Cochains and Homotopy Theory

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Abstract

The E_∞ algebra structure on the cochain complex of a space contains all the homotopy theoretic information about the space, but for partial information, less structure is needed. I will discuss some ideas and preliminary work in this direction.

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1. Distinguishing homotopy types
2. Homotopy algebras and operadic algebras
3. Formality in characteristic p
4. Generalizing AHAH   Anick   Hopf algebras up to Homology

Distinguishing Homotopy Types

Explanation through examples

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Example (1st Semester Algebraic Topology)

Can distinguish $\mathbb{C}P^2$ and S^4 by homology.

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Example (2nd Semester Algebraic Topology)

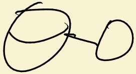
Cannot distinguish $\mathbb{C}P^2$ and $S^2 \vee S^4$ just through homology, but can distinguish them by their cohomology with **cup product**.

$\mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}$

$x \qquad y$

$x^2 = y \quad \mathbb{C}P^2$

$x^2 = 0 \quad S^2 \vee S^4$



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Example (3rd or 4th Semester Algebraic Topology)

Can classify homotopy types of all simply connected spaces with homology like $\mathbb{C}P^2$ or $S^2 \vee S^4$ by their cohomology with cup product.

$\mathbb{Z}\ 0\ \underset{x}{\mathbb{Z}}\ 0\ \underset{y}{\mathbb{Z}}$ $\quad x^2 = ny \quad \pi_3 S^2 = \mathbb{Z}$

Example (4th Semester Algebraic Topology)

Cannot distinguish $\Sigma\mathbb{C}P^2$ and $S^3 \vee S^5$ through their cohomology just with cup product

$\mathbb{Z} \quad 0 \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}$

Example (4th Semester Algebraic Topology)

Cannot distinguish $\Sigma\mathbb{C}P^2$ and $S^3 \vee S^5$ through their cohomology just with cup product, but can distinguish them using the Sq^2

Steenrod operation

$\mathbb{Z}/2 \quad 0 \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2$

$x \qquad y$

$Sq^2 x = \begin{cases} y & \Sigma\mathbb{C}P^2 \\ 0 & S^3 \vee S^5 \end{cases}$

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These are the only 2 homotopy types of simply connected spaces with this homology.

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Example (Advanced Graduate Algebraic Topology)

Homotopy types of spaces with homology like $S^n \vee S^{n+r}$ can be distinguished and classified using relations between higher cohomology "operations" implicit in the unstable Adams spectral sequence.

$\pi_{n+r-1} S^n$

Cohomology and Homotopy Types

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- McClure–Smith, "Multivariable Cochain Operations and Little n-Cubes", 2003

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This structure determined homotopy type for simply connected spaces. ~~Nice~~ spaces X and Y are homotopy equivalent if and only if C^*X and C^*Y are quasi-isomorphic as E_∞ algebras.

E_∞ Algebras and Homotopy Theory

Rational Homotopy Theory

p-adic Homotopy Theory

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Combining *p*-adic and rational theory to get integral theory

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p-adic Homotopy Theory $\mathbb{F}_p$ $\hat{\mathbb{Z}}_p$ $\mathbb{Z}/p$

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Combining *p*-adic and rational theory to get integral theory
Key step: André–Quillen cohomology computation

Practical Remarks

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- Cofibrant model still very big, not always easy to work with
- For spaces close to $K(\pi, n)$'s, practical to find cofibrant model
- No notion of "formal" space

Further Directions

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Approach: Weaken the algebraic structure

Look at an algebraic structure weaker than E_∞ and see what information is left.

Algebraic Structures

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What information is left when we view C^*X as an E_n algebra?

What is left in the E_n Structure?

- The E_{n-1} structure on a homotopy pullback

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  Can compute E_{n-1}-structure on $C^*(Y \times_X Z)$ as
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An approach to formality in characteristic p.

Formality in Characteristic Zero

Definition

A commutative differential graded $\mathbb{Q}$-algebra is *formal* if it is quasi-isomorphic to its cohomology though maps of commutative differential graded algebras.

Examples

- A $\mathbb{Q}$-CDGA with zero differential is formal
- A $\mathbb{Q}$-CDGA whose cohomology is a free gr. com. algebra is formal
- A $\mathbb{Q}$-CDGA whose cohomology is an exterior algebra is formal

Definition

A space is *rationally formal* if its polynomial De Rham complex is a formal $\mathbb{Q}$-CDGA.

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Cochains

Examples of Rationally Formal Spaces

Spheres

Cohomology is an exterior algebra.

Lie Groups / H-Spaces

Milnor-Moore: Cohomology is a free gr. comm. algebra.

Wedges and Products of Formal Spaces

Smooth Complex Algebraic Varieties

Deligne-Griffiths-Morgan-Sullivan / Morgan

(Mixed) Hodge structure on cohomology gives a (mixed) Hodge structure on the De Rham complex. Limits possibilities for differentials.

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Formality for E_∞ Algebras

Definition

An E_∞ algebra is *formal* if it is quasi-isomorphic to its cohomology though maps of E_∞ algebras.

Cohomology of an E_∞ algebra has E_∞ algebra from its graded commutative algebra structure.

In characteristic p, cohomology of E_∞ algebras have Steenrod / Dyer-Lashof operations. For commutative algebras, all but the p-th power operation are zero.

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For spaces, the zeroth operation is the identity.

The cochain algebra of a space cannot be formal unless the space has contractible components.

E_n Algebras

E_n algebras have operations on $x \in H^*A$

$$Sq^m x, Sq^{m-1} x, \ldots, Sq^{m-n+1} x \qquad p = 2, |x| = m$$

$$P^m x, P^{m-1} x, \ldots, P^{m-\lfloor (n-1)/2 \rfloor} x \qquad p > 2, |x| = 2m$$

$$P^m x, P^{m-1} x, \ldots, P^{m-\lfloor n/2 \rfloor} x \qquad p > 2, |x| = 2m+1, n > 1$$

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For $|x| \geq n$, Sq^0/P^0 not an E_n algebra operation on x.

If X is an $(n-1)$-connected space, no Sq^0/P^0 operation in E_n structure on cochains

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Which $(n-1)$-connected spaces are E_n formal?

Loops and Suspension

Recall: For any space X, Ω^n is an E_n space.

If X is an E_{n-1}-space, ΩX is an E_n space.

Because C^* is contravariant, $C^*\Sigma X$ is "like" loops of C^*X.
(Think $H\mathbb{Z}^{\Sigma X} \cong \Omega H\mathbb{Z}^X$)

Theorem

The E_{n-1} structure on C^*X determines the E_n structure on $C^*\Sigma X$.

Consequence

For any X, $\Sigma^n X$ is E_n formal.
S^n is E_n-formal but not E_{n+1} formal.

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S^n is E_n-formal but not E_{n+1} formal.

Loops and Suspension

Recall: For any space X, Ω^n is an E_n space.

If X is an E_{n-1}-space, ΩX is an E_n space.

Because C^* is contravariant, $C^*\Sigma X$ is "like" loops of C^*X.
(Think $H\mathbb{Z}^{\Sigma X} \cong \Omega H\mathbb{Z}^X$)

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The E_{n-1} structure on C^*X determines the E_n structure on $C^*\Sigma X$.

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Conjecture (Formality)

If X is rationally formal, then after inverting finitely many primes, C^*X is E_n formal.

Relationship to AHAH

- Anick, “Hopf Algebras up to Homotopy”, 1989

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Koszul dual translation (?)

$$
\begin{array}{ccc}
C^* X & \longleftrightarrow & C_* \Omega X \\
E_2 & \longleftrightarrow & \text{DG Hopf algebra} \\
\cup & & \\
\text{CDGA} & \longleftarrow & \text{cocommutative}
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Let R be a ring containing $1/m$ for $m < p$ and let X be an r-connected $\cancel{pr}$-dimensional simplicial complex. Then $\underline{C^*(X; R)}$ is quasi-isomorphic as an E_2-algebra to a commutative differential graded algebra.

$p(r-1)$

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For a 2-connected space, after inverting finitely many primes, $C^*(X)$ is quasi-isomorphic as an E_2-algebra to a commutative differential graded algebra.

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Can do homotopy theory with $\mathcal{C}om'$ algebras.

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Relative André-Quillen cohomology

$$H^*(A; M)$$

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Obstruction groups are relative André–Quillen cohomology groups.

The Linearity Hypothesis

Hypothesis. There exists and E_n operad $\mathcal{E}$ that acts on cochain complexes and satisfies the dimension bound

$$\dim \mathcal{E}(k) = (k-1)(n-1).$$

Highest chain-level k-ary operation occurs in degree $(k-1)(n-1)$.

Notes.

- This is the same degree as highest non-zero homology group.
- The standard E_n operads satisfy this bound for $k=2$.
- Standard E_1 and E_2 operads satisfy this bound for all n, k.
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A Weaker Linearity Hypothesis

At the cost of weakening the conjectures, the hypothesis can be weakened to a linearity hypothesis

$$\dim \mathcal{E}(k) = a(k-1)(n-1) \qquad \text{for } k \gg 0$$

The little n-cubes operad of spaces has k-th space a non-compact manifold with boundary, dimension $k(n+1)$.

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Consequences of the Linearity Hypothesis

Let X be r-reduced dimension d, so $\tilde{C}^*X = 0$ for $* \leq r$ and $* > d$

Look at E_n action.

$$\mathcal{E}(k) \otimes (\tilde{C}^*X)^{\otimes k} \to \tilde{C}^*X.$$

Left side is non-zero in range $k(r+1) - (k-1)(n-1)$ to kd.
Right side is non-zero in range $r+1$ to d.

$$k(r+1)-(k-1)(n-1) = k(r+1-(n-1))+(n-1) = k(r-n+2)+n-1$$

So if $k(r-n+2)+n-1 > d$ the map must be zero.

Limit dimension to $p(r-n+2)-n-2$ or even $p(r-n+2)$.

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Generalizing Anick's HAH Theorem

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