Power operations and Formal Group Laws in Complex Cobordism Theory

TONGTONG LIANG

October 15, 2022

ABSTRACT

This is a survey on Quillen's elementary proofs of some results of cobordism theory using power operations. We optimize the system of notations and clarify some vague arguments in Quillen's paper. Furthermore, we emphasize the relations among cobordism power operations, Landweber-Novikov operations and the formal group law associated to the complex cobordism theory. Particularly, we present a stable-homotopy-theoric construction of cobordism power operations in order to demonstrate the relations. Based on this, we give a different proof of Quillen's technical lemma by promoting a lemma in Rudyak's book from mod-2 case to mod-p cases for all primes p.

CONTENTS

Department of Mathematics, Southern University of Science and Technology, Shenzhen, China, 12132882@mail.sustech.edu.cn

1 INTRODUCTION

Complex oriented cohomology theories are in the central of homotopy theory. Many popular cohomology theories, such as integral ordinary cohomology theory, complex K-theory, and complex cobordism theory, are complex oriented. In few words, a complex oriented cohomology theory is a cohomology theory that assigns each complex vector bundle a corresponding Thom class (see Definition 2.1), and thus enable us to consider characteristic classes. Another way to characterize a complex oriented cohomology theory E is that

$$E^*(\mathbb{CP}^\infty) = E^*(pt)[[x]]$$

where x is the Euler class of the universal complex line bundle (see Definition 2.3, Proposition 2.4). From this perspective, we can deduce that for each complex oriented cohomology theory E, there exists a unique formal group law F_E such that

$$e_E(L_1 \otimes L_2) = F_E(e_E(L_1), e_E(L_2))$$

for line bundles L_1 and L_2.

Since the spectrum MU of the complex cobordism theory consists of Thom spaces of universal complex vector bundles, it is natural to regard MU as the universal complex oriented cobordism theory (see Proposition 2.10). Furthermore, we may ask whether the formal group law F_{MU} on MU is the universal formal group law in the sense of [Laz55].

$$\begin{tikzcd}
\boxed{\text{complex cobordism theory}} \arrow[r, "\text{universal in}"] \arrow[d, dashed] & \boxed{\text{complex oriented cohomology theories}} \arrow[d] \\
\boxed{\text{universal formal group law}} \arrow[r, "\text{universal in}"] & \boxed{\text{formal group laws}}
\end{tikzcd}$$

Quillen gave the answer positive answer in [Qui69] by using Adams spectral sequences. Later, he proved it again without using Adams spectral sequences in [Qui71], where he mainly use cohomology operations in cobordism theory and their relations. Quillen called the second proof the elementary proof. This survey aims to demonstrate some techniques that Quillen used in the elementary proof. In particular, we would like to emphasize two enlightening points in these techniques. The first one is that the Thom spectrum MU is interpreted geometrically as cobordism classes, see Section 3. Under this framework, we can see that MU is endowed with Gysin morphisms (or say Gysin transfers) for complex oriented maps, and we can construct two types of cobordism operations concretely, i.e. cobordism power operations and Landweber-Novikov operations (see Construction 6.2 and Construction 6.1), which are the crucial in Quillen's elementary proof. The second point is that the relation between cobordism power operations and Landweber-Novikov operations is illustrated by a formula in terms of Euler classes of some specified vector bundles (see Theorem 6.10). To derive this formula, we need to a Riemann-Roch type theorem (see Theorem 4.4) and apply it to the case of fix point locus (see Proposition 5.3), which is so called "localization at the fix point set".

$$\begin{tikzcd}
& \text{Riemann-Roch type theorem} \arrow[d] & \\
& \text{localization at the fix point set} \arrow[d] & \\
\boxed{\text{cobordism power operations}} \arrow[rr, leftrightarrow, "\text{Theorem 6.10}"'] & {} & \boxed{\text{Landweber-Novikov operations}}
\end{tikzcd}$$

The results that rely on the use of the technique are the structure theorems for the unoriented cobordism ring and the complex cobordism ring. In this paper, we focus on the structure theorem (see Theorem 8.3) for the complex cobordism theory asserting that for any compact manifold, the complex cobordism ring $U^*(X)$ is generated by non-negative elements of $U^*(X)$ over the ring generated by the coefficients of F_{MU}. The proof of the structure theorem is an inductive argument on degrees and the key step where we need Theorem 6.10 is how to proceed the induction. In this process, we still need a technical lemma (see Lemma 7.10 or [Qui71, Section 4]) to deduce that we just need to apply Theorem 6.10 for a simple case.

In the proof of the technical lemma, we do not follow Quillen's paper, which largely uses Gysin sequences. Instead, we generalize the technique in [Rud98, Chapter VII, Lemma 7.15] to the case of mod-p lens spaces for any prime p and use it to prove the technical lemma. In addition, we use a homotopical construction of cobordism power operations (see Definition 7.3, Construction 7.8) in the proof of the technical lemma, which is different from the geometric construction in Quillen's proof. Nevertheless, these two constructions of cobordism power operations are equivalent. One advantage of the homotopical construction is that it demonstrates how power operations related to the coefficients of the formal group law more explicitly (see Theorem 7.9). Another advantage is that the homotopical construction of cobordism power operations is a specific case in a general framework called H_∞-structures to utilize power operations, see [BMMS86] for more details. In other words, the cobordism power operations are derived from an H_∞-structure on MU. From this perspective, we expect that this survey can enlighten people to think the connection between power operations and formal group laws for an arbitrary complex oriented cohomology theory with an H_∞-structure.

The main references of this survey are [Qui71], [Lan70], [Rud98], and [Car17].

2 COMPLEX ORIENTED COHOMOLOGY THEORIES

Definition 2.1 A *complex oriented cohomology theory* is a generalized cohomology theory E which is multiplicative and for any complex vector bundle ξ of rank n, there exists a class $\Phi_\xi \in \widetilde{E}^{2n}(\mathrm{Th}(\xi))$ called *Thom class* such that

1. For any $x \in X$, the image of Φ_ξ of the following composition

$$E^{2n}(E(\xi), B(\xi)) \cong \widetilde{E}^{2n}(\mathrm{Th}(\xi)) \longrightarrow \widetilde{E}^{2n}(\mathrm{Th}(\xi|_x)) \longrightarrow \widetilde{E}^{2n}(S^{2n}) \longrightarrow E^0(pt)$$

   is the canonical identity element 1.

2. Thom classes is compatible with pullback, namely, $f^*\Phi_\xi = \Phi_{f^*\xi}$.

3. For any two vector bundles ξ, ι with the same base space, we have $\Phi_{\xi\oplus\iota} = \Phi_\xi \cdot \Phi_\iota$

Definition 2.2 Let E be a complex oriented cohomology theory and $\xi\colon E \to B$ a vector bundle bundle. Let $s\colon B \to \mathrm{Th}(\xi)$ be the zero section. Then the *Euler class* of ξ with respect to E is defined by

$$e_E(\xi) := s^*\Phi_\xi$$

There is an alternative equivalent definition of complex oriented cohomology theory as follows.

Definition 2.3 A complex oriented cohomology theory is a ring spectrum E with a chosen class $x \in \widetilde{E}^2(\mathbb{CP}^\infty)$ such that the following

$$\widetilde{E}^2(\mathbb{CP}^\infty) \to \widetilde{E}^2(\mathbb{CP}^1) = \widetilde{E}^2(S^2) \cong E^0(pt)$$

induced by inclusion $\mathbb{CP}^1 \to \mathbb{CP}^\infty$, $x \mapsto 1$ in $E^0(pt)$. The chosen class is called the *orientation class*. We may denote a complex oriented cohomology theory by (E, x).

Proposition 2.4 For any complex oriented cohomology theory E, we have

$$E^*(\mathbb{CP}^n) = E^*(pt)[x]/(x^{n+1})$$

where x is the Euler class of the tautological bundle ξ on $\mathbb{CP}^n$.

Definition 2.5 A (commutative) *formal group law* over a ring R is a power series $F(x, y) = \sum c_{ij}x^i y^j \in R[[x, y]]$ such that

1. $F(0, 0) = 0$
2. $F(x, 0) = x$ and $F(0, y) = y$
3. $F(x, F(y, z)) = F(F(x, y), z)$
4. $F(x, y) = F(y, x)$

Proposition 2.6 Given a complex oriented cohomology theory (E, t), there exists a unique formal group law $F_E(x, y) = \sum c_{ij}x^i y^j$ over the ring $E^*(pt)$ such that for any space X with any two line bundles L_1, L_2 on it, we have

$$e_E(L_1 \otimes L_2) = F_E(e_E(L_1), e_E(L_2))$$

in $E^*(X)$.

Construction 2.7 Let $\eta_n : EU(n) \to BU(n)$ be the universal complex vector bundle over the complex Grassmanian manifold $BU(n)$. Let $\mathbf{n}$ denote the trivial complex bundle of rank n on an evident space. Let $MU(n)$ be the Thom space of η_n. Then we have $\alpha_n : \mathrm{Th}(\eta_n \oplus \mathbf{1}) \cong \Sigma^2\mathrm{Th}(\eta) \to MU(n+1)$ induced by a classifying map of $\eta_n \oplus \mathbf{1}$. Then we may define *complex Thom spectrum* MU by

$$MU_{2q} := MU(q)$$
$$MU_{2q+1} := \Sigma MU(q)$$

and the structure maps are given by α_n. Let $\Phi \in MU^2(MU(1)) = [MU(1), MU(1)]$ be the class of the identity map, which is called the *universal Thom class* on MU and derives the Thom class of each vector bundle evidently. Let $i : \mathbb{CP}^\infty \to MU(1)$ be the zero section, then $i^*(\Phi) \in MU^2(\mathbb{CP}^\infty)$ offers the orientation of MU.

Proposition 2.8 There exists a unique formal group law $F_{MU}(x, y) = \sum c_{ij}x^i y^j$ over the ring MU^* such that for any smooth manifold X with any two line bundles L_1, L_2 on it, we have

$$e_{MU}(L_1 \otimes L_2) = F(e_{MU}(L_1), e_{MU}(L_2))$$

in $MU^*(X)$, where $c_{ij} \in MU^{2-2i-2j}$.

Convention 2.9 Let $C \subset MU^*$ be the subring of MU^* generated by the coefficients $\{c_{ij}\}_{i,j}$ in the formal group law.

Proposition 2.10 $(MU^*, i^*\Phi)$ is the universal complex oriented cohomology theory in the sense that for any complex oriented cohomology theory (E, x), there is a unique map (up to homotopy) $\phi : MU \to E$ that preserves the orientations $i^*\Phi \to x$.

Sketch proof. The Thom class of $\eta_n \in \widetilde{E}^{2n}(MU(n))$ provides us with a morphism between spectra, which is what we need. □

Convention 2.11 Let $f(x, y)$ be the formal group law, then for any non-negative integer n, we let $[n]_f(z) := f(z, [n-1]_f(z))$ and $[0]_f(z) = z$. In particular, $[n]_f(z) = nz +$ higher term. Additionally, we denote $\theta_n(z) = [n]_f(z)/z$.

3 GEOMETRIC FORMALISM OF COMPLEX COBORDISM THEORY

Let $\mathcal{M}$lfd be the category of compact smooth manifolds and smooth maps.

Definition 3.1 Suppose X is a compact smooth manifold, then *a complex oriented map to* X consists of a smooth proper map $f\colon M \to X$ with even relative dimension (i.e. $\dim f = \dim M - \dim X$ is even) and a continuous map $\nu\colon X \to BU$ such that f can be factored by

$$M \xrightarrow{i} X \times \mathbb{C}^n \xrightarrow{p} X$$

where i is a closed embedding, p is the evident projection and the normal bundle ν_i on M has a complex structure of rank $(2n - \dim f)/2$ that is classified by ν. An odd dimensional complex oriented map is a pair $(f, 0)\colon M \to X \times \mathbb{R}$, where f is a complex oriented map of even relative dimension.

Definition 3.2 Let $(f, \nu_i)\colon X \to Y$ be a complex oriented map with the embedding $i\colon X \to Y \times \mathbb{C}^n$ and let $(g, \nu_j)\colon Y \to Z$ be a complex oriented map with the embedding $j : Y \to Z \times \mathbb{C}^m$. Then the composition $g \circ f$ has an induced complex orientation with embedding

$$X \xrightarrow{i} Y \times \mathbb{C}^n \xrightarrow{j\times \mathrm{id}} Z \times \mathbb{C}^m \times \mathbb{C}^n$$

whose normal bundle is classified by $X \xrightarrow{\mathrm{id}\times f} X \times Y \xrightarrow{\nu_i \times \nu_j} BU \times BU \xrightarrow{m} BU$ (here m is the multiplication on BU) i.e. $i^*\nu_{j\times \mathrm{id}}$.

Remark 3.3 If $f : M \to X$ is complex oriented and $g : Y \to X$ is a smooth map which is transversal to f, then the pull-back $Y \times_X M \to Y$ has an induced complex orientation.

Example 3.4 Let X be a smooth manifold and let $E \to X$ be a complex vector bundle on X. The zero section $s : X \to E$ has an evident complex orientation, because the normal bundle of s is exactly E itself.

Definition 3.5 Two proper complex oriented maps $f_i : Z_i \to X$ for $i = 0, 1$ is said to be *corbordant* if there is a proper complex oriented map $h : W \to X \times \mathbb{R}$ such that the map $j_i : X \to X \times \mathbb{R}$, $x \mapsto (x, i)$ is transversal to h and the pull-back of h is isomorphic with the complex orientation of f_i for $i = 0, 1$.

Remark 3.6 The cobordant relation is indeed an equivalent relation, see[Tho54].

Definition 3.7 For any compact smooth manifold X, we define

$$U^n(X) = \{(f, \nu) \mid \text{complex oriented maps of relative dimension } n\}/\text{cobordant}$$

for each n. The addition structure on $U^n(X)$ is defined by

$$(f, \nu) + (f', \nu') := (f \sqcup f', \nu \sqcup \nu')$$

The external product on U^* is given by

$$\begin{array}{rccc} \times : & U^*(X) \otimes U^*(Y) & \longrightarrow & U^*(X \times Y) \\ & f \otimes g & \longmapsto & f \times g \end{array}$$

and the internal product is derived by

$$U^*(X) \otimes U^*(X) \xrightarrow{\times} U^*(X \times X) \xrightarrow{\Delta^*} U^*(X)$$

where $\Delta\colon X \to X \times X$ is the diagonal map.

We denote

$$U^*(X) := \bigoplus_{n \in \mathbb{Z}} U^n(X)$$

which forms a presheaf on $\mathcal{M}$lfd.

If A is a strong deformation retract of an open neighborhood V in X, we similarly define

$$U^*(X, X-A) = \{(f,\nu) \mid \text{complex oriented maps} \mid f(Z) \subset A\}/\text{cobordant}$$

In this way, the construction of $U^*(X)$ offer a geometric interpretation of MU. This geometric interpretation enable us to display some delicate operations on complex cobordism theory more manageable, such as the Gysin homomorphisms, Landweber-Novikov operations and power operations (the latter two items will be discussed in Section 6).

Remark 3.8 The presheaf U^* is actually a presheaf of graded rings. The addition on $U^n(X)$ is defined by disjoint unit and the multiplication on $U^*(X)$ is derived from the multiplicative property of BU. More specifically, U^* is a presheaf of $U^*(pt)$-algebras.

Proposition 3.9 For any $X \in \mathcal{M}\mathrm{lfd}$, we have a functorial isomorphism

$$U^*(X) \cong MU^*(X)$$

given by Pontrjagin-Thom construction. For the relative case, if A is a strong deformation retract of an open neighborhood V in X, then

$$U^*(X, X-A) \cong MU^*(X, X-A)$$

Proof. See [Tho54]. □

Definition 3.10 Given a proper complex oriented map $(g,\xi)\colon X \to Y$ of dimension d, we define the induced *Gysin homomorphism*

$$\begin{array}{cccc} g_*: & U^q(X) & \longrightarrow & U^{q+d}(Y) \\ & f & \longmapsto & g \circ f \end{array}$$

where $[g \circ f]$ is the cobordant class represented by the complex orientation of $g \circ f$ in the sense of Definition 3.2.

Proposition 3.11 The Gysin morphisms are additive and $U^*(pt)$-linear and given two composable complex oriented maps p, q, we have $(p \circ q)_* = p_* \circ q_*$.

Proposition 3.12 Let $i : Z \to X$ be a closed embedding of smooth manifolds of codimension d such that the normal bundle ν_i has a complex structure, then we have the *Gysin-Thom isomorphism*

$$i_*\colon U^*(Z) \xrightarrow{\sim} U^{*+d}(X, X-Z)$$

Proposition 3.13 Given a cartesian square of manifolds

$$\begin{tikzcd} Y \times_X Z \arrow[r, "g'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

where g is transversal to f and f is a complex oriented proper map and f' is endowed with the pull-back of the complex orientation of f, we have

$$g^* \circ f_* = f'_* \circ g'^*\colon U^*(Z) \to U^*(Y)$$

Remark 3.14 In summary, we conclude that U^* is a contravariant functor for all morphisms in $\mathcal{M}\mathrm{lfd}$ and is a covariant functor for all proper complex oriented maps. This structure can be roughly viewed as a bivariant structure.

Proposition 3.15 Let $f\colon Z \to X$ be a complex oriented map, and let $x \in U^*(X)$, then

$$f_*f^*(x) = f_*(1) \cdot x$$

Proposition 3.16 Let $E \to X$ be a complex vector bundle and let $s : X \to \mathrm{Th}(E)$ be its zero section to the Thom space. Then $s_*([\mathrm{id}_X]) \in U^*(\mathrm{Th}(E))$ is the Thom class and $s^*s_*([\mathrm{id}_X])$ is the Euler class $e_U(E)$, which coincides with $e_{MU}(E)$ via the identification. Moreover, under the identification between $U^*(X)$ and $MU^*(X)$, s_* is exactly the Thom isomorphism.

Proposition 3.17 Let $E \to X$ and $F \to Y$ be two smooth complex vector bundles, and let $p_X\colon X \times Y \to X$, $p_Y\colon X \times Y \to Y$ be the natural projections. Then we have

$$p_X^* e_U(E) \cdot p_Y^* e_U(F) = e_U(E) \cdot e_U(F)$$

where the left hand side is the internal product and the right hand side is the external product.

Sketch proof. We just need to check the zero section $p_X^*E \oplus p_Y^*F \to X \times Y$ and the zero section is indeed the product of the zero section of $E \to X$ and $F \to Y$. □

4 CLEAN INTERSECTION AND A RIEMANN-ROCH TYPE THEOREM

Definition 4.1 Suppose X is a smooth manifold, and Y, Z are closed submanifolds of X with submanifold $W = Y \cap Z$, then we have the pullback diagram

$$\begin{tikzcd} W \arrow[r, "j'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "j"] & X \end{tikzcd} \tag{1}$$

the *excess bundle* F *on* W is the cokernel of $TZ|_W \oplus TY|_W \to TX|_W$. If $TW = TZ|_W \cap TY|_W$, then the intersection of Y and Z is said to be a *clean intersection*.

Remark 4.2 If the excess bundle F is trivial, then Y, Z intersect transversally. From this perspective, the excess bundle is the obstruction to the transversality of the intersection.

Lemma 4.3 Suppose the intersection in Diagram 1 is clean, we have the following short exact sequence.

$$0 \longrightarrow \nu_{i'} \longrightarrow j'^*\nu_i \longrightarrow F \longrightarrow 0 \tag{2}$$

Theorem 4.4 With the same notation and assumption of Lemma 4.3, we assume that ν_i and $\nu_{i'}$ admit complex structures and the map $\nu_{i'} \to j'^*\nu_i$ is a morphism of complex vector bundles. Then for any $z \in U^*(Z)$,

$$j^*i_*(z) = i'^*(e(F) \cdot j'^*(z))$$

in $U^{*+d}(Y, Y - W) \cong U^*(W)$, where d is the codimension of W in Y.

5 EQUIVARIANT COBORDISM THEORY

Suppose G is a compact Lie group in this section.

Definition 5.1 (Equivariant G-bundle) A G-*manifold* is a smooth manifold M with a smooth action of G. An *equivariant* G-*bundle* is a bundle $\pi\colon E \to B$ where both total space E and base space B are G space and the fibration π is a G-continuous.

Example 5.2 For a trivial G-space X and a G-representation V, the natural projection $X \times V \to X$ forms a G-bundle on X.

Proposition 5.3 Suppose $i\colon Z \to X$ is an embedding of G-manifolds, then the following diagram is a clean intersection

$$\begin{tikzcd} Z^G \arrow[r, "r_Z"] \arrow[d, "i^G"] & Z \arrow[d, "i"] \\ X^G \arrow[r, "r_X"'] & X \end{tikzcd}$$

where $r_X\colon X^G \to X$ is the inclusion of the locus of fixed points, so is r_Z.

Since Z^G is a trivial G-manifold, then the G-bundle $r_Z^*(\nu_i)$ on it can be decomposed as $r_Z^*(\nu_i) = \nu_{i^G} \oplus \mu_i$ where μ_i is a subbundle with nontrivial G-action, namely it is formed by non-trivial representations of G. Note that ν_{i^G} is the fixed point space of $r_Z^*(\nu_i)$. Therefore, μ_i is the excess bundle of the clean intersection in Proposition 5.3.

Corollary 5.4 Using the notation in Proposition 5.3, we have

$$r_X^* i_*(z) = i_*^G(e_{MU}(\mu_i) \cdot r_Z^*(z)) \in U^*(Z^G)$$

for any $z \in U^*(Z)$.

Definition 5.5 (Equivariant complex oriented map) Suppose Z, X are two G-manifolds, an *equivariant complex oriented map* is a complex oriented map $f\colon Z \xrightarrow{i} X \times \mathbb{C}^n \xrightarrow{p} X$ where $X \times \mathbb{C}^n$ is a G-equivariant complex bundle over X and ν_i has a G-equivariant complex structure.

Proposition 5.6 Given a G-equivariant complex oriented map $f\colon Z \xrightarrow{i} X \times \mathbb{C}^n \xrightarrow{p} X$ and let $r_Z^*(\nu_i) = \nu_{i^G} \oplus \mu_i$ as we mention before. Let τ_i be the excess bundle of the following diagram of intersection

$$\begin{tikzcd} X^G \arrow[r, "r_X"] \arrow[d, "s^G"] & X \arrow[d, "s"] \\ (X \times \mathbb{C}^n)^G \arrow[r, "r_{X\times\mathbb{C}^n}"'] & X \times \mathbb{C}^n \end{tikzcd}$$

where $s\colon X \to X \times \mathbb{C}^n$ is the zero section. Then we have

$$e_{MU}(\tau_i) \cdot r_X^* f_*(z) = f_*^G(e_{MU}(\mu_i) \cdot r_Z^*(z)) \tag{3}$$

for any $z \in U^*(Z)$.

Sketch proof. Apply Corollary 5.4 on the diagrams

$$\begin{tikzcd} X^G \arrow[r, "r_X"] \arrow[d, "s^G"] & X \arrow[d, "s"] \\ (X \times \mathbb{C}^n)^G \arrow[r, "r_{X\times\mathbb{C}^n}"'] & X \times \mathbb{C}^n \end{tikzcd}$$

$$\begin{tikzcd} Z^G \arrow[r, "r_Z"] \arrow[d, "i^G"] & Z \arrow[d, "i"] \\ (X \times \mathbb{C}^n)^G \arrow[r, "r_{X\times\mathbb{C}^n}"'] & X \times \mathbb{C}^n \end{tikzcd}$$

and combine these two resulting formulas by using Proposition 3.12 to identify items. □

Remark 5.7 An alternative equivalent description of τ_i is the sum of the eigen bundles of $r_X^*(X \times \mathbb{C}^n)$ corresponding to the nontrivial irreducible representations of G. This description is used in [Qui71].

Construction 5.8 (Equivariant cobordism theorem) Given a principal G-bundle ξ, say $\pi_\xi \colon Q \to B$ over a manifold B and we let G act right on Q. Then for any G-space, we define the equivariant cobordism theory U_ξ^* twisted by ξ by

$$U_\xi^*(X) := U^*(Q \times_G X)$$

If ξ is the universal principal G-bundle, we denote it by U_G^*.

Remark 5.9 For a G-space X and a G-equivariant vector bundle $\eta \colon E \to X$ over X, we have

$$e_\xi(\eta) := e_{MU}(Q \times_G \eta \colon Q \times_G E \to Q \times_G X)$$

We also have

$$e_\xi(L_1 \otimes L_2) = F_{MU}(e_\xi(L_1), e_\xi(L_2))$$

because $Q \times_G (L_1 \otimes L_2) \cong (Q \times_G (L_1)) \otimes (Q \times_G (L_2))$.

Remark 5.10 If we replace U^* by U_ξ^* for any smooth principle G-bundle ξ and replace e_{MU} by e_ξ, Proposition 5.3 and Proposition 5.6 also hold, because the construction $Q \times_G -$ is functorial and preserves clean intersection. Furthermore, for any G-equivariant complex oriented map $f \colon M \to X$, $Q \times_G f$ is a complex oriented map.

6 OPERATIONS IN GEOMETRIC COBORDISM THEORY

In this section, we will discuss two kinds of operations in U^* and how they are related to each other, which plays an important role in the proof of the main theorem.

Construction 6.1 (Landweber-Novikov Operations) The *total Landweber-Steenrod operations* on X is defined to be

$$\begin{array}{rccc} s_t \colon & U^*(X) & \longrightarrow & U^*(X)[t_1, t_2, t_3, \dots] \\ & (f, \nu) & \longmapsto & \sum_\alpha t^\alpha f_*(c_\alpha(\nu)) \end{array}$$

where α runs over all the numerable sequences of non-negative integers with only finitely many integers are non-zero and c_α is the Conner-Floyd-Chern class indexed by α. We denote $s_\alpha(x) := f_*(c_\alpha(\nu))$.

Construction 6.2 (Power operations in cobordism) Given a principle $\mathbb{Z}/p$-bundle $\xi \colon Q \to B$, the *total Steenrod operation twisted by* ξ is defined to be

$$\begin{array}{ccccc} P_\xi \colon U^{-2q}(X) & \longrightarrow & U_\xi^{-2pq}(X^p) & \xrightarrow{\Delta^*} & U_\xi^{-2pq}(X) = U^{-2pq}(B \times X) \\ \langle Z \xrightarrow{f} X \rangle & \longmapsto & \langle Q \times_{\mathbb{Z}/p} Z^p \xrightarrow{\mathrm{id}_Q \times_{\mathbb{Z}/p} f^p} Q \times_{\mathbb{Z}/p} X^p \rangle & \longmapsto & \langle (Q \times_{\mathbb{Z}/p} Z^p)^{\mathbb{Z}/p} \to B \times X \rangle \end{array} \tag{4}$$

where $\mathbb{Z}/p$ acts on X^p by permuting factors and acts on X trivially; $\Delta \colon X \to X^p$ is the diagonal map.

Proposition 6.3 Let (ρ, V) be a representation of $\mathbb{Z}/p$ where

$$V = \{(z_1, \dots, z_p) \in \mathbb{C}^p \mid \sum_{i=1}^p z_i = 0\}$$

by permuting factors cyclically, then $X \times V \to X$ is a $\mathbb{Z}_p$-equivariant trivial vector bundle over a trivial $\mathbb{Z}/p$-space and we denote this vector bundle by X^ρ. Then we have

$$e_\xi(X^\rho)^n P_\xi(f,\nu) = f_*(e_\xi(M^\rho \otimes \nu_i)) \in U^*_\xi(X) = U^*(B \times X) \tag{5}$$

where $(f,\nu)\colon M \xrightarrow{i} X \times \mathbb{C}^n \xrightarrow{p} X$ is a complex oriented map in $U^*(X)$.

Sketch proof. Note that the diagonal map $\Delta\colon X \to X^p$ is the inclusion of the locus of the $\mathbb{Z}/p$-fixed points. Thus we apply Proposition 5.6 to the $\mathbb{Z}/p$-equivariant complex oriented map $f^p : M^p \to X^p$ and let the variable z in Proposition 5.6 be $[\mathrm{id}_M]$. According to Remark 5.7, $(X^\rho)^{\oplus n} \to X$ is indeed the excess bundle in

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "s"] & X^p \arrow[d, "s^p"] \\ X \times \mathbb{C}^n \arrow[r, "\Delta_{X\times\mathbb{C}^n}"'] & X^p \times \mathbb{C}^{np} \end{tikzcd}$$

and $e_\xi((X^\rho)^{\oplus n}) = e_\xi(X^\rho)^n$. Then we prove the proposition by decoding definitions and constructions. □

Remark 6.4 Note that $e_\xi(X^\rho) = e_{MU}(Q \times_{\mathbb{Z}/p} (X \times V) \to B \times X)$, since X is a trivial $\mathbb{Z}/p$-space. Furthermore, there is an evident identification

$$Q \times_{\mathbb{Z}/p} (X \times V) \cong (Q \times_{\mathbb{Z}/p} V) \times X \to B \times X$$

Then by Proposition 3.17, we have

$$e_\xi(X^\rho) = e_{MU}(Q \times_{\mathbb{Z}/p} V \to B) \times e_{MU}(X \xrightarrow{\mathrm{id}} X) = e_{MU}(Q \times_{\mathbb{Z}/p} V \to B) \times 1 \cong e_{MU}(Q \times_{\mathbb{Z}/p} V)$$

With this identification, we let $w = e_{MU}(Q \times_{\mathbb{Z}/p} V \to B)$, and we may rewrite Formula 5 into

$$w^n P_\xi(f) = f_*(e_\xi(\rho \otimes \nu_i)) \in U^*_\xi(X) = U^*(B \times X) \tag{6}$$

(Notice that we ignore the symbol of both external products and internal products.)

Remark 6.5 Proposition 6.3 offers a way to compute power operations in terms of Gysin homomorphisms and characteristic classes. Recall the construction of Landweber-Novikov operations (Construction 6.1), we may foresee that this proposition will help us demonstrate the connection between power operations and Landweber-Novikov operations.

Lemma 6.6 Let σ be a 1-dimensional representation of $\mathbb{Z}/p$ sending

$$[n] \mapsto \exp(2n\pi i/p) \in \mathbb{C}^*$$

then there is an isomorphism between representations

$$\bigoplus_{k=1}^{p-1} \sigma^{\otimes k} \cong \rho$$

where we define ρ in Proposition 5.3.

Corollary 6.7 Fix a principal $\mathbb{Z}/p$-bundle $\xi\colon Q \to B$ and a smooth manifold X. Let $\nu = e_\xi(\sigma) \in$ and $w = e_\xi(\rho)$, we have

$$w = \prod_{k=1}^{p-1} [k]_{F_{MU}}(\nu) = (p-1)!\nu^{p-1} + \sum_{j \geqslant p} d_j \nu^j \tag{7}$$

where $d_j \in C$ for all j.

Lemma 6.8 Fix a principal $\mathbb{Z}/p$-bundle $\xi\colon Q \to B$. Let $v = e_{MU}(Q \times_{\mathbb{Z}/p} \sigma \to B)$ and $w = e_{MU}(Q \times_{\mathbb{Z}/p} \rho \to B)$, then there exists a sequence of power series $a_j(t) \in C[[t]]$ such that for any smooth manifold X and any line bundle L on X, we have

$$e_\xi(X^\rho \otimes L) = w + \sum_{j\geqslant 1} a_j(v) e_{MU}(L)^j$$

Sketch proof. Note that the $e_\xi(X\rho \otimes L) = e_{MU}(Q \times_{\mathbb{Z}/p} (V \otimes L) \to B \times X)$. Let $p_B : B \times X \to B$ and $p_X\colon B \times X \to X$, then by checking fiber by fiber, we have

$$Q \times_{\mathbb{Z}/p} (V \otimes L) \cong p_B^*(Q \times_{\mathbb{Z}/p} V) \otimes p_X^* L \text{ over } B \times X$$

According to Lemma 6.6 and the definition of the formal group law F_{MU}, we have

$$\begin{aligned}
e_\xi(X^\rho \otimes L) =& e_{MU}(p_B^*(Q \times_{\mathbb{Z}/p} V) \otimes p_X^* L) \\
=& \prod_{k=1}^{p-1} e_{MU}(p_B^*(Q \times_{\mathbb{Z}/p} \sigma)^{\otimes k} \otimes p_X^* L) \\
=& \prod_{k=1}^{p-1} F_{MU}(e_{MU}(p_B^*(Q \times_{\mathbb{Z}/p} \sigma)^{\otimes k}), e_{MU}(p_X^* L)) \\
=& \prod_{k=1}^{p-1} F_{MU}\big(p_B^*[k]_{F_{MU}}(v), p_X^* e_{MU}(L)\big) \\
\text{Using Proposition 3.17} =& \prod_{k=1}^{p-1} \Big([k]_{F_{MU}}(v) + \sum_{j\geqslant 1} b_{jk}(v) e_{MU}(L)^j\Big) \\
=& \prod_{k=1}^{p-1} ([k]_{F_{MU}}(v)) + \sum_{j\geqslant 1} a_j(v) e_{MU}(L)^j \\
=& w + \sum_{j\geqslant 1} a_j(v) e_{MU}(L)^j
\end{aligned}$$

where $a_j(x), b_{jk}(x) \in C[[x]]$ by the definition. Note that a_j and b_{jk} are independent of the choice of L and only determined by the formal group law.

$\square$

Proposition 6.9 Fix a principal $\mathbb{Z}/p$-bundle $\xi : Q \to B$. Let $E \to X$ be a vector bundle on X, then we have

$$e_\xi(X^\rho \otimes E) = \sum_{l(\alpha)\leqslant r} w^{r-l(\alpha)} \Big(\prod_{j\geqslant 1} a_j(v)^{\alpha_j}\Big) c_\alpha(E)$$

Sketch proof. If $E = L_1 \oplus \cdots \oplus L_r$ is a sum of line bundles and let, then

$$\begin{aligned}
e_\xi(X^\rho \otimes E) =& \prod_{i=1}^{r} e_\xi(X^\rho \otimes L_i) \\
\text{Using Lemma 6.8} =& \prod_{i=1}^{r} \Big(w + \sum_{j\geqslant 1} a_j(v) e_{MU}(L_i)^j\Big) \\
=& \sum_{l(\alpha)\leqslant r} w^{r-l(\alpha)} \Big(\prod_{j\geqslant 1} a_j(v)^{\alpha_j}\Big) c_\alpha(E)
\end{aligned} \tag{8}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ a numerable sequence of natural numbers and $l(\alpha) = \sum_{j\geqslant 1} \alpha_j$. Then Formula 8 holds for any vector bundle E by the splitting principle.

$\square$

Theorem 6.10 Given an positive integer q, there exists an integer n such that the p-th power operation associated to a principle $\mathbb{Z}/p$-bundle $\xi\colon Q \to B$ is related to the Landweber-Novikov operations by the formula

$$w^{n+q}P_{\xi}x = \sum_{l(\alpha)\leqslant n} w^{n-l(\alpha)}(\prod_{j\geqslant 1} a_j(v)^{\alpha_j})s_{\alpha}(x)$$

Sketch proof. Assuming the normal bundle is a direct sum of line bundles, we just merge Proposition 6.3 and Equation 8 together, we will get

$$w^{n}P_{\xi}x = \sum_{l(\alpha)\leqslant n} w^{n-l(\alpha)}(\prod_{j\geqslant 1} a_j(v)^{\alpha_j})s_{\alpha}(x)$$

for $x = [f, \nu]$ and f factor through $i\colon M \to X \times \mathbb{C}^n$. In general case, we may use splitting principle. Since X is of finite dimensional, we may let n large enough, such as $n - q > \dim X$, then $s_{\alpha}(x) = 0$ for $l(\alpha) > n - q$. Then we have

$$w^{n-q+q}P_{\xi}x = \sum_{l(\alpha)\leqslant n-q} w^{n-q-l(\alpha)}(\prod_{j\geqslant 1} a_j(v)^{\alpha_j})s_{\alpha}(x)$$

□

7 QUILLEN'S TECHNIQUE LEMMA

The goal of this section is to prove a technical lemma, i.e. Lemma 7.10, which plays an important role in the proof of the structure theorem for the complex cobordism theory. Different from Quillen's proof of the technical lemma, we use power operations constructed in the homotopical formalism of MU^*, instead of the geometric formalism U^*. Despite of this, the two constructions are equivalent.

This section mainly refers to [Rud98, Chapter VII, Section 7], and we will extend the mod-2 constructions and related results in [Rud98, Chapter VII, Section 7] to mod-p cases for each prime p.

Let p be a prime and $\mathbb{Z}/p$ has an action on $S^{2n-1} \subset \mathbb{C}^n$ be multiplying $\exp(2\pi i/p)$. Note that this is a free action. Then we define the n^{th} mod-p lens space $L^n(p) := S^{2n-1}/p$ (for convenience, if Y is a $\mathbb{Z}/n$-space, the orbit space is denoted by Y/n). For a general space X, we let

$$\Gamma_n^p(X) := (S^{2n-1} \times X^p)/p$$

and for based space $(X, *)$, we denote

$$\Gamma_n^{p+}(X) := (S^{2n-1} \wedge X^{\wedge p})/p$$

where $\mathbb{Z}/p$ acts diagonally and particularly, $\mathbb{Z}/p$ permutes factors in X^p.

Construction 7.1 Let ξ be a complex vector bundle on X and $\pi\colon S^{2n-1} \times X^p \to X^p$ be the natural projection. Then we let ξ^p is defined to be the vector bundle on X^p given by external product of ξ. Note that ξ^p is an equivariant $\mathbb{Z}/p$-vector bundle by permuting factors cyclically. Since π is a $\mathbb{Z}/p$-equivariant morphism, $\pi^*(\xi^p)$ is also an equivariant $\mathbb{Z}/p$-vector bundle on $S^{2n-1} \times X^p$. By taking quotient by p, we have a vector bundle

$$\pi^*(\xi^p)/p \to \Gamma_n^p(X)$$

we denote this vector bundle by $\xi_n(p)$.

Lemma 7.2 By taking Thom spaces, we have

$$\mathrm{Th}(\xi_n(p)) \cong \Gamma_n^{p+}(\mathrm{Th}(\xi))$$

Sketch proof. This is true due to the community of small colimits.

$$\mathrm{Th}(\xi_n(p)) \cong \mathrm{Th}(S^{2n-1} \times \xi^p/p) \cong \mathrm{Th}(S^{2n-1} \times \xi^p)/p \cong (S^{2n-1} \wedge \mathrm{Th}(\xi)^{\wedge p})/p$$

Similarly, we have this construction for based cases. □

Definition 7.3 Given integer r, n and prime p, the *external power operation*

$$EP^{2r}_{n,p}\colon \widetilde{MU}^{2r}(X) \to \widetilde{MU}^{2pr}(\Gamma^{p+}_n(X))$$

is defined to be: for any $\alpha \in \widetilde{MU}^{2r}(X)$ that can be represented by $f\colon \Sigma^{2l}X \to MU_{2r+2l}$, we have

$$\Gamma f\colon \Gamma^{p+}_n \Sigma^{2l}X \to \Gamma^{p+}_n MU_{2r+2l}$$

Let γ_{r+l} be the universal complex vector bundle on BU_{r+l} and $\mathrm{Th}(\gamma_{r+l}) = MU_{2r+2l}$. Note that the Thom class of $\gamma_{r+l}(p)$ denoted by $\Phi_{\gamma_{r+l}(p)}$ is in $MU^{2p(r+l)}(\Gamma^{p+}_n MU_{2r+2l})$, then we define

$$EP^{2r}_{n,p}(\alpha) := \Gamma f^*(\Phi_{\gamma_{r+l}(p)}) \in \widetilde{MU}^{2p(r+l)}(\Gamma^{p+}_n \Sigma^{2l}X) \cong \widetilde{MU}^{2pr}(\Gamma^{p+}_n X)$$

Remark 7.4 If we consider the geometric model of the cobordism theory via Pontrjagin-Thom construction (see Construction 3.7), for smooth manifold X, $\alpha \in MU^{2r}(X)$ can be presented by a complex oriented map (X, f) and then $EP^{2r}_{n,p}(\alpha) = (\Gamma^p_n X, \Gamma^p_n f)$ (see Construction 6.2).

Proposition 7.5 The external power operations have the following properties:

1. They are natural with respect to X;
2. Let $i_n : S^{2n-1} \to S^{2n+1}$ be the inclusion induced by the natural inclusion $\mathbb{C}^n \to \mathbb{C}^{n+1}$, then $i^* EP^{2r}_{n+1,p}(\alpha) = EP^{2r}_{n,p}(\alpha)$;
3. Let $j\colon X^p \to \Gamma^{p+}_n X$ be a map induced by the evident based map $X^p \to S^{2n-1} \wedge X^p$, then $j^* EP^{2r}_{n,p}(\alpha) = \alpha^p$;
4. Let $\nabla\colon \Gamma^{p+}_n X \wedge Y \to \Gamma^{p+}_n X \wedge \Gamma^{p+}_n Y$ defined by
$$(s, x_1, y_1, \dots, x_p, y_p) \mapsto (s, x_1, \dots, x_r, s, y_1, \dots, y_p)$$
then ∇^* and the external power operations are compatible with the multiplication of $\widetilde{MU}$;
5. $EP^{2r}_{n,p}(\Phi_{\gamma_r}) = \Phi_{\gamma_r(p)}$, where γ_r is the universal complex vector bundle on $BU(r)$.

Sketch proof. We just decode the definitions. □

Definition 7.6 Given a positive integer n and a prime p, let $\Delta : X \to X^p$ be the diagonal map, then we have

$$\Delta\colon L^n(p)_+ \wedge X \to \Gamma^{p+}_n X$$

The *mod-*p *total power operation of degree* n is defined to be

$$\begin{array}{rccc} \mathcal{P}^{2r}_{n,p} : & \widetilde{MU}^{2r}(X) & \longrightarrow & \widetilde{MU}^{2pr}(L^n(p)_+ \wedge X) \\ & \alpha & \longmapsto & \Delta^* EP^{2r}_{n,p}(\alpha) \end{array}$$

for each r. For convenience, we may simply call it total power operations if the notations are clear.

Proposition 7.7 The total power operations have the following properties for any integers r, n, prime p and $\alpha \in \widetilde{MU}^{2r}(Y)$;

1. Let $f\colon X \to Y$ be a morphism, then $f^*\mathcal{P}^{2r}_{n,p}(\alpha) = \mathcal{P}^{2r}_{n,p}(f^*\alpha)$;
2. $i_n^*\mathcal{P}^{2r+2s}_{n,p}(\alpha) = \mathcal{P}^{2r}_{n,p}(\alpha)$;
3. $\mathcal{P}^{2r+2s}_{n,p}(xy) = \mathcal{P}^{2r}_n(x)\mathcal{P}^{2s}_n(y)$.
4. Let $u : pt \to L^n(p)$ be an arbitrary map, and let $l = u_+ \wedge 1\colon X \to L^n(p)_+ \wedge X$, then $l^*\mathcal{P}^{2r}_{n,p}(x) = x^p$.

Construction 7.8 Note that the lens space $L^\infty(p)$ is a model of $B\mathbb{Z}/p$. If we let $\mathcal{P}^{2r}_p = \mathcal{P}^{2r}_{\infty,p}$ and $X = Y_+$ for some space Y, then we have

$$\mathcal{P}^{2r}_p\colon MU^{2r}(Y) \to MU^{2pr}(B\mathbb{Z}/p \times Y)$$

which is called *mod-p total power operation*. Geometrically, we see that $\{\mathcal{P}^{2r}_{n,p}(\alpha)\}_{n\in\mathbb{N}}$ is a "filtration" of $\mathcal{P}^{2r}_p(\alpha)$.

Theorem 7.9 (Landweber) For finite complex X, we have

$$MU^*(B\mathbb{Z}/n \times X) \cong MU^*(X)[[z]]/[n]_{F_{MU}}(z)$$

where z is the Euler class of the complex line bundle $B\mathbb{Z}/n \times_{\mathbb{Z}/n} \mathbb{C}$ with $\mathbb{Z}/n$ acting on $\mathbb{C}$ by multiplying $\exp(2\pi i/n)$. In particular, $MU^*(B\mathbb{Z}/p) = MU^*(pt)[[z]]/[p]_{F_{MU}}(z)$.

Sketch proof. Let ξ_1 to be the universal line bundle on $\mathbb{CP}^\infty$, then $\mathbb{Z}/n$ acts on $\xi_1^{\otimes n}$ by permuting factors cyclically. In particular, this action is a free action. Note that ξ_1 is contractible, $\xi_1^{\otimes n}$ is also contractible. Therefore, $\xi_1^{\otimes n}$ is a model of $E\mathbb{Z}/n$ and $e_{MU}(\xi_1^{\otimes n}) = [n]_{F_{MU}}(e_{MU}(\xi_1))$. We let $t = e_{MU}(\xi_1) \in MU^*(\mathbb{CP}^\infty)$ and the Gysin-Thom sequence associated to $\xi_1^{\otimes n}$ derives that $MU^*(B\mathbb{Z}/n) = MU^*(pt)[[t]]/B\mathbb{Z}/n(t)$. For the general case and details, see [Lan70]. □

Therefore, the Steenrod tom-Dieck operation on $X = Y_+$ is of the form:

$$\mathcal{P}^*_p\colon MU^*(Y) \to MU^{p*}(Y)[[z]]/[p]_{F_{MU}}(z)$$

With these results at hand, we next prove a highly technical lemma, which plays an important role in the next section.

Let $\zeta_{n,p}$ be the complex line bundle $S^{2n-1} \times_{\mathbb{Z}/p} \mathbb{C} \to L^n(p)$ and let $z_n = e_{MU}(\zeta_{n,p})$. The following technical lemma is the goal of this section.

Lemma 7.10 Suppose $x \in MU^q(X \times L^n(p))$ such that $x \cdot z_n = 0$, then there exists $y \in MU^q(X)$ such that $y \cdot \theta_p(z_{n-1}) = j_n^*(x)$, where $j_n\colon X \times L^{n-1}(p) \to X \times L^n(p)$ is induced by the inclusion $i_n : L^{n-1}(p) \hookrightarrow L^n(p)$. (Note that θ_p is defined in Convention 2.11.)

Proof. Firstly, we have

$$Th(\zeta_{n-1,p}) \cong (S^{2n-1} \times D^1/S^{2n-1} \times S^1)/p \cong L^n(p)/L^1(p) \cong L^n(p)/S^1$$

Let $\phi\colon MU^i(L^n(p)) \to MU^{i+2}(L^n(p)/S^1)$ be the Thom isomorphism. In particular, $\phi(1)$ is the Thom class of $\zeta_{n-1,p}$.

Let $q\colon L^n(p) \to L^n(p)/S^1$ be the natural quotient map, then we have

$$q^*\phi\colon MU^i(L^{n-1}(p)) \to \widetilde{MU}^{i+2}(L^n(p))$$

If $i = 0$, then $q^*\phi(1)$ is the Euler class of $\zeta_{n,p}$ on $L^n(p)$, because the following diagram commutes

$$\begin{tikzcd}
\widetilde{MU}^2(L^n(p)) \arrow[r, equal] & \widetilde{MU}^2(L^n(p)) \\
\widetilde{MU}^2(Th(\zeta_{n-1,p})) \arrow[u, "q^*"] & \widetilde{MU}^2(Th(\zeta_{n,p})) \arrow[l, "i_n^*"'] \arrow[u, "s^*"'] \\
MU^0(L^{n-1}(p)) \arrow[u, "\phi"] & MU^0(L^n(p)) \arrow[l, "i_n^*"'] \arrow[u, "\phi"']
\end{tikzcd}$$

where $s\colon L^n(p) \to Th(\zeta_{n,p})$ is the zero section. Therefore, we may write $q^*\phi = s^*\phi(1) = z_n$, according to the conventions.

Next, we consider the following diagram

$$\begin{tikzcd}
\widetilde{MU}^1(S^1) \arrow[r, "\delta"] \arrow[d, "u"] & \widetilde{MU}^2(L^n(p)/S^1) \arrow[r, "q^*_{MU}"] \arrow[d, "u"] & \widetilde{MU}^2(L^n(p)) \arrow[d, "u"] \\
H^1(S^1) \arrow[r, "\delta_H"] \arrow[d, equal] & H^2(L^n(p)/S^1) \arrow[r, "q^*_H"] \arrow[d, equal] & H^2(L^n(p)) \arrow[d, equal] \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}/p
\end{tikzcd}$$

where the rows are exact sequences induced by the pair $(L^n(p), S^1)$ and u is the Steenrod-Thom homomorphism $MU \to H\mathbb{Z}$ induced the Thom classes in ordinary cohomology. Let $\iota \in \widetilde{MU}^1(S^1) \cong \mathbb{Z}$ be a generator. Note that $u(\iota)$ is also a generator of $H^1(S^1)$ and we have $\delta_H^* \circ u(\iota) = p\alpha$ for a generator $\alpha \in H^2(L^n(p)/S^1)$. Since that q_H^* is surjective, then $q_H^*(\alpha) \neq 0$ and we may choose a suitable orientation u such that $q_H^*(\alpha) = u(z_n)$ (recall Proposition 2.10).

Recall that $H^2(X) \cong [X, \mathbb{CP}^\infty]$ is the collection of isomorphic classes of complex line bundles over X. Then we let $Q_{n,p} \to L^n(p)/S^1$ be a complex line bundle such that $e_H(Q_{n,p}) = \alpha$. Since $q_H^*(\alpha) = u(z_n) = e_H(\zeta_{n,p})$, we have $q_H^*(Q_{n,p}) = \zeta_{n,p}$.

Note that $\zeta_{n,p}^{\otimes p}$ is trivial, we have $e_U(\zeta_{n,p}^{\otimes p}) = 0 = q_{MU}^*(e_U(Q_{n,p}^{\otimes p}))$, thus

$$e_U(Q_{n,p}^{\otimes p}) = \delta(m\iota)$$

for some $m \in \mathbb{Z}$.

Now we combine these results and we have

$$pm\alpha = \delta_H \circ u(m\iota) = u(e_U(Q_{n,p}^{\otimes p})) = e_H(Q_{n,p}^{\otimes p}) = pe_H(Q_{n,p}) = p\alpha$$

Therefore $m = 1$ and $\delta(\iota) = e_U(Q_{n,p}^{\otimes p})) = [p]_{F_{MU}}(e_U(Q_{n,p}))$. According to the Thom isomorphism theorem, there exists $x_{n-1} \in MU^0(L^{n-1}(p))$ such that $\delta(\iota) = \phi(x_{n-1})$. Let

$$i\colon L^{n-1}(p) \to L^n(p) \to L^n(p)/S^1$$

be the zero section.

Note that $i^*\phi(v) = z_{n-1} \cdot v$ for any $v \in MU^*(L^{n-1}(p))$ according to the definition of Thom isomorphism and the Euler class. Hence

$$z_{n-1} \cdot x_{n-1} = i^*\phi(x_{n-1}) = i^*e_U(Q_{n,p}^{\otimes p}) = e_U(i^*Q_{n,p}^{\otimes p}) = e_U(\zeta_{n-1,p}^{\otimes p}) = 0$$

Hence $z_{n-1} \cdot x_{n-1} = 0$. In particular, if we let $n \to \infty$, then we have $z_\infty \cdot x_\infty = 0$.

Now we let $\pi\colon MU^*(X)[[z]] \to MU^*(X \times L^\infty(p))$ be the quotient map such that $\pi(z) = z_\infty$ and there exists some $x \in MU^*(X)[[z]]$ such that $\pi(x) = x_\infty$. According to Theorem 7.9, we must have $z \cdot x = [p]_{F_{MU}}(z) \cdot (a_0 + \sum_{i>0}^{\infty} a_i z^i)$ in $MU^*(X)[[z]]$. Therefore

$$x_\infty = [p]_{F_{MU}}(z)/z(a_0 + \sum_{i>0} a_i z_\infty^i) = a_0\theta_p(z_\infty)$$

in $MU^*(X \times L^\infty(p))$.

Let $b_n : L^n(p) \to L^\infty(p)$ be the evident inclusion, then consider the following diagram

$$\begin{tikzcd}
\widetilde{MU}^1(S^1) \arrow[r, "\delta"] \arrow[d, equal] & \widetilde{MU}^2(L^\infty(p)/S^1) \arrow[d, "\bar{b}_n^*"] & \widetilde{MU}^0(L^\infty(p)) \arrow[l, "\phi"'] \arrow[d, "b_{n-1}^*"] \\
\widetilde{MU}^1(S^1) \arrow[r, "\delta"] & \widetilde{MU}^2(L^n(p)/S^1) & \widetilde{MU}^0(L^{n-1}(p)) \arrow[l, "\phi"']
\end{tikzcd}$$

Since $b_{n-1}^*(x_\infty) = x_{n-1}$, we have $x_{n-1} = a\theta_p(z_{n-1})$ for each n and some $a \in \widetilde{MU}^1(X_+ \wedge S^1) \cong MU^*(X)$. Therefore, $\phi^{-1} \circ \delta(\iota) = a\theta_p(z_{n-1})$.

Then we consider the commutative diagram of $MU^*(X)$-modules.

$$\begin{tikzcd}
\widetilde{MU}^q(X_+) \arrow[r, "\Sigma \simeq"] \arrow[d, "\bar{\delta}"] & \widetilde{MU}^{q+1}(X_+ \wedge S^1) \arrow[d, "\delta"] \\
\widetilde{MU}^q(X_+ \wedge L^{n-1}(p)_+) \arrow[r, "\phi \simeq"] & \widetilde{MU}^{q+2}(X_+ \wedge L^n(p), X_+ \wedge S^1) \arrow[d, "r"] \\
\widetilde{MU}^q(X_+ \wedge L^n(p)_+) \arrow[u, "j_n^*"] \arrow[r, "\cdot z_n"] & \widetilde{MU}^{q+2}(X_+ \wedge L^n(p))
\end{tikzcd}$$

where the right column is the exact sequence induced by the pair $(X_+ \wedge L^n(p), X_+ \wedge S^1)$ and $\bar{\delta} = \phi^{-1}\Sigma\delta$. For $x \in MU^q(X \times L^n(p)) = \widetilde{MU}^q(X_+ \wedge L^n(p)_+)$ such that $x \cdot z_n = 0$, we have $r \circ \phi \circ j_n^*(x) = 0$, then $\phi \circ j_n^*(x) = \delta(w)$ for some $w \in \widetilde{MU}^q(S^1)$. Thus

$$\begin{aligned}
j_n^*(x) = \bar{\delta}\Sigma^{-1}(w) =& \phi^{-1}\delta\Sigma\Sigma^{-1}(w) \\
=& \Sigma^{-1}(w) \cdot \phi^{-1}\delta\Sigma(1) \\
=& \Sigma^{-1}(w) \cdot \phi^{-1}\delta(\iota) \\
=& \Sigma^{-1}(w) a \cdot \theta_p(z_{n-1})
\end{aligned}$$

If we let $y = \Sigma^{-1}(w) \cdot a$, then we prove the lemma. □

8 THE STRUCTURE OF $U^*(X)$

The definition of $U^*(X)$ is for an unbased space X. For a based space (X, x_0), the reduced cobordism theory is defined to be

$$\widetilde{U}^*(X) := \ker(i^* : U^*(X) \to U^*(x_0))$$

where $i : x_0 \to X$ is the inclusion of based point.

Proposition 8.1 There are some facts about the reduced cobordism theory:

1. $U^{2j-1}(X) \cong \widetilde{U}^*(S^1 \times X/\{*\} \times X)$;
2. $U^{2j}(X) \cong \widetilde{U}^{2j+2}(S^2 \times X/\{*\} \times X)$;
3. $\widetilde{U}^{2j-1} \cong \widetilde{U}^{2j}(\Sigma X)$
4. $U^*(X) \cong \widetilde{U}^*(X) \oplus U^*(x_0)$

Lemma 8.2 $\widetilde{U}^0(X)$ is a nilpotent ideal of $U^0(X)$.

Sketch proof. Let $\Delta_n : X \to X \wedge X \wedge \cdots \wedge X$ be the diagonal map and Δ_n^* is null homotopic if $n > \dim X$, because $X \wedge X \cdots \wedge X$ has no non-trivial cells for dimension less than n and we can use cellular approximation to make Δ_n homotopic to a constant map. Second, the image of Δ_n^* is $\widetilde{U}^0(X)^n$. □

Theorem 8.3 If X is of the homotopy type of a compact smooth manifold, then

$$U^*(X) = C \cdot \sum_{q \geqslant 0} U^q(X)$$
$$\widetilde{U}^*(X) = C \cdot \sum_{q > 0} U^q(X)$$

Sketch proof. We give the outline of the proof here.

1. Reduce the case to even degrees, namely we prove the theorem with the assumption that
$$\widetilde{U}^{2*}(X) = C \cdot \bigoplus_{q>0} U^{2q}(X)$$

2. Now we set
$$R = C \cdot \bigoplus_{q>0} U^{2q}(X)$$
and we need to show $U^{2*}(X) = R$.
   a) The equation is true for $R^j = U^j(X)$ for $j > 0$.
   b) The inductive hypothesis: suppose $R^{-2j} = U^{-2j}(X)$ for any $j < q$, where q is a positive integer.
   c) With the inductive hypothesis, we just need to show $R^{-2q}_{(p)} = U^{-2q}(X)_{(p)}$ for any prime p-localization.
   d) (**Key Step**) Proceed the induction by using Steenrod operations on cobordism theory. The rough idea is to do operations on $x \in U^{-2q}(X)$ such that x can be decomposed to be a sum of elements in R.

We just show the details in the key step. For some $x \in \widetilde{U}^{-2q}(X)$ and for some large n, we have

$$w^{n+q} P_\xi x = \sum_{l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x) = w^n x + \sum_{0 < l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x) \tag{9}$$

in U_ξ for any principal $\mathbb{Z}/p$-bundle ξ, where $a(v)^\alpha = \prod a_j(v)^{\alpha_j}$ according to Theorem 6.10.

Now we let ε_m be principal $\mathbb{Z}/p$-bundle $S^{2m-1} \to L^m(p) = S^{2m-1}/p$.

By localization on p and Equation 7, we have

$$v^{p-1} = w \cdot \theta(v)$$

where θ is a power series with the coefficients in $C_{(p)}$ such that $\theta^{-1}(x) = (p-1)! + \sum_{j \geqslant 1} d_j x^{j-p+1}$.

Then we modify Equation 9 into

$$w^n(w^q P_{\varepsilon_m} x - x) = \sum_{0 < l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x) \tag{10}$$
$$(v^{p-1}\theta^{-1}(v))^n(w^q P_{\varepsilon_m} x - x) = \sum_{0 < l(\alpha) \leqslant n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x) \tag{11}$$
$$(v^{p-1})^n(w^q P_{\varepsilon_m} x - x) = \psi(v) \tag{12}$$

for any ε_m, where $\psi(x) \in R_{(p)}[[x]]$, since $s_\alpha(x) \in R_p$ according to the inductive hypothesis.

Now we let $m = n(p-1)$ and $m > 0$, then we have

$$v^m(w^q P_{\varepsilon_m} x - x) = \psi(v) \in U^*(L^m(p) \times X)_{(p)}$$

We may assume m is the minimal positive integer such that there indeed exists some formal power series $f(t) \in R_{(p)}[[t]]$ such that $v^m(w^q P_{\varepsilon_m} x - x) = f(v)$.

Let $i : X \to L^m(p) \times X$ be an inclusion for some point at $L^m(p)$ and $i^* v = 0$ because $i^* \sigma$ is a trivial bundle over X. Note that $i^*(\psi(v)) = \psi(0)$ and $\psi(0) = 0$ by previous equation. Therefore, $t \mid \psi(t)$ and we let $t\psi_1(t) = \psi(t)$:

$$v(v^{m-1}(w^q P_{\varepsilon_m} x - x) - \psi_1(v)) = 0$$

Note that

$$v^{m-1}(w^q P_{\varepsilon_m} x - x) - \psi_1(v) \in U^{2(m-1)-2q}(L^m(p) \times X).$$

By Lemma 7.10, there exists $y \in U^{2(m-1)-2q}(X)$ such that

$$j_m^*(v^{m-1}(w^q P_{\varepsilon_m} x - x) - \psi_1(v)) = y\theta_p(v) \in U^*(L^{m-1}(p) \times X)_{(p)}$$

In another way,

$$v^{m-1}(w^q P_{\varepsilon_{m-1}} x - x) = \psi_1(v) + y\theta_p(v)$$

(Warning: there exists abuse of notations. The definitions of v and w should adjust to the principal bundle ε_{m-1} automatically.)

By quotient the part on the base point, we may identify $y \in \widetilde{U}^{2(m-1)-2q}(X)_{(p)}$.

Here m must be 1, otherwise it against the minimality of m because $y \in R^{2(m-1)-2q}$ according to the inductive hypothesis.

For $m = 1$, we then have

$$w^q P_{\varepsilon_1} x - x = \psi_1(v) + y\theta_p(v) \in U^*(S^1/p \times X)_{(p)} \tag{13}$$

Let $i : X \to S^1/p \times X$ be a natural inclusion as we did it before and apply it to Equation 13, then we have

$$\begin{aligned} -x &= \psi_1(0) + py & q > 0 \quad (14) \\ x^p - x &= \psi_1(0) + py & q = 0 \quad (15) \end{aligned}$$

For the case $q > 0$: Since x is arbitrary, we have

$$\widetilde{U}^{-2q}(X) \subset R_{(p)}^{-2q} + p\widetilde{U}^{-2q}(X)_{(p)}$$

Then we have

$$\widetilde{U}^{-2q}(X)_{(p)} \subset R_{(p)}^{-2q} + p^n\widetilde{U}^{-2q}(X)_{(p)}$$

for any n, which means that $\widetilde{U}^{-2q}(X)_{(p)} \cong R_{(p)}^{-2q}$ p-locally. Since $\widetilde{U}^{-2q}(X)$ is a finitely generated abelian group, we have $\widetilde{U}^{-2q}(X)_{(p)} \subset R_{(p)}^{-2q}$.

For the case $q = 0$: Note that $x^p - x \in p\widetilde{U}^0(X) + R^0$, then let

$$\begin{array}{rccc} \gamma : & \widetilde{U}^0(X)/(p\widetilde{U}^0(X) + R^0) & \longrightarrow & \widetilde{U}^0(X)/(p\widetilde{U}^0(X) + R^0) \\ & z & \longmapsto & z^p \end{array}$$

be an endomorphism. Then $x \in \widetilde{U}^0(X)$ is a fixed point for γ. Since $\widetilde{U}^0(X)$ is a nilpotent ideal in $U^0(X)$ by Lemma 8.2, we conclude that $x \in p\widetilde{U}^0(X) + R^0$. Then we by using the techniques in the case $q > 0$, we deduce that $\widetilde{U}^0(X) = R^0$.

□

Corollary 8.4 Let L be the Lazard ring. The induced map $L \to MU^*$ is surjective.

9 QUILLEN'S THEOREM ON THE FORMAL GROUP LAW

Theorem 9.1 The induced map $L \to MU^*$ is bijective.

It remains to show the injective part. The rough idea is to use Landweber-Novikov operations to build a ring map $MU^* \to R$ for a simpler ring R with simple formal group law and show the composition $L \to MU^* \to R$ is eventually injective.

Let $\phi\colon MU \to H\mathbb{Z}$ be the canonical map by Proposition 2.10. Note that $\phi(e_U(L)) = e_H(L)$, where $e_U(L)$ (resp. $e_H(L)$) is the Euler class of a line bundle $L \to X$ in $MU^*(X)$ (resp .in $H^(X)$). Furthermore, ϕ also preserves Chern classes $c_\alpha(E)$ for vector bundle E. We define

$$\beta\colon U^*(X) \xrightarrow{s_t} U^*(X)[t_1,t_2,t_3,\dots] \xrightarrow{\phi} H^*(X)[t_1,t_2,t_3,\dots] \tag{16}$$

where s_t is the Landweber-Novikov operation.

Proposition 9.2 If L is a complex line bundle, then

$$\beta(e_U(L)) = \sum_{j\geqslant 0} t_j(e_H(L))^{j+1}$$

Proposition 9.3 The Lazard ring L is a polynomial ring over $\mathbb{Z}$ with a generator in degree q for each $q > 0$.

Let $\theta(x) = \sum_{j\geqslant 0} t_j x^{j+1}$, then we have

$$\begin{aligned}\beta F_{MU}(\theta(e_H(L_1)),\theta(e_H(L_2))) &= \sum_{i,j} \beta(c_{ij})\theta(e_H(L_1))\theta(e_H(L_2))\\ &= \beta F(e_U(L_1), e_U(L_2))\\ &= \beta(e_U(L_1\otimes L_2))\\ &= \theta(e_H(L_1\otimes L_2))\\ &= \theta(e_H(L_1)+e_H(L_2))\end{aligned}$$

then we have $(\beta F_{MU})(\theta(x),\theta(y)) = \theta(x+y)$. Note that $\theta(x) = x + \text{higher terms}$, there exists a power series $\theta^{-1}(x)$ such that $\theta\circ\theta^{-1}(x) = x$. Then we consider the following map

$$\begin{array}{ccccc} L & \xrightarrow{f} & U^* & \xrightarrow{\beta} & H^*[t_1,t_2,\dots] \cong \mathbb{Z}[t_1,t_2,\dots] \\ \\ F_{Univ} & \longmapsto & F_{MU} & \longmapsto & \theta^{-1*}G_a(x,y) \end{array}$$

where $G_a(x,y) = x+y$ the additive formal group law and θ^{-1*} means conjugation action of invertible power series on formal group law.

To prove the injectivity, we still need the following proposition.

Proposition 9.4 For each formal group law G over a $\mathbb{Q}$-algebra R, there exists a unique power series $\log_G(x)$ over G such that $G = \log_G^* G_a = G$.

Proposition 9.5 The map $\beta\circ f$ is injective.

Sketch proof. Since L is torsion free, we just need to should that $\mathbb{Q}\otimes\beta\circ f$ is injective.

Consider the natural transformation

$$\mathrm{Hom}_{\mathbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], -) \xrightarrow{(\beta\circ f)^*} \mathrm{Hom}_{\mathbf{Cring}}(L, -)$$

Given a ring R, there is an evident bijection between $\mathrm{Hom}_{\mathbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], R)$ and the set of power series in $R[[x]]$ divided by x. Specifically, the bijection is defined to be

$$u \in \mathrm{Hom}_{\mathbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], R) \mapsto \theta_u(x) := \sum u(t_j)x^{j+1}$$

For $\mathbb{Q}$-algebra R, we have

$$\mathrm{Hom}_{\mathbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], R) \cong \mathrm{Hom}_{\mathbb{Q}-\mathcal{A}\mathrm{lg}}(\mathbb{Q}[t_1, t_2, \dots], R)$$

by tensoring $\mathbb{Q}$. Similarly, we have

$$\mathrm{Hom}_{\mathbf{Cring}}(L, R) \cong \mathrm{Hom}_{\mathbb{Q}-\mathcal{A}\mathrm{lg}}(\mathbb{Q} \otimes_{\mathbb{Z}} L, R)$$

According to our convention and previous identifications, we have $(\beta \circ f)^*(\theta_u) = (\theta_u^{-1})^* G_a(x, y)$. Then Proposition 9.4 indicates that $\mathbb{Q} \otimes_{\mathbb{Z}} (\beta \circ f)^*$ is actually an isomorphism. Therefore, by Yoneda lemma, $\mathbb{Q} \otimes (\beta \circ f)$ is an isomorphism. $\square$

REFERENCES

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, H_∞ *ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986. MR 836132

[Car17] Christian Carrick, *An elementary proof of quillen's theorem for complex cobordism*, `https://www.math.ucla.edu/~carrick/Thesis.pdf`, 2017.

[Lan70] Peter S. Landweber, *Coherence, flatness and cobordism of classifying spaces*, Proc. Advanced Study Inst. on Algebraic Topology (Aarhus, 1970), Vol. II, Aarhus Univ., Matematisk Inst., Aarhus, 1970, pp. 256–269. MR 0271964

[Laz55] Michel Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France **83** (1955), 251–274. MR 73925

[Qui69] Daniel Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298. MR 253350

[Qui71] Daniel G. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29–56 (1971). MR 290382

[Rud98] Yuli B. Rudyak, *On Thom spectra, orientability, and cobordism*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, With a foreword by Haynes Miller. MR 1627486

[Tho54] René Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86. MR 61823