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# THE COHOMOLOGY STRUCTURE OF AN ASSOCIATIVE RING

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Let $A$ be an associative ring, $P$ be a two-sided $A$ module, and $H^n(A, P)$ denote the $n^{\text{th}}$ cohomology group of $A$ with coefficients in $P$; in particular, $P$ may be $A$ itself. Let $H^*(A, A)$, $H^*(A, P)$ denote the direct sum of the groups $H^n(A, A)$, $H^n(A, P)$, $n = 0, 1, 2, \cdots$, respectively. Then $H^*(A, A)$ is a graded ring in the cup product multiplication and $H^*(A, P)$ is a two-sided $H^*(A, A)$ module. In this paper, among others, the following results are proved:

1. The cup product is commutative, i.e., if $\alpha^m \in H^m(A, A)$, $\beta^n \in H^n(A, P)$ then

$$\alpha^m \smile \beta^n = (-1)^{mn}\beta^n \smile \alpha^m .$$

2. There is defined in $H^*(A, A)$ a second multiplication, the *bracket product* [ , ], in which $H^*(A, A)$ becomes a graded Lie ring, the grading being reduced by one from the usual. One has

$$(1) \qquad [H^m(A, A), H^n(A, A)] \subset H^{m+n-1}(A, A) ,$$

and for $\alpha^m$, $\beta^n$, $\gamma^p$ in $H^*(A, A)$ of dimensions $m$, $n$, $p$, respectively,

$$(2) \qquad [\alpha^m, \beta^n] = -(-1)^{(m-1)(n-1)}[\beta^n, \alpha^m] ,$$

and

$$(-1)^{(m-1)(p-1)}[[\alpha^m, \beta^n], \gamma^p] + (-1)^{(n-1)(m-1)}[[\beta^n, \gamma^p], \alpha^m] + (-1)^{(p-1)(m-1)}[[\gamma^p, \alpha^m], \beta^n] = 0 .$$

If $m = n = 1$, the bracket product in (1) and (2) is the ordinary Poisson bracket of derivations of $A$ into itself.

3. One has

$$[\alpha^m \smile \beta^n, \gamma^p] = [\alpha^m, \gamma^p] \smile \beta^n + (-1)^{m(p-1)}\alpha^m \smile [\beta^n, \gamma^p] ,$$

i.e., the additive endomorphism of $H^*(A, A)$ defined by $\alpha \longrightarrow [\alpha, \gamma^p]$ is a derivation of degree $p - 1$ of $H^*(A, A)$ considered as a ring under the cup product. (Somewhat more general statements are actually proved, for which see § 9.)

The present paper was originally intended as the first section of a paper in preparation on the deformation theory of algebras. There it will be shown that the bracket products $[\alpha^1, \beta^n]$, $[\alpha^2, \beta^n]$ of elements of $H^1(A, A)$ and of $H^2(A, A)$ with elements of $H^n(A, A)$ for any $n$ arise naturally, through an interpretation of $H^1(A, A)$ as the group of infinitesimal automorphism of $A$ and of $H^2(A, A)$ as the group of infinitesimal deformations. The ring $H^*(A, A)$ taken in the bracket may accordingly be called the *infinitesimal ring* of $A$.

It is assumed throughout this paper that a commutative ring $S$ with unit is given, and that all rings $A$, and the modules $P$ on which they act, are simultaneously $S$ modules, the operations of $A$ on itself and on $P$ being $S$ module homomorphisms. The role of $S$ is generally tacit. All tensor products will be assumed taken over $S$, and $S$ frequently will not be indicated.

## 1. Graded rings

A *graded ring* $A$ is one which, as an $S$ module, is a direct sum of submodules, $A = \sum A_\lambda$, indexed by the elements of an additive group $G$ and in which $A_\lambda A_\mu \subset A_{\lambda+\mu}$. In addition, *it will be assumed throughout this paper that there is specified in $G$ a subgroup $G^+$ of index two*, the elements of which will be called even; those of the complementary coset $G^-$ will be called odd. Commonly $G$ will be the additive group of integers, in which case $G^+$ is necessarily the even integers. The elements of $A_\lambda$ will be called homogeneous of degree $\lambda$. Setting $A^+ = \sum A_{\lambda'}$, $\lambda' \in G^+$, $A^- = \sum A_{\lambda''}$, $\lambda'' \in G^-$, we have $A^+A^+$, $A^-A^- \subset A^+$, $A^+A^-$, $A^-A^+ \subset A^-$; in particular, $A^+$ is a subring of $A$.

The groups $G$, $G^+$ being fixed, we define, for all $\lambda$ in $G$, $1^\lambda = 1$, and set $(-1)^\lambda = +1$ if $\lambda$ is even and $(-1)^\lambda = -1$ if $\lambda$ is odd. Also, we set $(-1)^{\lambda\mu} = ((-1)^\lambda)^\mu$ for all $\lambda$, $\mu$ in $G$. There is then an involutory automorphism $J$ of $A$ determined by setting for $a$ of degree $\lambda$, $Ja = (-1)^\lambda a$. Defining $J^\mu = 1$ for $\mu$ even, $J^\mu = J$ for $\mu$ odd, we have then, $J^\mu a = (-1)^{\lambda\mu} a$.

A graded ring $A = \sum A_\lambda$ will be called *commutative* if given elements $a$, $b$ in $A$ of degrees $\lambda$, $\mu$ respectively, we have

$$ab = (-1)^{\lambda\mu} ba ,$$

and will be called *skew* if

$$ab = -(-1)^{\lambda\mu} ba .$$

The *anti-isomorph* or *opposite* $A'$ of a ring $A$ is that ring which, as an $S$ module, is identical with $A$, but in which the product of $a$ and $b$ is defined to be $ba$. For ungraded rings, a commutative ring $A$ is identical with $A'$; while if $A$ is skew, the mapping $a \to -a$ is an isomorphism of $A$ with $A'$. For graded commutative and skew rings, because of the generality in which the definition has been given, it is conceivable that $A$ may fail to be isomorphic to $A'$. However, if the group $G$ is the integers, or more generally, if $G^+$ contains a subgroup $G^{++}$ such that $G/G^{++}$ is the cyclic group of four elements ($G^{++}$ being necessarily the multiples of four if $G$ is the integers) then $A$ is isomorphic to $A'$. For in this case, choosing a generator $u$ of $G/G^{++}$, we may define an element $\lambda$ of $G$ to be congruent to $n \bmod 4$, $n = 0, 1, 2$, or $3$, if $\lambda \equiv u^n \bmod G^{++}$. If $A$ is a graded commutative ring let an $S$ module automorphism $\sigma$ of $A$ be defined by setting, for $a^\lambda$ in $A_\lambda$,

$$\sigma(a^\lambda) = \begin{cases} a^\lambda & \text{if} \quad \lambda \equiv 0, 1 \bmod 4 \\ -a^\lambda & \text{if} \quad \lambda \equiv 2, 3 \bmod 4 \,. \end{cases}$$

It is then a straightforward matter to verify that, if $b^\mu$ is of degree $\mu$, then $\sigma(a^\lambda b^\mu) = (-1)^{\lambda\mu}\sigma(a^\lambda)\sigma(b^\mu) = \sigma(b^\mu)\sigma(a^\lambda)$, i.e., $\sigma$ is an anti-automorphism of $A$, or an isomorphism of $A$ onto $A'$. If $A$ is a graded skew ring, the requisite $\sigma$ is defined by

$$\sigma(a^\lambda) = \begin{cases} -a^\lambda & \text{if} \quad \lambda \equiv 0, 1 \bmod 4 \\ a^\lambda & \text{if} \quad \lambda \equiv 2, 3 \bmod 4 \,. \end{cases}$$

## 2. Graded Lie and pre-Lie rings

A ring $A$ is a *graded Lie ring* if it is a graded ring with a skew multiplication (usually denoted $[\ ,\ ]$) satisfying the graded Jacobi identity, i.e., if given elements $a^\lambda$, $b^\mu$, $c^\nu$ in $A$ of degrees $\lambda$, $\mu$, $\nu$, respectively, we have

$$(3) \qquad [a^\lambda, b^\mu] = -(-1)^{\lambda\mu}[b^\mu, a^\lambda] \,,$$

and

$$(4) \qquad (-1)^{\lambda\nu}[[a^\lambda, b^\mu]\, c^\nu] + (-1)^{\mu\lambda}[[b^\mu, c^\nu], a^\lambda] + (-1)^{\nu\mu}[[c^\nu, a^\lambda], b^\mu] = 0 \,.$$

If $A$ is a graded Lie ring, then it follows immediately from (3) and (4) that so is its anti-isomorph. Denoting by $R_a$ the module homomorphism of $A$ defined by $cR_a = [c, a]$, then (3) and (4) are equivalent to (3) and

$$(5) \qquad R_{a^\lambda}R_{b^\mu} - (-1)^{\lambda\mu}R_{b^\mu}R_{a^\lambda} = R_{[a^\lambda, b^\mu]} \,.$$

(Note: Generally transformations will be written on the left; right multiplications, like $R_a$, will be the only exceptions.)

A ring $A$ will be called a *graded right pre-Lie ring* if for elements $a^\lambda$, $b^\mu$, $c^\nu$ of $A$ of degrees $\lambda$, $\mu$, $\nu$, respectively, we have, denoting the product of $a$ and $b$ in $A$ by $a \circ b$,

$$(6) \qquad (c \circ a^\lambda) \circ b^\mu - (-1)^{\lambda\mu}(c \circ b^\mu) \circ a^\lambda = c \circ \big(a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda\big) \,.$$

(A graded left pre-Lie ring will be one whose anti-isomorph is a graded right pre-Lie ring. Henceforth all pre-Lie rings will be tacitly understood to be right pre-Lie rings unless otherwise specified.) Denoting by $R_a$ again the right multiplication by $a$, (6) is equivalent to

$$R_{a^\lambda}R_{b^\mu} - (-1)^{\lambda\mu}R_{b^\mu}R_{a^\lambda} = R_{a^\lambda\circ b^\mu - (-1)^{\lambda\mu}b^\mu\circ a^\lambda} \,. \tag{7}$$

It follows immediately from the defining identity (6), that a graded associative ring is also a graded pre-Lie ring. However, a graded Lie ring in general is not a graded pre-Lie ring.

THEOREM 1. *Let $A$ be a graded pre-Lie ring, and define for elements in $A$ a new multiplication by setting for $a^\lambda, b^\mu$ of degrees $\lambda, \mu$, respectively,*

$$[a^\lambda, b^\mu] = a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda \,.$$

*Then in the bracket product $A$ is a graded Lie ring.*

PROOF. It is evident that the bracket product is skew. We need therefore only verify that (4) holds. The first term in (4) is

$$(-1)^{\lambda\nu}\big[\big(a\circ b - (-1)^{\lambda\nu} b\circ a\big)\circ c - (-1)^{(\lambda+\mu)\nu} c\circ\big(a\circ b - (-1)^{\lambda\mu} b\circ a\big)\big] \,,$$

which by the hypothesis (6) is equal to

$$(-1)^{\lambda\nu}\big[\big(a\circ b - (-1)^{\lambda\nu} b\circ a\big)\circ c - (-1)^{(\lambda+\mu)\nu}\big((c\circ a)\circ b - (-1)^{\lambda\mu}(c\circ b)\circ a\big)\big] \,.$$

Applying similar transformations to the other terms in (6) the left side becomes

$$\begin{aligned}
&(-1)^{\lambda\nu}\big[\big(a\circ b - (-1)^{\lambda\mu} b\circ a\big)\circ c - (-1)^{(\lambda+\mu)\nu}\big((c\circ a)\circ b - (-1)^{\lambda\mu}(c\circ b)\circ a\big]\\
&+ (-1)^{\mu\lambda}\big[\big(b\circ c - (-1)^{\mu\nu} c\circ b\big)\circ a - (-1)^{(\mu+\nu)\lambda}\big((a\circ b)\circ c - (-1)^{\mu\nu}(a\circ c)\circ b\big]\\
&+ (-1)^{\nu\mu}\big[\big(c\circ a - (-1)^{\nu\lambda} a\circ c\big)\circ b - (-1)^{(\nu+\lambda)\mu}\big((b\circ c)\circ a - (-1)^{\nu\lambda}(b\circ a)\circ c\big] \,,
\end{aligned}$$

which vanishes identically. Therefore (4) holds, and this ends the proof.

It follows, in particular from Theorem 1, that the bracket product may be introduced in a graded associative ring, yielding a graded Lie ring.

## 3. Modules over associative, Lie and pre-Lie rings

Let $A = \sum A_\lambda$ be a graded ring and $P = \sum P_\lambda$ be a module which is a direct sum of modules indexed by the same group as indexes $A$. Suppose further that there is given a module homomorphism $\rho\colon P \otimes A \to P$ such that $\rho(P_\lambda \otimes A_\mu) \subset P_{\lambda+\mu}$. If $A$ is either an associative, Lie, or pre-Lie ring, then we shall say that $P$ is a *right $A$ module*, provided the following respective conditions are satisfied:

If $A$ is associative, then denoting for $z \in P$, $a \in A$, $\rho(z \otimes a)$ by $za$, we have

$$(za)b = z(ab) \,, \qquad z \in P;\ a, b \in A \,.$$

If $A$ is a graded Lie ring, denoting $\rho(z \otimes a)$ by $[z, a]$ we have, for $a^\lambda$, $b^\mu$ in $A$ of degrees $\lambda$, $\mu$, respectively,

$$(8) \qquad [[z, a^\lambda], b^\mu] - (-1)^{\lambda\mu}[[z, b^\mu], a^\lambda] = [z, [a^\lambda, b^\mu]] .$$

Denoting again by $R_a$ the endomorphism of $P$ defined by $zR_a = [z, a]$, (8) is formally equivalent to (5). Therefore, a graded Lie ring is a right module over itself.

If $A$ is a pre-Lie ring, denoting $\rho(z \otimes a)$ by $z \circ a$, we have for $a^\lambda$, $b^\mu$, in $A$ of degrees $\lambda$, $\mu$, respectively,

$$(9) \quad (z \circ a^\lambda) \circ b^\mu - (-1)^{\lambda\mu}(z \circ b^\mu) \circ a^\lambda = z \circ \left(a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda\right) .$$

Denoting now by $R_a$ the endomorphism $z \to z \circ a$, (9) is formally equivalent to (7), and a graded pre-Lie ring is therefore also a right module over itself.

If $P$ is a right module over an associative ring $A$, then it is also a right module over $A$ considered as a pre-Lie ring. We define $z \circ a = za$ for $z \in P$, $a \in A$.

If $P$ is a right module over a pre-Lie ring $A$, then it is also a right module over the associated Lie ring obtained by introducing the bracket product. We define

$$(10) \qquad [z, a] = z \circ a . \qquad \text{for} \quad z \in P, a \in A .$$

With respect to this definition, however, the reader must exercise a certain caution. If $A$ is a graded pre-Lie ring, then it is a right module over itself and therefore, using (10), it is also a right module over its associated graded Lie ring. However, taking in $A$ the graded Lie ring structure it is also a right module over itself, but *this is not the same module structure*. For example, suppose $A$ is associative, and for simplicity, ungraded (i.e., $A_\lambda = 0$ for $\lambda \neq 0$). Let $a, b, z$ be elements of $A$. Considering $A$ as a pre-Lie ring, we have $a \circ b = ab$, and the associated Lie structure is given by $[a, b] = ab - ba$. Under the definition given in (10), we have, considering the pre-Lie ring $A$ as a right $A$ module, $[z, a] = z \circ a = za$. It is then indeed the case that $[[z, a], b] - [[z, b], a] = [z, [a, b]]$, the left side being $zab - zba$ and the right $z(ab - ba)$. However, considering $A$ as a Lie ring in the multiplication $[a, b] = ab - ba$, we have $[z, a] = za - az$, and again $[[z, a], b] - [[z, b], a] = [z, [a, b]]$, but this is clearly in general not the same right module structure over the Lie ring $A$.

A *left module* $P$ over a graded associative, Lie, or left pre-Lie ring $A$ is a right module over its anti-isomorph $A'$. The notations are similar to those for right modules, the operation of the elements of $A$ on $P$ simply being written on the left.

If $P = \sum P_\lambda$ has both the structure of a right and left module over a graded ring $A$, the gradation of $P$ being the same for both structures, then the module direct sum $A + P$ is graded by setting $(A + P)_\lambda = A_\lambda + P_\lambda$, and, by defining the product of two elements of $P$ to be zero, may be made in a natural way into a ring containing $A$ as a subring, and $P$ as an ideal with $P^2 = 0$ and $A + P/A \cong A$. If $A$ is an associative or Lie ring and $A + P$ has the like type of structure, then $P$ is a *two-sided A module*. If $A$ is a graded commutative ring and $P$ a right $A$ module, then setting for $a \in A_\lambda$, $z \in P_\mu$,

$$a^\lambda z^\mu = (-1)^{\lambda\mu} z^\mu a^\lambda ,$$

$P$ becomes a two-sided $A$ module and $A + P$ is again commutative. Similarly, if $A$ is a graded Lie ring and $P$ a right $A$ module, setting

$$[a^\lambda, z^\mu] = -(-1)^{\lambda\mu}[z^\mu, a^\lambda] ,$$

$P$ becomes a two-sided $A$ module and $A + P$ is a graded Lie ring. For these cases, therefore, it is unambiguous to say that $P$ is a module over $A$.

If $P$ and $P'$ are right modules over a graded Lie or pre-Lie ring $A$, then $P \otimes P'$ may be given the structure of a right module over $A$ by setting

$$(P \otimes P')_\nu = \sum\nolimits_{\lambda+\mu=\nu} P_\lambda \otimes P'_\mu$$

and for $y^\mu \in P'_\mu$, $z \in P$, $a^\lambda \in A_\lambda$, by setting

$$[x \otimes y^\mu, a^\lambda] = (-1)^{\lambda\mu}[x, a^\lambda] \otimes y^\mu + x \otimes [y^\mu, a^\lambda]$$

in the Lie case, and

$$(x \otimes y^\mu) \circ a^\lambda = (-1)^{\lambda\mu}(x \circ a^\lambda) \otimes y^\mu + x \otimes (y^\mu \circ a^\lambda)$$

in the pre-Lie case (i.e., by using the identical formula).

If $P$ is a right module over a graded associative, Lie or pre-Lie ring $A$, given by a homomorphism $\rho$: $P \otimes A \to P$, and if $\varphi$ is a gradation preserving ring endomorphism of $A$, then we can define a new right $A$ module structure on $P$ by a homomorphism $\rho^\varphi$: $P \otimes A \to A$ given by

$$\rho^\varphi(z \otimes a) = \rho(z \times \varphi(a)) ,$$

for $z \in P$, $a \in A$. In particular, we may take $\varphi = J^\nu$ (i.e., $\varphi = J$ if $\nu$ odd, $\varphi = 1$ if $\nu$ even).

The concept of a module over a ring has been discussed only for those classes of rings arising here, but may generally be defined for any type of ring structure given in terms of identities in right multiplications, for example, for Jordan rings.

## 4. Derivations

If $P = \sum P_\lambda$, $P' = \sum P'_\lambda$ are direct sums of modules indexed by the same group $G$, then a module homomorphism $\varphi: P \to P'$ will be said to be homogeneous of degree $\nu \in G$, if $\varphi(P_\lambda) \subset P'_{\lambda+\nu}$ for all $\lambda$ in $G$. If $A = \sum A_\lambda$ is a graded ring, then a module endomorphism $D$ of $A$ will be called a *left derivation of degree* $\nu$ of $A$ if $D$ is of degree $\nu$ as a homomorphism of $A$ into itself and if, given $a^\lambda$, $b^\mu$ of degrees $\lambda$ and $\mu$ in $A$, we have

$$D(a^\lambda b^\mu) = (Da)b + (-1)^{\nu\lambda} a(Db) \; ; \tag{11}$$

$D$ is a *right derivation* of degree $\nu$ if it is of degree $\nu$ and

$$D(a^\lambda b^\mu) = (-1)^{\nu\mu}(Da)b + a(Db) \; . \tag{12}$$

A left derivation of $A$ is a right derivation of the anti-isomorph of $A$.

If $D$ and $D'$ are left derivations of $A$ of degrees $\nu$, $\nu'$, respectively, then

$$[D, D'] = DD' - (-1)^{\nu\nu'} D'D$$

is a left derivation of degree $\nu + \nu'$; the identical assertion is true for right derivations. Letting $\mathcal{D}_\nu = \mathcal{D}_\nu(A)$ denote the module of all left derivations of $A$ of degree $\nu$, $\mathcal{D} = \sum \mathcal{D}_\nu$ is therefore in a natural way a graded Lie algebra graded by the same group as $A$, and the same is true for right derivations. These algebras are anti-isomorphic to each other.

If $A$ is a graded associative ring and $a^\lambda$ an element of degree $\lambda$, then the module endomorphisms $D_a$, $D'_a$ of $A$, defined by setting for $b^\mu$ in $A$ of degree $\mu$,

$$D_{a^\lambda} b^\mu = a^\lambda b^\mu - (-1)^{\lambda\mu} b^\mu a^\lambda$$

and

$$D'_{a^\lambda} b^\mu = b^\mu a^\lambda - (-1)^{\lambda\mu} a^\lambda b^\mu$$

are left and right derivations, respectively, of $A$, of degree $\lambda$. Likewise, if $A$ is a grade Lie ring, then

$$D_{a^\lambda} b^\mu = [a^\lambda, b^\mu] \tag{13}$$

and

$$D'_{a^\lambda} b^\mu = [b^\mu, a^\lambda] \tag{14}$$

define left and right derivations, respectively, of degree $\lambda$. Such derivations are called inner.

If $A$ is a graded ring for which the concept of two-sided module has been defined, i.e., in the present case, if $A$ is a Lie or associative ring, and if $P$ is a two-sided module over $A$, then a module homomorphism

$D: A \to P$ is a left derivation, respectively right derivation, of degree $\nu$ of $A$ into $P$ if it is homogeneous of degree $\nu$ and if given $a^\lambda \in A_\lambda$, $b^\mu \in A_\mu$, (11), respectively (12), holds, where now the multiplication is the module operation of $A$ on $P$. Inner derivations are definable exactly as before, by (13) and (14), where now $b^\mu$ is taken in $P_\mu$. If $D$ is a left derivation of degree $\nu$ of $A$ into $P$, and if we define a new two-sided $A$ module structure on $P$ by defining the new product of an element $a$ of $A$ with an element $z$ of $P$ to be the old product of $J^\nu a$ with $z$, then the homomorphism $D: A \to P$, relative to the new module structure, becomes a right derivation of degree $\nu$. If $D$ is a derivation of $A$ either into itself or into a two-sided module over $A$, then the set of elements annihilated by $D$ is a subring of $A$.

## 5. Pre-Lie systems

By a right pre-Lie system $\{V_m, \circ_i\}$, we shall mean a sequence $\cdots V_{-1}, V_0, V_1, V_2, \cdots$ of $S$ modules and an assignment for every triple of integers $m, n, i \geqq 0$ with $i \leqq m$ of a homomorphism $\circ_i = \circ_i(m, n)$ of $V_m \otimes V_n$ into $V_{m+n}$ with properties described below. If $f \in V_m$, we may write $f = f^m$ to indicate its degree; and if $g \in V_n$, we shall denote $\circ_i(f \otimes g)$ by $f \circ_i g$. The prescribed properties are given by

$$(f^m \circ_i g^n) \circ_j h^p = \begin{cases} (f^m \circ_j h^p) \circ_{i+p} g^n & \text{if} \quad 0 \leqq j \leqq i-1 \\ f^m \circ_i (g^n \circ_{j-i} h^p) & \text{if} \quad i \leqq j \leqq n+1 \end{cases}, \tag{15}$$

$f, g, h$ in $V_m, V_n, V_p$, respectively. From the first case of (15), we may deduce further that

$$(f^m \circ_j h^p) \circ_{i+p} g^n = (f^m \circ_{i+p} h^p) \circ_{j+n} g^n \qquad \text{if} \quad 0 \leqq i + p \leqq j - 1 ,$$

or, interchanging the roles of $g$ and $h$, we have

$$(f^m \circ_i g^n) \circ_j h^p = (f^m \circ_{j-n} h^p) \circ_i g^n \qquad \text{if} \quad n + i + 1 \leqq j \leqq m + n . \tag{16}$$

Among the simple examples of (right) pre-Lie systems, we have the following.

1. Given any sequence of modules $V_{-m}$, $m = 1, 2, \cdots$, indexed by the negative integers, define $V_m = 0$ for $m \geqq 0$ and define all the $\circ_i$ to be zero.

2. Let $A$ be an associative ring and set $V_0 = A$, $V_m = 0$ for $m \neq 0$. Then $\circ_i(m, n)$ need only be defined for $i = m = n = 0$, in which case we take it to be the multiplication in $A$.

3. Let $V$ be an $S$ module, $V^m$, $m = 1, 2, \cdots$ denote the tensor product of $V$ with itself $m$ times, and let $V_m = V^m$, $m = 1, 2, \cdots$. Set $V_0 = S$, $V_{-1} = V_{-2} = \cdots = 0$. If $f \in V_m$, $g \in V_n$ are of the form $f = \alpha_1 \otimes \cdots \otimes \alpha_m$,

$g = \beta_1 \otimes \cdots \otimes \beta_n$, respectively, define $\circ_i = \circ_i(m, n)$ by

$$f^m \circ_i g^n = \alpha_1 \otimes \cdots \otimes \alpha_i \otimes \beta_1 \otimes \cdots \otimes \beta_n \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_m .$$

We may then extend $\circ_i$ linearly. In this example $\circ_i$ is always an isomorphism, and in fact $\circ_m(m, n)$ is the identity. If $m = 0$, in which case $f$ is just an element $s$ of $S$, then we identify $s \otimes g$ with $sg = gs$; $\circ_i$ is then defined only for $i = 0$ and $s \circ_0 g$ is again just $sg$. If $n = 0$, then $f \otimes s = sf = f \circ_i s$ for all $i = 0, 1, \cdots, m$. It is trivial that (15) holds.

4. Letting $V$, $V^m$ be as in 3, set $V_m = \mathrm{Hom}_S(V^m, S)$, $m = 1, 2, \cdots$, set $V_0 = \mathrm{Hom}_S(S, S)$, which we identify with $S$ itself, and set $V_{-1} = V_{-2} = \cdots = 0$. If $f \in V_m$, $g \in V_n$, we may define an element denoted $fg$ of $V_{m+n}$ as follows: Given $a \in V^m$, $b \in V^n$, set $fg(a \otimes b) = f(a)g(b)$, and extend the definition of $fg$ to all of $V^{m+n}$ linearly. Let $\circ_i' = \circ_i'(m, n)$ now denote the inverse of the isomorphism defined in 3 of $V^m \otimes V^n$ onto $V^{m+n}$. Then we may define homomorphisms $\circ_i$ of $V_m \otimes V_n$ into $V_{m+n}$ by setting $f \circ_i g(a \otimes b) = fg(a \circ_i' g)$, extended linearly. Again, it is not difficult to verify that (15) holds. Unlike the case in 3, in the present example the $\circ_i$ are not generally isomorphisms, nor is $V_m$ the tensor product of $V_1$ with itself $m$ times. For example, let $F_2$ be the field of 2 elements, $W$ be a 2-dimensional vector space over $F_2$, and let $F_2 + W$ be made into a ring $S$ by setting $W^2 = 0$. Let $V$ be a one dimensional vector space over $F_2$ on which $S$ operates by setting $WV = 0$. Then $V \otimes_S V \cong V$, whence $V^m = V$ for all positive $m$ and $V_m = \mathrm{Hom}_S(V, S) = W$ for all positive $m$, but $W \otimes_S W = W + W \not\cong W$.

The example in which we shall later be most interested is the following.

5. Let $V$, $V^m$ be as in 3, $W$ be another $S$ module and $\varphi$: $W \to V$ be an $S$ module homomorphism. Set $V_m = \mathrm{Hom}_S(V^{m+1}, W)$, $m = 0, 1, \cdots$, $V_{-1} = \mathrm{Hom}_S(S, W)$, which we may identify with $W$, and $V_{-2} = V_{-3} = \cdots = 0$. If $f \in V_m$, $g \in V_n$, $m, n \geqq 0$, then define $f \circ_i g \in V_{m+n} = \mathrm{Hom}_S(V^{m+n+1}, W)$ by setting

$$\begin{aligned} &f \circ_i g(a_0 \otimes \cdots \otimes a_{i-1} \otimes b_0 \otimes \cdots \otimes b_m \otimes a_{i+1} \otimes \cdots \otimes a_m) \\ &\quad = f\big(a_0 \otimes \cdots \otimes a_{i-1} \otimes \varphi g(b_0 \otimes \cdots \otimes b_m) \otimes a_{i+1} \otimes \cdots \otimes a_m\big) . \end{aligned}$$

The definition is extended to the case where $n = -1$ by interpreting $g$ to be simply an element of $W$ and setting

$$\begin{aligned} &f \circ_i g(a_0 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes \cdots \otimes a_m) \\ &\quad = f(a_0 \otimes \cdots \otimes a_{i-1} \otimes \varphi g \oplus a_{i+1} \otimes \cdots \otimes a_m) . \end{aligned}$$

That (16) holds is again trivial. Note that, while in the previous examples the module $\cdots + V_{-1} + V_0 + V_1 + \cdots$ was in a natural way an associative algebra, this is not the case here. When considering this example, we shall assume until § 9 that $W = V$ and that $\varphi$ is the identity.

## 6. The graded Lie ring of a pre-Lie system

In this section, $\lambda$, $\mu$, $\nu$ will denote integers.

Given an arbitrary pre-Lie system $\{V_m, \circ_i\}$, we now define for every $m$ and $n$ a new homomorphism $\circ$ of $V_m \otimes V_n$ into $V_{m+n}$ by setting for $f^m \in V_m$, $g^n \in V^n$, with $m \geqq 0$,

$$f^m \circ g^n = \textstyle\sum_{i=0}^{m} (-1)^{ni} f^m \circ_i g^n \ , \tag{17}$$

i.e.,

$$f^m \circ g^n = \begin{cases} f^m \circ_0 g^n + f^m \circ_1 g^n + \cdots + f^m \circ_m g^n & \text{if } n \text{ is even} \\ f^m \circ_0 g^n - f^m \circ_1 g^n + \cdots + (-1)^m f^m \circ_m g^n & \text{if } n \text{ is odd} \ , \end{cases} \tag{18}$$

and setting

$$f^m \circ g^n = 0 \qquad \text{if} \quad m < 0 \ . \tag{19}$$

THEOREM 2. *Let $\{V_m, \circ_i\}$ be a pre-Lie system and $f^m, g^n, h^p$ be elements of $V_m$, $V_n$, $V_p$ respectively. Then*

(i) $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = \sum' (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p$, *where the sum $\sum'$ is extended over those $i$ and $j$ with either $0 \leqq j \leqq i - 1$ or $n + i + 1 \leqq j \leqq m + n$.*

(ii) $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np}[(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)]$.

PROOF. If $(f^m \circ g^n) \circ h^p$ is expanded according to the definitions, (17)–(19), then the term $(f^m \circ_i g^n) \circ_j h^p$ occurs with coefficient $(-1)^{ni+pj}$. If all these terms are transformed according to the formulas (15), (16), then a term of the form $\pm f^m \circ_\lambda (g^n \circ_\mu h^p)$ occurs if and only if one can find $i \geqq 0$ and $j$ with $i \leqq j \leqq n + 1$, such that $\lambda = i$ and $\mu = j - i$. This says, however, that every term of the form $\pm f^m \circ_\lambda (g^n \circ_\mu h^p)$ occurs. It occurs further with coefficient $(-1)^{n\lambda + p(\lambda+\mu)} = (-1)^{p\mu + (n+p)\lambda}$, i.e., the same coefficient with which it appears in $f^m \circ (g^n \circ h^p)$. These terms being the transforms of those $(f^m \circ_i g^n) \circ_j h^p$ with $i \leqq j \leqq n + i$, assertion (i) follows. Observe further that $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p)$ can be expressed as a sum of terms of the form $\pm (f^m \circ_j h^p) \circ_{i+p} g^n$ with $0 \leqq j \leqq i - 1$ and of the form $\pm (f^m \circ_{j-n} h^p) \circ g^n$ with $n + i + 1 \leqq j \leqq m + n$, and all such occur. Therefore, a term of the form $(f^m \circ_\lambda h^p) \circ_\mu g^n$ occurs if and only if we can find either $i$ and $j$ with $i + p = \mu$, $j = \lambda$ and $0 \leqq j \leqq i - 1$, or $i$ and $j$ with $i = \mu$, $j - n = \lambda$ and $n + i + 1 \leqq j \leqq m + n$; i.e., as one sees readily, if and only if we do not have $\lambda \leqq \mu \leqq p + \lambda$. The terms which occur are therefore those occuring in $(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)$, up to sign. As for the sign, $(f^m \circ h^p) \circ_{i+p} g^n$ occurs in $(f^m \circ h^p) \circ g^n$ with sign $(-1)^{pj+n(i+p)} = (-1)^{ni+pj}(-1)^{np}$, and $(f^m \circ_{j-n} h^p) \circ_i g^n$ occurs with the same sign, which is in both cases $(-1)^{np}$ times the factor with which it appears in $(f^m \circ g^n) \circ h^p$. Therefore

$$(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np}[(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)] ,$$

which is assertion (ii).

COROLLARY. *Let $\{V_m, \circ_i\}$ be a pre-Lie system and let $A = \sum V_m$ be the direct sum of the modules $V_m$ made into a ring by extending $\circ$ to be an $S$ module homomorphism of $A \otimes A$ into $A$. Then with this multiplication, $A$ becomes a right pre-Lie ring graded by the integers.*

PROOF. This follows immediately from the fact that assertion (ii) of the theorem can be rewritten in the form

$$(f^m \circ g^n) \circ h^p - (-1)^{np}(f^m \circ h^p) \circ g^n = f^m \circ (g^n \circ h^p) - (-1)^{np} f^m \circ (h^p \circ g^n) ,$$

and this is identical with equation (6) defining pre-Lie rings.

We shall say that $A$ is the pre-Lie ring (graded by the integers) associated with $\{V_m, \circ_i\}$; the graded Lie ring derived from $A$ by Theorem 1 of § 2 will be called the graded Lie ring associated with $\{V_m, \circ_i\}$.

If $\{V_m, \circ_i\}$ is a pre-Lie system and $W = \sum W_m$ is a direct sum of modules indexed by the integers, then we shall say that $W$ has the structure of a right module over the given pre-Lie system if there exist module homomorphisms (which we shall again denote by $\circ_i(m, n)$) of $W_m \otimes V_n$ into $W_{m+n}$ such that, if $f = f^m \in W_m$, $g = g_n \in V_n$, $h = h^p \in V_p$, then (15) holds. If such a structure is defined on $W$, then one sees immediately that there may also be defined on $W$ the structure of a right module over the pre-Lie ring associated with $\{V_m, \circ_i\}$ by setting $f^m \circ g^n = \sum_{i=0}^{m} (-1)^{ni} f^m \circ_i g^n$, i.e., by using (17), where $f^m$ is now taken to be in $W_m$. It follows, therefore, from (10) of § 2 that $W$ also has the structure of a module over the graded Lie ring associated with $\{V_m, \circ_i\}$.

In brief, combining Theorems 1 and 2 we see that every pre-Lie system, in particular those of the examples, gives rise in a natural way to a graded Lie ring, and every right module over such a system gives rise to a right module over that graded Lie ring. The pre-Lie system with which we shall be mainly concerned will be that formed from the cochains of an associative ring with coefficients in the ring itself. This will be of the type of example 5 of § 5.

## 7. Cohomology of a ring and commutativity of the cup product

Let $A$ be an associative ring and $P$ be a two-sided $A$ module. Following the classical definition of Hochschild (and at the risk of seeming old-fashioned) we define an $m$-cochain $f^m$ of $A$ with coefficients in $P$ to be an

$S$ module homomorphism of the tensor product $A^{(m)}$ of $A$ with itself $m$ times into $P$. The module $\mathrm{Hom}_S\,(A^{(m)}, P)$ of all such $f^m$ will be denoted by $C^m(A, P)$, $S$ being understood. We identify $C^0(A, P)$ with $P$. For every $m$ there is defined a homomorphism $\delta_m$: $C^m(A, P) \to C^{m+1}(A, P)$ defined by setting

$$\begin{aligned}\delta_m f(a_1 \otimes \cdots \otimes a_{m+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{m+1}) \\ &\quad + \textstyle\sum_{i=1}^{m} (-1)^i f(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{m+1}) \\ &\quad + (-1)^{m+1} f(a_1 \otimes \cdots \otimes a_m) a_{m+1} ,\end{aligned}$$

and extending the definition of $\delta = \delta_m$ linearly. It is the case that $\delta_{m+1}\delta_m = 0$; $Z^m(A, P)$ is defined to be the kernel of $\delta_m$, $B^m(A, P)$ to be the image of $\delta_{m-1}$ for $m \geqq 1$, and to be zero for $m = 0$, one has $B^m(A, P) \subset Z^m(A, P)$, and $H^m(A, P)$ is defined to be $Z^m(A, P)/B^m(A, P)$. We will denote by $C^*(A, P)$, $Z^*(A, P)$, $H^*(A, P)$, respectively, the direct sums of the modules $C^m(A, P)$, $Z^m(A, P)$, $H^m(A, P)$ for $m = 0, 1, 2, \cdots$.

If $P$ is further an associative ring, then the multiplication in $P$ induces, for every $m$ and $n$, a homomorphism denoted $\smile$ and called the cup product of $C^m(A,P) \otimes C^n(A,P)$ into $C^{m+n}(A,P)$ defined by setting for $f^m \in C^m(A, P)$, $g^n \in C^n(A, P)$,

$$\begin{aligned}f^m \smile g^n(a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n) \\ = f^m(a_1 \otimes \cdots \otimes a_m) g^n(b_1 \otimes \cdots \otimes b_n) .\end{aligned}$$

Under this multiplication $C^*(A, P)$ becomes an associative ring graded by the integers. We have, trivially,

$$(f^m \smile g^n) = \delta f^m \smile g^n + (-1)^m f^m \smile \delta g^n \ , \tag{20}$$

i.e., $\delta$ is a left derivation of degree one of the ring $C^*(A, P)$. The formula (20) shows that if $f^m \in Z^m(A, P)$ and $g^n \in Z^n(A, P)$, then $f^m \smile g^n \in Z^{m+n}(A, P)$, whence $Z^*(A, P)$ is a subring of $C^*(A, P)$. Further, if either $f^m \in B^m(A, P)$ or $g^n \in B^n(A, P)$ then $f^m \smile g^n \in B^{m+n}(A, P)$, i.e., $B^*(A, P)$ is an ideal of $Z^*(A, P)$. We may therefore define the cup product of elements of $H^m(A, P)$ and $H^n(A, P)$ by choosing arbitrary representatives for them in $Z^m(A, P)$ and $Z^n(A, P)$, respectively. The induced multiplications will again be denoted by $\smile$, and makes $H^*(A, P) = Z^*(A, P)/B^*(A, P)$ into an associative ring, called the cohomology ring of $A$ with coefficients in $P$.

Now it is a familiar fact from algebraic topology that under rather general conditions the cohomology ring of a space with coefficients in a commutative ring is a commutative graded ring. However for arbitrary commutative coefficient rings $P$, it is not generally the case that $H^*(A, P)$ is commutative. For example, suppose $A$ is a zero ring (i.e., $A^2 = 0$)

acting trivially on $P$, i.e., $AP = PA = 0$. Then we may readily verify that every cochain is a cocycle, whence $C^m(A, P) = Z^m(A, P)$, $B^m(A, P) = 0$ for all $m$, and $H^*(A, P) = C^*(A, P)$. Let $f, g$ be elements of $C^1(A, P)$. Were $H^*(A, P)$ commutative then $f \smile g + g \smile f$ would have to be a coboundary, i.e., in the present case, would have to vanish. We would have, therefore, $f(a)g(b) + g(a)f(b) = 0$ for all $a$ and $b$ in $A$. But the multiplication here takes place in $P$, which is merely assumed to be commutative and not necessarily a zero ring; $f(a)g(b) + g(a)f(b)$ will generally fail to vanish.

Now let $P$ be, as at the start, a two-sided $A$ module. Then $C^*(A, P)$ is naturally endowed with the structure of a two-sided $C^*(A, A)$ module. The operation will again be denoted by $\smile$ and we define for $f \in C^m(A, A)$, $g \in C^n(A, P)$, $f \smile g \in C^{m+n}(A, P)$ by

$$\begin{aligned} f \smile g(a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n) \\ = f(a_1 \otimes \cdots \otimes a_m) g(b_1 \otimes \cdots \otimes b_n) , \end{aligned}$$

and similarly for $g \smile f$.

Note now that since $C^m(A, A) = \mathrm{Hom}_S(A^{(m)}, A)$, setting $A = V$ and considering only its $S$ module structure, and setting $A^{(m)} = V^m$, $V_m = C^{m+1}(A, A) = \mathrm{Hom}_S(V^{m+1}, V)$, then we have a system of modules of precisely the sort given in example 5 of § 5 (the homomorphism $\varphi$ of that example being taken to be the identity). However, the gradation of the system differs by one from the usual one by dimensions. The reader is cautioned in this respect when comparing formulas. We may say that an element of $C^m(A, A)$ has *dimension* $m$ but *degree* $m - 1$. Since $C^0(A, A)$ is identified with $A$, the elements of $A$ are to be understood as having degree $-1$. Repeating the definition, if we set for $f^m \in C^m(A, A)$, $g^n \in C^n(A, A)$

$$\begin{aligned} & f^m \circ_i g^n(a_0 \otimes \cdots \otimes a_{i-1} \otimes b_0 \otimes \cdots \otimes b_{n-1} \otimes a_{i+1} \otimes \cdots \otimes a_{m-1}) \\ (21) \qquad & = f\big(a_0 \otimes \cdots \otimes a_{i-1} \otimes g(b_0 \otimes \cdots \otimes b_{n-1}) \otimes a_{i+1} \otimes \cdots \otimes a_{m-1}\big) , \\ & \qquad i = 0, 1, \cdots, m-1 , \end{aligned}$$

and note that $f^m \circ_i g^n \in C^{m+n-1}(A, A)$, then it follows that *the structure of a right pre-Lie system is defined on the set of modules* $C^m(A, A)$. If now we take $f^m \in C^m(A, P)$, $g^n \in C^n(A, A)$, then (21) defines an element $f^m \circ_i g^n$ of $C^{m+n-1}(A, P)$, and we see immediately that $C^*(A, P)$ *has the structure of a right module over the right pre-Lie system* $\{C^m(A, A), \circ_i\}$.

From the results of § 6, it follows that $C^*(A, A)$ is also naturally endowed with the structure of a right pre-Lie ring and of a graded Lie ring. When it is necessary to state explicitly with respect to which product

$C^*(A, A)$ is being considered as a ring, we shall write $\{C^*(A, A), \smile\}$, $\{C^*(A, A), \circ\}$ or $\{C^*(A, A), [\ ,\ ]\}$. Only the first of these depends on the multiplicative structure of $A$, the others are defined exclusively in terms of the $S$ module structure. Further, from the results of § 6, one sees that $C^*(A, P)$ has the structure of a right module over $\{C^*(A, A), \circ\}$ and therefore of a two-sided module over $\{C^*(A, A), [\ ,\ ]\}$, in addition to being a two-sided module over $\{C^*(A, A), \smile\}$. Note again, however, the remarks following (10), § 3. If $f \in C^*(A, A)$, $g \in C^*(A, P)$, then we will write $f \circ g$ or $[f, g]$ (these being the same) for the operation of $f$ on $g$, and similarly if $f \in C^*(A, P)$, $g \in C^*(A, A)$.

One may readily verify that (20) holds not only when $f^m \in C^m(A, A)$ $g^n \in C^n(A, P)$, but also when either $f^m \in C^m(A, A)$, $g^n \in C^n(A, P)$ or $f^m \in C^m(A, P)$, $g^n \in C^n(A, A)$. It follows that $Z^m(A, A) \smile Z^n(A, P)$ and $Z^n(A, P) \smile Z^m(A, A)$ are both contained in $Z^{m+n}(A, P)$, and that $B^m(A, A) \smile Z^n(A, P)$ and $Z^m(A, A) \smile B^n(A, P)$ are both contained in $B^{m+n}(A, P)$. Therefore, $H^*(A, P)$ has the structure of a two-sided module over $\{H^*(A, A), \smile\}$. However, it is generally not the case that $Z^*(A, A)$ is closed under the pre-Lie multiplication, but it will be seen to be closed under the bracket product.

The associativity of $A$ implies that the element $\pi$ of $C^2(A, A)$ defined by $\pi(a, b) = ab$ is a 2-cocycle which may be called the canonical 2-cocycle of $A$. It is in fact a coboundary, being the coboundary of the identity cochain, i.e., of the cochain $\iota$ such that $\iota(a) = a$. Using $\pi$ and the operators of the system $\{C^m(A, A), \circ_i\}$, we may write, for $f^m \in C^m(A, A)$, $g^n \in C^n(A, A)$,

$$f^m \smile g^n = (\pi \circ_0 f^m) \circ_{m-1} g^n \ . \tag{22}$$

(Note that the degree of $\pi$ is one, of $f$ is $m - 1$, and of $g$ is $n - 1$.) The coboundary operator in $C^*(A, A)$ can also be expressed using $\pi$ and the pre-Lie product. One may readily verify that

$$\delta f^m = -\big(f \circ \pi - (-1)^{m-1} \pi \circ f\big) = (-1)^{m-1}\big(\pi \circ f - (-1)^{m-1} f \circ \pi\big) \ , \tag{23}$$

or, in terms of the bracket multiplication, since the degree of $\pi$ is one,

$$\delta f^m = [f, -\pi] = (-1)^{m-1}[\pi, f] \ .$$

Therefore, *the operator $\delta$ is a right inner derivation of degree* 1 *of the graded Lie ring* $\{C^*(A, A), [\ ,\ ]\}$. On the other hand, $\delta$ is generally not a derivation of any degree of the ring $\{C^*(A, A), \circ\}$, but we have the following fundamental result concerning its deviation from being a derivation of degree 1 and connecting its operation in $\{C^*(A, A), \circ\}$ with its operation in $\{C^*(A, A), \smile\}$.

THEOREM 3. *If $A$ is an associative ring and if $f^m, g^n \in C^m(A, A)$, $C^n(A, A)$, respectively, then*

$$f^m \circ \delta^n - \delta(f^m \circ g^n) + (-1)^{n-1}\delta f^m \circ g^n = (-1)^{n-1}[g^n \smile f^m - (-1)^{mn} f^m \smile g^n] .$$

PROOF. Using (23), the three terms on the left may be expanded to give

$$[(-1)^{n-1}f^m \circ (\pi \circ g^n) - f^m \circ (g^n \circ \pi)] - [(-1)^{m+n}\pi \circ (f^m \circ g^n) - (f^m \circ g^n) \circ \pi] + (-1)^{n-1}[(-1)^{m-1}(\pi \circ f^m) \circ g^n - (f^m \circ \pi) \circ g^n] .$$

Now from the defining relation for pre-Lie rings, (6) of § 2, with $c = f^m$, $a = g^n$, $b = n$, we have

$$(f^m \circ g^n) \circ \pi - (-1)^{n-1}(f^m \circ \pi) \circ g^n = f^m \circ [g^n \circ \pi - (-1)^{n-1}\pi \circ g^n] .$$

Therefore, of the six terms in the expansion, four cancel, leaving only $(-1)^{m+n}[(\pi \circ f^m) \circ g^n - \pi \circ (f^m \circ g^n)]$. On the other hand, using Theorem 2, (i) together with (22), we have

$$(\pi \circ f^m) \circ g^n - \pi \circ (f^m \circ g^n) = (-1)^{m(n-1)} f^m \smile g^n + (-1)^{m-1} g^n \smile f^m .$$

Multiplying by $(-1)^{m+n}$ proves the theorem.

It follows immediately from Theorem 3 that, if $f^m$, $g^n$ are cocycles, then

$$(24) \qquad (-1)^n[g^n \smile f^m - (-1)^{mn} f^m \smile g^n] = \delta(f^m \circ g^n) ,$$

whence we have

COROLLARY 1. *If $A$ is an associative ring, then the ring $\{H^*(A, A), \smile\}$ is a graded commutative ring, with grading given by dimension*, i.e., *if $\eta^m \in H^m(A, A)$, $\zeta^n \in H^n(A, A)$, then $\eta^m \smile \zeta^n = (-1)^{mn}\zeta^n \smile \eta^m$.*

Note here that the gradation taken on $H^*(A, A)$ is by dimensions. When dealing with the cup product we shall tacitly assume any gradation is by dimension; and that; when dealing with the bracket product, it is by degree unless otherwise stated. Observe that $H^0(A, A)$ is an ordinary commutative subring of $H^*(A, A)$, in fact, $H^0(A, A) = Z^0(A, A) =$ center of $A$.

In order to extend Theorem 3 to a statement about the module structure of $H^*(A, P)$ over $H^*(A, A)$, it is convenient to observe again that the module direct sum $A + P$ may be made, in a natural way into an associative ring over $S$ with $A$ as a subring and $P$ as an ideal with $P^2 = 0$ by setting, for $a, b \in A$, $x, y \in P$, $(a, x)(b, y) = (ab, ay + xb)$. There exist natural inclusions $C^m(A, A), C^m(A, P) \subset C^m(A + P, A + P)$ defined by setting for any $f^m$ in $C^m(A, A)$ or $C^m(A, P)$, $f^m(\alpha_1 \otimes \cdots \otimes \alpha_m) = 0$ if any $\alpha_i$ is in $P$. (Note, however, that the image of a cocycle under this map is not necessarily a cocycle.) Theorem 3 holds in particular for the ring $A + P$, from which we get the following stronger

COROLLARY 2. *Let $A$ be an associative ring, and $P$ be a two-sided $A$ module. If either $\eta^m \in H^m(A, A)$, $\zeta^n \in H^n(A, P)$ or $\eta^m \in H^m(A, P)$, $\zeta^n \in H^n(A, A)$, then*

$$\eta^m \smile \zeta^n = (-1)^{mn}\zeta^n \smile \eta^m .$$

## 8. The infinitesimal ring of a ring

Since $\delta$ is a derivation of $\{C^*(A, A), [\ ,\ ]\}$, it follows, exactly as for $\{C^*(A, A), \smile\}$, that $Z^*(A, A)$ is closed under the bracket multiplication, and in this multiplication $B^*(A, A)$ is an ideal. *Therefore, $H^*(A, A)$ becomes in a natural way a ring under the bracket multiplication,* which we have elected to call the infinitesimal ring of $A$. More generally, we have

THEOREM 4. *Let $A$ be an associative ring and $P$ be a two-sided $A$ module. Then $C^*(A, P)$ is, in a natural way, a two-sided module over $\{C^*(A, A), [\ ,\ ]\}$; we have*

$$[Z^*(A, P), Z^*(A, A)] \subset Z^*(A, P) ,$$

$$[B^*(A, P), Z^*(A, A)], [Z^*(A, P), B^*(A, A)] \subset B^*(A, P) ,$$

*and $H^*(A, P)$ is in a natural way a two-sided module over $\{H^*(A, A), [\ ,\ ]\}$.*

PROOF. That $C^*(A, P)$ is a two-sided module over $\{C^*(A, A), [\ ,\ ]\}$ has already been observed in § 7. Consider now the algebra $A + P$, and let $f^m$ be in $C^m(A + P, A + P)$, $g^n$ be an $C^n(A + P, A + P)$. Denoting by $\delta$ the coboundary operator in $C^*(A + P, A + P)$, we have, since $\delta$ is a right derivation of degree one of $\{C^*(A + P, A + P), [\ ,\ ]\}$,

$$\delta[f^m, g^n] = (-1)^{n-1}[\delta f^m, g^n] + [f^m, \delta g^n] .$$

Now since all terms of this equation are homomorphisms of the tensor power $(A + P)^{(m+n)}$ into $A + P$, the equation must hold *a fortiori* if the domains are restricted to $A^{(m+n)} \subset (A + P)^{(m+n)}$, and if either the range of $f^m$ is restricted to $P$ and the range of $g^n$ to $A$; or if the range of $f^m$ is restricted to $A$, and that of $g^n$ to $P$. But then $f^m$ becomes simply an element of $C^m(A, P)$ or $C^m(A, A)$, $g^n$ an element of $C^n(A, A)$ or $C^n(A, P)$, respectively, and $\delta$ becomes just the usual coboundary operator in $C^*(A,P)$ or $C^*(A, A)$, depending on the cochain on which it operates. The rest of the assertions of the theorem follow immediately.

Part of the information contained in the statement that $H^*(A, A)$ is a ring under the bracket multiplication is already quite familiar. In particular, the statement implies that $H^1(A, A)$ is an ordinary Lie ring. The elements of $Z^1(A, A)$ are just the ordinary derivations of $A$ into itself, and it is trivial that these form a Lie ring in which the inner derivations are an ideal. However, since $[H^1(A, A), H^n(A, P)] \subset H^n(A, P)$ for every $n$, we have the following more general statement.

COROLLARY. *Let $A$ be an associative ring and $P$ be a two-sided $A$ module. Then $H^n(A, P)$ is in a natural way a module over the (ordinary) Lie ring $H^1(A, A)$ for all $n = 0, 1, \cdots$.*

The operation of $H^1(A, A)$ on $H^0(A, A)$ is simple and illustrative. Recall that $H^0(A, A) = Z^0(A, A) =$ center of $A$. Suppose $f \in Z^1(A, A)$, i.e., given $a, b \in A$, $af(b) - f(ab) + f(a)b = 0$. If $\lambda \in$ center of $A$ is considered as an element of $Z^0(A, A)$, then $[f, \lambda] = f \circ \lambda - \lambda \circ f = f \circ \lambda$ is just the element $f(\lambda)$ of $Z^0(A, A)$. So it is asserted, in particular, that a derivation of $A$ carries the center of $A$ into itself. Indeed, if $\lambda$ is in the center of $A$ and $a$ arbitrary in $A$, then

$$\begin{aligned} af(\lambda) &= f(a\lambda) - f(a)\lambda = f(\lambda a) - \lambda f(a) \\ &= \lambda f(a) + f(\lambda)a - \lambda f(a) = f(\lambda)a \ . \end{aligned}$$

Further, if $f \in B^1(A, A)$, i.e., if $f(a)$ is of the form $ab - ba$ for some $b$ in $A$, then $f(\lambda) = 0$, i.e., $[f, \lambda] \in B^0(A, A)$.

It is natural to consider derivations of a ring $A$ into itself as being in some sense "infinitesimal automorphism." Since the automorphism group of $A$ operates in a natural way on $H^*(A, A)$, it is not surprising to find that $H^1(A, A)$ operates also. The close relationship between these will be examined in more detail in a forthcoming paper on the deformation of algebras, where it will be shown that $H^2(A, A)$ is to be interpreted as the set of "infinitesimal deformations" of $A$, explaining in some measure the mapping which $H^2(A, A)$ induces of $H^n(A, A)$ into $H^{n+1}(A, A)$ for every $n$. All the modules $H^n(A, A)$ should, in some way not yet entirely clear, be interpretable as modules of infinitesimal objects associated with $A$, with $\{H^*(A, A), [\ ,\ ]\}$ as the ring of infinitesimal operations associated with these objects.

We next investigate further the relationship between the product $\smile$ and the products $\circ$, $[\ ,\ ]$. To illustrate the method of the next computation in a simpler case, we give now another derivation of (24), which is a special case of Theorem 3, when $f$ and $g$ are cocycles. Let $f^m \in Z^m(A, P)$, $g^n \in Z^n(A, A)$ and $a_1, \cdots, a_{m+n} \in A$ be fixed. In what follows, in order to shorten certain formulas, we shall sometimes omit the argument from an expression of the form

$$f(a_\lambda \otimes a_{\lambda+1} \otimes \cdots \otimes a_{\lambda+m+1}) \quad \text{or} \quad g(a_\mu \otimes a_{\mu+1} \otimes \cdots \otimes a_{\mu+n-1}) \ .$$

When that is done, the argument should be understood; there will never be any difficulty in supplying the correct one as it depends on the first index only. We set now, for $1 \leqq i \leqq m$,

$$\begin{aligned} f_i = {} & a_1 f(a_2 \otimes \cdots \otimes a_i \otimes g \otimes a_{i+n+1} \otimes \cdots \otimes a_{m+n}) \\ & + \textstyle\sum_{\lambda=1}^{i-1} (-1)^\lambda f(a_1 \otimes a_2 \otimes \cdots \otimes a_{\lambda-1} \otimes a_\lambda a_{\lambda+1} \otimes a_{\lambda+2} \otimes \cdots \\ & \qquad \otimes a_i \otimes g \otimes a_{i+n+1} \otimes \cdots \otimes a_{m+n}) \\ & + (-1)^i f(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i g \otimes a_{i+n+1} \otimes \cdots \otimes a_{n+n}) , \end{aligned}$$

and

$$\begin{aligned} f_i' = {} & (-1)^{n+i-1} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g a_{i+n} \otimes a_{i+n+1} \otimes \cdots \otimes a_{m+n}) \\ & + \textstyle\sum_{\lambda=i+n}^{m+n-2} (-1)^\lambda f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g \otimes a_{i+n} \otimes \cdots \otimes a_\lambda a_{\lambda+1} \otimes \cdots \otimes a_{m+n}) \\ & + (-1)^{m+n-1} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1}) a_{m+n} . \end{aligned}$$

Since $g^n \in Z^n(A, A)$, we have

$$\begin{aligned} & (-1)^i a_i g(a_{i+1} \otimes \cdots \otimes a_{i+n}) + (-1)^{n+i-1} g(a_i \otimes \cdots \otimes a_{i+n-1}) \\ & \quad = \textstyle\sum_{\lambda=1}^{i+n-1} (-1)^\lambda g(a_i \otimes \cdots \otimes a_{\lambda-1} \otimes a_\lambda a_{\lambda+1} \otimes a_{\lambda+2} \otimes \cdots \otimes a_{i+n}) , \end{aligned}$$

whence

$$f_i + f_i' = \delta(f \circ_{i-1} g)(a_1 \otimes \cdots \otimes a_{m+n}) , \qquad i = 1, \cdots, m .$$

On the other hand, if $i < m$, then

$$f_i + (-1)^{n-1} f_{i+1}' = \delta f(a_1 \otimes \cdots \otimes a_i \otimes g \otimes a_{i+n+1} \otimes \cdots \otimes a_{m+n}) = 0 ,$$

while

$$(-1)^{n-1} g(a_1 \otimes \cdots \otimes a_n) f(a_{n+1} \otimes \cdots \otimes a_{m+n}) + f_1' = 0$$

and

$$f_m + (-1)^{m-1} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n}) = 0 .$$

It follows that

$$\begin{aligned} \delta(f \circ g) & = \textstyle\sum_{j=0}^{m-1} (-1)^{j(n-1)} \delta(f \circ_j g) = \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} (f_i + f_i') \\ & = f_1' + \big(f_1 + (-1)^{n-1} f_2'\big) + (-1)^{n-1}\big(f_2 + (-1)^{n-1} f_3'\big) + \cdots \\ & \qquad + (-1)^{(m-2)(n-1)}\big(f_{m-1} + (-1)^{n-1} f_m'\big) + (-1)^{(m-1)(n-1)} f_m \\ & = (-1)^n g f - (-1)^{(m-1)n} f g , \end{aligned}$$

i.e., considering once again $f$ and $g$ as functions rather than specific values,

$$\delta(f \circ g) = (-1)^n [g \smile f - (-1)^{mn} f \smile g] ,$$

which agrees with (24).

THEOREM 5. *Let $A$ be a ring, $P$ be a two-sided $A$ module, $h^p$ be in $Z^p(A, P)$, and $f^m$, $g^n$ be in $Z^m(A, A)$, $Z^n(A, A)$, respectively, and set*

$$H = \sum_{i=0}^{p-2} \sum_{j=m+1}^{m+p-2} (-1)^{(m-1)i+(n-1)j}(h^p \circ_i f^m) \circ_j g^n .$$

*Then $H \in C^{m+n+p-2}(A, P)$, and*

$$\delta H = (-1)^{(m-1)n}[h \circ (f \smile g) - (-1)^{n(p-1)}(h \circ f) \smile g - f \smile (h \circ g)] .$$

PROOF. Let $a_1, \cdots, a_{m+n+p-1} \in A$ be fixed. As before, we sometimes omit the argument from expressions of the form

$$f^m(a_\lambda \otimes a_{\lambda+1} \otimes \cdots \otimes a_{\lambda+m-1}) \quad \text{and} \quad g^n(a_\mu \otimes \cdots \otimes a_{\mu+m-1}) ,$$

and in addition, if $i \leqq j$, will write $a_{i,j}$ for $a_i \otimes a_{i+1} \otimes \cdots \otimes a_j$. Set, for $1 \leqq i \leqq p-1$, $m + i \leqq j \leqq m + p - 1$,

$$\begin{aligned} h_{ij} = {} & a_1 h(a_{2,i} \otimes f \otimes a_{i+m+1,j} \otimes g \otimes a_{j+n+1,m+n+p-1}) \\ & + \sum_{\lambda=1}^{i-1} (-1)^\lambda h(a_{1,\lambda-1} \otimes a_\lambda a_{\lambda+1} \otimes a_{\lambda+2,i} \otimes f \\ & \qquad \otimes a_{i+m+1,j} \otimes g \otimes a_{j+n+1,m+n+p-1}) \\ & + (-1)^i h(a_{1,i-1} \otimes a_i f \otimes a_{i+m+1,j} \otimes g \otimes a_{j+n+1,m+n+p-1}) , \\ h'_{ij} = {} & (-1)^{m+i-1} h(a_{1,i-1} \otimes f a_{i+m} \otimes a_{i+m+1,j} \otimes g \otimes a_{j+n+1,m+n+p-1}) \\ & + \sum_{\lambda=m+i}^{j-1} (-1)^\lambda h(a_{1,i-1} \otimes f \otimes a_{i+m,\lambda-1} \otimes a_\lambda a_{\lambda+1} \\ & \qquad \otimes a_{\lambda+2,j} \otimes g \otimes a_{j+n+1,m+n+p-1}) \\ & + (-1)^j h(a_{1,i-1} \otimes f \otimes a_{i+m,j-1} \otimes a_j g \otimes a_{j+n+1,m+n+p-1}) , \end{aligned}$$

and

$$\begin{aligned} h''_{ij} = {} & (-1)^{j+n-1} h(a_{1,i-1} \otimes f \otimes a_{i+m,j-1} \otimes g a_{j+n} \otimes a_{j+n+1,m+n+p-1}) \\ & + \sum_{\lambda=j+n}^{m+n-2} (-1)^\lambda h(a_{1,i-1} \otimes f \otimes a_{i+m,j-1} \otimes g \\ & \qquad \otimes a_{j+m,\lambda-1} \otimes a_\lambda a_{\lambda+1} \otimes a_{\lambda+2,m+n+p-1}) \\ & + (-1)^{m+n-1} h(a_{1,i-1} \otimes f \otimes a_{i+m,j-1} \otimes g \otimes a_{j+m,m+n+p-2}) a_{m+n+p-1} . \end{aligned}$$

Then

$$h_{ij} + h'_{ij} + h''_{ij} = \delta\big((h \circ_{i-1} f) \circ_{j-1} g\big) . \tag{25}$$

On the other hand, if $m + i + 1 \leqq j \leqq m + p - 2$, then

$$\begin{aligned} & h_{ij} + (-1)^{m-1} h'_{i+1,j} + (-1)^{(m-1)+(n+1)} h''_{i+1,j+1} \\ & \qquad = \delta h(a_{1,i} \otimes f \otimes a_{i+m+1,j} \otimes g \otimes a_{j+n+1,m+n+p-1}) = 0 . \end{aligned} \tag{26}$$

It is not difficult to see that one may extend the range of indices for which (26) is valid by setting (the right side in each equation being evaluated at $a_{1,m+n+p-1}$),

$$h_{0j} = f \smile (h \circ_{j-m} g) \qquad \text{for} \quad j = m, m+1, \cdots, m+p-1 \tag{27}$$

$$h'_{i,m+i-1} = (-1)^{m+i-1} h \circ_{i-1} (f \smile g) \qquad \text{for} \quad i = 1, 2, \cdots, p \tag{28}$$

$$h''_{i,m+p} = (-1)^{m+n+p-1} (h \circ_{i-1} f) \smile g \qquad \text{for} \quad i = 1, 2, \cdots, p\,. \tag{29}$$

With these definitions, if any of the three quantities $h_{ij}$, $h'_{i+1,j}$, $h''_{i+1,j+1}$ is defined, then so are the other two, and (26) holds. Therefore,

$$\sum\nolimits_{i,j} (-1)^{(m-1)(i-1)+(n-1)(j-1)} [h_{ij} + (-1)^{m-1} h'_{i+1,j} + (-1)^{(m-1)+(n-1)} h''_{i+1,j+1}] = 0\,, \tag{30}$$

the sum being taken over all pairs $(i, j)$ for which $h_{ij}$, $h'_{i+1,j}$, $h''_{j+1,j+1}$ are now defined, namely, those $(i, j)$ with $0 \leqq i \leqq p-1, m+i \leqq j \leqq m+p-1$. Therefore, from (25),

$$\delta H = \sum\nolimits_{i=1}^{p-1} \sum\nolimits_{j=m+i}^{m+p-1} (-1)^{(m-1)(i-1)+(n-1)(j-1)} (h_{ij} + h'_{ij} + h''_{ij})\,,$$

all terms of which are contained, with their proper signs, in the expression on the left in (30). Those terms on the left in (30) not appearing in $\delta H$ are, up to sign, those defined in (27), (28), and (29), whence (30) reads

$$\begin{aligned} \delta H &+ \sum\nolimits_{j=m}^{m+p-1} (-1)^{(m-1)(-1)+(n-1)(j-1)} h_{0j} \\ &+ \sum\nolimits_{i=1}^{p-1} (-1)^{(m-1)(i-1)+(n-1)(m+i-2)} h'_{i,m+i-1} \\ &+ \sum\nolimits_{i=1}^{p-1} (-1)^{(m-1)(i-1)+(n-1)(m+p-1)} h''_{i,m+p} = 0\,, \end{aligned}$$

whence, by (27), (28), (29),

$$\begin{aligned} \delta H &+ (-1)^{(m-1)n} f \smile (h \circ g) + (-1)^{(m-1)n+1} h \circ (f \smile g) \\ &+ (-1)^{(m-1)n+n(p-1)} (h \circ f) \smile g = 0\,, \end{aligned}$$

which proves the theorem.

It is much simpler to compute $(f \smile g) \circ h$ when $h^p \in C^p(A, A)$ and $f^m \in C^m(A, A)$, $g^n \in C^n(A, P)$; in fact, one may verify immediately from the definitions that

$$(f \smile g) \circ h = (f \circ h) \smile g + (-1)^{m(p-1)} f \smile (g \circ h)\,.$$

In the case when $P = A$, combining this with the preceding theorem and using the definitions of $[f, g]$ and $[g, h]$ yields

COROLLARY 1. *If $f^m$, $g^n$, $h^p \in Z^m(A, A)$, $Z^n(A, A)$, $Z^p(A, A)$, respectively, then*

$$\begin{aligned} [f^m \smile g^n, h^p] - [f^m, h^p] \smile g^n - (-1)^{m(p-1)} f^m \smile [g^n, h^p] \\ = (-1)^{mn-n+1} \delta H\,. \end{aligned}$$

Passing to the cohomology modules, we may assert

COROLLARY 2. *Let $A$ be an associative ring and $\xi^p$, $\eta^m$, $\zeta^n$ be elements* *of* $H^p(A, A)$, $H^m(A, A)$, $H^n(A, A)$, *respectively. Then*

$$[\eta^m \smile \zeta^n, \xi^p] = [\eta^m, \xi^p] \smile \zeta^n + (-1)^{m(p-1)} \eta^m \smile [\zeta^n, \xi^p] \ . \tag{31}$$

One sees from Corollary 2 that, if $\xi^p \in H^p(A, A)$, then the module endomorphism $D_\xi$ of $H^*(A, A)$ defined by setting for all $\eta \in H^*(A, A)$,

$$D_\xi \eta = [\eta, \xi]$$

is a left derivation of $H^*(A, A)$, of degree $p - 1$, and that the mapping $\xi \to D_\xi$ is an anti-homomorphism of $\{H^*(A, A), [\ ,\ ]\}$ into the ring of left derivations of $\{H^*(A, A), \smile\}$, where if $D$, $D'$ are derivations of degree $\lambda$, $\lambda'$, their product $[D, D']$ as usual is defined to be $DD' - (-1)^{\lambda\lambda'} D'D$. That the mapping $\xi \to D_\xi$ is an anti-homomorphism is purely fortuitous; we could have taken in $\{H^*(A, A), [\ ,\ ]\}$ the anti-isomorphic structure, in which case the mapping would have been a homomorphism.

## 9. Generalizations and comments

In order not to confuse the ideas, we have, in § 7 and § 8 not attempted to state the broadest possible results, but certain useful generalizations nevertheless deserve mention. We give them here without detailed proof.

Let $A$, as usual, be an associative ring and $P$ be a two-sided $A$ module. We have seen that $P$ is a two-sided module over both $\{H^*(A, A), \smile\}$ and $\{[H^*(A, A), [\ ,\ ]\}$. Suppose now that $\eta^m$, $\zeta^n$, $\xi^p$ are elements of dimensions $m$, $n$, $p$, respectively, two of which are in $H^*(A, A)$ and the remaining one of which is in $H^*(A, P)$. Then both sides of (31) are well-defined, but the equation in general does not hold. This stems from the fact that if $f^m \in C^m(A, A)$, $g^n \in C^n(A, P)$, then $g^n \circ f^m$ is well-defined, which, ultimately makes it possible to define the module structure of $H^*(A, P)$ over $\{H^*(A, A), [\ ,\ ]\}$, but $f^m \circ g^n$ is not defined. (See in this respect again the remarks following (10), § 3.) If we assume now that there exists an $A$ module homomorphism $\varphi$: $P \to A$ (whence $\varphi(P) = I$ is necessarily an ideal of $A$), then we may set $f^m \circ g^n = f^m \circ \varphi(g^n)$. It then becomes possible to define in $H^*(A, P)$ a bracket product in which it becomes a graded Lie ring (see again example 5 of § 5), and to give it a module structure over $\{H^*(A, A), [\ ,\ ]\}$ such that (31) will hold if one of $\eta^m$, $\zeta^n$, $\xi^p$ is in $H^*(A, P)$ and the remaining ones in $H^*(A, A)$.

Suppose now that $P$ is a two-sided $A$ module (but no $\varphi$: $P \to A$ is given) and that $P$ is an associative ring, right and left multiplication in $P$ being assumed to be $A$ module homomorphisms. We have shown by example that $\{H^*(A, P), \smile\}$ is generally not commutative. However, if there exists an $A$ module homomorphism $\varphi$: $P \to A$ such that

$$\varphi(x)y = xy = x\varphi(y)$$

for all $x, y$ in $P$, then we can again define $f^m \circ g^n$ for $f^m \in C^m(A, P)$, $g^n \in C^n(A, P)$, and it will follow both that $\{H^*(A, P), \smile\}$ is commutative and that (31) holds with one, two, or all of $\eta^m$, $\zeta^n$, $\xi^p$ in $H^*(A, P)$, and the remaining ones in $H^*(A, A)$.

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