ANNALS OF MATHEMATICS
Vol. 79, No. 1, January, 1964
Printed in Japan

ON THE DEFORMATION OF RINGS AND ALGEBRAS

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(Received December 17, 1962)
(Revised June 17, 1963)

CONTENTS

Introduction

This paper contains the definitions and certain elementary theorems of a deformation theory for rings and algebras. For the present we consider mainly associative rings and algebras, with only brief allusions to the Lie case, but the definitions hold for wider classes of algebras.

The present paper is divided into three chapters. In the first, it is shown that the second cohomology group $H^2(A, A)$ of an algebra A with coefficients in itself may be interpreted as the group of infinitesimal

* The author gratefully acknowledges the support of the National Science Foundation through contract no. G-24412 during the preparation of this paper.

deformations of A in precisely the same way that the first cohomology group (derivations of A into itself modulo inner derivations) is interpreted as the group of infinitesimal automorphisms. This simple observation already gives as a sufficient condition for the "rigidity" of A, the vanishing of $H^2(A, A)$. (The question of whether this condition is necessary is open.) It follows immediately that separable semi-simple algebras are rigid. It is also shown that, if a field K is a (possibly infinite) extension of a field k, then K is separable over k (i.e., K and $k^{1/p}$ are linearly disjoint over k) if and only if the second commutative cohomology group of K (considered as an algebra over k) with coefficients in itself vanishes. It follows, in particular, that a field, considered as a commutative algebra over its prime field, is rigid in the deformation theory for commutative algebras. The same chapter contains an account of the formal obstruction theory of both derivations and infinitesimal deformations, and of the global deformation induced by the primary obstruction to a derivation in characteristic p. This obstruction is a cohomology operation mapping $H^1(A, A)$ into $H^2(A, A)$, and it is likely that the cohomology ring of a ring possesses further cohomology operations analogous in various respects to the Steenrod operations.

The second chapter is devoted principally to a discussion of the set of structure constants for associative algebras of dimension n as a parameter space for the deformation theory of these algebras. It contains a simple but fundamental upper semi-continuity theorem, a special case of which asserts that the dimension of the radical of an algebra is an upper semi-continuous function of the algebra.

The last chapter develops a deformation theory for graded and filtered rings and gives sufficient conditions that a complete filtered ring be isomorphic to its associated graded ring. In particular, it is seen that an algebra A with unit over an infinite field k is rigid if and only if the power series ring in one or several variables with coefficients in A is rigid. (The hypotheses can probably be considerably relaxed.) The rigidity of a field as a ring over its prime field implies I.S. Cohen's theorem asserting that a complete equi-characteristic regular local ring is a power series ring.

Certain aspects of the present deformation theory parallel closely those of the Froelicher-Kodaira-Nijenhuis-Spencer theory. Comparing the algebraic and analytic theories, a deformation theory seems to have at least the following aspects:

1. A definition of the class of objects within which deformation takes place, and identification of the infinitesimal deformations of a given object with the elements of a suitable cohomology group.

2. A theory of the obstructions to the integration of an infinitesimal deformation.

3. A parameterization of the set of objects obtainable by deformation from a fixed one, and the construction of a fiber space over this space, the fibers of which are the objects. There is an upper semi-continuity theorem for certain natural functions on the parameter space. For example, in the algebraic case the dimension of the radical is, as remarked, an upper semi-continuous function of the algebra.

4. A determination of the natural automorphisms of the parameter space (the modular group of the theory) and determination of the rigid objects. In some cases almost all points of a parameter space will represent the same rigid object, degenerating in various ways to objects admitting proper deformations.

Some of the questions raised in this outline are resolved much more simply in the algebraic than in the analytic theory. For example, for finite dimensional algebras, the set of structure constants is an algebraic set in a finite dimensional space and may be taken as parameter space. On the other hand, certain questions so far seem equally inaccessible in either theory, like that of determining when the tangent space at a simple point of the parameter space is indeed identical with the space of infinitesimal deformations of the object corresponding to the simple point.

Again comparing the algebraic and analytic theories, the interpretation of certain cocycles as infinitesimals, whenever this is possible, seems to be within the province of a "deformation theory." For an algebra A we discuss $H^n(A, A)$ from this point of view for $n = 1$ and $n = 2$, but it is reasonable to conjecture that the higher cohomology groups also have natural interpretations as groups of infinitesimal objects. In any case, the deformation theory for algebras has shown that the direct sum of the groups $H^n(A, A)$ possesses a much richer structure than had previously been exhibited. Certain aspects of this structure, such as the existence of a graded Lie as well as a graded commutative and associative product, have been discussed in [3]. A cursory knowledge of the latter paper will, in certain places, be helpful to the reader of the present paper. The reader is assumed familiar with the Hochschild cohomology theory for algebras as given in [8].

CHAPTER I. THE DEFORMATION THEORY FOR ALGEBRAS

1. Infinitesimal deformations of an algebra

Let A be an associative, not necessarily finite dimensional algebra over a field k, and V be the underlying vector space of A. Let $k[[t]] = R$

denote the power series ring in one variable, t, $K = k((t))$ be the quotient power series field of R, and let V_K denote the vector space obtained from V by extending the coefficient domain from k to K, i.e., $V_K = V \otimes_k K$. Any bilinear function f: $V \times V \to V$ (in particular, the multiplication in A) can be extended to a function bilinear over K from $V_K \times V_K$ to V_K. A bilinear function from $V_K \times V_K$ to V_K which is such an extension is "defined over k." Suppose that there is given a bilinear function f_t: $V_K \times V_K \to V_K$ expressible in the form

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \cdots , \tag{1}$$

where F_i is a bilinear function defined over k and where we may set $F_0(a, b) = ab$, the product in A. Suppose further that f_t is associative, i.e.,

$$f_t\big(f_t(a, b), c\big) = f_t\big(a, f_t(b, c)\big) \tag{2}$$

for all a, b, c in V_K (or equivalently, for all a, b, c in V). Then intuitively we may consider the algebra A_t whose underlying vector space is V_K and multiplication is f_t as the generic element of a "one-parameter family of deformations of A." (The family is analytic — see a forthcoming paper. If A is infinite dimensional, then the multiplication f_t given by (1) may have no specialization defined over $\bar{k}$ other than the trivial one defined by setting $t=0$, i.e., the original multiplication. If A is finite dimensional, in which case non-trivial specializations defined over $\bar{k}$ necessarily exist, then the dimension of the space of these may be greater than one.) The "infinitesimal deformation" or "differential" of this family is the bilinear function F_1, considered as a function from $V \times V$ to V.

The condition (2) that f_t be associative is equivalent to having for all a, b, c in V and all $\nu = 0, 1, 2, \cdots$

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu\geqq 0}} F_\lambda\big(F_\mu(a, b), c\big) - F_\lambda\big(a, F_\mu(b, c)\big) = 0 \,. \tag{3_ν}$$

For $\nu = 0$ this is just the associativity of the original multiplication. For $\nu = 1$, the condition (3_1) may be expressed in the form

$$aF_1(b, c) - F_1(ab, c) + F_1(a, bc) - F_1(a, b)c = 0 \,, \tag{3_1}$$

i.e., in terms of the Hochschild theory, F_1 is an element of the group $Z^2(A, A)$ of two-cocycles of A with coefficients in A.

In the analytic theory [9], an infinitesimal deformation of a manifold M is a one-cocycle of M with coefficients in the sheaf of germs of holomorphic tangent vectors on M. The conventions in the algebraic and analytic cohomology theories have so arisen that functions of n arguments in the former theory have dimension n and in the latter, dimension $n-1$.

The infinitesimal deformations themselves are actually completely analogous. That they are both cocycles should not be considered as accidental, but rather as forcing on one, had none existed previously, the definition of a suitable cohomology theory. Suppose, for example, that A is assumed to be a Lie algebra rather than an associative one, and that f_t defines a one-parameter family of deformations of A the generic element of which is again a Lie algebra. Then analogous to (2) we would have

$$(2') \qquad \begin{aligned} &f_t(a, b) = -f_t(b, a) ; \\ &f_t(f_t(a, b), c) + f_t(f_t(b, c), a) + f_t(f_t(c, a), b) = 0 . \end{aligned}$$

The condition analogous to (3_ν) is

$$(3_\nu') \qquad F_\nu(a, b) = -F_\nu(b, a)$$

and

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu\geqq 0}} F_\lambda(F_\mu(a, b), c) + F_\lambda(F_\mu(b, c), a) + F_\lambda(F_\mu(c, a), b) = 0 .$$

For $\nu = 0$ this expresses again the assumption that A is a Lie algebra, and for $\nu = 1$ may be written

$$(3_1) \qquad F_1(a, b) = -F_1(b, a)$$

and

$$\begin{aligned} F_1([a, b], c) + F_1([b, c], a) - F_1([a, c], b) \\ - [a, F_1(b, c)] + [b, F_1(a, c)] - [c, F_1(a, b)] = 0 . \end{aligned}$$

Considering F_1 as an element of $C^2(A, A)$, the group of 2-cochains of the Lie algebra A with coefficients in itself, the quantity on the left in the last equation is just $-\delta F_1(a, b, c)$ and (3_1) therefore asserts that F_1 is again a 2-cocycle. (The interested reader may perform the analogous computation for Jordan algebras.)

For an associative ring A, the definition of the Lie product in the direct sum $H^*(A, A)$ of the groups $H^n(A, A)$, $n = 0, 1, 2, \cdots$ introduced in [3], was suggested by the deformation theory. An analogous operation is definable for Lie rings [Nijenhuis and Richardson (unpublished)].

2. Obstructions

Given a Lie or associative ring A, an arbitrary element F_1 in $Z^2(A, A)$ need not be the differential of a one-parameter family of deformations of A. If it is such, then we shall say that F_1 is *integrable*. The integrability of F_1 implies an infinite sequence of relations which may be interpreted as the vanishing of the "obstructions" to the integration of F_1. In the associative case, these obstructions are all deducible from (3_ν)

by rewriting that equation in the form

$$(4_\nu)\qquad \textstyle\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}} F_\lambda(F_\mu(a, b), c) - F_\lambda(a, F_\mu(b, c)) = \delta F_\nu(a, b, c)\ .$$

Setting $\nu = 2$ gives

$$(4_2)\qquad F_1(F_1(a, b), c) - F_1(a, F_1(b, c)) = \delta F_2(a, b, c)\ .$$

The function of three variables on the left is the "associator" of F_1, and is an element of $Z^3(A, A)$ whenever F_1 is in $Z^2(A, A)$. The cohomology class of this element is the first obstruction to the integration of F_1; if F_1 is integrable, this must be the zero class.

The associator of F is quadratic in F. Linearizing, one finds that if F, G are in $Z^2(A, A)$, then the 3-cochain $[F, G]$ defined by

$$[F,G](a,b,c) = F(G(a,b),c) - F(a,G(b,c)) + G(F(a,b),c) - G(a,F(b,c))$$

is an element of $Z^3(A, A)$. This is a special case of the Lie product introduced in [3]. It will be shown in § 5 that if $F_1, \cdots, F_{n-1}$ satisfy (4_ν) for $\nu = 1, \cdots, n-1$, then the left side of (4_n) in fact defines an element of $Z^3(A, A)$, the $n-1^{\text{st}}$ "obstruction cocycle" which, however, is a function not of F_1 only, but of the sequence $F_1, \cdots, F_{n-1}$. In analogy with the analytic deformation theory, if $H^3(A, A) = 0$, then all obstructions vanish and every F_1 in $Z^2(A, A)$ is integrable. Here this is a trivial theorem, but in the analytic theory it is a deep result because of the convergence questions involved [10].

Assertions analogous to the foregoing hold in the Lie theory, using $(3_\nu')$ instead of (3_ν). Henceforth we shall not refer explicitly to the Lie case, but it will be readily seen that our considerations will work equally well for it.

3. Trivial deformations

A one-parameter family of deformations of an associative algebra defined by a multiplication g_t is *trivial* if there is a non-singular linear mapping Φ_t of $V_{k((t))} = V_K$ onto itself (an automorphism of V_K) of the form

$$(5)\qquad \Phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \cdots\ ,$$

where all the φ_i: $V_K \to V_K$ are linear maps defined over k, such that

$$(6)\qquad g_t(a, b) = \Phi_t^{-1}(\Phi_t a \cdot \Phi_t b)\ .$$

The algebra A_t so obtained is obviously isomorphic to $A_K = A \otimes_k K$, the isomorphism being in fact the linear mapping Φ_t considered as a mapping from A_t to A_K. This mapping is defined over the power series ring $R =$

$k[[t]]$, therefore *a fortiori* over $K = k((t))$, and does not involve any algebraic extension of K. Writing for this multiplication,

$$g_t(a, b) = ab + tG_1(a, b) + \cdots ,$$

one has $G_1 = \delta\varphi_1$. More generally, if given a one-parameter family of deformations of A as in (1), we set

$$(6') \qquad g_t(a, b) = \Phi_t^{-1} f_t(\Phi_t a \cdot \Phi_t b)$$

and write, as before, $g_t(a, b) = ab + tG_1(a, b) + \cdots$, then $G_1(a, b) = F_1(a, b) + \delta\varphi_1(a, b)$. It follows that the integrability of an element F_1 of $Z^2(A, A)$ depends only on its cohomology class; and that, if any element in a class is integrable, then the elements of that class may be integrated to give one-parameter families of algebras whose generic elements are isomorphic, the isomorphism being defined over $R = k[[t]]$. We may therefore interpret the classes, i.e., the elements of $H^2(A, A)$, as being in fact the infinitesimal deformations. Note, however, that it is possible for the differential F_1 of a one-parameter family to be zero while the generic element A_t of the family is not isomorphic to A_K over any field. The deformation may "start" in a trivial direction and then "veer off."

One-parameter families f_t and g_t of deformations of an algebra A will be called *equivalent* if there exists a non-singular linear automorphism Φ_t of V_K of the form (5) such that (6′) holds. The family f_t is trivial if it is equivalent to the identity deformation g_t defined by $g_t(ab) = ab$.

Suppose now that f_t is a one-parameter family of deformations of A for which $F_1 = \cdots = F_{n-1} = 0$. Then it follows from (3_ν) that $\delta F_n = 0$, i.e., F_n is in $Z^2(A, A)$. If further F_n is in $B^2(A, A)$, so that $F_n = -\delta\varphi_n$ for some φ_n in $C^1(A, A)$, then setting $\Phi_t(a) = a + t\varphi_n(a)$ we have

$$\Phi_t^{-1} f_t(\Phi_t a \cdot \Phi_t b) = ab + t^{n+1}F_{n+1}(a, b) + t^{n+2}F_{n+2}(a, b) + \cdots ;$$

and again F_{n+1} is in $Z^2(A, A)$. We may therefore assert the following trivial but fundamental result.

PROPOSITION 1. *Let f_t be a one-parameter family of deformations of an algebra A. Then f_t is equivalent to a family $g_t(a, b) = ab + t^nF_n(a, b) + t^{n+1}F_{n+1}(a, b) + \cdots$, where the first non-vanishing cochain F_n is in $Z^2(A, A)$, and is not cohomologous to zero.*

COROLLARY. *If $H^2(A, A) = 0$, then A is rigid.*

For any finite dimensional separable semi-simple algebra A and two-sided A module P, it is known that $H^n(A, P) = 0$ for $n > 0$. It follows, in particular, that finite dimensional separable semi-simple algebras are rigid. When the algebraic set of structure constants is introduced as parameter space for the finite dimensional algebras (Ch. II), it will be seen that every rigid algebra determines a component of that set. The problem of determining all rigid finite dimensional algebras presently seems to offer some hope of solution.

4. Obstructions to derivations and the squaring operation

Given an algebra A, which for the moment need not be associative, it is natural to consider the elements of $Z^1(A, A)$ (the derivations of A into itself) as infinitesimal automorphisms. The corresponding objects in the analytic theory, namely holomorphic tangent vector fields, are always integrable on compact manifolds in the sense that, given any such, there is a one-parameter family of automorphisms of the manifold (i.e., analytic homeomorphisms of the manifold onto itself) whose differential is the given tangent vector field. For finite dimensional algebras of characteristic zero, the analogous statement holds for derivations but it fails in characteristic $p \neq 0$. One may exhibit obstructions, now lying in $Z^2(A, A)$, in a fashion analogous to that already given, as follows. Suppose that $A_K = A \otimes_k k((t))$ possesses an automorphism Φ_t expressible in the form

$$\Phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \cdots , \tag{7}$$

the φ_i being linear functions from A_K to A_K defined over k, and φ_0 being interpreted as the identity mapping, I. We may consider Φ_t of (7) to be the generic element of a "one-parameter family of automorphisms" of A. (The same caution must be observed as earlier. If A is infinite dimensional, then the "generic automorphism" Φ_t of this family may have no specialization to an automorphism of A or even of $A \otimes_k \bar{k}$ except the identity, and if A is finite dimensional, then the dimension of the family of automorphisms obtained by specializing Φ_t over k may be greater than one.) Analogous to (3_ν), the condition that Φ_t be an automorphism is

$$\sum\nolimits_{\substack{\lambda+\mu=\nu \\ \lambda,\mu \geqq 0}} \varphi_\lambda(a)\varphi_\mu(b) = \varphi_\nu(ab) , \tag{8_ν}$$

or, analogous to (4_ν),

$$\sum\nolimits_{\substack{\lambda+\mu=\nu \\ \lambda,\mu > 0}} \varphi_\lambda(a)\varphi_\mu(b) = -\delta\varphi_\nu(a, b) . \tag{9_ν}$$

The condition is vacuous for $\nu = 0$. For $\nu = 1$ it asserts that $\delta\varphi_1(a, b) = a\varphi_1(b) - \varphi_1(ab) + \varphi_1(a)b = 0$, i.e., φ_1 is a derivation of A into itself, or an element of $Z^1(A, A)$. For $\nu = 2$, we have

$$\varphi_1(a)\varphi_1(b) = -\delta\varphi_2(a, b) . \tag{9_2}$$

The function of two variables on the left is just the cup product, $\varphi_1 \smile \varphi_1$ of φ_1 with itself, and is an element of $Z^2(A, A)$. It is the "first obstruction" to the integration of φ_1, and from (9_2) must represent the zero cohomology class if φ_1 is the differential of a one-parameter family of automorphisms. Now if the characteristic of A is not 2, then in fact $\varphi_1 \smile \varphi_1 = -\frac{1}{2}\delta\varphi_1^2$, and the first obstruction vanishes. Linearizing, we find that for φ_1, ψ_1 in $Z^1(A, A)$, $\varphi_1 \smile \psi_1 + \psi_1 \smile \varphi_1$ is a 2-coboundary, suggesting that the cohomology ring of a ring with coefficients in itself is a graded commutative ring. This is indeed the case, regardless of characteristic, as was shown in [3]. If the characteristic of A is p, then setting $\varphi_i = (i!)^{-1}\varphi_1^i$ for $i = 1, 2, \cdots, p-1$, the elements of $C^1(A, A)$ so defined satisfy (9_ν) for $\nu = 1, \cdots, p-1$, and the expression on the left in (9_p) is an element of $Z^2(A, A)$, the *primary obstruction* to the integration of φ_1. It will be shown that the primary obstruction induces an additive mapping of $H^1(A, A)$ into $H^2(A, A)$. The obstruction is analogous in some respects to a p^{th} power and will be denoted $\mathrm{Sq}_p\varphi_1$. If k is the prime field F_p of p elements, then Sq_p is linear.

Being an element of $Z^2(A, A)$, $\mathrm{Sq}_p\varphi_1$ may be interpreted as an infinitesimal deformation. It will be shown that such an infinitesimal deformation is always integrable. In fact, letting $K = k((t))$, there exists an associative multiplication f_t on V_K of the form

$$f_t = ab + t\,\mathrm{Sq}_p\varphi_1(a, b) + t^2F_2(a, b) + \cdots ,$$

such that if A_t denotes, as usual, the corresponding algebra, then $A_t \otimes_K K(t^{1/p})$ is isomorphic to $A \otimes_k K(t^{1/p})$, but A_t is generally not isomorphic to $A_K = A \otimes_k K$. It is interesting to note that while the cohomology groups of an algebra are essentially independent of the ground field, the deformation theory may distinguish between non-isomorphic algebras even if they are isomorphic over some larger field.

Since all the obstructions to the integration of an element of $Z^1(A, A)$ will be elements of $Z^2(A, A)$, and since only their cohomology classes are of interest, the vanishing of $H^2(A, A)$ implies not only the rigidity of A but also the integrability of any φ_1 in $Z^1(A, A)$ to a one-parameter family of automorphisms of A.

5. Obstructions are cocycles

While we have been so far considering only the case of an algebra over a field, the definitions of the obstruction cochains given in the preceding sections do not require the existence of a ground field. It is shown in this section that the obstruction cochains are cocycles under the assumption only that A is an associative ring, a fact which will be useful later.

Consider first the case of derivations. Given a one-parameter family of linear automorphisms of A as in (7), $\Phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \cdots$, the necessary and sufficient conditions (9_ν) that this be a one-parameter family of multiplicative automorphisms may be written

$$(9_\nu') \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \varphi_\lambda \smile \varphi_\mu = -\delta\varphi_\nu \, .$$

Suppose given $\varphi_1, \cdots, \varphi_{m-1}$ satisfying $(9_\nu')$ for $\nu = 1, \cdots, m-1$. Then the cohomology class of $F = \sum_{\substack{\lambda+\mu=m \\ \lambda,\mu>0}} \varphi_\lambda \smile \varphi_\mu$ may be viewed as the obstruction to prolonging the sequence $\varphi_1, \cdots, \varphi_{m-1}$ to a sequence $\varphi_1, \cdots, \varphi_{m-1}, \varphi_m$ satisfying $(9_\nu')$ for $\nu = 1, \cdots, m$. We show now that F is a cocycle.

PROPOSITION 2. *Let A be an associative ring, and $\varphi_1, \cdots, \varphi_{m-1}$ be elements of $C^1(A, A)$ such that $\sum_{\lambda+\mu=\nu} \varphi_\lambda \smile \varphi_\mu = -\delta\varphi_\nu$ for $\nu = 1, \cdots, m-1$. Set $F = \sum_{\lambda+\mu=m} \varphi_\lambda \smile \varphi_\mu$. Then $\delta F = 0$,* i.e., *F is in $Z^2(A, A)$.*

PROOF. Computing δF explicitly, noting that if φ, ψ are in $C^1(A, A)$, then $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi - \varphi \smile \delta\psi$, one has

$$\delta F = \sum_{\lambda+\mu=m} \left(-\sum_{\alpha+\beta=\lambda} \varphi_\alpha \smile \varphi_\beta\right) \smile \varphi_\mu - \sum_{\lambda+\mu=m} \varphi_\lambda \smile \left(-\sum_{\alpha+\beta=\mu} \varphi_\alpha \smile \varphi_\beta\right) .$$

Since the cup product is associative, this vanishes identically, both double sums being, up to difference in sign, $\sum_{\alpha+\beta+\gamma=m} \varphi_\alpha \smile \varphi_\beta \smile \varphi_\gamma$. This ends the proof.

Consider now the case of deformations. Given a one-parameter family of multiplications on the underlying vector space of an algebra A as in (1), $f_t(a, b) = ab + tF_1(a, b) + \cdots$, the necessary and sufficient conditions (4_ν) that this be a family of associative multiplications may in the notation of [3] be expressed in the form

$$(4_\nu') \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu = \delta F_\nu \, .$$

Suppose given $F_1, \cdots, F_{m-1}$ satisfying $(4_\nu')$ for $\nu = 1, \cdots, m-1$. Then setting $G = \sum_{\lambda+\mu=m} F_\lambda \circ F_\mu$, the cohomology class of G may be viewed as the obstruction to the construction of an F_m such that $F_1, \cdots, F_m$ satisfy $(4_\nu')$ for $\nu = 1, \cdots, m$, and it is this element G of $C^3(A, A)$ which will be shown to be a cocycle. To this end it is convenient to use certain results of [3], together with the following, only a special case of which will actually be needed.

LEMMA 1. *Let f, g be elements of a right graded pre-Lie ring B, with g homogeneous of odd degree. Suppose either*

(i) *B has no elements of order 2,* i.e., *$x \in B$ and $2x = 0$ implies $x = 0$, or*

(ii) *B is the pre-Lie ring of a pre-Lie system $\{V_m, \circ_i\}$. Then $(f \circ g) \circ g = f \circ (g \circ g)$.*

PROOF. Observe first that, if f, g, h are elements of a graded pre-Lie ring with g, h homogeneous of odd degree, then one has immediately from the axioms, [3, § 2], that $2(f\circ g)\circ g = 2f\circ(g\circ g)$, and in case (i) the assertion follows immediately. In case (ii), note that by [3, § 6, Th. 2], we have $(f\circ g)\circ g - f\circ(g\circ g) = \sum' (-1)^{i+j}(f\circ_i g)\circ_j g$, $\sum'$ being extended over all i and j such that either $0 \leqq j \leqq i-1$ or $n+i+1 \leqq j \leqq m+n$. But by the definition of a pre-Lie system, [3, § 5], if $0 \leqq j \leqq i-1$ then $(f\circ_i g)\circ_j g = (f\circ_j g)\circ_{i+n} g$. Therefore, the sum taken over those i and j with $0 \leqq j \leqq i-1$ is, up to sign, the same as that over those i and j with $n+i+1 \leqq j \leqq m+n$, and since n is odd, the signs are different. Therefore, the total sum vanishes. This ends the proof.

One may note that the lemma gives a necessary condition that a right pre-Lie ring graded by the integers be the pre-Lie ring associated with a pre-Lie system.

Observe that the *degree* of an element of $C^2(A, A)$ is one, and that the pre-Lie structure on $C^*(A, A)$ (the direct sum of the groups $C^n(A, A)$) has been obtained from a pre-Lie system.

PROPOSITION 3. *Let A be an associative ring, and $F_1, \cdots, F_{m-1}$ be elements of $C^2(A, A)$ such that $\sum_{\lambda+\mu=\nu} F_\lambda \circ F_\mu = \delta F_\nu$ for $\nu = 1, \cdots, m-1$. Let an element G of $C^3(A, A)$ be defined by $G = \sum_{\lambda+\mu=m} F_\lambda\circ F_\mu$. Then $\delta G = 0$, i.e., G is in $Z^3(A, A)$.*

PROOF. Since F_λ and F_μ are in $C^2(A, A)$, one has from [3, § 7, Th. 3], that

$$\delta(F_\lambda\circ F_\mu) = F_\lambda\circ\delta F_\mu - \delta F_\lambda\circ F_\mu + (F_\mu \smile F_\lambda - F_\lambda \smile F_\mu)\ .$$

It follows that $\delta G = \sum_{\lambda+\mu=m}\delta(F_\lambda\circ F_\mu) = \sum_{\lambda+\mu=m}[F_\lambda\circ\delta F_\mu - (\delta F_\lambda)\circ F_\mu]$. By the hypothesis, this equals $\sum_{\alpha+\beta+\gamma=m}[F_\alpha\circ(F_\beta\circ F_\gamma) - (F_\alpha\circ F_\beta)\circ F_\gamma]$. Now by the preceding lemma, any term for which $\beta = \gamma$ must vanish. Therefore, we may write

$$\delta G = \sum_{\alpha+\beta+\gamma=m}[F_\alpha\circ(F_\beta\circ F_\gamma + F_\gamma\circ F_\beta) - ((F_\alpha\circ F_\beta)\circ F_\gamma + (F_\alpha\circ F_\gamma)\circ F_\beta)]\ .$$

But by the observation immediately preceding Prop. 3, every term of this sum vanishes. Therefore $\delta G = 0$, ending the proof.

6. Additivity and integrability of the square

The definition of the squaring operation given in § 4 was made, for convenience, to depend on the assumption that A is an algebra over a field of characteristic p. While this is necessary in order that $\mathrm{Sq}_p\varphi$ should be interpretable as the primary obstruction to the derivation φ, given any associative ring A and element φ in $Z^1(A, A)$, one can define for every prime power $q = p^\alpha$ an element $\mathrm{Sq}_q \varphi \in Z^2(A, A)$ by setting formally $\mathrm{Sq}_q \varphi = -(1/p)\delta\varphi^q$. Since

$$-\delta\varphi^q = \binom{q}{1}\varphi^{q-1} \smile \varphi + \binom{q}{2}\varphi^{q-2} \smile \varphi^2 + \cdots + \binom{q}{1}\varphi \smile \varphi^{q-1} ,$$

and every coefficient is divisible by p, the coefficients in $\mathrm{Sq}_q\varphi$ are all integers, and Sq_q is therefore well-defined. It is immediate from the formal definition that $\mathrm{Sq}_q\varphi \in Z^2(A, A)$.

It is easy to see that the definition of $\mathrm{Sq}_q\varphi$ given in § 4 coincides with the present one over fields of characteristic p. For in § 4, $\mathrm{Sq}_q\varphi$ is defined in effect as $\sum_{\lambda=1}^{p-1}(1/(\lambda!(p-\lambda)!))\varphi^{p-\lambda} \smile \varphi^\lambda$, and here is defined as $\sum_{\lambda=1}^{p-1}(1/p)\binom{p}{\lambda}\varphi^{p-\lambda} \smile \varphi^\lambda$, but the coefficients in the former sum are congruent to those in the latter modulo p.

We show next that Sq_q induces a mapping, which we shall continue to denote by Sq_q of $H^1(A, A)$ into $H^2(A, A)$.

LEMMA 2. $\mathrm{Sq}_q(B^1(A, A)) \subset B^2(A, A)$.

PROOF. Suppose $\varphi \in B^1(A, A)$, i.e., $\varphi = \delta a$ for some a in A. Then $\mathrm{Sq}_q\varphi = -(1/p)\delta(\delta a)^q$. Now if b is an arbitrary element of A, then noting that $(\delta a)(b) = ab - ba$, we have

$$\begin{aligned}(\delta a)^q(b) &= [a, [\cdots, [a, [a, b]] \cdots]] \\ &= a^q b - \binom{q}{1}a^{q-1}ba + \binom{q}{2}a^{q-2}ba^2 - \cdots + (-1)^q ba^q ,\end{aligned}$$

while $(\delta a^q)(b) = a^q b - ba^q$. It follows that $(\delta a)^q - \delta a^q$ is of the form pf for some $f \in C^1(A, A)$, namely that f defined by $f(b) = -(1/p)\binom{q}{1}a^{q-1}ba + (1/p)\binom{q}{2}a^{q-2}ba^2 - \cdots$. Therefore,

$$\mathrm{Sq}_q \varphi = -\frac{1}{p}\delta(\delta a)^q = -\frac{1}{p}\delta pf - \frac{1}{p}\delta\delta a^q = -\delta f .$$

The latter being in $B^2(A, A)$, this ends the proof.

LEMMA 3. *Considered as a mapping from* $H^1(A, A)$ *to* $H^2(A, A)$, Sq_q *is additive*, i.e., *if* $\eta, \zeta \in H^1(A, A)$, *then* $\mathrm{Sq}_q(\eta + \zeta) = \mathrm{Sq}_q\eta + \mathrm{Sq}_q\zeta$.

PROOF. Let φ, ψ be representatives of η, ζ, respectively, in $Z^1(A, A)$. It must be shown that $-(1/p)[\delta(\varphi+\psi)^q - \delta\varphi^q - \delta\psi^q] = -(1/p)\delta[(\varphi+\psi)^q - \varphi^q - \psi^q]$ is a coboundary. Now $(\varphi + \psi)^q - \varphi^q - \psi^q$ is a sum of terms $\mu_1 + \mu_2 + \cdots + \mu_{q-1}$, where μ_i is the sum of all monomials in φ and ψ of degree i in ψ and $q - i$ in φ; in particular, $\mu_1 = \varphi^{q-1}\psi + \varphi^{q-2}\psi\varphi + \cdots + \psi\varphi^{q-1}$. We make use of the formal identity

$$x^{m-1}y + x^{m-2}yx + \cdots + yx^{m-1}$$
$$= \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}[y, x] + \binom{m}{3}x^{m-3}[[y, x], x] + \cdots$$
$$+ \binom{m}{m}[[\cdots[y, x]\cdots, x], x] ,$$

where $[y, x] = yx - xy$. Since $\psi, \varphi \in Z^1(A, A)$ and $Z^1(A, A)$ is closed under the Lie product, it follows that

$$\delta\mu_1 = \delta\Big\{\binom{q}{1}\varphi^{q-1}\psi + \binom{q}{2}\varphi^{q-2}[\psi, \varphi] + \cdots$$
$$+ \binom{q}{q-1}\varphi[[\cdots[\psi, \varphi]\cdots, \varphi], \varphi]\Big\} .$$

All the coefficients being divisible by p, it follows that $-(1/p)\delta\mu_1$ is in $B^2(A, A)$.

Now $\mu_i = \mu_i(\varphi, \psi)$, and if t is any indeterminate, we have

$$\mu_1(\varphi + t\psi, \psi) = \mu_1(\varphi, \psi) + t\mu_2(\varphi, \psi) + \cdots + t^{q-1}\mu_{q-1}(\varphi, \psi) .$$

It follows that $-(1/p)\delta\mu_i$ is in $B^2(A, A)$ for $i = 1, \cdots, q-1$, whence $\mathrm{Sq}_q(\varphi + \psi) - \mathrm{Sq}_q\varphi - \mathrm{Sq}_q\psi$ is in $B^2(A, A)$. This ends the proof.

It is trivial that, if λ is any element in the center of A, then $\mathrm{Sq}_q\lambda\varphi = \lambda^q\,\mathrm{Sq}_q\varphi$. Collecting results, we have

THEOREM 1. *Let A be an associative ring and $q = p^\alpha$ be a power of a prime p. Then the mapping Sq_q: $H^1(A, A) \to H^2(A, A)$ is additive and has the property that, if λ is in the center of A, then $\mathrm{Sq}_q\lambda\eta = \lambda^q\,\mathrm{Sq}_q\eta$ for $\eta \in H^1(A, A)$. If A is an algebra over the prime field $\mathbf{F}_p$, then Sq_q is linear.*

For the rest of this section we shall again suppose that A is an algebra over a field k of characteristic p, in which case $\mathrm{Sq}_p\varphi$ is interpretable as the primary obstruction to the integration of an element $\varphi \in Z^1(A, A)$. To prove the integrability of the primary obstruction to a derivation, we need certain lemmas. The following is trivial.

LEMMA 4. *Let A be an algebra of characteristic p over a field k, and F_1 be an element of $Z^2(A, A)$. Suppose there exists a one-parameter family of linear automorphisms Ψ_t of $A \otimes_k k((t))$ of the form $\Psi_t(a) = a + t\psi_1(a) + t^2\psi_2(a) + \cdots$, ψ_i being defined over k, such that $\Psi_t^{-1}(\Psi_t a \cdot \Psi_t b) = ab + t^pF_1(a, b) + t^{2p}F_2(a, b) + \cdots$, for certain $F_2, F_3, \cdots$ in $C^2(A, A)$. Then F_1 is an integrable element of $Z^2(A, A)$ and $f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \cdots$ is a one-parameter family of deformations of A such that denoting by A_t its generic element, $A_t \otimes_{k((t))} k((t^{1/p}))$ is isomorphic* *to* $A \otimes_k k((t^{1/p}))$.

Note in the foregoing that it need not be the case that A_t is isomorphic to $A \otimes_k k((t))$.

We shall show that if F_1 is of the form $\mathrm{Sq}_p \varphi$ for some $\varphi \in Z^1(A, A)$, then a one-parameter family Ψ_t of the sort in Lemma 4 must necessarily exist. To this end, we examine the ordinary exponential power series in characteristic zero.

LEMMA 5. *If the power series* $e^x = \sum_{n=0} x^n/n!$ *is factored into a product of power series of the form* $e^x = e_p(x)e_0(x)$ *with* $e_p(x) = \sum_{m=0} x^{mp}/(mp)!$, *p any fixed prime, then the coefficients of* $e_0(x)$ *are all integral at p,* i.e., *when expressed as rational fractions in lowest terms, have denominators not divisible by p.*

PROOF. Let η denote a primitive p^{th} root of unity. Then we may write $e_p(x) = p^{-1}(e^x + e^{\eta x} + \cdots + e^{\eta^{p-1}x})$, whence $e_0^{-1}(x) = e_p(x)e^{-x} = p^{-1}(1 + e^{(\eta-1)x} + \cdots + e^{(\eta^{p-1}-1)x})$. It is clearly sufficient to show that the coefficients of the power series for $e_0^{-1}(x)$ are integral at p. The coefficient of x^m in the latter is, for $m \geqq 1$,

$$p^{-1}(m!)^{-1}\big((\eta - 1)^m + (\eta^2 - 1)^m + \cdots + (\eta^{p-1} - 1)^m\big)\,.$$

Let Q denote the rationals and v_p denote the p-adic valuation of Q so normalized that $v_p(p) = 1$. The valuation v_p ramifies completely in $Q(\eta)$, and if we extend v_p to $Q(\eta)$, preserving the normalization $v_p(p) = 1$, then $v_p(\eta - 1) = 1/(p - 1)$. It follows that $v_p(\mathrm{tr}(\eta - 1)^m) \geqq m/(p - 1)$. Setting $e_0^{-1}(x) = 1 + \sum_{m=1} a_m x^m$, we have therefore, $v_p(a_m) \geqq (p - 1)^{-1}m - 1 - v_p(m!)$. However donoting the greatest integer contained in a real number α by $[\alpha]$, we have $v_p(m!) = [m/p] + [m/p^2] + \cdots$. This is strictly less than $\sum_{n=1} m/p^n = m/(p - 1)$. Therefore, $v_p(a_m) > -1$. But a_m is rational, and therefore $v_p(a_m)$ is an integer. It follows that $v_p(a_m) \geqq 0$, which is, in fact, what has been asserted. This ends the proof.

We continue to suppose now that A is an algebra over a field k of characteristic p.

THEOREM 2. *Let* φ *be an element of* $Z^1(A, A)$. *Then there exists a one-parameter family* $\psi_t = 1 + t\psi_1 + t^2\psi_2 + \cdots$ *of linear automorphisms of* $A \otimes k((t))$ *with* ψ_i *defined over k such that* $\Psi_t^{-1}(\Psi_t a \cdot \Psi_t b) = ab + t^p \mathrm{Sq}_p \varphi(a, b) + t^{2p}F_2(a, b) + \cdots$ *for certain elements* $F_i \in C^2(A, A)$, $i = 2, 3, \cdots$.

PROOF. Consider formally an algebra A of characteristic zero. If φ is a derivation of A into itself, then $e^{t\varphi}$ is a one-parameter family of automorphisms of A, and one has $e^{t\varphi}(ab) = e^{t\varphi}a \cdot e^{t\varphi}b$. From the factorization

$e^x = e_0(x)e_p(x)$, it follows that $e_p^{-1}(t\varphi)[e_p(t\varphi)a \cdot e_p(t\varphi)b] = e_0(t\varphi)[e_0^{-1}(t\varphi)a \cdot e_0^{-1}(t\varphi)b]$. Since $e_p(x)$ is a power series in x^p, it follows that the expression on the left, and therefore also on the right, is a power series in t^p, and is of the form $ab + t^pF_1(a, b) + t^{2p}F_2(a, b) + \cdots$, $F_i \in C^2(A, A)$. Further, since $e_p(t\varphi) = 1 + t^p\varphi^p/p! + \cdots$, one has (cf. § 3) that $F_1 = (p!)^{-1}\delta\varphi^p = (1/(p-1)!)\cdot(1/p)\delta\varphi^p = -(1/(p-1)!)\,\mathrm{Sq}_p\,\varphi$. It follows, therefore, that $e_0(-t\varphi)[e_0^{-1}(-t\varphi)a \cdot e_0^{-1}(-t\varphi)b]$ is of the form $ab - (1/(p-1)!)\,\mathrm{Sq}_p\,\varphi(a, b) + t^{2p}F_2(a, b) + \cdots$. Now observe that all the coefficients of e_0 are integral at p, and may therefore be reduced modulo p. The latter equality must therefore also hold in characteristic p. Since $(p-1)!$ is congruent to -1 modulo p, we may set $\Psi_t = e_0(-t\varphi)$. This proves the theorem.

COROLLARY. If $\varphi \in Z^1(A, A)$, then $\mathrm{Sq}_p\,\varphi$ is integrable.

Another proof of the present result is possible, which, although not as explicit, may have wider applicability. It rests on the computation of the cohomology groups of a certain complex related to that of Heaton and Whaples, [5], [6], and [7].

7. Restricted deformation theories and their cohomology theories

Up to this point we have been taking, as the domain of our deformation theory, the set of all associative algebras of some fixed dimension over a field k. Returning to the first point in the outline of a deformation theory, and also to equation (1), we may observe that if the algebra A has certain special algebraic properties expressible by polynomial equations in the structure constants, then it may be required that these properties be preserved under deformation. This limits the class of objects obtainable from a given one by deformation. For example, if A is commutative, then we may require that the one-parameter families of deformations of A have commutative generic elements. This is equivalent to asserting that the bilinear functions F_i in (1) satisfy, in addition to (3_ν), the condition $F_i(a, b) = F_i(b, a)$ for all a, b in A. In particular, this must hold for $i = 1$, and the infinitesimal deformations of this restricted theory are the "commutative" 2-cocycles F_1 in $Z^2(A, A)$, that is, those such that $F_1(a,b) = F_1(b, a)$. Since a 2-coboundary of a commutative algebra is always commutative, it follows that we can define, for a commutative algebra, a "commutative" 2-cohomology group (which will be a subgroup of the full group $H^2(A, A)$). Also, since a trivial deformation of a commutative algebra A of the type given by (6) must again produce a commutative algebra (since except for extending the ground field it does not change A), it follows that the integrability of a commutative 2-cocycle of A to a one-parameter family of deformations whose generic element is commutative, depends only on the cohomology class of the element. Further, different elements in the same integrable class may be integrated to one-parameter families with isomorphic generic elements. Therefore we may, as expected, identify the commutative 2-cohomology group of A with the infinitesimal deformations of A when the theory is restricted to commutative algebras. (In the case of characteristic p, a 2-cocycle obtained as a primary obstruction to a derivation of a commutative algebra must also always be commutative.)

These observations suggest that for a commutative algebra it must be possible to define a restricted cohomology theory containing a definition of commutative cocycle in every dimension. This has in fact been done by Harrison [4], and it has been shown by Barr that the natural homomorphisms of the commutative cohomology groups into the usual ones are injections in low dimensions. Since the composition of commutative infinitesimal objects should again be a commutative infinitesimal object, the direct sum of the Harrison groups must also admit a Lie or "bracket" product of the sort introduced in [3].

An outline of the commutative theory through dimension 3, in which dimension the obstructions take place, is the following. Let A be a commutative ring. A two-sided A module P is called commutative if $ax = xa$ for all a in A and x in P. The coboundary of an n-cochain of A with coefficients in P is defined exactly as in the Hochschild theory, and it is only necessary to define for every dimension the commutative cochains. In dimensions 0 and 1 all cochains are defined to be commutative. In dimension 2, a cochain f is commutative if $f(a \otimes b) = f(b \otimes a)$. (We use the tensor notation because A is not assumed to be an algebra over a field.) In dimension 3, a cochain f is commutative if $f(a \otimes b \otimes c) = -f(c \otimes b \otimes a)$, and if $f(a \otimes b \otimes c) + f(b \otimes c \otimes a) + f(c \otimes a \otimes b) = 0$. One readily verifies that the coboundary of a commutative 2-cochain is a commutative 3-cochain, whence the commutative 3-cohomology group is definable. One may also observe that, if F_1 is a commutative 2-cocycle, then its associator G defined by $G(a \otimes b \otimes c) = F_1(F_1(a \otimes b) \otimes c) - F_1(a \otimes F_1(b \otimes c))$ is in fact a commutative 3-cocycle. When A is an algebra, the first obstruction to the integration of F_1 is now the commutative cohomology class of its associator. As an illustration of the first point in the outline of the deformation theory, the commutative theory shows that a limitation of the class within which deformation takes place may have a non-trivial effect. We shall denote the commutative cohomology groups of A with coefficients in a commutative module P by $H_c^i(A, P)$, $i = 0, 1, \cdots$; for $i = 0, 1$ these coincide with the usual ones.

Another important deformation theory obtained by a limitation of the general one is the deformation theory of nilpotent algebras. Given a nilpotent algebra A, we may ask what deformations are possible in which the generic element is nilpotent, nilpotent of the same index as A, or even satisfies more stringent conditions. Correspondingly, we will be led to the definition of suitable 2-dimensional cohomology groups which serve as groups of infinitesimal deformations. For example, if A is nilpotent of index n, and we have a one-parameter family of deformations of A in which the generic element has index n, and if F_1 is the differential of the family, then

$$\begin{aligned}(10)\qquad F_1(a_1, a_2a_3 \cdots a_n) + a_1F_1(a_2, a_3 \cdots a_n) + \cdots \\ + a_1a_2 \cdots a_{n-2}F_1(a_{n-1}, a_n) = 0\end{aligned}$$

for all $a_1, \cdots, a_n$ in A. The elements of $Z^2(A, A)$ of "index n" are those satisfying (10). All elements of $B^2(A, A)$ must satisfy (10), and the group of infinitesimal deformations for the restricted theory is the quotient.

An example of a theory which may be considered both as a restriction of the theory given in the preceding sections, and as a generalization of it, is the deformation theory of an algebra with a filtration, where it is required that the deformation preserve the filtration. We devote Ch. III to the deformation of rings with a decreasing filtration by ideals.

It is indeed true that every restricted deformation theory generates its proper cohomology theory. In the classical Hochschild theory, given an algebra A and two-sided A module P, the isomorphism classes of extensions of A by P in which $P^2 = 0$ are in one-one correspondence with the elements of $H^2(A, P)$. Moreover, in any restricted theory the analogous result will hold. This is certainly the case, for example, for the commutative theory. If A is a commutative algebra over a field, P a commutative A module, and f an element of $Z_c^2(A, P)$, i.e., a commutative 2-cocycle of A with coefficients in P, then the vector space direct sum $A + P$ becomes a commutative algebra under the multiplication $(a, x)(b, y) = (ab, ax + by + f(a, b))$, $a, b \in A$, $x, y \in P$, and indeed, cohomologous cocycles give isomorphic multiplications. Conversely, suppose B is a commutative algebra containing an ideal P such that $P^2 = 0$ and such that B/P is isomorphic to A. Let q be any linear mapping of A into B such that, denoting the projection of B onto A by p, we have $pq = I$, the identity. Then P becomes a commutative A module by setting $a \cdot x = q(a)x$, $x \cdot a = xq(a)$, for $a \in A$, $x \in P$. (Since $P^2 = 0$, this does not depend on the choice of q.) The function $f\colon A \times A \to P$ defined by $f(a, b) = q(a)q(b) - q(ab)$ is further a commutative 2-cocycle, and B, as a vector space, is identifiable with $A + P$ and has a multiplication of the type already given. Thus, for a commutative algebra A and commutative A module P, the commutative extensions of A by P with $P^2 = 0$ are easily seen to be in natural one-one correspondence with the elements of $H_c^2(A, P)$. General definitions and results will appear in another paper.

8. Rigidity of fields in the commutative theory

If K, k are extensions of a field k_0 of characteristic $p \neq 0$, and if $k^p \subset k_0$, then there exists a unique compositum kK of k and K inside the algebraic closure $\bar{K}$ of K. If c in k and a in K are given, then ca is well-defined in $\bar{K}$ and is an element of kK. Further K is said to be *linearly disjoint* from k over k_0 if this is so inside $\bar{K}$.

A field K is a *separable* extension of k if K and $k^{1/p}$ are linearly disjoint over k, where $k^{1/p}$ is the field of p^{th} roots of elements of k. If the characteristic is zero, then the condition is vacuous and every extension is separable.

The main result of the present section, Theorem 3, asserts that if K is an extension of k, then considering K as an algebra over k, one has $H_c^2(K, K) = 0$ if and only if K is separable over k. This gives in particular a sufficient condition for the rigidity, in the commutative theory, of a field as an algebra over a subfield. It is conjectured that the condition is sufficient.

An extension B of an algebra A by an ideal I *splits* if there exists an isomorphism $q: A \to B$ which, if followed by the natural homomorphism $\pi: B \to A$ gives the identity map on A. Note that $H^2(A, P) = 0$ for a two-sided A module P if and only if every extension of A by P in which $P^2 = 0$ splits; the analogous statement holds in the commutative case. It is a classic observation (not needed in this section) that if $H^2(A, P) = 0$ for all P, then every extension of A by an ideal I with $I^m = 0$ for some finite m also splits. This proposition is generalized in Ch. III, § 2.

LEMMA 6. *Let K, k be extension of a field k_0 of characteristic $p \neq 0$, and suppose that $k^p \subset k_0$. Set $B = k \otimes_{k_0} K$ and let π be the homomorphism of B onto K given by $\pi(c \otimes a) = ca$, $c \in k$, $a \in K$, extended linearly. Then the kernel I of π is the set of all non-units of B, and $I^p = 0$. Further, B is a field if and only if k and K are linearly disjoint over k_0. If B is not a field, then I contains an element of nilindex exactly p.*

PROOF. Let s be the endomorphism of B given by $s(b) = b^p$ for all b in B. Then $s(c \otimes a) = c^p \otimes a^p = 1 \otimes c^p a^p$ since $c^p \in k^p \subset k_0$. Therefore, $s(B) = B^p = 1 \otimes K^p$, a field isomorphic to K^p, and the restriction of π to B^p is an isomorphism. Let s' be the isomorphism of K into itself given by $s'(a) = a^p$. Then $s'\pi = \pi s$, whence the kernel of $s'\pi$, which is that of π, is the same as the kernel of πs, which is that of s. The latter is just the set of all b in B such that $b^p = 0$. On the other hand, given $b \in B$ such that $b^p \neq 0$, since B^p is a field we have $b^p b' = 1$ for some b', whence $b(b^{p-1}b') = 1$ and b is a unit. Therefore, the kernel I of π is at the same time the set of all non-units and the set of all b such that $b^p = 0$. Further, since no element of I^p is a unit but $I^p \subset B^p$, a field, it follows that $I^p = 0$.

As to the last assertion, if B is a field then π is an isomorphism, whence k and K are linearly disjoint over k_0 in K, and conversely, if they are linearly disjoint, then π is an isomorphism onto the field kK, whence B is a field. It remains, therefore, only to prove that, if B is not a field, then it contains an element b with $b^{p-1} \neq 0$, $b^p = 0$. If B is not a field, i.e., if k is not linearly disjoint from K over k_0, then there exist elements $\alpha_1, \cdots, \alpha_m$ in finite number in k such that $k_0(\alpha_1, \cdots, \alpha_m)$ is not linearly disjoint from K over k_0, and it is sufficient to demonstrate that $B' = k_0(\alpha_1, \cdots, \alpha_m) \otimes_{k_0} K$ contains an element of nilindex exactly p. We may therefore set $B' = B$ and assume that $k = k_0(\alpha_1, \cdots, \alpha_m)$. We may further assume that m is minimal, whence $k_0(\alpha_1, \cdots, \alpha_{m-1})$ is linearly disjoint from K over k_0 and therefore that $k = k_0(\alpha_1, \cdots, \alpha_m)$ is not linearly disjoint from $K(\alpha_1, \cdots, \alpha_{m-1})$ over $k_0(\alpha_1, \cdots, \alpha_{m-1})$. Set now $k_0(\alpha_1, \cdots, \alpha_{m-1}) = k_0'$, $\alpha_m = \alpha$, whence $k = k_0'(\alpha)$, and $K(\alpha_1, \cdots, \alpha_{m-1}) = k_0'K = K'$. Observe that $B = k \otimes_{k_0} K = k \otimes_{k_0'} k_0' \otimes_{k_0} K$, and that therefore there exists a homomorphism π' of B onto $k \otimes_{k_0'} K'$ given by $\pi'(c \otimes c' \otimes a) = c \otimes c'a$. The composition of this with the natural homomorphism of $k \otimes_{k_0'} K'$ onto $kK' = kK$ is just π. It is sufficient, therefore, to show that $k \otimes_{k_0'} K'$ contains a nilpotent element of index exactly p, i.e., it is sufficient to consider the case where k is obtained by adjoining to k_0 the p^{th} root α of some element c of k_0. Since k is not linearly disjoint from K over k_0, it follows that K also contains a p^{th} root of c which will again be denoted by α. Then $k \otimes_{k_0} K$ contains $\alpha \otimes 1 - 1 \otimes \alpha$, which is easily seen to be nilpotent of index exactly p. This ends the proof.

Let K again be an extension of a field k of characteristic $p \neq 0$. Then $B' = k \otimes_{k^p} K$ is a commutative ring with unit containing the field $k \otimes 1$, and so may be considered as an algebra over k. The natural homomorphism $\pi'\colon B' \to kK = K$ is a k-homomorphism. Further, B' contains $k \otimes_{k^p} K^p$ which is isomorphic as a ring to $k^{1/p} \otimes_k K$ and is therefore either a field or, by Lemma 6, contains an element b' nilpotent of index p according as K is separable over k or not. Suppose now the latter case. Letting I denote the kernel of π', by Lemma 6, $I^p = 0$, whence b' is not in I^2. Set $B'' = B'/I^2$, $I'' = I/I^2$, and let b'' be the image of b' in B''. Note now that B' is also an algebra over $1 \otimes K$ (which is isomorphic to K and contains $1 \otimes k$), whence B'' is also an algebra over $1 \otimes K$. If c is in k, then $1 \otimes c - c \otimes 1$ is in I, whence $(1 \otimes c - c \otimes 1)b'$ is in I^2, so $(1 \otimes c)b'' = (c \otimes 1)b''$. Therefore $(1 \otimes K)b''$ is also a k (i.e., $k \otimes 1$) subspace of B''. It is therefore an ideal. Let J'' be any K (i.e., $1 \otimes K$) subspace of $I'' = I/I^2$ complementary to $(1 \otimes K)b''$. By the same argument J'' is also an ideal. Set $B = B''/J''$, $P = I''/J''$. There is a natural homomorphism π of B onto K induced by π', and B is thus a commutative extension of K by an ideal P with $P^2 = 0$. As a K-module, P is naturally isomorphic to K, and B therefore represents an element of $H_c^2(K, K)$. Note further that B contains k (=the image of $k \otimes 1$), and that the compositum kB^p of k and B^p in B (=the image of $k \otimes_{k^p} K^p$) still contains a non-zero nilpotent element, the image of b'. Therefore kB^p is not a field, and it will be seen that this implies that B does not split. Therefore, B will represent a non-zero element of $H_c^2(K, K)$.

LEMMA 7. *Let B be a commutative k-algebra with an ideal P such that $P^2 = 0$ and $B/P \cong K$. Let π denote that natural projection of B onto K. Then*

(i) *there exists a unique idempotent e in B such that $\pi(e) = 1$,*

(ii) *there exists a unique k-isomorphism q of k into B such that πq is the identity on k, and*

(iii) *given a subfield L of K and a k-isomorphism q of L into B such that πq is the identity on L, if $a \in K$ is either transcendental or separably algebraic over L, then q can be extended to such an isomorphism of $L(a)$ into B.*

PROOF. (i) If f is any element of B such that $\pi(f) = 1$, then $e = f + (f - f^2)$ is an idempotent since $(f - f^2)^2 \in P^2 = 0$, and further $\pi(e) = 1$. If e' is another such idempotent, then $(e - e')^2 = 0$, whence $e + e' = 2ee'$. Multiplying by e gives $e = ee'$; similarly $e' = ee'$, so $e = e'$, proving the unicity.

(ii) Given $c \in k$, set $q(c) = ce$.

(iii) Identify L with qL, and choose α in B such that $\pi\alpha = a$. If a is transcendental over K then $\alpha \notin P$, hence is a unit, and $L(\alpha)$ is a field projecting onto $L(a)$. If a is separable over L, let f be the irreducible polynomial which it satisfies over L, and set $\beta = \alpha - [f(\alpha)/f'(\alpha)]$ (Newton's approximation). Then $f(\beta) \in P^2$, whence $f(\beta) = 0$, $\pi(\beta) = \alpha$, and $L(\beta)$ is a field projecting onto $L(a)$. This ends the proof.

In the case of characteristic zero, the lemma implies that every commutative extension B of K by an ideal of square zero splits, while K is separable over k by definition. Note that the hypothesis $P^2 = 0$ may, by continuing the Newton approximation, be replaced by the hypothesis that $P^m = 0$ for any finite m, or even by the hypothesis that $\bigcap_{m=1}^{\infty} P^m = 0$ (cf., again, Ch. III, § 2). Assume now that the characteristic, p, is not zero.

LEMMA 8. *Let B be a commutative k-algebra with an ideal P such that $P^2 = 0$ and $B/P \cong K$, and π denote the natural projection of B onto K. Then B^p is a field projecting onto K^p by π, and B splits if and only if kB^p is a field. In that case, if q: $K \rightarrow B$ is an isomorphism such that πq is the identity on K, then $q(kK^p) = kB^p$.*

PROOF. Note that in any case B^p is a field and projects onto K^p. If B splits, then $q(K^p)$ must be just B^p, whence $q(kK^p) = kB^p$, showing that the latter is a field. Conversely, if kB^p is a field, then setting $kK^p = L$, there exists an isomorphism q: $L \rightarrow B$ with πq equal to the identity on L, whence applying Lemma 7, (iii), and observing that K may be obtained from L by a succession of transcendental or separable algebraic extensions, it follows that B splits.

As before, the hypothesis that $P^2 = 0$ can be replaced by $\bigcap P^m = 0$.

THEOREM 3. *Let K be an extension of a field k. Then K is separable over k if and only if $H_c^2(K, K) = 0$; and when that is so, $H_c^2(K, P) = 0$ for all K-modules P.*

PROOF. In the case of characteristic zero, K is always separable, while every commutative extension of K by an ideal with square zero has been shown to split, whence $H_c^2(K, P) = 0$ for all K-modules P. The theorem is therefore true in this case, and we assume now that the characteristic, p, is different from zero. Suppose that K is separable over k. To show that $H_c^2(K, P) = 0$ for all P, it is sufficient to show that every commutative k-algebra B which is an extension of K by an ideal P with square zero splits, and by the preceding lemma, this is equivalent to kB^p being a field. But there is a natural homomorphism of $k \otimes_{k^p} K^p$ onto kB^p, and as K is separable over k, the former is a field by Lemma 6, whence so is the latter. Conversely, if K is not separable over k, then the extension exhibited in the remarks following Lemma 6 can not split by Lemma 8, and therefore represents a non-zero element of $H_c^2(K, K)$. This ends the proof.

COROLLARY 1 (Harrison [4]). *If K is an extension of a perfect field k, then considering K as an algebra over k, one has $H_c^2(K, P) = 0$ for all K-modules P.*

PROOF. Since $k^{1/p} = k$, any extension of k is separable.

COROLLARY 2. *If K is separable over k, then considered as an algebra* *over k, K is rigid in the commutative theory; in particular, K is always rigid as an algebra over its prime field.*

COROLLARY 3. *Let K be separable over k, and φ be a derivation of K over k into itself. Then there exists an automorphism over k of $K[[t]]$ of the form $a \to a + t\varphi(a) + t^2\varphi_2(a) + \cdots$.*

PROOF. Since $H_c^2(K, K) = 0$, all obstructions to the integration of a k-derivation of K into itself vanish. As to the condition that the automorphism leave k fixed, this is equivalent to having $\varphi_\nu(1) = 0$ for all ν. This is true for $\nu = 1$ by hypothesis, and making the inductive assumption that it is true for all $\mu < \nu$, § 4, (9_ν) shows that $\delta\varphi_\nu(1, 1) = \varphi_\nu(1) = 0$, ending the proof.

COROLLARY 4. *Let K be an extension of a field k of characteristic p, and let K be considered as an algebra over k. Then every non-zero element of $H_c^2(K, K)$ can be represented by a homomorphic image of $k \otimes_{k^p} K$.*

PROOF. It must be shown that if B is a commutative k-algebra which is an extension of K by an ideal P isomorphic as a K-module to K with $P^2 = 0$, and if B does not split, then B is a homomorphic image of $k \otimes_{k^p} K$. Considered as an algebra over the prime field of characteristic p (which is perfect) B splits, whence B contains a subfield K' (which is not a k-subalgebra) projecting onto K by the natural homomorphism $\pi: B \to K$. Then $(K')^p = B^p$ and contains k^p (considered as a subfield of K) whence B indeed contains a homomorphic image of $k \otimes_{k^p} K'$, or what is the same thing, of $k \otimes_{k^p} K$. This image is not a field since it contains kB^p which is not a field by Lemma 6, and therefore is of dimension at least 2 considered as an algebra over K'. Since B is only of dimension 2 over K' (being in effect an extension of K by K), the image is all of B, ending the proof.

The algebra $k \otimes_{k^p} K$ is in a respect analogous to the regular representation of a finite group; the former contains all non-trivial extensions of K by K, the latter all irreducible representations. It is conjectured that, if K is not separable over k, then indeed it is not rigid, and if it is further finitely generated, then an integrable element of $H_c^2(K, K)$ will be found in the image of Sq_p.

CHAPTER II. THE PARAMETER SPACE

1. The set of structure constants as parameter space for the deformation theory

In this chapter we consider algebras of a fixed dimension over a field and, in the present section, give an intrinsic definition of the algebraic set of all structure constants of n dimensional associative algebras. This set will serve as parameter space for the deformation theory and after introducing it, we shall take the point of view that the objects being deformed are not merely algebras, but essentially algebras with a fixed basis. The points of the parameter space clearly fail to be in one-one correspondence with the isomorphism classes of algebras of the fixed dimension, but the general linear group of that dimension operates on the parameter space, identifying points corresponding to isomorphic algebras. When considering a restricted deformation theory, it may be desirable to take, as parameter space, a subspace of the space of all structure constants on which further certain identifications may be made.

A characteristic being fixed, let $\mathbf{F}$ be the prime field in the given characteristic, Ω be a universal domain over $\mathbf{F}$ in the sense of Weil [11], and x_ι, $\iota \in I$, be a set of objects indexed by a set I of cardinality n. Then the collection of formal finite linear combinations $\sum \alpha_\iota x_\iota$ of elements x_ι with coefficients α in $\mathbf{F}$ form a vector space V over $\mathbf{F}$. Let A be an algebra of dimension n over a field k (where by the Weil conventions k is a subfield of Ω, and Ω is algebraically closed and has infinite transcendence degree over k). Then there exist linear isomorphisms φ of $V \otimes_{\mathbf{F}} k$ onto A, i.e., linear maps such that every element of A is uniquely expressible in the form $\sum \alpha_\iota \varphi(x_\iota)$, $\alpha_\iota \in k$. Let S be the set of all pairs (A, φ) where A is an algebra of the fixed dimension n over some field k and φ: $V \to A$ is such a map. We will denote by V_Ω and A_Ω, respectively, the vector space $V \otimes_{\mathbf{F}} \Omega$ and the algebra $A \otimes_k \Omega$. (If V is a vector space defined over a field k_0 and K is an extension of k, then we shall generally denote by V_k the space $V \otimes_{k_0} k$, and similarly for algebras.) Then φ has a natural extension to a linear isomorphism of V_Ω onto A_Ω. The extension will continue to be denoted by φ. We consider two elements (A, φ), (A', φ') of S to be equivalent if $\varphi'\varphi^{-1}$: $A_\Omega \to A'_\Omega$ is an algebra isomorphism of A_Ω onto A'_Ω. (Instead of Ω we could have used any subfield of Ω over which A and A' were both defined.) The points of the parameter space C are the equivalence classes of S under this relation.

Now in the pair (A, φ), it is clear that the mapping φ serves to pick a basis of A, namely $\varphi(x_\iota)$. Further, if ι, κ, θ are in the indexing set I, then since A is defined over k, there exist "structure constants" $c_{\iota\kappa\theta}$ in k (relative to the basis $\varphi(x_\iota)$) such that $\varphi(x_\iota)\varphi(x_\kappa) = \sum c_{\iota\kappa\theta}\varphi(x_\theta)$. It follows that (A', φ') is equivalent to (A, φ) if and only if the structure constants $c'_{\iota\kappa\theta}$ of A' relative to the basis $\varphi'(x_\iota)$ are the same as those of A relative to $\varphi(x_\iota)$. The parameter space C is therefore, indeed, just the set of structure constants. Further, if (A, φ) is equivalent to (A', φ'), and A'

is given as an algebra over a field k', then both k and k' contain the field $k_0 = \mathbf{F}(c)$ generated over the prime field by the common set of structure constants $(c) = (c_{\iota\kappa\theta})$. We shall say that the point P of C corresponding to the equivalence class (A, φ) has k_0 as a minimal field of definition, and that it is defined or rational over any extension of k_0. We shall denote the minimal field of definition by $k_0(P)$, and if k is any subfield of Ω, then the compositum of k and $k_0(P)$ will be denoted by $k(P)$. There is an algebra in the class P defined over any extension k of k_0, namely that obtained by defining on V_k the multiplication $x_\iota x_\kappa = \sum c_{\iota\kappa\theta} x_\theta$. Fixing a field k, let C_k denote the set of points of C which are rational over k. To every point of C_k there corresponds in a natural way a unique isomorphism class of n dimensional algebras over k, namely, given a point, let (A, φ) be an element of the equivalence class it represents such that A is an algebra over k. Then A is completely determined up to isomorphism. In particular, if $P \in C$ and k_0 is the minimal field of definition for P, then to P corresponds a unique isomorphism class of algebras over k_0. This class contains the algebra obtained by defining on V_{k_0} the multiplication whose structure constants are those associated with the point P. The latter algebra will be denoted by $A(P)$.

Given a field K, let $G(K)$ denote the group of all linear automorphisms of $V_K = V \otimes_{\mathbf{F}} K$. This group operates on C in the following way. Let ψ be an element of $G(K)$. If a point P in C is the equivalence class of a pair (A, φ), where A is an algebra over k, set $L = Kk$, let φ be extended to a linear isomorphism of V_L onto A_L, and ψ be extended to a linear automorphism of V_L. Then $\varphi\psi^{-1}$ is also a linear isomorphism of V_L onto A_L, and we define ψP to be the equivalence class of $(A_L, \varphi\psi^{-1})$. It is clear that this does not depend on the choice of (A, φ) in the equivalence class P. Further, $G(K)$ carries C_K onto itself. If $P \in C_K$, then the isotropy group in $G(K)$ of the point P (i.e., the group of all ψ in $G(K)$ such that $\psi P = P$) may be identified with the automorphism group of any algebra in the isomorphism class of algebras over K represented by P. Since $G(K)$ operates on C_K, the quotient $C_K/G(K)$ is defined, and this set of equivalence classes is clearly in one-one correspondence with the isomorphism classes of algebras over K of dimension n. However, this quotient may not carry any reasonable structure. On the other hand, since the condition that an n dimensional algebra defined by a set of structure constants (c) be associative is expressible by the vanishing of certain quadratic polynomials in these constants, for finite n, the set C is an algebraic set (in fact a bunch of varieties, in the sense of Weil [11], normal over the prime field $\mathbf{F}$) in a space of dimension n^3.

Note that if an algebra A has structure constants $(c) = (c_{\iota\kappa\theta})$ relative to some choice of basis, then multiplying the basis elements by a common constant replaces the structure constants by $(\lambda c_{\iota\kappa\theta}) = (\lambda c)$, λ a non-zero constant. The parameter space $\mathcal{C}$ is therefore a cone and gives rise to an algebraic set in projective space of dimension $n^3 - 1$. Note that the function $F_1(a, b) = ab$, namely, the multiplication in A, is an element of $Z^2(A, A)$ and is in fact in $B^2(A, A)$, being the coboundary of the identity mapping of A onto itself. It follows that F_1 is integrable; the replacement of (c) by (λc) may be obtained by integrating F_1.

It is of interest at this point to compare the construction of the parameter set in a special case of the analytic theory with the construction given here. Teichmueller's definition of the parameter space for the set of compact Riemann surfaces of a fixed genus g, translated into reasonable mathematical terms, reads as follows. Let S be a fixed compact oriented topological surface of genus g, and $\mathcal{S}$ be the collection of all pairs (R, φ) where R is a compact Riemann surface of genus g, and $\varphi: S \rightarrow R$ is an orientation-preserving homeomorphism of S onto R. Pairs (R, φ) and (R', φ') will be considered equivalent if $\varphi'\varphi^{-1}: R \rightarrow R'$ is homotopic to a conformal map. The set $\mathcal{T}$ of equivalence classes is the parameter space of the Teichmueller theory. The group G of orientation-preserving homeomorphisms of S onto itself operates on $\mathcal{T}$ so: If $\psi \in G$ and P in $\mathcal{T}$ is represented by (R, φ), then ψP is the point represented by $(R, \varphi\psi^{-1})$. (The normal subgroup G_0 of G consisting of those ψ homotopic to the identity leaves every P fixed, so that here in fact the discrete group G/G_0 operates on $\mathcal{T}$.) This definition gives, of course, only the set of points of $\mathcal{T}$, not its deeper analytic structure, either local or global. For these see [1].

Given points P and P' of $\mathcal{C}$ associated with which are, respectively, sets (c) and (c') of structure constants, we shall say that P' is a specialization of P relative to a field k if (c') is a specialization of (c) relative to k. (In the infinite dimensional case, (c') is a specialization of (c) if every finite subset is a specialization of the corresponding subset of (c).) We may similarly define generic specializations. Suppose P and P' are generic specializations of each other. Then their minimal fields of definition, $k = \mathbf{F}(c)$ and $k' = \mathbf{F}(c')$ are isomorphic, but the algebras $A(P)$ and $A(P')$ generally are not because they are defined over different fields. However, there does exist an isomorphism σ of k onto k' (namely that induced by the specialization) and a ring isomorphism $\bar{\sigma}$ of $A(P)$ onto $A(P')$ such that if $\lambda \in k$, $a \in A$, then $\bar{\sigma}(\lambda a) = \sigma(\lambda)\bar{\sigma}(a)$. Given a pair of algebras over isomorphic fields which stand in the latter relation to each other, we shall say that they are quasi-isomorphic, and that the pair $(\sigma, \bar{\sigma})$ is a quasi-isomorphism. (Quasi-isomorphic algebras are isomorphic as algebras over the prime field.)

Returning once more to the first point in our outline, we see that the objects being deformed are not merely algebras but elements of $\mathcal{C}$. Suppose given an element P of $\mathcal{C}$ defined over a field k, and let (A, φ) be an element in the equivalence class P, where A is an algebra over k. Let A_t be the generic element of a one-parameter family of deformations of A. The mapping φ defines a basis of A, namely $\{\varphi(x_\iota)\}$, and if we consider the underlying vector space of A as contained in that of A_t, this continues to be a basis of A_t. Let $(c(t))$ be the set of structure constants of A_t relative to this basis, and L be the field generated over the prime field $\mathbf{F}$ by the $c(t)$. Let B be the linear space over L spanned by the elements $\varphi(x_\iota)$ of A_t. Then B is an algebra over L and $A_t \cong B \otimes_L k((t))$. If the dimension n is finite, then L is isomorphic to some subfield K of Ω, where $K \supset k$, and this will be true in every case as long as we assume that the transcendence degree of Ω over the prime field $\mathbf{F}$ is at least n^3. (Note that Ω is assumed algebraically closed.) It follows that B is quasi-isomorphic to an algebra $\bar{A}$ over K. The image under the quasi-isomorphism of the basis $\varphi(x_\iota)$ of B will be a basis of $\bar{A}$ and determines uniquely a linear isomorphism $\bar{\varphi}$ of V_K onto $\bar{A}$. The equivalence class of $(\bar{A}, \bar{\varphi})$ is some point $\bar{P}$ of $\mathcal{C}$ which is not uniquely determined by P and the one-parameter family, but it is easy to see that any other point $\bar{P}'$ of $\mathcal{C}$ obtainable by the process so described is a generic specialization of $\bar{P}$. It follows that, if A is any algebra over k whose structure constants, relative to some basis, are those associated with P, then a one-parameter family of deformations of A determines uniquely a subvariety W of $\mathcal{C}$ containing P and defined over k, namely that subvariety whose generic point over k is $\bar{P}$. (Note that $k(\bar{P}) = K$ is a regular extension of k because it is isomorphic to L which is a subfield containing k of a regular extension of k, namely $k((t))$.) The following trivial example shows that the transcendence degree of K over k (i.e., if n is finite, the dimension of W) may be greater than one. Let A be the algebra over k generated by elements a, b such that $a^2 = b^2 = ab = ba = 0$, i.e., A is the direct sum of two zero algebras of order one. Fixing the basis a, b determines a linear isomorphism φ: $V_k \to A$, and therefore also a point P of $\mathcal{C}$. Let $A(t)$ be the algebra over $k((t))$ with the same basis a, b but with multiplication defined by $ab = ba = 0$, $a^2 = t$, $b^2 = \gamma(t)$, where γ is a power series in t transcendental over $k(t)$. It is then trivial that $\dim W = 2$.

In the trivial case where the product of basis elements $\varphi(x_\iota)$ in $A(t)$ is defined to be just their product in A, we have $\dim W = 0$, i.e., W is reduced to P. In all other cases dim $W \geqq 1$, and we will say of the points in W that they are "obtainable from P by continuous deformation". Note that we might have $A(P) \cong A(P')$ for some point $P \neq P'$. On the other hand, if $(A(P), \varphi)$ is in the equivalence class P and $(A(P'), \varphi')$ in that of P', then $\varphi'\varphi^{-1}$ is not an isomorphism.

Suppose now that P is a point of C rational over some field k, and $\bar{P}$ a point of C having P as a specialization over k and such that $k(\bar{P})$ is a regular extension of k. We may regard P and $\bar{P}$ as sets of structure constants (c), $(\bar{c})$, respectively. Suppose further that $k[\bar{c}]$ is embeddable in the power series ring $k[[t]]$ in such a way that the place $t \rightarrow 0$ of $k[[t]]$ induces on $k[\bar{c}]$ the place $k[\bar{c}] \rightarrow k[c]$. Then it is easy to see that there exists a one-parameter family of deformations of $A(P)$ such that the subvariety W of C associated with this family is just that subvariety, defined over k, whose generic point is $\bar{P}$. In particular, if the dimension (transcendence degree) of $k(\bar{P})$ over k is one, then the necessary embedding of $k[\bar{c}]$ in $k[[t]]$ is always possible. It follows that, if P is rational over a field k and W a one-dimensional subvariety of C defined over k and containing P, then the points of W are obtainable from P by continuous deformation. Now suppose that the dimension n is finite, so that C is an algebraic set in some space of dimension n^3. Then given any point P' of C in the same component of C as P, there exists a one-dimensional subvariety of C defined over a field of definition k for P and containing P'. Therefore, in the finite dimensional case, any point in a given component C_i of C is continuously deformable into any other, which is precisely what one would have expected under any reasonable definition of deformation.

At this point it is possible to describe in greater detail one of the principal open questions of the theory. Let P be a point of C rational over a field k, and let T_P be the linear variety attached to C at P. If (A, φ) is in the equivalence class P, and A is an algebra over k, then every element of T_P may be considered as a bilinear function from $A_\Omega \times A_\Omega$ to A_Ω, and it is trivial that this bilinear function is an element of $Z^2(A_\Omega, A_\Omega)$. This linear space may be identified with $Z^2(A, A)_\Omega$ by means of the basis selected in A by φ, and the elements of T_P which are rational over k may be identified with $Z^2(A, A)$. The elements of the linear variety attached to C at a point P may therefore be interpreted as infinitesimal deformations. It is an open question whether given a pair (A, φ), every element of $Z^2(A, A)$ actually is an element of T_P, where P is the equivalence class of (A, φ). For many purposes it would be sufficient to know that this is the case when P is a simple point of some component C_i of C (the dimension in this case being assumed to be finite).

If P is simple on C_i, then C_i has a tangent linear variety T at P, and given any element of that variety, i.e., given any tangent vector to C_i at P, there exists a one-dimensional subvariety of C_i passing through P and tangent at P to that tangent vector. It is easy to see that this implies that the given tangent vector is integrable as an element of $Z^2(A, A)$. If P lies on a unique component of C and is simple, then it would follow that every element of $Z^2(A, A)$ is integrable. The obstruction theory would then be concerned only with the singularities of C. It would further follow that an algebra A is rigid if and only if $H^2(A, A) = 0$. This suggests (if the answer to the open question is in the affirmative) that, if $H^2(A, A) \neq 0$, then there should be some formal means to exhibit an integrable element of $H^2(A,A)$. It will be seen that the corresponding question for $H^1(A, A)$ has a trivial answer. In characteristic zero every element of $H^1(A, A)$ is integrable, and in characteristic $p \neq 0$ there are elements of $H^1(A, A)$ which do not correspond to tangent vectors of the automorphism group. Note, however, the fundamental fact that the dimension of $Z^2(A, A)$ is at least equal to the dimension of the linear variety attached to C at P. Since every element of $B^2(A, A)$ is integrable, it follows that, if $H^2(A, A) = 0$, then P is a simple point of C, in particular, it is contained in only one component of C. Further, if P' is any other point on this component, and (A', φ') an element of the equivalence class P' where A' is defined over the same field as A, then A' is isomorphic to A.

2. Central algebras and an example justifying the choice of parameter space

It is clear that all the foregoing considerations will hold also for a restricted deformation theory as long as the associated parameter space is still an algebraic set. This is the case for the commutative deformation theory, and in the finite dimensional case, for the nilpotent theory.

The reason for taking as objects of a deformation theory the points of C, which are roughly algebras with fixed bases, rather than the algebras themselves, is that "deformation" of an algebra will usually imply an extension of the field over which the algebra is defined. Now if an algebra A over a field k is given, and K is an extension of k, then A is generally not equivalent in any reasonable sense to $A_K = A \otimes_k K$, for we may have non-isomorphic algebras A and A' over the same field k such that A_K is isomorphic to A'_K. This is a familiar fact in the study of semi-simple algebras. To emphasize it further, we show that it is possible for nilpotent algebras as well. To give an example, it is convenient to introduce first the general notion of an algebra *central* over a field k; A is central over

k if k is the unique (up to isomorphism) maximal field over which A can be considered an algebra. (In general, there may be no field over which a given algebra is central.)

Let A be an algebra over a field k; A is indecomposable if it is not a direct sum $A_1 + A_2$ of subalgebras over k. Suppose now that K', K'' are extensions of k, and that each is representable by linear transformations T on A in such a way that elements of k are represented by multiplication by themselves and such that, if T is any transformation of the representation, then $T(ab) = (Ta)b = a(Tb)$ for all a, b in A, i.e., suppose that A is also a K' and a K'' algebra over k. Then $K' \otimes_k K''$ also operates on A. Suppose that A is finite dimensional over k. Then so are K' and K''. Further, if k is perfect or of characteristic zero, then $K' \otimes_k K''$ is a direct sum of fields. If A is indecomposable, then all but one of the direct summands of $K' \otimes_k K''$ act as a set of zero operators on A, and A is an algebra over the remaining one, which is some compositum of K' and K''. We have, therefore,

LEMMA 1. *Let A be a finite dimensional indecomposable algebra over a field k which is either perfect or of characteristic zero. Then there is an extension of k, unique up to isomorphism, over which A is central.*

Suppose now that A is a finite dimensional indecomposable algebra over a field k which is either perfect or of characteristic zero. Then the extension K of k whose existence is asserted by the lemma will be called the excenter of A. If A' is isomorphic to A over K, then the excenter of A is isomorphic to that of A'. It is easy now to give examples of nilpotent algebras over the rationals, Q, which are not isomorphic over Q, but which are isomorphic over some extension. Observe that, if B is any algebra over a field k, then the vector space direct sum of B with itself may be made into a nilpotent algebra of index 3 over k by setting $(a, b)(c, d) = (0, ac)$. Denote this algebra by $\bar{B}$. Now let $k = Q$, the rationals, set $B = Q(i)$, $i = \sqrt{-1}$, set $A = \bar{B}$, a four-dimensional algebra of index 3 over Q, set $B' = Q(\sqrt{2})$, and $A' = \bar{B}'$. Let K be any extension of Q containing both i and $\sqrt{2}$. Then B_K is isomorphic to B'_K, whence A_K is isomorphic to A'_K. On the other hand, A is not isomorphic to A', for A and A' are indecomposable, and it is trivial that $Q(i)$ is the excenter of A and $Q(\sqrt{2})$ is the excenter of A'. Their excenters being non-isomorphic, A and A' are likewise.

The foregoing example shows that the difficulties attendant on extension of the ground field are present even for nilpotent algebras. Suppose now that A and A' are algebras over a field k and that K is an extension of k. If bases in A and A' are fixed and the linear transformation of A_K onto

A'_K obtained by mapping the given basis of A into that of A' and extending linearly is an isomorphism, then indeed A is isomorphic to A' over k. Therefore, if a basis is fixed, we may consider A as equivalent to $A \otimes_k K$.

3. The automorphism group as a parameter space, and examples of obstructions to derivations

We consider now briefly, in a fashion similar to that of § 1, the space of automorphisms of an algebra A defined over a field k. The generic element Φ_t of a one-parameter family of automorphisms of A is not an automorphism of A but of $A \otimes_k k((t))$. Letting a basis of A be fixed, if the dimension of A is n, then an automorphism σ of A is determined by a set of n^2 elements of k (the coefficients in the matrix of σ relative to the given basis) and these satisfy certain polynomial relations which may be taken to define an algebraic set, $\mathrm{Aut}(A)$. The points in the latter set are considered as having coefficients in Ω. Those which are rational over a field $K \supset k$ form a subgroup, $\mathrm{Aut}_K(A)$, which is isomorphic to the group of automorphisms of A_K. If the dimension of Ω over the prime field $\mathbf{F}$ is sufficiently large, then the automorphism Φ_t of $A \otimes_k k((t))$ determines a collection of points $\bar{P}$ of $\mathrm{Aut}(A)$ which are all generic specializations of each other, have the point I of $\mathrm{Aut}(A)$ corresponding to the identity automorphism of A as a specialization, and have the property that $k(\bar{P})$ is regular over k. Therefore, as in the case for deformations, a one-parameter family of automorphisms of A determines uniquely, once a basis for A is fixed, a subvariety W of $\mathrm{Aut}(A)$ which contains the identity. If Φ_t had been taken of the form $\varphi_0 + t\varphi_1 + t^2\varphi_2 + \cdots$, then φ_0 would be an automorphism of A, we would be considering one-parameter families whose value for $t = 0$ is φ_0, and the variety W in question would contain φ_0 instead of (or possibly together with) the identity. We will say that we can pass continuously from φ_0 to any element of W. As before, any one-dimensional subvariety of $\mathrm{Aut}(A)$ containing φ_0 can be induced by a one-parameter family, and we can, in the finite dimensional case, pass continuously from any point in some component of $\mathrm{Aut}(A)$ to any other in the same component. (Again, in general, the dimension of the subvariety W of $\mathrm{Aut}(A)$ determined by a one-parameter family of automorphisms of A will be greater than one.)

We continue to assume for the moment that the dimension of A is finite. A basis of A being fixed, $\mathrm{Aut}(A)$ is an algebraic matrix group. Every point P belongs to one and only one component and is simple there. If A is an algebra over k, then the tangent vectors at I which are rational over k may be identified with elements of $Z^1(A, A)$. If the characteristic is zero, then conversely every element φ of $Z^1(A, A)$ is a tangent vector at I, as may easily be seen from the fact that φ is integrable with $e^{t\varphi}$ as the generic element of a one-parameter family whose differential is φ. Suppose now that k is a field of characteristic two, and let A be the two-dimensional algebra over k with a unit element and an element η such that $\eta^2 = 0$. (Then A is the group algebra over k of the group Z_2 of two elements.) The linear mapping $\varphi\colon A \to A$ determined by setting $\varphi(1) = 0$, $\varphi(\eta) = 1$, is a derivation which is obstructed, i.e., not integrable, whence not every element of $H^1(A, A)$ actually represents a tangent vector to Aut (A) at I. (For every prime p, the group algebra of Z_p over a field of characteristic p always has an obstructed derivation.)

In the infinite dimensional case, both for deformations and automorphisms, as remarked earlier, a variety defined over a field k may not have very many algebraic points over k (whereas, in the finite dimensional case, it is determined by them). As an example of the difficulties this may cause, let k be of characteristic zero, A be an algebra over k, and φ be a derivation of A into itself. Then $e^{t\varphi} = I + t\varphi + (t^2/2!)\varphi^2 + \cdots$ is the generic element of a one-parameter family of automorphisms of A, and is in fact an automorphism of $A_{k((t))}$. Suppose now that k is the field $\mathbf{C}$ of complex numbers and A is the rational function field $\mathbf{C}(x)$ in one variable over $\mathbf{C}$. This is an algebra over $\mathbf{C}$ of countably infinite dimension. If $g(x)$ is any rational function of x, then $x \to g(x)$ may be extended uniquely to a derivation φ of $\mathbf{C}(x)$. In particular, we may take $g(x) = x^3$. It is then trivial to show that

$$e^{t\varphi} = \sum_{n=0} \binom{2n}{n} \left(\frac{t}{2}\right)^n x^{2n+1} .$$

If the subvariety W of Aut $(\mathbf{C}(x))$ determined by this one-parameter family had any point algebraic over $\mathbf{C}$ (and therefore rational over $\mathbf{C}$), there would have to exist an element $\alpha \in \mathbf{C}$ such that

$$x \longrightarrow e^{\alpha\varphi}x = \sum \binom{2n}{n} \left(\frac{\alpha}{2}\right)^n x^{2n+1} = f(x)$$

could be extended to an automorphism of $\mathbf{C}(x)$. Now all automorphisms of $\mathbf{C}(x)$ are of the form $x \to (ax + b)/(cx + d)$, and the function f can not be of this form for any $\alpha \neq 0$, as one may see readily by observing that it is an odd function of x. Therefore, the only algebraic point of W is the identity.

4. A fiber space over the parameter space, and the upper semi-continuity theorem

In § 1, the parameter space C was defined to be a collection of pairs (A, φ) consisting of an algebra A and a linear isomorphism $\varphi: V \to A$ of a fixed n-dimensional space V onto A, with certain identifications. It followed that C could be naturally represented as the set of structure constants and was operated upon naturally by $\mathrm{GL}(n, \Omega) = G$. Form now the algebraic set $C \times S^n$ contained in $S^{n^3} \times S^n$. Considering A as an algebra over Ω, we may identify V with S^n and let ψ in G operate on $C \times S^n$ by setting $(c, x)\psi = (c\psi, x\psi)$. It is convenient to consider $C \times S^n$ as a fiber space over C the fiber over a point (c) being the algebra (over Ω) whose structure constants are (c), for if (c) represents (A, φ) then we may identify the (x) in (c, x) with $(x\varphi)$ in A.

An algebraic subset of C invariant under G will be called an algebraic set of algebras, or variety of algebras if it is a variety. Let $\mathcal{B}$ be such a set, and $\mathcal{A}$ be an algebraic subset of $C \times S^n$ invariant under G (considered as operating on $C \times S^n$) with the properties that

(i) the projection of $\mathcal{A}$ on the first factor is $\mathcal{B}$, and

(ii) the intersection of $\mathcal{A}$ with the fiber over a point of $\mathcal{B}$ is a linear space.

Then $\mathcal{A}$ will be said to define an algebraic space over $\mathcal{B}$ or for every algebra in $\mathcal{B}$ (i.e., having a set of structure constants in $\mathcal{B}$). If the intersection is always an ideal of the fiber, the latter being considered as an algebra, then $\mathcal{A}$ will be said to define an algebraic ideal.

It is easy to verify that the union of all points (c, x) in $C \times S^n$ with x in the radical of the algebra represented by (c) is an algebraic subset of $C \times S^n$ and defines an algebraic ideal. Other algebraic spaces are the left and right annihilators of an algebra in itself, and its center. An example of a non-algebraic ideal of an algebra is its square.

Suppose that $\mathcal{A}$ defines an algebraic space over an algebraic set of algebras $\mathcal{B}$. Then for every algebra in $\mathcal{B}$, there is defined an integer, the dimension of the algebraic subspace defined by $\mathcal{A}$. If $\mathcal{B}$ is defined over a field k and (c), (c') are in $\mathcal{B}$ with (c') a specialization of (c) relative to k, then it is trivial that the dimension of the algebraic space defined by $\mathcal{A}$ of the algebra represented by (c') is greater than or equal to that for (c). This is the fundamental upper semi-continuity theorem, here limited to a specific integer-valued function, but the principle may be employed more generally.

As an example of the applicability of this theorem, observe that the dimension of the radical of a finite dimensional algebra is an upper semi-continuous function of the algebra. This implies again the rigidity of a separable semi-simple algebra, for the generic element of a one-parameter family of deformations of such an algebra must, by the foregoing, again be semi-simple; there being only finitely many non-isomorphic semi-simple algebras of given dimension over an algebraically closed field, the generic element of the family must, up to enlargement of the coefficient field, be the same as the original semi-simple algebra. One does not yet obtain in this way, however, the assertion that $H^2(A, A) = 0$ for a separable semi-simple algebra.

5. An example of a restricted theory and the corresponding modular group

The natural parameter set C for the deformation theory is sometimes too large. There are circumstances in which a subvariety or bunch of subvarieties $\mathscr{B}$ will serve, the points of $\mathscr{B}$ representing a given algebra being a zero dimensional set and $\mathscr{B}$ itself being operated on by a zero dimensional group M transforming into each other the points of such a set. In such a case, M is to be considered a modular group. (Even more generally, $\mathscr{B}$ may be a quotient of a subvariety or bunch of subvarieties of C by some group.) Rather than discuss in general the circumstances under which one can expect a discrete modular group, we close this chapter with a discussion of the deformation theory of a rather trivial nilpotent algebra which illustrates, however, the concept of a modular group associated with the deformations of that algebra.

Let A be the algebra of 3×3 matrices with zeros on and below the diagonal. This algebra is of dimension 3 and is nilpotent of index 3. We restrict consideration to those nilpotent algebras of index 3 into which A is continuously deformable. The algebra A has a natural basis $a = e_{12}$, $b = e_{23}$, $c = e_{13}$, and may be considered as an algebra over the prime field $\mathbf{F}$. The infinitesimal deformations in this restricted theory are (cf. I. (10)) those $F_1 \in Z^2(A, A)$ such that

$$(1) \qquad F_1(x, yz) + xF_1(y, z) = 0 , \qquad \text{or} \qquad F_1(z, xy) + zF_1(x, y) = 0 .$$

Since F_1 is a 2-cocycle, we have also

$$(2) \qquad F_1(xy, z) + F_1(x, y)z = 0 .$$

If $xy = 0$, then (1) implies that $F_1(x, y)$ is in the right annihilator of A, and (2) implies that it is in the left annihilator. Together these imply that $F_1(x, y)$ is a multiple of c. It follows that, if x and y are chosen from among the basis elements a, b, c, then only $F_1(a, b)$ can be anything other than a multiple of c. Further, $F_1(c, a)$, $F_1(b, c)$, and $F_1(c, c)$ all vanish.

To see the first of these, set $x = a$, $y = b$, and $z = a$ in (1) and note that a is in the right annihilator of A. The rest follow similarly.

We show next that $H^2(A, A)$ is in fact one-dimensional and that any F_1 in $Z^2(A, A)$ is cohomologous to an F_1' such that $F_1'(x, y) = 0$ whenever x and y are chosen from among the basis a, b, c, except possibly when $x = b$ and $y = a$, and that $F_1'(b, a)$ is a multiple of c. Now F_1' must be of the form $F_1 - \delta\varphi$ for some $\varphi \in C^1(A, A)$, and $\delta\varphi(x, y) = x\varphi(y) - \varphi(xy) + \varphi(y)x$. Therefore, $\delta\varphi(a, a) = a\varphi(a)$, $\delta\varphi(b, b) = \varphi(b)b$, and $\delta\varphi(a, b) = a\varphi(b) - \varphi(c) + \varphi(a)b$. Since $F_1(a, a)$ and $F_1(b, b)$ are multiples of c, we can so choose $\varphi(a)$ and $\varphi(b)$ that $\delta\varphi(a, a) = F_1(a, a)$ and $\delta\varphi(b, b) = F_1(b, b)$, and then we may choose $\varphi(c)$ so that $\delta\varphi(a, b) = F_1(a, b)$. Then $F_1'(a, a) = F_1'(b, b) = F_1'(a, b) = 0$. Setting $x = a$, $y = b$, $z = b$ in (1) gives $F_1'(c, b) = 0$. Similarly, $F_1'(a, c) = 0$. This, together with what has already been shown, proves the assertion.

Now every element F_1' of $Z^2(A, A)$ of the type described is integrable, for the algebra A_t with basis a, b, c and multiplication given by $a^2 = b^2 = c^2 = ac = bc = ca = cb = 0$, $ab = c$, $ba = tc$ is associative. Observe now that A_t is isomorphic to $A_{t'}$ if and only if $t' = t^{-1}$. It will follow, after a suitable definition of the modular group associated with A, that this group is just the group Z_2 of two elements, here exhibited as the group consisting of the identity and the transformation $t \to t^{-1}$. The matter is simple in this case because there has been exhibited a cross section of $Z^2(A, A)$ over $H^2(A, A)$ which is invariant under the automorphisms of A. There are two fixed points of the modular group (except if the characteristic is two, when they coincide). These correspond to the commutative algebra with multiplication $a^2 = b^2 = c^2 = ac = bc = 0$, $ab = ba = c$, and the skew algebra with multiplication $a^2 = b^2 = c^2 = ac = bc = 0$, $ab = -ba = c$.

Chapter III. The Deformation Theory for Graded and Filtered Rings

1. Graded, filtered, and developable rings

Throughout this chapter, by a graded ring A we shall mean a ring which is isomorphic, as an additive group, with a direct *product* $A = A^{(0)} \times A^{(1)} \times \cdots$ of a sequence of groups indexed by the non-negative integers, such that identifying $A^{(m)}$ with its image under the natural inclusion map into A, we have $A^{(m)}A^{(n)} \subset A^{(m+n)}$. A two sided graded module P over A will be one of the form $P = P^{(0)} \times P^{(1)} \times \cdots$ with $A^{(m)}P^{(n)}$, $P^{(m)}A^{(n)} \subset P^{(m+n)}$. The ring $A_d = A^{(d)} \times A^{(d+1)} \times \cdots$ and the module $P_d = P^{(d)} \times P^{(d+1)} \times \cdots$ have natural inclusions into A and P, respectively, and their images will again be denoted by A_d and P_d. We have then the filtrations of A and P associated with their gradings, namely, $A = A_0 \supset A_1 \supset \cdots$ and $P = P_0 \supset P_1 \supset \cdots$.

It may be noted that if A is a ring and P a two-sided A module (which may be A itself), both endowed with topologies such that the usual algebraic operations in A, as well as the module operations of A on P, are continuous, then we may define, in analogy with the usual Hochschild theory, the groups of continuous cochains and cocycles of A with coefficients in P. Since the boundary of a continuous cochain will obviously be continuous, we can define the continuous cohomology groups of A with coefficients in P. When the topology is understood, these groups may be denoted by the usual symbols $C^m(A, P)$, $Z^m(A, P)$, $B^m(A, P)$, and $H^m(A, P) = Z^m(A, P)/B^m(A, P)$.

If $A = A^{(0)} \times A^{(1)} \times \cdots$ and $P = P^{(0)} \times P^{(1)} \times \cdots$ are a graded ring and a two-sided module over that ring, respectively, then taking in each $A^{(m)}$ and $P^{(n)}$ any prescribed topology (here the discrete topology), A and P are endowed with product topologies, and the continuous cohomology groups of A with coefficients in P are defined. Elements of A and of P may be written as infinite sums of elements in the $A^{(m)}$ and $P^{(n)}$, these converging in the product topolopy. Assuming in each $A^{(m)}$ and $P^{(n)}$ the discrete topology, an r-cochain F of A with coefficients in P is continuous (in the product topology) if and only if for every integer $n \geqq 0$ there exists an $N \geqq 0$ such that $F(A^{(m_1)} \times A^{(m_2)} \times \cdots \times A^{(m_r)}) \subset P_n$ whenever $\sum m_i \geqq N$. If there exists a d such that $F(A^{(m_1)} \times \cdots \times A^{(m_r)}) \subset P^{(m+d)}$, where $m = \sum m_i$, then F will be called homogeneous of degree d; d may be negative, in which case we set $P^{(m)} = 0$ for $m < 0$. A homogeneous cochain is continuous.

Any cochain may be expressed uniquely as a sum of homogeneous ones, and will be continuous if the degrees of the non-zero homogeneous parts are bounded below, i.e., $F = \sum_{d \geqq 0} F^{(d)}$, $F^{(d)}$ homogeneous of degree d. Letting $C^{r,d}$ denote the group of homogeneous r cochains of degree d, and setting $C_d^r = C^{r,d} \times C^{r,d+1} \times \cdots$, there exists a natural inclusion $C_{d-1}^r \supset C_d^r$, and we have $C^r = \bigcup_d C_d^r$. In particular, therefore, C^r has a natural filtration, and as each C_d^r carries a natural product topology and the inclusions are homeomorphisms, C^r carries a natural topology. If F is homogeneous of degree d then so is its coboundary δF. It follows that we may define similarly $Z^{r,d}$, $B^{r,d}$, Z_d^r, B_d^r, and obtain $Z^r = \bigcup_d Z_d^r$, $B^r = \bigcup_d B_d^r = \delta Z^{r-1}$. Since $B^{r,d}$ is closed in $Z^{r,d}$, both being discrete, it follows that B_d^r is closed in Z_d^r, and hence that B^r is closed in Z^r. Therefore, H^r has naturally the structure of a topological group. Setting $H^{r,d} = Z^{r,d}/B^{r,d}$

and $H_d^r = H^{r,d} \times H^{r,d+1} \times \cdots = Z_d^r/B_d^r$, we have $H^r = \bigcup_d H_d^r$.

Note that if A is a ring and t an indeterminate, then the ring $A[[t]]$ of formal power series in t with coefficients in A is a graded ring whose additive group is isomorphic to the product of a set of copies of A indexed by the non-negative integers. Most of the formal manipulations to follow will imitate precisely those with power series.

A ring with a decreasing filtration is a ring $\mathcal{F}$ together with a descending sequence of ideals $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$ such that $\mathcal{F}_i\mathcal{F}_j \subset \mathcal{F}_{i+j}$. Set $\mathcal{G}^{(i)} = \mathcal{F}_i/\mathcal{F}_{i+1}$, and denote by p_i the natural projection of $\mathcal{F}_i$ onto $\mathcal{G}^{(i)}$. By the associated graded ring of $\mathcal{F}$, we shall mean the ring whose additive group is the direct product $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$ with multiplication defined so: Given $\alpha \in \mathcal{G}^{(i)}$, $\beta \in \mathcal{G}^{(j)}$, let x, y by representatives for α, β in $\mathcal{F}_i, \mathcal{F}_j$, respectively; then $\alpha\beta = p_{i+j}(xy)$.

Suppose that $\bigcap \mathcal{F}_i = 0$. Then $\mathcal{F}$ possesses a natural topology defined by taking as neighborhoods of zero the ideals $\mathcal{F}_i$. If $\mathcal{F}$ is complete in this topology, then $\mathcal{F}$ is called a complete filtered ring. Suppose now that $\mathcal{F}$ is a filtered ring such that for each i there exists an additive mapping q_i of $\mathcal{G}^{(i)}$ into $\mathcal{F}^{(i)}$ such that $p_iq_i(\alpha) = \alpha$ for all α in $\mathcal{G}^{(i)}$. (This will be the case, for example, if $\mathcal{F}$ is an algebra over a field k or an algebra over a ring S with each $\mathcal{G}^{(i)}$ a projective S module.) Then $\mathcal{F}$ will be called a pre-developable ring. If $\mathcal{F}$ is both pre-developable and complete then it will be called developable. The completion of a pre-developable ring is developable.

If $\mathcal{F}$ is pre-developable, then we may define additive maps $T_i: \mathcal{F} \to \mathcal{G}^{(i)}$ by setting $T_0(x) = p_0(x)$, $T_i(x) = p_i(1 - q_{i-1}p_{i-1})(1 - q_{i-2}p_{i-2}) \cdots (1 - q_0p_0)x$. These may be combined to a single map $T: \mathcal{F} \to \mathcal{G}$ by setting $T(x) = (T_0x, T_1x, \cdots)$.

Proposition 1. *If $\mathcal{F}$ is a pre-developable ring, then T is an additive isomorphism and a homeomorphism of $\mathcal{F}$ into $\mathcal{G}$; T is onto if and only if $\mathcal{F}$ is complete,* i.e., *developable.*

Proof. It obviously suffices to assume that $\mathcal{F}$ is complete. Since all the T_i are additive, so is T. As one may readily see by induction on n, if $T_ix = 0$ for $i = 1, \cdots, n$, then x lies in $\mathcal{F}_{n+1}$. Therefore, $Tx = 0$ implies $x \in \bigcap \mathcal{F}_n = 0$. Setting $\mathcal{G}_i = \mathcal{G}^{(i)} \times \mathcal{G}^{(i+1)} \times \cdots$, considered as an ideal of $\mathcal{G}$, one sees that $T\mathcal{F}_i \subset \mathcal{G}_i$ and $T^{-1}\mathcal{G}_i \subset \mathcal{F}_i$. Therefore T is a homeomorphism. The burden of the assertion is that, when $\mathcal{F}$ is complete, T is onto. Suppose now that $y = y_i \in \mathcal{G}_i$. Then there exists an $x = x_i \in F_i$ such that $Tx_i = y_i \bmod \mathcal{G}_{i+1}$, namely, $x_i = q_iy_i$. Define $y_{i+1} = y_i - Tx_i$. Then given $y = y_0$ in $\mathcal{G}$, set $x = x_0 + x_1 + \cdots$. Since x_i lies in $\mathcal{F}_i$ the series converges because of the assumed completeness of $\mathcal{F}$, and $Tx = y$. This ends the proof.

When $\mathcal{F}$ is complete, we shall in the future refer to the system of additive mappings q_i, or equivalently, the additive isomorphism T, as a development of $\mathcal{F}$.

2. The Hochschild theory for developable rings

The classical cohomology theory of Hochschild, in its original formulation [8], is concerned exclusively with finite dimensional algebras over a field, and with modules over such algebras which are finite dimensional vector spaces. If B is such an algebra over a field k, containing an ideal I, and if we set $A = B/I$, then it is said that B is an extension of A by I. Letting p denote the natural projection of B onto A, the extension is *split* if there exists an isomorphism q of A into B such that pq (q followed by p) is the identity. (Hochschild called such extensions "segregated".) In this case B is a vector space direct sum, $B = q(A) + I$, and I is a $q(A)$, and hence in a natural way, an A module. It is a fundamental theorem that A is segregated in every extension if and only if $H^2(A, P) = 0$ for all two-sided A modules P.

The foregoing theorem of Hochschild fails as soon as one considers an extension of A by I in which I may be infinite dimensional over k. We give a simple example (adapted from one due to Goldman). Observe first that, if k is considered as a one-dimensional algebra over itself, then $H^2(k, P) = 0$ for every vector space P over k. Now take $A = k$, let $\{x_\alpha\}$ be a set of indeterminates indexed by the non-zero elements α of k, and let B be the ring of polynomials in the x_α with coefficients in k but without constant terms. There is a natural homomorphism of B onto A defined by setting $p(x_\alpha) = \alpha$, but B contains no isomorphic copy of k since in fact it contains no idempotents. (The algebra B need not be very large in this example. If, say, k is the field $\mathbf{F}_2$ of two elements, then B is just $x\mathbf{F}_2[x]$.)

The proof of the theorem of Hochschild divides essentially into two steps; the first, a reduction to the case in which I is nilpotent, based essentially on the fact that in the contrary case I contains an idempotent effectively permitting a reduction in the dimension of B; and the second a consideration of the nilpotent case. The first step is not readily extendable to the infinite dimensional case, while the second may be carried out almost verbatim for developable rings.

THEOREM 1. *Let $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$ be a developable ring, $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \mathcal{G}^{(2)} \times \cdots$ be its associated graded ring, and $p_i\colon \mathcal{F}_i \to \mathcal{G}^{(i)} = \mathcal{F}_i/\mathcal{F}_{i+1}$, $i = 0, 1, 2, \cdots$, be the natural projections. If $H^2(\mathcal{G}^{(0)}, \mathcal{G}^{(\lambda)}) = 0$ for $\lambda = 1, 2, \cdots$, then there exists an isomorphism q of $\mathcal{G}^{(0)}$ into $\mathcal{F}$ such* *that p_0q is the identity.* (We say then that $\mathcal{G}^{(0)}$ is segregated in $\mathcal{F}$ or that it has a ring of representatives.)

PROOF. By the hypotheses $\bigcap \mathcal{F}_\lambda = 0$, $\mathcal{F}$ is complete in the natural topology, and there exist additive mappings $q_i\colon \mathcal{G}^{(i)} \to \mathcal{F}_i$ such that $p_iq_i = I$. Suppose $a, b \in \mathcal{G}^{(0)}$. Then $q_0(a)q_0(b) - q_0(ab)$ is in $\mathcal{F}_1$ and therefore, setting $F(a, b) = p_1[q_0(a)q_0(b) - q_0(ab)]$ defines an element of $C^2(\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$. It is easy to see that F is in fact a 2-cocycle. Nothing that for a in $\mathcal{G}^{(0)}$, x in $\mathcal{G}^{(1)}$, ax is defined as $p_1(q_0(a)q_1(x))$, we have, $\delta F(a, b, c) = aF(b, c) - F(ab, c) + F(a, bc) - F(a, b)c = p_1\{q_0(a)[q_0(b)q_0(c) - q_0(bc)] - [q_0(ab)q_0(c) - q_0(abc)] + [q_0(a)q_0(bc) - q_0(abc)] - [q_0(a)q_0(b) - q_0(ab)]q_0(c)\} = p_1(0) = 0$.

By hypothesis, there exists an element $\varphi \in C^1(\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ such that $F = \delta\varphi$. Now $q_1\varphi$ is an additive mapping of $\mathcal{G}^{(0)}$ into $\mathcal{F}_1$, whence setting $q_0' = q_0 - q_1\varphi$, we have $p_0q_0' = p_0q_0 = I$. The fact that $F = \delta\varphi$ implies immediately that $q_0(a)q_0'(b) - q_0'(ab) \in \mathcal{F}_2$ for all $a, b \in \mathcal{G}^{(0)}$, and from this one has that $F'(a, b) = p_2[q_0'(a)q_0'(b) - q_0'(ab)]$ is an element of $Z^2(\mathcal{G}^{(0)}, \mathcal{G}^{(2)})$. Therefore, $F'(a, b) = \delta\varphi'$ for some $\varphi' \in C^2(\mathcal{G}^{(0)}, \mathcal{G}^{(2)})$, and we may proceed to define $q_0'' = q_0 - q_1\varphi - q_2\varphi'$. Continuing in this way, there exist elements $\varphi^{(i)}$ of $C^1(\mathcal{G}^{(0)}, \mathcal{G}^{(i)})$, $i = 1, 2, \cdots$, such that setting $q_0^{(i)} = q_0 - q_1\varphi - q_2\varphi' - \cdots - q_{i-1}\varphi^{(i-1)}$ we have $q_0^{(i)}(a)q_0^{(i)}(b)$ congruent $q_0^{(i)}(ab)$ modulo $\mathcal{F}_{i+1}$ for a, b in $\mathcal{G}^{(0)}$. Since $\mathcal{F}$ is complete, the sequence of mappings $q_0^{(i)}$ in fact converges to some additive map q of $\mathcal{G}^{(0)}$ into $\mathcal{F}$ and this q is an isomorphism of $\mathcal{G}^{(0)}$ into $\mathcal{F}$ with $p_0q = I$. This ends the proof.

It is important to observe that the foregoing considerations can be carried over to suitable restricted theories, most significantly, to the commutative theory. By following precisely the proof of Hochschild, one may establish

PROPOSITION 2. *Let A be a finite dimensional commutative algebra. Then A is segregated in every finite dimensional commutative extension if and only if $H_c^2(A, P) = 0$ for every finite dimensional commutative A module P.*

Similarly, in the present case we have,

THEOREM 2. *Let $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ be a commutative developable ring with associated graded ring $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$. If $H_c^2(\mathcal{G}^{(0)}, \mathcal{G}^{(\lambda)}) = 0$ for $\lambda = 1, 2, \cdots$, then $\mathcal{G}^{(0)}$ is segregated in $\mathcal{F}$.*

While we do not consider here the analogs of this theorem for other restricted theories, they are important and will be discussed elsewhere.

3. Developable rings as deformations of their associated graded rings

Let $\mathcal{F}$ be a developable ring with associated graded ring $\mathcal{G}$, and a development $T\colon \mathcal{F} \to \mathcal{G}$ be given. The development $Tx = (T_0x, T_1x, \cdots)$ of an element x of $\mathcal{F}$ is analogous to a formal power series representation of x. Extending the theory of the previous section, we measure the deviation of T from being a multiplicative isomorphism. Suppose that $x_i \in \mathcal{F}_i$, $x_j \in \mathcal{F}_j$ are given. Then $T(x_i x_j) - Tx_i \cdot Tx_j$ lies in $\mathcal{G}_{i+j+1}$. If $\mathcal{F}$ is a module over a commutative ring S with unit, and all additive mappings are assumed to be module homomorphisms, then this defines a module homomorphism of $\mathcal{F}_i \otimes_S \mathcal{F}_j$ into $\mathcal{G}_{i+j+1}$. Letting π_λ denote the projection of $\mathcal{G}_\lambda$ onto $\mathcal{G}^{(\lambda)}$, the latter being considered as contained in $\mathcal{G}$, we may now define homogeneous module homomorphisms $\bar{F}_\lambda$ of degree λ, $\lambda = 1, 2, \cdots$, of $\mathcal{F} \otimes \mathcal{F}$ into $\mathcal{G}$ by setting for x_i in $\mathcal{F}_i$, x_j in $\mathcal{F}_j$,

$$\bar{F}_\lambda(x_i \otimes x_j) = \pi_{i+j+\lambda}[T(x_i x_j) - Tx_i \cdot Tx_j] .$$

(The definition is then extended linearly.) We may now write,

$$T(x \otimes y) = Tx \cdot Ty + \bar{F}_1(x \otimes y) + \bar{F}_2(x \otimes y) + \cdots .$$

Defining $\bar{F}_0$ by setting $\bar{F}_0(x \otimes y) = Tx \cdot Ty$, one then has $\bar{F}_\lambda \in C^{2,\lambda}(\mathcal{F}, \mathcal{G})$, $\lambda = 0, 1, 2, \cdots$. The mapping T being an additive isomorphism, we may finally define cochains $F_\lambda \in C^{2,\lambda}(\mathcal{G}, \mathcal{G})$ by setting, for a, b in $\mathcal{G}$, $F_\lambda(a \otimes b) = \bar{F}_\lambda(T^{-1}a \cdot T^{-1}b)$. Then $F_0(a \otimes b)$ is just ab, and the associativity of the multiplication in $\mathcal{F}$ gives, exactly as in Ch. I, § 1,

$$(1_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda(F_\mu(a, b), c) - F_\lambda(a, F_\mu(b, c)) = \delta F_\nu(a, b, c) .$$

(One may note that since the F_λ are homogeneous, they are continuous.) It is on the other hand trivial to demonstrate that, if $\mathcal{G}$ is a graded ring (by our conventions complete, i.e., with additive group isomorphic to a direct product $\mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$) and if $F_\lambda \in C^{2,\lambda}(\mathcal{G}, \mathcal{G})$ are given satisfying (1_ν) for all ν, then the new multiplication $\times$ introduced on $\mathcal{G}$ by setting

$$(2) \qquad a \times b = ab + F_1(a \otimes b) + F_2(a \otimes b) + \cdots$$

is associative and defines a filtered ring $\mathcal{F}$ whose associated graded ring is $\mathcal{G}$. Further, letting q_i be the identity mapping of $\mathcal{G}^{(i)}$ onto itself considered as a mapping of $\mathcal{F}_i/\mathcal{F}_{i+1}$ into $\mathcal{F}_i$, $\mathcal{F}$ is seen to be developable, and the development T associated with these q_i gives rise precisely to the cochains F_λ. Making the natural identification of isomorphic filtered rings over a given graded ring, we may therefore assert

THEOREM 3. *The developments of developable filtered rings with a fixed associated graded ring $\mathcal{G}$ are in one-one correspondence with the sequences of cochains $F_\lambda \in C^{2,\lambda}(\mathcal{G}, \mathcal{G})$ satisfying* (1_ν).

Because of the strong analogy between the formulae of Ch. I, § 1, and those of the present section, it will be our point of view that a *developable* *filtered ring is obtained by some deformation of its associated graded ring.* The set of all developable rings with a fixed associated graded ring will be called a class of developable rings, and of all rings in the class, the graded ring itself is to be considered as having the simplest structure. If $\mathcal{G}$ is a graded ring and a deformation $\mathcal{F}$ of $\mathcal{G}$ (i.e., developable filtered ring $\mathcal{F}$ with associated graded ring $\mathcal{G}$ together with a development $T: \mathcal{F} \to \mathcal{G}$) is given, then the cocycle F_1 will be called the differential of the deformation.

4. Trivial deformations and a criterion for rigidity

Since from this point the deformation theory for graded rings follows in many respects almost exactly that for algebras, certain of the fundamental propositions will be stated without proof.

Let $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$ be as before a graded ring and $\Psi: \mathcal{G} \to \mathcal{G}$ be an automorphism of the additive group of $\mathcal{G}$ which preserves the filtration, i.e., setting again $\mathcal{G}_d = \mathcal{G}^{(d)} \times \mathcal{G}^{(d+1)} \times \cdots$, it is assumed that $\Psi(\mathcal{G}_d) \subset \mathcal{G}_d$. It then follows readily by induction on d that $\Psi(\mathcal{G}_d) = \mathcal{G}_d$ for all d, and that Ψ induces an additive automorphism of $\mathcal{G}^{(d)} = \mathcal{G}_d/\mathcal{G}_{d+1}$. The collection of these additive automorphisms constitutes an additive automorphism ψ_0 of $\mathcal{G}$ of degree zero, i.e., an automorphism of $\mathcal{G}$ as a graded group. Since Ψ preserves the filtration, it can be written as an infinite sum $\Psi = \psi_0 + \psi_1 + \cdots$, where ψ_i is an additive endomorphism of $\mathcal{G}$ of degree i, i.e., $\psi(\mathcal{G}^{(j)}) \subset \mathcal{G}^{(i+j)}$. We now assume that the automorphism ψ_0 is the identity, I. Then there can be introduced on the group $\mathcal{G}$ a new multiplication completely analogous to (6) of Ch. I, § 3, by setting $a \times b = \Psi^{-1}(\Psi a \cdot \Psi b)$. It is then trivial that $\mathcal{G}_i \times \mathcal{G}_j \subset \mathcal{G}_{i+j}$, i.e., in the $\times$ multiplication $\mathcal{G}$ is an associative ring filtered by the $\mathcal{G}_i$, and the identity map of $\mathcal{G}$ onto itself, considered as a mapping from $\mathcal{G}$ in the $\times$ multiplication to $\mathcal{G}$ in its original multiplication is a development of $\mathcal{G}$. Writing as in (2), $a \times b = ab + F_1(a \otimes b) + \cdots$, we have then $F_1 = \delta\psi_1$. In the $\times$ multiplication $\mathcal{G}$ becomes a filtered ring, which we may denote by $\mathcal{F}$, which is developable and has $\mathcal{G}$ as associated graded ring. The additive automorphism Ψ has therefore given rise to a deformation $\mathcal{F}$ of $\mathcal{G}$, but $\mathcal{F}$ is clearly isomorphic to $\mathcal{G}$. Such a deformation will be considered trivial. More generally, given a deformation of $\mathcal{G}$, which by the preceding section is equivalent to giving a multiplication of the form $a \times b = ab + F_1(a \otimes b) + \cdots$ on $\mathcal{G}$, if $\Psi = I + \psi_1 + \cdots$ is an additive automorphism of $\mathcal{G}$, then the new multiplication $\times'$ defined by $a \times' b = \Psi^{-1}(\Psi a \times \Psi b)$ is again a deformation of $\mathcal{G}$. We have for the corresponding F_1' that $F_1' = F_1 + \delta\psi_1$, in complete analogy with Ch. I, § 3, (6′). The deformation defined by the

$\times'$ multiplication will be said to be equivalent to that defined by $\times$. Just as in Ch. I, § 3, we may assert

PROPOSITION 3. *Let $\mathcal{G}$ be a graded ring, and a deformation of $\mathcal{G}$ be given, represented by a multiplication $\times$ such that $a \times b = ab + F_1(a \otimes b) + F_2(a \otimes b) + \cdots$, where $F_\lambda \in C^{2,\lambda}(\mathcal{G}, \mathcal{G})$. Then there exists an equivalent multiplication $\times'$, $a \times' b = ab + F'_\mu(a \otimes b) + F'_{\mu+1}(a \otimes b) + \cdots$, where $F'_\lambda \in C^{2,\lambda}(\mathcal{G}, \mathcal{G})$, and the first non-vanishing cochain F'_μ is in $Z^{2,\mu}(\mathcal{G}, \mathcal{G})$ and is not cohomologous to zero.*

COROLLARY. *If $H^{2,\lambda}(\mathcal{G}, \mathcal{G}) = 0$ for $\lambda = 1, 2, \cdots$, then $\mathcal{G}$ is rigid,* i.e., *if $\mathcal{F}$ is a developable filtered ring with $\mathcal{G}$ as associated graded ring, then there exists a filtration preserving ring isomorphism $T: \mathcal{F} \to \mathcal{G}$.*

Note that if the continuous cohomology group $H^2(\mathcal{G}, \mathcal{G})$ vanishes, then *a fortiori* so does $H^{2,d}(\mathcal{G}, \mathcal{G})$ for all positive d. We shall say that a graded ring $\mathcal{G}$ is rigid if any developable filtered ring $\mathcal{F}$ with $\mathcal{G}$ as associated graded ring is in fact isomorphic to $\mathcal{G}$. The theorem gives a sufficient condition for rigidity. One may therefore state, in analogy with the Corollary of Proposition 1, Ch. I, § 3, that a sufficient condition for the rigidity of $\mathcal{G}$ is the vanishing of $H^2(\mathcal{G}, \mathcal{G})$. This is, however, weaker than what has been shown.

5. Restriction to the commutative theory

It is important to note that considerations exactly analogous to those of Ch. I, § 7 hold for the present deformation theory of graded rings. In particular, observe that if $\mathcal{F}$ is a commutative filtered ring, then its associated graded ring $\mathcal{G}$ is also commutative. If further $\mathcal{F}$ is developable, then considerations analogous to those of §2 show that a sufficient condition for $\mathcal{G}^{(0)} = \mathcal{F}_0/\mathcal{F}_1$ to be segregated in $\mathcal{F}$ is that $H_c^2(\mathcal{G}^{(0)}, \mathcal{G}^{(d)}) = 0$ for $d = 1, 2, \cdots$. In particular, Theorem 3 of Ch. I, § 8 shows that this is the case whenever $\mathcal{G}^{(0)}$ is a field. We may state this as

THEOREM 4. *Let $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ be a developable ring with associated graded ring $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$. If $\mathcal{G}^{(0)}$ is a field then it has a field of representatives in $\mathcal{F}$.*

The foregoing theorem is due essentially to I. S. Cohen [2], who showed that an equi-characteristic complete local ring has a field of representatives (for its residue class field).

Finally, note that the considerations of § 4 can be carried through for the commutative theory (and indeed for other restricted theories), yielding an exact analog of Proposition 3, of which we state only the corollary as

PROPOSITION 4. *Let $\mathcal{F}$ be a commutative developable ring with associated graded ring $\mathcal{G}$, and suppose that $H_c^{2,d}(\mathcal{G}, \mathcal{G}) = 0$ for $d = 1, 2, \cdots$. Then $\mathcal{F}$ is isomorphic to $\mathcal{G}$ as a filtered ring.*

6. Deformations of power series rings

An important special case of the deformation theory is that in which $\mathcal{F}$ is a filtered ring whose associated graded ring is a power series ring with coefficients in some ring $\mathcal{G}^{(0)}$. A power series ring $\mathcal{G}^{(0)}[[t_1, \cdots, t_n]]$ in n variables $t_1, \cdots, t_n$ over $\mathcal{G}^{(0)}$ may be defined as the collection of all formal (including infinite) linear combinations $\sum g_\alpha M_\alpha$ of monomials M_α in $t_1, \cdots, t_n$ with coefficients g_α in $\mathcal{G}^{(0)}$, with the obvious addition and multiplication. Such a ring has a natural gradation by total degrees. A monomial $M = t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$ has total degree $m_1 + \cdots + m_n$. (It is allowed that $m_1 = \cdots = m_n = 0$.) Denoting the formal finite linear combinations $\sum g_\alpha M_\alpha$ of the monomials M_α of degree d by $\mathcal{G}^{(d)}$, as a group $\mathcal{G}^{(0)}[[t_1, \cdots, t_n]]$ is just the direct product $\mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$, and $\mathcal{G}^{(\lambda)}\mathcal{G}^{(\mu)}$ is contained in $\mathcal{G}^{(\lambda+\mu)}$.

Let $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ now be an arbitrary filtered ring, and $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$ be its associated graded ring. Suppose that for every $d = 0, 1, \cdots$, $\mathcal{G}^{(d)}$ is a finite direct sum of modules isomorphic to $\mathcal{G}^{(0)}$ and that $\mathcal{G}^{(0)}$ is itself a cyclic $\mathcal{G}^{(0)}$ module, i.e., is a module with a single generator (the case, in particular, if $\mathcal{G}^{(0)}$ has a unit element). In this case, if there exists an additive mapping $q_0: \mathcal{G}^{(0)} \to \mathcal{F}$ such that $p_0 q_0 = I$ (where p_i is, as usual, the natural projection of $\mathcal{F}_i$ onto $\mathcal{G}^{(i)}$), then $\mathcal{F}$ is predevelopable, i.e., has additive maps $q_i: \mathcal{G}^{(i)} \to \mathcal{F}_i$ such that $p_i q_i = I$ for all $i = 0, 1, \cdots$. In fact, if $\mathcal{G}^{(d)}$ is the direct sum of modules $\mathcal{G}^{(0)} M_\alpha^{(d)}$, $M_\alpha^{(d)}$ elements of $\mathcal{G}^{(d)}$, and if $\bar{M}_\alpha^{(d)}$ are representatives for the $M_\alpha^{(d)}$ in $\mathcal{F}_d$, then setting $q_d(\sum_\alpha g_\alpha M_\alpha^{(d)}) = \sum_\alpha q_0(g_\alpha)\bar{M}_\alpha^{(d)}$ defines the required mappings. In particular, if $\mathcal{G}$ is a power series ring $\mathcal{G}^{(0)}[[t_1, \cdots, t_n]]$ and $\mathcal{G}^{(0)}$ contains a unit element, then every $\mathcal{G}^{(d)}$ is the direct sum of the modules generated by the monomials $M_\alpha^{(d)}$, and the preceding applies. The resulting development, which is completely determined by q_0, will be called a *consistent* one.

Suppose now that $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ is a developable ring with associated graded ring $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$, and suppose that $\mathcal{G}$ is in fact a power series ring $\mathcal{G} = \mathcal{G}^{(0)}[[t_1, \cdots, t_n]]$ over $\mathcal{G}^{(0)}$ with the usual grading. If $\mathcal{G}^{(0)}$ has a unit then we may choose a consistent development of $\mathcal{F}$. It is then trivial that, if $F_d \in C^{2,d}(\mathcal{G}, \mathcal{G})$ are the cochains defined in § 3, then for arbitrary monomials M, M' in $t_1, \cdots, t_n$, we have $F_d(Ma, M'b) = MM'F_d(a, b)$, $a, b \in \mathcal{G}$. This says that we may regard F_d as being simply an element of $C^2(\mathcal{G}^{(0)}, \mathcal{G}^{(0)})$, "blown up" to an element of $C^{2,d}$ in the trivial way. If $\mathcal{F}$ is an algebra over a field k (implying that $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are also algebras) and if $n = 1$, then the conditions (1_ν) of § 3 become identical with those of (4_ν) of Ch. I, § 2, with $\mathcal{G}^{(0)}$ here taking the place of the algebra A of that section. One has, therefore,

PROPOSITION 5. *Let A be an algebra with a unit over a field k. Then the equivalence classes of one-parameter families of deformations of A are in one-one correspondence with the isomorphism classes of complete filtered rings which are algebras over k and have $A[[t]]$ as associated graded algebra* (i.e., *with the deformations of $A[[t]]$ which preserve the structure of $A[[t]]$ as a vector space over k*).

COROLLARY. *If A is rigid as an algebra over k, then $A[[t]]$ is rigid as a graded ring.*

The foregoing proposition and corollary can be generalized to n variables by introducing the concept of an *n-parameter family* of deformations of an algebra A over a field k. This is simply an associative multiplication in the vector space $A \otimes_k k[[t]]$ such that for a, b in A we have

$$(3) \qquad f(a, b) = ab + \sum F_{1,M'}(a, b)M' + \sum F_{2,M''}(a, b)M'' + \cdots ,$$

where $M^{(i)}$ ranges over the monomials in $t_1, \cdots, t_n$ of degree i, and for every $M^{(i)}$, $F_{i,M^{(i)}}$ is a bilinear function from $A \times A$ to A. Letting $A^{(1)}$ denote the vector space direct sum of A with itself n times, and $A^{(d)}$ denote the symmetric tensor product of $A^{(1)}$ with itself d times, considered in the natural way as an A module, $A^{(d)}$ is isomorphic to the direct sum of A with itself $(d!)^{-1}n(n + 1) \cdots (n + d - 1)$ times, and the collection of bilinear maps $F_{d\ M^{(d)}}$ may be construed as an element F_d of $C^2(A, A^{(d)})$. Equivalent n-parameter families of deformations may be defined just as in the one-dimensional case. We then have the following more general form of Proposition 5 which, like the latter, asserts nothing more than the existence of a consistent development.

THEOREM 5. *Let A be an algebra with a unit over a field k. Then the equivalence classes of n-parameter families of deformations of A are in one-one correspondence with the isomorphism classes of complete filtered rings which are algebras over k and have $A[[t_1, \cdots, t_n]]$ as associated graded algebra.*

Just as in the case for a one-parameter family of deformations, one verifies that if (3) is an n-parameter family of deformations of A, then F_1 is an element of $Z^2(A, A^{(1)})$. Further, if the given n-parameter family is non-trivial, then there exists an equivalent one for which the first non-vanishing coefficient F_d is an element of $Z^2(A, A^{(d)})$ not cohomologous to zero. Conversely, such an n-parameter family can not be trivial. Note that if F_d is in $Z^2(A, A^{(d)})$, then the various bilinear functions $F_{d,M^{(d)}}$ are elements of $Z^2(A, A)$. If F_d is cohomologous to zero, then so are all the $F_{d,M^{(d)}}$.

In order to be able to assert a proposition analogous to the Corollary to Proposition 3, we shall make now the overly restrictive assumption that the field k is infinite, in which case it is easy to prove that, if F_d is an element of $Z^2(A, A^{(d)})$ not cohomologous to zero, then by taking new generators for $A[[t_1, \cdots, t_n]]$ which are linear combinations of $t_1, \cdots, t_n$ with coefficients in k, one may cause F_{d,t_1^d} to be not cohomologous to zero. Suppose now that A has a non-trivial n-parameter family of deformations as in (3), in which the first non-vanishing coefficient is F_d. Then setting $t_2 = \cdots = t_n = 0$, one sees that the sequence of coefficients F_{i,t_1^i}, $i = 1, 2, \cdots$, (in which the first $d-1$ terms are zero), defines a one-parameter family of deformations of A which is not trivial because $F_{d\,t_1^d}$ is not cohomologous to zero. In view of Theorem 5, we may assert

THEOREM 6. *Let A be an algebra with a unit over an infinite field k. Then $A[[t_1, \cdots, t_n]]$ is rigid as a graded ring if and only if A is rigid as an algebra over k.*

It has so far been shown that under suitable conditions (actually more general than those we have considered), the rigidity of an algebra implies the rigidity of a power series ring over that algebra, it being tacitly presumed that the vector space structure of the power series ring is preserved under the deformation of that ring. On the other hand, the most useful criterion we have for the rigidity of an algebra A is the vanishing of $H^2(A, A)$. Making this assumption, we need not even assume that A is an algebra.

THEOREM 7. *Let $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ be a developable filtered ring with associated graded ring $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$. Suppose that $\mathcal{G}^{(0)}$ has a unit, that $H^2(\mathcal{G}^{(0)}, \mathcal{G}^{(0)}) = 0$, and that $\mathcal{G}$ is a power series ring $\mathcal{G}^{(0)}[[t_1, \cdots, t_n]]$ in a finite number of variables over $\mathcal{G}^{(0)}$. (It is assumed that the grading of $\mathcal{G}$ as the associated graded ring of $\mathcal{F}$ coincides with its grading as a power series ring.) Then $\mathcal{F}$ is isomorphic to $\mathcal{G}$. In other words, a power series ring $A[[t_1, \cdots, t_n]]$ with coefficients in a ring A with unit such that $H^2(A, A) = 0$ is rigid.*

PROOF. Choosing a consistent development $T: \mathcal{F} \to \mathcal{G}$, if $F_d \in C^{2,d}(\mathcal{G}, \mathcal{G})$ are the cochains associated with that development, the remarks preceding Proposition 5 show that these are in fact derived from elements of

$C^2(\mathcal{G}^{(0)}, \mathcal{G}^{(0)})$. The rest is trivial.

All of the considerations of this section hold equally well for various restricted theories, in particular, for the commutative theory. The analog for that theory of the preceding theorem is

THEOREM 8. *Let $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ be a commutative filtered ring whose associated graded ring $\mathcal{G} = \mathcal{G}^{(0)} \times \mathcal{G}^{(1)} \times \cdots$ is a power series ring over $\mathcal{G}^{(0)}$. If $\mathcal{G}^{(0)}$ has a unit and $H_c^2(\mathcal{G}^{(0)}, \mathcal{G}^{(0)}) = 0$, then $\mathcal{F}$ is isomorphic to $\mathcal{G} = \mathcal{G}^{(0)}[[t_1, \cdots, t_n]]$.*

In view of Theorem 3 of Ch. I, § 8, we have the following

COROLLARY. *A power series ring over a field is rigid in the commutative theory.*

In this corollary one may recognize another theorem of I. S. Cohen [2]. The corollary implies that if $\mathcal{F}$ is an equi-characteristic complete local ring whose associated graded ring is a power series ring, then $\mathcal{F}$ is itself in fact a power series ring, being identical with its associated graded ring. Since the associated graded ring of a regular local ring is a power series ring, it follows (assuming this) that a complete equi-characteristic regular local ring is a power series ring over its residue class field, a theorem first proved by Cohen.

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