Methods of Homotopy Theory in Algebraic Geometry

from the Viewpoint of Cohomology Operations

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ABSTRACT

This article is an exposition on the homotopy-theoretic tools of cohomology operations applied to algebraic geometry and inner workings. We first survey on the construction and mechanism of cohomology operations in classical homotopy theory. Particularly, we aim to explain how these operations detect elements and relations in homotopy classes along the Adams spectral sequences. Next, in order to explore analogous mechanism of cohomology operations in algebraic geometry, we introduce the framework of motivic homotopy theory as constructed by Morel and Voevodsky. Based on this framework, we study the constructions and properties of motivic power operations and related spectral sequences in motivic stable homotopy theory. Specifically, we would like to understand how motivic power operations exhibit the coherence encoded by norms in motivic homotopy theory and how motivic extended powers emerge in the motivic Adams spectral sequences.

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1 INTRODUCTION

1.1 Overview

Morel and Voevodsky constructed motivic homotopy theory in the 1990s in order to extend the framework of homotopy theory to algebraic geometry [MV99]. In such a framework, many techniques of homotopy theory can be implemented in the study of algebro-geometric objects. One of the most effective tools is power operations in motivic cohomology [Voe03b, Voe10] in analogy to Steenrod operations in singular cohomology, which plays a central role in Voevodsky's proofs of the Milnor conjecture on mod-2 norm residue maps [Voe03a] and of the Bloch-Kato conjecture on mod-ℓ norm residue maps [Voe11]. Based on the work of motivic power operations, Dugger and Isaksen invented motivic Adams spectral sequences [DI10] in the 2010s, which have facilitated computations in both algebraic geometry and algebraic topology up to this day [IO20, IWX20]. Nevertheless, the systematic use of motivic power operations and spectral sequences is not yet as mature as the corresponding applications in classical homotopy theory. In particular, much of the mechanism still waits to be explored of how motivic power operations detect structures of $\mathbb{A}^1$-homotopy classes . Here we intend to begin an investigation in this direction, by giving a self-contained, motivated account of necessary background and stating several specific questions.

1.2 What are methods of homotopy theory?

Classifying objects up to a specified equivalence relation is central to nearly all of geometry and topology. Many beautiful theorems are solutions to particular classification problems, such as the classification theorem of closed surfaces, or they are motivated by classifications, such as partial solutions to the generalized Poincaré conjecture. Some of the deepest results related to classification problems made essential use of the methods of algebraic topology, translating geometric questions to computations with algebra.

Algebraic topology carries out such translations from a geometric problem into an algebraic problem by taking invariants. The effective working of this type of methods depends on the following two aspects:

- the associated algebraic problem captures the essential features of the geometric problem;
- the associated algebraic problem is sufficiently simple to solve.

Actually, these two aspects are reciprocal to each other: the more geometric information an algebraic problem encodes, the more difficult it is to solve. In this case, homotopy theory plays a central role in reconciling these two aspects. The strategy of homotopy theory to resolve a classification problem is to convert a task of classifying objects (spaces, manifolds, etc.) into a task of classifying related (stable) homotopy classes. This method works effectively because in the first place, with proper set-up, homotopy classes are able to capture sufficient geometric features. Here are some examples.

Theorem 1.2.1 (Thom [Tho54]) Let G be a subgroup of $GL(F, k)$ for $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Suppose X is an n-dimensional manifold. Then there is a bijection between the set of cobordism classes of submanifolds of X with a G-structure (on their tangent bundle) and the set of homotopy classes of continuous maps from X to a Thom space MG:

$$\{G\text{-cobordism classes in } X\} \xleftrightarrow{\text{bijection}} [X, MG]$$

Theorem 1.2.2 (Pontryagin [Pon59]) There is a bijection between the set of cobordism classes of framed k-dimensional submanifolds of $\mathbb{R}^{n+k}$ and the set of homotopy classes of continuous maps between spheres:

$$\{k\text{-dimensional framed cobordism classes in } \mathbb{R}^{n+k}\} \xleftrightarrow{\text{bijection}} [S^{n+k}, S^n]$$

Theorem 1.2.3 (Steenrod [Ste51]) Given a topological group G, there is a bijection between the set of isomorphic classes of principal G-bundles over a paracompact space X and the set of homotopy classes of continuous maps from X to a classifying space BG:

$$\{\text{isomorphism classes of principal } G\text{-bundles on } X\} \xleftrightarrow{\text{bijection}} [X, BG]$$

These theorems demonstrate a general principle that classifying geometric objects of a specific type is equivalent to classifying homotopy classes of maps to a corresponding object. The homotopical structure of this "classifying object" largely determines the classification in question.

In addition to the capability of homotopy classes to encode geometric information, there are many tools to effectively address the associated algebraic problems with homotopy classes, making them easier to solve. The most significant ones are homological and cohomological machineries with operations, which we discuss in the next subsection.

1.3 The yoga of cohomology operations and spectral sequences

The method of detecting homotopy classes by cohomology operations originated from Steenrod's work [Ste47]. In this work, Steenrod constructed a device called cup-i products, for $i \geqslant 1$, as a higher-order analogue of cup product to give some results on the classification of homotopy classes of maps from an $(n+1)$-dimensional complex to the n-dimensional sphere. Specifically, Steenrod used cup-i products to derive a family of cohomology operations called Steenrod squares on mod-2 cohomology. These are the first examples of cohomology operations. From the 1950s to the 1960s, Steenrod developed the theory of such cohomology operations [Ste52, Ste53b, Ste53a, Ste57]. On the mod-p cohomology, these cohomology operations along with the Bockstein operations became known as Steenrod operations.

They were widely applied to solve various problems in topology and geometry. For example, Borel and Serre proved that S^{2n}, for $n \geqslant 4$, do not admit an almost complex structure [BS53]. Around the same time, Thom solved the Steenrod problems of determining when an integral or mod-2 homology class of a finite-dimensional polyhedron can be realized as a manifold [Tho54].

In the previous examples, the crux is to exploit the actions of Steenrod operations on cohomology rings. From this viewpoint, it is natural to use homological methods to analyze these actions. Specifically, mod-p stable cohomology operations form an algebra called the Steenrod algebra $\mathcal{A}_p$. In the 1950s, Adem discovered a set of relations in $\mathcal{A}_p$ [Ade52, Ade57]; Serre showed that Steenrod operations and their Adem relations fully determine the algebra $\mathcal{A}_p$ as generators and relations [Ser53]; Milnor showed the Hopf algebra structure of Steenrod algebras and their duals [Mil58]. In the same period, Adams invented his famous spectral sequences [Ada58] to show the existence of Hopf elements in $\pi_{2n-1}(S^n)$ and $\pi_{4n-1}(S^{2n})$ for $n \leqslant 4$. The significance of the Adams spectral sequences is to exhibit how Steenrod operations detect homotopy classes and illustrate the extent to which the information is detected. In particular, Greenlees explained how the Adams spectral sequences "cure the blindness" of a cohomology theory in his enlightening article [Gre88].

In the 1960s, the monograph by Steenrod and Epstein was published and it gives a comprehensive introduction to cohomology operations [Ste62]. Specifically, the authors presented a systematic method to construct power operations by using transfers and extended powers. It is tempting to think of and desirable in practice that such power operations be applicable to cohomology theories other than mod-p ordinary cohomology and this systematic construction work more generally.

This is indeed the case. In fact, power operations on other generalized cohomology theories led to even deeper results than their ordinary analogues did. For example, Adams and Atiyah constructed power operations in K-theory, called the Adams operations [Ati66]. Equipped with these, Adams solved the problem of vector fields on spheres completely [Ada62], obtaining stronger results than those in [SW51] with Steenrod squares. Adams and Atiyah also presented an elegant solution to the Hopf invariant one problem [AA66], which is conceptually much simpler than Adams's proof using secondary cohomology operations on ordinary cohomology [Ada60]. Besides K-theory, tom Dieck constructed power operations in cobordism theory [tD68]. Quillen then used these operations to show that the cobordism ring $MU^*(X)$ is generated by $\bigoplus_{i \geqslant 0} MU^i(X)$ as an $MU^*(pt)$-module, and deduced his theorem on formal group laws that the complex cobordism ring is isomorphic to the Lazard ring [Qui71]. As demonstrated above, the structure of cohomology operations is prevalent and carries a wealth of information through various cohomology theories. For deeper investigations, we need a framework to conceptualize cohomology theories in order to exploit this intrinsic structure of power operations. In fact, stable homotopy theory serves as a desired framework.

1.4 Cohomology theories and spectra

In [Bro62], E.H. Brown showed that for each generalized cohomology theory h^*, there exists a sequence of spaces $\{E_n\}$ with structure maps $\varepsilon_n : E_n \to \Omega E_{n+1}$ such that $h^n(Y) \cong [Y, E_n]$ and the suspension isomorphisms are induced by the adjoint maps of $\{\varepsilon_n\}$. These spaces with structure maps form a spectrum in the sense of [Whi60] and we say h^* is represented by the spectrum $E = \{E_n, \epsilon_n\}$. For example, ordinary cohomology theory with coefficient ring R is represented by the Eilenberg-MacLane spectrum HR, complex K-theory is represented by the K-theory spectrum KU, and G-cobordism theory is represented by the Thom spectrum MG for a classical group G [Swi75]. In general, the representability theorem indicates that the study of the generalized cohomology theories is equivalent to the study of spectra, which are central objects in stable homotopy theory. Notably, the manipulation of Steenrod operations can be simplified in the context of stable homotopy theory. For example, Rudyak showed how to simplify Thom's method using Steenrod operations [Tho54] by an approach with stable homotopy theory [Rud92]. More applications of stable homotopy theory are documented in May's review [May99].

Here, we focus on how power operations are present at the level of spectra. Given a cohomology theory E, its degree-n cohomology operations are natural transformations from E^* to E^{*+n}. By the Yoneda lemma and Brown's representability theorem, $E^*E = [E, E]_{-*}$ is the algebra of cohomology operations on E. If we take $E = H\mathbb{F}_p$, the mod-p Steenrod algebra $\mathcal{A}_p \cong H\mathbb{F}_p^* H\mathbb{F}_p$. Recall that $\mathcal{A}_p$ is generated by mod-p Steenrod operations subject to Adem relations [Ser53], and Steenrod operations are induced by extended powers [Ste62]. Therefore, in order to study power operations in stable homotopy theory, we need to define extended powers for spectra.

1.5 How power operations work

In the 1970s, May and his collaborators built a theory of multiplicative E_∞-structures in spaces and spectra through a series of works [May72, CLM76, May77]. Furthermore, May demonstrated that an E_∞-structure produces power operations [May70]. In particular, HR, KU, and Thom spectra are all E_∞-spectra [May77], which illuminates why ordinary cohomology, complex K-theory, and cobordism theory each possess power operations. Conversely, the existence of power operations does not imply the existence of an E_∞-structure, which means that E_∞-structures may be too stringent for utilizing power operations. A more suitable structure to supply power operations is an H_∞-structure, a weaker notion than E_∞, which was introduced by May in the 1980s [BMMS86]. There, May used equivariant half-smash products to define extended powers of ring spectra and then defined the notion of H_∞-structure in terms of maps related to extended powers. Bruner showed that every H_∞-ring spectrum admits an associated generalized Adams spectral sequence and explained how an H_∞-structure converts cohomology operations into homotopy operations, which is the essence of Adams-type spectral sequences. McClure analyzed the connection between H_∞-structures and power operations and showed that the power operations in mod-p ordinary cohomology, complex K-theory, and cobordism theory coincide with the respective operations derived from H_∞-structures.

1.6 Homotopy theory of smooth schemes

Besides within topology, algebraic geometry is a field which makes extensive use of cohomological methods. We would naturally expect that the model of homotopy theory and cohomology operations can be modified in a suitable way so that they function well in algebraic geometry. To achieve this, we need to address the following two questions:

- How can we carry out homotopy theory in a general setting beyond topology?
- Cohomology theories in algebraic geometry are defined by sheaves, while cohomology theories in algebraic topology are defined by spectra. How can we generalize the homotopical framework to incorporate these two types of cohomology?

For the first question, Quillen built a framework called homotopical algebra, which distills the essential features for working with homotopy theoretic tools in terms of axiomatic properties possessed by a model category [Qui67]. For the second question, K.S. Brown used sheaves of spectra (or simplicial sets) to generalize sheaf cohomology and related spectral sequences [Bro73]. These set the stage for performing homotopy theory in algebraic geometry that is compatible with cohomology theories.

In the 1990s, Morel and Voevodsky constructed $\mathbb{A}^1$-homotopy theory of schemes, also called motivic homotopy theory [MV99]. Under this framework, Voevodsky constructed motivic power operations [Voe03b] and these operations led to an elegant solution of the Milnor conjecture [Voe03a] and the Bloch-Kato conjecture [Voe11] (the earlier proofs of these conjectures had been given by Voevodsky in the 1990s, but they were lengthy as the framework of motivic homotopy theory had not been well developed at that time). Apart from the settlement of these famous conjectures, the following are more evidences showing why Morel and Voevodsky's approach is a reasonable and fruitful one.

As an analogue of Theorem 1.2.3, Morel [Mor12], Asok, Hoyois, and Wendt [AHW17] proved that given a smooth affine scheme X over a Noetherian commutative ring of a particular class, isomorphic classes of rank-r algebraic vector bundles over X are in bijection with $\mathbb{A}^1$-homotopy classes of maps from X to the infinite Grassmannian of r-planes.

As an analogue of Brown's representability theorem from Section 1.4, Voevodsky also constructed motivic stable homotopy theory to represent cohomology theories in algebraic geometry [Voe98]. For example, motivic cohomology theory (analogous to singular cohomology theory) is represented by the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}_{\mathrm{mot}}$, algebraic K-theory (analogous to complex K-theory) is represented by KGL, and algebraic cobordism theory (analogous to complex cobordism) is represented by MGL.

Moreover, motivic stable homotopy theory is deeply related to classical stable homotopy theory. To be more concrete, let k be a field with an embedding $k \hookrightarrow \mathbb{C}$. There is a realization functor

$$t_{\mathbb{C}} \colon \mathcal{SH}(k) \to \mathcal{SH}$$

where $\mathcal{SH}(k)$ is the motivic stable homotopy category over k and $\mathcal{SH}$ is the classical stable homotopy category. The striking coincidences are

$$H\mathbb{Z}_{mot} \xmapsto{t_{\mathbb{C}}} H\mathbb{Z} \qquad KGL \xmapsto{t_{\mathbb{C}}} KU \qquad MGL \xmapsto{t_{\mathbb{C}}} MU$$

The properties of the realization functor $t_{\mathbb{C}}$ indicate that motivic (stable) homotopy theory is an adequate version of (stable) homotopy theory in algebraic geometry. It is worth noting that $t_{\mathbb{C}}$ plays a central role in Voevodsky's earlier unpublished proof of the Milnor conjecture [Voe96]. Voevodsky proved a purely topological result on MU and $H\mathbb{Z}/\ell$. He then applied the result and the realization functor $t_{\mathbb{C}}$ to prove a motivic version of the result on motivic cohomology and algebraic cobordism, which is essential to the proof of the Milnor conjecture. Specifically, the proof of the topological results relies on the use of the Steenrod algebra, while the proof of the motivic result relies on a motivic Steenrod algebra and the realization functor $t_{\mathbb{C}}$ which preserves the structure of these algebras. This technique demonstrates the significance and efficacy of the methods of homotopy theory in algebraic geometry via cohomology operations.

This profound connection between classical homotopy theory and motivic homotopy theory has not only advanced the research in algebraic geometry and number theory, but also facilitated the study of classical stable homotopy theory. Isaksen and Dugger constructed motivic Adams spectral sequences in the 2010s and used them to improve the computations of the classical stable stems [DI10]. More recently, the discovery of a deep relationship between the motivic Adams spectral sequence and the algebraic Novikov spectral sequence has led to great extensions of the computations to higher dimensions [IWX20, GWX21, BKWX22]. Central to this relationship is an element τ in the mod-p cohomology of a point, which serves as a parameter for a deformation between motivic and classical stable homotopy categories. This element featured in Voevodsky's computations with the motivic algebra and its dual.

1.7 Recent work on motivic power operations

In [BH21], Bachmann and Hoyois constructed norm structures in motivic homotopy theory. With this set-up, they defined a notion of normed motivic spectrum, as an analogue of a structured ring spectrum in classical homotopy theory. As examples, they proved motivic cohomology, algebraic K-theory, and algebraic cobordism can be represented by normed motivic spectra.

The significance of their work is the encapsulation of the coherence data in a multiplicative structure which was elusive previously. Note that encoding coher-

Cohomology theories	Classical E_∞-ring spectra	Normed motivic spectra
singular cohomology	$H\mathbb{Z}$	$\mathbf{H}\mathbb{Z}_{\mathrm{mot}}$
K-theory	KU	KGL
cobordism theory	MU	MGL

ence data is also the *raison d'être* of E_∞-structures and H_∞-structures in classical homotopy theory. Therefore, in view of the following diagram, the norm structures should give rise to motivic power operations (see [BH21, Example 7.25]).

$$\begin{tikzcd}
\begin{array}{c} E_\infty\text{-structure} \\ \scriptstyle E_\infty\text{-operad action} \end{array} \arrow[d, "\text{pass to homotopy}"] & \text{multiplicative coherence data} \arrow[l, "\text{encoded in}"'] \arrow[r, "\text{encoded in}"] & \text{motivic norm structure} \arrow[dd, dashed, "\text{Bachmann-Hoyois}"] \\
\begin{array}{c} H_\infty\text{-structure} \\ \scriptstyle E_\infty\text{-operad homotopy action} \end{array} \arrow[d, "\text{cohomological presentation}"] & \boxed{\text{intrinsic symmetry}} \arrow[u, "\text{encoded in}"'] & \\
\text{power operations} & & \text{motivic power operations}
\end{tikzcd}$$

Similar to the case of E_∞-structures, norm structures may be too strict for the implementation of power operations. We expect that there is a motivic analogue of H_∞-structures and a preliminary for such a notion is the construction of motivic extended powers. In [BEH21], Bachmann, Elmanto, and Heller defined a notion of motivic colimit and used it to construct motivic extended powers as a generalization of the formalism of motivic Thom spectra. This is the first in a series of papers they plan to write on power operations in motivic stable homotopy theory.

2 POWER OPERATIONS IN CLASSICAL HOMOTOPY THEORY AND MOTIVIC HOMOTOPY THEORY

2.1 Generalized cohomology theories in classical homotopy theory

2.1.1 *Cohomology theories and Brown's representability*

Let **Ab** be the category of abelian groups, $\mathcal{CW}$ be the category of CW-complexes, and $\mathcal{CW}^2$ be the category of CW-pairs. Let $\rho\colon \mathcal{CW}^2 \to \mathcal{CW}^2$ be a functor defined by

$$\begin{array}{rccc} \rho: & \mathcal{CW}^2 & \longrightarrow & \mathcal{CW}^2 \\ & (X, A) & \longmapsto & (A, \emptyset) \end{array}$$

Definition 2.1.1 (Cohomology theory) An *unreduced cohomology theory* on $\mathcal{CW}^2$ is a sequence of pairs $\{h^n, \delta^n\}_{n\in\mathbb{Z}}$, where $h^n : \mathcal{CW}^2 \to \mathbf{Ab}$ is a contravariant functor and $\delta^n\colon h^n \circ \rho \to h^{n+1}$ is a natural transformation, such that the following axioms are satisfied:

- **Homotopy axiom:** The functors h^n factor through the homotopy category $h\mathcal{CW}^2$;
- **Exactness axiom:** For every pair $(X, A) \in \mathcal{CW}^2$, the sequences

$$\cdots \longrightarrow h^{n-1}(A, \emptyset) \xrightarrow{\delta^{n-1}} h^n(X, A) \longrightarrow h^n(X, \emptyset) \longrightarrow h^n(A, \emptyset) \longrightarrow \cdots$$

- **Excision axiom:** For every pair $(X, A) \in \mathcal{CW}^2$ and an open subset $U \subset A$ with $\bar{U} \subset \bar{A}$, the inclusion $j : (X - U, A - U) \to (X, A)$ induces an isomorphism $h_n(j) \colon h_n(X - U, A - U) \to h_n(X, A)$ for all $n \in \mathbb{Z}$.

Let $\mathcal{CW}_*$ be the category of pointed CW-complexes.

Definition 2.1.2 A *reduced cohomology theory* on $\mathcal{CW}_*$ is a sequence of contravariant functors $\tilde{h}^n \colon \mathcal{CW} \to \mathbf{Ab}$ together with natural isomorphisms

$$\sigma_m \colon \tilde{h}^n \to \tilde{h}^{n+1} \circ \Sigma$$

where Σ is the suspension functor, satisfying the following axioms

- **Homotopy axiom:** The functors $\tilde{h}^n$ factor through the homotopy category $h\mathcal{CW}_*$;
- **Exactness axiom:** For subcomplex $A \subset X \in \mathcal{CW}_*$, the following sequence is exact:
$$\tilde{h}^n(X/A) \to \tilde{h}^n(X) \to \tilde{h}^n(A).$$

Remark 2.1.3 Unreduced cohomology theories and reduced cohomology theories are related. There is a functor

$$\begin{array}{ccc} \mathcal{CW}_* & \longrightarrow & \mathcal{CW}^2 \\ (X, x_0) & \longmapsto & (X, x_0) \end{array}$$

where we just take the base point x_0 as a subcomplex. With this functor, an unreduced cohomology theory can be viewed as a reduced cohomology theory.

Definition 2.1.4 Let $F \colon \mathcal{CW}_* \to \mathrm{Set}$ be a homotopy functor that factors through $h\mathcal{CW}$. Then F is said to be a *Brown functor* if it satisfies the following two axioms

- **Wedge axiom:** If $X = \wedge_\alpha X_\alpha$ and $i_\alpha \colon X_\alpha \hookrightarrow X$ is the inclusion, then
$$\prod F(i_\alpha) \colon F(X) \to \prod_\alpha F(X_\alpha)$$
is a bijection.
- **Mayer-Vietoris axiom:** Given a subcomplexes A of $X \in \mathcal{CW}_*$ with inclusion $i_A \colon A \hookrightarrow X$, for any $x \in F(X)$, we denote $x|_A := F(i_A)(x) \in A$.

  Suppose A, B are two subcomplexes of $X \in \mathcal{CW}_*$ and $(a, b) \in F(A) \times F(B)$ such that $a|_{A\cap B} = b|_{A\cap B}$, then there exists $c \in F(A \cup B)$ such that $c|_A = a$ and $c|_B = b$.

Theorem 2.1.5 (Brown's representability theorem [Bro62]) Any Brown functor $H\colon \mathcal{CW}_* \to \mathrm{Set}$ is representable (in the homotopy category).

Proposition 2.1.6 Suppose $\tilde{h}^*$ is a reduced cohomology theory, then for each $n \in \mathbb{Z}$, $\tilde{h}^n$ is a Brown functor.

More specifically, there exists a CW-complex $E_n \in \mathcal{CW}_*$ and $\iota_n \in \tilde{h}^n(E_n)$ such that for any $X \in \mathcal{CW}$,

$$\begin{aligned} [X, E_n] &\longrightarrow \tilde{h}^n(X) \\ [f] &\longmapsto f^*\iota_n \end{aligned}$$

is a bijection for each n. Notably, the suspension isomorphism induced

$$[X, \Sigma E_n]_* \cong [\Sigma X, E_{n+1}]_* \cong [X, \Omega E_{n+1}]_*$$

for each $X \in \mathcal{CW}_*$. Then there exists a homotopy equivalence

$$f_n\colon E_n \to \Omega E_{n+1}$$

In summary, the data of a equivalent to a sequence of CW-complexes together with some structures. The objects to characterize these features are called *spectra*.

2.1.2 Spectra and the stable homotopy category

Definition 2.1.7 (Classical definition of spectrum) A pair $E = \{E_n, \varepsilon_n\}_{n\in\mathbb{Z}}$ of a sequence of pointed topological spaces indexed by integers $\{E_n\}_{n\in\mathbb{Z}}$ and basepoint-preserving maps

$$\varepsilon_n\colon \Sigma E_n \to E_{n+1}$$

is called a *spectrum*.

If the adjoint of each ε_n

$$E_n \to \Omega E_{n+1}$$

is a weak homotopy equivalence, it is called an *Ω-spectrum*.

A morphism between classical spectra $f\colon E \to F$ consists of pointed continuous maps $f_n\colon E_n \to F_n$ that are compatible with structure maps. The category of spectra is called stable category.

Given a pointed space X, we define $E \wedge X_n := E_n \wedge X$ and the suspension spectrum $\Sigma^\infty X_n := \Sigma^n X$. The notion of homotopy in the stable category is defined on $E \wedge I_+$ as an analogy to the notion of homotopy for spaces. By quotient the homotopy relation, we then have the naive stable homotopy category.

Given two spectra X, Y, the mapping space between X and Y in the naive stable homotopy category is defined to be

$$[X, Y]^* := \bigoplus_{n\in\mathbb{Z}} [\Sigma^n X, Y]$$

where $[\Sigma^j X, Y]$ is the set of morphisms from $\Sigma^j X$ to Y modulo the homotopy and it is an abelian group.

Remark 2.1.8 Given an Ω-spectrum E, then we can defined an associated reduced cohomology $\tilde{E}^*$ to be

$$\tilde{E}^*(X) := [\Sigma^n X, E]^*$$

where X is a pointed space.

There are some issues for this classical definition: the definition of smash-product in stable category is very complicated, see [Ada74]. To deal with the issue of smash product, there are three different approaches: orthogonal spectra [MMSS01, MM02], symmetric spectra [HSS00] and EKMM spectra [EKMM97]. A survey [Gre07] written by Greenlees introduces these three approaches. Their advantages and disadvantages are dicussed by Dugger in [BGH22]. In this proposal, we used the approach of Elmendorf, Kriz, Mandell, and May.

Definition 2.1.9 Let $\mathcal{U}$ be an infinite-dimensional vector space with inner product over $\mathbb{R}$ with countable basis; let $\mathcal{F}(\mathcal{U})$ be the category of finite dimensional subspaces of $\mathcal{U}$ and inclusion maps.

A *prespectrum* (in the sense of EKMM) is a rule to associate a pointed space EV to each object V in $\mathcal{F}(\mathcal{U})$ and a continuous map

$$\sigma_{V,W} \colon \Sigma^{W-V} EV \to EW$$

to a morphism $V \to W$ in $\mathcal{F}(\mathcal{U})$, where $W - V$ is the orthonormal subspace of V in W and S^V is the one-point compactification of V and $\Sigma^V X = X \wedge S^V$ for a pointed space X.

A prespectrum (in the sense of EKMM) is a spectrum if the adjoint of each structure map is a homeomormphism for all V,W.

A map of prespectra $f \colon E \to F$ is a collection $\{f_V \colon EV \to F_V\}$ compatible with the structure map.

Let $\mathcal{P}(\mathcal{U})$ (resp. $\mathcal{S}(\mathcal{U})$) denote the category of prespecta (resp. spectra) indexed by $\mathcal{U}$.

A common choice of $\mathcal{U}$ is $\mathbb{R}^\infty = 0 \oplus \mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots \subset \mathbb{R}^n \subset \cdots$

Proposition 2.1.10 The forgetful functor $F \colon \mathcal{S}(\mathcal{U}) \to \mathcal{P}(U)$ has a left adjoint

$$L \colon \mathcal{P}(U) \to \mathcal{S}(\mathcal{U})$$

Given a pointed space X, $\Sigma^\infty X \in \mathcal{P}(U)$ is defined to be

$$(\Sigma^\infty X)(V) = \Sigma^V X$$

and $\Sigma^\infty X \in \mathcal{S}(\mathcal{U})$ is defined to be $L(\Sigma^\infty X)$. Let $\mathbb{S} := \Sigma^\infty S^0$ be the sphere spectrum.

Given a pointed space X and $E \in \mathcal{P}(U)$, we defien $E \wedge X(V) := EV \wedge X$; in $\mathcal{S}(\mathcal{U})$, we denote $L(E \wedge X)$ by $E \wedge X$. (This is an abuse of notation)

Define $\mathrm{Map}(X, E) \in \mathcal{P}(U)$ by

$$\mathrm{Map}(X, E)(V) := \mathrm{Map}_*(X, E(V))$$

The homotopy in $\mathcal{P}(U)$ and $\mathcal{S}(\mathcal{U})$ is exhibited by $E \wedge I_+$.

The cone $CE := E \wedge I$; the path spectrum $PE = \mathrm{Map}(I, E)$

Proposition 2.1.11 The category $S(U)$ is complete and cocomplete; i.e. it is closed under arbitrary small limit and colimit.

Definition 2.1.12 A map of spectra $f\colon E \to F$ is called a weak equivalence if each f_V is a weak homotopy equivalence of topological spaces.

The homotopy category (quotient relation in the morphism spaces) of spectra is denoted by $h\mathcal{S}(\mathcal{U})$; the category defined from $hS(U)$ by formally inverting morphisms represented by weak equivalences is denoted by $\bar{h}\mathcal{S}(\mathcal{U})$ and it is called the *stable homotopy category* of spectra.

Definition 2.1.13 For $E \in \mathcal{S}(\mathcal{U})$ and $E' \in \mathcal{S}(U')$,,define $E \wedge E' \in \mathcal{S}(U \oplus U')$ by

$$E \wedge E'(V \oplus V') = EV \wedge EV'$$

Definition 2.1.14 Let $f\colon U \to U'$ be a linear isometry, then given $E \in \mathcal{S}(\mathcal{U})$, we define

$$f_* E(V) := E(f^{-1}V)$$

for $V \in \mathcal{F}(U')$.

Definition 2.1.15 Let $\mathcal{I}(U, U')$ be the space of linear isometry from U to U' with compact-open topology.

Fix a universe U, we define

$$\mathcal{L}(i) := \mathcal{I}(U^{\oplus i}, U)$$

Proposition 2.1.16 The collection $\mathcal{L} = \{\mathcal{L}(i)\}_{i \in \mathbb{N}}$ has the following properties:

1. Each $\mathcal{L}(i)$ is contractible;

2. The maps
$$\gamma\colon \mathcal{L}(k) \times \mathcal{L}(j_1) \times \cdots \times \mathcal{L}(j_k) \to \mathcal{L}(j_1 + \cdots + j_k)$$
given by $(g; f_1, \ldots, f_k) \mapsto g \circ (f_1 \oplus \cdots \oplus f_k)$ make them form an E_∞-operad.

Proposition 2.1.17 (Hopkins) For $i \geqslant 1$ and $j \geqslant 1$, the diagram

$$\begin{tikzcd}
\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times \mathcal{L}(i) \times \mathcal{L}(j) \arrow[d, shift right, "\gamma\times\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}\times\gamma^2"] \\
\mathcal{L}(2) \times \mathcal{L}(i) \times \mathcal{L}(j) \arrow[d, "\gamma"] \\
\mathcal{L}(i+j)
\end{tikzcd}$$

is a split coequalizer of spaces. Therefore,

$$\mathcal{L}(i+j) \cong \mathcal{L}(2) \times_{\mathcal{L}(1)^2} \mathcal{L}(i) \times \mathcal{L}(j)$$

Note that the contruction of such twisted semi-product is very complicated.

Proposition 2.1.18 Given a topological space A, for any continuous map $\alpha\colon A \to \mathcal{I}(U, U')$, there exists a functor:

$$A \ltimes (-)\colon \mathcal{S}(\mathcal{U}) \to \mathcal{S}(U')$$

such that

1. If α is an identity map, then so is $A \ltimes -$;
2. For continuous maps:

   $$\alpha\colon A \to \mathcal{I}(U, U')\beta\colon B \to \mathcal{I}(U', U'')$$

   define a composition by

   $$(A \times B) \to \mathcal{I}(U, U') \times \mathcal{I}(U', U'') \to \mathcal{I}(U, U'')$$

   Then

   $$(A \times B) \ltimes (E_1 \wedge E_2) \cong (A \ltimes E_1) \wedge (B \ltimes E_2).$$

3. $A \ltimes (E \wedge X) \cong (A \ltimes E) \wedge X.$
4. $A \ltimes \Sigma^\infty X \cong \Sigma^\infty(A_+ \wedge X)$ for any pointed space X.

Proof. See [EKMM97, Appendix A]. □

Therefore, we have an $\mathcal{L}(1)$ action on $E \in \mathcal{S}(\mathcal{U})$:

$$\mathcal{L}(1) \ltimes E \to E$$

Definition 2.1.19 For a spectrum E, we define

$$\mathbb{L}E := \mathcal{L}(1) \ltimes E.$$

An $\mathbb{L}$-*spectrum* is a spectrum E together with

$$\eta\colon E \to \mathbb{L}E, \xi\colon \mathbb{L}E \to E$$

such that the following diagrams commute:

$$\begin{tikzcd} \mathbb{L}\mathbb{L}E \arrow[r, "\mu"] \arrow[d, "\mathbb{L}\xi"] & \mathbb{L}E \arrow[d, "\xi"] \\ \mathbb{L}E \arrow[r, "\xi"] & E \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[dr, equal] & \mathbb{L}M \arrow[d, "\xi"] \\ & E \end{tikzcd}$$

The category of $\mathbb{L}$-spectra index by the university $\mathcal{U}$ is denoted by $\mathcal{S}(\mathcal{U})[\mathbb{L}]$.

Definition 2.1.20 For $E, E' \in \mathcal{S}(\mathcal{U})$, define $E \wedge_{\mathcal{L}} E'$ to be the following coequalizer diagram:

$$\begin{array}{ccc} (\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (E \wedge E') & \xrightarrow{\gamma \ltimes \mathrm{id}} & \mathcal{L}(2) \ltimes (E \wedge E') \\ \downarrow{\scriptstyle \mathrm{id} \ltimes (\xi \times \xi)} & & \downarrow \\ \mathcal{L}(2) \ltimes (E \wedge E') & \longrightarrow & E \wedge_{\mathcal{L}} E' \end{array}$$

Proposition 2.1.21 For any j-tuple $M_1, \ldots, M_j$ of $\mathbb{L}$-spectra, there is a canonical isomorphism

$$M_1 \wedge_{\mathcal{L}} M_2 \wedge_{\mathcal{L}} \cdots \wedge_{\mathcal{L}} M_j \cong \mathcal{L}(i) \wedge_{\mathcal{L}(1)^j} (M_1 \wedge \cdots \wedge M_j).$$

Proposition 2.1.22 The functor

$$\mathbb{L} \colon \mathcal{S}(\mathcal{U}) \to \mathcal{S}(\mathcal{U})[\mathbb{L}]$$

is the left adjoint to the forgetful functor

$$\mathcal{S}(\mathcal{U})[\mathbb{L}] \to \mathcal{S}(\mathcal{U})$$

Proposition 2.1.23 Given any linear isometry $f \colon U \oplus U \to U$, we have

$$\mathbb{L}E \wedge_{\mathcal{L}} \mathbb{L}E' \cong \mathcal{L}(2) \ltimes (E \wedge E') \cong \mathbb{L}f_*(E \wedge E')$$

Proposition 2.1.24 In the category of $\mathbb{L}$-spectra, we have the following natural isomorphisms:

$$\tau \colon E \wedge_{\mathcal{L}} E' \cong E' \wedge_{\mathcal{L}} E, (E_1 \wedge_{\mathcal{L}} E_2) \wedge_{\mathcal{L}} E_3 \cong E_1 \wedge_{\mathcal{L}} (E_2 \wedge_{\mathcal{L}} E_3)$$

There is an issue: Since the sphere spectrum $\mathbb{S}$ is an $\mathbb{L}$-spectrum, it is expected to be a unit for the smash product, however, $\mathbb{S} \wedge_{\mathcal{L}} E$ may not be isomorphic to E. That is why we need the concept of $\mathbb{S}$-module.

Proposition 2.1.25 For an $\mathbb{L}$-spectrum E, we have a natural map of $\mathbb{L}$-spectra

$$\lambda_E \colon \mathbb{S} \wedge_{\mathcal{L}} E \to E$$

satisfying the following conditions:

1. The following diagram commutes:

$$\begin{tikzcd} E \wedge_{\mathcal{L}} \mathbb{S} \wedge_{\mathcal{L}} E' \arrow[r, "\tau\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\tau"'] \arrow[dr, "\mathrm{id}\wedge\lambda_{E'}"] & \mathbb{S} \wedge_{\mathcal{L}} E \wedge_{\mathcal{L}} E' \arrow[d, "\lambda_E\wedge\mathrm{id}"] \\ \mathbb{S} \wedge_{\mathcal{L}} E \wedge_{\mathcal{L}} E' \arrow[r, "\lambda_E\wedge\mathrm{id}"] & E \wedge_{\mathcal{L}} E' \end{tikzcd}$$

2. λ is a weak equivalence of $\mathbb{L}$-spectra.

3. $\lambda \colon \mathbb{S} \wedge_{\mathcal{L}} \mathbb{S} \to \mathbb{S}$ is an isomorphism of $\mathbb{L}$-spectra.

Definition 2.1.26 An A_∞-*ring spectrum* is a monoid in the category of $\mathbb{L}$-spectra and an E_∞-ring spectrum is a commutative monoid in the category of $\mathbb{L}$-spectra.

Remark 2.1.27 Suppose M is an $\mathbb{L}$-spectrum and let M^j be the j-fold power with respect to $\wedge_{\mathcal{L}}$, then we define the monad of tensor algebra on $\mathcal{S}(\mathcal{U})$:

$$\mathbb{T}M := \bigvee_{j\geqslant 0} M^j$$

and the monad of polynomial algebra on $\mathcal{S}(\mathcal{U})$:

$$\mathbb{P}M := \bigvee_{j\geqslant 0} M^j/\Sigma_j$$

where Σ_j act on M^j by permuting factors. The structure of an A_∞ ring spectrum M is exhibited by $\mathbb{T}M \to M$ that with suitable commutative properties; The structure of an E_∞-ring spectrum M is exhibited by $\mathbb{P}M \to M$ that with suitable commutative properties.

For a (pre)spectra X, let X^j be its j-fold external smash product, then we may define

$$\mathbb{B}X := \bigvee_{j\geqslant 0} \mathcal{L}(j) \ltimes X^j$$

and

$$\mathbb{C}X := \bigvee_{j\geqslant 0} (\mathcal{L}(j) \ltimes X^j)/\Sigma_j$$

With this setting, we can defined A_∞ ring (pre)spectra and E_∞-ring (pre)spectra for general (pre)spectra. Note that $\mathbb{B} \cong \mathbb{T}\mathbb{L}$ and $\mathbb{C} \cong \mathbb{P}\mathbb{L}$, according to Proposition 2.1.21. In some context, $(\mathcal{L}(j) \ltimes X^j)/\Sigma_j$ is denoted by $(X^j)_{h\Sigma_j}$.

Definition 2.1.28 An $\mathbb{L}$-spectrum is called an $\mathbb{S}$-*module* if $\lambda : \mathbb{S} \wedge_{\mathcal{L}} E \to E$ is an isomorphism of $\mathbb{L}$-spectra.

For $\mathbb{S}$-modules M, N, we denote

$$M \wedge_{\mathbb{S}} N := M \wedge_{\mathcal{L}} N$$

The category of $\mathbb{S}$-modules indexed by the university $\mathcal{U}$ is denoted by $\mathcal{M}_{\mathbb{S}}(\mathcal{U})$.

Proposition 2.1.29 The category of $\mathbb{S}$-modules is complete and cocomplete.

Definition 2.1.30 An $\mathbb{S}$-*algebra* is an algebra in the symmetric monoidal category $\mathcal{M}_{\mathbb{S}}(U)$.

Example 2.1.31 The Thom spectrum $MO_{\mathcal{U}}$ indexed by the university $\mathcal{U}$ is defined to be

$$MO_{\mathcal{U}}(U) := Gr_{|U|}(U \oplus \mathcal{U})^{\gamma_{|U|}}$$

where $|U| = N$ is the dimension of U, $Gr_{|U|}(U \oplus \mathcal{U})$ is the Grassmannian space of N-planes in $U \oplus \mathcal{U}$ and $Gr_{|U|}(U \oplus \mathcal{U})^{\gamma_{|U|}}$ is the Thom space of the universal bundle over the Grassmannian. Note that $MO_{\mathcal{U}}$ is a commutative $\mathbb{S}$-algebra, namely an E_∞-ring spectra. Then we have the natural map:

$$MO(U)_{\mathcal{U}} \wedge MO(V)_{\mathcal{U}} \longrightarrow MO(U \oplus V)_{\mathcal{U}\oplus\mathcal{U}}$$

Given an element in $\mathcal{L}(2)$, there is a map $MO(U \oplus V)_{\mathcal{U} \oplus \mathcal{U}} \to MO(U \oplus V)_{\mathcal{U}}$. Then combine these maps, we have

$$\mathcal{L}(2) \wedge MO(U)_{\mathcal{U}} \wedge MO(V)_{\mathcal{U}} \longrightarrow MO(U \oplus V)_{\mathcal{U}}$$

2.1.3 *Other models for stable homotopy theory*

In addition to the EKMM formalism of stable homotopy category, we still have another choices of models for stable homotopy theory. For example,

- A *symmetric spectrum* consists of a sequences of spaces X_n for $n \geqslant 0$ and structure maps $\sigma_n \colon S^1 \wedge X_n \to X_{n+1}$ such that X_n is equipped with an action of Σ_n and the iterated structure maps $\sigma^p : (S^1)^{\wedge p} \wedge X_q \to X_{p+q}$ is $\Sigma_p \times \Sigma_q$-equivariant. This notion is introduced by Hovey, Shipley and Smith [HSS00].
- An *orthogonal spectrum* is to assign each finite-dimensional real inner product space V a pointed $O(V)$-space X_V, together with structure maps $\sigma_{V,W} \colon S^V \wedge X_W \to X_{V \oplus W}$. This notion is introduced by Mandell, May, Schwede and Shipley [MMSS01, MM02].

$\mathbb{S}$-modules, symmetric spectra and orthogonal spectra are three popular models for stable homotopy theory. Actually, they are equivalent in the sense of Quillen. A Quillen equivalence between $\mathbb{S}$-modules and orthogonal spectra can be found in [MM02], and a Quillen equivalence between $\mathbb{S}$-modules and symmetric can be found in [Sch01]. The following table demonstrates their advantages and disadvantages respectively.

	Advantages	Disadvantages
$\mathbb{S}$-modules	All the objects are fibrant. Weak equivalences are easy to understand.	The unit is not cofibrant. The definition is complicated.
Symmetric spectra	The unit is cofibrant. The objects are easy to define.	Fibrant replacement is required. Weak equivalences are hard to describe.
Orthogonal spectra	The unit is cofibrant. Weak equivalences are easy to understand.	Fibrant replacement is required.

2.2 Power operations in topology

2.2.1 *Steenrod operations in ordinary cohomology*

Definition 2.2.1 (Cohomology operations) Let n, m be two integers and let π, G be two abelian groups, a *cohomology operation* of type $(n, \pi; m, G)$ is a collection of functions $\varphi_X : H^n(X; \pi) \to H^m(X; G)$ for each CW-complex X such that for any continuous map $f \colon X \to Y$, the following diagram commutes

$$\begin{array}{ccc} H^n(X;\pi) & \xrightarrow{\varphi_X} & H^q(X;G) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ H^n(Y;\pi) & \xrightarrow[\varphi_Y]{} & H^m(Y;G) \end{array}$$

Clearly, the sum of two cohomology operations of the same type is still a cohomology operations. We denote the group of cohomology operations of type $(n, \pi; m, G)$ by $\mathcal{O}(n, \pi; m, G)$.

A *stable cohomology operation* of type (r, π, G) is a sequence of cohomology operations $\varphi_n \in \mathcal{S}tab(n, \pi; n + r; G)$ for $n = 1, 2, 3, \dots$ such that for every X and every n, the following diagram commutes

$$\begin{tikzcd} H^n(X;\pi) \arrow[r, "(\varphi_n)_X"] \arrow[d, "\Sigma"'] & H^{n+r}(X;G) \arrow[d, "\Sigma"] \\ H^{n+1}(\Sigma X;\pi) \arrow[r, "(\varphi_{n+1})_{\Sigma X}"'] & H^{n+r+1}(\Sigma X;G) \end{tikzcd}$$

where Σ is the suspension isomorphism.

Let $\mathcal{S}tab(r; \pi, G)$ be the collection of stable cohomology operations of type $(r, \pi; G)$.

Definition 2.2.2 The i-th Steenrod square Sq^i consists of stable cohomology operations

$$Sq^i : H^n(X; \mathbb{F}_2) \to H^{n+i}(X; \mathbb{F}_2)$$

for each $n \in \mathbb{N}$ satisfying the following axioms

1. For any cocycle α, we have

$$Sq^i\alpha = \begin{cases} 0, & i > \dim \alpha, \\ \alpha^2, & i = \dim \alpha, \\ \alpha, & i = 0. \end{cases}$$

2. The following Cartan's multiplication formula holds:

$$Sq^i(uv) = \sum_{j+k=i} Sq^j(u)Sq^k(v)$$

Proposition 2.2.3 Sq^1 is exactly the Bockstein operation associated with the short exact sequence on the coefficient

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

Definition 2.2.4 Let p be an odd prime and let $\beta : H^*(X; \mathbb{F}_p) \to H^{*+1}(X; \mathbb{F}_p)$ be the Bockstein operation associated with the short exact sequence on the coefficient

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

The i-th mod-p reduced power operations P^i consists of stable cohomology operations

$$P^i_p : H^n(X; \mathbb{F}_p) \to H^{n+2i(p-1)}(X; \mathbb{F}_p)$$

for each $n \in \mathbb{N}$ satisfying the following axioms

1. For any cocycle α, we have

$$P_p^i \alpha = \begin{cases} 0, & 2i > \dim \alpha, \\ \alpha^p, & 2i = \dim \alpha, \\ \alpha, & i = 0. \end{cases}$$

2. The following Cartan's multiplication formula holds:

$$P_p^i(uv) = \sum_{j+k=i} P_p^j(u) P^k(v)$$

Theorem 2.2.5 The cohomology operations in Definition 2.2.2 and Definition 2.2.4 exist uniquely.

Theorem 2.2.6 (Adem relation) For Steenrod squares, if $0 < a < 2b$, then,

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

For odd prime p, if $a < pb$, then

$$P_p^a P_p^b = \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} P_p^{a+b-j} P_p^j$$

if $a \leqslant b$, then

$$\begin{aligned} P_p^a \beta P_p^b = & \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} \beta P_p^{a+b-j} P_p^j \\ & + \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \beta P_p^{a+b-j} P_p^j \end{aligned}$$

The mod-p Steenrod operation St_p^i is defiend as

$$St_p^i = \begin{cases} P_p^k, & i = 2k(p-1) \\ \beta P_p^k, & i = 2k(p-1)+1 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2.7 (Steenrod algebra) The mod-p Steenrod algebra $\mathcal{A}_p$ is the graded associative $\mathbb{F}_p$-algebra generated by P_p^i (Sq^i if $p = 2$) and the Bockstein operation β, subject to the Adem relations, where $\dim Sq^i = i$, $\dim P_p^i = 2i(p-1)$ and $\dim \beta = 1$.

A *2-admissible sequence* I is an ordered sequence with finitely many positive integers $\{i_1, \dots, i_r\}$ such that $i_1 \geqslant 2i_2, \dots, i_{r-1} \geqslant 2i_r$. We denote $Sq^I = Sq^{i_1} \cdots Sq^{i_r}$. The *total degree* of the sequence I is defined to be $\sum_m i_m$. The *length* of I is the number of non-zero elements in I.

Proposition 2.2.8 The iterated Steenrod squares Sq^I for all admissible sequences form a basis of $\mathcal{A}_2$.

The case of odd prime is more complicated: a p-*admissible sequence* I is an order sequence

$$(\varepsilon_0, s_1, \varepsilon_1, s_2, \dots, \varepsilon_k, s_k, 0, 0, \dots)$$

where $\varepsilon = 0, 1$ and s_i is a positive integer such that $s_i \geqslant ps_{i+1} + \varepsilon_i$ for each $i \geqslant 1$. We denote

$$P_p^I = \beta^{\varepsilon_0} P_p^{s_1} \beta^{\varepsilon_1} P_p^{s_2} \beta^{\varepsilon_2} \cdots P_p^{s_k} \beta^{\varepsilon_k}$$

Proposition 2.2.9 $\mathcal{A}_p$ is linearly generated by P_p^I for all p-admissible sequences.

Proposition 2.2.10 (Milnor [Mil58]) The Steenrod algebra $\mathcal{A}_p$ is a Hopf algebra.

2.2.2 *Algebraic Steenrod operations*

This subsubsection mainly refers to May's algebraic approach to Steenrod operations [May70].

Let $\pi \subset \Sigma_r$ be a subgroup and $C_p \subset \Sigma_p$ be the cyclic group generated by $(12\dots p)$. Let V be a free $k[\Sigma_r]$ resolution of k, W be a free $k[\pi]$-resolution of k and $j\colon W \to V$ be an inclusion induced by $\pi \subset \Sigma_r$. In particular, if $\pi = C_P$, then we let W be the Tate resolution of k. Let $k[C_p] = k[T]/(T^p)$, where T can be identified as $(12\cdots p)$, the Tate resolution W is defined to be

$$W_i = k[C_p] \cdot e_i,\ \ d(e_{2i}) = (1-T)e_{2i-1},\ \ d(e_{2i+1}) = (1 + T + \cdots + T^{p-1})e_{2i}$$

Given a $\mathbb{Z}$-graded homotopy associated and commutative differential k-algebra K, we let Σ_r act on K^r by permuting factors and let Σ_r act on K trivially. Note that we assign the degree of $W_i \otimes_K (K^r)_n$ to be $n - i$.

Definition 2.2.11 A (π, k)-pair (K, θ) consists of a $\mathbb{Z}$-graded homotopy associated and commutative differential k-algebra K and a $k[\pi]$-equivariant map $\theta\colon W \otimes_k K^r \to K$ such that

1. $\theta(e_0 \otimes_k x_1 \otimes_k \cdots \otimes_k x_r) = x_1 \dots x_r$.
2. There exists $k[\Sigma_r]$-morphism $\phi\colon V \otimes_k K^r \to K$ such that $\theta = \phi \circ (j \otimes_k \mathrm{id})$.

A morphism between (K, θ) and (K', θ') is a morphism of k-complexes $f\colon K \to K'$ such that $f \circ \theta \simeq \theta' \circ (\mathrm{id} \otimes_k f^r)$.

Let $\mathcal{C}(\pi, k)$ be the category of (π, k)-pairs.

The tensor product $(K, \theta) \otimes_k (K', \theta')$ is defined to be $(K \otimes_k K', \tilde{\theta})$, where $\tilde{\theta}$ is defined to be the following composition

$$W \otimes_k (K \otimes_k K')^r \xrightarrow{\psi \otimes_k \sigma} W \otimes_k W \otimes_k K^r \otimes_k K'^r \xrightarrow{\mathrm{id} \otimes_k \tau \otimes_l \mathrm{id}} W \otimes_k K \otimes_k W' \otimes_k K' \xrightarrow{\theta \otimes \theta'} K \otimes_k K'$$

where $\psi : W \to W \otimes_k W$ is a $k[\pi]$-homomorphism covering $k \cong k \otimes_k k$, σ is the shuffle permutation and τ is the transposition.

Construction 2.2.12 For any $(K, \theta) \in \mathcal{C}(C_p, \mathbb{F}_p)$, θ is a mod-p total Steenrod operation. If the product $K \otimes K \to K$ can be extended to $(K, \theta) \otimes (K, \theta) \to (K, \theta)$, then the morphism in $\mathcal{C}(C_p, \mathbb{F}_p)$ exhibits Cartan formula for the Steenrod operations.

Let U be a $k[\Sigma_{p^2}]$-resolution of k and let $\tau = C_p \wr C_p \subset \Sigma_{p^2}$ be the p-Sylow subgroup. Note that $\tau = C_p \wr C_p$ acts on $W \otimes_k W^p$ "diagonally", where one of

C_p acts naturally on W and the other C_p acts on W^p by permuting factors. Let $\omega\colon W \otimes W^p \to U$ be a $k[\tau]$-homormophism extending the identity $k \to k$. Then the following homotopy commutative diagram demonstrates the Adem relations.

$$\begin{tikzcd}
(W \otimes_k W^p) \otimes_k K^{p^2} \arrow[r, "\omega\otimes_k \mathrm{id}"] \arrow[dd, "\text{shuffle}"] & U \otimes_k K^{p^2} \arrow[dr, "\xi"] & \\
 & & K \\
W \otimes_k (W \otimes_k K^p)^p \arrow[r, "1\otimes_k \theta^p"] & W \otimes_k K^p \arrow[ur, "\theta"] &
\end{tikzcd}$$

Example 2.2.13 For any CW-complex X, let $K_{-n} = C^n(X;\mathbb{F}_p)$. Then there exists a map θ such that $(K,\theta) \in \mathcal{C}(C_p,\mathbb{F}_p)$ canonically. In other words, there exists a functor from the category of CW-complexes to $\mathcal{C}(C_p,\mathbb{F}_p)$. In particular, such θ is exactly a mod-p total Steenrod operation at cochain complex level.

If $p = 2$ and $x \in H_q(K)$, then the n^{th} Steenrod squares are defined as follows.

$$Sq^{-n}(x) := Sq_n(x) \begin{cases} 0 & n < q \\ \theta_*(e_{n-q} \otimes x^p) & n > q \end{cases}$$

If p is an odd prime, we define

$$\begin{cases} P^i(x) = (-1)^i \nu(t-s)\theta_*(e_{(2i-t+s)(p-1)} \otimes_k x^p) & 2i \geqslant t-s \\ \beta P^i(x) = (-1)^i \nu(t-s)\theta_*(e_{(2i-t+s)(p-1)-1} \otimes_k x^p) & 2i > t-s \end{cases}$$

where $\nu(n) = (-1)^j (m!)^\varepsilon$ for $n = 2j+\varepsilon$ and $m = (p-1)/2$.

More details can be found in [Ste62, Chapter VII].

Example 2.2.14 Fixed a commutative unital ring k, given an arbitrary topological space X, $C^*(X;k)$ is in $\mathcal{C}(\Sigma_n,k)$ for any n. In other words, $C^*(-;k)$ is a functor from the category of spaces to $\mathcal{C}(\Sigma_n,k)$.

2.2.3 *Power operations on H_∞-ring spectra*

Construction 2.2.15 Given an $\mathbb{S}$-module M, the j'th *extended power* of M is defined to be

$$D_j M = (E\Sigma_j)_+ \wedge M^j)/\Sigma_j$$

where $E\Sigma_j$ is the universal principal-Σ_j bundle and M^j is the j-fold power of M with respect to $\wedge_{\mathbb{S}}$.

There are natural maps associated to extended powers:

- $\iota_j\colon M^j \to D_j M$;
- $\alpha_{j,k}\colon D_j M \wedge D_k M \to D_{j+k} M$ induced by the inclusion $\Sigma_j \times \Sigma_k \hookrightarrow \Sigma_{j+k}$;
- $\beta_{j,k}\colon D_j D_k M \to D_{jk} M$ induced by the wreath product $\Sigma_j \wr \Sigma_k \to \Sigma_{jk}$;
- $\delta_j\colon D_j(M \wedge N) \to D_j M \wedge D_j N$.

The extended powers and these maps for pointed spaces are defined similarly and they compatible with the suspension functor. Given two pointed spaces X, Y, the following diagrams commutative up to homotopy:

$$\begin{tikzcd} & D_j\Sigma^\infty X \arrow[dd] \\ \Sigma^\infty(X^j) \arrow[ur, "\iota_j"] \arrow[dr, "\Sigma^\infty \iota_j"'] & \\ & \Sigma^\infty D_j X \end{tikzcd}$$

$$\begin{tikzcd} D_j(\Sigma^\infty X \wedge \Sigma^\infty Y) \arrow[r, "\delta_j"] \arrow[d] & D_j(\Sigma^\infty X) \wedge D_j(\Sigma^\infty Y) \arrow[d] \\ \Sigma^\infty D_j(X \wedge Y) \arrow[r, "\Sigma^\infty \delta_j"] & \Sigma^\infty(D_j X \wedge D_j Y) \end{tikzcd}$$

$$\begin{tikzcd} D_j D_k \Sigma^\infty X \arrow[r, "\beta_{j,k}"] \arrow[d] & D_{jk}\Sigma^\infty X \arrow[d] \\ \Sigma^\infty D_j D_k X \arrow[r, "\Sigma^\infty \beta_{j,k}"] & \Sigma^\infty D_{jk} X \end{tikzcd}$$

$$\begin{tikzcd} D_j\Sigma^\infty X \wedge D_k \Sigma^\infty X \arrow[r, "\alpha_{j,k}"] \arrow[d] & D_{i+k}\Sigma^\infty X \arrow[d] \\ \Sigma^\infty(D_j X \wedge D_k X) \arrow[r, "\Sigma^\infty \alpha_{j,k}"] & \Sigma^\infty D_{j+k} X \end{tikzcd}$$

Let $\tau \colon E \wedge F \to F \wedge E$ denote the commutative isomorphism in $\mathcal{M}_S(\mathcal{U})$. The following lemmas demonstrate the homotopy coherence data carried by extended powers.

Lemma 2.2.16 $\{\alpha_{j,k}\}$ is a commutative and associative system up to homotopy, namely for any i, j, k, we have

- $\alpha_{j,k} \circ \tau \simeq \alpha_{k,j}$;
- $\alpha_{i+j,k} \circ (\alpha_{i,j} \wedge \mathrm{id}) \simeq \alpha_{i,j+k} \circ (\mathrm{id} \wedge \alpha_{j,k})$;

Lemma 2.2.17 $\{\beta_{j,k}\}$ is an associative system up to homotopy, namely for any i, j, k, we have

$$\beta_{ij,k} \circ \beta_{i,j} \simeq \beta_{i,jk} \circ D_i \beta_{j,k}.$$

Lemma 2.2.18 Each δ_j is commutative and associative with respect to the smash product up to homotopy:

- $\tau \circ \delta_j \simeq \delta_j \circ D_j \tau$;
- $(\delta_j \wedge \mathrm{id}) \circ \delta_j \simeq \delta_j \circ (\mathrm{id} \wedge \delta_j)$

Lemma 2.2.19 The following diagrams commute up to homotopy:

$$
\begin{tikzcd}
M^j \wedge M^k \arrow[r, equal] \arrow[d, "\iota_j \wedge \iota_k"'] & M^{j+k} \arrow[d, "\iota_{j+k}"] \\
D_j M \wedge D_k M \arrow[r, "\alpha_{j,k}"] & D_{j+k} M
\end{tikzcd}
\qquad
\begin{tikzcd}
(D_k M)^j \arrow[dr, "\alpha_{k,\dots,k}"] \arrow[d, "\iota_j"'] & \\
D_j D_k M \arrow[r, "\beta_{j,k}"'] & D_{jk} M
\end{tikzcd}
$$

Lemma 2.2.20 Let ν_j be the evident shuffle isomorphism, then the following diagram commute up to homotopy

$$
\begin{tikzcd}
(E \wedge F)^j \arrow[r, "\nu_j"] \arrow[d, "\iota_j"'] & E^j \wedge F^j \arrow[d, "\iota_j \wedge \iota_j"] \\
D_j(E \wedge F) \arrow[r, "\delta_j"] & D_j E \wedge D_j F
\end{tikzcd}
$$

Lemma 2.2.21 The following diagram commutes up to homotopy:

$$
\begin{tikzcd}
D_i D_k M \wedge D_j D_k M \arrow[r, "\beta_{i,k} \wedge \beta_{j,k}"] \arrow[d, "\alpha_{i,j}"'] & D_{ik} M \wedge D_{jk} M \arrow[d, "\alpha_{ik,jk}"] \\
D_{i+j} D_k M \arrow[r, "\beta_{i+j,k}"] & D_{ik+jk} M
\end{tikzcd}
$$

Lemma 2.2.22 The following diagrams commute up to homotopy

$$
\begin{tikzcd}
D_j(M \wedge N) \wedge D_k(M \wedge N)) \arrow[rr, "\alpha_{j,k}"] \arrow[d, "\delta_j \wedge \delta_k"] & & D_{j+k}(M \wedge N) \arrow[d, "\delta_{j+k}"] \\
D_j M \wedge D_j N \wedge D_k M \wedge D_k N \arrow[r, "id \wedge \tau \wedge id"] & D_j M \wedge D_k M \wedge D_j N \wedge D_k F \arrow[r, "\alpha_{j,k} \wedge \alpha_{j,k}"] & D_{j+k} M \wedge D_{j+k} N
\end{tikzcd}
$$

$$
\begin{tikzcd}
D_j D_k(M \wedge N) \arrow[rr, "\beta_{j,k}"] \arrow[d, "D_j \delta_k"] & & D_{jk}(M \wedge N) \arrow[d, "\delta_{jk}"] \\
D_j(D_k M \wedge D_k N) \arrow[r, "\delta_j"] & D_j D_k M \wedge D_j D_k N \arrow[r, "\beta_{j,k} \wedge \beta_{j,k}"] & D_{jk} M \wedge D_{jk} N
\end{tikzcd}
$$

Lemma 2.2.23 The following diagram commutes up to homotopy:

$$
\begin{tikzcd}
D_i(D_j M \wedge D_k M) \arrow[r, "\delta_i"] \arrow[d, "D_i \alpha_{j,k}"'] & D_i D_j M \wedge D_i D_k M \arrow[r, "\beta_{i,j} \wedge \beta_{i,k}"] & D_{ij} M \wedge D_{ik} M \arrow[d, "\alpha_{ij,ik}"] \\
D_i D_{j+k} M \arrow[rr, "\beta_{i,j+k}"] & & D_{ij+ik} M
\end{tikzcd}
$$

Definition 2.2.24 An H_∞-ring spectrum is a $\mathbb{S}$-module M together with $\xi_j : D_j M \to M$ for $j \geqslant 0$ such that ξ_1 is the identity map and the following diagrams commute for $j, k \geqslant 0$ up to homotopy

$$\begin{tikzcd} D_jM \wedge D_kM \arrow[rr, "\alpha_{j,k}"] \arrow[d, "\xi_j \wedge \xi_k"'] & & D_{j+k}M \arrow[d, "\xi_{j+k}"] \\ M \wedge M \arrow[r, "\iota_2"] & D_2M \arrow[r, "\xi_2"] & M \end{tikzcd} \qquad \begin{tikzcd} D_jD_kM \arrow[r, "\beta_{j,k}"] \arrow[d, "D_j\xi_k"'] & D_{jk}M \arrow[d, "\xi_{jk}"] \\ D_jM \arrow[r, "\xi_j"] & M \end{tikzcd}$$

Construction 2.2.25 Let E be an H_∞-*ring spectrum* and $\tilde{E}$ be the associated cohomology theory, the j-*total power operation* $\mathcal{P}_j$ is defined to be

$$\begin{tikzcd} \tilde{E}^*(X) \arrow[rrr, "\mathcal{P}_j"] \arrow[d, equal] & & & \tilde{E}^*(B\Sigma_{j+} \wedge X) \arrow[d, equal] \\ {[\Sigma^\infty X, E]} \arrow[r, "D_j"] & {[D_j(\Sigma^\infty X), D_jE]} \arrow[r, "\circ \xi_j"] & {[\Sigma^\infty(D_jX), E]} \arrow[r, "\Delta^*"] & {[\Sigma^\infty(B\Sigma_{j+} \wedge X), E]} \end{tikzcd}$$

where Δ^* is induced by the diagonal map $X \to X^j$.

If we have $\tilde{E}^*(B\Sigma_{j+} \wedge X) \cong \tilde{E}^*(B\Sigma_{j+}) \otimes_{\pi_* E} \tilde{E}^*(X)$, then given $\alpha \in E^*B\Sigma_j$ and $x \in \tilde{E}^r(X)$, we can define $\alpha^*(x)$ to be $D\alpha\mathcal{P}(x)$, where $D\alpha^* : \tilde{E}^r(X) \to \pi_* E$ is the dual function of α.

2.3 The Adams spectral sequences

The outline of this subsection is presented as follows.

1. **Subsubsection 2.3.1**: We study Hopf algebroids and the associated cohomological setting. In particular, we concern the notion of normalized canonical resolution $C(\Gamma, M)$ and the notion of canonical complex $C(N, \Gamma, M)$ (Definition 2.3.9), which provide us with explicit computable resolutions and complexes. Note that the cohomology of $C(N, \Gamma, M)$ is exactly $\mathrm{Ext}_\Gamma(N, M)$.

2. **Subsubsection 2.3.2**: We show that for each suitable triple (N, Γ, M) over $\mathbb{F}_p$, $C(\Gamma, M)$ or $C(N, \Gamma, M)$ is a $(\pi, \mathbb{F}_p)$-pair in the sense of Definition 2.2.11, where we may simply call this structure *algebraic extended power* informally. Thus there exists power operations on the cohomology of $C(N, \Gamma, M)$.

3. **Subsubsection 2.3.3**: We introduce the notion of Adams resolution of a spectrum and the associated spectral sequences called Adams spectral sequences. The key construction is the canonical Adams resolution (Construction 2.3.20). Specifically, given a spectrum X and Y, the associated complex of the canonical Adams resolution of Y with respect to E and X is exactly the canonical complex $C(E_*X, E_*E, E_*Y)$, so that we can apply the tools in the previous two subsubsections on Adams spectral sequences. In this way, we have power operations in the E_2-page of an Adams spectral sequence.

4. **Subsubsection 2.3.4**: Given an H_∞-ring spectrum Y, the unique (up to homotopy) algebraic extended power structures on $C(\pi_*E, E_*E, E_*Y)$ mentioned

in Subsubsection 2.3.2 can be derived from the extended powers in the H_∞-structure on Y. In detail, we construct a "filtration" of $\xi\colon D_\pi Y \to Y$ (See Theorem 2.3.33) and show this "filtration" induced the algebraic extended power structure on $C(\pi_* E, E_* E, E_* Y)$ (See Corollary 2.3.35).

5. **Subsubsection 2.3.5**: In order to demonstrate how the power operations detect the homotopy operations on $\pi_*(Y)$ associated to the extended power, we introduce a notion of generalized Adams spectral sequences (see Theorem 2.3.38). Then there is such a spectral sequence $E^{*,*}(\mathbb{S}, \Xi)$ associated to a filtration Ξ of $D_\pi \mathbb{S}^n$, and each $x \in \pi_n(Y)$ determines a morphism of spectral sequences from $E^{*,*}(\mathbb{S}, \Xi)$ to $E^{*,*}(\mathbb{S}, Y)$, which exhibits $E^{*,*}(\mathbb{S}, Y)$ is a module over $E^{*,*}(\mathbb{S}, \Xi)$. This module structure indicates that how the power operations detect the homotopy operations.

2.3.1 Cohomology of Hopf algebroid

Definition 2.3.1 A *Hopf algebroid* over a commutative ring k is a pair of commutative k-algebras (A, Γ) endowed with maps

- $\eta_L\colon A \to \Gamma$ called left unit or source,
- $\eta_R\colon A \to \Gamma$ called right unit or target,
- $\Psi\colon \Gamma \to \Gamma \otimes_A \Gamma$ called coproduct or composition,
- $\epsilon\colon \Gamma \to A$ called counit or identity,
- $c\colon \Gamma \to \Gamma$ called conjugation or inverse.

and the data satisfies the following rules:

1. η_L is flat.
2. $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_A$.
3. $\Gamma \xrightarrow{\Psi} \Gamma \otimes_A \Gamma \xrightarrow{q} \Gamma \otimes_A A \cong \Gamma$ is the identity map, where $q = \mathrm{id}_\Gamma \otimes \epsilon$ or $\epsilon \otimes \mathrm{id}_\Gamma$.
4. $(\mathrm{id}_\Gamma \otimes \Psi) \circ \Psi = (\Psi \otimes \mathrm{id}_\Gamma) \circ \Psi$.
5. $c \circ \eta_R = \eta_L$ and $c\eta_L = \eta_R$.
6. $c \circ c = \mathrm{id}_\Gamma$.
7. There exists maps such that the following diagram commutes

$$\begin{tikzcd}
\Gamma & \Gamma \otimes_k \Gamma \arrow[l, "c\circ \mathrm{id}"'] \arrow[r, "\mathrm{id}\circ c"] \arrow[d] & \Gamma \\
& \Gamma \otimes_A \Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\
A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\epsilon"] \arrow[r, "\epsilon"'] \arrow[u, "\Psi"] & A \arrow[uu, "\eta_L"']
\end{tikzcd}$$

Remark 2.3.2 (η_L, η_R) exhibits Γ as an A-bimodule and Γ is ab A-comodule.

Remark 2.3.3 Given a Hopf algebroid $(A, \Gamma, \eta_L, \eta_R, \Psi, \epsilon, c)$ and a commutative k-algebra R, there is a groupoid whose objects are $\mathrm{Hom}_k(A, R)$ and its morphisms are $\mathrm{Hom}_k(\Gamma, R)$. In summary, a Hopf algebroid determines a functor from the category of k-algebras to the category of groupoids.

Definition 2.3.4 Given a Hopf algebroid (A, Γ), a *right* Γ*-comodule* M is an right A-module M together with an A-linear map $\psi_M \colon M \to M \otimes_A \Gamma$ such that $(\mathrm{id}_M \otimes \epsilon) \circ \psi_M = \mathrm{id}_M$ and $(\mathrm{id}_M \otimes \Psi) \circ \psi_M = (\psi_M \otimes \mathrm{id}_A) \circ \psi_M$. The category of Γ-comodules is denoted by Comod_Γ.

Remark 2.3.5 Given a right A-module M, $M \otimes_A \Gamma$ is a right Γ-comodule naturally, which is called the extened comodule of M.

Suppose P, Q are two graded right A-modules, then we define

$$\mathrm{Hom}_R^t(P, Q) := \{f \colon P^* \to Q^{*+t} \mid f \text{ is } A\text{-linear}\}$$

In this way, $\mathrm{Hom}_A(P, Q)$ is a graded A-module.

If P, Q are right Γ-comodules, then let $\mathrm{Hom}_\Gamma(P, Q)$ be the submodule of $\mathrm{Hom}_A(P, Q)$ consisting of Γ-comodule morphisms.

Theorem 2.3.6 (Comparison theorem) If $X = \{0 \to M \to X_0 \to X_1 \to \cdots\}$ is an A-split exact sequence of right Γ-comodules and $Y = \{0 \to N \to Y_0 \to Y_1 \to \cdots\}$ is a complex of injective right Γ-comodules, then for each Γ-homomorphism $f \colon M \to N$, there is a unique chain homotopy class of Γ-homomorphisms $F \colon X \to Y$ extended f, where $X = \{X_0 \to X_1 \to \cdots\}$ and $Y = \{Y_0 \to Y_1 \to \cdots\}$.

Let Ext_A^i be the i^{th} right derived functor of Hom_Γ relative to injective and Γ-split comodules resolution.

Definition 2.3.7 Given two right Γ-comodules, the right Γ-comodule structure on $M \otimes_A N$ is defined to be

$$M \otimes_A N \xrightarrow{\psi_M \otimes \psi_N} M \otimes_A \Gamma \otimes_A N \otimes \Gamma \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} M \otimes_A N \otimes_A \Gamma \otimes_A \Gamma \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \phi} M \otimes_A N \otimes_A \Gamma$$

where $\phi \colon \Gamma \otimes_A \Gamma \to \Gamma$ is the morphism induced by the multiplication in Γ.

Remark 2.3.8 Note that we can always embed the category of left (or right) A-modules into the category of A-bimodules by setting $a \cdot m = (-1)^{|m||a|} m \cdot a$. Thus the tensor product between two right A-modules can be identified with the tensor product between the associated A-bibmodules. This can be done because A is commutative (in the sense of graded algebras).

Let Γ_R (resp. Γ_L) be a right A-module (resp. left A-module) by forgetting the left A-action induced by η_L (resp. the right A-action induced by η_R). The A-bibmodule structure of Γ and Γ_R are different with this setting. However, given a right A-comodule M, $M \otimes_A \Gamma \cong M \otimes_A \Gamma_R$, see [BMMS86, P92]. Let $\theta \colon M \otimes_A \Gamma_R \to M \otimes_A \Gamma$

be the isomorphism (as right Γ-comodules). Specifically, the following diagram commutes.

$$\begin{tikzcd} & M \otimes_A \Gamma_R \arrow[dd, "\theta"] \\ M \arrow[ur, "\mathrm{id}\otimes\eta_R"] \arrow[dr, "\psi_M"'] & \\ & M \otimes_A \Gamma \end{tikzcd}$$

Let $p\colon \Gamma_R \to \bar{\Gamma}$ be the cokernel of η_R and given an element $x \in \Gamma$, let $\bar{x} := p(x)$. Define $t\colon \bar{\Gamma} \to \Gamma_R$ by $\bar{x} \mapsto x - \eta_R \circ \epsilon(x)$. Note that t is well-defined, since if $x = \eta_R(y)$, then $t(\bar{x}) = \eta_R(y) - \eta_R \circ \epsilon\eta_R(y) = 0$. Then for any right Γ-comodule M, we have the following Γ-split short exact sequence

$$p \longrightarrow M \xrightarrow{\mathrm{id}\otimes_A\eta_R} M \otimes_A \Gamma_R \xrightarrow{1\otimes_A p} M \otimes_A \bar{\Gamma} \longrightarrow 0$$

where a section of $\mathrm{id} \otimes_A \eta_R$ is $\mathrm{id} \otimes_A \epsilon$ and a section of $\mathrm{id} \otimes_A p$ is $\mathrm{id} \otimes_A t$.

Definition 2.3.9 Let M be a right A-comodule, then the *normalized canonical resolution* $C(\Gamma, M)$ of M is the Γ-split differential graded right Γ-comodule

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots$$

where $C_s = M \otimes_A \bar{\Gamma}^{\otimes s} \otimes_A \Gamma_R$, $d_s = (1 \otimes_A \eta_R) \circ (1 \otimes_A p)$ and a section σ_s of d_s is $(\mathrm{id} \otimes_A t) \circ (\mathrm{id} \otimes_A \epsilon)$. We denote

$$m|a_1|\cdots|a_s|a := m \otimes_A \bar{a_1} \otimes_A \cdots \otimes_A \bar{a_s} \otimes_A a$$

and we assign it *homological degree* s, *internal degree* $t = |m| + \sum |a_i| + |a|$, *bidegree* (s, t) and *total degree* $t - s$.

If N is a right Γ-comodule, the *canonical complex* $C(N, \Gamma, M)$ is defined to be

$$C_{s,t}(N, \Gamma, M) := \mathrm{Hom}^t_\Gamma(N, C_s(\Gamma, M))$$

Proposition 2.3.10 $\mathrm{Ext}^{s,t}_\Gamma(N, M) \cong H_{s,t}(C(N, \Gamma, M))$.

Proof. See [BMMS86, Chapter IV, Proposition 1.2]. □

2.3.2 *Power operations in* Ext

Definition 2.3.11 Let $\mathcal{C}$ be the category whose objects are triples (N, Γ, M) such that

1. (A, Γ) is a Hopf algebroid over k,
2. M is a commutative unital algebra in Comod_Γ (let $\eta_M\colon \Gamma \to M$ be the unit),
3. N is a cocommutative unital coalgebra in Comod_Γ (let $\epsilon_N\colon \Gamma \to N$ be the counit).

and whose morphism $(N, \Gamma, M) \to (N', \Gamma', M')$ are triples (f, λ, g) such that

1. $\lambda\colon (A, \Gamma) \to (A', \Gamma')$ is a morphism of Hopf algebroids,
2. $f\colon M \to M'$ is an λ-equivariant morphism of algebras preserving units,

3. $g\colon N' \to N$ is a λ-equivariant morphism of coalgebras preserving counits.

Given a triple $(N, A, M) \in \mathcal{C}$, let $\phi : M^n \to M$ be the iterated product and $\Delta : N \to N^n$ be the iterated coproduct. Then by using the comparison theorem 2.3.6, we can extend $\phi\colon M^n \to M$ to

$$\tilde{\phi}\colon C(\Gamma, M)^n \to C(\Gamma, M)$$

and this extension is unique up to homotopy. In this way, $C(\Gamma, M)$ is a homotopy associative and commutative differential graded algebra in Comod_Γ. Furthermore, the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Gamma(N, C(\Gamma, M))^n \arrow[r, equal] \arrow[d, "\otimes"] & C(N, \Gamma, M)^n \arrow[dd] \\
\mathrm{Hom}_\Gamma(N^n, C(\Gamma, M)^n) \arrow[d, "\mathrm{Hom}(\Delta, \tilde{\phi})"] & \\
\mathrm{Hom}_\Gamma(N, C(\Gamma, M)) \arrow[r, equal] & C(N, A, M)
\end{tikzcd}$$

also characterizes $C(N, \Gamma, M)$ as a homotopy associative and commutative differential graded algebra in Comod_Γ.

Proposition 2.3.12 Let (N, Γ, M) be a triple over $\mathbb{F}_p$ and $\pi \subset \Sigma_p$, then there is a unique chain homotopy class of $\mathbb{F}_p[\pi]$-equivariant maps $\Phi : W \otimes_k C(\Gamma, M) \to C(N, \Gamma, M)$ such that $(C(\Gamma, M), \Phi)$ is a $(\pi, \mathbb{F}_p)$-pair.

Corollary 2.3.13 Let (N, Γ, M) be a triple over $\mathbb{F}_p$, then $C(N, \Gamma, M)$ has a $(\pi, \mathbb{F}_p)$-pair structure.

$$\begin{tikzcd}
W \otimes \mathrm{Hom}_\Gamma(N, C(\Gamma, M))^p \arrow[r, equal] \arrow[d, "\otimes"] & W \otimes C(N, \Gamma, M)^p \arrow[dd] \\
\mathrm{Hom}_\Gamma(N^p, W \otimes_k C(\Gamma, M)^p) \arrow[d, "\mathrm{Hom}(\Delta, \Phi)"] & \\
\mathrm{Hom}_\Gamma(N, C(\Gamma, M)) \arrow[r, equal] & C(N, A, M)
\end{tikzcd}$$

Construction 2.3.14 (Steenrod operations in Ext) Note that $\mathrm{Ext}^{s,t}_\Gamma(N, M)$ is the homology of $C(N, \Gamma, M)$ and here we let $\pi = C_p$. Let $x \in \mathrm{Ext}^{s,t}_\Gamma(N, M)$.

If $p = 2$, we define

$$P^i = Sq^i(x) := \theta_*(e_{i-t+s} \otimes_k x^2), \text{ if } i \geqslant t - s$$

If p is an odd prime, we define

$$\begin{cases} P^i(x) = (-1)^i \nu(t-s)\theta_*(e_{(2i-t+s)(p-1)} \otimes_k x^p) & 2i \geqslant t - s \\ \beta P^i(x) = (-1)^i \nu(t-s)\theta_*(e_{(2i-t+s)(p-1)-1} \otimes_k x^p) & 2i > t - s \end{cases}$$

where $\nu(n) = (-1)^j (m!)^\varepsilon$ for $n = 2j + \varepsilon$ and $m = (p-1)/2$.

Let (N, Γ, M) be a triple over $\mathbb{Z}b$ such that N, Γ and M are torsion free. Let $\bar{N} = N \otimes \mathbb{Z}/p$, $\bar{\Gamma} = \Gamma \otimes \mathbb{Z}/p$ and $\bar{M} = M \otimes \mathbb{Z}/p$. Then $(\bar{N}, \bar{\Gamma}, \bar{M})$ is a triple over $\mathbb{F}_p$. The exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

induces the Bockstein operation

$$\beta\colon \mathrm{Ext}^{s,t}_{\bar{\Gamma}}(\bar{N}, \bar{M}) \to \mathrm{Ext}^{s+1,t}_{\bar{\Gamma}}(\bar{N}, \bar{M})$$

Theorem 2.3.15 The Steenrod operations in previous definition has the following properties.

1. $\beta^{\varepsilon}P^i\colon \mathrm{Ext}^{s,t}_{\Gamma} \to \mathrm{Ext}^{s+(t-2i)(p-1)+\varepsilon,pt)}_{\Gamma}$ where $\varepsilon = 0$ if $p = 2$.
2. When $p = 2$, $P^i = 0$ unless $t - s \geqslant i \geqslant t$. When p is an odd prime, $P^i = 0$ unless $t - s \geqslant 2i \geqslant t$, and $\beta P^i = 0$ unless $t - s + 1 \geqslant 2i \geqslant t$.
3. $P^i(x) = x^p$ if $p = 2$ and $i = t - s$ or if p is an odd prime and $2i = t - s$.
4. The Catan formulas hold:
$$P^n(xy) = \sum_i P^i(x)P^{n-i}(y)$$
$$\beta P^n(xy) = \sum_i \beta P^i(x)P^{n-i}(y) + \sum_i (-1)^{|x|}P^i(x)\beta P^{n-i}(y)$$
5. The Adem relations hold: if $p = 2$ and $0 < a < 2b$, then
$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j$$
If p is an odd prime and $a < pb$, then
$$P^a P^b = \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j$$
and if $a \leqslant b$, then
$$\begin{aligned} P^a\beta P^b = &\sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} \beta P^{a+b-j}P^j \\ &+ \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \beta P^{a+b-j}P^j \end{aligned}$$
6. Let $f\colon (N, \Gamma, M) \to (N', \Gamma', M')$ and $g\colon (N', \Gamma', M') \to (N'', \Gamma'', M'')$ be two morphisms of triples such that the following sequence is exact
$$0 \longrightarrow C(N, \Gamma, M) \xrightarrow{C(f)} C(N', \Gamma', M') \xrightarrow{C(g)} C(N'', \Gamma'', M'') \longrightarrow 0$$

and let $\delta\colon \mathrm{Ext}_A^{s,t}(N,M) \to \mathrm{Ext}_{A''}^{s,t}(N'',M'')$ be the boundary map in the associated long exact sequence. Then $\delta \circ P^i = P^i \circ \delta$ and $\delta \circ \beta P^i = -\beta P^i \circ \delta$.

7. If (N,Γ,M) is the mod-p reduction of torsion free triple over $\mathbb{Z}$, then $\beta \circ Sq^{i+1} = iSq^i$ and $\beta \circ P^i = \beta P^i$ if p is an odd prime.

Proof. See [BMMS86, Chapter IV, Theorem 2.5]. □

2.3.3 *The Adams spectral sequences*

Construction 2.3.16 Given inverse sequences of CW-spectra

$$Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow Y_3 \longleftarrow \cdots$$

where i_s is the inclusion of a subcomplex. (Any inverse sequence can be replaced by an equivalent one of this form by taking CW approximation and mapping telescopes.) Then we define $i_{s,r} = i_s \circ i_{s+1} \circ \cdots \circ i_{s+r-1} : Y_{s+r} \to Y_s$ and $Y_{s,r} = Y_s/Y_{s+r} = Ci_{s,r}$ and let

$$\begin{tikzcd} Y_{s+r} \arrow[rr, "i_{s,r}"] & & Y_s \arrow[dl, "p_{s,r}"] \\ & Y_{s,r} \arrow[ul, "\partial_{s,r}"] & \end{tikzcd}$$

be a cofiber sequence with $\partial_{s,r}$ of degree -1.

Given a spectrum X, we obtain an exact couple

$$\begin{tikzcd} \bigoplus_{s,t}[X,Y_s]_{t-s} \arrow[rr, "i_*"] & & \bigoplus_{s,t}[X,Y_s]_{t-s} \arrow[dl, "p_*"] \\ & \bigoplus_{s,t}[X,Y_{s,1}]_{t-s} \arrow[ul, "\partial_*"] & \end{tikzcd}$$

and the $E_r^{s,t}$-term of the associated spectral sequence is

$$E_r^{s,t} = \frac{\mathrm{im}([X,Y_{s,r}]_{t-s} \to [X,Y_{s,1}]_{t-s})}{\ker([X,Y_{s,1}]_{t-s} \to [X,Y_{s-r+1,r}]_{t-s})}$$

Suppose E is a commutative ring spectrum with unit $\eta : \mathbb{S} \to E$ and product $\mu\colon E \wedge E \to E$, we assume the induced map $\eta_*\colon \pi_*E \to E_*E$ is flat. Then (π_*E, E_*E) is a Hopf algebroid.

Proposition 2.3.17 With the assumption that η_* is flat, the natural map

$$E_*E \otimes_{\pi_*E} E_*X \to \pi_*(E \wedge E \wedge X)$$

is an isomorphism.

In particular, we have $E_*(E \wedge E) \cong E_*E \otimes_{\pi_*E} E_*E$. Now we let $A = \pi_*E$ and $\Gamma = E_*E$. The coproduct $\Psi\colon \Gamma \to \Gamma \otimes_A \Gamma$ is

$$x \in [\mathbb{S}, E \wedge E]_* \mapsto \eta \wedge x \in [\mathbb{S}, E \wedge E \wedge E]_*$$

For any spectrum X, E_*X is a right A-comodule in a similar way. (we just replace the suitable E by X in previous formula.)

Definition 2.3.18 An *Adams resolution* of a spectrum X of is an inverse sequence

$$Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow Y_3 \longleftarrow \cdots$$

such that for each s

1. $Y_{s,1}$ is a wedge of (suspensions) of E or a retract of $X_s \wedge E$ for some spectrum X_s.
2. $E_*Y_s \to E_*Y_{s,1}$ is an A-split monomorphism.

Remark 2.3.19 If we splice an Adams resolution of Y, we obtain an injective resolution of E_*Y

$$\begin{tikzcd} 0 \arrow[r] & E_*Y \arrow[r] & E_*Y_{0,1} \arrow[r] \arrow[d] & E_*\Sigma Y_{1,1} \arrow[r] \arrow[d] & E_*\Sigma^2 Y_{2,1} \arrow[r] \arrow[d] & \cdots \\ & & E_*\Sigma Y_1 \arrow[ur] & E_*\Sigma^2 Y_2 \arrow[ur] & \cdots & \end{tikzcd} \tag{1}$$

Construction 2.3.20 (Canonical Adams resolution) Let $i : \bar{E} \to S$ be the fiber of the unit $\eta : S \to E$. Since a cofiber sequence in the stable homotopy category is a fiber sequence, the cofiber of i is exactly the unit. We defined the *canonical Adams resolution* inductively by setting $Y_0 = Y$, $Y_{s+1} = Y_s \wedge \bar{E}$ and $i_s = \mathrm{id} \wedge i : Y_s \wedge \bar{E} \to Y_s \wedge S \cong Y_s$. Note that the cofiber Ci_s is $Y_s \wedge E$ according to the definition.

The *Adams spectral sequence* for $[X, Y]_*$ with respect to E is the associated spectral sequence of the exact couple obtained by applying $[X, -]_*$ to an Adams resolution of Y. We denote it by $E_r^{*,*}(X, Y)$.

Remark 2.3.21 If we smash E on the cofiber sequence

$$S \xrightarrow{\eta} E \longrightarrow \Sigma\bar{E}$$

then we have another cofiber sequence

$$S \wedge E \xrightarrow{\eta\wedge\mathrm{id}} E \wedge E \longrightarrow \bar{E} \wedge E \tag{2}$$

Note that $\eta \wedge \mathrm{id}$ has a section $\mu : E \wedge E \to E \cong S \wedge E$. Then we have a π_*E-split exact sequence by applying π_* on the cofiber sequence (2)

$$0 \longrightarrow \pi_*E \xrightarrow{\eta_*} E_*E \longrightarrow E_*\Sigma\bar{E} \longrightarrow 0.$$

Therefore $E_*\Sigma\bar{E}$ is isomorphic to the cokernel of $\eta : \pi_*E \to E_*E$, which means that $E_*\Sigma\bar{E}$ play a role as $\bar{\Gamma}$ in the normalized resolution resolution in Definition 2.3.9.

Lemma 2.3.22 The spliced resolution in the form of (1) obtained from the canonical Adams resolution is the normalized canonical resolution $C(E_*E, E_*Y)$ in Definition 2.3.9.

Condition 2.3.23 For any Y and π_*E-projective E_*X, we have the following isomorphisms.

$$[X, Y \wedge E]_* \cong \mathrm{Hom}_{E_*E}(E_*X, E_*(Y \wedge E)) \cong \mathrm{Hom}_{\pi_*E}(E_*X, E_*Y)$$

Remark 2.3.24 Given Condition 2.3.23, for any Adams resolution in Definition 2.3.18, we have $[X, Y_{s,1}] \cong \mathrm{Hom}_{E_*E}(E_*E, E_*Y_{s,1})$. Therefore, the E_2-term in the associated Adams spectral sequence is $\mathrm{Ext}_{E_*E}(E_*X, E_*Y)$.

Suppose the Condition 2.3.23 is satisfied, then we have the following lemmas.

Lemma 2.3.25 If E_*X is π_*E-projective, then

$$E_1^{s,t}(X, Y) = C_{s,t}(E_*X, E_*E, E_*Y)$$

Proof. See [BMMS86, Chapter IV, Lemma 3.7]. □

Corollary 2.3.26 If E_*X is π_E-projective, then

$$E_2^{s,t}(X, Y) = \mathrm{Ext}^{s,t}_{E_*E}(E_*X, E_*Y)$$

Proposition 2.3.27 The Condition 2.3.23 holds for $E = \mathbb{S}, H\mathbb{Z}/p, MO, MU, MSp, K, KO,$ and BP.

Proof. See [Ada74, Proposition 13.4]. □

Theorem 2.3.28 (Adams) Given a commutative ring spectrum E and two spectra X, Z satisfying the condition in [Ada74, Theorem 15.1], the Adams spectral sequence $E_r^{s,t}(X, Z)$ converges to $[X, Z]_*^E$, where $[X, Z]_*^E$ is the graded group of homotopy classes in the E-localized stable homotopy category.

Proof. See [Ada74, Part III, Chapter 15]. □

Remark 2.3.29 Adams's conditions for the convergence of $E_r^{s,t}(X, Z) \Rightarrow [X, Z]_*^E$ are

1. Z is bounded below,
2. E is connective and $\mu_* \colon \pi_0(E) \otimes \pi_0(E) \to \pi_0(E)$ is an isomorphism,
3. if $R \subset \mathbb{Q}$ is maximal such that the natural ring homomorphism $\mathbb{Z} \to \pi_0(E)$ extends to $R \to \pi_0 E$, then $H_r E$ is finitely generated as an R-module for all r;

Theorem 2.3.30 There is a pairing of Adams spectral sequences

$$E_r^{*,*}(X, Y) \otimes E_r^{*,*}(X', Y') \to E_r^{*,*}(X \wedge X', Y \wedge Y')$$

converging to the smash product

$$[X, Y]_*^E \otimes [X', Y']_*^E \to [X \wedge X', Y \wedge Y']_*^E.$$

If E_*X and E_*X' are π_*E-projective, then the pairing on the E_2-pages is the external product

$$\mathrm{Ext}(E_*X, E_*Y) \otimes \mathrm{Ext}(E_*X', E_*Y') \to \mathrm{Ext}(E_*X \otimes E_*X', E_*Y \otimes E_*Y')$$

composed with the homomorphisms

$$E_*(X \wedge X') \xrightarrow{\sim} E_*X \otimes E_*X'$$

$$E_*Y \otimes E_*Y' \to E_*(Y \wedge Y')$$

(Here the Kunneth homomorphism $E_*X \otimes E_*X' \to E_*(X \wedge X')$ is an isomorphism because E_*X and E_*X' are π_*E-projective.)

Corollary 2.3.31

1. $\{E_r^{*,*}(\mathbb{S}, \mathbb{S})\}$ is a spectral sequence of bigraded commutative algebras.
2. $E_r^{*,*}(X, Y)$ is a differential $E_r^{*,*}(\mathbb{S}, \mathbb{S})$-module.
3. If X is a suspension spectral and Y is a commutative ring spectrum, then $\{E_r^{/*,*}\}$ is a spectral sequence of bigraded commutative $\{E_r^{*,*}(\mathbb{S}, \mathbb{S})\}$-algebras whose product converges to the smash product on $[X, Y]$ defined by the diagonal map $\Delta \colon X \to X \wedge X$ and the product $\mu \colon Y \wedge Y \to Y$.

2.3.4 Extended powers in the Adams spectral sequence

Suppose Y is an H_∞-ring spectrum, we let

$$\xi \colon D_r Y \to Y$$

be its extended power.

Let $\pi \subset \Sigma_r$ and let $E\pi_n$ be the n-skeleton of a contractible π-free CW-complex $E\pi$. We assume $W_0 = \pi$. Then we let $D_\pi^i Y := ((E\pi_i)_+ \wedge Y^r)/\pi$, which is a subcomplex of $D_\pi Y$. This construction induces a filtration of $D_\pi Y$.

$$D_\pi^0 Y \subset D_\pi^1 Y \subset D^2\pi Y \subset \cdots \subset D_\pi Y$$

Now we let E be a ring spectrum satisfying Condition 2.3.23 and (π_*E, E_*E) forms a Hopf algebroid. Let

$$Y \simeq Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \ldots$$

be an Adams resolution with respect to E. Then we let

$$F_s = (Y_s)^r$$

$$Z = D_\pi Y_0 = ((E\pi)_+ \wedge Y_0^r)/\pi$$

$$Z_{i,s} = ((E\pi_i)_+ \wedge F_s)/\pi$$

Lemma 2.3.32 Let $B_i = E\pi_i/\pi$.

1. $Z_{i-1,s}$ and $Z_{i,s+1}$ are subcomplex of $Z_{i,s}$.
2. $\frac{Z_{i,s}}{Z_{i-1,s}} \simeq \frac{B_i}{B_{i-1}} \wedge F_s$.
3. $\frac{Z_{i,s}}{Z_{i-1,s} \cup Z_{i,s+1}} \simeq \frac{B_i}{B_{i-1}} \wedge \frac{F_s}{F_{s+1}}$

4. The following diagram commutes.

$$\begin{tikzcd}
\frac{Z_{i,s}}{Z_{i-1,s}\cup Z_{i,s+1}} \arrow[r, "\sim"] \arrow[d, "\partial"] & \frac{B_i}{B_{i-1}} \wedge \frac{F_s}{F_{s-1}} \arrow[d, "\partial\wedge \mathrm{id}\vee \mathrm{id}\wedge\partial"] \\
\frac{Z_{i-1,s}}{Z_{i-2,s}\cup Z_{i-1,s+1}} \vee \frac{Z_{i,s+1}}{Z_{i-1,s+1}\cup Z_{i,s+2}} \arrow[r, "\sim"] & \frac{B_{i-1}}{B_{i-2}} \wedge \frac{F_s}{F_{s+1}} \wedge \frac{B_i}{B_{i-1}} \wedge \frac{F_{s+1}}{F_{s+2}}
\end{tikzcd}$$

Proof. See [BMMS86, Chapter IV, Lemma 5.1]. □

Theorem 2.3.33 If E_*Y_s is π_*E-projective for each s, then there exists maps $\xi_{i,s} : Z_{i,s} \to Y_{s-i}$ such that the following diagrams commute

$$\begin{tikzcd}
D_\pi Y \arrow[d, "\xi"'] & Z_{i,s} \arrow[l] \arrow[d, "\xi_{i,s}"] \\
Y & Y_{s-i} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
Z_{i,s-1} \arrow[d, "\xi_{i,s-1}"] & Z_{i,s} \arrow[l] \arrow[d, "\xi_{i,s}"] & Z_{i-1,s} \arrow[l] \arrow[d, "\xi_{i-1,s}"] \\
Y_{s-i-1} & Y_{s-i} \arrow[l] & Y_{s-i+1} \arrow[l]
\end{tikzcd}$$

Proof. See [BMMS86, Chapter IV, Theorem 5.2]. □

Remark 2.3.34 The mix $\{Z_{i,s}\}$ of the skeleton filtration on $E\pi$ and the Adams resolution of Y together with $\{\xi_{i,s}\}$ is the “resolution" of $\xi : D_\pi Y \to Y$. We will see how $\{\xi_{i,s}\}$ “converge" to ξ along a generalized Adams spectral sequence later (see Theorem 2.3.38).

Let $W_k = \pi_k(E\pi_k/E\pi_{k-1})$ and $d : W_k \to W_{k-1}$ be the map induced by the geometric boundary map. Then we have a $\mathbb{Z}[\pi]$-resolution of $\mathbb{Z}$ with $W_0 = \mathbb{Z}[\pi]$. Let $C_{s,t} = E_{t-s}Y_{s,1}$. Then

$$0 \longrightarrow C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots$$

is the resolution of E_*Y associated to the Adams resolution. (Here the index i of C_i is the total degree of elements in $\bigoplus_{s,t} C_{s,t}$.) Note that $C_{s,t}$ coincides with $E_1^{s,t}(\mathbb{S}, Y)$, and $C_{s,t} = C_{s,t}(\pi_*E, E_*E, E_*Y)$.

Note that if each E_*Y_s is π_*E-projective, the Kunneth homomorphism is an isomorphism from C^r to the resolution associated to $\{F_s\}$. Let $h_E : \pi_* \to E_*$ be the Hurewicz homomorphism, κ the Kunneth homomorphism.

Corollary 2.3.35 If $\pi_0 E = \mathbb{Z}/p$ and the chain map $\Phi' : W \otimes_k C^r \to C$ is defined to make the following diagram commute

$$\begin{tikzcd}
W_k \otimes (\bigotimes_i C_{s_i,t_i}) \arrow[r, "h_E\otimes \mathrm{id}"] \arrow[dd, dashed, "\Phi'"'] & E_k(E\pi_k/E\pi_{k-1}) \otimes (\bigotimes_i E_{t_i-s_i}Y_{s_i,1}) \arrow[d, "\kappa"] \\
& E_{t-s+k}(E\pi_k/E\pi_{k-1} \wedge F_{s,1}) \arrow[d, "\xi_{k,s}"] \\
C_{s-k,t} \arrow[r, equal] & E_{t-s+k}Y_{s-k,1}
\end{tikzcd}$$

where $t = t_1 + \cdots + t_r$ and $s = s_1 + \cdots + s_r$, then Φ' is chain homotopic to Φ in Proposition 2.3.12 (Here we let $\Gamma = E_*E$, $M = E_*Y$, and $N = \pi_*E$).

Corollary 2.3.36 Suppose X is a spectrum with a coproduct $\Delta\colon X \to X \wedge X$ and E_*X is π_*E-projective. Let $e \in W_k$ and $f_j \in [X, Y_{s_j,1}]_{t_j-s_j}$, then $\Phi_*(e \otimes f_{1*} \otimes \cdots \otimes f_{r*})$ is represented by the composite

$$\begin{tikzcd}
\Sigma^{t-s+k}X \arrow[d, "\Sigma^{t-s+k}\Delta^r"] \arrow[rr, dashed] & & Y_{s-k,1} \\
\Sigma^{t-s+k}X^r \arrow[r] & \Sigma^k(\bigwedge_j \Sigma^{t_j-s_j}X) \arrow[r, "e\wedge(\bigwedge_j f_j)"] & E\pi_k/E\pi_{k-1} \wedge (\bigwedge_j Y_{s_j,1}) \arrow[u, "\xi_{k,s}"']
\end{tikzcd}$$

Remark 2.3.37 With the extended powers internalized by H_∞-structures, the total power operations can be write down explicitly. Furthermore, we can use them to study differentials of the form $d_r\beta^\epsilon P^i x$ and related homotopy operations.

2.3.5 *How power operations detect homotopy operations*

Theorem 2.3.38 Suppose E is a commutative ring spectrum such that (π_*E, E_*E) is a Hopf algebroid which satisfies Condition 2.3.23. Let

$$\mathcal{Z} := (Z = Z_0 \xleftarrow{f_0} Z_1 \xleftarrow{f_1} Z_2 \xleftarrow{f_2} \cdots)$$

be an inverse sequence such that E_*Z_i is π_*E-projective and E_*f_i is a π_*E-split monomorphism for each i. Then

1. there exists a spectral sequence $E_*^{*,*}(X, \mathcal{Z})$, natural with respect to maps of such sequences, such that

$$E_2^{s,t}(X, \mathcal{Z}) = \bigoplus_i E_2^{s-i,t-i}(X, Cf_i)$$

where $E_*^{*,*}(X, Cf_i)$ is the Adams spectral sequence converging to $[X, Cf_i]$ (recall Theorem 2.3.28);

2. if E_*Y' is π_*E-projective and we let

$$\mathcal{Z} \wedge Y' := (Z \wedge Y' = Z_0 \wedge Y' \xleftarrow{f_0\wedge \mathrm{id}} Z_1 \wedge Y' \xleftarrow{f_1\wedge \mathrm{id}} Z_2 \wedge Y' \xleftarrow{f_2\wedge \mathrm{id}} \cdots)$$

there is a pairing

$$\begin{tikzcd}
E_r^{*,*}(X, \mathcal{Z}) \otimes E_r^{*,*}(X', Y') \arrow[r] \arrow[d, Rightarrow] & E_r^{*,*}(X \wedge X', \mathcal{Z} \wedge Y') \arrow[d, Rightarrow] \\
{[X, Z]_*^E \otimes [X', Y']_*^E} \arrow[r] & {[X \wedge X', Z \wedge Y']_*^E}
\end{tikzcd}$$

3. if

$$\begin{tikzcd} & Z_0 \arrow[d, "c_0"'] & Z_1 \arrow[l, "f_0"'] \arrow[d, "c_1"'] & \cdots \arrow[l, "f_1"'] \\ Y \arrow[r, equal] & Y_0 & Y_1 \arrow[l, "i_0"'] & \cdots \arrow[l, "i_1"'] \end{tikzcd}$$

is an inverse-sequence morphism from $\mathcal{Z}$ to an Adams resolution of Y, then there is a homomorphism c of spectral sequences

$$\begin{tikzcd} E_r^{*,*}(X,\mathcal{Z}) \arrow[r, Rightarrow] \arrow[d, "c"'] & {[X,Z]_*^E} \arrow[d, "c_{0*}"] \\ E_r^{*,*}(X,Y) \arrow[r, Rightarrow] & {[X,Y]_*^E} \end{tikzcd}$$

which maps the pairing in (2) to the smash product pairing

$$\begin{tikzcd} E_r^{*,*}(X,\mathcal{Z}) \otimes E_r^{*,*}(X',Y') \arrow[r] \arrow[d, "c\otimes \mathrm{id}"'] & E_r^{*,*}(X\wedge X', \mathcal{Z}\wedge Y') \arrow[d, "c"] \\ E_r^{*,*}(X,Y) \otimes E_r^{*,*}(X',Y') \arrow[r] & E_r^{*,*}(X\wedge X', Y\wedge Y') \end{tikzcd}$$

4. the spectral sequence $E_r^{*,*}(X,\mathcal{Z})$ converges to $[X,Z]_*^E$ if E and Z satisfies the Adams condition in Remark 2.3.29 and $E_*(\mathrm{Mic}\,\mathcal{Z}) = 0$, where $\mathrm{Mic}\,\mathcal{Z}$ is the microscope, or homotopy limit of the inverse sequence $\mathcal{Z}$.

Proof. See [BMMS86, Chapter IV, Section 6]. □

Let $x \in \pi_n(Y)$ be detected by $\bar{x} \in E_2^{s,n+s}(\mathbb{S}, Y)$ (this means that the element $x : \mathbb{S}^n \to Y$ is detected by $\bar{x} : \mathbb{S}^n \to Y_s$ and the later one is an element in some E_*-homology group), the Adams spectral sequence with respect to a commutative ring spectrum E satisfying the condition in Theorem 2.3.38. Let Ξ be the sequence

$$\Xi = (D_\pi^{ps}\mathbb{S}^n \longleftarrow D_\pi^{ps-1}\mathbb{S}^n \longleftarrow D_\pi^{ps-2}\mathbb{S}^n \longleftarrow \cdots \longleftarrow D_\pi^{1}\mathbb{S}^n \longleftarrow \mathbb{S}^{np})$$

where $\pi = C_p$ and $D_\pi^i\mathbb{S}^n = ((W_i)_+ \wedge \mathbb{S}^{np})/\pi$ is the extended power of $\mathbb{S}^n$ based on the i-skeleton W_i of the standard free π-CW-complex, i.e. the universal cover of the mod-p lens space where $W_{2i-1} = S^{2i-1}$. By Theorem 2.3.33, if E_*Y_j is π_*E-projective, then we have a morphism from Ξ to the canonical Adams resolution of Y.

$$\begin{tikzcd} D_\pi^{ps}\mathbb{S}^n \arrow[d, "D_\pi\bar{x}"] & D_\pi^{ps-1}\mathbb{S}^n \arrow[l] \arrow[d, "D_\pi\bar{x}"] & \cdots \arrow[l] & D_\pi^{1}\mathbb{S}^n \arrow[l] \arrow[d, "D_\pi\bar{x}"] & \mathbb{S}^{np} \arrow[l] \arrow[d, "D_\pi\bar{x}"] \\ D_\pi^{ps}Y_s \arrow[d, "\xi_{ps,ps}"] & D_\pi^{ps-1}Y_s \arrow[l] \arrow[d, "\xi_{ps-1,ps}"] & \cdots \arrow[l] & D_\pi^{1}Y_s \arrow[l] \arrow[d, "\xi_{1,\xi}"] & Y_s^p \arrow[l] \arrow[d, "\xi_{0,ps}"] \\ Y_0 & Y_1 \arrow[l] & \cdots \arrow[l] & Y_{ps-1} \arrow[l] & Y_{ps} \arrow[l] \end{tikzcd}$$

By Theorem 2.3.38 3, we have a homomorphism

$$\mathcal{P}(x)\colon E_r^{*,*}(\mathbb{S},\Xi) \to E_r^{*,*}(\mathbb{S},Y)$$

of spectral sequences (here we assume the domain spectral sequences exists). Similarly, we have compatible maps

$$D^1_\pi \mathbb{S}^n \wedge Y \to Y_{ps-i}$$

and a homomorphism

$$\mathcal{P}(x)\colon E_r^{*,*}(\mathbb{S}, \Xi \wedge Y) \to E_r^{*,*}(\mathbb{S}, Y)$$

Proposition 2.3.39 If $E_* D^{i-1}_\pi \mathbb{S}^n \to E_* D^i_\pi \mathbb{S}^n$ is a $\pi_* E$-split monomorphism for each $i \geqslant ps$, then the spectral sequence $E_r^{*,*}(\mathbb{S}, \Xi)$ exists and $E_2(\mathbb{S}, \Xi)$ is free over $E_2(\mathbb{S}, \mathbb{S})$ on generators $e_i \in E_2^{ps-i, ps+pn}(D^{ps}_\pi \mathbb{S}^n, \Xi)$. Similarly, $E_2(\mathbb{S}^n, \Xi \wedge Y)$ is free over $E_2(\mathbb{S}, Y)$ on the image of the e_i under the map induced by the unit $\mathbb{S} \to Y$.

Proof. See [BMMS86, Chapter IV, Proposition 7.5]. □

Remark 2.3.40 We may take e_i as the $np + i$-cell of $D_\pi \mathbb{S}^n$ (the smash product among p copyies of S^n and the unique i-cell of W).

Theorem 2.3.41 Suppose the hypothesis of Proposition 2.3.39 holds and $E_* Y$ is $\pi_* E$-projective. Then $\mathcal{P}(x)$ sends e_i to $\Phi_*(e_i \otimes \bar{x}^p)$.

Remark 2.3.42 If $p = 2$, then $\mathcal{P}(x)$ sends e_i to $Sq^{i+n}(\bar{x})$. If p is an odd prime, then $\mathcal{P}(x)$ sends $(-1)\nu(n) e_i$ to $\beta^\varepsilon P^j \bar{x}$ if $i = (2j - n)(p-1) - \varepsilon$. $\mathcal{P}(x)$ sends elements to 0 if i is not of this form.

Definition 2.3.43 (Homotopy operations) Suppose Y is an H_∞-ring spectrum. Given $\alpha \in Y_m(D_{j_1} \mathbb{S}^{n_1} \wedge \cdots \wedge D_{j_k} \mathbb{S}^{n_k})$, the associated *homotopy operation*

$$\alpha^*\colon \pi_{n_1} Y \times \cdots \times \pi_{n_k} Y \to \pi_m(Y)$$

is defined by sending $f_1 \times \cdots \times f_k \in \pi_{n_1} Y \times \cdots \times \pi_{n_k} Y$ to the composite

$$\mathbb{S}^m \xrightarrow{\alpha} D_{j_1} \mathbb{S}^{n_1} \wedge \cdots \wedge D_{j_k} \mathbb{S}^{n_k} \wedge Y \xrightarrow{D_{j_1} f_1 \wedge \cdots \wedge D_{j_k} f_k \wedge \mathrm{id}} D_{j_1} Y \wedge \cdots \wedge D_{j_k} Y \wedge Y \xrightarrow{\xi} Y$$

Now we show how Steenrod operations on the E_2-page detect homotopy operations. If we assume

$$E_r^{*,*}(\mathbb{S}, \Xi) \Rightarrow \pi_* D^{ps}_\pi \mathbb{S}^n$$

then any $\alpha \in \pi_* D^{ps}_\pi \mathbb{S}^n$ can be detected by an element $\sum a_k e_k \in E_2(\mathbb{S}, \Xi)$, where $a_k \in E_2(\mathbb{S}, \mathbb{S})$. Applying $\mathcal{P}(x)$, we see that $\alpha^*(x)$ is detected by $\sum a_k \Phi_*(e_k \otimes \bar{x}^p) \in E_2(\mathbb{S}, \mathbb{S})$. Similarly, if $E_r^{*,*}(\mathbb{S}, \Xi \wedge Y)$ converges to $Y_* D^{ps}_\pi \mathbb{S}^n$, any $\alpha \in Y_* D^{ps}_\pi \mathbb{S}^n$ is detected by $\sum a_k e_k \in E_2(\mathbb{S}, \Xi \wedge Y)$, where $a_k \in E_2(\mathbb{S}, Y)$, then $\alpha^*(x)$ is detected by $\mathcal{P}(x)$.

2.4 Motivic homotopy theory

In this subsection, we will give a brief introduction to motivic homotopy theory. We mainly follow Morel and Voevodsky's method [MV99]. Roughly speaking, this method has two stages as follows.

1. Consider the injective model structures on the category of simplicial sheaves (or sheaves of spectra) over the Nisnevich site. In this stage, we set up the foundation of homotopy theory by using simplicial homotopy theory and address the related descent problem by choosing the Nisnevich topology. The resulting homotopy category in this stage is called the simplicial homotopy category of schemes.

2. Based on the simplicial homotopy category, we invert all the natural projection $X \times \mathbb{A}^1 \to X$ by Bousfield localization, which exhibits $\mathbb{A}^1$ as an interval. The resulting homotopy category is the desired motivic homotopy category.

However, an issue in this construction is that not all objects are fibrant. Therefore, we need a functorial fibrant replacement. The construction of such functors is sketched in Subsubsection 2.4.2. Actually, the construction of a fibrant resolution functor is very complicated, on which Morel and Voevodsky spent dozens of pages in [MV99]. Furthermore, we discuss other models for motivic homotopy theory in Subsubsection 2.4.6. In particular, the fibrant issue will be more manageable, if we use projective model structure on the category of simplicial presheaves.

2.4.1 The unstable motivic homotopy category

Let S be a regular separated Noetherian scheme of finite Krull dimension. Let Sm/S be the category of smooth S-schemes of finite type. Let $s\mathcal{P}re(Sm/S)$ be the category of simplicial presheaves over Sm/S, which acts as a platform to build homotopy theory of schemes.

Example 2.4.1 By Yoneda embedding theorem, $X \in Sm/S$ can be embedding in $s\mathcal{P}re(Sm/S)$ as the presheaf $h_X : Z \mapsto \mathrm{Hom}_{Sm/S}(Y, X)$, where $\mathrm{Hom}_{Sm/S}(Y, X)$ is a discrete simplicial set.

Let $U \hookrightarrow X$ be an open embedding, then X/U is defined to be the quotient presheaf h_X/h_Z.

Example 2.4.2 Given a simplicial set Y, then Y can be taken as a constant presheaf valued at Y on Sm/S.

The first modification on $s\mathcal{P}re(Sm/S)$ is about Grothendieck topologies. We need to choose a suitable topology T such that it satisfies the following properties.

- **Property A:** For any $X \in Sm/S$, the cohomological dimension with respect to T of X is the same as the Krull dimension of X.

- **Property B:** If $i\colon Z \hookrightarrow X$ is a closed embedding, then i is T-locally of the form $\mathbb{A}_S^{\dim Z} \hookrightarrow \mathbb{A}_S^{\dim X}$. (This properties implies cohomology purity.)

- **Property C:** For any $X \in Sm/S$, $\mathcal{K}(X)$ should satisfy T-descent.

Now we recall some candidates for such properties.

Example 2.4.3 Let X be an object in Sm/S.

An *Zariski covering* on X is a family of open embedding $\{p_i : U_i \to X\}$ such that $\cup f_i(U_i) = X$. The Grothendieck topology generated by Zariski coverings is exactly the Zariski topology.

An *étale covering* on X is a family of finite étale morphisms $\{p_i \colon U_i \to X\}$ such that $\cup f_i(U_i) = X$. The Grothendieck topology generated by étale coverings is called the étale topology.

However, neither Zariski topology or étale topology satisfies all of these there properties.

Properties	Zariski topology	Étale topology
Property A	Yes	No
Property B	No	Yes
Property C	Yes	No

Definition 2.4.4 A *Nisnevich covering* on X is an étale covering $\{p_i : U_i \to X\}$ such that for any $x \in X$, there exists an $p_i : U_i \to X$ and $y \in U_i$ such that $p_i(y) = x$ and $p_{i*} : \kappa(y) \to \kappa(x)$ is an isomorphism, where $\kappa(x)$ is the residue field of $x \in X$. The Grothendieck topology generated by Nisnevich covering is called the *Nisnevich topology*.

Proposition 2.4.5 The Nisnevich topology satisfies all of the mentioned properties.

Remark 2.4.6 The Nisnevich topology is finer than the Zariski topology and coarser than the étale topology.

Proposition 2.4.7 The Nisnevich topology is generated by the squares of the following form.

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where p is an étale morphism and i an open embedding such that $p^{-1}(X - i(U)) \xrightarrow{p} X - i(U)$ is an isomorphism on the reduced locus. Such diagram is called *elementary distinguished squares*.

Corollary 2.4.8 A presheaf $\mathcal{F}$ on Sm/S is a *Nisnevich sheaf* if it send any elementary distinguished square to a pull-back diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \times_X V) \end{tikzcd}$$

Proposition 2.4.9 Given any $X \in Sm/S$ and $x \in X$,

1. the stalk at x with respect to Zariski neighbourhood is $\mathcal{O}_{X,x}$,
2. the stalk at x with respect to Nisnevich neighbourhood is $\mathcal{O}^h_{X,x}$, the henselization of $\mathcal{O}_{X,x}$,
3. the stalk at x with respect to Nisnevich neighbourhood is $\mathcal{O}^{sh}_{X,x}$, the strict henselization of $\mathcal{O}_{X,x}$.

Next we model homotopy theory on Nisnevich sheaves.

Construction 2.4.10 Let $s\mathcal{S}hv_{Nis}(S)$ be the category of Nisnevich simplicial sheaves over Sm/S. Then there exists a simplicial closed model category with

1. $f\colon \mathcal{X} \to \mathcal{Y}$ is a weak equivalence if f is a stalkwise weak equivalence, i.e. for any $U \in \mathrm{Sm}/S$ and any $u \in U$,

$$f_*\colon \mathcal{X}(\mathrm{Spec}(\mathcal{O}_{U,u})) \to \mathcal{Y}(\mathrm{Spec}(\mathcal{O}_{U,u}))$$

   is a weak equivalence between simplicial sets,

2. f is a cofibration if f is a monomorphism,

3. f is a fibration if it has the right lifting property with respect to any acyclic cofibration. Such fibrations are called *global fibrations*.

The existence of such simplicial model category can be found in [MV99, Theorem 1.4]. Let $\mathcal{H}_s(S)$ be the resulting homotopy category. We denote this model category by $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)_{Joyal}$, because the original idea of this construction is given by Joyal [Joy84]. We denote

$$[\mathcal{X}, \mathcal{Y}]_s := \mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{X}, \mathcal{Y})$$

Given $\mathcal{X}, \mathcal{Y} \in s\mathcal{S}hv_{Nis}(S)$, the mapping simplicial set $\mathcal{S}(\mathcal{X} \to \mathcal{Y})$ is defined to be

$$\mathcal{S}(\mathcal{X}, \mathcal{Y})_n := \mathrm{Hom}_{s\mathcal{S}hv_{Nis}(S)}(\mathcal{X} \times \Delta^n \to \mathcal{Y})$$

Suppose $f, g\colon \mathcal{X} \to \mathcal{Y}$ are maps in $\mathcal{S}(\mathcal{X}, \mathcal{Y})_0$, a *simplicial homotopy* from f to g is an 1-simplex in $S(\mathcal{X}, \mathcal{Y})_1$ with boundary $\{f, g\}$. $f\colon \mathcal{X} \to \mathcal{Y}$ is a *simplicial homotopy equivalence* if there exists $q\colon \mathcal{Y} \to \mathcal{X}$ such that $q \circ f$ and $f \circ q$ are simplicially homotopic $\mathrm{id}_{\mathcal{X}}$ and $\mathrm{id}_{\mathcal{Y}}$ respectively.

The *internal function sheaf* $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ is defined to be right adjoint $\mathcal{Y} \mapsto \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ of the functor

$$\mathcal{Z} \mapsto \mathcal{Z} \times \mathcal{X}$$

Let $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)_\bullet$ be the pointed model category obtained from $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)_{Joyal}$ and $\mathcal{H}_{s,\bullet}(S)$ be the resulting pointed homotopy category. We also denote

$$[\mathcal{X}, \mathcal{Y}]_{s,\bullet} := \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(S)}(\mathcal{X}, \mathcal{Y})$$

Remark 2.4.11 Note that Δ^1 is a cylinder object in this model category. Then if $\mathcal{X}$ and $\mathcal{Y}$ are cofibrant and fibrant, we have

$$\pi_0 S(\mathcal{X}, \mathcal{Y}) \cong [\mathcal{X}, \mathcal{Y}]_s$$

Definition 2.4.12 Given a simplicial presheaf $\mathcal{X}$, let $\pi_n\mathcal{X} = \coprod_{x \in X_0} \pi_n(X, x)$ be its n^{th}-homotopy presheaf for $n > 0$ and let $\tilde{\pi}_n\mathcal{X}$ be the associated sheaf (with respect to the Nisnevich topology) of $\pi_n(X)$. Similarly, $\tilde{\pi}_0\mathcal{X}$ is the sheafification of $\pi_0\mathcal{X}$ with respect to the Nisnevich topology. $f\colon \mathcal{X} \to \mathcal{Y}$ is a *local weak equivalence* if

1. the map $\tilde{\pi}_0(\mathcal{X}) \to \tilde{\pi}_0(\mathcal{Y})$ is an bijection of sheaves.

2. the diagram of sheaf morphisms

$$\begin{tikzcd} \tilde{\pi}_n\mathcal{X} \arrow[r] \arrow[d] & \tilde{\pi}_n\mathcal{Y} \arrow[d] \\ \tilde{\mathcal{X}}_0 \arrow[r] & \tilde{\mathcal{Y}}_0 \end{tikzcd}$$

is a pullback for all $n \geqslant 1$.

Remark 2.4.13 In short, a morphism is a local weak equivalence if and only if it induces an isomorphism in all possible sheaves of homotopy group at all local choices of base points.

Lemma 2.4.14 A morphism between simplicial sheaves is a local weak equivalence if and only if it is a stalkwise weak equivalence.

Proof. See [MV99, Section 3, Lemma 1.11]. □

Remark 2.4.15 Note that $f: \mathcal{X} \to \mathcal{Y}$ is a sectionwise weak equivalence if for any $U \in Sm/S$, $f: \mathcal{X}(U) \to \mathcal{Y}(U)$ is a weak equivalence. A sectionwise equivalence is a stalkwise equivalence. However, a stalkwise equivalence may not be a sectionwise weak equivalence. That is because $\pi_n(\mathcal{X})$ may not be a sheaf.

Proposition 2.4.16 Given a global fibration $f: \mathcal{X} \to \mathcal{Y}$, the induced map

$$\mathcal{X}(U) \to \mathcal{Y}(U)$$

is a Kan fibration for any $U \in Sm/S$. In particular, if $\mathcal{X}$ is fibrant, then $\mathcal{X}(U)$ is a Kan complex.

Proof. See [Jar07, P10]. □

Remark 2.4.17 The converse proposition is false.

Proposition 2.4.18 A stalkwise weak equivalence $f: \mathcal{X} \to \mathcal{Y}$ between fibrant simplicial sheaves is a simplicial homotopy equivalence (an isomorphism in $\pi_0 S(\mathcal{X}, \mathcal{Y})$). Moreover, f is a sectionwise weak equivalence.

Proof. See [MV99, Section 2, Lemma 1.10]. □

Corollary 2.4.19 Let $(\mathcal{X}, x)$ be a pointed fibrant simplicial sheaf. Then for any $U \in Sm/S$ and any non-negative integer n, we have

$$\pi_n(\mathcal{X}(U)) \cong [S^n \wedge U_+, (\mathcal{X}, x)]_{s,\bullet}$$

Remark 2.4.20 Roughly speaking, the slogan is that fibrant objects are representable in the corresponding homotopy category.

Remark 2.4.21 There exists a proper closed simplicial model category on $s\mathcal{P}re((Sm/S)_{Nis})$ such that

1. $f: \mathcal{X} \to \mathcal{Y}$ is a weak equivalence if and only if f is a local weak equivalence.

2. f is a cofibration if and only if f is a monomorphism.

3. f is a fibration if it has the right lifting property with respect to acyclic cofibration in previous sense. Such fibrations are called *global fibrations*.

The proof can be found in [Jar87] and we denote this model category by $s\mathcal{P}re((Sm/S)_{Nis})_{Jardine}$. Let $i\colon s\mathcal{P}re((Sm/S)_{Nis}) \to s\mathcal{S}hv_{Nis}(Sm/S)$ be the inclusion of the full subcategory. The model structure on $s\mathcal{S}hv_{Nis}(Sm/S)$ inherited from $s\mathcal{P}re((Sm/S)_{Nis})_{Jardine}$ is exactly $s\mathcal{S}hv_{Nis}(Sm/S)_{Joyal}$. Furthermore, the inclusion $i : s\mathcal{S}hv_{Nis}(Sm/S) \hookrightarrow s\mathcal{P}re((Sm/S)_{Nis})_{Jardine}$ and the sheafification functor $\mathscr{L}_{Nis}\colon s\mathcal{P}re((Sm/S)_{Nis})_{Jardine} \to s\mathcal{S}hv_{Nis}(Sm/S)_{Joyal}$ form a pair of Quillen equivalences [Jar07, Theorem 5]. Therefore, we are free to use the shefification functor without worrying about the issues about model categories.

Remark 2.4.22 Since the right adjoint functor $\mathscr{L}_{Nis}$ preserves fibrations, the characterization of fibrant object in $s\mathcal{P}re((Sm/S)_{Nis})_{Jardine}$ (see [DHI04, Theorem 1.1]) is the same as the one in $s\mathcal{S}hv_{Nis}(Sm/S)_{Joyal}$.

The next modification on simplicial presheaves is to localize $\mathbb{A}^1$. Intuitively speaking, we need to make $\mathbb{A}^1$ contractible and play a role as an interval to parametrize "homotopies".

Construction 2.4.23 $\mathcal{X} \in s\mathcal{P}re(Sm/S)$ is $\mathbb{A}^1$*-local*, if the projection $\mathcal{Y} \times \mathbb{A}^1 \to \mathcal{Y}$ induces a bijection $\mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y}, \mathcal{X}) \to \mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$ for any $\mathcal{Y} \in Sm/S$. A morphism $\mathcal{Y} \to \mathcal{Z}$ is an $\mathbb{A}^1$*-weak equivalence* if, for any $\mathbb{A}^1$-local, simplicially fibrant sheaf $\mathcal{X}$, the induced map $S(\mathcal{Z}, \mathcal{X}) \to S(\mathcal{Y}, \mathcal{X})$ is a weak equivalence between simplicial sets. Note that there exists a proper model category structure on $s\mathcal{S}hv_{Nis}(Sm/S)$ such that

1. $f\colon \mathcal{Y} \to \mathcal{X}$ is a weak equivalence if f is an $\mathbb{A}^1$-weak equivalence,
2. f is a cofibration if f is a monomorphism,
3. f is fibration if it has the right lifting property with respect to any acyclic $\mathbb{A}^1$-cofibration. Such fibrations are called $\mathbb{A}^1$*-fibrations*.

The existence is proved in [MV99, Theorem 2.5, Theorem 2.7] and we denote this model category by $s\mathcal{S}hv_{Nis}(Sm/S)_{\mathbb{A}^1}$ and the resulting homotopy category is the *motivic homotopy category (or say* $\mathbb{A}^1$*-homotopy category)* denoted by $\mathcal{H}(S)$. The associated pointed homotopy category is denoted by $\mathcal{H}_\bullet(S)$. We denote

$$[\mathcal{X}, \mathcal{Y}]^{\mathbb{A}^1} := \mathrm{Hom}_{\mathcal{H}(S)}(\mathcal{X}, \mathcal{Y})$$

$$[\mathcal{X}, \mathcal{Y}]^{\mathbb{A}^1}_\bullet := \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathcal{X}, \mathcal{Y})$$

Let $\mathcal{H}_{s,\mathbb{A}^1}(S) \subset \mathcal{H}_s(S)$ be the full subcategory of $\mathbb{A}^1$-local simplicial sheaves. The inclusion functor $\mathcal{H}_{s,\mathbb{A}^1}(S) \hookrightarrow \mathcal{H}_s(S)$ has a left adjoint $L_{\mathbb{A}^1}$ as $\mathbb{A}^1$-localization. Then we have

$$L_{\mathbb{A}^1}\mathcal{H}_s(S) \simeq \mathcal{H}(S)$$

Proposition 2.4.24 Let $\mathcal{X}$ be a fibrant simplicial sheaf in $s\mathcal{S}hv_{Nis}(Sm/S)_{Joyal}$. $\mathcal{X}$ is $\mathbb{A}^1$-local if and only if $\mathcal{X}$ is $\mathbb{A}^1$-local.

Proof. See [MV99, Section 2, Proposition 2.28]. □

Definition 2.4.25 Let $f, g\colon \mathcal{X} \to \mathcal{Y}$ be morphisms between simplicial sheaves. Let $i_0, i_1 : S \to \mathbb{A}^1_S$ be the embeddings corresponding to $\mathbb{Z}[x] \xrightarrow{x \mapsto 0} \mathbb{Z}$ and $\mathbb{Z}[x] \xrightarrow{x \mapsto 1} \mathbb{Z}$. A *naive* $\mathbb{A}^1$*-homotopy* from f to g is a morphism $H : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Two morphisms are $\mathbb{A}^1$*-homotopic* if they are connected by a sequence of naive $\mathbb{A}^1$-homotopies. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is called a *strict* $\mathbb{A}^1$*-homotopy equivalence* if there is a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ such that $f \circ g$ and $g \circ f$ are $\mathbb{A}^1$-homotopic to $\mathrm{id}_{\mathcal{X}}$ and $\mathrm{id}_{\mathcal{Y}}$ respectively.

Proposition 2.4.26 Any strict $\mathbb{A}^1$-homotopy equivalence is an $\mathbb{A}^1$-weak equivalence.

Proof. See [MV99, Section 2, Lemma 3.6]. □

Proposition 2.4.27 Suppose $(\mathcal{X}, x)$ is an $\mathbb{A}^1$-invariant and fibrant pointed simplicial sheaf. Then for any $U \in \mathrm{Sm}/S$ and any non-negative integer n, we have

$$\pi_n(\mathcal{X}(U)) = [S^n \wedge U_+, (\mathscr{L}_{Nis}\mathcal{X}, x))]^{\mathbb{A}^1}_{\bullet}$$

2.4.2 Fibrant resolution functors

Since every object in $s\mathcal{S}hv_{Nis}(S)$ is cofibrant, it is desired to characterize fibrant objects. Furthermore, we have shown that the homotopy sheaf of a fibrant simplicial sheaf is representable in the corresponding homotopy category. In this subsection, we will construct fibrant resolution functors to replace objects by fibrant objects functorially. First, we define and construct fibrant resolution functors in $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)_{Joyal}$. Secondly, we proceed the same procedure in $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)_{\mathbb{A}^1}$ in terms of previously resulting functors.

Definition 2.4.28 A *fibrant resolution functor* on $s\mathcal{S}hv_{Nis}(S)$ is a pair (Ex, θ) consisting of a functor $Ex\colon s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S) \to s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)$ and a natural transformation $\theta\colon \mathrm{id} \to Ex$ such that for any $\mathcal{X}$, the object $Ex(\mathcal{X})$ is fibrant and the morphism $\mathcal{X} \to Ex\mathcal{X}$ is a trivial cofibration.

Theorem 2.4.29 There exists a resolution functor on $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)$.

Proof. A construction is given by the composition of Godement resolution, Nisnevich sheafification and Ex^∞, where Ex^∞ is a fibrant resolution in Joyal's model category of simplicial sets. See [MV99, Section 2, Theorem 1.66]. □

Definition 2.4.30 A simplicial presheaf $\mathcal{X}$ is $\mathbb{A}^1$*-invariant* if $\mathcal{X}(U) \to \mathcal{X}(U \times \mathbb{A}^1)$ is a weak equivalence of simplicial sets for any $U \in \mathrm{Sm}/S$.

Proposition 2.4.31 Let $\mathcal{X}$ be a fibrant simplicial sheaf. Then $\mathcal{X}$ is $\mathbb{A}^1$-invariant if and only if $\mathcal{X}$ is $\mathbb{A}^1$-local.

Proof. See [MV99, Section 2, Proposition 3.19]. □

Construction 2.4.32 Given a commutative ring R, we can construct a cosimplicial scheme by setting

$$\Delta^n_R := \mathrm{Spec} R[x_0, \cdots, x_n]/(x_0 + \cdots + x_n = 1)$$

For the scheme S, we define a cosimplicial object Δ_S^* in Sm/S by setting

$$\Delta_S^n := S \times_{\mathbb{Z}} \Delta_S^n$$

Given a simplicial presheaf $\mathcal{X}$, we define a simplicial presheaf $\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{X}$ to be

$$\mathrm{Sing}_n^{\mathbb{A}^1}\mathcal{X} := \underline{\mathrm{Hom}}(\Delta_S^n, \mathcal{X}_n)$$

In other words, $\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{X}$ is the diagonal of the bisimplicial presheaf $\mathcal{X}(- \times \Delta_S^*)$.

Proposition 2.4.33 The $\mathrm{Sing}^{\mathbb{A}^1}$-functor has the following properties.

1. Let $f, g \colon \mathcal{X} \to \mathcal{Y}$ be two morphisms and H be a naive $\mathbb{A}^1$-homotopy from f to g. Then there exists a simplicial homotopy from $\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{X} \to \mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{Y}$.
2. For any simplicial sheaf $\mathcal{X}$, the canonical $\mathcal{X} \mapsto \mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{X}$ is an $\mathbb{A}^1$-weak equivalence.
3. $\mathrm{Sing}_*^{\mathbb{A}^1}$ preserves $\mathbb{A}^1$-fibrations.
4. For any simplicial sheaf $\mathcal{X}$, $\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{X}$ is $\mathbb{A}^1$-invariant.

Proof. See [MV99, P89]. □

Remark 2.4.34 The first assertion in Proposition 2.4.33 demonstrates that the $\mathrm{Sing}_*^{\mathbb{A}^1}$-functor converts $\mathbb{A}^1$-homotopies into simplicial homotopies. In this way, $\mathbb{A}^1$-homotopies are encoded by simplicial homotopies via $\mathrm{Sing}_*^{\mathbb{A}^1}$, and the homotopy theory with respect to simplicial homotopies is easier to manipulate. This is the magic of the $\mathrm{Sing}_*^{\mathbb{A}^1}$-functor.

Remark 2.4.35 It is not true that for any simplicial sheaf F, $\mathrm{Sing}_*^{\mathbb{A}^1}F$ is $\mathbb{A}^1$-local. Recall that if F is simplicially fibrant, $\mathrm{Sing}_*^{\mathbb{A}^1}F$ is $\mathbb{A}$-local according to 2.4.31. Based this observation, the issue happens when F is not global fibrant. A concrete example can be found in [MV99, Section 3, Example 2.7]. Therefore, we need to modify $\mathrm{Sing}_*^{\mathbb{A}^1}$ to address the simplicially fibrant issue.

Construction 2.4.36 ($\mathbb{A}^1$-fibrant resolution functor) Let (Ex, θ) be a fibrant resolution functor in the sense of Definition 2.4.28. Then we define

$$Ex_{\mathbb{A}^1} = Ex \circ (Ex \circ \mathrm{Sing}_*^{\mathbb{A}^1})^{\mathbb{N}} \circ Ex$$

which is called an $\mathbb{A}^1$-fibrant resolution functor.

Proposition 2.4.37 For any simplicial sheaf $\mathcal{X}$, $Ex_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-fibrant.

Proof. See [MV99, Section 3, Lemma 2.6]. □

Therefore, we may take $Ex_{\mathbb{A}^1}$ as an explicit presentation of $L_{\mathbb{A}^1}$.

Definition 2.4.38 For any pointed simplicial sheaf $(\mathcal{X}, x)$ and any $i \geqslant 0$, we define a presheaf of i^{th} $\mathbb{A}^1$-*homotopy groups*

$$\underline{\pi}_i^{\mathbb{A}^1}(\mathcal{X}, x)(U) = \pi_i(Ex_{\mathbb{A}^1}(\mathcal{X})(U), x)$$

Recall that $\tilde{\underline{\pi}}_i^{\mathbb{A}^1}(\mathcal{X}, x)$ is the sheaf associated to the presheaf $\underline{\pi}_i^{\mathbb{A}^1}(\mathcal{X}, x)$. $\mathcal{X}$ is said to be $\mathbb{A}^1$-*connected* if $\tilde{\underline{\pi}}_i^{\mathbb{A}^1}(\mathcal{X}, x)$ is the constant sheaf valued at a single point.

Theorem 2.4.39 Let $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ be a morphism of $\mathbb{A}^1$-connected pointed simplicial sheaves. Then the following assertions are equivalent:

1. f is an $\mathbb{A}^1$-weak equivalence;
2. for any $i \geqslant 0$, the morphism of the presheaves of $\mathbb{A}^1$-homotopy groups $\underline{\pi}_i^{\mathbb{A}^1}(\mathcal{X}, x) \to \underline{\pi}_i^{\mathbb{A}^1}(\mathcal{Y}, y)$ is an isomorphism;
3. for any $i \geqslant 0$, the morphism of the sheaves of $\mathbb{A}^1$-homotopy groups $\tilde{\underline{\pi}}_i^{\mathbb{A}^1}(\mathcal{X}, x) \to \tilde{\underline{\pi}}_i^{\mathbb{A}^1}(\mathcal{Y}, y)$ is an isomorphism.

This is an $\mathbb{A}^1$-version of Whitehead theorem.

Definition 2.4.40 A *motivic space* is an $\mathbb{A}^1$-fibrant object in $s\mathcal{S}hv_{\mathbb{A}^1}(Sm/S)_{\mathbb{A}^1}$.

Remark 2.4.41 Since we have an $\mathbb{A}^1$-fibrant resolution functor $Ex_{\mathbb{A}^1}$, we can identify each scheme in Sm/S as a motivic space via $Ex_{\mathbb{A}^1}$.

2.4.3 Basic in the unstable motivic homotopy category

Given $(\mathcal{X}, x), (\mathcal{Y}, y) \in \mathcal{H}_\bullet(S)$, the wedge $(\mathcal{X}, x) \vee (\mathcal{Y}, y)$ and the smash product $(\mathcal{X}, x) \wedge (\mathcal{Y}, y)$ are defined sectionwisely. Let $S^0 \in \mathcal{H}(S)$ be the constant sheaf valued at S^0. Then $(\mathcal{H}_\bullet(S), \wedge, S^0)$ forms a symmetric monoidal category. Note that the smash product is characterized by

$$X_+ \wedge Y_+ = (X \times_S Y)_+$$

for $X, Y \in Sm/S$.

Consider the following convention in $s\mathcal{S}hv_{Nis}(Sm/S)_\bullet$:

- **the simplicial circle** S^1_s: the constant simplicial sheaf valued at the pointed simplicial circle $\Delta^1/\partial\Delta^1$;
- **the Tate circle** S^1_t: the sheaf represented by $\mathbb{A}^1 - \{0\}$ pointed by 1;
- **the Tate sphere** T: the quotient sheaf $\mathbb{A}^1/(\mathbb{A}^1 - \{0\})$.

We introduce the following notations:

- $S^n_s = (S^1_s)^{\wedge n}$;
- $S^n_t = (S^1_t)^{\wedge n}$;
- $T^n = T^{\wedge n}$;
- $S^{p,q} = S^{p-q}_s \wedge S^q_t$.

Lemma 2.4.42 There is a canonical isomorphism in $\mathcal{H}_\bullet(S)$

$$S^1_s \wedge S^1_t \cong T$$

We define three kinds of suspensions as follows.

1. $\Sigma_s(\mathcal{X}, x) = S^1_s \wedge (\mathcal{X}, x)$;
2. $\Sigma_t(\mathcal{X}, x) = S^1_t \wedge (\mathcal{X}, x)$;

3. $\Sigma_T(\mathcal{X}, x) = T \wedge (\mathcal{X}, x)$.

Definition 2.4.43 Let $X \in Sm/S$ and E a vector bundle over X. The Thom space of E is the pointed sheaf

$$\mathrm{Th}(E) = \mathrm{Th}(E/X) = E/(E - z(X))$$

where $z \colon X \to E$ is the zero section of E.

Proposition 2.4.44 For any vector bundle E over X, let $\mathbb{P}(E) \to X$ be the corresponding projective bundle over X and $\mathcal{O}_X^n$ be the trivial bundle of dimension n. Then we have

1. $\mathrm{Th}(E_1 \times E_2/X_1 \times X_2) = \mathrm{Th}(E_1/X_1) \wedge \mathrm{Th}(E_2/X_2)$.
2. $\mathrm{Th}(\mathcal{O}_X^n) = \Sigma_T^n X_+$.
3. Let $\mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus \mathcal{O}_X)$ be the closed embedding at infinity. Then the canonical morphism of pointed sheaves $\mathbb{P}(E \oplus \mathcal{O}_X)/\mathbb{P}(E) \to \mathrm{Th}(E)$ is an $\mathbb{A}^1$-weak equivalence.

Corollary 2.4.45 $\mathbb{P}^n/\mathbb{P}^{n-1} \cong T^n$ in $\mathcal{H}_\bullet(S)$. In particular, one has $(\mathbb{P}^1, *) \cong T$.

Proposition 2.4.46 There is a conical isomorphism in $\mathcal{H}_\bullet(S)$

$$\mathbb{A}^n - \{(0, \cdots, 0)\} \cong (S_s^1)^{n-1} \wedge (S_t^1)^n = S^{2n-1,n}.$$

Let $f \colon S_1 \to S_2$ be a morphism between schemes, then we have a pair of adjoint functors

$$f^* \colon \mathcal{H}_\bullet(S_2) \rightleftarrows \mathcal{H}_\bullet(S_1) : f_*$$

such that f^* is obtained by left Kan extension of the functor $f^* : Sm/S_2 \to Sm/S_1$, $X_+ \mapsto X \times \times_{S_2} S_1$ and taking the total derived functors [MV99].

Let $f \colon S_1 \to S_2$ be a smooth morphism, then we have a pair of adjoint functors

$$f_\# \colon \mathcal{H}_\bullet(S_2) \rightleftarrows \mathcal{H}_\bullet(S_1) : f^*$$

Theorem 2.4.47 (Localization theorem) Let $i : Z \to S$ be a closed embedding and $j \colon U \to S$ be the complimentary open embedding. Then for any simplicial sheaf $\mathcal{X}$, the square

$$\begin{array}{ccc} j_\# j^* \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & i_* i^* \mathcal{X} \end{array}$$

is homotopy cocartesian in $\mathcal{H}(S)$.

Proof. See [MV99, Section 3, Theorem 2.21]. □

Remark 2.4.48 The proof of this theorem relies on the choice of Nisnevich topology and the choice of $\mathbb{A}^1$ as an interval object. Furthermore, the proof in [MV99] also relies on the condition of smoothness.

Theorem 2.4.49 (Homotopy purity) Let $i\colon Z \to X$ be a closed embedding of smooth schemes over S. Let $N_{X/Z}$ be the normal vector bundle to i. Then there is a canonical isomorphism in $\mathcal{H}_\bullet(S)$

$$X/(X - i(Z)) \cong \mathrm{Th}(N_{X/Z})$$

Proof. See [MV99, Section 3, Theorem 2.23]. □

Remark 2.4.50 The homotopy purity theorem is an analogy of tubular neighbourhood theorem in differential geometry.

Remark 2.4.51 Here we show the connection between homotopy purity and cohomological purity. Let H^* be an oriented cohomology theory on Sm/S factored through $\mathcal{H}_\bullet(S)$. Note that Thom isomorphism theorem holds for H^*, since H^* is oriented. Let $i : Z \to X$ be a closed embedding of smooth schemes and let $U = X \backslash i(Z) \subset X$. Then we have long exact sequence for the pair (X, U)

$$\cdots \longrightarrow H^*(X,U) \longrightarrow H^*(X) \longrightarrow H^*(U) \longrightarrow H^{*+1}(X,U) \longrightarrow \cdots$$

By the homotopy purity theorem, we have $H^*(X,U) \cong H^*(X/U) \cong H^*(\mathrm{Th}(N_{X/Z}))$. Suppose Z is of codimension d, then the Thom isomorphism implies that $H^*(\mathrm{Th}(N_{X/Z})) \cong H^{*-d}(Z)$. Now we combine these results together to get a Gysin long exact sequence

$$\cdots \longrightarrow H^{*-d}(Z) \longrightarrow H^*(X) \longrightarrow H^*(U) \longrightarrow H^{*-d+1}(Z) \longrightarrow \cdots$$

which implies the cohomological purity isomorphism

$$H^{*-d}(Z) \xrightarrow{\sim} H^*_Z(X)$$

Note that Gysin structures is essential for Riemann-Roch type theorems [Pan04]. From this point of view, homotopy purity is necessary for building a suitable homotopy theory for schemes.

More applications of homotopy purity can be found in [Mor12, Chapter 4].

2.4.4 Representability of K-theory

Definition 2.4.52 Let F be a simplicial presheaf over Sm/S. F is said to have the B.G. property if

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \times_X V) \end{tikzcd}$$

is a homotopy cartesian diagram for any elementary distinguished square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

Example 2.4.53 Note that a cartesian diagram of simplicial sets may not be a homotopy cartesian diagram. Thus not every Nisnevich simplicial sheaf has B.G. property. However, if we assume a Nisnevich simplicial sheaf $\mathcal{X}$ is fibrant with respect to Joyal's model, then $\mathcal{X}$ has B.G property [MV99, Section 3, Remark 1.15].

Theorem 2.4.54 Suppose $\mathcal{X}, \mathcal{Y}$ are two simplicial presheaves that have B.G. property. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism such that

$$\mathscr{L}_{Nis}(f) \colon \mathscr{L}_{Nis}\mathcal{X} \to \mathscr{L}_{Nis}\mathcal{Y}$$

is a stalkwise weak equivalence. Then $\mathscr{L}_{Nis}(f)$ is a sectionwise weak equivalence.

Corollary 2.4.55 Let $(\mathcal{X}, x)$ be a pointed simplicial presheaf with B.G properties. Then for any $U \in Sm/S$ and any non-negative integer n, we have

$$\pi_n(\mathcal{X}(U)) = \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(S)}(S^n \wedge U_+, (\mathscr{L}_{Nis}\mathcal{X}, x))$$

Furthermore, if we assume $\mathcal{X}$ is $\mathbb{A}^1$-local, then

$$\pi_n(\mathcal{X}(U)) = [S^n \wedge U_+, (\mathscr{L}_{Nis}\mathcal{X}, x))]^{\mathbb{A}^1}_{\bullet}$$

Definition 2.4.56 Let $\mathcal{X}$ be simplicial presheaf (resp. sheaf). $\mathcal{X}$ is said to *satisfy descent* if there is a fibrant replacement

$$j \colon \mathcal{X} \to \tilde{\mathcal{X}}$$

such that for any $U \in Sm/S$

$$j(U) \colon \mathcal{X}(U) \to \tilde{\mathcal{X}}(U)$$

is a weak equivalence of simplicial sets.

Theorem 2.4.57 Suppose $\mathcal{X} \in s\mathcal{S}hv_{Nis}(Sm/S)$ is a simplicial sheaf that is valued at Kan complexes. Then $\mathcal{X}$ satisfies descent if and only if $\mathcal{X}$ has B.G. property.

Proof. See [MV99, Section 3, Proposition 1.16]. □

Theorem 2.4.58 Given $X \in Sm/S$, let $K(X)$ be the K-theory space (a Kan complex). Then the presheaf of K-theory $X \mapsto K(X)$ satisfies descent and is $\mathbb{A}^1$-local.

Proof. See [TT90]. □

Remark 2.4.59 In other words, the associated sheaf of K-theory spaces is a motivic space.

Corollary 2.4.60 The n^{th} K-functor $K_n(X) = \pi_n(K(X))$ is representable in $\mathcal{H}_{\bullet}(S)$

$$K_n(X) = [S^n \wedge X_+, K]^{\mathbb{A}^1}_{\bullet}$$

2.4.5 Stable motivic homotopy theory

Definition 2.4.61 An S^1_s*-spectrum* over Sm/S is a collection $\{E_n, \sigma_n\}_{n \in \mathbb{N}}$, where E_n is a pointed simplicial sheaf and $\sigma_n \colon E_n \wedge S^1_s \to E_{n+1}$ of simplicial sheaves for each

$n \geqslant 0$. A morphism between S^1_s-spectra is a collection of morphisms of pointed simplicial sheaves which are compatible with σ_n. Denote the category of S^1_s-spectra over Sm/S by $\mathcal{S}p^{S^1}(S)$.

Definition 2.4.62 Let E be an S^1_s-spectrum. Let $n \in \mathbb{Z}$ be an integer. The n^{th} stable homotopy sheaf of E is the sheaf of abelian groups

$$\pi_n(E) := \mathrm{colim}_{r\to\infty}\pi_{n+r}(E_r)$$

An S^1_s-spectrum is an Ω-*spectrum* if the adjunction map of σ_n

$$\tilde{\sigma}_n \colon E_n \to \underline{\mathrm{Hom}}(S^1_s, E_{n+1}) =: \Omega(E_{n+1})$$

is a weak equivalence in $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)_{Joyal}$ (i.e. a stalkwise equivalence) for each n.

Definition 2.4.63 A morphism of S^1_s-spectra $f : E \to F$ is a *stable simplicial weak equivalence* if it induces an isomorphism

$$\pi_(E) \cong \pi_n(F)$$

for any integer n.

f is a *stable cofibration* if the morphism

$$E_0 \to F_0$$

and

$$E_{n+1} \vee_{E_n \wedge S^1_s} F_n \wedge S^1_s \to F_{n+1}$$

is a simplicial cofibration (i.e. a monomorphism) for each $n \geqslant 0$.

Construction 2.4.64 There exists a a proper closed simplicial model category on $\mathcal{S}p^{S^1_s}(S)$ such that stable weak equivalences are weak equivalences and stable cofibrations are cofibrations in this model category [MV99, Jar00]. We denote the resulting homotopy category by $\mathcal{SH}^{S^1_s}_s(S)$ and

$$\mathrm{Hom}_{\mathcal{SH}^{S^1_s}_s(S)}(\mathcal{X}, \mathcal{Y}) = [\mathcal{X}, \mathcal{Y}]^{S^1_s}_s$$

Lemma 2.4.65 An S^1_s-spectrum E is stably fibrant if E is an Ω-spectrum and each E_n is fibrant in $s\mathcal{S}hv_{Nis}(\mathrm{Sm}/S)_{Joyal}$.

Lemma 2.4.66 The functor

$$\begin{array}{rccc} \Sigma : & \mathcal{SH}^{S^1_s}(S) & \longrightarrow & \mathcal{SH}^{S^1_s}(S) \\ & E & \longmapsto & S^1_s \wedge E \end{array}$$

is an equivalence between categories.

Definition 2.4.67 Let E be an S^1_s-spectrum. E is said to be $\mathbb{A}^1$*-local*, if for any $X \in \mathcal{S}p^{S^1_s}$, the projection

$$X \wedge \mathbb{A}^1 \to X$$

induces a bijection:

$$[X, E]_s^{S_s^1} \to [X \wedge \mathbb{A}^1, E]_s^{S_s^1}$$

A morphism $f\colon X \to Y$ in $\mathcal{S}p^{S_s^1}(S)$ is said to be a *stable* $\mathbb{A}^1$*-weak equivalence* if for any $\mathbb{A}^1$-local S_s^1-spectrum E, the map

$$[Y, E]_s^{S_s^1} \to [X, E]_s^{S_s^1}$$

is bijective.

Proposition 2.4.68 There exists a model category on $\mathcal{S}p^{S_s^1}(S)$ such that weak equivalences are exactly stable $\mathbb{A}^1$-weak equivalences. The resulting homotopy category is denoted by $\mathcal{SH}^{S_s^1}(S)$ called *stable motivic homotopy category of* S_s^1*-spectra*.

Proof. See [Jar00]. □

Theorem 2.4.69 Let $\mathcal{SH}_{s,\mathbb{A}^1}^{S_s^1}(S) \subset \mathcal{SH}_s^{S_s^1}(S)$ be the full subcategory of $\mathbb{A}^1$-local S_s^1-spectra. Then the inclusion admits a left adjoint $L_{\mathbb{A}^1}\colon \mathcal{SH}_s^{S_s^1}(S) \to \mathcal{SH}_{s,\mathbb{A}^1}^{S_s^1}(S)$ called $\mathbb{A}^1$*-localization functor*. Furthermore, it induces an equivalence

$$\mathcal{SH}^{S_s^1}(S) \simeq \mathcal{SH}_{s,\mathbb{A}^1}^{S_s^1}(S)$$

Proof. The concrete construction can be found in [Mor04, Proposition 3.2.2]. □

Construction 2.4.70 Let $\mathcal{A}b((Sm/S)_{Nis})$ be the category of sheaves of abelian groups on the site $(Sm/S)_{Nis}$. Then given $M \in \mathcal{A}b((Sm/S)_{Nis})$, its *Eilenberg-MacLane* S_s^1*-spectrum* is defined to be

$$HM := \{M, K(M,1), K(M,2), \dots, K(M,n), \dots\}$$

where $K(M,n)(U) := K(M(U),n)$ for each U (the simplicial set $K(M(U),n)$ is given by iterated bar constructions, see [May99, Section 16.5]).

Remark 2.4.71 For any $X \in Sm/S$, we have

$$[\Sigma^\infty(X_+), S^n \wedge HM]_s^{S_s^1} \cong H^n_{Nis}(X; M)$$

Definition 2.4.72 A sheaf of abelian groups $M \in \mathcal{A}b((Sm/S)_{Nis})$ is said to be *strictly* $\mathbb{A}^1$*-invariant* if for all $X \in Sm/S$ and all $n \geqslant 0$, the projection $X \times \mathbb{A}^1 \to X$ induced an isomorphism

$$H^n_{Nis}(X; M) \cong H^n_{Nis}(X \times \mathbb{A}^1; M)$$

Lemma 2.4.73 For any $M \in \mathcal{A}b((Sm/S)_{Nis})$, M is strictly $\mathbb{A}^1$-invariant if and only if HM is $\mathbb{A}^1$-local.

Definition 2.4.74 A $\mathbb{P}^1$*-spectrum* over Sm/S is a collection $\{E_n, \sigma_n\}_{n\in\mathbb{N}}$, where E_n is a pointed simplicial sheaf and $\sigma_n\colon E_n \wedge \mathbb{P}^1 \to E_{n+1}$ of simplicial sheaves for each $n \geqslant 0$. A morphism between $\mathbb{P}^1$-spectra is a collection of morphisms of pointed simplicial sheaves which are compatible with σ_n. Denote the category of $\mathbb{P}^1$-spectra over Sm/S by $\mathcal{S}p^{\mathbb{P}^1}(S)$.

A T-*spectrum* over Sm/S is a collection $\{E_n, \sigma_n\}_{n\in\mathbb{N}}$, where E_n is a pointed simplicial sheaf and $\sigma_n : E_n \wedge T \to E_{n+1}$ of simplicial sheaves for each $n \geqslant 0$. A morphism between T-spectra is a collection of morphisms of pointed simplicial sheaves which are compatible with σ_n.

Remark 2.4.75 Note that $\mathbb{P}^1/\mathbb{A}^1 = T$, which means that $\mathbb{P}^1$ is $\mathbb{A}^1$-weak equivalent to T. Therefore, the notion of T-spectrum is equivalent to the notion of $\mathbb{P}^1$-spectrum.

Example 2.4.76 (Thom spectrum) For any $X \in \mathrm{Sm}/S$, let $\mathcal{E}$ be a vector bundle on X and let $\mathcal{O}$ be the rank 1 trivial vector bundle over X. Then we have

$$\mathrm{Th}(\mathcal{E} \oplus \mathcal{O}) = \mathrm{Th}(E) \wedge T$$

where $\oplus$ is the Whitney sum. Now we let $\mathbb{G}r_n$ be the infinite Grassmanian for rank n vector bundle and let $\gamma_n \to \mathbb{G}r_n$ be the universal vector bundle. Note that $\gamma_n \oplus \mathcal{O}$ is a vector bundle of rank $n+1$ and we let $f_n : \mathbb{G}r_n \to \mathbb{G}r_{n+1}$ be the classifying map of $\gamma_n \oplus \mathcal{O}$. The classifying map induces

$$\mathrm{Th}(\gamma_n) \wedge T \to \mathrm{Th}(\gamma_{n+1})$$

Then the *Thom spectrum* $\mathrm{M}\mathbb{G}\mathrm{l}$ is a T-spectrum defined by

$$\{T, \mathrm{Th}(\gamma_1), \mathrm{Th}(\gamma_2), \dots, \mathrm{Th}(\gamma_n), \dots\}$$

Remark 2.4.77 In the motivic context, a spectrum usually means a $\mathbb{P}^1$-spectrum. We simply use $\mathcal{S}p(S) := \mathcal{S}p^{\mathbb{P}^1}(S)$.

Definition 2.4.78 For each $U \in \mathrm{Sm}/S$ and any pair $(n, m) \in \mathbb{Z}^2$ of integers and any $\mathbb{P}^1$-spectrum E, we define

$$\tilde{\pi}_n(E)_m(U) := \mathrm{colim}_{r\to\infty}[U_+ \wedge S^{n+m} \wedge (\mathbb{P}^1)^{\wedge(r-m)}, E_r]_{\bullet}^{\mathbb{A}^1}$$

Definition 2.4.79 A morphism of $\mathbb{P}^1$-spectra $f : E \to F$ is a *stable* $\mathbb{A}^1$ *weak equivalence* if it induces an isomorphism

$$\tilde{\pi}_n(E)_m \cong \tilde{\pi}_n(F)_m$$

for any pair $(n, m) \in \mathbb{Z}^2$.

f is a *stable cofibration* if the morphism

$$E_0 \to F_0$$

and

$$E_{n+1} \vee_{E_n \wedge \mathbb{P}^1} F_n \wedge \mathbb{P}^1 \to F_{n+1}$$

is a simplicial cofibration (i.e. a monomorphism) for each $n \geqslant 0$.

Proposition 2.4.80 There exists a model category on $\mathcal{S}p^{\mathbb{P}^1}(S)$ such that stable $\mathbb{A}^1$-weak equivalences are weak equivalence and stable $\mathbb{A}^1$-cofibrations are cofibrations. The resulting homotopy category is the motivic stable homotopy category $\mathcal{SH}(S)$.

For any $E, F \in \mathcal{SH}(S)$, we denote

$$[E, F]^{\mathbb{P}^1} := \mathrm{Hom}_{\mathcal{SH}(S)}(E, F)$$

Remark 2.4.81 Similarly, the suspension construction gives a functor

$$\Sigma^{\infty}_{\mathbb{P}^1} : \mathcal{H}_{\bullet}(S) \to \mathcal{SH}(S)$$

Proposition 2.4.82 The functor

$$\begin{array}{rccc} \Sigma_{\mathbb{P}^1} : & \mathcal{SH}(S) & \longrightarrow & \mathcal{SH}(S) \\ & E & \longmapsto & E \wedge \mathbb{P}^1 \end{array}$$

is an equivalence.

In this sense, $\mathbb{P}^1$ is "invertible" in the $\mathcal{SH}(S)$.

Construction 2.4.83 For any spectrum E and for any $n, i \in \mathbb{Z}$, we set

$$E(i)[n] := E \wedge S^{n,i}$$

For any $\mathcal{X} \in s\mathcal{S}hv_{Nis}(S)_{\bullet}$, we set

$$\tilde{E}^{n,i} := [\Sigma^{\infty}_{\mathbb{P}^1}(\mathcal{X}), E(i)[n]]^{\mathbb{P}^1}$$

Then we have a functor $\tilde{E}^{*,*}$ from $s\mathcal{S}hv_{Nis}(Sm/S)^{op}_{\bullet}$ to the category of bigraded abelian groups, which is *the reduced cohomology theory associated to* E. For any $\mathcal{X} \in s\mathcal{S}hv_{Nis}(S)$, we set

$$E^{n,i} := [\Sigma^{\infty}_{\mathbb{P}^1}(\mathcal{X}_+), E(i)[n]]^{\mathbb{P}^1}$$

Then we have a functor $E^{*,*}$ from $s\mathcal{S}hv_{Nis}(Sm/S)^{op}$ to the category of bigraded abelian groups, which is *the cohomology theory associated to* E.

Remark 2.4.84 Recall the models for stable homotopy theory in Subsubsection 2.1.3, Jardine construct motivic symmetric spectra in [Jar00] in analogy to symmetric spectra. Hu construct motivic $\mathbb{S}$-modules in [Hu03] in analogy to $\mathbb{S}$-modules.

2.4.6 Other models for motivic homotopy theory

Construction 2.4.85 (Dugger, [Dug01]) There is a *projective model structure* on $s\mathcal{P}re(Sm/S)$ such that

1. $f : \mathcal{X} \to \mathcal{Y}$ is a weak equivalence if $f : \mathcal{X}(U) \to \mathcal{Y}(U)$ is an equivalence of simplicial sets for all $U \in Sm/S$,

2. f is a fibration if $f : \mathcal{X}(U) \to \mathcal{Y}(U)$ is a fibration of simplicial sets for all $U \in Sm/S$,

3. f is a cofibration if it has left lifting property with respect to acyclic fibrations.

and this model category is a left proper combinatorial simplicial model category. This model category is a *universal model category* of Sm/S in the sense of [Dug01], and we denote it by $U(Sm/S)$ (it is a "homotopy cocompletion" of Sm/S).

Let $X \in \mathrm{Sm}/S$ and suppose U_* is a simplicial presheaf with a morphism of presheaves $U_* \to X$. This map is called a *Nisnevich hypercover* if

1. Each U_n is a coproduct of representable presheaves,
2. $U_0 \to X$ is a Nisnevich cover,
3. For every integer $n \geqslant 1$, the component $U^{\Delta^n} \to U^{\partial\Delta^n}$ in degree 0 is a cover.

A *Čech cover* is a hypercover $U_* \to X$ such that each $U^{\Delta^n} \to U^{\partial\Delta^n}$. Given a Nisnevich cover $U \to X$ in Sm/S, the associated Čech cover $(U)_*$ is defined to be

$$\check{C}(U)_n := U \times_X U \times_X \cdots \times_X U$$

Let $\mathcal{N}is$ be the class of the natural map

$$\mathrm{hocolim} U_* \to X$$

where X runs through all objects of Sm/S and U_* runs through all Nisnevich hypercovers of $\mathcal{X}$. Let $U(\mathrm{Sm}/S)/\mathcal{N}is$ be the model category obtained by Bousfield localization with respect to $\mathcal{N}is$ from $U(\mathrm{Sm}/S)$. Finally, let $U(\mathrm{Sm}/S)_{\mathbb{A}^1}$ be the model category obtained by Bousfield localization with respect to $X \times \mathbb{A}^1 \to X$ for all $X \in \mathrm{Sm}/S$ from $U(\mathrm{Sm}/S)/\mathcal{N}is$.

Proposition 2.4.86 (Dugger) There is a Quillen equivalence

$$U(\mathrm{Sm}/S)_{\mathbb{A}^1} \xrightarrow{\sim} L_{\mathbb{A}^1}\mathcal{S}\mathrm{pc}_{\mathrm{Nis}}(S)_{\mathrm{Joyal}}.$$

Proof. According to [Dug01, Proposition 7.3], we have the following Quillen equivalence

$$U(\mathrm{Sm}/S)/\mathcal{N}is \rightleftarrows s\mathcal{P}\mathrm{re}(\mathrm{Sm}/S)_{\mathrm{Jardine}} \rightleftarrows s\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}/S)_{\mathrm{Joyal}}$$

Then see [Dug01, Proposition 8.1]. □

Remark 2.4.87 One advantage of $U(\mathrm{Sm}/S)/\mathcal{N}is$ over $s\mathcal{P}\mathrm{re}(\mathrm{Sm}/S)_{\mathrm{Jardine}}$ is that the fibrant objects are much easier to describe. Referring to [DHI04, Theorem 1.3], a motivic space $\mathcal{F}$ is fibrant in $U(\mathrm{Sm}/S)/\mathcal{N}is$ if

1. $\mathcal{F}(X)$ is fibrant for any $X \in \mathrm{Sm}/S$,
2. for any Nisnevich hypercover $U_* \to X$, the natural map $\mathcal{F}(X) \to \mathrm{holim}_n \mathcal{F}(U_n)$ is a weak equivalence.

Another advantage of $U(\mathrm{Sm}/S)_{\mathbb{A}^1}$ is that it exhibit the $\mathcal{H}(S)$ as a universal homotopy category in the following sense: given a functor F from Sm/S to a model category $\mathcal{M}$ such that the natural map $F(X \times \mathbb{A}^1) \to F(X)$ is a weak equivalence for any $X \in \mathrm{Sm}/S$ and sends any elementary distinguished square to a homotopy push-out diagram, then it factor through $U(\mathrm{Sm}/S)_{\mathbb{A}^1}$ uniquely up to “homotopy". Therefore, we can characterize (unstable) motivic homotopy theory by its unisersal property. The rigorous definitions and proof can be found in [Dug01, Section 8]. See also Nardin’s MO answer [Nar18].

Remark 2.4.88 The constructions in Construction 2.4.10, Remark 2.4.21 and Construction 2.4.85 do not rely on the choice of the Nisnevich topology and so do the Quillen equivalences among them. They hold for any general (small) site, see [Dug01, Jar87, Jar07, Jar15, MV99].

Remark 2.4.89 Actually, the construction of motivic homotopy category is independent of the choice of model structures. To be more specific, all these model categories have equivalent underlying ∞-categories, which is the ultimate reason why their homotopy categories are equivalent. More advantages of ∞-categories are analyzed in Robalo's thesis [Rob15, Section 2.2]. We will show the ∞-categorical formalism of motivic homotopy category later.

3 SOME QUESTIONS FOR FURTHER INVESTIGATION

To study methods of homotopy theory in algebraic geometry from the viewpoint of cohomology operations, we first study the ∞-categorical formalism of homotopy theory in order to avoid the issues about different choices of models (see Subsubsection 2.1.3 and Subsubsection 2.4.6). Meanwhile we study the essential workings of power operations and multiplicative structures in classical homotopy theory [May77, BMMS86]. Finally, we try to combine the two and study a framework in algebraic geometry analogous to the classical framework via the contemporary formalism [BEH21, BH21] in the motivic setting.

3.1 Infinity-categorical formalism of motivic homotopy theory

As a part of this thesis, we plan to study Robalo's ∞-categorical formalism of motivic stable homotopy theory [Rob15]. In particular, we would like to understand how to characterize motivic stable homotopy theory by universal properties. The key point of this characterization is to present monoidal structures in the context of homotopy theory, but it is extremely hard to realize this in terms of model categories. In Robalo's work, he solved this problem by using the framework of ∞-categories. The foundation of this framework is established in Lurie's work on higher topos theory [Lur09] and higher algebra [Lur17].

3.2 How the Adams spectral sequences exhibit H_∞-structures

In the 1980s, May and his collaborators studied H_∞-structures (see Definition 2.2.24), a fundamental class of multiplicative structure [BMMS86]. In the monograph, Bruner revealed how the power operations present in the E_2-page reflect the multiplicative information of the homotopy classes in the E_∞-page along an Adams spectral sequence. Given an H_∞-ring spectrum E, a suspension spectrum X, and a ring spectrum Y, we summarize the mechanism of H_∞-structures studied by Bruner in the following diagram.

$$\begin{tikzcd}
\text{power operations} \arrow[d, "\text{displayed}"'] & H_\infty\text{-structure} \arrow[l, "\text{induces at}", "\text{cohomology level}"'] \arrow[r, "\text{induces at}", "\text{homotopy level}"'] & \text{extended powers} \arrow[d, "\text{displayed}"] \\
 & \boxed{\text{How to convert cohomology operations into homotopy operations}} \arrow[u, Rightarrow, "\text{encoded}"'] \arrow[d, Rightarrow, "\text{presented}"] & \\
\underset{E_2\text{-page}}{\mathrm{Ext}^{s,t}_{E^*E}(E^*Y, E^*X)} \arrow[rr, Rightarrow, "\text{differentials in the Adams spectral sequence}", "\text{converges along the Adams spectral sequence}"'] & & \underset{E_\infty\text{-page}}{[X,Y]^E_{t-s}}
\end{tikzcd}$$

We would like to understand the diagram through examples, centering around the boxed part, and try to extend this mechanism to motivic homotopy theory.

3.3 How motivic extended powers emerge in the motivic Adams spectral sequences

Recently, Bachmann and Hoyois have formally introduced the notion of a normed motivic spectrum [BH21], which is a refinement of a motivic E_∞-ring spectrum. Since one can derive H_∞-structures from E_∞-structures by taking the homotopy action of the E_∞-operad in classical homotopy theory, we expect that a *motivic* H_∞-*structure* should arise from this norm structure. In [BEH21], Bachmann, Elmanto, and Heller constructed motivic extended powers by using motivic colimits. Both of these two new notions give rise to power operations in motivic cohomology. It is still unknown how they are related to each other and to motivic Adams spectral sequences. Our goal is to investigate these potential relations based on their classical analogue as in Section 3.2, especially the dashed arrows and the ovalbox in the following diagram.

$$\begin{tikzcd}
\text{normed motivic spectrum} \arrow[r, dashed] \arrow[dr, "\text{Bachmann-Hoyois}"] & \boxed{\text{motivic } H_\infty\text{-structure?}} \arrow[d, dashed] & \text{motivic extended powers} \arrow[l, dashed] \arrow[dl, "\text{Bachmann-Elmanto-Heller}"'] \\
 & \text{motivic power operations} \arrow[d, "\text{Dugger-Isaksen [DI10]}"] & \\
 & \text{motivic Adams spectral sequences} \arrow[uul, dashed, bend left, leftrightarrow, "?"] \arrow[uur, dashed, bend right, leftrightarrow, "?"'] &
\end{tikzcd}$$

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