(w \smile_{i-1} w + w \smile_i \delta w) = \delta w \smile_i \delta w - [(-1)^i + (-1)^p](w \smile_{i-2} w + w \smile_{i-1} \delta w). \tag{6.3}$$

THEOREM. *If $p - i$ is odd, and $u, v \in Z^p(K, G)$, then*

$$u \smile_i v + v \smile_i u \sim 0 \tag{6.4}$$

$$\delta(u \smile_i u) = 0 \tag{6.5}$$

$$2u \smile_i u \sim 0 \tag{6.6}$$

$$u \sim 0 \textit{ implies } u \smile_i u \sim 0 \tag{6.7}$$

$$u \sim v \textit{ implies } u \smile_i u \sim v \smile_i v \tag{6.8}$$

$$(u + v) \smile_i (u + v) \sim u \smile_i u + v \smile_i v. \tag{6.9}$$

PROOF. (6.4) follows from (6.1), (6.5) from (6.2), (6.6) from (6.4) with $u = v$, (6.7) from (6.3) with $u = \delta w$, (6.9) from the bilinearity of $\smile_i$ and (6.4). To prove (6.8), apply (6.7) to $u - v$:

$$(u - v) \smile_i (u - v) = u \smile_i u - v \smile_i v + 2v \smile_i v - (u \smile_i v + v \smile_i u) \sim 0.$$

The third and fourth terms are ~ 0 by (6.6) and (6.4). Therefore $u \smile_i u - v \smile_i v \sim 0$.

An immediate consequence of (6.5), (6.8), (6.9), and (6.6) is

THEOREM 6.10. *If $p - i$ is odd, the operation $u \to u \smile_i u$ maps cocycles into cocycles, cohomologous cocycles into such, and thereby induces a homomorphic mapping*

$$\mathrm{Sq}_i : H^p(K, G) \to H^{2p-i}(K, G')$$

called the i^{th} square. Each image under Sq_i *has order* 2.

If $p - i$ is even, the situation is more complex. Let $2G'$ denote the subgroup of elements of G' divisible by 2 and let $\xi: G' \to G'/2G'$ be the natural homomorphism. Then ξ induces a homomorphism $\xi^*: C^q(K, G') \to C^q(K, G'/2G')$ defined by $\xi^*(\sum g_j' \sigma_j^q) = \sum \xi(g_j')\sigma_j^q$. It follows readily that $\xi^*\delta = \delta\xi^*$ and, for a simplicial map f, $f^*\xi^* = \xi^* f^*$. The operation ξ^* is called reduction modulo 2. The relation $u \sim v$ mod 2 means $\xi^*u \sim \xi^*v$.

THEOREM 6.11. *If $p - i$ is even, and $u, v \in Z^p(K, G)$, then the conclusions of the statements* (6.4) *to* (6.9) *all hold* mod 2.

The proofs parallel those already given in the case $p - i$ is odd. Analogous to (6.10):

THEOREM 6.12. *If $p - i$ is even, the operation $u \to \xi^*(u \smile_i u)$ maps cocycles into cocycles, cohomologous coycles into such and thereby induces a homomorphism*

$$\mathrm{Sq}_i: H^p(K, G) \to H^{2p-i}(K, G'/2G')$$

called the i^{th} square mod 2.

THEOREM 6.13. *If $f: K' \to K$ is an order preserving simplicial map, then, for each i, $f^*\mathrm{Sq}_i = \mathrm{Sq}_i f^*$.*

PROOF. If u is a p-cocycle, and $p - i$ is odd, then, by (3.1) and the properties of f^*:

$$f^*\mathrm{Sq}_i\{u\} = f^*\{u \smile_i u\} = \{f^*(u \smile_i u)\} = \{f^*u \smile_i f^*u\} = \mathrm{Sq}_i\{f^*u\} = \mathrm{Sq}_i f^*\{u\}.$$

If $p - i$ is even, the proof differs only in the use of $f^*\xi^* = \xi^* f^*$.

REMARK. In case p is even and $i = 0$, it is not necessary to reduce mod 2 in order that S should be defined. Without such reduction the operation need not be homomorphic. In the Pontrjagin extension theorem (see (21.1)), $f^*s^2 \smile_0 f^*s^2$ is not reduced mod 2. In the generalized form of the theorem $f^*s^n \smile_{n-2} f^*s^n$ need not be reduced mod 2 since, as will appear, its coefficients are in $\pi_{n+1}(S^n)$ which is cyclic of order 2 for $n > 2$.

7. The product complex $K \times I$

Until now the products have all been based on a fixed order α in K. The present objective is to show that Sq_i is independent of the choice of α. This will be prove by exhibiting a cochain homotopy connecting products based on different orders. This leads to the consideration of the product space $K \times I$ where I = unit interval [0, 1]. This space has a simple cellular division into product cells. Since it is necessary to form cup products in $K \times I$, and no rules are at hand for computing these in a cell space, a simplicial division must be introduced. The required simplicial complex is easily defined. Since it is both awkward and unnecessary to prove it homeomorphic to $K \times I$, this point is skipped.

Let (A_0), (A_1) be two disjoint sets of vertices each in 1–1 correspondence with the vertices (A) of K. Let $f_0(A) = A_0$, $f_1(A) = A_1$ be the correspondences. The union of (A_0) and (A_1) form the set of vertices of $K \times I$. Let α be an order in K. A set of vertices $A_0^0 \cdots A_0^k A_1^{k+1} \cdots A_1^p$ are those of a p-simplex in $K \times I$ if, in the order α, $A^0 < A^1 < \cdots < A^k \leqq A^{k+1} < \cdots < A^p$, and these are the vertices of a p or $(p-1)$-simplex of K.

The maps f_0, f_1 define unique simplicial maps of K in $K \times I$, these are also denoted by f_0, f_1. The map $g\colon K \times I \to K$, defined by $g(A_0) = g(A_1) = A$ for each A, is a simplicial map, and

$$gf_0 = gf_1 = \text{the identity map of } K. \tag{7.1}$$

If $u \in C^p(K \times I, G)$, $p > 0$ define $Du \in C^{p-1}(K, G)$ by

$$Du(A^0 \cdots A^{p-1}) = \sum_{k=0}^{p-1} (-1)^k u(A_0^0 \cdots A_0^k A_1^k \cdots A_1^{p-1}) \tag{7.2}$$

where $A^0 \cdots A^{p-1}$ is a $(p-1)$-simplex of K in the order α. The deformation operator D is a homomorphism

$$D\colon C^p(K \times I, G) \to C^{p-1}(K, G), \qquad p > 0.$$

Since $f_0, f_1\colon K \to K \times I$, they induce homomorphisms

$$f_0^*, f_1^*\colon C^p(K \times I, G) \to C^p(K, G).$$

These three operations are related by

$$Du = f_1^*u - f_0^*u - D\delta u, \quad u \in C^p(K \times I, G), p > 0, \tag{7.3}$$

$$0 = f_1^*u - f_0^*u - D\delta u. \qquad u \in C^0(K \times I, G). \tag{7.4}$$

In proof of (7.4), $D\delta u(A) = \delta u(A_0 A_1) = u(A_1) - u(A_0) = f_1^*u(A) - f_0^* u(A)$. To prove (7.3), evaluate $D\delta u$ and δDu on the p-simplex $A^0 \cdots A^p$:

$$D\delta u(A^0 \cdots A^p) = \sum_{k=0}^{p} (-1)^k \delta u(A_0^0 \cdots A_0^k A_1^k \cdots A_1^p)$$

$$= \sum_{k=0}^{p} (-1)^k \Bigg[\sum_{j=0}^{k} (-1)^j u(A_0^0 \cdots \hat{A}_0^j \cdots A_0^k A_1^k \cdots A_1^p)$$

$$- \sum_{j=k}^{p} (-1)^j u(A_0^0 \cdots A_0^k A_1^k \cdots \hat{A}_1^j \cdots A_1^p) \Bigg],$$

$$\delta D u(A^0 \cdots A^p) = \sum_{j=0}^{p} (-1)^j D u(A^0 \cdots \hat{A}^j \cdots A^p)$$

$$= \sum_{j=0}^{p} (-1)^j \Bigg[\sum_{k=0}^{j-1} (-1)^k u(A_0^0 \cdots A_0^k A_1^k \cdots \hat{A}_1^j \cdots A_1^p)$$

$$- \sum_{k=j+1}^{p} (-1)^k u(A_0^0 \cdots \hat{A}_0^j \cdots A_0^k A_1^k \cdots A_1^p) \Bigg].$$

The symbol $\hat{A}$ means that A has been deleted from the array. Each term of δDu is identical with a term of $-D\delta u$. The excess terms of the latter cancel in pairs save for two terms

$$u(A_1^0 \cdots A_1^p) - u(A_0^0 \cdots A_0^p) = f_1^* u(A^0 \cdots A^p) - f_0^* u(A^0 \cdots A^p).$$

Since, for any $u \in C^p(K, G)$, g^*u is zero on simplexes of the form appearing on the right of (7.2), it follows that

(7.5) $$Dg^* = 0.$$

8. Change of order in K

Let α_0, α_1 be two orders in K. Define $K \times I, f_0, f_1, g$ using the order α_0 as in §7. Let

$$g^*: C^p(K, G) \rightarrow C^p(K \times I, G)$$

be the cochain mapping induced by g. The orders α_0, α_1 define two products $\smile_i^0$, $\smile_i^1$ in K.

An order (α_0, α_1) is defined in $K \times I$ as follows. Order (A_0) as their correspondents (A) are ordered by α_0, order (A_1) as their correspondents (A) are ordered by α_1, and agree that, on any simplex of $K \times I$, a vertex of $K \times 0$ precedes one of $K \times 1$. Then (α_0, α_1) defines a product $\smile_i$ in $K \times I$. Since $f_0\,(f_1)$ preserves $\alpha_0\,(\alpha_1)$, it follows from (3.1) that $f_0^*\ (f_1^*)$ maps $\smile_i$ into $\smile_i^0\ (\smile_i^1)$.

Corresponding to the orders α_0, α_1 define a new product in K by

(8.1) $$u \vee_i v = D(g^*u \smile_i g^*v), \qquad u \in C^p(K, G), v \in C^q(K, G).$$

This product lies in $C^{p+q-i-1}(K, G')$. Since D, g^* are linear and $\smile$ is bilinear, it follows that $\vee$ is bilinear. To obtain its coboundary formula, apply δ to (8.1), use (7.3), (7.1), (5.1) and then (8.1) with i replaced by $i - 1$. This yields

(8.2) $$\delta(u \vee_i v) = u \smile_i^1 v - u \smile_i^0 v - [(-1)^{p+q-i} u \vee_{i-1} v + (-1)^{pq+p+q} v \vee_{i-1} u + \delta u \vee_i v + (-1)^p u \vee_i \delta v].$$

Take now the case where $u = v$ is a cocycle, then

(8.3) $$\delta(u \vee_i u) = u \smile_i^1 u - u \smile_i^0 u - [(-1)^i + (-1)^p]\, u \vee_{i-1} u.$$

This proves

THEOREM 8.4. *If* $u \in Z^p(K, G)$ *and* $p - i$ *is odd, the cohomology class of* $u \smile_i u$ *is independent of the order defining* $\smile$. *If* $p - i$ *is even, the cohomology class* mod 2 *of* $u \smile_i u$ *is independent of the order. Thus* Sq_i *is independent of the order used to define it.*

THEOREM 8.5. *If* $f\colon K' \to K$ *is simplicial, then* $f^*\mathrm{Sq}_i = \mathrm{Sq}_i f^*$.

Since, for any order α in K, there exists an order in K' such that f is order preserving (§3), (8.5) follows from (8.4) and (6.13).

THEOREM 8.6. *If the orders* α_0, α_1 *coincide, then* $u \vee_i v = 0$.

In this case, g is order preserving, so that $g^*u \smile_i^0 g^*v = g^*(u \smile_i v)$ by (3.1). Now apply (7.5) to (8.1)

9. The relative case

Let L be a closed subcomplex of K. The p-cochains of K which are zero on simplexes of L form a subgroup $C^p(K, L, G)$ of $C^p(K, G)$. Since L is closed, the coboundary of such a cochain is also zero on L:

$$\delta\colon C^p(K, L, G) \to C^{p+1}(K, L, G).$$

Therefore $Z^p(K, L, G)$, $B^p(K, L, G)$ and $H^p(K, L, G) = Z^p/B^p$ can be defined in the usual way. If $f\colon K' \to K$ is simplicial and such that $f(L') \subset L$ where L' is a closed subcomplex of K', then the induced cochain mapping f^* maps $C^p(K, L, G) \to C^p(K', L', G')$. It follows that f induces a homomorphism

$$f^*\colon H^p(K, L, G) \to H^p(K', L', G). \tag{9.1}$$

If σ, τ are simplexes in $K - L$, then $\sigma \smile_i \tau$ is either zero or a simplex of $K - L$. Thus, if the cochains u, v are zero on L, so also is $u \smile_i v$. This is likewise true of $u \vee_i v$. It follows now that the operations Sq_i can be defined for the groups $H^p(K, L, G)$ and they are independent of the order used to define them. Just as before, Sq_i will commute with the f^* of (9.1).

If $w \in C^p(L, G)$, it may be regarded as an element of $C^p(K, G)$ by defining it to be zero on simplexes of $K - L$. This is an isomorphic imbedding of $C^p(L, G)$ into $C^p(K, G)$. Then w has two coboundaries $\delta_L w$ and $\delta_K w$; and $\delta_K w = \delta_L w + v$ where $v \in C^{p+1}(K, L, G)$. In case $w \in Z^p(L, G)$, $\delta_L w = 0$ so that

$$\delta_K\colon Z^p(L, G) \to Z^{p+1}(K, L, G) \tag{9.2}$$

homomorphically. Since $0 = \delta_K\delta_K w = \delta_K\delta_L w + \delta_K v$, it follows that δ_K maps $B^p(L, G)$ into $B^{p+1}(K, L, G)$. Therefore (9.2) preserves cohomology classes and thereby induces a homomorphism

$$\delta^*\colon H^p(L, G) \to H^{p+1}(K, L, G). \tag{9.3}$$

Since, for a simplicial f, $\delta_K f^* = f^*\delta_K$ it follows that

$$f^*\delta^* = \delta'^*(f \mid L')^* \tag{9.4}$$

where f^* is (9.1) and δ'^* is the analogue of (9.3) in K', L'.

Suppose now that L has the property that it contains any simplex of K whose vertices are in L. If σ, τ are simplexes in L, it follows from this hypothesis on L that the products $\sigma \smile_i \tau$ calculated in K coincide with the corresponding products calculated in L (using, of course, the same order as in K).

Order the vertices of K so that each vertex of $K - L$ precedes each vertex of L. If σ is in L and τ is in $K - L$, then $(\sigma_i\ \tau)$ is not i-regular since the first vertex of τ is not in σ. Thus if $w \in C^p(L, G)$ and $v \in C^q(K, L, G)$ it follows that $w \smile_i v = 0$. In particular, if $w \in Z^p(L, G)$, then $w \smile_i \delta_K w = 0$. Apply this to (6.3) to obtain

$$\delta_K(w \smile_{i-1} w) = \delta_K w \smile_i \delta_K w - [(-1)^i + (-1)^p] w \smile_{i-2} w. \tag{9.5}$$

This proves

THEOREM 9.6. $\mathrm{Sq}_i \delta^* = \delta^* \mathrm{Sq}_{i-1}$, $i \geqq 1$.

10. Products in a space

Let X be a topological space and A a closed subset of X. Let $H^p(X, A, G)$ denote the Čech cohomology group of X mod A with coefficients in G (see [11]). An element $\{u\} \in H^p(X, A, G)$ is represented by $u \in H^p(K, L, G)$ where K is the nerve of some finite covering of X by closed sets and L consists of the simplexes of K that meet A. If $u' \in H^p(K', L', G')$ for a second covering complex K', and $\{u\} = \{u'\}$; then there exists a common refinement of the two coverings with nerve K'' such that

$$g^* u = g'^* u' \qquad \text{in } H^p(K'', L'', G) \tag{10.1}$$

where $g\colon (K'', L'') \to (K, L)$ and $g'\colon (K'', L'') \to (K', L')$ are simplicial projections determined by inclusion relations among the closed sets of the various coverings. From (10.1) and (8.5) it follows that $\{\mathrm{Sq}_i u\} = \{\mathrm{Sq}_i u'\}$. Therefore

$$\mathrm{Sq}_i\{u\} = \{\mathrm{Sq}_i u\} \tag{10.2}$$

defines a homomorphic map $\mathrm{Sq}_i\colon H^p(X, A, G) \to H^{2p-i}(X, A, G')$ or $H^{2p-i}(X, A, G'/2G')$ according as $p - i$ is odd or even.

If $f\colon X' \to X$ is continuous and $f(A') \subset f(A)$, then f induces a homomorphism

$$f^*\colon H^p(X, A, G) \to H^p(X', A', G). \tag{10.3}$$

It is determined as follows. Let $u \in H^p(K, L, G)$, represent $\{u\} \in H^p(X, A, G)$, where K is the nerve of the covering $[U]$. Then $[f^{-1}(U)]$ is a covering of X' with nerve K'. Let $f_K\colon K' \to K$ be the simplicial map which attaches the vertex $f^{-1}(U)$ of K' to the vertex U of K. Then (10.3) is obtained by

$$f^*\{u\} = \{f_K^* u\}. \tag{10.4}$$

by (10.4), (10.2) and (8.5),

$$f^* \mathrm{Sq}_i = \mathrm{Sq}_i f^*. \tag{10.5}$$

The homomorphism

$$\delta^*\colon H^p(A, G) \to H^{p+1}(X, A, G)$$

is defined by $\delta^*\{w\} = \{\delta^* w\}$ where $w \in H^p(L, G)$. If X is a normal space, the relation

$$\delta^* \mathrm{Sq}_{i-1} = \mathrm{Sq}_i \delta^* \tag{10.6}$$

holds. This follows from (9.6) and the proposition: Any covering complex (K, L) of (X, A) has a refinement (K', L') such that L' contains each simplex of K' whose vertices are in L'. To prove this, let the vertices of K be the closed sets $U_1, \cdots, U_m$. Form the union B of all the intersections $U_{i_1} \cdots U_{i_K}$ which do not meet A. Then B is closed and does not meet A. By normality, there exists an open set W such that $A \subset W$ and $\overline{W}B = 0$. Let $U'_i = U_i W$, and $U''_i = U_i(X - W)$. The closed sets $[U'_i, U''_i]$ cover X and none of the sets U''_i meet A. Suppose $U'_{i_1}, \cdots, U'_{i_K}$ meet A and $V' = U'_{i_1} \cdots U'_{i_K} \neq 0$. It follows that $V = U_{i_1} \cdots U_{i_K} \neq 0$ and is not in B. Therefore V meets A. But $V' = V\overline{W}$, so V' meets A. Thus the covering $[U'_i, U''_i]$ has the required property.

Suppose now that X is the space of a complex K and A corresponds to the subcomplex L of K. Then K is the nerve of the covering of X by the closed stars of the vertices of K in the first barycentric sudivision K' of K. If $u \in H^p(K, L, G)$, then $\psi u = \{u\} \in H^p(X, A, G)$ is known to be an isomorphism $\psi: H^p(K, L, G) \approx H^p(X, A, G)$. It is the function ψ which proves the topological invariance of $H^p(K, L, G)$. Since $\psi \mathrm{Sq}_i u = \{\mathrm{Sq}_i u\} = \mathrm{Sq}_i\{u\} = \mathrm{Sq}_i \psi u$, by (10.2) it follows that $\psi \mathrm{Sq}_i = \mathrm{Sq}_i \psi$. Therefore the operation Sq_i as defined in a complex has a topologically invariant meaning.

11. The join operation

Let K be a simplicial complex, A a vertex not in K, and let $\hat{K}$ denote the join of A and K. Since $\hat{K}$ is contractible in itself to A, $H^p(\hat{K}, G) = 0$ for $p > 0$ and $H^0(\hat{K}, G) \approx G$. It follows now from the exactness of the cohomology sequence of $(\hat{K}, K)$ that

$$\delta^*: H^p(\hat{K}, G) \approx H^{p+1}(\hat{K}, K, G), \qquad p \geqq 0. \tag{11.1}$$

Let B be a second vertex not in K and let $\mathring{K}$ denote the join of K with the pair of vertices A, B. Let L denote the join of K with B so that L is a closed subcomplex of $\mathring{K}$. The identity map $h: (\hat{K}, K) \to (\mathring{K}, L)$ is an ismorphism of the open complexes $\hat{K} - K$ and $\mathring{K} - L$. Therefore h induces isomorphisms.

$$h^*: H^p(\mathring{K}, L, G) \approx H^p(\hat{K}, K, G), \qquad p \geqq 0. \tag{11.2}$$

Since L is contractible on itself to B, the two pairs $(\mathring{K}, L)$ and $(\mathring{K}, B)$ are homotopically equivalent. Therefore, the identity map $k: (\mathring{K}, B) \to (\mathring{K}, L)$ induces isomorphisms:

$$k^*: H^p(\mathring{K}, L, G) \approx H^p(\mathring{K}, B, G), \qquad p \geqq 0. \tag{11.3}$$

Finally, since B is a point, the identity map $l: \mathring{K} \to (\mathring{K}, B)$ induces isomorphisms

$$l^*: H^p(\mathring{K}, B, G) \approx H^p(\mathring{K}, G), \qquad p \geqq 1. \tag{11.4}$$

The isomorphisms (11.1, 2, 3, 4) combine to form an isomorphism

(11.5) $$\phi: H^p(K, G) \approx H^{p+1}(\mathring{K}, G), \qquad p \geqq 0.$$

By (10.5), Sq_i commutes with h^*, k^* and l^*. Then, by (10.6),

(11.6) $$\phi \mathrm{Sq}_{i-1} = \mathrm{Sq}_i \phi.$$

It should be observed that $p - (i - 1) = (p + 1) - i$ so that the odd and even cases in the definition of Sq_i agree for the two sides of (11.6).

In case p is even, $\mathrm{Sq}_0\{u^p\} = \{u^p \smile_0 u^p\}$ reduced mod 2. Therefore $\phi\{u \smile_0 u^p\} = \phi \mathrm{Sq}_0\{u^p\}$ mod 2. This and (11.6) give

(11.7) $$\phi\{u^p \smile_0 u^p\} = \mathrm{Sq}_1 \phi\{u^p\} \bmod 2, \qquad p \text{ is even.}$$

12. The existence of non-trivial squares

Since there are examples of cocycles u such that $u \smile_0 u$ is not a coboundary, an inductive argument, using (11.5, 6, 7), shows that there exist examples of non-zero squares for each order i. However, by (2.1),

THEOREM 12.1. *$\mathrm{Sq}_i\{u^p\} = 0$ if $i > p$.*

THEOREM 12.2. *$\mathrm{Sq}_{p-2k-1}\{u^p\} = 0$ if and only if there exists a cocycle $v \in Z^{p+2k}(K, L, G')$ such that $u^p \smile_{p-2k} u^p \sim v$ mod 2.*

Suppose $\delta w = u \smile_{p-2k-1} u$ mod 2. Let $v = u \smile_{p-2k} u - (-1)^p 2w$. Then by (6.2), $\delta v = 0$. For the converse, suppose the cocycle v exists. Then there is cochain w such that

$$\delta w = u \smile_{p-2k} u - v - 2v'$$

for some cochain v'. Form δ of both sides, apply (6.2) and obtain $\delta v' = (-1)^p u \smile_{p-2k-1} u$.

COROLLARY 12.3. *If $\mathrm{Sq}_{p-2k}\{u^p\} = 0$, then $\mathrm{Sq}_{p-2k-1}\{u^p\} = 0$.*

LEMMA 12.4. *For any p-simplex σ, $\sigma \smile_p \sigma = \sigma$.*

A check of the definition shows that (σ, σ) is p-regular. Furthermore the ζ spanned by the union of the vertices of σ, σ is σ. The sign of the permutation is $+$ since all the τ' terms are empty.

THEOREM 12.5. *If u is a p-cochain with integer coefficients then $u \smile_p u = u$ mod 2.*

If σ, τ are distinct p-simplexes, then (σ, τ) is not p-regular since the common face has dimension $< p$. Thus, by (12.4), $u \smile_p u$ is obtained from u by squaring the coefficient of each simplex. The coefficient of each simplex of $u \smile_p u - u$ has the form $n^2 - n \equiv 0$ mod 2.

THEOREM 12.6. *If u is a p-cocycle with integer coefficients, then $\mathrm{Sq}_p\{u\} = \{u\}$ mod 2, and $\mathrm{Sq}_{p-1}\{u\} = 0$.*

The first proposition follows from (12.5), the second from the first in view of (12.2).

COROLLARY 12.7. *If u^1 is a 1-cocycle with integer coefficients, then $u^1 \smile_0 u^1$ is a coboundary.*

COROLLARY 12.8. *In an orientable n-manifold, the self-intersection of any* $(n - 1)$*-cycle with integer coefficients is homologous to zero.*

This follows from (12.7) and standard duality theorems [18, §19].

PART II. APPLICATIONS OF PRODUCTS TO EXTENSION AND HOMOTOPY PROBLEMS

13. Preliminaries

Let K, Y be topological spaces, L a closed subset of K, and $f: L \to Y$ be continuous. A continuous map $f': K \to Y$ such that $f' \mid L = f$ (i.e. $f'(x) = f(x)$ for $x \in L$) is called an extension of f to K. The general problem is to determine conditions on K, L, Y and f which are necessary or sufficient or both for the existence of an f'.

As a special case, let $L = Y$ and $f =$ identity. If an extension f' exists, L is called a retract of K and the map f' is called a retraction of K into L.

13.1. Suppose L is a retract of K and g is a retraction. Then, for any Y and any map $f: L \to Y$, the extension $f'(x) = f(g(x))$ exists.

If f_0, f_1 are maps of K into Y, a map $F: K \times I \to Y$ ($I =$ unit interval $0 \leqq t \leqq 1$) is called a homotopy of f_0 into f_1 if $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. The statement $f_0 \cong f_1$ means that the homotopy F exist. The statement $f_0 \cong f_1$ rel. L means that an F exists such that $F(x, t) = f_0(x)$ for $x \in L$. For a fixed t, the map $f_t: K \to Y$ is defined by $f_t(x) = F(x, t)$.

The general problem is to determine conditions on K, Y, f_0, f_1 which are necessary or sufficient or both for the existence of an F. If $L = K \times 0 + K \times 1$, then F is already prescribed on K, and the problem is to extend to $K \times I$. Thus a homotopy problem gives rise to a special type of extension problem.

Henceforth K will be a finite cell complex and L will be a subcomplex of K. This is a customary restriction; results obtained in the simpler case usually extend to more general cases by a limiting process.

It is known [2, p. 501] that $K \times 0 + L \times I$ is a retract of $K \times I$. This proposition has two useful consequences:

13.2. If $f: K \to Y$, a homotopy of $f \mid L$ can be extended to a homotopy of f.

13.3. Let $f_0, f_1: L \to Y$ and let F be a homotopy of f_0 into f_1. If f_0 admits an extension f_0' to K, there exists an extension f_1' of f_1 to K, and an extension F' of F to $K \times I$ such that F' is a homotopy of f_0' into f_1'.

A consequence of (13.3) is that the existence or non-existence of an extension f' depends only on the homotopy class of f. As an application suppose $f_0: L \to Y$ is homotopic to a constant map (i.e. $f_1(L) =$ a point $y_0 \in Y$. Since a constant map can be extended $f_1'(K) = y_0$, it follows that f_0 can be extended to K. In particular if Y is contractible on itself to a point y_0 (e.g. $Y = n$-cell), the contraction gives a homotopy of f_0 to a constant map; so any map of L into Y can be extended to K.

Since the n-spheres S^n ($n = 1, 2, \cdots$) are perhaps the simplest spaces not contractible on themselves to points, it will be supposed henceforth that $Y = S^n$.

14. The obstructions to an extension

The *q-dimensional section* K^q of K is composed of the cells of K of dimensions $\leqq q$. It is a closed subcomplex of L. Let $\bar{K}^q = K^q + L$. Let $f: \bar{K}^{q-1} \to S^n$. Let the homotopy group $\pi_{q-1}(S^n)$ be defined as the homotopy classes of maps of a fixed oriented $(q-1)$-sphere S^{q-1} into S^n. For each oriented q-cell σ of K choose a map g of S^{q-1} on the boundary $\partial\sigma$ of σ of degree $+1$, and let $c(f, \sigma)$ be the element of $\pi_{q-1}(S^n)$ determined by fg. It is independent of the choice of g. The *q-dimension obstruction* $c^q(f)$ to the extension of f is the q-cochain with coefficients in $\pi_{q-1}(S^n)$ which has the value $c(f, \sigma)$ on each σ. If σ is in L, f is defined on σ, so fg is homotopic to a constant. Therefore $c^q(f)$ is zero on L and represents a cochain of $K - L$.

The obstruction was first defined by Eilenberg [4]; he proved the following of its properties:

1. $\delta c^q(f) = 0$.
2. If $f_0 \cong f_1$, then $c^q(f_0) = c^q(f_1)$.
3. If $f_0 \mid \bar{K}^{q-2} \cong f_1 \mid \bar{K}^{q-2}$, then $c^q(f_0) \sim c^q(f_1)$ in $K - L$.
4. If $c^q \sim c(^qf_0)$ in $K - L$, there exists an $f_1: \bar{K}^{q-1} \to S^n$ such that $f_1 \mid \bar{K}^{q-2} = f_0 \mid \bar{K}^{q-2}$ and $c^q(f_1) = c^q$.
5. $c^q(f) = 0$ if and only if f extends over $\bar{K}^q$.
6. $c^q(f) \sim 0$ in $K - L$ if and only if $f \mid \bar{K}^{q-2}$ extends over $\bar{K}^q$.

There is an additional property which will be needed. Let $f: K \to K'$ map each cell of K into a cell of equal or lower dimension. Let $L \subset K$, $L' \subset K'$ be subcomplexes such that $f(L) \subset L'$. Let $h: \bar{K}'^{q-1} \to S^n$. Then $hf: \bar{K}^{q-1} \to S^n$, so that $c^q(h)$ and $c^q(hf)$ are defined. If f maps the q-cell σ on a cell of lower dimension, both $c^q(hf)$ and $f^*c^q(h)$ are zero on σ. Suppose f maps σ on the q-cell σ' with degree m. Let $g: S^{q-1} \to \partial\sigma$ and $g': S^{q-1} \to \partial\sigma'$ have degree 1, and let $k: S^{q-1} \to S^{q-1}$ have degree m. Since $fg: S^{q-1} \to \partial\sigma'$ has degree m, $fg \cong g'k$ and $hfg \cong hg'k$. By definition of addition in $\pi_{q-1}(S^n)$, $hg'k$ represents m times the element represented by hg'. Therefore $c(hf, \sigma) = mc(h, \sigma')$. This proves

7. $f^*c^q(h) = c^q(hf)$.

If $f: \bar{K}^{q-1} \to S^n$ can be extended to $f': \bar{K}^q \to S^n$ (i.e. $c^q(f) = 0$), then $c^{q+1}(f')$ is defined. According to (3) above, the cohomology class $\{c^{q+1}(f')\} = z^{q+1}(f)$ in $K - L$ is independent of the choice of the extension f' of f. This class is called the *secondary obstruction* to the extension of f. It is defined if and only if the *primary* obstruction $c^q(f) = 0$.

Suppose, in the situation to which property (7) applies, the map h has an extension $h': \bar{K}'^q \to S^n$. Then hf has the extension $h'f$; and, by (7), $f^*c^{q+1}(h') = c^{q+1}(h'f)$. This proves

8. $f^*z^{q+1}(h) = z^{q+1}(hf)$.

15. The stepwise extension process

Let $f: L \to S^n$ and suppose an extension of f to K is sought. By defining f on the vertices of $K - L$ arbitrarily, an extension $f_0: \bar{K}^0 \to S^n$ is obtained. Since

S^n is arcwise connected $(n > 0)$, f can be extended over the edges of $K - L$ giving an extension $f_1: \bar{K}^1 \rightarrow S^n$. If $f_q: \bar{K}^q \rightarrow S^n$ and $\pi_q(S^n) = 0$, then $c^{q+1}(f_q) = 0$; so an extension $f_{q+1}: \bar{K}^{q+1} \rightarrow S^n$ can be found.

Since $\pi_i(S^n) = 0$ for $i < n$, it follows that an extension $f_n: \bar{K}^n \rightarrow S^n$ of f can be found. Thus the first possibly non-zero obstruction which can be met is $c^{n+1}(f_n)$. If this is zero, f_{n+1} can be defined. If it is cohomologous to zero in $K - L$, by (14.6), f_n can be redefined on the n-cells of $K - L$ so that $c^{n+1}(f_n) = 0$. If f_{n+1} exists, then the secondary obstruction $z^{n+2}(f_n) = \{c^{n+2}(f_{n+1})\}$ depends only on f_n. The question to be settled in the following sections is the extent of this dependence, i.e., when can f_n be redefined on the n-cells of $K - L$ so that z^{n+2} is still defined and is zero?

16. Normal maps

Let S^n be represented as a cell complex composed of a single vertex E^0 and a single oriented n-cell E^n (whose point set boundary is E^0). A map $f: K \rightarrow S^n$ is said to be *normal* if $f(K^{n-1}) = E^0$. As shown by Whitney [17], any map f_0 of K in S^n is homotopic to a normal map f_1. Even more, if $f_0 \mid L$ is already normal, the homotopy can be chosen to leave L fixed. In particular, if F is a homotopy connecting two normal maps f_0, f_1, then $F \cong F'$ rel. $K \times 0 + K \times 1$ where F' is a normal map (relative to the division of $K \times I$ into product cells $\sigma \times I$). Thus if two normal maps are homotopic, they can be connected by a normal homotopy. Thus, for maps into S^n, the study of extension and homotopy problems can be carried out within the domain of normal maps and normal homotopies. Henceforth all maps and homotopies will be assumed to be normal.

17. The characteristic cocycle f^*s^n.

Let $\bar{E}^n$ be a closed oriented n-cell, and let $\pi_n(S^n)$ be represented as the set of homotopy classes of normal maps of $\bar{E}^n$ in S^n. Let $f: K \rightarrow S^n$ be normal. For each oriented n-cell σ, choose a map $g: \bar{E}^n \rightarrow$ the closure of σ so as to map the interior of $\bar{E}^n$ on σ with degree $+1$. The element $a(f, \sigma) \in \pi_n(S^n)$ determined by fg is independent of the choice of g. The cochain $a^n(f)$ which attaches the coefficient $a(f, \sigma)$ to σ is called the *characteristic cocycle* of the map f.

If $g: K' \rightarrow K$ is a cell mapping, then fg is a normal map. It is easily verified that $a^n(fg) = g^*a^n(f)$.

The identity map $h: S^n \rightarrow S^n$ is normal. It is clear that $a(h, E^n)$ is a generator of $\pi_n(S^n)$. Therefore the cocycle

$$s^n = a^n(h)$$

generates $H^n(S^n, \pi_n(S^n))$. It follows that

$$a^n(f) = f^*s^n$$

for any normal $f: K \rightarrow S^n$. Since s^n is a cocycle, so also is f^*s^n.

18. The Hopf extension theorem

Let E^{n+1} be a closed $(n + 1)$-cell with boundary S^n so oriented that $\partial E^{n+1} = E^n$ (S^n is the cell complex E^0, E^n as in §16). Let $h: S^n \to S^n$ be the identity map. The problem of extending h to a map of E^{n+1} in S^n leads to an obstruction $c^{n+1}(h)$ such that $c(h, E^{n+1})$ is the generator of $\pi_n(S^n)$ defined by the identity map of S^n. Therefore

$$\delta s^n = c^{n+1}(h). \tag{18.1}$$

If $f_n: \bar{K}^n \to S^n$, it can be extended to a map $f_n': \bar{K}^{n+1} \to E^{n+1}$ because E^{n+1} is contractible to a point. Then (14.7) and (18.1) yield

$$\delta f_n^* s^n = c^{n+1}(f_n). \tag{18.2}$$

This leads to the Hopf extension theorem [8] as modified by Whitney [17] and Eilenberg [4].

THEOREM 18.3. *The map $f: L \to S^n$ can be extended to $\bar{K}^{n+1}$ if and only if the cocycle f^*s^n in L can be extended to a cocycle z^n in K. For each such z^n there exists an extension f' of f to $\bar{K}^{n+1}$ such that $f'^*s^n = z^n$.*

If f_{n+1} exists, then $f_{n+1}^*s^n$ is the desired extension of the cocycle f^*s^n. Suppose z^n is an extension of f^*s^n and $\delta z^n = 0$. Define f_n on L to be f and $f_n(x) = E^0$ for $x \in \bar{K}^{n-1} - L$. For each n-cell σ of $K - L$, let f_n map σ on E^n with a degree so chosen that $a(f_n, \sigma) = z^n(\sigma)$. Then $f_n^*s^n = z^n$. Since $\delta z^n = 0$, it follows from (18.2) that $c^{n+1}(f_n) = 0$. Therefore, by (14.5), f_{n+1} exists.

REMARK. The condition that f^*s^n can be extended to a cocycle of K means that f^*s^n is the image h^*z^n of some n-cocycle of K under the cochain mapping $h^*: C^n(K) \to C^n(L)$ induced by the identity map $h: L \to K$. Due to the exactness of the homology sequence of (K, L) this is equivalent to the statement $\delta^*f^*\{s^n\} = 0$ where δ^* is as defined in (9.3). This leads to a form of the Hopf theorem suitable for generalization to a general space: If X is a topological space, A a closed subset, $\dim (X - A) \leqq n + 1$, and $f: A \to S^n$, then f extends to X if and only if $\delta^*f^*\{s^n\} = 0$.

19. The complex projective plane

In the space of triples of complex numbers $[z_0, z_1, z_2]$, not all zero, two such triples are said to be equivalent if they are proportional i.e. $[z_0, z_1, z_2] \sim [z_0', z_1', z_2']$ if there exists a complex number k such that $z_i = kz_i'$ $(i = 0, 1, 2)$). The set of equivalence classes forms a 4-dimensional manifold M^4 known as the complex projective plane.

A representation of M^4 as a cell complex composed of three cells E^0, E^2, E^4 is obtained as follows. The equation $z_2 = 0$ defines a complex projective line S^2 in M^4. It is a 2-sphere. The vertex E^0 is defined by $z_1 = z_2 = 0$, and E^2 is the remainder of S^2. That $E^4 = M^4 - S^2$ is a 4-cell is seen as follows. In the space of two non-homogeneous complex variables (z_0, z_1) define a 4-cell

$$\bar{E}^4: |z_0|^2 + |z_1|^2 \leqq 1,$$

and define a map $\psi: \bar{E}^4 \to M^4$ by

$$\psi(z_0, z_1) = [z_0, z_1, (1 - |z_0|^2 - |z_1|^2)^{\frac{1}{2}}]. \tag{19.1}$$

The 3-sphere boundary of $\bar{E}^4$ is denoted by $\bar{S}^3$: $|z_0|^2 + |z_1|^2 = 1$. The following properties of ψ are easily verified:

(1) ψ is continuous,

(2) ψ maps $\bar{E}^4 - \bar{S}^3$ homeomorphically onto $M^4 - S^2$,

(3) $\psi \mid \bar{S}^3: \bar{S}^3 \to S^2$ is the Hopf map of the 3-sphere on the 2-sphere [7, 9].

With this cell structure on M^4 the homology and cohomology groups are readily computed. There is no torsion. The 1 and 3-dimensional Betti numbers are zero. The 0, 2 and 4-dimensional Betti numbers are each 1. The 2-cocycle s^2 on S^2 (§17) is a cocycle in M^4 and generates $H^2(M^4, \pi_2(S^2))$.

Now $E^2 \smile_0 E^2$ is not directly defined by any of the definitions of the cup product. However, passing to cohomology classes (integer coefficients), $\{E^2\} \smile_0 \{E^2\}$ is defined and is a generator of $H^4(M^4)$. This follows by duality from the fact that any two distinct projective lines intersect in a point so that the self-intersection of the 2-cycle S^2 in M^4 has the Kronecker index ± 1. Now orient E^4 so that $\{E^4\} = \{E^2\} \smile_0 \{E^2\}$, then define $E^2 \smile_0 E^2 \doteq E^4$.

Orient $\bar{E}^4$ so that ψ has degree $+1$. Then orient $\bar{S}^3$ so that, algebraically, $\partial \bar{E}^4 = \bar{S}^3$. Since $\psi \mid \bar{S}^3$ is the Hopf map, its homotopy class β is a generator of $\pi_3(S^2)$ [10]. Define s^4 to be the 4-cocycle which attaches to E^4 the coefficient β. Let $\alpha \in \pi_2(S^2)$ be the coefficient that s^2 assigns to E^2. Since α generates $\pi_2(S^2)$, the formula

$$\alpha \cdot \alpha = \beta \tag{19.2}$$

and the requirement of bilinearity defines a pairing of $\pi_2(S^2)$ with itself with products in $\pi_3(S^2)$. It follows now that

$$s^2 \smile_0 s^2 = s^4. \tag{19.3}$$

Let $h: S^2 \to S^2$ be the identity map. The problem of extending h to a map of M^4 in S^2 leads to a 4-dimensional obstruction $c^4(h)$ with coefficients in $\pi_3(S^2)$. By construction $c(h, E^4) = \beta$. Therefore $c^4(h) = s^4$ and by (19.3),

$$c^4(h) = s^2 \smile_0 s^2. \tag{19.4}$$

Thus the Pontrjagin extension theorem [14] is verified in this special case.

20. The complexes M^n

Starting with the complex projective plane M^4, we define inductively a sequence of cell complexes. Define M^5 to be the join of M^4 with a pair of points M^6 the join of M^5 with a pair of points, etc. In general, M^{n+2} ($n \geqq 2$) is a cell complex composed of a 0-cell E^0, an n-cell E^n forming with E^0 an n-sphere S^n, and an $(n+2)$-cell E^{n+2}. M^{n+2} is the join of M^{n+1} with a pair of points A, B; its subset S^n is the join of A, B with S^{n-1}, and E^{n+2} is the join of A, B with E^{n+1}. Moreover

$$\psi_n: \bar{E}^{n+2} \to M^{n+2}$$

is a map of an $(n + 2)$-cell $\bar{E}^{n+2}$ in M^{n+2} which maps $\bar{S}^{n+1} = \partial\bar{E}^{n+2}$ on S^n and $\bar{E}^{n+2} - \bar{S}^{n+1}$ homeomorphically on $M^{n+2} - S^n$. E^{n+2} is defined inductively as the join of $\bar{E}^{n+1}$ with a pair of points A', B', and ψ_n is the Einhangung [Freudenthal, 5] of ψ_{n-1} (i.e. $\psi_n(A') = A$, $\psi_n(B') = B$, and, for each point $y \in \bar{E}^{n+1}$, ψ_n maps the line segment $A'y$ $(B'y)$ on the segment $A\psi_{n-1}(y)$ $(B\psi_{n-1}(y))$).

Freudenthal has shown that the Einhangung operation maps $\pi_i(S^{n-1})$ isomorphically onto $\pi_{i+1}(S^n)$ for $i < 2n - 3$ and homomorphically onto when $i = 2n - 3$. Since $\psi_2 \mid \bar{S}^3$ represents the generator $\beta_2 \in \pi_3(S^2)$, it follows that $\psi_3 \mid \bar{S}^4$ represents the sole non-zero element $\beta_3 \in \pi_4(S^3)$. And, inductively, $\psi_n \mid \bar{S}^{n+1}$ represents the only non-zero element $\beta_n \in \pi_{n+1}(S^n)$.

Let α_n be the generator of $\pi_n(S^n)$ determined by the identity map of S^n. Define a pairing of $\pi_n(S^n)$ with itself with products in $\pi_{n+1}(S^n)$ by

$$\alpha_n \cdot \alpha_n = \beta_n . \tag{20.1}$$

Comparing with (19.2), products are defined so as to commute with the operation of Einhangung.

If it is assumed, inductively, that $\{E^{n-1}\} \smile_{n-3} \{E^{n-1}\} = \{E^{n+1}\}$ in M^{n+1}, it follows by (11.7) that $\{E^n\} \smile_{n-2} \{E^n\} = \{E^{n+2}\}$ in M^{n+2}. In agreement with this, define

$$E^n \smile_{n-2} E^n = E^{n+2}. \tag{20.2}$$

The cocycle s^n in S^n is a cocycle in M^{n+2} and attaches the coefficient α_n to E^n.

If h: $S^n \to S^n$ is the identity, the problem of extending h to a map of M^{n+2} on S^n leads to an obstruction $c^{n+2}(h)$ which, by construction, attaches the coefficient β_n to E^{n+2}. It follows from (20.1, 2) that

$$c^{n+2}(h) = s^n \smile_{n-2} s^n. \tag{20.3}$$

REMARK. It is to be noted that the Einhangung homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ reduces the infinite cyclic group $\pi_3(S^2)$ mod 2. Correspondingly, the join operation maps $s^2 \smile_0 s^2$ (which has invariant meaning without reduction mod 2) into $s^3 \smile_1 s^3$ which must be reduced mod 2 to have invariant meaning. Since its coefficients are in the group $\pi_4(S^3)$ of order 2, reduction mod 2 is not necessary. The author has no explanation for this fortuitous circumstance.

21. The extension theorem

THEOREM 21.1. *Let f: $K^n \to S^n$ be such that $\delta f^*s^n = 0$ in K (see 18.3), then the secondary obstruction $z^{n+2}(f) = \{f^*s^n\} \smile_{n-2} \{f^*s^n\}$.*

Choose an extension f': $K^{n+1} \to S^n$ which exists by (18.3). S^n is assumed to be a subset of M^{n+2} as in §20. An extension f'': $K^{n+2} \to M^{n+2}$ of f' exists as follows. Lt σ be an $(n + 2)$-cell. If $c(f', \sigma) = 0$ f'' can be defined to map σ on S^n. If $n > 2$ and $c(f', \sigma) \neq 0$ then $c(f', \sigma) = \beta_n$ (see §20). Choose a map g: $\partial\sigma \to \bar{S}^{n+1}$ of degree 1. Since $\psi_n \mid \bar{S}^{n+1}$ also represents β_n it follows that $f' \mid \partial\sigma$ is homotopic to $\psi_n g$. Now g can be extended to a map g': $\sigma \to \bar{E}^{n+2}$ of degree 1. Therefore $\psi_n g$ can be extended to $\psi_n g'$: $\sigma \to M^{n+2}$. It follow, as in

§13, that $f' \mid \partial\sigma$ can be extended to a map f'' of σ in M^{n+2} so as to have degree 1 on E^{n+2}. If $n = 2$, then $c(f', \sigma) = m\beta_2$ for some integer m. In this case choose g to have degree m. In a similar manner, the extension f'' of f' over σ exists and maps σ on E^4 with degree n. In either case, if $c^{n+2}(h)$ is defined as in §19, 20,

$$f''^*c^{n+2}(h) = c^{n+2}(f'), \tag{21.2}$$

see (14.7). By (10.5),

$$\{f''^*s^n\} \smile_{n-2} \{f''^*s^n\} = \{f''^*(s^n \smile_{n-2} s^n)\}.$$

This together with (19.4), (20.3) and (21.2) yields (21.1).

22. The difference of two maps

Suppose K, K' are cell complexes, L a subcomplex of K and $f, g: K \to K'$ are cellular maps such that $f \mid L = g \mid L$ (i.e., f, g are two extensions of a map $L \to K'$) For any cochain u on K', f^*u and g^*u are cochains in K which have equal values on each cell of L. Therefore

$$(f - g)^*u = f^*u - g^*u \tag{22.1}$$

is zero on L and defines thereby homomorphisms

$$(f - g)^*: C^q(K', G) \to C^q(K, L, G), \qquad q \geqq 0. \tag{22.2}$$

It is easily verified that $(f - g)^*\delta' = \delta(f - g)^*$ so that $(f - g)^*$ maps cocycles into cocycles, cohomology classes into such and defines thereby homomorphisms

$$(f - g)^*: H^q(K', G) \to H^q(K, L, G), \qquad q \geqq 0. \tag{22.3}$$

If f, g, h are cellular maps of K in K' which agree on L, then

$$(f - g)^* + (g - h)^* = (f - h)^* \tag{22.4}$$

where $+$ means the usual addition of homomorphisms. This is proved first for the cochain mapping (22.2) using (22.1). This in turn implies its truth for the homomorphisms (22.3).

Suppose now that $h: (K_1, L_1) \to (K_2, L_2)$ and $f, g: K_2 \to K'$ so that $f \mid L_2 = g \mid L_2$. Then $fh, gh: K_1 \to K'$ and $fh \mid L_1 = gh \mid L_1$. It follows immediately that

$$h^*(f - g)^* = (fh - gh)^* \tag{22.5}$$

and these may be interpreted as mappings of cochains or cohomology classes.

Suppose next $f, g: K \to K_1'$ so that $f \mid L = g \mid L$, and $h: K_1' \to K_2'$. Then also

$$(f - g)^*h^* = (hf - hg)^*. \tag{22.6}$$

In order to determine the behavior of products under $(f - g)^*$, let K, K', f, g be simplicial. Let products in K' be based on an order β. Let α, α' be orders in K preserved by f, g respectively. Since $f \mid L = g \mid L$, it can be supposed that α, α' coincide on L. The orders α, α' define products $\smile$, $\smile'$. For

p, q-cocycles u, v in K', it follows from (22.1) and (3.1) that

$$(22.7)\qquad \begin{aligned}(f - g)^*u \cup_i (f - g)^*v + (f - g)^*u \cup_i g^*v + g^*u \cup_i (f - g)^*v \\ = (f - g)^*(u \cup_i v) + [g^*u \cup_i' g^*v - g^*u \cup_i g^*v].\end{aligned}$$

Since α, α' agree on L, the term in brackets is zero on each simplex of L. Thus (22.7) is a relation in $Z^{p+q-i}(K, L, G')$.

Consider first the case $i = 0$. If $\vee$ is the operation of §8 corresponding to the pair of orders α, α' it follows from (8.6) that $g^*u \vee_0 g^*v$ is zero on L. Therefore, by (8.2), the term of (22.7) in brackets is ~ 0 in $K - L$. Thus (22.7) yields the following cohomology in $K - L$:

$$(22.8)\qquad \begin{aligned}(f - g)^*(u \cup_0 v) \sim (f - g)^*u \cup_0 (f - g)^*v + (f - g)^*u \cup_0 g^*v \\ + g^*u \cup_0 (f - g)^*v.\end{aligned}$$

Consider next the case $i \geqq 0$ and $u = v$. Just as above $g^*u \vee_i g^*u$ is zero on L; and, therefore, the term in brackets in (22.7) is ~ 0 in $K - L$. Since $(f - g)^*u$ is zero on L, its products with g^*u are zero on L. Therefore, by (6.1),

$$(22.9)\qquad \begin{aligned}\delta[(f - g)^*u \cup_{i+1} g^*u] = -(-1)^i(f - g)^*u \cup_i g^*u \\ + (-1)^p g^*u \cup_i (f - g)^*u\end{aligned}$$

is a coboundary relation in $K - L$. Thus (22.7) reduces to

$$(22.10)\qquad \begin{aligned}(f - g)^*(u \cup_i u) \sim (f - g)^*u \cup_i (f - g)^*u \\ + [1 + (-1)^{p-i}]g^*u \cup_i (f - g)^*u \text{ in } K - L.\end{aligned}$$

Passing to cohomology classes (and reducing mod 2 when $p - i$ is even),

$$(22.11)\qquad (f - g)^*\mathrm{Sq}_i = \mathrm{Sq}_i(f - g)^*.$$

23. The topological invariance of $(f - g)^*$

Let X, Y be topological spaces, A a closed set in X and $f, g: X \to Y$ such that $f \mid A = g \mid A$. Using cohomology groups in the sense of Čech [11], homomorphisms

$$(23.1)\qquad (f - g)^*: H^q(Y, G) \to H^q(X, A, G)$$

will be defined. In the case of complexes (23.1) will be proved to coincide with (22.3). This will establish the invariant character of the operation $(f - g)^*$.

Let $\phi = (V^1, \cdots, V^n)$ be a finite covering of Y by open sets. Let K'_ϕ be its nerve. The *associated* (f, g)*-covering* of X consists of the open sets.

$$U^i = f^{-1}(V^i)\cdot g^{-1}(V^i),\ U^{ij} = f^{-1}(V^i)\cdot g^{-1}(V^j)\cdot(X - A), \qquad (i, j = 1, \cdots, n).$$

Let K_ϕ be its nerve, and L_ϕ the subcomplex of simplexes whose vertices intersect on A. The vertices of L_ϕ are included among the non-vacuous U^i. Define the associated simplicial projections f_ϕ, g_ϕ: $K_\phi \to K'_\phi$ by the vertex assignments

$$f_\phi(U^i) = g_\phi(U^i) = V^i,\ f_\phi(U^{ij}) = V^i,\ g_\phi(U^{ij}) = V^j.$$

Then $f_\phi \mid L_\phi = g_\phi \mid L_\phi$.

Suppose ϕ' is a refinement of ϕ and $h: K'_{\phi'} \to K'_\phi$ is an admissible projection. Denote by $V^{h(i)}$ the open set of ϕ which contains the open set V'^i of ϕ' and corresponds to it under h. Then the vertex assignment

$$\bar{h}(U'^i) = U^{h(i)},\ \bar{h}(U'^{ij}) = U^{h(i)h(j)}$$

defines an admissible projection $h: (K_{\phi'}, L_{\phi'}) \to (K_\phi, L_\phi)$. By construction,

$$hf_{\phi'} = f_\phi \bar{h}, \qquad hg_{\phi'} = g_\phi \bar{h}.$$

Then, for any cochain u in K'_ϕ, it follows from (22.5, 6) that

$$\bar{h}^*(f_\phi - g_\phi)^*u = (f_{\phi'} - g_{\phi'})^*h^*u. \tag{23.2}$$

Now $H^q(Y)$ is the limit group of the direct system of groups $\{H^q(K'_\phi, G)\}$ using all coverings ϕ. By (23.2), the collection of homomorphisms $\{(f - g)^*\}$ maps this direct system homomorphically into the direct system $\{H^q(K_\phi, L_\phi, G)\}$ which is a subsystem of the direct system whose limit group is $H^q(X, A, G)$. A homomorphism of a direct system into a subsystem of another induces, in a natural manner a homomorphism of the limit group of the first into that of the second. The homomorphism (23.1) is the one so determined by $\{(f_\phi - g_\phi)^*\}$.

Suppose now that $X = K$, $A = L$, $Y = K'$ and f, g are simplicial. Let φ (ψ) be the covering of K' (K) by the stars of its vertices. It is known that $K' = K'_\phi$, $K = K_\psi$ under the correspondence attaching to each vertex its star. The invariance of the cohomology groups of K' and of (K, L) is expressed by the known proposition that the projections

$$\pi_\phi: H^q(K', G) \to H^q(Y, B),\ \pi_\psi: H^q(K, L, G) \to H^q(X, A, G)$$

(of terms of direct systems into their limit groups) are isomorphisms. Let $A^1, \cdots, A^n$ be the vertices of K' and $V^i = \text{Star}(A^i)$. Define a simplicial map h: $(K, L) \to (K_\phi, L_\phi)$ as follows. If B is a vertex of L and $f(B) = g(B) = A^i$, then $h(B) = U^i$. If B is not in L and $f(B) = A^i$, $g(B) = A^j$, then $h(B) = U^{ij}$. Then h is an admissible projection, and

$$f = f_\phi h, \qquad g = g_\phi h.$$

Using these, it can now be proved that

$$\pi_\psi(f - g)^*u = (f - g)^*\pi_\phi u$$

for any $u \in H^q(K', G)$. The $(f - g)^*$ on the left is (22.3) and that on the right is (23.1). Thus the two definitions of$(f - g)^*$ correspond under the isomorphisms π_ψ, π_ϕ.

Remark. It is almost certainly true that the properties of $(f - g)^*$ stated in §22 for complexes and cellular maps also hold for more general spaces and maps. This question is not decided here since it is not needed.

24. The relative extension theorem

THEOREM 24.1. *If f, g are maps of $\bar{K}^n$ in S^n which coincide on L and possess extensions to $\bar{K}^{n+1}$ (i.e. $\delta f^* s^n = \delta g^* s^n = 0$, see (18.3)), then their secondary obstrutions (14.8) are related as follows*

$$z^{n+2}(f) - z^{n+2}(g) = \begin{cases} \lambda^n \smile_{n-2} \lambda^n, & n > 2, \\ 2\lambda^2 \smile_0 f^*\{s^2\} - \lambda^2 \smile_0 \lambda^2, & n = 2, \end{cases} \tag{24.2}$$

where

$$\lambda^n = (f - g)^*\{s^n\} \in H^n(K, L, \pi_n(S^n)). \tag{24.3}$$

Conversely, if f and λ^n are given, there exists a map $g: \bar{K}^n \to S^n$ such that $g \mid L = f \mid L$, $\delta g^* s^n = 0$, and (24.3) holds.

As in the proof of (21.1), let f', $g': \bar{K}^{n+1} \to S^n$ be extensions of f, g; and let f'', $g'': \bar{K}^{n+2} \to M^{n+2}$ be extensions of f', g'. By (21.2),

$$c^{n+2}(f') = f''^* c^{n+2}(h), \qquad c^{n+2}(g') = g''^* c^{n+2}(h).$$

Therefore

$$c^{n+2}(f') - c^{n+2}(g') = (f'' - g'')^* c^{n+2}(h).$$

Passing to cohomology classes

$$\begin{aligned} z^{n+2}(f) - z^{n+2}(g) &= (f'' - g'')^* z^{n+2}(h) \\ &= (f'' - g'')^*\{s^n \smile_{n-2} s^n\} && \text{by (19.4), (20.3),} \\ &= \begin{cases} \lambda^n \smile_{n-2} \lambda^n, & n > 2, \text{ by (22.11),} \\ \lambda^2 \smile_0 \lambda^2 + 2g^*\{s^2\} \smile_0 \lambda^2, & n = 2, \text{ by (22.10).} \end{cases} \end{aligned}$$

To complete the proof in the case $n = 2$, substitute $g^*\{s^2\} = f^*\{s^2\} - \lambda^2$, and apply the distributive and commutation laws for the $\smile_0$ product.

For the converse part of the theorem, let $\lambda^n = \{v^n\}$ where v^n is a cocycle in $K - L$. Then $z^n = f^* s^n - v^n$ is a cocycle in K. By (18.3), there exists a map $g: \bar{K}^n \to S^n$ such that $g^* s^n = z^n$ and $g \mid L = f \mid L$. Then (24.3) follows directly.

25. Cohomology relations in $K \times I$

The unit interval I is regarded as a complex composed of the two vertices 0, 1 and the oriented 1-cell (0, 1). Let I^1 be the 1-cocycle which attaches $+1$ to (0, 1). If K is a cell complex, $K \times I$ will be regarded as the cell complex composed of the cells $\sigma \times 0 = \sigma_0$, $\sigma \times 1 = \sigma_1$, and $\sigma \times I$ for all cells σ of K. Let $K_0 = K \times 0$, $K_1 = K \times 1$ and

$$L^* = K_0 + K_1 + L \times I.$$

THEOREM 25.1. *The cochain mapping $\phi(u) = u \times I^1$ defines isomorphisms*

$$\phi: C^q(K, L, G) \to C^{q+1}(K \times I, L^*, G), \qquad q \geqq 0$$

such that $\phi\delta = \delta\phi$ *and thereby induces isomorphisms*

$$\phi^*: H^q(K, L, G) \to H^{q+1}(K \times I, L^*, G), \qquad q \geqq 0.$$

Furthermore, $\phi^*\mathrm{Sq}_{i-1} = \mathrm{Sq}_i\phi^*$ for $i \geqq 1$.

That ϕ is a cochain isomorphism follows from the fact that each cell of $K \times I - L^*$ is of the form $\sigma \times I^1$ for some σ in $K - L$. Since $\delta I^1 = 0, \delta(\sigma \times I^1) = \delta\sigma \times I^1$. Therefore the incidence number of σ_0 and $\sigma \times I^1$ is $(-1)^{q+1}$ where $\dim \sigma = q$. Therefore, if u is a q-cocycle of $K - L$ and $u_0 = u \times 0$, then

$$u \times I^1 = (-1)^{q+1}\delta u_0. \tag{25.2}$$

Now $H^q(L^*, L \times I, G)$ is the direct sum of $H^q(K_0, L_0, G)$ and $H^q(K_1, L_1, G)$. The operation δ^* of §9 for the pair $(K \times I, L^*)$ therefore maps $H^q(K_0, L_0, G)$ into $H^{q+1}(K \times I, L^*, G)$. By (25.2), δ^* is equivalent to $(-1)^{q+1}\phi^*$ under the correspondence between K and K_0. The last statement of the theorem now follows from (9.6) and the fact that a square is always of order 2.

26. The separation cocycle $d^{n+1}(f, g)$ of a normal pair

Two normal maps $f, g: \bar{K}^{n+1} \to S^n$ such that $f \mid L = g \mid L$ form a normal pair if $f \mid \bar{K}^n = g \mid \bar{K}^n$. In this case,

$$f^*s^n = g^*s^n. \tag{26.1}$$

If f, g form a normal pair, define a map

$$F: \bar{K}^n \times I + K_0 + K_1 \to S^n$$

$$F(x, 0) = f(x), F(x, 1) = g(x), F(x, t) = f(x) \text{ if } x \in \bar{K}^n. \tag{26.2}$$

Then the obstruction $c^{n+2}(F)$ is in $K \times I - L^*$; and, by (25.1),

$$c^{n+2}(F) = d^{n+1}(f, g) \times I^1 \tag{26.3}$$

defines uniquely the cochain $d^{n+1}(f, g)$ in $K - L$ with coefficients in $\pi_{n+1}(S^n)$. It is called the *separation cocycle* of the normal pair f, g. Eilenberg [4] has proved the following of its properties:

(26.4) $\delta d^{n+1}(f, g) = c^{n+2}(f) - c^{n+2}(g)$.

(26.5) $d^{n+1}(f, g) + d^{n+1}(g, h) = d^{n+1}(f, h)$.

(26.6) If f and d^{n+1} are given, there exists a g such that $d^{n+1}(f, g) = d^{n+1}$.

(26.7) $d^{n+1}(f, g) = 0$ if and only if $f \cong g$ rel. $\bar{K}^n$.

(26.8) $d^{n+1}(f, g) \sim 0$ in $\bar{K}^{n+1} - L$ if and only if $f \cong g$ rel. $\bar{K}^{n-1}$.

27. The deformation cochain $e^{n-1}(F)$ of a normal homotopy

Let $f_0, f_1 : \bar{K}^{n+1} \to S^n$ be normal maps such that $f_0 \mid L = f_1 \mid L$; and let F be a normal homotopy rel. L connecting f_0 and f_1. Since $F \mid K_0 = f_0$, etc., the decomposition

$$F^*s^n = f_0^*s^n \times 0 + f_1^*s^n \times 1 - e^{n-1}(F) \times I^1 \tag{27.1}$$

defines uniquely the cochain $e^{n-1}(F)$ in $K - L$ with coefficients in $\pi_n(S^n)$. It is called the *deformation cochain* of the homotopy F. Since $\delta F^*s^n = 0$,

$$\delta e^{n-1}(F) = (-1)^n(f_1 - f_0)^*s^n. \tag{27.2}$$

Thus, if f_0, f_1 form a normal pair, it follows from (26.1) that $e^{n-1}(F)$ is a cocycle.

28. The homotopy classification theorem

THEOREM 28.1. *Let $f_0, f_1 : \bar{K}^{n+1} \to S^n$ be normal maps such that $f_0 \mid \bar{K}^n = f_1 \mid \bar{K}^n$. Then $f_0 \cong f_1$ rel. L if and only if there exists an $(n-1)$-cocycle e^{n-1} in $K - L$ with coefficients in $\pi_n(S^n)$ such that*

$$\{d^{n+1}(f_0, f_1)\} = \begin{cases} \{e^{n-1}\} \smile_{n-3} \{e^{n-1}\} & n > 2, \\ 2\{e^1\} \smile_0 \{f_0^*s^2\} & n = 2. \end{cases} \tag{28.2}$$

Suppose $f_0 \cong f_1$ rel. L, and F' is a normal homotopy rel. L connecting them. Let $e^{n-1} = e^{n-1}(F')$. Let F be the map (26.2) using f_0, f_1 in place of f, g. Since F maps each $\sigma \times I$ of dimension $\leqq n$ into a point, it follows that $e^{n-1}(F) = 0$. Now apply (24.1) to the maps F, F' of $(K \times I)^n$. Since $c^{n+2}(F') = 0$,

$$\{c^{n+2}(F)\} = \begin{cases} \lambda^n \smile_{n-2} \lambda^n, & n > 2, \\ 2\lambda^2 \smile_0 F^*\{s^2\} - \lambda^2 \smile_0 \lambda^2, & n = 2, \end{cases}$$

where

$$\lambda^n = (F - F')^*\{s^n\} = \{e^{n-1} \times I^1\}.$$

The last equality follows from $F = F'$ on $K_0 + K_1$ and $e^{n-1}(F) = 0$. If $n > 2$, by (25.1),

$$\lambda^n \smile_{n-2} \lambda^n = [\mathrm{Sq}_{n-3}\{e^{n-1}\}] \times \{I^1\}.$$

This proves the necessity for $n > 2$. In the case $n = 2$, the rules devised by Whitney [18] for calculating $\smile_0$ in a product complex can be applied. Then

$$(e^1 \times I^1) \smile_0 (e^1 \times I^1) = -(e^1 \smile_0 e^1) \times (I^1 \smile_0 I^1) = 0,$$

since $I^1 \smile_0 I^1 = 0$. Therefore $\lambda^2 \smile_0 \lambda^2 = 0$. For the other term,

$$(e^1 \times I^1) \smile_0 (f_0^*s^2 \times 0 + f_1^*s^2 \times 1 - e^1 \times I^1) = (e^1 \smile_0 f_1^*s^2) \times I^1$$

since $I^1 \smile_0 0 = 0$, $I^1 \smile_0 1 = I^1$, and $I^1 \smile_0 I^1 = 0$.

To prove the sufficiency, suppose e^{n-1} exists. Define a map

$$F: \bar{K}_0^{n+1} + L \times I \to S^n$$

by $F(x, t) = f_0(x)$ if $x \in L$ or if $t = 0$. Then F is a normal map, and

$$F^*s^n = f_0^*s^n \times 0 + (f_1 \mid L)^*s^n \times 1.$$

Now

$$z^n = f_0^* s^n \times 0 + f_1^* s^n \times 1 - e^{n-1} \times I^1$$

is an extension of $F^* s^n$ to a cocycle in $K \times I$. By (18.3), F has a normal extension

$$F' : \bar{K}_0^{n+1} + \bar{K}^n \times I \to S^n$$

such that $F'^* s^n = z^n$. By (13.2), F' extends to a map $F'' : \bar{K}^{n+1} \times I \to S^n$. Define $f_1'(x) = F''(x, 1)$ so that $f_0 \cong f_1'$ rel. L. By the part of the theorem already proved, $\{d^{n+1}(f_0, f_1')\}$ equals the right side of (28.2). Therefore

$$d^{n+1}(f_0, f_1') \sim d^{n+1}(f_0, f_1).$$

It follows now from (26.4) that $d^{n+1}(f_1', f_1) \sim 0$, and therefore $f_1' \cong f_1$ rel. L by (26.7).

29. Problems

As it stands the results of this paper settle the general homotopy and extension problem for a single dimensional stage. Special problems arising at the next stage can be formulated as follows:

Suppose K is a 5-complex and $f : K^2 \to S^2$ can be extended to K^4 (i.e. $f^* s^2$ is a cocycle in K and $f^* s^2 \smile_0 f^* s^2 \sim 0$). If f' is such an extension, then $c^5(f')$ is defined and has coefficients in $\pi_4(S^2)$, a cyclic group of order 2. In general, $c^5(f')$ may be non-zero for every choice of f'. For example, this is true if K is the complex obtained from a 5-cell by reducing (upper semi-continuously) its boundary S^4 to a 2-sphere S^2 by means of an essential map $S^4 \to S^2$. Two questions arise:

(1) Is the cohomology class $c^5(f')$ independent of the choice of the extension f' of f?

(2) If (1) is true, is there an effective rule for calculating its cohomology class in terms of $f^* s^2$?

Suppose now that K is an $(n + 4)$-complex $(n > 2)$, and $f : K^n \to S^n$ is such that f extends to $f' : K^{n+2} \to S^n$. Since $\pi_{n+2}(S^n) = 0$, then $c^{n+3}(f') = 0$. Therefore, f' extends to a map $f'' : K^{n+3} \to S^n$. Then $c^{n+4}(f')$ is defined and has coefficients in $\pi_{n+3}(S^n)$. This latter group has not been determined in general. However it is not zero for $n > 3$, [9]. The questions (1) and (2) can now be asked of $c^{n+4}(f'')$.

Since S^n is n-dimensional, $s^n \smile_i s^n = 0$ for $i = 0, \cdots, n - 1$. If $f : K^n \to S^n$ is extendable to a map $f' : K \to S^n$ (dimension of K is arbitrary), it follows that $f^* s^n \smile_i f^* s^n \sim 0$ in K for $i = 0, \cdots, n - 1$, regardless of the rule for multiplying coefficients. Are these conditions sufficient for the existence of f' if dim $K \leqq 2n$?

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Bibliography

1. J. W. Alexander, *On the connectivity ring of an abstract space*. Ann. of Math., 37 (1937), 698–708.
2. P. Alexandroff and H. Hopf, Topologie I. Berlin: J. Springer, 1935.
3. E. Čech, *Multiplications on a complex*. Ann. of Math., 37 (1936), 681–697.
4. S. Eilenberg, *Cohomology and continuous mappings*. Ann. of Math., 41 (1940), 231–251. See also, *Lectures in Topology*, Univ. of Mich. Press, (1941), 57–100.
5. H. Freudenthal, *Über die Klassen der Spharenabbildungen* I. Compositio Math., 5 (1937), 299–314.
6. ———, *Neue Erweiterungs und Überfuhrungssätze*. Proc., Akad. Amsterdam, 42 (1939), 139–140.
7. H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelflache*. Math. Ann., 104 (1939), 639–665.
8. ———. *Die Klassen der Abbildungen der n-dimensionalen Polyeder auf die n-dimensionale Sphäre*. Comm. Math. Helv., 5 (1933), 39–54.
9. ———, *Über die Abbildungen von Spharen auf Spharen niedrieger Dimension*. Fund. Math., 25 (1935), 427–440.
10. W. Hurewicz, *Beitrage zur Topologie der Deformationen* I–IV. Proc., Akad. Amsterdam, 38 (1935), 112–119, 521–528, also 39 (1936), 117–126, 215–224.
11. S. Lefschetz, *Algebraic Topology*, Amer. Math. Soc., New York, 1942.
12. L. Pontrjagin, *Classification des transformations d'un complexe $(n + 1)$-dimensionnel dans une sphere n-dimensionelle*. C. R., Paris, 206 (1938), 1436–1438.
13. ———, *A classification of continuous transformations of a complex into a sphere*, 1, 2. C. R., Akad. nauk, U.R.S.S., 19 (1938), 147–149, 361–363.
14. ———, *A classification of mappings of the 3-dimensional complex into the 2-dimensional sphere*. Rec. Math. [Mat. Sbornik] N.S. 9 (51) (1941), 331–363.
15. ———, *Mappings of the 3-dimenional sphere into an n-dimensional complex*. C. R., Akad. nauk, U.R.S.S., 34 (1942), 35–37.
16. G. Whitehead, *On the homotopy groups of spheres and rotation groups*. Ann. of Math., 43 (1942), 634–640.
17. H. Whitney, *The maps of an n-complex into an n-sphere*. Duke Math. J., 3 (1937), 51–55.
18. ———, *On products in a complex*. Ann. of Math., 39 (1938), 397–432.