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The Bousfield-Kuhn functor and topological André-Quillen cohomology

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Abstract We construct a natural transformation from the Bousfield-Kuhn functor evaluated on a space to the Topological André-Quillen cohomology of the $K(n)$-local Spanier–Whitehead dual of the space, and show that the map is an equivalence in the case where the space is a sphere. This results in a method for computing unstable v_n-periodic homotopy groups of spheres from their Morava E-cohomology (as modules over the Dyer-Lashof algebra of Morava E-theory). We relate the resulting algebraic computations to the algebraic geometry of isogenies between Lubin–Tate formal groups.

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1 Introduction

Let X be a simply connected space. Quillen [41] introduced two algebraic models of its rational homotopy type: a rational differential graded Lie algebra $\mathcal{L}(X)$ and a rational cocommutative differential graded coalgebra $\mathcal{C}(X)$. The rational homotopy groups of X are given by the homology of $\mathcal{L}(X)$ and the rational homology is given by the homology of $\mathcal{C}(X)$. The differential graded Lie algebra $\mathcal{L}(X)$ can be obtained by taking the derived primitives of the differential graded coalgebra $\mathcal{C}(X)$. Sullivan [47] reformulated this theory in concrete terms in the case where X is of finite type. In Sullivan's theory, one takes a minimal model $\Lambda(X)$ of the differential graded algebra given by the dual of $\mathcal{C}(X)$. The underlying graded commutative algebra of the minimal model $\Lambda(X)$ is free, and the rational homotopy groups of X are recovered from the dual of the indecomposables of $\Lambda(X)$:

$$\pi_*(X)_{\mathbb{Q}} \cong (Q\Lambda(X))^\vee . \tag{1.1}$$

Thus the rational homotopy groups of a space can be computed by taking the dual of the derived indecomposables of a commutative algebra model of its rational cochains.

Work of Kriz [27], Goerss [21], Mandell [36], and Dwyer–Hopkins [36, Appendix C] shows that the unstable p-adic homotopy type of a simply connected finite type space is similarly encoded in its $\bar{\mathbb{F}}_p$-valued singular cochains. Unfortunately, the p-adic analog of (1.1) fails: the space of derived indecomposables of the resulting E_∞-algebra is contractible [37, Thm. 3.4].

One can nevertheless ask if there are other localizations of unstable homotopy groups for which the analog of (1.1) holds. Unstable chromatic homotopy theory [14] suggests that the p-primary unstable v_h-periodic homotopy groups are a likely candidate, as rational homotopy is the $h = 0$ case of this theory. Integral homotopy groups are assembled out of the v_h-periodic homotopy groups for all h and p. The purpose of this paper is to show that an analog of (1.1) sometimes holds in the case of v_h-periodic homotopy groups.

To state our results precisely, we will need to establish some notation. Fix a prime p and a height $h \geq 1$. Let E denote the Morava E-theory spectrum E_h associated to the Lubin–Tate universal deformation $\mathbb{G}$ of the Honda height h formal group $\mathbb{G}_0/\mathbb{F}_{p^h}$. Let $\mathfrak{m}$ denote the maximal ideal of E_*, and let K denote the even periodic variant of Morava K-theory $K(h)$ with

$$K_* = E_*/\mathfrak{m}.$$

We shall use $(-)_K$ to denote localization with respect to K. In this paper, unless explicitly stated otherwise, $E_*(-)$ will denote completed E-homology

$$E_*Y = \pi_*(E \wedge Y)_K.$$

Let $T = T(h)$ denote the telescope of a v_h-self map on a type h complex. Bousfield and Kuhn (see, for example, [33]) constructed functors Φ_T (for each $h \geq 1$) from pointed spaces to spectra with the properties that

$$\begin{aligned} \Phi_T(\Omega^\infty Y) &\simeq Y_T, \\ \pi_*\Phi_T(X) &\cong v_h^{-1}\pi_*(X)^\wedge. \end{aligned}$$

Here, $v_h^{-1}\pi_*(X)^\wedge$ are the completed v_h-periodic homotopy groups of X.[1] In this paper, we will be interested in the K-local variant

$$\Phi(X) := \Phi_T(X)_K.$$

If the telescope conjecture is true, then $\Phi = \Phi_T$, but this conjecture is widely believed to be false for $h \geq 2$ [40]. The homotopy groups

$$\pi_*\Phi(X)$$

thus constitute a competing definition of "completed unstable v_h-periodic homotopy groups" which are potentially more computable than $\pi_*\Phi_T(X)$.

In this paper we construct a natural transformation between functors from pointed spaces to K-local spectra

$$c^{S_K} : \Phi(X) \to \mathrm{TAQ}_{S_K}(S_K^{X_+})$$

(the "comparison map") which relates $\Phi(X)$ to the topological André-Quillen cohomology of the augmented commutative S_K-algebra $S_K^{X_+}$. This relates unstable v_h-periodic homotopy to the dual of the derived indecomposables of the E_∞ algebra of cochains with values in the K-local sphere, which generalizes (1.1) in the case of $h = 0$. Our main theorem (Theorem 8.1) states that the comparison map is an equivalence when $X = S^q$ for q odd. The case of $h = 1$ is closely related to the thesis of Jennifer French [20].

[1] These should not be confused with the "uncompleted" unstable v_h-periodic homotopy groups studied by Bousfield, Davis, Mahowald, and others. These are given as the homotopy groups of the nth telescopic monochromatic layer of $\Phi_T(X)$ (see [32]).

In general, there is a class of finite spaces for which the comparison map is an equivalence. This is discussed further in [16], where such spaces are called "$\Phi_{K(h)}$-good". Thus the main result of this paper shows odd dimensional spheres are $\Phi_{K(h)}$-good. In [16], we show one can easily deduce from this that even dimensional spheres, the groups $SU(n)$, and the groups $Sp(n)$, are $\Phi_{K(h)}$-good. Also, finite products of finite $\Phi_{K(h)}$-good spaces are $\Phi_{K(h)}$-good. However, the work of Langsetmo and Stanley [34] (in the case of $h = 1$) and Brantner and Heuts [11] (h arbitrary) shows that mod p Moore spaces are not $\Phi_{K(h)}$-good. Brantner and Heuts also show in [11] that wedges of spheres of dimension greater than 1 are not $\Phi_{K(h)}$-good.

By the work of Ching [18], $\mathrm{TAQ}_{S_K}(S_K^{X_+})$ has the structure of an algebra over the operad formed by the Goodwillie derivatives $\partial_*(\mathrm{Id})$ of the identity functor on pointed spaces. As this operad is Koszul dual to the commutative operad in spectra, it may be regarded as a topological analog of the (shifted) Lie operad. Thus, at least for the class of finite $\Phi_{K(h)}$-good spaces, one could regard $\mathrm{TAQ}_{S_K}(S_K^{X_+})$ as a Lie algebra model for the unstable v_h-periodic homotopy type of X, mimicking in higher heights Quillen's Lie algebra model $\mathcal{L}(X)$ for the rational homotopy type. This idea has since been successfully pursued by Heuts [22], and is outlined in [16].

We apply our main theorem to understand the v_h-periodic Goodwillie tower of the identity evaluated on odd spheres. This constitutes a step in the program begun by Arone and Mahowald [6] to generalize the Mahowald–Thompson approach to unstable v_1-periodic homotopy groups of spheres [35,50]. Work of Arone–Mahowald [6] and Arone–Dwyer [2] shows that applying Φ to the Goodwillie tower of the identity evaluated on S^q (q odd) gives a (finite) resolution

$$\Phi(S^q) \to (L(0)_q)_K \to (L(1)_q)_K \to (L(2)_q)_K \to \cdots \to (L(h)_q)_K. \tag{1.2}$$

Here $L(k)_q$ denotes the Steinberg summand of the Thom spectrum of q copies of the reduced regular representation of $(\mathbb{Z}/p)^k$, as described in §5.

We show (Theorem 9.1) that the E-homology of the resolution (1.2) is isomorphic to the dual of the Koszul resolution of the (degree q) Dyer-Lashof algebra Δ^q for Morava E-theory (2.9). This results (Corollary 9.2) in a spectral sequence having the form

$$\mathrm{Ext}^s_{\Delta^q}(\widetilde{E}^q(S^q), \bar{E}_t) \Rightarrow E_{q+t-s}\Phi(S^q). \tag{1.3}$$

This is related to unstable v_h-periodic homotopy groups of spheres by the homotopy fixed point spectral sequence [19]

$$H^s_c(\mathbb{S}_h; E_t\Phi(S^q))^{Gal} \Rightarrow \pi_{t-s}\Phi(S^q).$$

In [43], the second author defined the *modular isogeny complex*, a cochain complex geometrically defined in terms of finite subgroups of the formal group $\mathbb{G}$, mimicking the structure of the building for $GL_h(\mathbb{F}_p)$. We show that the cohomology of the modular isogeny complex is the dual of the Koszul resolution for Δ^q, and use this to give a modular description of the differentials in the Koszul resolution. This gives a modular interpretation of the E_2-term of the spectral sequence (1.3). Presumably this modular interpretation is related to the "pile" interpretation of unstable chromatic homotopy, proposed by Ando, Morava, Salch, and others, but we do not pursue this here (see [44]).

The results of this paper were first announced in 2012, however it took many years to resolve some thorny technical issues which emerged, the most significant of which involved the structure of the E-cohomology of QX as an algebra over the Morava E-theory Dyer-Lashof algebra. In fact, the first preprint version of this paper made an erroneous claim about this structure, and the authors are indebted to Nick Kuhn for discovering this error.

During the evolution of the present form of this paper, many interesting developments have emerged. The first author's Ph.D. student Guozhen Wang used some of the techniques of this paper to give a complete computation of $\pi_*\Phi(S^3)$ for $h = 2$ and $p \geq 5$ [52]. Yifei Zhu [54] has used our techniques to compute $E_*\Phi(S^q)$ for q odd and $h = 2$. Finally, Arone-Ching [1] and Heuts [22] have recently announced alternative approaches to prove of Theorem 8.1 which are more conceptual than our computational approach, and more general, in the sense that they apply to the functor Φ_T as well as the functor Φ. Both of these alternative approaches are are outlined in [16].

Organization of the paper

In Sect. 2 we summarize the results about the Morava E-theory Dyer-Lashof algebra Δ^q we will need for the rest of the paper.

In Sect. 3 we introduce a form of André-Quillen homology for unstable algebras over Δ^q, as well as a Grothendieck-type spectral sequence which relates this homology to $\mathrm{Tor}^{\Delta^q}_*$.

In Sect. 4 we introduce a bar construction model for Kuhn's filtration on topological André-Quillen homology. The layers of this filtration, as well as the layers of the Goodwillie tower of the identity, are equivalent to the spectra $L(k)_q$. We also construct a Morava K-theory analog of a spectral sequence of Basterra, which relates the Morava K-homology of topological André-Quillen homology to the algebraic André-Quillen homology groups introduced in Sect. 3.

In Sect. 5 we show the E-homology of the spectrum $L(k)_q$ is dual to the k-th term of the Koszul resolution for Δ^q.

In Sect. 6, we define the comparison map, and investigate its behavior on infinite loop spaces. This requires a technical result on the structure of the E-cohomology of QX as a Δ^*-algebra, which is relegated to Appendix B.

In Sect. 7, we discuss a K-local analog of Weiss's orthogonal calculus.

In Sect. 8, we prove that the comparison map is an equivalence on odd spheres, by using K-local Weiss calculus to play the Goodwillie tower off of the Kuhn filtration.

In Sect. 9 we use the identification of the Goodwillie tower with the Kuhn filtration to compute the E-homology of the k-invariants of the Goodwillie tower. From these results we establish the spectral sequence (1.3).

In Sect. 10 we give our modular description of the Koszul resolution for Δ^q, by showing that it is given by the cohomology of the modular isogeny complex.

There are two appendices. Appendix A contains a summary of results on norms, transfers, and Euler classes. These are needed in Appendix B, which is devoted to a detailed study of the Morava E-cohomology of QX.

Conventions

- $(-)^\vee$ denotes the E_0-linear dual when applied to an E_0-module, and the Spanier–Whitehead dual when applied to a spectrum.
- Sp denotes the category of symmetric spectra with the positive stable model structure [38], and shall simply refer to these as "spectra".
- If R is a commutative S-algebra, A is a commutative augmented R-algebra, and M is an R-module, we will let $\mathrm{TAQ}^R(A; M)$ denote topological André-Quillen homology of A (relative to R) with coefficients in M. Similarly we let $\mathrm{TAQ}_R(A; M)$ denote the corresponding topological André-Quillen cohomology. If $M = R$, we shall omit it from the TAQ-notation.
- For an endofunctor F of Top_*, we shall let $\{P_n(F)\}$ denote its Goodwillie tower, with fibers $D_n(F) = \Omega^\infty \mathbb{D}_n(F)$ and derivatives $\partial_n(F)$.

2 Recollections on the Dyer-Lashof algebra for Morava E-theory

Morava E-theory of symmetric groups

Strickland studied the Hopf ring

$$E^0(\amalg_n B\Sigma_n),$$

where the two products $\cdot$ and $*$ are given respectively by the cup product and transfers associated to the inclusions

$$\Sigma_n \times \Sigma_m \to \Sigma_{n+m} \tag{2.1}$$

and the coproduct is given by the restrictions associated to the above inclusions. Note that there are actually inclusions (2.1) for every partition of the set $\{1, \ldots, n\}$ into two pieces. We shall refer to the stabilizers of such partitions as *partition subgroups*.

Strickland [46] proved that $E^0(\amalg_n B\Sigma_n)$ is a formal power series ring (with respect to the $*$ product) with indecomposables

$$\prod_{k \geq 0} E^0(B\Sigma_{p^k})/\operatorname{Tr}(\text{proper partition subgroups}).$$

Let

$$\operatorname{Sub}_{p^k}(\mathbb{G}) = \operatorname{Spf}(\mathcal{S}_{p^k})$$

be the (affine) formal scheme of subgroups of $\mathbb{G}$ of order p^k. For a Noetherian complete local E_0-algebra R, the R-points of $\operatorname{Sub}_{p^k}(\mathbb{G})$ are given by

$$\operatorname{Sub}_{p^k}(\mathbb{G})(R) = \{H < \mathbb{G} \times_{\operatorname{Spf}(E_0)} \operatorname{Spf}(R) \ : \ |H| = p^k\}.$$

Strickland also shows that there is a canonical isomorphism

$$E^0(B\Sigma_{p^k})/\operatorname{Tr}(\text{proper partition subgroups}) \cong \mathcal{S}_{p^k} \tag{2.2}$$

of rings, where the product on the left-hand-side is induced from the $\cdot$ product.

Let

$$s : E_0 \to \mathcal{S}_{p^k}$$

be the map induced topologically from the map

$$B\Sigma_{p^k} \to *,$$

which gives $\mathcal{S}_{p^k}$ an E_0-algebra-structure. We regard $\mathcal{S}_{p^k}$ as a left module over E_0 by the module structure induced by s. With respect this module structure, $\mathcal{S}_{p^k}$ is free of finite rank [42, Prop. 6.3].

Give the ring $\mathcal{S}_{p^k}$ the structure of a right E_0-module via the ring map

$$t : E_0 \to \mathcal{S}_{p^k} \tag{2.3}$$

which associates to an R-point $H < \mathbb{G} \times_{\operatorname{Spf}(E_0)} \operatorname{Spf}(R)$ the deformation

$$(\mathbb{G} \times_{\operatorname{Spf}(E_0)} \operatorname{Spf}(R))/H.$$

The map t arises topologically from the total power operation

$$E^0(*) \to E^0(B\Sigma_{p^k})$$

coming from the E_∞ structure of E [4, Sec. 12.4].

Morava E-theory of extended powers

For an E-module Y, define

$$\mathbb{P}_E(Y) := \bigvee_{i\geq 0} Y^{\wedge_E i}_{h\Sigma_i}.$$

In [42, §4], the second author defined a monad

$$\mathbb{T} : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$$

and a natural transformation

$$\mathbb{T}\pi_* Y \to \pi_*(\mathbb{P}_E Y)_K$$

which induces an isomorphism

$$[\mathbb{T}\pi_*(Y)]^\wedge_{\mathfrak{m}} \xrightarrow{\cong} \pi_*(\mathbb{P}_E Y)_K \tag{2.4}$$

if $\pi_* Y$ is flat as an E_*-module [42, Prop. 4.9]. There is a decomposition

$$\mathbb{T} = \bigoplus_{i\geq 0} \mathbb{T}\langle i\rangle$$

so that if $\pi_* Y$ is finite and flat, we have

$$\pi_*[Y^{\wedge_E i}_{h\Sigma_i}]_K \cong \mathbb{T}\langle i\rangle \pi_* Y_K. \tag{2.5}$$

The monad $\mathbb{T}$ comes equipped [42, Prop. 4.7] with natural isomorphisms

$$\mathbb{T}(M) \otimes_{E_*} \mathbb{T}(N) \xrightarrow{\cong} \mathbb{T}(M \oplus N). \tag{2.6}$$

In particular, if A is a $\mathbb{T}$-algebra, then A is a graded-commutative E_*-algebra in the following strong sense: not only do elements of odd degree anticommute, but also elements of odd degree square to 0 (see [42, Prop. 3.4] for an explanation of this phenomenon).

A convenient summary of the most important properties of the $\mathbb{T}$ construction is given in Section 3.2 of [45]. In particular, we note that if R is a $K(n)$-local commutative E-algebra, then $\pi_* R$ canonically admits the structure of a $\mathbb{T}$-algebra.

Lemma 2.7 *If M is a free E_*-module, then $\mathbb{T}M$ is a free graded commutative E_*-algebra in the above sense.*

Proof The rank 1 cases $M = E_*$ and $M = \Sigma E_*$ are discussed in the proof of Proposition 7.2 of [42]. The general case then follows from 2.6. □

Specializing to the case where $Y = \Sigma^q E$ (for $q \in \mathbb{Z}$), and $i = p^k$, we have

$$[\mathbb{T}\langle p^k \rangle E_*(S^q)]_q = [E_* S^{qp^k}_{h\Sigma_{p^k}}]_q = E_0(B\Sigma_{p^k})^{q\bar{\rho}_k}$$

where $\bar{\rho}_k$ denotes the reduced standard real representation of Σ_{p^k}, and $(B\Sigma_{p^k})^{q\bar{\rho}_k}$ denotes the associated Thom spectrum.

Consider the sub- and quotient modules

$$\mathrm{Prim}_q[k] \hookrightarrow E_0(B\Sigma_{p^k})^{q\bar{\rho}_k} \twoheadrightarrow \mathrm{Ind}_q[k]$$

where $\mathrm{Prim}_q[k]$ denotes the intersection of the kernels of transfers to proper partition subgroups, and $\mathrm{Ind}_q[k]$ denotes quotient by the sum of the images of the restrictions from proper partition subgroups. Both $\mathrm{Prim}_q[k]$ and $\mathrm{Ind}_q[k]$ are finite free E_0-modules, and (2.2) implies that there is a canonical isomorphism

$$\mathrm{Prim}_0[k] \cong \mathcal{S}^\vee_{p^k}.$$

Let ι_q denote the composite

$$\iota_q : \mathrm{Prim}_q[k] \to \mathrm{Ind}_q[k].$$

The suspension σ is shown in [42] to fit these modules together to give a diagram

$$\begin{tikzcd}
 & \mathrm{Prim}_{2m-1}[k] \arrow[d, "\iota_{2m-1}", "\cong"'] & \mathrm{Prim}_{2m}[k] \arrow[d, hook, "\iota_{2m}"] & \mathrm{Prim}_{2m+1}[k] \arrow[d, "\iota_{2m+1}", "\cong"'] & \cdots \\
\cdots \arrow[ur, "\sigma", "\cong"'] & \mathrm{Ind}_{2m-1}[k] \arrow[ur, "\sigma", "\cong"'] & \mathrm{Ind}_{2m}[k] \arrow[ur, "\sigma", "\cong"'] & \mathrm{Ind}_{2m+1}[k] \arrow[ur, "\sigma", "\cong"'] &
\end{tikzcd} \tag{2.8}$$

where ι_q is an isomorphism for q odd, and an inclusion with torsion cokernel for q even.

The Dyer-Lashof algebra for Morava E-theory

The algebra of additive power operations acting on cohomological degree q is given by

$$\Gamma^q = \bigoplus_k \mathrm{Prim}_{-q}[k].$$

This is contained (via the map ι_{-q}) in the larger algebra of indecomposable power operations

$$\Delta^q = \bigoplus_k \mathrm{Ind}_{-q}[k]. \tag{2.9}$$

In both rings, the ring E_0 is not central, and thus Γ^q and Δ^q have distinct left and right E_0-module structures. In the case of Γ^0, these left and right module structures are induced respectively from the left and right module structures of E_0 on $\mathcal{S}_{p^k}$ under the isomorphism

$$\Gamma^0[k] \cong \mathcal{S}_{p^k}^\vee. \tag{2.10}$$

The algebra Γ^q is the algebra of natural endomorphisms of the functor

$$\begin{aligned} U^q : \mathrm{Alg}_{\mathbb{T}} &\to \mathrm{Mod}_{E_0}, \\ A_* &\mapsto A_{-q}; \end{aligned}$$

see [45, §3.8]. It follows that the underlying E_*-module of a $\mathbb{T}$-algebra carries the structure of a graded Γ^*-module. The morphism (2.6) gives this Γ^*-module the structure of a graded-commutative Γ^*-algebra. The functors U^q thus assemble to give a functor

$$U^* : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{Alg}_{\Gamma^*}.$$

The algebra Δ^q is the algebra of natural endomorphisms of the functor

$$\begin{aligned} V^q : \mathrm{Alg}_{\mathbb{T}} \downarrow E_* &\to \mathrm{Mod}_{E_0}, \\ A_* &\mapsto [I(A)/I(A)^2]_{-q}.; \end{aligned} \tag{2.11}$$

see [45, §3.10].

The non-canonical natural isomorphisms $U^q \cong U^{q+2}$ and $V^q \cong V^{q+2}$ given by multiplication by a unit in E_{-2} induce non-canonical isomorphisms of algebras

$$\Gamma^q \cong \Gamma^{q+2}, \tag{2.12}$$

$$\Delta^q \cong \Delta^{q+2}. \tag{2.13}$$

The suspension induces canonical isomorphisms of algebras

$$\sigma : \Delta^q \xrightarrow{\cong} \Gamma^{q-1}. \tag{2.14}$$

In particular, all of the E_0-algebras Γ^q and Δ^q, for all q, are non-canonically isomorphic to each other.

For an E_*-module M we define[2]

$$\widehat{\mathbb{T}}M = \prod_{i \geq 0} \mathbb{T}\langle i \rangle M. \tag{2.15}$$

Note that by (2.6), Lemma 2.7, and (2.9), for M a free module over E_*, there is a non-canonical isomorphism

$$\mathbb{T}M \cong \mathrm{Sym}_{E_*}(\Delta^* \otimes_{E_*} M). \tag{2.16}$$

From this perspective, $\widehat{\mathbb{T}}$ is non-canonically isomorphic to the corresponding formal power series ring

$$\widehat{\mathbb{T}}M := \widehat{\mathrm{Sym}}_{E_*}(\Delta^* \otimes_{E_*} M). \tag{2.17}$$

Here $\widehat{\mathrm{Sym}}$ denotes the completed graded symmetric algebra. For an E-module Y, let

$$\widehat{\mathbb{P}}_E Y = \prod_{i \geq 0} Y^{\wedge_E i}_{h\Sigma_i}$$

denote the completed free commutative E-algebra. The we have the following lemma.

Lemma 2.18 *Suppose that Y is an E-module, and that $\pi_* Y$ is flat over E_*. Then* (2.4) *induces an isomorphism*

$$[\widehat{\mathbb{T}}\pi_*(Y)]^\wedge_{\mathfrak{m}} \xrightarrow{\cong} \pi_*(\widehat{\mathbb{P}}_E Y)_K.$$

Proof This follows from (2.5), together with the fact that K-localization commutes with products of E-local spectra [9, Cor. 6.1.3], and [42, Prop. 3.6-7]. □

[2] Warning: the functor $\widehat{\mathbb{T}}$ defined here is *different* from the one defined by Barthel-Frankland [10].

The Koszul resolution

Observe that the augmentation

$$\epsilon : \Delta^q = \bigoplus_{k \geq 0} \Delta^q[k] \to \Delta^q[0] = E_0 \tag{2.19}$$

sending $\Delta^q[k]$ to 0 for $k \geq 0$ endows E_0 with the structure of a Δ^q bi-module: we shall use the notation $\overline{E}_0$ to denote this Δ^q-bimodule. Let $\widetilde{\Delta}^q$ denote the kernel of the augmentation ϵ; it is likewise a Δ^q-bimodule.

Consider the normalized bar complex $B_*(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0)$ with

$$B_s(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0) = \overline{E}_0 \otimes_{E_0} (\widetilde{\Delta}^q)^{\otimes_{E_0} s} \otimes_{E_0} \overline{E}_0 \cong (\widetilde{\Delta}^q)^{\otimes_{E_0} s}.$$

Endow the complex $B_*(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0)$ with a grading by setting

$$B_s(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0)[k] := \bigoplus_{\substack{k = k_1 + \cdots + k_s \\ k_i > 0}} \Delta^q[k_1] \otimes_{E_0} \cdots \otimes_{E_0} \Delta^q[k_s].$$

Observe that since $\Delta^q[k]$ acts trivially on $\overline{E}_0$ for $k > 0$, the differential in the bar complex preserves this grading. Thus there is a decomposition of chain complexes

$$B_*(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0) = \bigoplus_{k \geq 0} B_*(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0)[k].$$

In [45], the second author proved that the algebras Δ^q are Koszul, as summarized in the following theorem.

Theorem 2.20 ([45], Prop. 4.6) *For each k, the kth graded part of the chain complex has homology concentrated in top degree:*

$$H_s(B_*(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0)[k]) = 0, \qquad s \neq k.$$

The top degree homology

$$C[k]_{-q} := H_k(B_*(\overline{E}_0, \widetilde{\Delta}^q, \overline{E}_0)[k])$$

is finitely generated and free as a right E_0-module; furthermore, $C[k]_{-q} = 0$ if $k > h$.

Thus we have

$$\mathrm{Tor}^k_{\Delta^q}(\overline{E}_0, \overline{E}_0) \cong C[k]_{-q},$$

$$\mathrm{Ext}^k_{\Delta^q}(\overline{E}_0, \overline{E}_0) \cong C[k]^\vee_{-q}.$$

Remark 2.21 Actually, in [45], it is proven that Δ^0 is Koszul, but using the isomorphisms (2.13) and (2.14), there are non-canonical isomorphisms $\Delta^0 \cong \Delta^q$. Therefore Δ^q is also Koszul.

If M is a left Δ^q-module, then the *Koszul complex* $C^{\Delta^q}_*(M)$ associated to M is the chain complex

$$C^{\Delta^q}_*(M) = \left(C[0]_{-q} \otimes_{E_0} M \xleftarrow{\delta_0} C[1]_{-q} \otimes_{E_0} M \xleftarrow{\delta_1} \cdots \right)$$

with differentials δ_k induced from the following diagram.

$$\begin{tikzcd}
B_{k+1}(\overline{E}_0, \Delta^q, M)[k+1] \arrow[r, "d_{k+1}"] \arrow[d, equal] & B_k(\overline{E}_0, \Delta^q, M)[k] \arrow[d, equal] \\
\Delta^q[1]^{\otimes(k+1)} \otimes_{E_0} M & \Delta^q[1]^{\otimes k} \otimes_{E_0} M \\
C[k+1]_{-q} \otimes_{\overline{E}_0} M \arrow[r, "\delta_k"'] \arrow[u, hook] & C[k]_{-q} \otimes_{\overline{E}_0} M \arrow[u, hook]
\end{tikzcd} \tag{2.22}$$

Here, the map d_{k+1} is the last face map in the bar complex $B_\bullet(\overline{E}_0, \Delta^q, M)$. We have

$$H_s(C^{\Delta^q}_*(M)) \cong \mathrm{Tor}^{\Delta^q}_s(\overline{E}_0, M).$$

Recall that the E_0-modules $C[k]_{-q}$ are projective. It follows that if M is projective as an E_0-module, the dual cochain complex computes Ext:

$$H^s(C^{\Delta^q}_*(M)^\vee) \cong \mathrm{Ext}^s_{\Delta^q}(M, \overline{E}_0).$$

3 Barr-Beck homology

Augmented $\mathbb{T}$-algebras

Consider the adjunction

$$\mathbb{T} : \mathrm{Mod}_{E_*} \leftrightarrows \mathrm{Alg}_{\mathbb{T}} : U^*.$$

A free $\mathbb{T}$-algebra $\mathbb{T}M$ is augmented over E_* by the map

$$\mathbb{T}M \to \mathbb{T}\langle 0\rangle M = E_*.$$

Thus the above adjunction restricts to an adjunction for augmented $\mathbb{T}$-algebras

$$\mathbb{T} : \mathrm{Mod}_{E_*} \leftrightarrows \mathrm{Alg}_{\mathbb{T}} \downarrow E_* : I(-)$$

where $I(-)$ is the kernel of the augmentation. The monad $\mathbb{T}$ contains a "non-unital" summand

$$\bar{\mathbb{T}} := \bigoplus_{i>0} \mathbb{T}\langle i\rangle.$$

Note that there is a natural isomorphism

$$I(\mathbb{T}M) \cong \bar{\mathbb{T}}M.$$

In particular, if A is an augmented $\mathbb{T}$-algebra, then $I(A)$ is a $\bar{\mathbb{T}}$-algebra.

Trivial $\mathbb{T}$-algebras

The monad $\bar{\mathbb{T}}$ is augmented over the identity functor via the projection

$$\bar{\mathbb{T}} \to \mathbb{T}\langle 1\rangle = \mathrm{Id}.$$

If M is an E_*-module, then via the augmentation we can give M the *trivial* $\bar{\mathbb{T}}$-algebra structure. We shall denote the resulting $\bar{\mathbb{T}}$-algebra by $\overline{M}$.

Remark 3.1 It may appear that this notation conflicts in with the definition of the Δ^q-module $\bar{E}_0$ immediately following Eq. (2.19). Note, however, that in the case of $M = E_*$, the corresponding $\mathbb{T}$-algebra $\bar{E}_*$ as defined above is a square-zero algebra, and hence is a Δ^*-module. The Δ^0 module structures on $\bar{E}_0$ then agree in both cases.

If X is an E-module spectrum, write $\overline{X}$ for this spectrum endowed with the structure of a non-unital E-algebra spectrum with trivial multiplication. We have the following.

Proposition 3.2 *If X is a K-local E-module spectrum, the evident identification $\pi_*\overline{X} \approx \overline{\pi_* X}$ is an isomorphism of $\bar{\mathbb{T}}$-algebras.*

Cotriple homology

Suppose that we are given a functor $F : \mathrm{Alg}_{\mathbb{T}} \downarrow E_* \to \mathcal{A}$ for $\mathcal{A}$ an abelian category. Barr and Beck [8] define a "cotriple homology" associated to F relative to the comonad $\mathbb{T}I(-)$ on $\mathrm{Alg}_{\mathbb{T}} \downarrow E_*$, which we shall simply denote $\mathbb{L}_* F$, as it could be viewed as a kind of left derived functor. Explicitly it may be computed in terms of the monadic bar construction as

$$\mathbb{L}_s F(A) \cong H_s(F(B_*(\mathbb{T}, \bar{\mathbb{T}}, I(A)))).$$

Derived functors of $\mathbb{T}$-indecomposables

Let N be an E_0-module. Consider the functor

$$\begin{gathered}\Omega^q_{\mathbb{T}/E_*}(-; N) : \mathrm{Alg}_{\mathbb{T}} \downarrow E_* \to \mathrm{Mod}_{E_0}, \\ A \mapsto \overline{N} \otimes_{\Delta^q} V^q(A).\end{gathered}$$

where V^q is the functor (2.11).

If $N = E_0$, we shall simply write

$$\Omega^q_{\mathbb{T}/E_*} A := \Omega^q_{\mathbb{T}/E_*}(A; E_0).$$

Combining (2.6), Lemma 2.7, and the definition of Δ^*, we have the following lemma.

Lemma 3.3 *Suppose that M is a free E_*-module. Then there is a natural isomorphism*

$$V^q(\mathbb{T}M) \approx \Delta^q \otimes_{E_0} M_{-q},$$

and hence a natural isomorphism

$$\Omega^q_{\mathbb{T}/E_*}(\mathbb{T}M; N) \cong M_{-q} \otimes_{E_0} N.$$

Corollary 3.4 *If $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$ is free as an E_*-module, then there is an isomorphism*

$$\mathbb{L}_s \Omega^q_{\mathbb{T}/E_*}(A; N) \cong H_s(B_*(\mathrm{Id}, \bar{\mathbb{T}}, I(A))_{-q} \otimes_{E_0} N).$$

A Grothendieck spectral sequence

Proposition 3.5 *Suppose that A is an augmented $\mathbb{T}$-algebra which is free as an E_*-module. Then there is a Grothendieck-type spectral sequence*

$$E^2_{s,t} = \operatorname{Tor}^{\Delta^q}_s(\overline{N}, \mathbb{L}_t V^q(A)) \Rightarrow \mathbb{L}_{s+t}\Omega^q_{\mathbb{T}/E_*}(A; N).$$

Proof Consider the double complex

$$C_{s,t} := B_s(\overline{N}, \Delta^q, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))).$$

Computing the spectral sequence for the double complex by running s-homology, then t-homology, we have

$$H_t H_s C_{s,t} \Rightarrow H_{s+t} \operatorname{Tot} C_{s,t}.$$

Using (2.6), Lemma 2.7, and the definition of Δ^q, we have

$$\begin{aligned}
H_t H_s C_{s,t} &= H_t H_s B_s(\overline{N}, \Delta^q, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\
&\cong H_t \operatorname{Tor}^{\Delta^q}_s(\overline{N}, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\
&\cong H_t \operatorname{Tor}^{\Delta^q}_s(\overline{N}, \Delta^q \otimes_{E_0} \bar{\mathbb{T}}^{\circ t} I(A)) \\
&\cong \begin{cases} H_t(B_t(\mathrm{Id}, \bar{\mathbb{T}}, I(A)) \otimes_{E_0} N), & s = 0, \\ 0, & s \neq 0. \end{cases}
\end{aligned}$$

The isomorphism of the second line uses the fact that $V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))$ is a free E_0-module when A is one, using Lemma 2.7. The isomorphism of the third line uses Lemma 3.3.

The spectral sequence therefore collapses to give an isomorphism

$$H_i \operatorname{Tot} C_{*,*} \cong \mathbb{L}_i \Omega^q_{\mathbb{T}/E_*}(A; N).$$

Running t-homology followed by s-homology therefore gives a spectral sequence

$$H_s H_t C_{s,t} \Rightarrow \mathbb{L}_{s+t}\Omega^q_{\mathbb{T}/E_*}(A; N).$$

Using the fact that Δ^q is free over E_0, we compute

$$\begin{aligned}
H_s H_t C_{s,t} &= H_s H_t B_s(\overline{N}, \Delta^q, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\
&\cong H_s B_s(\overline{N}, \Delta^q, H_t V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\
&\cong H_s B_s(\overline{N}, \Delta^q, \mathbb{L}_t V^q A)
\end{aligned}$$

$$\cong \mathrm{Tor}_s^{\Delta^q}(\overline{N}, \mathbb{L}_t V^q A).$$

□

The homology groups $\mathbb{L}_t V^q A$ appearing in the E^2-term of the Grothendieck spectral sequence are demystified by the following lemma. We write "$\mathbb{L}_* \Omega_{(-)/E_*}$" for the André-Quillen homology of augmented graded commutative E_*-algebras, where as in §2, graded commutativity implies that odd degree elements square to 0.

Lemma 3.6 *Suppose that $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$ is free as an E_*-module. Then there are isomorphisms*

$$\mathbb{L}_i V^q A \cong [\mathbb{L}_i \Omega_{A_*/E_*}]_{-q}.$$

Proof By Lemma 2.7, the bar resolution

$$B_\bullet(\mathbb{T}, \bar{\mathbb{T}}, I(A)) \to A$$

is a simplicial resolution of A by free graded commutative algebras. Since $V^*(-) = I(-)/I(-)^2$, the result follows. □

Corollary 3.7 *Suppose that $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$ is free as an augmented graded commutative E_*-algebra. Then the Grothendieck spectral sequence collapses to give an isomorphism*

$$\mathbb{L}_s \Omega^q_{\mathbb{T}/E_*}(A; N) \cong \mathrm{Tor}_s^{\Delta^q}(\overline{N}, V^q(A)).$$

Linearization

The definition of Δ^* gives rise to natural transformations

$$\Delta^* \otimes_{E_*} M \to V^*(\mathbb{T}M) = \bar{\mathbb{T}}(M)/(\bar{\mathbb{T}}(M))^2 \leftarrow \bar{\mathbb{T}}(M)$$

of functors. We have noted (Lemma 3.3) that if M is a free E_*-module, then $\Delta^* \otimes_{E*} M \to V^*(\mathbb{T}M)$ is an isomorphism. Hence, on the full subcategory of free E_*-modules, we obtain a natural transformation of monads

$$\mathcal{L} : \bar{\mathbb{T}}M \to \Delta^* \otimes_{E_*} M$$

on Mod_{E_*}. In [45], this transformation is observed to be linearization for projective M.[3] For $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$, the natural transformation $\mathcal{L}$ induces a map of chain complexes

[3] We warn the reader that our notation here differs slightly from that used in [45]: there the notation $\mathcal{L}_F$ is used for the linearization of a functor F, and $\epsilon : F \to \mathcal{L}_F$ is used for the natural transformation from a functor to its linearization.

$$\mathcal{L} : B(\mathrm{Id}, \bar{\mathbb{T}}, I(A))_{-q} \to B_*(\overline{E}_0, \Delta^q, V^q(A)) \tag{3.8}$$

and therefore a map

$$\mathcal{L} : \mathbb{L}_s \Omega^q_{\mathbb{T}/E_*} A \to \mathrm{Tor}_s^{\Delta^q}(\overline{E}_0, V^q(A)).$$

Lemma 3.9 *If A is free as a graded commutative E_*-algebra, the map (3.8) is a quasi-isomorphism.*

Proof This essentially follows Corollary 3.7 from an identification of the map (3.8) with the edge homomorphism of the Grothendieck spectral sequence. Specifically, consider the following commutative diagram of maps of chain complexes.

$$\begin{tikzcd}
\bigoplus_{s+t=n} B_s(\bar{E}_0, \Delta^q, V^q B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A))) \arrow[rr, "\mathrm{aug}_t", "\simeq"'] \arrow[d, "\mathrm{aug}_s"', "\simeq"] \arrow[drr, "\mathcal{L}"] & & B_n(\bar{E}_0, \Delta^q, V^q A) \\
\bar{E}_0 \otimes_{\Delta^q} V^q(B_n(\mathbb{T}, \bar{\mathbb{T}}, I(A))) \arrow[d, equal] & & \bigoplus_{s+t=n} B_s(\bar{E}_0, \Delta^q, B_t(\Delta^q, \Delta^q, V^q A)) \arrow[u, "\simeq", "\mathrm{aug}_t"'] \arrow[d, "\simeq"', "\mathrm{aug}_s"] \\
B_n(\mathrm{Id}, \bar{\mathbb{T}}, I(A))_{-q} \arrow[rr, "\mathcal{L}"'] & & B_n(\bar{E}_0, \Delta^q, V^q A)
\end{tikzcd} \tag{3.10}$$

Here the maps labeled aug_s and aug_t are the augmentations of the corresponding bar complexes, and $\mathcal{L}$ are the maps induced by linearization. All of the augmentation maps are edge homomorphisms of appropriate spectral sequences of double complexes, with E_2-terms:

$$\begin{aligned}
{}^{I}E^2_{s,t} &= H_t H_s B_s(\bar{E}_0, \Delta^q, V^q B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A))),\\
{}^{II}E^2_{s,t} &= H_s H_t B_s(\bar{E}_0, \Delta^q, V^q B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A))),\\
{}^{III}E^2_{s,t} &= H_t H_s B_s(\bar{E}_0, \Delta^q, B_t(\Delta^q, \Delta^q, V^q A)),\\
{}^{IV}E^2_{s,t} &= H_s H_t B_s(\bar{E}_0, \Delta^q, B_t(\Delta^q, \Delta^q, V^q A)).
\end{aligned}$$

Each of these spectral sequences collapses: the case of $\{{}^{I}E^r_{s,t}\}$ is discussed in the proof of Proposition 3.5, the case of $\{{}^{II}E^r_{s,t}\}$, the Grothendieck spectral sequence, is handled by Corollary 3.7, and the spectral sequences $\{{}^{III}E^2_{s,t}\}$ and $\{{}^{IV}E^2_{s,t}\}$ collapse for trivial reasons. It follows that each of the augmentation maps in Diagram (3.10) are quasi-isomorphisms, as indicated. It follows that the bottom arrow in (3.10) is a quasi-isomorphism, as desired. □

4 Topological André-Quillen homology

Definitions

Suppose that R is a commutative S-algebra, and that A is an augmented commutative R-algebra. Topological André-Quillen homology of A (relative to R) was defined by Basterra [7] as a suitably derived version of the cofiber of the multiplication map on the augmentation ideal:

$$\mathrm{TAQ}^R(A) = I(A)/I(A)^{\wedge 2}.$$

If M is an R-module, then topological André-Quillen homology and cohomology of A with coefficients in M are defined respectively as

$$\mathrm{TAQ}^R(A; M) = \mathrm{TAQ}^R(A) \wedge_R M,$$
$$\mathrm{TAQ}_R(A; M) = F_R(\mathrm{TAQ}^R(A), M).$$

As with TAQ^R, we define $\mathrm{TAQ}_R(A) := \mathrm{TAQ}_R(A; R)$.

The augmentation ideal functor gives an equivalence

$$I(-) : \mathrm{Alg}_R \downarrow R \xrightarrow{\simeq} \mathrm{Alg}_R^{nu}$$

between the homotopy category of augmented commutative R-algebras and the category of non-unital commutative R-algebras [7, Prop. 2.2]. These categories are tensored over pointed spaces. Basterra–McCarthy [12] show that $\mathrm{TAQ}^R(-)$ is the stabilization: there is an equivalence

$$\mathrm{TAQ}^R(A) \simeq \operatorname*{hocolim}_n \Omega^n(S^n \otimes IA). \tag{4.1}$$

The Kuhn filtration

Kuhn [29] endows the topological André-Quillen homology $\mathrm{TAQ}^S(A)$ of an augmented commutative S-algebra A with an increasing filtration

$$F_1 \mathrm{TAQ}^S(A) \to F_2 \mathrm{TAQ}^S(A) \to \cdots . \tag{4.2}$$

We shall use the simplicial presentation of TAQ^S to give a point set level construction of Kuhn's filtration.

Remark 4.3 Our construction of the filtration is different from the construction given by Kuhn in [29]. Kuhn and Pereira have recently explained to the authors an argument which shows that the two filtrations are equivalent [26].

Let $\mathbb{P}$ denote the free E_∞-ring monad on Sp

$$\mathbb{P}(Y) := \bigvee_{n \geq 0} Y^{\wedge n}_{h\Sigma_n},$$

and let $\widetilde{\mathbb{P}}$ denotes the "non-unital" version

$$\widetilde{\mathbb{P}}(Y) := \bigvee_{n \geq 1} Y^{\wedge n}_{h\Sigma_n}.$$

Note that the monad $\widetilde{\mathbb{P}}$ is augmented over the identity. Basterra [7, §5] shows that TAQ admits a simplicial presentation using the monadic bar construction:

$$\mathrm{TAQ}^S(A) \simeq \left| B_\bullet(\mathrm{Id}, \widetilde{\mathbb{P}}, I(A)) \right|. \tag{4.4}$$

For a non-unital operad $\mathcal{O}$ in Sp, let $\mathcal{F}_{\mathcal{O}}$ denote the free $\mathcal{O}$-algebra monad in Sp:

$$\mathcal{F}_{\mathcal{O}} Y := \bigvee_{n \geq 1} \mathcal{O}_n \wedge_{\Sigma_n} Y^{\wedge n}.$$

Let Comm denote the (non-unital) commutative operad in spectra, with

$$\mathrm{Comm}_n = S.$$

Viewed as an endofunctor of spectra, we have (using [38, Lem. 15.5])

$$\mathbb{L}\mathcal{F}_{\mathrm{Comm}} \simeq \widetilde{\mathbb{P}}.$$

We therefore have, for A positive cofibrant:

$$\mathrm{TAQ}^S(A) \simeq \left| B_\bullet(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A)) \right|.$$

Observe for fixed s there is a splitting

$$\begin{aligned} B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A)) &= \mathcal{F}^s_{\mathrm{Comm}} I(A) \\ &\cong \mathcal{F}_{[\mathrm{Comm}^{\circ s}]} I(A) \\ &= \bigvee_{i \geq 1} [\mathrm{Comm}^{\circ s}]_i \wedge_{\Sigma_i} I(A)^{\wedge i}. \\ &=: \bigvee_{i \geq 1} B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A))\langle i \rangle. \end{aligned} \tag{4.5}$$

Here, $\circ$ denotes the composition product of symmetric sequences. Consider the filtration

$$F_n B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A)) = \bigvee_{1 \le i \le n} B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A))\langle i \rangle.$$

This filtration is compatible with the simplicial structure, and therefore induces a filtration on realizations. For positive cofibrant R, we have:

$$F_n \, \mathrm{TAQ}^S(A) \simeq |F_n B_\bullet(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A))| \, .$$

The layers of the Kuhn filtration

We now compute the structure of the layers of the Kuhn filtration. These layers have the same structure as that computed by Kuhn for the filtration he defined on TAQ in [29], which seems to suggest the two filtrations agree. The (pointed) partition poset complex $\mathcal{P}(n)_\bullet$ is defined to be the pointed simplicial Σ_n-set whose set of s-simplices is the set

$$\left\{ \lambda_0 \le \lambda_1 \le \cdots \le \lambda_s \ : \ \begin{array}{l} \lambda_i \text{ is a partition of } \underline{n}, \\ \lambda_0 = \{1, \dots, n\}, \\ \lambda_s = \{1\} \cdots \{n\} \end{array} \right\} \amalg \{*\}.$$

The face and degeneracy maps send the disjoint basepoint $*$ to the disjoint basepoint, and are given on the other elements by the formulas

$$d_i(\lambda_0 \le \cdots \le \lambda_s) = \begin{cases} \lambda_0 \le \cdots \le \widehat{\lambda_i} \le \cdots \le \lambda_s, & i \notin \{0, s\}, \\ *, & i \in \{0, s\}, \end{cases}$$
$$s_i(\lambda_0 \le \cdots \le \lambda_s) = \lambda_0 \le \cdots \le \lambda_i \le \lambda_i \le \cdots \le \lambda_s.$$

Note that we have

$$\mathcal{P}(n)_0 = \begin{cases} \{\{1\}, *\}, & n = 1, \\ \{*\}, & n > 1. \end{cases}$$

Proposition 4.6 (c.f. [29]) *We have*

$$F_n \, \mathrm{TAQ}^S(A)/F_{n-1} \, \mathrm{TAQ}^S(A) \simeq |\mathcal{P}(n)_\bullet| \wedge_{h\Sigma_n} I(A)^{\wedge n}. \tag{4.7}$$

Proof. Let $\overline{I(A)}$ denote the spectrum $I(A)$ endowed with the trivial $\mathcal{F}_{\mathrm{Comm}}$-algebra structure. We have

$$F_n \, \mathrm{TAQ}^S(A)/F_{n-1} \, \mathrm{TAQ}^S(A) \simeq B(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, \overline{I(A)})\langle n \rangle$$

$$\cong \left|[\mathrm{Comm}^{\circ\bullet}]_n \wedge_{\Sigma_n} I(A)^{\wedge n}\right|$$
$$\cong |B_\bullet(1, \mathrm{Comm}, 1)_n| \wedge_{\Sigma_n} I(A)^{\wedge n}.$$

Here, 1 denotes the unit symmetric sequence. The lemma now follows from the isomorphism of simplicial Σ_n-spectra (see [18])

$$B_\bullet(1, \mathrm{Comm}, 1)_n \cong \mathcal{P}(n)_\bullet. \quad \square$$

A Basterra spectral sequence

We conclude this section with the construction of a spectral sequence which computes the K-homology of TAQ.

Proposition 4.8 *Suppose that A is an augmented commutative S_K-algebra such that E_*A is flat over E_*. Then there is a spectral sequence*

$$E^2_{s,*} = \mathbb{L}_s\Omega^*_{\mathbb{T}/E_*}(E_*A; K_0) \Rightarrow K_{s+*}\,\mathrm{TAQ}^{S_K}(A).$$

Proof We have

$$\begin{aligned} K \wedge \mathrm{TAQ}^{S_K}(A) &\simeq K \wedge \left|B_\bullet(\mathrm{Id}, \widetilde{\mathbb{P}}_{S_K}, I(A))\right| \\ &\simeq \left|K \wedge \widetilde{\mathbb{P}}^\bullet_{S_K} I(A)\right| \\ &\simeq \left|K \wedge_E E \wedge \widetilde{\mathbb{P}}^\bullet_{S_K} I(A)\right| \\ &\simeq \left|K \wedge_E \widetilde{\mathbb{P}}^\bullet_E(E \wedge I(A))\right|. \end{aligned}$$

Associated to this simplicial presentation is a Bousfield-Kan spectral sequence

$$E^1_{s,t} = \pi_t K \wedge_E \widetilde{\mathbb{P}}^s_E(E \wedge I(A)) \Rightarrow K_{s+t}\,\mathrm{TAQ}^{S_K}(A). \tag{4.9}$$

Using (2.4), this E^1 term can therefore be described as

$$\begin{aligned} E^1_{s,*} &= \left(\bar{\mathbb{T}}^s E_* I(A)\right)/\mathfrak{m} \\ &= B_s(\mathrm{Id}, \bar{\mathbb{T}}, E_*I(A)) \otimes_{E_0} K_0 \end{aligned}$$

with d_1 differential given by the alternating sum of the face maps of the bar construction. We therefore deduce that

$$E^2_{s,*} = \mathbb{L}_s\Omega^*_{\mathbb{T}/E_*}(E_*A; K_0).$$

$\square$

5 The Morava E-theory of $L(k)$

$L(k)$-spectra

Let $L(k)_q$ denote the spectrum given by [49]

$$L(k)_q := \epsilon_{st}(B\mathbb{F}_p^k)^{q\bar{\rho}_k}. \tag{5.1}$$

Here, $\bar{\rho}_k$ denotes the reduced regular real representation of the elementary abelian p-group $\mathbb{F}_p^k$, and $(B\mathbb{F}_p^k)^{q\bar{\rho}_k}$ denotes the Thom spectrum of the q-fold direct sum of $\bar{\rho}_k$. We write ϵ_{st} for the Steinberg idempotent, acting on this spectrum, so that $L(k)_q$ is the Steinberg summand.

Mitchell and Priddy [39] showed that there are equivalences

$$\mathrm{Sp}^{p^k}(S)/\mathrm{Sp}^{p^{k-1}}(S) \simeq \Sigma^k L(k)_1$$

where $\mathrm{Sp}^n(S)$ is the nth symmetric product of the sphere spectrum.

The Goodwillie derivatives of the identity functor

$$\mathrm{Id} : \mathrm{Top}_* \to \mathrm{Top}_*$$

are given by (see [6])

$$\partial_n(\mathrm{Id}) \simeq \left(\Sigma^\infty |\mathcal{P}(n)_\bullet|\right)^\vee . \tag{5.2}$$

Arone and Dwyer [2, Cor. 9.6] establish mod p equivalences (for q odd)

$$L(k)_q \simeq_p \Sigma^{k-q}[\partial_{p^k}(\mathrm{Id}) \wedge S^{qp^k}]_{h\Sigma_{p^k}} = \Sigma^{k-q}\mathbb{D}_{p^k}(\mathrm{Id})(S^q). \tag{5.3}$$

Here $\mathbb{D}_{p^k}(\mathrm{Id})$ is the infinite delooping of the (p^k)th layer of the Goodwillie tower of the identity functor on Top_*.

Remark 5.4 For the purposes of the rest of the paper, one could take (5.3) as the definition of the p-adic homotopy type of $L(k)_q$, instead of (5.1). All of the computations and properties of the spectra $L(k)_q$ in what follows are really aspects of the partition poset model of $\mathbb{D}_{p^k}(\mathrm{Id})(S^q)$.

The E-homology calculation

We now turn our attention to computing the E-homology of the spectra $L(k)_q$ using (5.3). We do this with a sequence of lemmas. Recall from §3 that for an E_*-module M, we write $\overline{M}$ for the $\bar{\mathbb{T}}$-algebra obtained by endowing M with the trivial action.

Lemma 5.5 *If Y is a spectrum with E_*Y finite and flat as an E_*-module, then there is an isomorphism of simplicial E_*-modules*

$$E_*(\mathcal{P}(n)_\bullet \wedge_{h\Sigma_n} Y^{\wedge n}) \cong B_\bullet(\mathrm{Id}, \bar{\mathbb{T}}, \overline{E_*Y})\langle n\rangle.$$

Proof Replacing Y with a cofibrant replacement in the positive model structure for symmetric spectra, this follows immediately from applying (2.5) to the isomorphisms

$$\begin{aligned}\mathcal{P}(n)_\bullet \wedge_{h\Sigma_n} Y^{\wedge n} &\cong B_\bullet(1, \mathrm{Comm}, 1)_n \wedge_{\Sigma_n} Y^{\wedge n} \\ &\cong B_\bullet(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, \overline{Y})\langle n\rangle.\end{aligned}$$

□

Recall from Theorem 2.20 that $C[*]_q$ denotes the Koszul complex for Δ^{-q}.

Lemma 5.6 *For q odd, there is a canonical isomorphism*

$$E_0(\Sigma^{-k-q}\left|\mathcal{P}(p^k)_\bullet\right| \wedge_{h\Sigma_{p^k}} S^{qp^k}) \cong C[k]_q.$$

Proof Consider the Bousfield-Kan spectral sequence:

$$E^1_{s,t} = E_t(\mathcal{P}(p^k)_s \wedge_{h\Sigma_{p^k}} S^{qp^k}) \Rightarrow E_{t+s}(\left|\mathcal{P}(p^k)_\bullet\right| \wedge_{h\Sigma_{p^k}} S^{qp^k}). \tag{5.7}$$

We compute, using Lemmas 5.5 and 3.9

$$\begin{aligned}E_{q+*}(\mathcal{P}(p^k)_\bullet \wedge_{h\Sigma_{p^k}} S^{qp^k}) &\cong E_* \otimes_{E_0} B_\bullet(\mathrm{Id}, \bar{\mathbb{T}}, \overline{E_*S^q})\langle p^k\rangle_q \\ &\xrightarrow[\cong]{\mathcal{L}} E_* \otimes_{E_0} B_\bullet(\bar{E}_0, \Delta^{-q}, \bar{E}_0)[k].\end{aligned}$$

By Theorem 2.20, the spectral sequence (5.7) collapses to give the desired result. □

Remark 5.8 This can also be proven directly from the work of Arone, Dwyer, and Lesh [3].

Theorem 5.9 *For q odd, there are canonical isomorphisms of E_*-modules*

$$E_0L(k)_q \cong C[k]^\vee_{-q}$$

and

$$E^0L(k)_q \cong C[k]_{-q}.$$

Proof By (5.3) and (5.2) there are equivalences

$$\begin{aligned} L(k)_q &\simeq \Sigma^{k-q} \partial_{p^k}(\mathrm{Id}) \wedge_{h\Sigma_{p^k}} S^{qp^k} \\ &\simeq \Sigma^{k-q} \left|\mathcal{P}(p^k)_\bullet\right|^\vee \wedge_{h\Sigma_{p^k}} S^{qp^k}. \end{aligned}$$

Since $\mathcal{P}(p^k)_\bullet$ is a finite complex, the results of [30] imply that there are equivalences

$$\begin{aligned} \left([\Sigma^{k-q} \left|\mathcal{P}(p^k)_\bullet\right|^\vee \wedge S^{qp^k}]_{h\Sigma_{p^k}}\right)_K &\xrightarrow[\text{norm}]{\simeq} \left[(\Sigma^{k-q} \left|\mathcal{P}(p^k)_\bullet\right|^\vee \wedge S^{qp^k})_K\right]^{h\Sigma_{p^k}} \\ &\simeq F(\Sigma^{-k+q} \left|\mathcal{P}(p^k)_\bullet\right| \wedge S^{-qp^k}, S_K)^{h\Sigma_{p^k}} \\ &\simeq F((\Sigma^{-k+q} \left|\mathcal{P}(p^k)_\bullet\right| \wedge S^{-qp^k})_{h\Sigma_{p^k}}, S_K). \end{aligned}$$

Now apply the universal coefficient theorem, using the fact that $C[k]_{-q}$ is free as a module over E_0, to deduce the result from Lemma 5.6. □

Remark 5.10 Arone and Dwyer actually give another identification of the spectrum $L(k)_q$, dual to (5.3): they prove that there is an equivalence

$$L(k)_q \simeq \Sigma^{-k-q}[\left|\mathcal{P}(p^k)_\bullet\right| \wedge S^{qp^k}]_{h\Sigma_{p^k}}.$$

Thus Lemma 5.6 gives the following alternative to Theorem 5.9: for q odd we have

$$E_0 L(k)_q \cong C[k]_q.$$

This description of the E-homology of $L(k)_q$ is less well suited to the perspective of the present paper.

6 The Bousfield-Kuhn functor and the comparison map

The Bousfield-Kuhn functor

Let T denote any v_h-telescope on a type h finite complex. The Bousfield-Kuhn functor Φ_T factors localization with respect to T. We are mainly interested in the K-localization of Φ_T, which we shall denote Φ. Thus we have a diagram of functors commuting up to natural weak equivalence.

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, "(-)_T"] \arrow[rr, bend left, "(-)_K"] \arrow[d, "\Omega^\infty"'] & \mathrm{Sp} \arrow[r, "(-)_K"] & \mathrm{Sp} \\
\mathrm{Top}_* \arrow[ur, "\Phi_T"] \arrow[urr, "\Phi"'] & &
\end{tikzcd}$$

The completed unstable v_h-periodic homotopy groups of X are the homotopy groups of $\Phi_T(X)$:

$$v_h^{-1}\pi_*(X) \cong \pi_*\Phi(X)^\wedge.$$

If the telescope conjecture for height h is true, then the functors Φ_T and Φ are equivalent.

See [33] for a detailed summary of the construction and properties of these functors. The main additional property we will need is that Φ commutes with finite homotopy limits [15], and thus in particular $\Phi\Omega \to \Omega\Phi$ is a natural weak equivalence.

Applying Φ to the unit of the adjunction

$$X \to \Omega^\infty\Sigma^\infty X,$$

we get a natural transformation

$$\eta_X : \Phi(X) \to (\Sigma^\infty X)_K.$$

The comparison map

Let R be a commutative S-algebra, and consider the functor

$$R^{(-)_+} : \mathrm{Top}_*^{op} \to \mathrm{Alg}_R \downarrow R.$$

Here, the R-algebra structure on R^{X_+} comes from the diagonal on X, with unit given by the map $X \to *$, and augmentation coming from the basepoint on X.

The augmentation ideal $I(R^{X_+})$ is identified with R^X, the R-module of maps from $\Sigma^\infty X$ to R. As the functor $\mathrm{Top}_*^{op} \to \mathrm{Alg}_R^{nu}$ given by $X \mapsto R^X$ is a pointed homotopy functor, there is are natural transformations

$$S^n \otimes R^X \to R^{\Omega^n X}.$$

Assume that R is K-local. We define a natural transformation

$$c_R\colon \operatorname{TAQ}^R(R^{X_+}) \to R^{\Phi(X)}$$

of functors $\mathrm{Top}_*^{op} \to \mathrm{Mod}_R$ as follows (using (4.1)):

$$\begin{aligned} c_R : \operatorname{TAQ}^R(R^{X_+}) &\simeq \operatorname*{hocolim}_n \Omega^n(S^n \otimes R^X) \\ &\to \operatorname*{hocolim}_n \Omega^n R^{\Omega^n X} \\ &\simeq \operatorname*{hocolim}_n \Omega^n R^{(\Sigma^\infty \Omega^n X)_K} \\ &\xrightarrow{\eta^*_{\Omega^n X}} \operatorname*{hocolim}_n \Omega^n R^{\Phi(\Omega^n X)} \\ &\simeq \operatorname*{hocolim}_n \Omega^n R^{\Sigma^{-n}\Phi(X)} \\ &\simeq R^{\Phi(X)}. \end{aligned}$$

Taking the R-linear dual of c_R and composing with the evident map $R \wedge \Phi(X) \to \operatorname{Hom}_R(R^{\Phi(X)}, R)$ gives a natural transformation

$$c^R : (R \wedge \Phi(X))_K \to \operatorname{TAQ}_R(R^{X_+}).$$

We shall refer to c_R and c^R as the *comparison maps*.

The comparison map on infinite loop spaces

Let Y be a spectrum. The counit of the adjunction

$$\epsilon : \Sigma^\infty\Omega^\infty \to \mathrm{Id}$$

induces a natural transformation

$$\epsilon^* : S_K^Y \to S_K^{\Omega^\infty Y}.$$

Regarding $S_K^{\Omega^\infty Y}$ as a non-unital commutative S_K-algebra, this induces a map of augmented commutative S_K-algebras

$$\widetilde{\epsilon}^* : \mathbb{P}_{S_K} S_K^Y \to S_K^{\Omega^\infty Y_+}.$$

The following property of $c_{S_K}\colon \operatorname{TAQ}^{S_K}(S_K^{\Omega^\infty Y_+}) \to S_K^{\Phi(\Omega^\infty Y)}$ will be all that we need to know about it.

Lemma 6.1 *The composite*

$$S_K^Y \simeq \mathrm{TAQ}^{S_K}(\mathbb{P}_{S_K} S_K^Y) \xrightarrow{\mathrm{TAQ}^{S_K}(\widetilde{\epsilon}^*)} \mathrm{TAQ}^{S_K}(S_K^{\Omega^\infty Y_+}) \xrightarrow{c_{S_K}} S_K^{\Phi(\Omega^\infty Y)} \simeq S_K^Y$$

is the identity.

Proof The lemma is proved by the commutativity of the following diagram

$$\begin{tikzcd}
S^Y \arrow[r, "\simeq"] \arrow[d, hook] \arrow[ddddr, bend right=60, "\epsilon^*"'] & \mathrm{TAQ}^{S_K}(\mathbb{P}_{S_K} S_K^Y) \arrow[r, "\widetilde{\epsilon}^*"] \arrow[d, equal] & \mathrm{TAQ}^{S_K}(S_K^{\Omega^\infty Y_+}) \arrow[d, equal] \arrow[dddd, bend left=60, "c_{S_K}"] \\
\mathbb{P}_{S_K} S_K^Y \arrow[r] \arrow[dr] \arrow[dddr, "\widetilde{\epsilon}^*"'] & \varinjlim \Omega^n (S^n \otimes \widetilde{\mathbb{P}}_{S_K} S_K^Y) \arrow[r, "\widetilde{\epsilon}^*"'] \arrow[d] & \varinjlim \Omega^n (S^n \otimes S_K^{\Omega^\infty Y}) \arrow[d] \\
& \varinjlim \Omega^n (\widetilde{\mathbb{P}}_{S_K} S_K^{\Sigma^{-n} Y}) \arrow[r, "\widetilde{\epsilon}^*"'] & \varinjlim \Omega^n (S_K^{\Omega^\infty \Sigma^{-n} Y}) \arrow[d, "\eta^*_{\Omega^\infty \Sigma^{-n} Y}"] \\
& & \varinjlim \Omega^n (S_K^{\Sigma^{-n} Y}) \\
& S_K^{\Omega^\infty Y} \arrow[r, "\eta^*_{\Omega^\infty Y}"'] \arrow[uur] & S_K^Y \arrow[u, "\simeq"']
\end{tikzcd}$$

together with the fact that $\eta^*_{\Omega^\infty Y} \circ \epsilon^* \simeq \mathrm{Id}$ [33, Sec. 7]. □

Remark 6.2 Since the the previous lemma implies that the comparison map c^R is the inclusion of a wedge summand for K-locally dualizable infinite loop spaces Y, it provides an interesting mechanism for computing the effect of the Bousfield-Kuhn functors on maps between infinite loop spaces using TAQ, even if we do not know the comparison map is an equivalence for these spaces (see [51], where this observation is employed to give a determination of the effect of the James-Hopf map on E_*-homology which is independent of Theorem 10.3).

The comparison map on QX

The previous lemma allows us to deduce the following.

Proposition 6.3 *There is a non-negative integer N such that for all pointed spaces of the form $X = \Sigma^N X'$ with finite free E-homology, the comparison map for QX*

$$c_{S_K} : \mathrm{TAQ}^{S_K}(S_K^{QX_+}) \to S_K^{\Phi QX} \simeq S_K^X$$

is an equivalence.

Proof We will argue that $\mathrm{TAQ}^{S_K}(S_K^{QX_+})$ has finite K-homology, of rank equal to the rank of the K-homology of S_K^X. The proposition then follows from Lemma 6.1.

We will use the Basterra spectral sequence (Prop. 4.8):

$$E^2_{s,*} = \mathbb{L}_s \Omega^*_{\mathbb{T}/E_*}(E_* S_K^{QX_+}; K_0) \Rightarrow K_{s+*} \mathrm{TAQ}^{S_K}(S_K^{QX_+}). \tag{6.4}$$

The calculation of this E^2 term requires a thorough understanding of the E-homology of $S_K^{QX_+}$, as a $\mathbb{T}$-algebra, modulo the maximal ideal $\mathfrak{m}$. This is the subject of Appendix B. We shall freely refer to results in this appendix for the remainder of this proof.

We first note that (B.7) and Corollary B.12 imply that $E_* S_K^{QX_+}$ satisfies the flatness hypotheses required by Proposition 4.8 for the E_2-term to take the desired form. The main result of Appendix B is Lemma B.2, which implies that for N sufficiently large, there is a simplicial isomorphism of bar constructions

$$B_\bullet(\mathrm{Id}, \bar{\mathbb{T}}, E_* S_K^{QX})/\mathfrak{m} \cong B_\bullet(\mathrm{Id}, \bar{\mathbb{T}}, \widehat{\mathbb{T}} \widetilde{E}^* X)/\mathfrak{m}$$

where $\widehat{\mathbb{T}}$ is the functor (2.15). Therefore there is an isomorphism

$$\mathbb{L}_s \Omega^*_{\mathbb{T}/E_*}(E_* S_K^{QX_+}; K_0) \cong \mathbb{L}_s \Omega^*_{\mathbb{T}/E_*}(\widehat{\mathbb{T}} \widetilde{E}^* X; K_0).$$

The Grothendieck spectral sequence of Proposition 3.5

$$E^2_{s,t} = \mathrm{Tor}_s^{\Delta^*}(\bar{K}_0, \mathbb{L}_t V^*(\widehat{\mathbb{T}} \widetilde{E}^* X)) \Rightarrow \mathbb{L}_{s+t} \Omega^*_{\mathbb{T}/E_*}(\widehat{\mathbb{T}} \widetilde{E}^* X; K_0)$$

collapses to give

$$\mathbb{L}_s \Omega^*_{\mathbb{T}/E_*}(\widehat{\mathbb{T}} \widetilde{E}^* X; K_0) \cong \begin{cases} \widetilde{K}^* X, & s = 0, \\ 0, & s > 0. \end{cases}$$

We conclude that spectral sequence (6.4) converges and collapses to give an isomorphism

$$K_* \mathrm{TAQ}^{S_K}(S_K^{QX_+}) \cong \widetilde{K}^* X.$$

□

Remark 6.5 The authors would like to believe that Proposition 6.3 is an equivalence for all connected X with finite free E-homology. Ideally, some kind of weak convergence of the K-based cohomological Eilenberg-Moore spectral sequence for the path-loop fibration for $Q\Sigma^N X'$ would give a K-local equivalence

$$S_K \wedge_{S_K^{Q\Sigma^N X'_+}} S_K \xrightarrow{\simeq} S_K^{Q\Sigma^{N-1} X'_+}.$$

It would then follow that there is a K-local equivalence

$$\Sigma\, \mathrm{TAQ}^{S_K}(S_K^{Q\Sigma^N X'_+}) \simeq \mathrm{TAQ}^{S_K}(S_K^{Q\Sigma^{N-1} X'_+}).$$

The general result would then follow from downward induction on N.

7 Weiss towers

In this section we freely use the language of Weiss's orthogonal calculus [53].

Definition 7.1 Let F be a reduced homotopy functor from complex vector spaces to K-local spectra. We shall say that a tower

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1$$

of functors under F is a *finite K-local Weiss tower* if

(1) the fiber of $F_n \to F_{n-1}$ is equivalent to the K-localization of a homogeneous degree n functor from complex vector spaces to spectra, and
(2) The map $F \to F_n$ is an equivalence for $n \gg 0$.

Remark 7.2 Suppose that $\{F_n\}$ is a finite K-local Weiss tower for F. We record the following observations.

(1) The functor F_n is n-excisive. This is because the localization of a homogeneous degree n functor is n-excisive.
(2) If $\{G_n\}$ is a finite K-local Weiss tower for G, and $F \to G$ is a natural transformation, there is a homotopically unique induced compatible system of natural transformations

$$F_n \to G_n.$$

This is because if D_n is a homogeneous degree n functor which is K-locally equivalent to the fiber $F_n \to F_{n-1}$, the space of natural transformations

$$\mathbf{Nat}((D_n)_K, G_m) \simeq \mathbf{Nat}(D_n, G_m)$$

is contractible for $m < n$. It follows that the natural map

$$\mathbf{Nat}(F_m, G_m) \xrightarrow{\simeq} \mathbf{Nat}(F, G_m)$$

is an equivalence.

(3) It follows from (2) that if F admits a finite K-local Weiss tower, such a tower is homotopically unique.

We will construct finite K-local Weiss towers of the following functors from complex vector spaces to spectra:

$$V \mapsto \Phi(\Sigma S^V),$$
$$V \mapsto \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}).$$

In each of these cases, the towers will only have non-trivial layers in degrees p^k for $k \le h$.

Proposition 7.3 *The tower* $\{\Phi(P_n(\mathrm{Id})(\Sigma S^V))\}_n$ *is a finite* K*-local Weiss tower for* $\Phi(\Sigma S^V)$.

Proof The fibers of the tower $\{\Phi(P_n(\Sigma S^V))\}_n$ are given by

$$\mathbb{D}_n(\mathrm{Id})(\Sigma S^V)_K \to \Phi(P_n(\mathrm{Id})(\Sigma S^V)) \to \Phi(P_{n-1}(\mathrm{Id})(\Sigma S^V)).$$

By [33, Thm. 8.9], the map

$$\Phi(\Sigma S^V) \to \Phi(P_{p^h}(\mathrm{Id})(\Sigma S^V))$$

is an equivalence. □

Proposition 7.4 *The tower* $\{F_n \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+})\}$ *obtained by taking the* K*-local Spanier–Whitehead dual of the Kuhn filtration* $\{F_n \mathrm{TAQ}^{S_K}(S_K^{(\Sigma S^V)_+})\}$ *is a finite* K*-local Weiss tower.*

Proof By 4.7, the fibers of the tower are given by

$$\begin{aligned} F(\partial_n(\mathrm{Id})^\vee \wedge_{h\Sigma_n} (S_K^{\Sigma S^V})^{\wedge_{S_K} n}, S_K) &\simeq F(\partial_n(\mathrm{Id})^\vee \wedge_{h\Sigma_n} (S^{\Sigma S^V})^n, S_K) \\ &\simeq F(\partial_n(\mathrm{Id})^\vee \wedge (S^{\Sigma S^V})^{\wedge n}, S_K)^{h\Sigma_n} \\ &\simeq (F(\partial_n(\mathrm{Id})^\vee \wedge (S^{\Sigma S^V})^{\wedge n}, S_K)^{h\Sigma_n})_K \\ &\simeq (((\partial_n(\mathrm{Id}) \wedge S^n \wedge S^{nV})_K)^{h\Sigma_n})_K \\ &\simeq ((\partial_n(\mathrm{Id}) \wedge S^n \wedge S^{nV})_{h\Sigma_n})_K. \end{aligned}$$

Thus they are equivalent to K-localizations of homogeneous degree n functors. Since we have

$$\mathrm{TAQ}^{S_K}(S_K^{(\Sigma V)_+}) \simeq \operatorname*{hocolim}_n F_n \, \mathrm{TAQ}^{S_K}(S_K^{(\Sigma V)_+}),$$

we have

$$\mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}) \simeq \operatorname*{holim}_n F_n \, \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}).$$

Since the layers are equivalent to $\mathbb{D}_n(\mathrm{Id})(\Sigma S^V)_K$, they are acyclic for $n > p^h$ [6]. $\square$

8 The comparison map on odd spheres

Fix q to be an odd positive integer. The main result of this section is the following theorem.

Theorem 8.1 *The comparison map*

$$c^{S_K} : \Phi(S^q) \to \mathrm{TAQ}_{S_K}(S_K^{S^q_+})$$

is an equivalence.

We shall begin with its dual, and establish the following weaker statement.

Lemma 8.2 *For* $\dim V \gg 0$*, the natural transformation*

$$c_{S_K} : \mathrm{TAQ}^{S_K}(S_K^{(\Sigma S^V)_+}) \to S_K^{\Phi(\Sigma S^V)}$$

of functors from complex vector spaces to K-local spectra has a weak section: there is a natural transformation

$$s : S_K^{\Phi(\Sigma S^V)} \to \mathrm{TAQ}^{S_K}(S_K^{(\Sigma S^V)_+})$$

so that $c_{S_K} \circ s$ is an equivalence.

Proof Using work of Arone-Mahowald, Kuhn shows that the map

$$\Phi(X) \to \Phi(P_{p^h}(\mathrm{Id})(X))$$

is an equivalence [33, Thm. 8.9]. Let $X = \Sigma S^V$, and let

$$X \to Q^{\bullet+1} X$$

denote the Bousfield-Kan cosimplicial resolution. Consider the diagram:

$$\begin{tikzcd} \mathrm{TAQ}^{S_K}(S_K^{X_+}) \arrow[r, "c_{S_K}"] & S_K^{\Phi(X)} \arrow[d, equal] \\ & S_K^{\Phi P_{p^h}(\mathrm{Id})(X)} \\ \mathrm{TAQ}^{S_K}(S_K^{P_{p^h}(Q^{\bullet+1})(X)_+}) \arrow[uu] \arrow[r, "c_{S_K}"] & S_K^{\Phi P_{p^h}(Q^{\bullet+1})X} \arrow[u] \end{tikzcd} \tag{8.3}$$

It is well known (see [5]) that there is an equivalence of cosimplicial Σ_n-spectra:

$$\partial_n(Q^{\bullet+1}) \simeq \Sigma^\infty \mathcal{P}(n)^\vee_\bullet$$

so that the induced map

$$\partial_n(\mathrm{Id}) \xrightarrow{\simeq} \mathrm{Tot}\, \partial_n(Q^{\bullet+1}) \simeq \mathrm{Tot}\, \Sigma^\infty \mathcal{P}(n)^\vee_\bullet \simeq \Sigma^\infty\, |\mathcal{P}(n)_\bullet|^\vee$$

is equivalence (5.2). For a fixed s, the iterated Snaith splitting implies that the Goodwillie tower for Q^{s+1} splits, giving equivalences

$$\begin{aligned} P_{p^h}(Q^{\bullet+1})(X) &\simeq \prod_{1\le i\le p^h} Q(\mathcal{P}(i)_s \wedge_{h\Sigma_i} X^{\wedge i}) \\ &\simeq Q\left(\bigvee_{1\le i\le p^h} \mathcal{P}(i)_s \wedge_{h\Sigma_i} X^{\wedge i}\right). \end{aligned}$$

In particular, for $\dim V \gg 0$, the spaces above satisfy the hypotheses of Prop. 6.3, and the comparison map

$$\mathrm{TAQ}^{S_K}(S_K^{P_{p^h}(Q^{\bullet+1})(X)_+}) \xrightarrow{c_{S_K}} S_K^{\Phi P_{p^h}(Q^{\bullet+1})X}$$

is a levelwise equivalence of simplicial spectra. It follows from Diagram (8.3) that the natural map

$$\left| S_K^{\Phi P_{p^h}(Q^{\bullet+1})(X)} \right|_K \to S_K^{\Phi P_{p^h}(\mathrm{Id})(X)} \simeq S_K^{\Phi(X)}$$

factors through c_{S_K}:

$$\left| S_K^{\Phi(P_{p^h}(Q^{\bullet+1})(X))} \right|_K \to \mathrm{TAQ}^{S_K}(S_K^{X_+}) \xrightarrow{c_{S_K}} S_K^{\Phi(X)} \simeq S_K^{\Phi(P_{p^h}(\mathrm{Id})(X))}.$$

The lemma will be proven if we can show that the natural map

$$\left| S_K^{\Phi(P_{p^h}(Q^{\bullet+1})(X))} \right|_K \to S_K^{\Phi(P_{p^h}(\mathrm{Id})(X))}$$

is an equivalence. To do this, we will prove that for all n the map

$$\left| S_K^{\Phi(P_n(Q^{\bullet+1})(X))} \right|_K \to S_K^{\Phi(P_n(\mathrm{Id})(X))}$$

is an equivalence, by induction on n. The map of fiber sequences

$$\begin{tikzcd}
D_n(\mathrm{Id})(X) \arrow[r] \arrow[d] & P_n(\mathrm{Id})(X) \arrow[r] \arrow[d] & P_{n-1}(\mathrm{Id})(X) \arrow[d] \\
D_n(Q^{\bullet+1})(X) \arrow[r] & P_n(Q^{\bullet+1})(X) \arrow[r] & P_{n-1}(Q^{\bullet+1})(X)
\end{tikzcd}$$

gives a map of fiber sequences

$$\begin{tikzcd}
\left| S_K^{\Phi(P_{n-1}(Q^{\bullet+1})(X))} \right|_K \arrow[r] \arrow[d] & \left| S_K^{\Phi(P_n(Q^{\bullet+1})(X))} \right|_K \arrow[r] \arrow[d] & \left| S_K^{\Phi(D_n(Q^{\bullet+1})(X))} \right|_K \arrow[d] \\
S_K^{\Phi(P_{n-1}(\mathrm{Id})(X))} \arrow[r] & S_K^{\Phi(P_n(\mathrm{Id})(X))} \arrow[r] & S_K^{\Phi(D_n(\mathrm{Id})(X))}
\end{tikzcd}$$

The induction on n therefore rests on proving that the natural map

$$\left| S_K^{\mathbb{D}_n(Q^{\bullet+1})(X)} \right|_K \simeq \left| S_K^{\Phi(D_n(Q^{\bullet+1})(X))} \right|_K \to S_K^{\Phi(D_n(\mathrm{Id})(X))} \simeq S_K^{\mathbb{D}_n(\mathrm{Id})(X)}$$

is an equivalence.

Using the finiteness of X and $\mathcal{P}(n)_\bullet$, together with the vanishing of K-local Tate spectra [30], we have the following diagram of equivalences

$$\begin{tikzcd}
\left| S_K^{\mathbb{D}_n(Q^{\bullet+1})(X)} \right|_K \arrow[r] & S_K^{\mathbb{D}_n(\mathrm{Id})(X)} \\
\left| S_K^{\Sigma^\infty \mathcal{P}(n)_\bullet^\vee \wedge_{h\Sigma_n} X^{\wedge n}} \right|_K \arrow[r] \arrow[u, "\simeq"'] & S_K^{\Sigma^\infty |\mathcal{P}(n)_\bullet|^\vee \wedge_{h\Sigma_n} X^{\wedge n}} \arrow[u, "\simeq"'] \\
\left| \left(S_K^{\Sigma^\infty \mathcal{P}(n)_\bullet^\vee \wedge X^{\wedge n}} \right)^{h\Sigma_n} \right|_K \arrow[r] \arrow[u, "\simeq"'] & \left(S_K^{\Sigma^\infty |\mathcal{P}(n)_\bullet|^\vee \wedge X^{\wedge n}} \right)^{h\Sigma_n} \arrow[u, "\simeq"'] \\
\left| \left(S_K^{\Sigma^\infty \mathcal{P}(n)_\bullet^\vee \wedge X^{\wedge n}} \right)_{h\Sigma_n} \right|_K \arrow[r] \arrow[u, "\simeq"'] & \left[\left(S_K^{\Sigma^\infty |\mathcal{P}(n)_\bullet|^\vee \wedge X^{\wedge n}} \right)_{h\Sigma_n} \right]_K \arrow[u, "\simeq"'] \\
\left| \left(\Sigma^\infty \mathcal{P}(n)_\bullet \wedge S_K^{X^{\wedge n}} \right)_{h\Sigma_n} \right|_K \arrow[r] \arrow[u, "\simeq"'] & \left[\left(\Sigma^\infty |\mathcal{P}(n)_\bullet| \wedge S_K^{X^{\wedge n}} \right)_{h\Sigma_n} \right]_K \arrow[u, "\simeq"']
\end{tikzcd}$$

The bottom arrow in this diagram is an equivalence, since realizations commute past homotopy colimits and smash products. Therefore the top arrow in the diagram is an equivalence, as desired. □

The final ingredient we will need to prove Theorem 8.1 will be a result which will allow us to dualize Lemma 8.2.

Proposition 8.4 *The spectrum $\Phi(S^q)$ is K-locally dualizable.*

Proof It suffices to show that its completed Morava E-homology is finitely generated [25]. Since $\Phi(S^q) \simeq \Phi(P_{p^h}(\mathrm{Id})(S^q))$ [33, Sec. 7], one can prove this by proving $\Phi(P_{p^k}(\mathrm{Id})(S^q))$ has finitely generated completed Morava E-homology by induction on k. This is done using the fiber sequences

$$\mathbb{D}_{p^k}(\mathrm{Id})(S^q)_K \to \Phi(P_{p^k}(\mathrm{Id})(S^q)) \to \Phi(P_{p^{k-1}}(\mathrm{Id})(S^q))$$

together with our computation $E_0 L(k)_q \cong C[k]^\vee_{-q}$. Note that $C[k]^\vee_{-q}$ is finitely generated by [45, Prop. 4.6]. □

Proof of Theorem 8.1 For $\dim V \gg 0$, we can take the K-local Spanier–Whitehead dual of the retraction

$$S_K^{\Phi(\Sigma S^V)} \to \mathrm{TAQ}^{S_K}(S_K^{(\Sigma S^V)_+}) \xrightarrow{c_{S_K}} S_K^{\Phi(\Sigma S^V)}$$

provided by Lemma 8.2 to obtain a retraction of functors from complex vector spaces to K-local spectra:

$$\Phi(\Sigma S^V) \xrightarrow{c^{S_K}} \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}) \to \Phi(\Sigma S^V).$$

We therefore get a retraction of the K-local Weiss towers of these functors, restricted to $\dim V \gg 0$ (see Propositions 7.3 and 7.4)

$$\{\Phi(P_n(\mathrm{Id})(\Sigma S^V))\}_n \xrightarrow{c^{S_K}} \{F_n\,\mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+})\}_n \to \{\Phi(P_n(\mathrm{Id})(\Sigma S^V))\}_n.$$

However, the layers of both of these towers are equivalent to the spectra $\mathbb{D}_n(\mathrm{Id})(\Sigma S^V)_K$. Since the Morava K-theory of these layers is finite, it follows that the map c^{S_K} induces an equivalence on the layers of the K-local Weiss towers for $\dim V \gg 0$. It follows that the comparison map induces an equivalence on Weiss derivatives, from which it follows that the comparison map is an equivalence on layers for *all* V. Since the K-local Weiss towers are themselves finite, we deduce that the natural transformation

$$\Phi(\Sigma S^V) \xrightarrow{c^{S_K}} \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+})$$

is an equivalence by inducting up the towers. □

Actually, the method of proof gives the following corollary, which allows us to compare Φ applied to the Goodwillie tower of the identity with the much easier to understand Kuhn tower.

Corollary 8.5 *The comparison map induces an equivalence of towers*

$$\{\Phi(P_n(\mathrm{Id})(S^q))\} \xrightarrow[\simeq]{c^{S_K}} \{F_n\,\mathrm{TAQ}_{S_K}(S_K^{S^q_+})\}.$$

9 The Morava E-homology of the Goodwillie attaching maps

Fix q to be an odd positive integer. Let α_k denote the attaching map connecting the p^k and p^{k+1}-layers of the Goodwillie tower for S^q.

$$\begin{array}{ccc} \alpha_k : D_{p^k}(\mathrm{Id})(S^q) & \longrightarrow & BD_{p^{k+1}}(\mathrm{Id})(S^q) \\ \| & & \| \\ \Omega^\infty \Sigma^{q-k} L(k)_q & & \Omega^\infty \Sigma^{q-k} L(k)_q \end{array}$$

Applying Φ and desuspending, we get a map

$$\Phi(\alpha_k) : (L(k)_q)_K \to (L(k+1)_q)_K$$

which should be regarded as the corresponding attaching map between consecutive non-trivial layers in the v_h-periodic Goodwillie tower of the identity.

Note that since $E^{S^q_+}$ is a commutative E-algebra, the reduced cohomology group

$$\widetilde{E}^q(S^q) = V^q \pi_* E^{S^q_+}$$

is a Δ^q-module. Under the isomorphisms

$$E_0 L(k)_q \cong \left(C[k]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q)\right)^\vee$$

obtained by tensoring the isomorphism of Theorem 5.9 with the fundamental class in $\widetilde{E}^q(S^q)$, there is an induced map

$$E_0\Phi(\alpha_k) : \left(C[k]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q)\right)^\vee \to \left(C[k+1]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q)\right)^\vee.$$

We have the following more refined version of Theorem 5.9.

Theorem 9.1 *There is an isomorphism of cochain complexes*

$$(E_0 L(k)_q, E_0\Phi(\alpha_k)) \cong (C_k^{\Delta^q}(\widetilde{E}^q(S^q))^\vee, \delta_k^\vee)$$

where $C_k^{\Delta^q}(\widetilde{E}^q(S^q))$ is the Koszul complex for the Δ^q-module $\widetilde{E}^q(S^q)$.

Proof By Corollary 8.5, it suffices to show that the E-homology of the attaching maps in the Kuhn tower

$$\alpha'_k : (L(k)_q)_K \simeq \left(\frac{F_{p^k}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^{k-1}}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}\right)^\vee \to \left(\frac{F_{p^{k+1}}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^k}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}\right)^\vee$$

$$\simeq (L(k+1)_q)_K$$

has the desired description (here the $(-)^\vee$ notation above denotes the K*-local* Spanier–Whitehead dual). The result is obtained by dualizing the following diagram

$$\begin{tikzcd}
C[k+1]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q) \arrow[r, "\delta_k"] \arrow[d, hook] & C[k]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q) \arrow[d, hook] \\
B_{k+1}(\bar{E}_0, \Delta^q, \widetilde{E}^q(S^q))[k+1] \arrow[r, "d_{k+1}"] & B_k(\bar{E}_0, \Delta^q, \widetilde{E}^q(S^q))[k] \\
B_{k+1}(\mathrm{Id}, \bar{\mathbb{T}}, \widetilde{E}^*(S^q))\langle p^{k+1}\rangle_{-q} \arrow[r, "d_{k+1}\langle p^k\rangle"] \arrow[u, "\mathcal{L}"] \arrow[d, equal] & B_k(\mathrm{Id}, \bar{\mathbb{T}}, \widetilde{E}^q(S^q))\langle p^k\rangle_{-q} \arrow[u, "\mathcal{L}"'] \arrow[d, equal] \\
E_{-q} B_{k+1}(\mathrm{Id}, \widetilde{\mathbb{P}}, I(S^{S^q_+}))\langle p^{k+1}\rangle \arrow[r, "d_{k+1}\langle p^k\rangle"] & E_{-q} B_k(\mathrm{Id}, \widetilde{\mathbb{P}}, I(S^{S^q_+}))\langle p^k\rangle
\end{tikzcd}$$

which identifies the E-homology of the attaching map

$$\frac{F_{p^{k+1}} \operatorname{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^k} \operatorname{TAQ}^{S_K}(S_K^{S^q_+})} \to \frac{F_{p^k} \operatorname{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^{k-1}} \operatorname{TAQ}^{S_K}(S_K^{S^q_+})}.$$

In this diagram, the maps d_{k+1} are the last face maps in the corresponding bar complexes, and the maps $d_{k+1}\langle p^k\rangle$ are the projections of the face maps on to the $\langle p^k\rangle$-summands. $\square$

Corollary 9.2 *The spectral sequence obtained by applying E_* to the tower $\{\Phi(P_n(\mathrm{Id})(S^q))\}$ takes the form*

$$\operatorname{Ext}^s_{\Delta^q}(\widetilde{E}^q(S^q), \bar{E}_t) \Rightarrow E_{q+t-s}\Phi(S^q).$$

10 A modular description of the Koszul complex

Reduction to the case of $q = 1$

In this section we give a modular interpretation of the Koszul complex $C_*^{\Delta^q}(\widetilde{E}^q(S^q))$ in the case of $q = 1$. Since the suspension gives inclusions of bar complexes (see (2.8))

$$B(\bar{E}_0, \Delta^q, \widetilde{E}^q(S^q)) \hookrightarrow B(\bar{E}_0, \Delta^1, \widetilde{E}^1(S^1))$$

we deduce that we have an induced map of Koszul complexes

$$\begin{tikzcd} C[k]_{-q} \arrow[r, hook] \arrow[d, dotted, "\sigma^{q-1}"'] & \Delta^q[1]^{\otimes k} \arrow[d, hook] \\ C[k]_{-1} \arrow[r, hook] & \Delta^1[1]^{\otimes k} \end{tikzcd}$$

Furthermore, the map σ^{q-1} above must be an inclusion. We deduce that there is an inclusion of Koszul complexes

$$\sigma^{q-1} : C_*^{\Delta^q}(\widetilde{E}^q(S^q)) \hookrightarrow C_*^{\Delta^1}(\widetilde{E}^1(S^1)).$$

It follows that the modular description of the Koszul complex we shall give for $q = 1$ will extend to a modular description for arbitrary odd q provided we have a good understanding of the inclusions of lattices

$$\Delta^q[1] \subseteq \Delta^1[1],$$
$$\Delta^q[2] \subseteq \Delta^1[2].$$

This amounts to having a concrete understanding of the second author's "Wilkerson Criterion" [42].

The modular isogeny complex

We review the definition of the modular isogeny complex $\mathcal{K}^*_{p^k}$ of [43] associated to the formal group $\mathbb{G}$.

For $(k_1, \ldots, k_s)$ a sequence of positive integers, let

$$\mathrm{Sub}_{p^{k_1},\ldots,p^{k_s}}(\mathbb{G}) = \mathrm{Spf}(\mathcal{S}_{p^{k_1},\ldots,p^{k_s}})$$

be the (affine) formal scheme whose R-points are given by

$$\mathrm{Sub}_{p^{k_1},\ldots,p^{k_s}}(\mathbb{G})(R) = \{H_1 < \cdots < H_s < \mathbb{G} \times_{\mathrm{Spf}(E_0)} \mathrm{Spf}(R) \ : \ |H_i/H_{i-1}| = p^{k_i}\}.$$

Lemma 10.1 *There is a canonical isomorphism of E_0-algebras*

$$\mathcal{S}_{p^{k_1},\ldots,p^{k_s}} \cong \mathcal{S}_{p^{k_1}} \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_{p^{k_s}}.$$

Proof An R point of $\mathrm{Spf}(\mathcal{S}_{p^{k_1},\ldots,p^{k_s}})$ corresponds to a chain of finite subgroups

$$(H_1 < \cdots < H_s)$$

in $\mathbb{G}_1 := \mathbb{G} \times_{\mathrm{Spf}(E_0)} \mathrm{Spf}(R)$ with $|H_i/H_{i-1}| = p^i$. Define $\mathbb{G}_i := \mathbb{G}_1/H_{i-1}$. Then, defining, $\widetilde{H}_i := H_i/H_{i-1}$, we get a collection of R-points

$$(\mathbb{G}_i, \widetilde{H}_i) \in \mathrm{Spf}(\mathcal{S}_{p^i})(R)$$

and isomorphisms $\mathbb{G}_i/\widetilde{H}_i \cong \mathbb{G}_{i+1}$. This is precisely the data of an R-point of

$$\mathrm{Spf}(\mathcal{S}_{p^{k_1}} \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_{p^{k_s}}).$$

Conversely, given such a sequence $(\mathbb{G}_i, \widetilde{H}_i)$ with isomorphisms $\mathbb{G}_i/\widetilde{H}_i \cong \mathbb{G}_{i+1}$, there is an associated chain of subgroups $(H_1, \ldots, H_s)$ of $\mathbb{G}_1$ obtained by pulling back the subgroup $\widetilde{H}_i$ over the isogeny:

$$\mathbb{G}_1 \to \mathbb{G}_1/\widetilde{H}_1 \cong \mathbb{G}_2 \to \mathbb{G}_2/\widetilde{H}_2 \cong \mathbb{G}_3 \to \cdots \to \mathbb{G}_{i-1}/\widetilde{H}_{i-1} \cong \mathbb{G}_i.$$

$\square$

For $k > 0$ we define

$$\mathcal{K}^s_{p^k} = \begin{cases} \prod_{\substack{k_1+\cdots+k_s=k \\ k_i>0}} \mathcal{S}_{p^{k_1},\ldots,p^{k_s}}, & 1 \le s \le k, \\ 0, & \text{otherwise.} \end{cases}$$

We handle the case of $k = 0$ by defining

$$\mathcal{K}^s_1 = \begin{cases} E_0, & s = 0, \\ 0, & s > 0. \end{cases}$$

For $1 \le i \le s$ and a decomposition $k_i = k'_i + k''_i$ with $k'_i, k''_i > 0$, define maps

$$u_i : \mathcal{S}_{p^{k_1},\ldots,p^{k_s}} \to \mathcal{S}_{p^{k_1},\ldots,p^{k'_i},p^{k''_i},\ldots,p^{k_s}}$$

on R points by

$$u_i^* : (H_1 < \cdots < H_{s+1}) \mapsto (H_1 < \cdots < \widehat{H}_i < \cdots < H_{s+1}).$$

The maps u_i, under the isomorphism of Lemma 10.1, all arise from the maps

$$u_1 : \mathcal{S}_{p^{k'+k''}} \to \mathcal{S}_{p^{k'}} \otimes_{E_0} \mathcal{S}_{p^{k''}}.$$

In [42], it is established that the maps u_1 above are dual to the algebra maps

$$\Gamma[k'] \otimes_{E_0} \Gamma[k''] \to \Gamma[k' + k''].$$

Taking a product over all possible such decompositions of $k_i = k_i' + k_i''$ gives a map

$$u_i : \mathcal{K}^s_{p^k} \to \mathcal{K}^{s+1}_{p^k}.$$

The differentials

$$\delta : \mathcal{K}^s_{p^k} \to \mathcal{K}^{s+1}_{p^k}, \quad 1 \le s < k$$

in the cochain complex $\mathcal{K}^*_{p^k}$ are given by

$$\delta(x) = \sum_{1 \le i \le s} (-1)^i u_i(x).$$

The cohomology of the modular isogeny complex

The key observation of this section is the following.

Proposition 10.2 *There is an isomorphism of cochain complexes*

$$B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k]^\vee \cong \mathcal{K}^*_{p^k}.$$

It follows that we have

$$H^s(\mathcal{K}^*_{p^k}) \cong \begin{cases} C[k]^\vee_{-1}, & s = k, \\ 0, & s \ne k. \end{cases}$$

Proof The suspension isomorphism (2.14)

$$\sigma : \Delta^1 \xrightarrow{\cong} \Gamma^0$$

induces an isomorphism of chain complexes

$$B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k] \cong B_*(\bar{E}_0, \widetilde{\Gamma}^0, \bar{E}_0)[k].$$

The isomorphisms (2.10) together with those of Lemma 10.1 induce isomorphisms

$$B_s(\bar{E}_0, \widetilde{\Gamma}^0, \bar{E}_0)[k]^\vee = \left(\bigoplus_{\substack{k_1+\cdots+k_s=k \\ k_i>0}} \widetilde{\Gamma}^0[k_1] \otimes_{E_0} \cdots \otimes_{E_0} \widetilde{\Gamma}^0[k_s] \right)^\vee$$

$$
\begin{aligned}
&\cong \bigoplus_{\substack{k_1+\cdots+k_s=k\\ k_i>0}} \widetilde{\Gamma}^0[k_1]^\vee \otimes_{E_0} \cdots \otimes_{E_0} \widetilde{\Gamma}^0[k_s]^\vee \\
&\cong \bigoplus_{\substack{k_1+\cdots+k_s=k\\ k_i>0}} \mathcal{S}_{p^{k_1}} \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_{p^{k_s}} \\
&\cong \prod_{\substack{k_1+\cdots+k_s=k\\ k_i>0}} \mathcal{S}_{p^{k_1},\ldots,p^{k_s}}
\end{aligned}
$$

since all of the E_0-modules involved are finite and free. Using the facts that $\Gamma^0[t]$ acts trivially on $\bar{E}_0$ for $t > 0$, and that the differential in the modular isogeny complex is an alternating sum of maps dual to the multiplication maps in Γ^0, our isomorphisms yield the desired isomorphism of cochain complexes

$$B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k]^\vee \cong \mathcal{K}^*_{p^k}.$$

Again, appealing the the fact that these cochain complexes are free E_0-modules in each degree, and that the modules $C[k]_{-1}$ are free (see [45, Prop. 4.6]), we have

$$
\begin{aligned}
H^s(\mathcal{K}^*_{p^k}) &\cong H^s(B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k]^\vee) \\
&\cong H_s(B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k])^\vee \\
&\cong \begin{cases} C[k]^\vee_{-1}, & s = k, \\ 0, & s \neq k. \end{cases}
\end{aligned}
$$

□

Modular description of the Koszul differentials

What remains is to give a modular description of the Koszul differentials

$$H^k(\mathcal{K}^*_{p^k}) \cong C_k^{\Delta^1}(\widetilde{E}^1(S^1))^\vee \xrightarrow{\delta_k^\vee} C_{k+1}^{\Delta^1}(\widetilde{E}^1(S^1))^\vee \cong H^{k+1}(\mathcal{K}^*_{p^{k+1}}).$$

Consider the map

$$u_{k+1} : \mathcal{S}_{\underbrace{p, \ldots, p}_{k}} \to \mathcal{S}_{\underbrace{p, \ldots, p}_{k+1}}$$

whose effect on R-points is given by

$$u_{k+1}^* : (H_1 < \cdots < H_{k+1} < \mathbb{G}) \mapsto (H_2/H_1 < \cdots < H_{k+1}/H_1 < \mathbb{G}/H_1).$$

Theorem 10.3 *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{S}\underbrace{p, \ldots, p}_{k} \arrow[r, "u_{k+1}"] \arrow[d] & \mathcal{S}\underbrace{p, \ldots, p}_{k+1} \arrow[d] \\
C_k^{\Delta^1}(\widetilde{E}^1(S^1))^\vee \arrow[r, "\delta_k^\vee"'] & C_{k+1}^{\Delta^1}(\widetilde{E}^1(S^1))^\vee
\end{tikzcd}$$

Proof Under the suspension isomorphism $\sigma : \Delta^1 \cong \Gamma^0$, the Δ^1-module $\widetilde{E}^1(S^1)$ is isomorphic to the Γ^0-module $\widetilde{E}^0(S^0) = E_0$. Moreover the action map

$$\Gamma^0[1] \cong \Gamma^0[1] \otimes_{E_0} E_0 \to E_0$$

is dual to the map t of (2.3)

$$t : E_0 \to \mathcal{S}_p$$

whose effect on R points is given by

$$t^* : (H < \mathbb{G}) \mapsto \mathbb{G}/H.$$

The result follows from the isomorphisms

$$\begin{aligned}
\mathcal{S}\underbrace{p, \ldots, p}_{k} &\cong \underbrace{\mathcal{S}_p \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_p}_{k} \\
&\cong B_k(\bar{E}_0, \widetilde{\Gamma}^0, E_0)[k] \\
&\cong B_k(\bar{E}_0, \widetilde{\Delta}^0, \widetilde{E}^1(S^1))[k]
\end{aligned}$$

and (2.22). □

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Appendix A: Borel equivariant stable homotopy theory

The technical aspects of Appendix B will necessitate a detailed understanding of homotopy orbit and fixed point spectra, including norm and transfer maps in the stable homotopy category, and a theory of Euler classes. Everything in this appendix should be well known, but it seems difficult to fully track down in the literature. The first author learned of this particular perspective on norms from some lectures of Jacob Lurie.

Functors induced from homomorphisms

For a finite group G, let Sp_G denote the category of G-spectra (G-equivariant objects in Sp, with weak equivalences given by those equivariant maps which are equivalences on underlying non-equivariant spectra), and $\mathrm{Ho}(\mathrm{Sp}_G)$ the corresponding homotopy category.

Given a homomorphism $f : H \to G$, the associated restriction functor

$$f^* : \mathrm{Ho}(\mathrm{Sp}_G) \to \mathrm{Ho}(\mathrm{Sp}_H)$$

has a left adjoint

$$f_! : \mathrm{Ho}(\mathrm{Sp}_H) \to \mathrm{Ho}(\mathrm{Sp}_G)$$

and a right adjoint

$$f_* : \mathrm{Ho}(\mathrm{Sp}_H) \to \mathrm{Ho}(\mathrm{Sp}_G).$$

In the case where $f : H \to G$ is the inclusion of a subgroup, these functors are given by induction and coinduction

$$f_! Y = \mathrm{Ind}_H^G Y = G_+ \wedge_H Y,$$
$$f_* Y = \mathrm{CoInd}_H^G Y = \mathrm{Map}_H(G, Y).$$

In this special case, since finite products are equivalent to finite wedges in Sp, the natural map

$$f_!Y = \mathrm{Ind}_H^G Y \xrightarrow[\cong]{\psi_f} \mathrm{CoInd}_H^G Y = f_*Y$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G)$, and thus $f_!$ is also right adjoint to f^*.

If f is the unique map to the trivial group $f : G \to 1$, then these functors are given by homotopy orbits and homotopy fixed points:

$$\begin{aligned} f_!Y &= Y_{hG}, \\ f_*Y &= Y^{hG}. \end{aligned}$$

In general, these functors are compatible with composition:

$$\begin{aligned} (fg)^* &= g^* f^*, \\ (fg)_! &= f_! g_!, \\ (fg)_* &= f_* g_*. \end{aligned}$$

For Y_1 and Y_2 in Sp_G, let $Y_1 \wedge Y_2 \in \mathrm{Sp}_G$ denote the smash product with diagonal G-action. For $f : H \to G$, $Y \in \mathrm{Sp}_H$, and $Z \in \mathrm{Sp}_G$, there is a projection formula

$$Y \wedge (f_!Z) \cong f_!((f^*Y) \wedge Z).$$

Finally, if

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "g"'] & G \arrow[d, "g'"] \\ H' \arrow[r, "f'"'] & G' \end{tikzcd}$$

is a pullback, and g' is surjective, then for $Y \in \mathrm{Sp}_G$ there is an isomorphism

$$g_! f^* Y \cong (f')^* (g')_! Y.$$

For example, if $f : G \to G/N$ is a quotient, then for $Y \in \mathrm{Sp}_G$, $f_!Y$ is a G/N-equivariant model for Y_{hN}. Indeed, this can be seen formally by considering the following diagram.

$$\begin{tikzcd} N \arrow[r, "k"] \arrow[d, "i"'] & 1 \arrow[d, "j"] \\ G \arrow[r, "f"] \arrow[dr, "h"'] & G/N \arrow[d, "g"] \\ & 1 \end{tikzcd}$$

Since the square in the above diagram is a pullback, we deduce

$$j^* f_! Y \cong k_! i^* Y = Y_{hN}.$$

Furthermore, we get an iterated homotopy orbit theorem

$$(Y_{hN})_{hG/N} = g_! f_! Y = h_! Y = Y_{hG}.$$

Norm and transfer maps

In the language introduced in the previous subsection, norm and transfer maps have a particularly nice description. Suppose that H is a subgroup of G, and consider the diagram:

$$\begin{tikzcd} H \arrow[rr, "i"] \arrow[dr, "f"'] & & G \arrow[dl, "g"] \\ & 1 & \end{tikzcd}$$

For $Y \in \mathrm{Sp}_G$, the transfer is given by the composite

$$\mathrm{Tr}_H^G : Y_{hG} = g_! Y \xrightarrow{\eta_H^G} g_! i_! i^* Y \cong f_! i^* Y = Y_{hH}.$$

where η_H^G is the composite

$$\eta_H^G : Y \to i_* i^* Y \underset{\cong}{\xrightarrow{\psi_i^{-1}}} i_! i^* Y \tag{10.4}$$

arising from the unit of the adjunction.

If H is normal in G, there is a refinement of the transfer Tr_e^H which is G-equivariant. Consider the diagram

$$\begin{tikzcd}
G \arrow[drr, "\mathrm{Id}"] \arrow[dr, "\Delta"] \arrow[ddr, "\mathrm{Id}"'] & & \\
& G \times_{G/H} G \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & G \arrow[d, "f"] \\
& G \arrow[r, "f"'] & G/H
\end{tikzcd}$$

Using the fact that the square in the diagram is a pullback, we define the G-equivariant transfer to be the composite

$$\mathrm{Tr}_e^H : f^*Y_{hH} = f^*f_!Y = (\pi_1)_!(\pi_2)^*Y \xrightarrow{\eta_G^{G\times_{G/H}G}} (\pi_1)_!\Delta_!\Delta^*(\pi_2)^*Y = Y. \tag{10.5}$$

The adjoint of this map gives a G/H-equivariant norm map

$$N_H : Y_{hH} \to f_*Y = Y^{hH}.$$

The equivariant transfer maps (10.5) can be constructed more generally: for subgroups

$$K \le H \le G$$

with K and H normal in G we can construct the G/K equivariant transfer Tr_K^H as the composite

$$\mathrm{Tr}_K^H : Y_{hH} \simeq (Y_{hK})_{hH/K} \xrightarrow{\mathrm{Tr}_e^{H/K}} Y_{hK}.$$

We end this section with a lemma which we will need to make use of later.

Lemma 10.6 *Given* $X, Y \in \mathrm{Sp}_G$*, the following diagram commutes in* $\mathrm{Ho}(\mathrm{Sp}_G)$.

$$\begin{tikzcd}
X_{hG} \wedge Y_{hG} \arrow[r, "\cong"] \arrow[d, "\cong"'] & (X \wedge Y_{hG})_{hG} \arrow[d, "1\wedge \mathrm{Tr}_e^G"] \\
(X_{hG} \wedge Y)_{hG} \arrow[r, "\mathrm{Tr}_e^G \wedge 1"'] & (X \wedge Y)_{hG}
\end{tikzcd}$$

Proof With respect to the maps:

$$\begin{tikzcd}
G \arrow[dr, "\delta"] \arrow[drr, bend left, "="] \arrow[ddr, bend right, "="'] & & \\
& G \times G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, "g"] & G \arrow[d, "f"] \\
& G \arrow[r, "f"'] & 1
\end{tikzcd}$$

the lemma follows from the following commutative diagram.

$$\begin{tikzcd}
f_!X \wedge f_!Y \arrow[rr, "\cong"] \arrow[dd, "\cong"] \arrow[dr, "\cong"] & & f_!(X \wedge f^* f_!Y) \arrow[r, "\cong"] & f_!(X \wedge (\pi_1)_!\pi_2^*Y) \arrow[dd, bend left] \\
& g_!(\pi_1^*X \wedge \pi_2^*Y) \arrow[r, "\cong"] \arrow[d, "\cong"] \arrow[dr] & f_!(\pi_1)_!(\pi_1^*X \wedge \pi_2^*Y) \arrow[ur, "\cong"] & f_!(\pi_1)_!\delta_!(\delta^*\pi_1^*X \wedge \delta^*\pi_2^*Y) \arrow[d, "\cong"] \\
f_!(f^* f_!X \wedge Y) \arrow[d, "\cong"] & f_!(\pi_2)_!(\pi_1^*X \wedge \pi_2^*Y) \arrow[dl, "\cong"] & g_!\delta_!\delta^*(\pi_1^*X \wedge \pi_2^*Y) \arrow[ur, "\cong"] \arrow[dl, "\cong"] & f_!(X \wedge (\pi_1)_!\delta_!\delta^*\pi_2^*Y) \arrow[d, "\cong"] \\
f_!((\pi_2)_!\pi_1^*X \wedge Y) \arrow[rr, bend right] & f_!(\pi_2)_!\delta_!(\delta^*\pi_1^*X \wedge \delta^*\pi_2^*Y) \arrow[r, "\cong"'] & f_!((\pi_2)_!\delta_!\delta^*\pi_1^*X \wedge Y) \arrow[r, "\cong"'] & f_!(X \wedge Y)
\end{tikzcd}$$

□

Thom isomorphism and Euler classes

For the purposes of this subsection, let E be a complex orientable ring spectrum, with a fixed choice of complex orientation. We may regard E as a G-spectrum with trivial action.

For a d-dimensional complex representation V, its Thom class can be represented by a map

$$[V] : S^V_{hG} \to \Sigma^{2d} E$$

in Ho(Sp)[4]. By adjointness, this map corresponds to an equivariant map

$$\widetilde{[V]} : S^V \to \Sigma^{2d} E.$$

[4] Alternatively, the reader may assume that E is even periodic, and that the Thom class lies in dimension 0—the reader then may just set $d = 0$ in the formulas that follow.

in $\mathrm{Ho}(\mathrm{Sp}_G)$. By the definition of a Thom class, the induced map of E-modules

$$\widetilde{[V]}_E : E \wedge S^V \to \Sigma^{2d} E$$

is an underlying equivalence of spectra. It follows that for any $Y \in \mathrm{Sp}_G$, there is an equivalence

$$\widetilde{[V]}_E : E \wedge S^V \wedge Y \to E \wedge \Sigma^{2d} Y.$$

It follows that there is an equivalence of non-equivariant spectra

$$\Phi_V : E \wedge (S^V \wedge Y)_{hG} \to E \wedge (\Sigma^{2d} Y)_{hG},$$

and thus isomorphisms

$$(\Phi_V)_* : E_{*+2d}(S^V \wedge Y)_{hG} \xrightarrow{\cong} E_* Y_{hG}, \tag{10.7}$$

$$\Phi_V^* : E^{*+2d}(S^V \wedge Y)_{hG} \xleftarrow{\cong} E^* Y_{hG}. \tag{10.8}$$

These isomorphisms are instances of the classical Thom isomorphism if Y is of the form $\Sigma^\infty X_+$ for a G-space X.

Observe that, in general, $E^* Y_{hG}$ is a module over the ring $E^*(BG)$. Indeed, given classes

$$\alpha \in E^n(BG),$$
$$\beta \in E^m Y_{hG}$$

represented by stable maps

$$\alpha : S^0_{hG} \to \Sigma^n E,$$
$$\beta : Y_{hG} \to \Sigma^m E,$$

we can take the smash of their adjoints

$$\widetilde{\alpha} : S^0 \to \Sigma^n E,$$
$$\widetilde{\beta} : Y \to \Sigma^m E$$

to get a map

$$\widetilde{\alpha} \wedge \widetilde{\beta} : Y \to \Sigma^{n+m} E \wedge E.$$

Postcomposing with the product for E, and taking the adjoint, gives a map

$$\alpha \cdot \beta : Y_{hG} \to \Sigma^{n+m} E$$

which represents the desired product

$$\alpha \cdot \beta \in E^{n+m} Y_{hG}.$$

The composite

$$S^0_{hG} \to S^V_{hG} \xrightarrow{[V]} \Sigma^{2d} E$$

represents the *Euler class*

$$e_V \in E^{2d}(BG).$$

Lemma 10.9 *The composite*

$$E^* Y_{hG} \xrightarrow[\cong]{\Phi_V^*} E^{*+2d}(S^V \wedge Y)_{hG} \to E^{*+2d} Y_{hG}$$

induced by the inclusion $Y_{hG} \hookrightarrow (S^V \wedge Y)_{hG}$ *is given by multiplication by* e_V.

Appendix B. The H_∞ structure of $S_K^{QX_+}$

To state the main result of this appendix, we shall need the following.

Definition B.1 We shall say that a pair of $\mathbb{T}$-algebras A, B are *isomorphic mod* $\mathfrak{m}$ (and write $A \cong_{\mathfrak{m}} B$) if there is a map of E_*-modules

$$f : A \to B$$

such that

(1) the map

$$\bar{f} : A/\mathfrak{m} \to B/\mathfrak{m}$$

is an isomorphism, and

(2) the following diagram commutes

$$\begin{tikzcd} (\mathbb{T}A)/\mathfrak{m} \arrow[r, "\overline{\mathbb{T}f}"] \arrow[d] & (\mathbb{T}B)/\mathfrak{m} \arrow[d] \\ A/\mathfrak{m} \arrow[r, "\bar{f}"'] & B/\mathfrak{m} \end{tikzcd}$$

where the vertical maps are the mod $\mathfrak{m}$ reductions of the $\mathbb{T}$-algebra structure maps.

In this appendix we prove the following technical lemma needed in the proof of Proposition 6.3.

Lemma B.2 *There is an $N \gg 0$ such that for all pointed spaces $X = \Sigma^N X'$ whose suspension spectra are K-locally strongly dualizable, with E^*X flat as an E_*-module, there is an isomorphism of $\mathbb{T}$-algebras mod $\mathfrak{m}$:*

$$E_* S_K^{QX_+} \cong_{\mathfrak{m}} \widehat{\mathbb{T}} \widetilde{E}^* X.$$

The mod $\mathfrak{m}$ isomorphism in Lemma B.2 is given by the sequence of mod $\mathfrak{m}$ isomorphisms:

$$\begin{aligned}
E_* S_K^{QX_+} &\cong E_* S_K^{\mathbb{P}(X)^{\mathrm{mult}}} && \text{(Lemma B.6)} \\
&\cong_{\mathfrak{m}} E^* \mathbb{P}(X)^{\mathrm{mult}} && \text{(Lemma B.8)} \\
&\cong_{\mathfrak{m}} E^* \mathbb{P}(X)^{\mathrm{add}} && \text{(Lemma B.16)} \\
&\cong_{\mathfrak{m}} E_* S_K^{\mathbb{P}(X)^{\mathrm{add}}} && \text{(Lemma B.8)} \\
&\cong E_* \widehat{\mathbb{P}}_{S_K}(S_K^X) && \text{(Lemma B.18)} \\
&\cong_{\mathfrak{m}} \pi_* \widehat{\mathbb{P}}_E(E^X) && \text{(Lemma B.8)} \\
&\cong_{\mathfrak{m}} \widehat{\mathbb{T}} \widetilde{E}^* X. && \text{(Lemma 2.18)}
\end{aligned}$$

Here, $\mathbb{P}(X)^{\mathrm{add}}$ and $\mathbb{P}(X)^{\mathrm{mult}}$ denote two different H_∞-coalgebra structures on $\mathbb{P}(X)$, which will be defined in the next section.

Many parts of this appendix apply more generally to aspects of the H_∞-R-algebra structure of R^{QX_+}, where R is a commutative S-algebra, and X is a connected pointed space such that $R \wedge X$ is strongly dualizable as an R-module. Therefore, we shall always implicitly assume X and R satisfy these hypotheses throughout this appendix. At times we shall have to specifically take R to be S_K or E. Observe that the dualizability of X implies that the natural map

$$(R^X)^{\wedge_R i} \to R^{(X^{\wedge i})}$$

is an equivalence. We therefore may simply use R^{X^i} to unambiguously refer to either of these equivalent spectra.

H_∞ coalgebras in spectra

By an H_∞ coalgebra C in spectra, we shall mean a spectrum C equipped with Σ_k-equivariant comultiplication maps:

$$\psi_k : C \to C^{\wedge k}$$

for all $k \geq 0$, such that for all k and ℓ the diagram

$$\begin{tikzcd} C \arrow[r, "\psi_2"] \arrow[dr, "\psi_{k+\ell}"'] & C^{\wedge 2} \arrow[d, "\psi_k \wedge \psi_\ell"] \\ & C^{\wedge k+\ell} \end{tikzcd}$$

commutes in $\mathrm{Ho}(\mathrm{Sp}_{\Sigma_k \times \Sigma_\ell})$, and

$$\begin{tikzcd} C \arrow[r, "\psi_k"] \arrow[d, "\psi_{k\ell}"'] & C^{\wedge k} \arrow[d, "(\psi_\ell)^k"] \\ C^{\wedge kl} \arrow[r, "\cong"'] & (C^{\wedge \ell})^{\wedge k} \end{tikzcd}$$

commutes in $\mathrm{Ho}(\mathrm{Sp}_{\Sigma_k \wr \Sigma_\ell})$.

The H_∞ coalgebra structures $\mathbb{P}(X)^{\mathrm{add}}$ and $\mathbb{P}(X)^{\mathrm{mult}}$ on $\mathbb{P}(X)$ will be encoded in structure maps

$$\psi_k : \mathbb{P}(X) \to \mathbb{P}(X)^k$$

where the maps ψ_k are maps of E_∞ ring spectra (thus making $\mathbb{P}(X)$ some kind of bialgebra). The structure maps ψ_k of such H_∞-coalgebra structures on $\mathbb{P}(X)$ are determined by their restrictions to $\Sigma^\infty X$, given by the composites

$$\psi_k|X : \Sigma^\infty X \hookrightarrow \mathbb{P}(X) \xrightarrow{\psi_k} \mathbb{P}(X)^{\wedge k}.$$

For a spectrum Y, the zig-zag

$$Y \xrightarrow{\Delta} \prod_{i=1}^{k} Y \xleftarrow{\simeq} \bigvee_{i=1}^{k} Y$$

determines a canonical map

$$w_k : Y \to \bigvee_{i=1}^{k} Y$$

in $\mathrm{Ho}(\mathrm{Sp}_{\Sigma_k})$, where Σ_k acts trivially on Y, and by permuting the wedge factors of $\bigvee_{i=1}^{k} Y$.

The coproducts ψ_k^{add} giving rise to $\mathbb{P}(X)^{\mathrm{add}}$ are determined (in the sense described above) by the composites

$$\Sigma^\infty X \xrightarrow{w_k} \bigvee_{i=1}^{k} \Sigma^\infty X_i \to \mathbb{P}(X)^{\wedge k}$$

where $X_i = X$, regarded as a wedge summand of the ith term of the smash product $\mathbb{P}(X)^k$. The coproducts ψ_k^{mult} giving rise to $\mathbb{P}(X)^{\mathrm{mult}}$ are determined by the composites

$$\Sigma^\infty X \xrightarrow{w_{2^k-1}} \bigvee_{\emptyset \neq S \subseteq \underline{k}} \Sigma^\infty X \xrightarrow{\bigvee \Delta^S} \bigvee_{\emptyset \neq S \subseteq \underline{k}} \bigwedge_{s \in S}^{k} \Sigma^\infty X_s \to \mathbb{P}(X)^{\wedge k},$$

where the wedge ranges over non-empty subsets of $\underline{k} = \{1, \ldots, k\}$, and Δ^S is the S-fold diagonal.

Remark B.3 We explain why we call these the "additive" and "multiplicative" coalgebra structures. Consider first the additive formal group, represented by $\mathbb{Z}[[x]]$, with coproduct given by

$$\begin{aligned} \psi^{\mathrm{add}} : \mathbb{Z}[[x]] &\to \mathbb{Z}[[x_1, x_2]], \\ x &\mapsto x_1 + x_2. \end{aligned}$$

The k-fold coproduct

$$\psi_k^{\mathrm{add}} : \mathbb{Z}[[x]] \to \mathbb{Z}[[x_1, \ldots, x_k]]$$

is then given by

$$\psi_k^{\mathrm{add}}(x) = x_1 + \cdots + x_k.$$

Consider now the multiplicative formal group, again represented by $\mathbb{Z}[[x]]$, but now with coproduct

$$\psi^{\mathrm{mult}}(x) = x_1 + x_2 + x_1 x_2.$$

The k-fold coproduct is then given by

$$\psi_k^{\mathrm{mult}}(x) = \sum_{\emptyset \neq S \subseteq \underline{k}} \prod_{s \in S} x_s.$$

Lemma B.4 *Suppose R is an H_∞ ring spectrum, and C is an H_∞ coalgebra in spectra. Then R^C inherits an H_∞-R-algebra structure.*

Proof The H_∞-R-algebra structure of R^C is given by structure maps

$$\xi_k : (R^C)^{\wedge_R k}_{h\Sigma_k} \to R^C$$

whose adjoints are given by the composites (see, e.g. [13, Lem. II.3.3])

$$\begin{aligned} \widetilde{\xi}_k : (R^C)^{\wedge_R k}_{h\Sigma_k} \wedge C &\simeq \left((R^C)^{\wedge_R k} \wedge C\right)_{h\Sigma_k} \\ &\xrightarrow{1 \wedge \psi_k} \left((R^C)^{\wedge_R k} \wedge C^{\wedge k}\right)_{h\Sigma_k} \xrightarrow{\mathrm{ev}^{\wedge k}} R^{\wedge_R k}_{h\Sigma_k} \xrightarrow{\mu_k} R. \end{aligned}$$

Here μ_k comes from the H_∞-R-algebra structure of R itself: under the isomorphism $R^{\wedge_R k} \cong R$ the composite

$$R_{h\Sigma_k} = R^{\wedge_R k}_{h\Sigma_k} \xrightarrow{\mu_k} R$$

is the restriction coming from the map of groups $\Sigma_k \to 1$. Therefore the map $\widetilde{\xi}_k$ is also given by the composite

$$\begin{aligned} \widetilde{\xi}_k : (R^C)^{\wedge_R k}_{h\Sigma_k} \wedge C \to (R^{C^{\wedge k}})_{h\Sigma_k} \wedge C &\simeq \left(R^{C^{\wedge k}} \wedge C\right)_{h\Sigma_k} \\ &\xrightarrow{1 \wedge \psi_k} \left(R^{C^{\wedge k}} \wedge C^{\wedge k}\right)_{h\Sigma_k} \xrightarrow{\mathrm{ev}} R_{h\Sigma_k} \xrightarrow{\mathrm{Res}^1_{\Sigma_k}} R. \end{aligned}$$

□

The coalgebra structure of $\Sigma^\infty QX_+$

The H_∞ structure of R^{QX_+} comes from the cocommutative coalgebra structure on $\Sigma^\infty QX_+$ associated to the diagonal map

$$QX_+ \xrightarrow{\Delta} (QX_+)^{\wedge 2}.$$

Understanding how this diagonal map interacts with the Kahn splitting is key to everything else in this appendix. This was worked out by Kuhn in [28];

because our language and setting differs somewhat from his, we recall some details.

Recall the convenient point-set level description of the Kahn stable splitting of QX_+ given in [31].

Lemma B.5 ([31]) *The map*

$$s_X : \mathbb{P}(X) \to \Sigma^\infty QX_+$$

of E_∞ ring spectra adjoint to the natural inclusion of spectra

$$\Sigma^\infty X \to \Sigma^\infty QX_+$$

is a weak equivalence when X is connected.

Lemma B.6 ([28]) *The equivalence*

$$s_X : \mathbb{P}(X)^{\mathrm{mult}} \xrightarrow{\simeq} \Sigma^\infty QX_+$$

is a map H_∞ coalgebras.

Proof We need to show that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{P}(\Sigma^\infty X) \arrow[r, "\psi_k^{\mathrm{mult}}"] \arrow[d, "s_X"', "\simeq"] & \mathbb{P}(\Sigma^\infty X)^{\wedge k} \arrow[d, "\simeq"', "s_X^{\wedge k}"] \\
\Sigma^\infty QX_+ \arrow[r, "\Delta"'] & \Sigma^\infty (QX_+)^{\wedge k}
\end{tikzcd}$$

By adjointness, this follows from the commutativity of the diagram:

$$\begin{tikzcd}
\Sigma^\infty X \arrow[rrr, "\psi_k^{\mathrm{mult}}|X"] \arrow[dr, hook] \arrow[dd, hook] & & & \mathbb{P}(\Sigma^\infty X)^{\wedge k} \arrow[dd, "s_X^{\wedge k}"] \\
& \Sigma^\infty X_+ \arrow[r, "\Delta"'] \arrow[dl, hook] & \Sigma^\infty (X_+)^{\wedge k} \arrow[ur, hook] \arrow[dr, hook] & \\
\Sigma^\infty QX_+ \arrow[rrr, "\Delta"'] & & & \Sigma^\infty (QX_+)^{\wedge k}
\end{tikzcd}$$

□

E-homology of products

Disregarding multiplicative structure, for X connected, the following K-local spectra are all equivalent:

$$S_K^{QX_+} \simeq S_K^{\mathbb{P}(X)} \simeq \prod_i (S_K^{X^i})^{h\Sigma_i} \simeq \prod_i [(S_K^{X^i})_{h\Sigma_i}]_K \simeq \widehat{\mathbb{P}}_{S_K}(S_K^X)_K \qquad \text{(B.7)}$$

(where the second to last equivalence is given by the product of norm maps, and the last equivalence is by [9, Cor. 6.1.3]). We would like to study multiplicative structures on these equivalent spectra by means of E-homology, but this is complicated by the fact that homology, in general, does not commute with products. The following lemma indicates that in our setting, there is an instance where completed Morava E-homology does commute with a certain infinite product.

We shall say that a K-local spectrum Y is *strongly dualizable* if the natural map

$$S_K^Y \wedge Y \to Y^Y$$

is a K-local equivalence.

Lemma B.8 *Suppose that Y is a strongly dualizable K-local spectrum, such that E_*Y is flat. Then the natural map*

$$E_*\widehat{\mathbb{P}}_{S_K} Y \to \pi_*\widehat{\mathbb{P}}_E(E \wedge_{S_K} Y) = \widehat{\mathbb{T}}(E_*Y)$$

is an isomorphism.

As in this paper we are working under the convention that E_* denotes completed Morava E-homology, we shall let E_*^{uc} denote *uncompleted* Morava E-homology. Lemma B.8 will be proven using a variant of a spectral sequence of Hovey [24] for the homology of a product:

$$E_2^{s,*} = R^s \prod_{E_*^{uc}E} E_*^{uc} X_\alpha \Rightarrow E_{*-s}^{uc} \prod X_\alpha. \qquad \text{(B.9)}$$

In (B.9), the E_2-term is given in terms of derived functors of the product of $E_*^{uc}E$-comodules (which in general is different from the product of sets/modules), and Hovey proves that if the spectra X_α are E-local, then the spectral sequence converges strongly.

Note that if $I \subset E_*$ is an invariant regular ideal, the Hopf algebroid structure on $(E_*, E_*^{uc}E)$ descends to the quotient $(E_*/I, E_*E/I)$. Let E/I denote the associated spectrum. We will need to use the following slight variant of spectral sequence (B.9).

Lemma B.10 *For a set of spectra $\{X_\alpha\}$ with $E^{uc}_* X_\alpha$ flat over E_*, there is a spectral sequence*

$$E_2^{s,*} = R^s \prod_{E_*E/I} (E/I)_* X_\alpha \Rightarrow (E/I)_{*-s} \prod X_\alpha.$$

If all of the spectra X_α are E-local, then this spectral sequence converges strongly.

Proof Hovey constructed spectral sequence (B.9) from the exact couple obtained by taking the E-homology of the product of modified E-based Adams resolutions of each of the X_α. The desired spectral sequence is obtained by instead taking the E/I-homology of this product of E-based Adams resolutions. This spectral sequences converges strongly if each of the X_α are E-local, for the same reasons the original one did. □

The E_2-term of this spectral sequence is in general quite mysterious, but Hovey does prove two key facts about it.

- If $\{N_\alpha\}_\alpha$ is a collection of E_*/I-modules, then the derived functors of the product of the extended comodules $E_*E/I \otimes_{E_*/I} N_\alpha$ are computed to be

$$R^s \prod_{E_*E/I} E_*E/I \otimes_{E_*/I} N_\alpha \cong \begin{cases} E_*E/I \otimes_{E_*/I} \prod N_\alpha, & s = 0, \\ 0, & s > 0. \end{cases}$$

- If $\{M_\alpha\}_\alpha$ is a collection of E_*E/I-comodules, and

$$M_\alpha \to J^0_\alpha \to J^1_\alpha \to \cdots$$

  are resolutions of each of these comodules by extended comodules, then the derived functors of product may be computed as

$$R^s \prod_{E_*E/I} M_\alpha \cong H^s \left(\prod_{E_*E/I} J^*_\alpha \right).$$

We now assume the invariant ideal I is of the form $(p^{i_0}, v_1^{i_1}, \ldots, v_{h-1}^{i_{h-1}})$. Let $\mathbb{S}$ denote the extended Morava stabilizer group. Then we have [23]

$$E_*E/I \cong \mathrm{Map}^c(\mathbb{S}, E_*/I).$$

An E_*E/I-comodule structure on an E_*/I-module M is the same thing as a (twisted E_*/I-linear) continuous action of $\mathbb{S}$ on M. Here, M is given the discrete topology, so that continuity of the action is equivalent to the statement that every element $m \in M$ has an open stabilizer $\mathrm{Sta}_{\mathbb{S}}(m)$. For a collection $\{M_\alpha\}$ of E_*E/I-comodules, the product is easily seen to be

$$\prod_{E_*E/I} M_\alpha = \{(m_\alpha) \in \prod M_\alpha \ : \ \bigcap_\alpha \mathrm{Sta}_{\mathbb{S}}(m_\alpha) \text{ is open}\}.$$

In particular, if there is a fixed open subgroup $U \leq \mathbb{S}$ (independent of α) such that the $\mathbb{S}$-action on each of the modules M_α restricts to the trivial action on U, then the product in E_*E/I-comodules agrees with the ordinary product:

$$\prod_{E_*E/I} M_\alpha = \prod M_\alpha.$$

Less obviously, the following lemma holds, whose proof seems to require cohomological finiteness properties of $\mathbb{S}$.

Lemma B.11 *Let $\{M_\alpha\}$ be a collection of E_*E/I-comodules, and suppose that there is a fixed open subgroup $U \leq \mathbb{S}$ (independent of α) such that the $\mathbb{S}$-action on each of the modules M_α restricts to the trivial action on U. Then*

$$R^s \prod_{E_*E/I} M_\alpha = \begin{cases} \prod M_\alpha, & s = 0, \\ 0, & s > 0. \end{cases}$$

Proof By [48, Thm. 5.1.2], there is a resolution of the trivial $\mathbb{S}$-module $\mathbb{Z}_p$ by finitely generated free $\mathbb{Z}_p[[\mathbb{S}]]$-modules:

$$\mathbb{Z}_p \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots.$$

Write

$$P_i = \mathbb{Z}_p[[\mathbb{S}]] \otimes_{\mathbb{Z}_p} N_i$$

where N_i are finite free $\mathbb{Z}_p$-modules. Then for any E_*E/I-comodule M, we get an induced resolution by extended comodules:

$$M \to E_*E/I \otimes_{E_*/I} \mathrm{Hom}_{\mathbb{Z}_p}(N_0, M) \to E_*E/I \otimes_{E_*/I} \mathrm{Hom}_{\mathbb{Z}_p}(N_1, M) \to \cdots.$$

(Note that without the finiteness conditions on the modules N_i, the Hom's above would have to be replaced with Hom^c's — continuous homomorphisms;

the argument that follows would fail in this more general context.) Denote this resolution

$$C^*(M; P_*) = (E_*E/I \otimes_{E_*/I} \mathrm{Hom}_{\mathbb{Z}_p}(N_0, M) \\ \to E_*E/I \otimes_{E_*/I} \mathrm{Hom}_{\mathbb{Z}_p}(N_1, M) \to \cdots)$$

to emphasize its functoriality in E_*E/I-comodules M. Since our hypotheses ensure $\prod M_\alpha$ is an E_*E/I-comodule, the isomorphisms

$$\prod_{E_*E/I} E_*E/I \otimes_{E_*/I} \mathrm{Hom}(N_i, M_\alpha) \cong E_*E/I \otimes_{E_*/I} \mathrm{Hom}(N_i, \prod M_\alpha)$$

extend to give isomorphisms

$$\prod_{E_*E/I} C^*(M_\alpha; P_*) \cong C^*\left(\prod M_\alpha; P_*\right).$$

We therefore have

$$\begin{aligned} R^s \prod_{E_*E/I} M_\alpha &\cong H^s\left(\prod_{E_*E/I} C^*(M_\alpha; P_*)\right) \\ &\cong H^s\left(C^*\left(\prod M_\alpha; P_*\right)\right) \\ &\cong \begin{cases} \prod M_\alpha, & s = 0, \\ 0, & s > 0. \end{cases} \end{aligned}$$

□

Proof of Lemma B.8 Let $I = (p^{i_0}, \ldots, v_{h-1}^{i_{h-1}}) \subset E_*$ be an invariant ideal. By Lemma B.10, there is a convergent spectral sequence

$$E_2^{s,*} = R^s \prod_{E_*E/I} (E/I)_* Y^i_{h\Sigma_i} \Rightarrow (E/I)_{*-s} \widehat{\mathbb{P}}_{S_K} Y.$$

Moreover, since Y is strongly dualizable, $(E/I)_*Y$ is finite, and there is therefore an open subgroup $U \leq \mathbb{S}$ which acts trivially on $(E/I)_*Y$. Since $\Delta^*[1]$ is finitely generated as an E_*-module (and $\Delta^*[1]$ generates Δ^*), it follows that there is an open subgroup $U' \leq U$ which acts trivially on

$$\left(\mathrm{Sym}_{E_*}(\Delta^* \otimes_{E_*} E_*Y)\right)/I \cong \bigoplus_i (\mathbb{T}\langle i\rangle E_*Y)/I.$$

In particular, U' acts trivially on

$$(E/I)_* Y^i_{h\Sigma_i} = (\mathbb{T}\langle i\rangle E_* Y)/I.$$

Using Lemma B.11, we conclude that the natural map

$$(E/I)_* \prod_i Y^i_{h\Sigma_i} \to \prod_i (\mathbb{T}\langle i\rangle E_* Y)/I$$

is an isomorphism. Viewed as an inverse system indexed on I, the above system is Mittag-Leffler. Taking inverse limits over I (and using the facts that E_* is Noetherian, and $\mathbb{T}\langle i\rangle E_* Y$ is flat over E_*), we therefore obtain an isomorphism

$$E_* \widehat{\mathbb{P}}_{S_K} Y \cong \varprojlim_I (E/I)_* \prod_i Y^i_{h\Sigma_i} \cong \widehat{\mathbb{T}} E_* Y.$$

□

Corollary B.12 *Suppose that Y is a strongly dualizable K-local spectrum, such that $E_* Y$ is flat. Then $E_* \widehat{\mathbb{P}}_{S_K} Y$ is flat.*

Proof Since E_* is Noetherian, it is coherent, and hence products of flat E_*-modules are flat [17]. □

The $\mathbf{H}_\infty$ structure of $\mathbf{R}^{\mathbb{P}(X)^{\mathbf{add}}}$

We return to the motivating analogy (see Remark B.3) of the bialgebra representing the additive coproduct. The additive coproduct ψ_k^{add} on $\mathbb{Z}[[x]]$ satisfies

$$\begin{aligned}
\psi_k^{\mathrm{add}}(x^i) &= (x_1 + \cdots + x_k)^i \\
&= \sum_{\substack{I=(i_1,\dots,i_k)\\ \|I\|=i}} \frac{i!}{i_1!\cdots i_k!} x_1^{i_1}\cdots x_k^{i_k}
\end{aligned}$$

where, for a sequence $I = (i_1, \dots, i_k)$ of non-negative integers, we define

$$\|I\| := i_1 + \cdots + i_k.$$

We wish to give a homotopy theoretic refinement of the above formula in the case of $\mathbb{P}(X)^{\mathrm{add}}$. Define

$$\Sigma_I := \Sigma_{i_1} \times \cdots \times \Sigma_{i_k},$$

and let $\Sigma_{(I)}$ denote the subgroup of Σ_k which preserves the sequence I, and define $\Sigma_{[I]}$ to be the subgroup of Σ_i given by

$$\Sigma_{[I]} := \Sigma_{(I)} \ltimes \Sigma_I.$$

The H_∞-R-algebra structure of $R^{\mathbb{P}(X)^{\mathrm{add}}}$ is given by structure maps

$$\xi_k^{\mathrm{add}} : (R^{\mathbb{P}(X)^{\mathrm{add}}})^{\wedge_R k}_{h\Sigma_k} \to R^{\mathbb{P}(X)^{\mathrm{add}}}$$

whose adjoints are given by the composites (see Lemma B.4)

$$\begin{aligned}\widetilde{\xi}_k^{\mathrm{add}} : (R^{\mathbb{P}(X)})^{\wedge_R k}_{h\Sigma_k} \wedge \mathbb{P}(X) \to (R^{(\mathbb{P}(X)^{\wedge k})})_{h\Sigma_k} \wedge \mathbb{P}(X) \simeq \left(R^{(\mathbb{P}(X)^{\wedge k})} \wedge \mathbb{P}(X)\right)_{h\Sigma_k} \\ \xrightarrow{1\wedge\psi_k^{\mathrm{add}}} \left(R^{(\mathbb{P}(X)^{\wedge k})} \wedge \mathbb{P}(X)^{\wedge k}\right)_{h\Sigma_k} \xrightarrow{\mathrm{ev}} R_{h\Sigma_k} \xrightarrow{\mathrm{Res}^1_{\Sigma_k}} R.\end{aligned}$$

There are Σ_k-equivariant equivalences

$$\mathbb{P}(X)^{\wedge k} \simeq \bigvee_{I=(i_1,\dots,i_k)} X^{\|I\|}_{h\Sigma_I} \tag{B.13}$$

where Σ_k acts on the indexing set by permuting the sequences. It follows that there are equivalences

$$R^{(\mathbb{P}(X)^{\wedge k})} \simeq \prod_i \bigvee_{\substack{I=(i_1,\dots,i_k)\\ \|I\|=i}} R^{X^{\|I\|}_{h\Sigma_I}}.$$

Note that there is are equivalences

$$\left(\bigvee_{\substack{I=(i_1,\dots,i_k)\\ \|I\|=i}} R^{X^{\|I\|}_{h\Sigma_I}}\right)_{h\Sigma_k} \simeq \bigvee_{[I]\in\mathcal{I}_i^k} \left(\mathrm{Ind}_{\Sigma_{(I)}}^{\Sigma_k} R^{X^{\|I\|}_{h\Sigma_I}}\right)_{h\Sigma_k} \simeq \bigvee_{[I]\in\mathcal{I}_i^k} \left(R^{X^{\|I\|}_{h\Sigma_I}}\right)_{h\Sigma_{(I)}}$$

where $\mathcal{I}_i^k$ is the set of Σ_k-orbits:

$$\mathcal{I}_i^k := \{I = (i_1, \dots, i_k) \ :\ \|I\| = i\}/\Sigma_k.$$

Lemma B.14 *The map ξ_k^{add} is given by the composite*

$$
\begin{aligned}
\xi_k^{\mathrm{add}} : (R^{\mathbb{P}(X)})^{\wedge_R k}_{h\Sigma_k} &\to (R^{(\mathbb{P}(X)^{\wedge k})})_{h\Sigma_k} \\
&\simeq \left(\prod_i \bigvee_{\substack{I=(i_1,\dots,i_k) \\ \|I\|=i}} R^{X^{\|I\|}_{h\Sigma_I}} \right)_{h\Sigma_k} \\
&\to \prod_i \left(\bigvee_{\substack{I=(i_1,\dots,i_k) \\ \|I\|=i}} R^{X^{\|I\|}_{h\Sigma_I}} \right)_{h\Sigma_k} \\
&\simeq \prod_i \bigvee_{[I]\in \mathcal{I}_i^k} \left(R^{X^{\|I\|}_{h\Sigma_I}} \right)_{h\Sigma_{(I)}} \\
&\xrightarrow{(\xi^{\mathrm{add}}_{I,i})_{I,i}} \prod_i R^{X^i_{h\Sigma_i}} \\
&\simeq R^{\mathbb{P}(X)}.
\end{aligned}
$$

where the only non-zero matrix coefficients

$$\xi^{\mathrm{add}}_{I,i} : (R^{X^{\|I\|}_{h\Sigma_I}})_{h\Sigma_{(I)}} \to R^{X^i_{h\Sigma_i}}$$

occur when $i = \|I\|$, for which they are adjoint to the composites

$$
\begin{aligned}
\widetilde{\xi}^{\mathrm{add}}_{I,i} : (R^{X^{\|I\|}_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_{\|I\|}} &\simeq (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^{\|I\|}_{h\Sigma_{\|I\|}})_{h\Sigma_{(I)}} \\
&\xrightarrow{1\wedge \mathrm{Tr}_{\Sigma_I}^{\Sigma_{\|I\|}}} (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^{\|I\|}_{h\Sigma_I})_{h\Sigma_{(I)}} \xrightarrow{\mathrm{ev}} R_{h\Sigma_{(I)}} \xrightarrow{\mathrm{Res}^1_{\Sigma_{(I)}}} R.
\end{aligned}
$$

Proof Using (B.13), the map $\widetilde{\xi}^{\mathrm{add}}_k$ is given by the composite

$$
\begin{aligned}
&(R^{\mathbb{P}(X)})^{\wedge_R k}_{h\Sigma_k} \wedge \mathbb{P}(X) \to \left(R^{(\mathbb{P}(X)^{\wedge k})} \wedge \mathbb{P}(X)\right)_{h\Sigma_k} \simeq \left(R^{\bigvee_I X^{\|I\|}_{h\Sigma_I}} \wedge \textstyle\bigvee_i X^i_{h\Sigma_i}\right)_{h\Sigma_k} \\
&\xrightarrow{1\wedge \bigvee_i \sum_I \mathrm{Tr}_{\Sigma_I}^{\Sigma_i}} \left(R^{\bigvee_I X^{\|I\|}_{h\Sigma_I}} \wedge \textstyle\bigvee_i \bigvee_{\|I\|=i} X^i_{h\Sigma_I}\right)_{h\Sigma_k} \xrightarrow{\mathrm{ev}} R_{h\Sigma_k} \xrightarrow{\mathrm{Res}^1_{\Sigma_k}} R.
\end{aligned}
$$

The result follows. □

The H_∞ structure of $R^{\mathbb{P}(X)^{\mathrm{mult}}}$

We now turn to a similar analysis of the H_∞ structure of $R^{\mathbb{P}(X)^{\mathrm{mult}}}$. This is essentially done in [28], except that there the computation is of the product, rather than the full H_∞ structure. In order to facilitate the understandability of the combinatorics, we again return to the motivating analogy (see Remark B.3) of the bialgebras representing the multiplicative groups. We compute the multiplicative coproduct ψ_k^{mult} on $\mathbb{Z}[[x]]$ to be

$$\begin{aligned}\psi_k^{\mathrm{mult}}(x^i) &= \left(\sum_{\emptyset\neq S\subseteq \underline{k}} \prod_{s\in S} x_s\right)^i \\ &= \sum_{\substack{J=(j_S)\\ \|J\|=i}} \frac{i!}{\prod_S j_S!} \prod_S \prod_{s\in S} x_s^{i_S}\end{aligned}$$

where the last sum now ranges over sequences

$$J = (j_S \ : \ \emptyset \neq S \subseteq \underline{k})$$

of non-negative integers indexed by subsets of $\underline{k}$, and

$$\|J\| := \sum_S j_S.$$

Note that the exponent i_s of x_s in the Jth summand of $\psi_k^{\mathrm{mult}}(x^i)$ is given by

$$i_s = \sum_{\substack{S\\ s\in S}} i_S.$$

We therefore define $I(J)$ to be the sequence

$$I(J) = (i_1, \ldots, i_k)$$

with i_s defined as above. Note that

$$\|I(J)\| = \sum_S |S|\, j_S.$$

The homotopical refinement of the formula for $\psi_k^{\mathrm{mult}}(x^i)$ proceeds as follows. For such a sequence $J = (j_S)$ of non-negative integers, define

$$\Sigma_J := \prod_S \Sigma_{j_S}.$$

Let $\Sigma_{(J)}$ denote the subgroup of Σ_k which preserves the sequence J:

$$\Sigma_{(J)} = \{\sigma \in \Sigma_k \ : \ j_S = j_{\sigma(S)}\}$$

and define $\Sigma_{[J]}$ to be the subgroup of $\Sigma_{\|J\|}$ given by

$$\Sigma_{[J]} := \Sigma_{(J)} \ltimes \Sigma_J.$$

Lemma B.15 *The H_∞ structure maps ξ_k^{mult} for $R^{\mathbb{P}(X)^{\mathrm{mult}}}$ are given by the composites*

$$\begin{aligned}
\xi_k^{\mathrm{mult}} : (R^{\mathbb{P}(X)})^{\wedge_R k}_{h\Sigma_k} &\to (R^{(\mathbb{P}(X)^{\wedge k})})_{h\Sigma_k} \\
&\simeq \left(\prod_i \bigvee_{\substack{I=(i_1,\dots,i_k) \\ \|I\|=i}} R^{X^{\|I\|}_{h\Sigma_I}} \right)_{h\Sigma_k} \\
&\to \prod_i \left(\bigvee_{\substack{I=(i_1,\dots,i_k) \\ \|I\|=i}} R^{X^{\|I\|}_{h\Sigma_I}} \right)_{h\Sigma_k} \\
&\simeq \prod_i \bigvee_{[I]\in\mathcal{I}_i^k} (R^{X^{\|I\|}_{h\Sigma_I}})_{h\Sigma_{(I)}} \\
&\xrightarrow{(\xi^{\mathrm{mult}}_{I,i})_{I,i}} \prod_i R^{X^i_{h\Sigma_i}} \\
&\simeq R^{\mathbb{P}(X)}.
\end{aligned}$$

Here the matrix coefficients

$$\xi^{\mathrm{mult}}_{I,i} : (R^{X^{\|I\|}_{h\Sigma_I}})_{h\Sigma_{(I)}} \to R^{X^i_{h\Sigma_i}}$$

are now given by sums

$$\xi^{\mathrm{mult}}_{I,i} = \sum_{[J]\in\mathcal{J}(I,i)} \xi^{\mathrm{mult}}_J$$

where

$$\mathcal{J}(I,i) = \{J = (j_S) \ : \ \|J\| = i, I(J) = I\}/\Sigma_{(I)},$$

ξ_J^{mult} *is adjoint to the composite*

$$
\begin{aligned}
\widetilde{\xi}_J^{\text{mult}} : (R^{X^{\|I\|}_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_i} \simeq (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^i_{h\Sigma_i})_{h\Sigma_{(I)}} \\
& \xrightarrow{\mathrm{Tr}^{\Sigma_{(I)}}_{\Sigma_{(J)}}} (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^i_{h\Sigma_i})_{h\Sigma_{(J)}} \\
& \xrightarrow{1 \wedge \mathrm{Tr}^{\Sigma_i}_{\Sigma_J}} (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^i_{h\Sigma_J})_{h\Sigma_{(J)}} \\
& \xrightarrow{1 \wedge \Delta_J} (R^{X^{\|I\|}_{h\Sigma_J}} \wedge X^{\|I\|}_{h\Sigma_J})_{h\Sigma_{(J)}} \\
& \xrightarrow{1 \wedge \mathrm{Res}^{\Sigma_I}_{\Sigma_J}} (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^{\|I\|}_{h\Sigma_I})_{h\Sigma_{(J)}} \\
& \xrightarrow{\mathrm{Res}^{\Sigma_{(I)}}_{\Sigma_{(J)}}} (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^{\|I\|}_{h\Sigma_I})_{h\Sigma_{(I)}} \\
& \xrightarrow{\mathrm{ev}} R_{h\Sigma_{(I)}} \\
& \xrightarrow{\mathrm{Res}^{1}_{\Sigma_{(I)}}} R,
\end{aligned}
$$

and Δ_J *is induced by the diagonal map*

$$
X^i = \bigwedge_S X^{j_S} \xrightarrow{\bigwedge \Delta_{|S|}} \bigwedge_S X^{|S| j_S} = X^{\|I\|}.
$$

Proof This result follows from the fact that the map $\widetilde{\xi}_k^{\text{mult}}$ is given by the composite

$$
\begin{aligned}
(R^{\mathbb{P}(X)})^{\wedge_R k}_{h\Sigma_k} \wedge \mathbb{P}(X) \to \left(R^{(\mathbb{P}(X)^{\wedge k})} \wedge \mathbb{P}(X) \right)_{h\Sigma_k} \\
& \simeq \left(R^{\bigvee_I X^{\|I\|}_{h\Sigma_I}} \wedge \bigvee_i X^i_{h\Sigma_i} \right)_{h\Sigma_k} \\
& \xrightarrow{1 \wedge \bigvee_i \sum_J \Delta_J \mathrm{Tr}^{\Sigma_i}_{\Sigma_J}} \left(R^{\bigvee_I X^{\|I\|}_{h\Sigma_I}} \wedge \bigvee_i \bigvee_{\|J\|=i} X^{\|I(J)\|}_{h\Sigma_J} \right)_{h\Sigma_k}
\end{aligned}
$$

$$\xrightarrow{1\wedge\bigvee_i\bigvee_J \mathrm{Res}_{\Sigma_J}^{\Sigma_{I(J)}}} \left(R^{\bigvee_I X_{h\Sigma_I}^{\|I\|}} \wedge \bigvee_I X_{h\Sigma_I}^{\|I\|}\right)_{h\Sigma_k}$$

$$\xrightarrow{\mathrm{ev}} R_{h\Sigma_k}$$

$$\xrightarrow{\mathrm{Res}_{\Sigma_k}^1} R.$$

□

The mod $\mathfrak{m}$ E-cohomology of $\mathbb{P}(X)^{\mathrm{add}}$ and $\mathbb{P}(X)^{\mathrm{mult}}$

We will now argue the crucial point, namely, as long as X is a sufficiently large suspension, the mod $\mathfrak{m}$ $\mathbb{T}$-algebra structures on $E^*\mathbb{P}(X)^{\mathrm{add}}$ and $E^*\mathbb{P}(X)^{\mathrm{mult}}$ are indistinguishable.

Lemma B.16 *Suppose that $X = \Sigma^N X'$ is a pointed space whose suspension spectrum is K-locally strongly dualizable. Then for $N \gg 0$ the identity map gives an isomorphism of $\mathbb{T}$-algebras mod $\mathfrak{m}$:*

$$E^*\mathbb{P}(X)^{\mathrm{mult}} \cong_{\mathfrak{m}} E^*\mathbb{P}(X)^{\mathrm{add}}.$$

Proof Comparing Lemmas B.14 and B.15 (with $R = S_K$), the H_∞ structure map ξ_k^{add} is the map obtained from ξ_k^{mult} by only including the terms ξ_J^{mult} for $J = (j_S)$ with $j_S = 0$ for $|S| > 1$. Since the $\mathbb{T}$-algebra structure is completely determined by the maps ξ_2^{mult} and ξ_p^{mult}, we may assume that k is either 2 or p. It then suffices to show that if j_S is not zero for some $S \subseteq \underline{k}$ with $|S| > 1$, then for N sufficiently large the induced map

$$\xi_J^{\mathrm{mult}} : E_*\left(\left(S_K^{X_{h\Sigma_{I(J)}}^{\|I(J)\|}}\right)_{h\Sigma_{(I(J))}}\right)/\mathfrak{m} \to E_*\left(S_K^{X_{h\Sigma_{\|J\|}}^{\|J\|}}\right)/\mathfrak{m}$$

is zero. It suffices to show that the map

$$\Delta_J^* : E_*\left(\left(S_K^{X_{h\Sigma_J}^{\|I(J)\|}}\right)_{h\Sigma_{(J)}}\right)/\mathfrak{m} \to E_*\left(\left(S_K^{X_{h\Sigma_J}^{\|J\|}}\right)_{h\Sigma_{(J)}}\right)/\mathfrak{m}$$

is zero, as the above map factors through this. Since X is K-locally dualizable, and K-locally $\Sigma_{(J)}$ homotopy orbits are equivalent to homotopy fixed points, this map is isomorphic to the map

$$\Delta_J^* : \widetilde{E}^*\left((X_{h\Sigma_J}^{\|I(J)\|})_{h\Sigma_{(J)}}\right)/\mathfrak{m} \to \widetilde{E}^*\left((X_{h\Sigma_J}^{\|J\|})_{h\Sigma_{(J)}}\right)/\mathfrak{m}. \tag{B.17}$$

The group $\Sigma_{(J)}$ naturally acts by permutation on the representations

$$\rho = \bigoplus_{\substack{0 \neq S \subseteq \underline{k} \\ j_S \neq 0}} \mathbb{R},$$
$$\rho' = \bigoplus_{\substack{0 \neq S \subseteq \underline{k} \\ j_S \neq 0}} \mathbb{R}^{|S|}.$$

The diagonal map

$$\Delta_J : \rho \to \rho'$$

is $\Sigma_{(J)}$-equivariant. Let $\rho^\perp$ be the orthogonal complement of ρ in ρ'. Our assumption on J ensures that $\rho^\perp$ has positive dimension. Writing $X = \Sigma^N X'$, the map (B.17) now can be interpreted as a composite

$$\begin{aligned}
&\widetilde{E}^*\left(\left(S^{N\rho+N\rho^\perp} \wedge (S^{\|I(J)\|-N|\rho'|} \wedge X'^{\|I(J)\|})_{h\Sigma_J}\right)_{h\Sigma_{(J)}}\right)/\mathfrak{m} \\
&\xrightarrow{1\wedge\Delta_J} \widetilde{E}^*\left(\left(S^{N\rho+N\rho^\perp} \wedge (S^{\|J\|-N|\rho|} \wedge X'^{\|J\|})_{h\Sigma_J}\right)_{h\Sigma_{(J)}}\right)/\mathfrak{m} \\
&\xrightarrow{\Delta_J\wedge 1} \widetilde{E}^*\left(\left(S^{N\rho} \wedge (S^{\|J\|-N|\rho|} \wedge X'^{\|J\|})_{h\Sigma_J}\right)_{h\Sigma_{(J)}}\right)/\mathfrak{m}.
\end{aligned}$$

It therefore suffices to show that the map $\Delta_J \wedge 1$ above is trivial. We may assume that $N = 2M$ is even. To simplify notation let Y denote the $\Sigma_{(J)}$-space

$$Y := (S^{\|J\|} \wedge X'^{\|J\|})_{h\Sigma_J}.$$

We wish to show that the map

$$Y_{h\Sigma_{(J)}} \hookrightarrow \left(S^{2M\rho^\perp} \wedge Y\right)_{h\Sigma_{(J)}}$$

is zero on mod $\mathfrak{m}$ E-cohomology.

Using the fact that E is complex orientable (and taking a fixed complex orientation), Lemma 10.9 implies that the effect on mod $\mathfrak{m}$ E-cohomology of the above map, under the Thom isomorphism (10.8), is multiplication by a power of the Euler class

$$e^M_{2\rho^\perp} : \widetilde{E}^*\left(Y_{h\Sigma_{(J)}}\right)/\mathfrak{m} \to \widetilde{E}^*\left(Y_{h\Sigma_{(J)}}\right)/\mathfrak{m}.$$

Since $\rho^\perp$ is not zero dimensional, the colimit

$$\varinjlim_M S^{2M\rho^\perp}$$

is non-equivariantly contractible, which implies that

$$\varinjlim_M E\Sigma_{(J)+} \wedge S^{2M\rho^\perp}$$

is equivariantly contractible, and therefore the localization is trivial:

$$e^{-1}_{2\rho^\perp} E^*\left(B\Sigma_{(J)}\right)/\mathfrak{m} = 0.$$

Since G is finite, the K_*-vector space $E^*\left(B\Sigma_{(J)}\right)/\mathfrak{m}$ is finite dimensional. It follows (e.g. using Jordon canonical form after base change to $\bar{\mathbb{F}}_p$) that the element $e_{2\rho^\perp}$ is nilpotent in $E^*(B\Sigma_{(J)})$, and therefore

$$e^M_{2\rho^\perp} : \widetilde{E}^*\left(Y_{h\Sigma_{(J)}}\right)/\mathfrak{m} \to \widetilde{E}^*\left(Y_{h\Sigma_{(J)}}\right)/\mathfrak{m}$$

is zero for M sufficiently large.

The value of M above depends on only on k (which is either 2 or p), the subgroup $\Sigma_{(J)} \le \Sigma_k$, and the representation $\rho^\perp$. As there are only finitely many k and subgroups, and associated representations in play, there is a global bound on the values of M for the various subgroups showing up for different sequences J. As such, M can be taken sufficiently large to handle all of these different terms simultaneously. □

The norm equivalence of $\mathbf{H}_\infty$-ring spectra

Lemma B.18 *Suppose that X is a space whose suspension spectrum is K-locally strongly dualizable. Then the norm induces an equivalence*

$$\widehat{\mathbb{P}}_{S_K}(S_K^X) = \prod_i [(S_K^{X^i})_{h\Sigma_i}]_K \xrightarrow[\simeq]{\prod N_{\Sigma_i}} \prod_i [S_K^{X^i}]^{h\Sigma_i} \simeq S_K^{\mathbb{P}(X)^{\mathrm{add}}}$$

of H_∞-S_K-algebras.

Observe that since K-localization commutes with products when the factors involved are E-local (see [9, Cor. 6.1.3]) there is an equivalence

$$\prod_i [(S_K^{X^i})_{h\Sigma_i}]_K \simeq \left[\prod_i (S_K^{X^i})_{h\Sigma_i}\right]_K.$$

Therefore, Lemma B.18 follows from the following slightly more general observation.

Lemma B.19 *The composite*

$$\widehat{\mathbb{P}}_R(R^X) := \prod_i (R^{X^i})_{h\Sigma_i} \xrightarrow{\prod N_{\Sigma_i}} \prod_i (R^{X^i})^{h\Sigma_i} \simeq R^{\mathbb{P}(X)^{\mathrm{add}}}$$

is a map of H_∞-R-algebras.

The H_∞-R-algebra structure of $\widehat{\mathbb{P}}_R(R^X)$ has structure maps

$$\zeta_k : \widehat{\mathbb{P}}_R(R^X)^{\wedge_R k}_{h\Sigma_k} \to \widehat{\mathbb{P}}_R(R^X)$$

which are given by the composites

$$\begin{aligned}
\zeta_k : \widehat{\mathbb{P}}_R(R^X)^{\wedge_R k}_{h\Sigma_k} &= \left(\prod_i (R^{X^i})_{h\Sigma_i}\right)^{\wedge_R k}_{h\Sigma_k} \\
&\to \left(\prod_i \bigvee_{\substack{I=(i_1,\dots,i_k)\\ \|I\|=i}} (R^{X^{\|I\|}})_{h\Sigma_I}\right)_{h\Sigma_k} \\
&\to \prod_i \left(\bigvee_{\substack{I=(i_1,\dots,i_k)\\ \|I\|=i}} (R^{X^{\|I\|}})_{h\Sigma_I}\right)_{h\Sigma_k} \\
&\simeq \prod_i \bigvee_{[I]\in\mathcal{I}_i^k} \left((R^{X^{\|I\|}})_{h\Sigma_I}\right)_{h\Sigma_{(I)}} \\
&\simeq \prod_i \bigvee_{[I]\in\mathcal{I}_i^k} (R^{X^{\|I\|}})_{h\Sigma_{[I]}} \\
&\xrightarrow{\prod_i \sum_I \mathrm{Res}^{\Sigma_i}_{\Sigma_{[I]}}} \prod_i (R^{X^i})_{h\Sigma_i} \\
&= \widehat{\mathbb{P}}_R(R^X).
\end{aligned}$$

Lemma B.19 therefore follows from the following lemma.

Lemma B.20 *Fix a sequence $I = (i_1, \ldots, i_k)$ with $\|I\| = i$. Then the following diagram commutes.*

$$\begin{tikzcd}
(R^{X^i})_{h\Sigma_{[I]}} \arrow[r, "\mathrm{Res}^{\Sigma_i}_{\Sigma_{[I]}}"] \arrow[d, "N_{\Sigma_I}"'] & (R^{X^i})_{h\Sigma_i} \arrow[d, "N_{\Sigma_i}"] \\
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \arrow[r, "\xi^{\mathrm{add}}_{I,i}"'] & R^{X^i_{h\Sigma_i}}
\end{tikzcd}$$

Proof The construction of the norm as the adjoint to the equivariant transfer implies that the following diagram commutes.

$$\begin{tikzcd}
(R^{X^i})_{h\Sigma_{[I]}} \arrow[rr, "\mathrm{Res}^{\Sigma_i}_{\Sigma_{[I]}}"] \arrow[d, "N_{\Sigma_I}"'] \arrow[dr, "N_{\Sigma_{[I]}}"] & & (R^{X^i})_{h\Sigma_i} \arrow[d, "N_{\Sigma_i}"] \\
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \arrow[r, "N_{\Sigma_{(I)}}"'] & R^{X^i_{h\Sigma_{[I]}}} \arrow[r, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] & R^{X^i_{h\Sigma_i}}
\end{tikzcd}$$

Therefore it suffices to show the commutativity of the diagram below.

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \arrow[r, "\xi^{\mathrm{add}}_{I,i}"] \arrow[d, "N_{\Sigma_{(I)}}"'] & R^{X^i_{h\Sigma_i}} \\
R^{X^i_{h\Sigma_{[I]}}} \arrow[ur, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] &
\end{tikzcd}$$

By adjointness this is equivalent to showing the commutativity of the diagram:

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_i} \arrow[r, "\tilde{\xi}^{\mathrm{add}}_{I,i}"] \arrow[d, "N_{\Sigma_{(I)}} \wedge 1"'] & R \\
R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_i} \arrow[r, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}} \wedge 1"'] & R^{X^i_{h\Sigma_i}} \wedge X^i_{h\Sigma_i} \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

By Lemma B.14, the commutativity of the above diagram is seen in the following commutative diagram.

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_i} \arrow[r, equal] \arrow[ddd, "N_{\Sigma_{(I)}} \wedge 1"'] \arrow[ddr, "1\wedge \mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"] & (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_i})_{h\Sigma_{(I)}} \arrow[r, "1\wedge \mathrm{Tr}^{\Sigma_i}_{\Sigma_I}"] \arrow[d, "1\wedge \mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] & (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[r, "\mathrm{ev}"] & R_{h\Sigma_{(I)}} \arrow[r, "\mathrm{Res}^1_{\Sigma_{(I)}}"] \arrow[d, phantom, "(2)"] & R \\
 & (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_{[I]}})_{h\Sigma_{(I)}} \arrow[ur, "1\wedge \mathrm{Tr}^{\Sigma_{[I]}}_{\Sigma_I}"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & (R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[u, "\mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I} \wedge 1"'] \arrow[d, equal] & & \\
 & (R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_{[I]}} \arrow[r, "N_{\Sigma_{(I)}} \wedge 1"] & R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_{[I]}} \arrow[uurr, "\mathrm{ev}"'] & & \\
R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_i} \arrow[urr, "1\wedge \mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] \arrow[rrrr, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}} \wedge 1"'] & & & & R^{X^i_{h\Sigma_i}} \wedge X^i_{h\Sigma_i} \arrow[uuu, "\mathrm{ev}"']
\end{tikzcd} \tag{B.21}$$

With the exception of regions (1) and (2) in the above diagram, all of the faces of the diagram clearly commute.

The commutativity of (1) is seen below, making use of Lemma 10.6.

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}} \wedge (X^i_{h\Sigma_I})_{h\Sigma_{(I)}})_{h\Sigma_{(I)}} \arrow[rr, "1\wedge \mathrm{Tr}_1^{\Sigma_{(I)}}"] \arrow[dd, equal] & & (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \\
 & ((R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[ur, "\mathrm{Tr}_1^{\Sigma_{(I)}} \wedge 1"] \arrow[r, "N_{\Sigma_{(I)}} \wedge 1"] \arrow[dl, equal] & (R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[u, "\mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I} \wedge 1"'] \arrow[d, equal] \\
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_{[I]}} \arrow[rr, "N_{\Sigma_{(I)}} \wedge 1"'] & & R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_{[I]}}
\end{tikzcd}$$

By adjointness, the commutativity of region (2) in Diagram (B.21) is equivalent to the commutativity of the following diagram in $\mathrm{Ho}(\mathrm{Sp}_{\Sigma_{(I)}})$, which clearly commutes.

$$\begin{tikzcd}
R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_I} \arrow[r, "\mathrm{ev}"] & R \\
R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_I} \arrow[u, "\mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I} \wedge 1"] \arrow[r, "1\wedge \mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I}"'] & R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_{[I]}} \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

$\square$

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