A Proof of Quillen's Theorem on Formal Group Laws using Power Operations

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Outline

Cohomology Theory with Characteristic Classes

$$
\begin{tikzcd}
\boxed{\text{vector bundles}} \arrow[r, dashed] \arrow[d] & \boxed{\text{characteristic classes}} \arrow[d, dashed] \\
\boxed{\text{geometric objects}} \arrow[r, "\text{via cohomology machinery}"] & \boxed{\text{algebraic invariants}}
\end{tikzcd}
$$

We focus on complex vector bundles, therefore we expect a fruitful cohomology theory to endow characteristic classes for each complex vector bundles. The desired notion is called **complex orientation**.

Complex Oriented Cohomology Theory

Definition

A **complex oriented cohomology theory** is a ring spectrum E with a chosen class $x \in \widetilde{E}^2(\mathbb{CP}^\infty)$ such that the following

$$\widetilde{E}^2(\mathbb{CP}^\infty) \to \widetilde{E}^2(\mathbb{CP}^1) = \widetilde{E}^2(S^2) \cong E^0(pt)$$

induced by inclusion $\mathbb{CP}^1 \to \mathbb{CP}^\infty$, $x \mapsto 1$ in $E^0(pt)$. The chosen class is called the **orientation class**. We may denote a complex oriented cohomology theory by (E, x).

This definition makes sense due to the splitting principle and the classification theorem of vector bundles.

Characteristics Classes on COCT

Definition

Given a line bundle $L \to X$ classified by $f : X \to \mathbb{CP}^\infty$ and a COCT E with orientation x, its **Euler class** $e_E(L)$ is defined to be f^*x.

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Proposition

For any complex oriented cohomology theory E, we have

$$E^*(\mathbb{CP}^n) = E^*(pt)[x]/(x^{n+1})$$

where x is the Euler class of the tautological bundle ξ on $\mathbb{CP}^n$.

An Alternative Definition of COCT

Definition

A **complex oriented cohomology theory** is a generalized multiplicative cohomology theory E such that for any complex vector bundle ξ of rank n, there exists a class $\Phi_\xi \in \widetilde{E}^{2n}(\mathrm{Th}(\xi))$ called **Thom class** such that

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1. For any $x \in X$, the image of Φ_ξ of the following composition

$$\widetilde{E}^{2n}(\mathrm{Th}(\xi)) \longrightarrow \widetilde{E}^{2n}(\mathrm{Th}(\xi|_x)) \longrightarrow \widetilde{E}^{2n}(S^{2n}) \longrightarrow E^0(pt)$$

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is the canonical identity element 1.
2. Thom classes is compatible with pullback, namely, $f^*\Phi_\xi = \Phi_{f^*\xi}$.
3. For any two vector bundles ξ, ι with the same base space, we have $\Phi_{\xi \oplus \iota} = \Phi_\xi \cdot \Phi_\iota$

An Alternative Definition of the Euler Class

Definition

Let E be a complex oriented cohomology theory and $\xi\colon E \to B$ a vector bundle bundle. Let $s\colon B \to \mathrm{Th}(\xi)$ be the zero section. Then the **Euler class** of ξ with respect to E is defined by

$$e_E(\xi) := s^*\Phi_\xi$$

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Remark

These two definitions of complex oriented cohomology theories and Euler classes are equivalent and each of them has its own benefits. The former one is simpler, while the latter one is more essential.

Construction

Let $\eta_n\colon EU(n) \to BU(n)$ be the universal complex vector bundle over the complex Grassmanian manifold $BU(n)$. Let **n** *denote the trivial complex bundle of rank n on an evident based space. Let $MU(n)$ be the Thom space of η_n. Then we have $\alpha_n\colon \mathrm{Th}(\eta_n \oplus \mathbf{1}) \cong \Sigma^2\mathrm{Th}(\eta) \to MU(n+1)$ induced by a classifying map of $\eta_n \oplus \mathbf{1}$. Then we may define* **complex Thom spectrum** *MU by*

$$MU_{2q} := MU(q)$$
$$MU_{2q+1} := \Sigma MU(q)$$

and the structure maps are given by α_n. The class of the identity map in $MU^2(MU(1)) = [MU(1), MU(1)]$ is the **universal Thom class** *Φ on MU and derives the Thom class of each vector bundle evidently.*

Universal Complex Oriented Cohomology Theory

Proposition

Let $i\colon \mathbb{CP}^\infty \to MU(1)$ be the zero section. Then $i^*(\Phi) \in MU^2(\mathbb{CP}^\infty)$ offers an orientation of MU such that $(MU^*, i^*\Phi)$ is the universal complex oriented cohomology theory in the sense that for any complex oriented cohomology theory (E, x), there is a unique map (up to homotopy) $\phi\colon MU \to E$ that preserves the orientations $i^*\Phi \to x$.

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Sketch proof.

The Thom class of $\eta_n \in \widetilde{E}^{2n}(MU(n))$ provides us with a morphism between spectra, which is what we need. $\square$

Formal Group Laws on COCT

Definition

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2. $F(x, F(y,z)) = F(F(x,y), z)$
3. $F(x,y) = F(y,x)$

Proposition

Given a complex oriented cohomology theory (E,t), there exists a unique formal group law $F_E(x,y) = c_{ij}x^i y^j$ over the ring $E^*(pt)$ such that for any space X and any two line bundles L_1, L_2 on X, we have $e_E(L_1 \otimes L_2) = F_E(e_E(L_1), e_E(L_2))$ in $E^*(X)$.

Universal Formal Group Law

Theorem (Lazard)

There exists a ring L called Lazard ring with a universal formal group law ℓ such that for any ring R with any formal group law $g(x, y) \in R[[x, y]]$ there exits a unique ring homomorphism $f : L \to R$ that sends ℓ to g.

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Motivation

$$\begin{tikzcd}
\boxed{\textit{complex cobordism theory}} \arrow[r, "\textit{universal in}"] \arrow[d, dashed, "?"'] & \boxed{\textit{complex oriented cohomology theories}} \arrow[d] \\
\boxed{\textit{universal formal group law}} \arrow[r, "\textit{universal in}"] & \boxed{\textit{formal group laws}}
\end{tikzcd}$$

Quillen's Theorem on the Formal Group Laws

Theorem (Quillen)

Let F_{MU} be the formal group law associated to MU. Then the map $L \to MU^*$ classifying F_{MU} is a ring isomorphism. In particular, (MU^*, F_{MU}) is exactly the universal formal group law.

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Outline of the proof.

The structure theorem on $MU \longrightarrow L \to MU^*$ is surjective

The properties of $MU \to H\mathbb{Z} \longrightarrow L \to MU^*$ is injective

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Outline of the proof.

The structure theorem on MU $\longrightarrow$ $L \to MU^*$ is surjective

The properties of $MU \to H\mathbb{Z}$ $\longrightarrow$ $L \to MU^*$ is injective

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The Structure Theorem

Let $F_{MU}(x,y) = \sum c_{ij}x^i y^j$ be the formal group law on MU, where $c_{ij} \in MU^{2-2i-2j}$. Let $C \subset MU^*$ be the subring of MU^* generated by $\{c_{ij}\}$.

Theorem (structure theorem of MU^*)

If X is of the homotopy type of a compact smooth manifold, then

$$MU^*(X) = C \cdot \sum_{q \geq 0} MU^q(X)$$

$$\widetilde{MU}^*(X) = C \cdot \sum_{q > 0} MU^q(X)$$

Since $MU^* = MU^*(pt)$ has trivial negative part, we conclude that $MU^* = C$ and thus $L \to MU^*$ is surjective.

Outline

Complex Oriented Maps

Definition

Let X be a compact smooth manifold. **A complex oriented map to** X consists of a smooth proper map $f: M \to X$ with even relative dimension and a continuous map $\nu: X \to BU$ such that f can be factored by

$$M \xrightarrow{i} X \times \mathbb{C}^n \xrightarrow{p} X$$

where i is a closed embedding, p is the evident projection and the normal bundle ν_i on M has a complex structure of rank $(2n - \dim f)/2$ that is classified by ν.

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Example

Let X be a smooth manifold and let $E \to X$ be a complex vector bundle on X. The zero section $s\colon X \to E$ has an evident complex orientation, because the normal bundle of s is exactly E itself.

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Remark

The notion of complex oriented maps is analogous to the notion of projective maps in algebraic geometry. This insight enables us to consider "algebraic cobordism", the algebro-geometric version of cobordism theory.

Definition

Two proper complex oriented maps $f_i \colon Z_i \to X$ for $i = 0, 1$ is said to be **corbordant** if there is a proper complex oriented map $h \colon W \to X \times \mathbb{R}$ such that the map $j_i \colon X \to X \times \mathbb{R}$, $x \mapsto (x, i)$ is transversal to h and the pull-back of h is isomorphic with the complex orientation of f_i for $i = 0, 1$.

$$\begin{tikzcd}
Z_0 \arrow[d, "f_0"] & W \arrow[d, "h"] & Z_1 \arrow[d, "f_1"] \\
X \arrow[r, "j_0"] & X \times \mathbb{R} & X \arrow[l, "j_1"']
\end{tikzcd}$$

Geometric Cobordism Theory

Definition

For any compact smooth manifold X, we define

$$U^n(X) = \{(f, \nu) \mid \text{complex oriented maps of dim} n\}/\text{cobordant}$$

for each n.
We denote

$$U^*(X) := \bigoplus_{n \in \mathbb{Z}} U^n(X)$$

If A is a strong deformation retract of an open neighborhood V in X, we similarly define

$$U^*(X, X - A) = \{(f, \nu) \mid \text{complex oriented maps} \mid f(Z) \subset A\}/\text{cobordant}$$

Pontrjagin-Thom Isomorphism

Theorem

For any compact smooth manifold X, we have a functorial isomorphism

$$U^*(X) \cong MU^*(X)$$

given by Pontrjagin-Thom construction. For the relative case, if A is a strong deformation retract of an open neighborhood V in X, then

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Remark

The operations on U^* display more explicitly and more intuitively, which enables us to utilize them more conveniently.

Outline

Operations on the Geometric Cobordism Theory I

The addition on $U^n(X)$ is defined by

$$(f,\nu)+(f',\nu'):=(f\sqcup f',\nu\sqcup\nu')$$

The external product on U^* is given by

$$\begin{array}{rccc} \times : & U^*(X)\otimes U^*(Y) & \longrightarrow & U^*(X\times Y) \\ & f\otimes g & \longmapsto & f\times g \end{array}$$

and the internal product is derived by

$$U^*(X)\otimes U^*(X)\xrightarrow{\times} U^*(X\times X)\xrightarrow{\Delta^*} U^*(X)$$

where $\Delta\colon X\to X\times X$ is the diagonal map.

Gysin homomorphisms

Definition

Given a proper complex oriented map $(g,\xi)\colon X \to Y$ of dimension d, we define the induced *Gysin homomorphism*

$$\begin{array}{rccc} g_* : & U^q(X) & \longrightarrow & U^{q+d}(Y) \\ & f & \longmapsto & g \circ f \end{array}$$

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Proposition

The Gysin morphisms are additive and $U^*(pt)$-linear and given two composable complex oriented maps p, q, we have $(p \circ q)_* = p_* \circ q_*$.

Digression: the Geometry of Thom Classes

Proposition

Let $i\colon Z \to X$ be a closed embedding of smooth manifolds of codimension d such that the normal bundle ν_i has a complex structure. Then we have the **Gysin-Thom isomorphism**

$$i_*\colon U^*(Z) \xrightarrow{\sim} U^{*+d}(X, X-Z) = U_Z^{*+d}(X)$$

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Remark

We have $(X, X-Z) \simeq (\mathrm{Th}(N_{X/Z}), *)$ using tubular neighbourhood theorem, and i_* can be identified as the Thom isomorphism for $N_{X/Z}$ on Z. The equivalence $\mathrm{Th}(N_{X/Z}) \simeq X/(X-Z)$ is so called "homotopy purity". In particular, this holds in Morel-Voevodsky's $\mathbb{A}^1$-homotopy category.

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Remark

We may call this kinds of isomorphisms "purity isomorphisms". Here "purity" comes from the theorem of absolute purity in étale cohomology, which states a similar phenomenon for closed embedding of regular schemes of pure codimension.

Digression: the Geometry of Thom classes

Proposition

Let $E \to X$ be a complex vector bundle and let $s \colon X \to \mathrm{Th}(E)$ be its zero section to the Thom space. Under the identification between $U^*(X)$ and $MU^*(X)$, $s_*([\mathrm{id}_X]) \in U^*(\mathrm{Th}(E))$ is the Thom class, $s^*s_*([\mathrm{id}_X])$ is the Euler class $e_U(E)$, and s_* is exactly the Thom isomorphism.

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Remark

From this perspective, if one can define Gysin homomorphisms properly for a given cohomology theory, then we may expect that the cohomology theory should be endowed with Thom classes, namely the cohomology theory is oriented in the previous sense. Furthermore, this viewpoint also makes sense in algebro-geometric context.

Landweber-Novikov Operations

Construction (Landweber-Novikov Operations)

The total Landweber-Steenrod operations on X is defined to be

$$\begin{array}{rccc} s_t : & U^*(X) & \longrightarrow & U^*(X)[t_1, t_2, t_3, \ldots] \\ & (f, \nu) & \longmapsto & \sum_\alpha t^\alpha f_*(c_\alpha(\nu)) \end{array}$$

where α runs over all the numerable sequences of non-negative integers with only finitely many integers are non-zero and c_α is the Conner-Floyd-Chern class indexed by α. We denote $s_\alpha(x) := f_* c_\alpha(\nu)$ if x is represented by (f, ν).

Equivariant Setting on U^*

Construction (Equivariant cobordism theorem)

Given a principal G-bundle ξ, say $\pi_\xi : Q \to B$ over a manifold B and we let G act right on Q. Then for any G-space X, we define the equivariant cobordism theory U^*_ξ twisted by ξ by

$$U^*_\xi(X) := U^*(Q \times_G X)$$

If ξ is the universal principal G-bundle, we denote it by U^*_G simply.

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If ξ is the universal principal G-bundle, we denote it by U^*_G simply.

Remark

For a G-equivariant vector bundle $\eta\colon E \to X$ over X, we define

$$e_\xi(\eta) := e_{MU}(Q \times_G \eta\colon Q \times_G E \to Q \times_G X)$$

and we have

$$e_\xi(L_1 \otimes L_2) = F_{MU}(e_\xi(L_1), e_\xi(L_2))$$

Power Operations in U^* (Geometric Construction)

Construction (Power operations in cobordism)

Given a principle $\mathbb{Z}/p$-bundle $\xi\colon Q \to B$, the **total power operation twisted by** ξ *is defined to be*

$$P_\xi\colon U^{-2q}(X) \longrightarrow U_\xi^{-2pq}(X^p) \xrightarrow{\Delta^*} U_\xi^{-2pq}(X) = U^{-2pq}(B \times X)$$

$$\langle Z \xrightarrow{f} X\rangle \longmapsto \langle Q \times_{\mathbb{Z}/p} Z^p \xrightarrow{\mathrm{id}_Q \times_{\mathbb{Z}/p} f^p} Q \times_{\mathbb{Z}/p} X^p\rangle \longmapsto \langle (Q \times_{\mathbb{Z}/p} Z^p)^{\mathbb{Z}/p} \to B \times X\rangle$$

where $\mathbb{Z}/p$ acts on X^p by permuting factors and acts on X trivially; $\Delta\colon X \to X^p$ is the diagonal map.

Note that $E\mathbb{Z}/p \to B\mathbb{Z}/p$ has a model whose skeleton filtration consists of mod-p lens spaces and related $\mathbb{Z}/p$-bundles. If we take the inverse limit according to the filtration, we then have the desired mod-p total power operation on U^* resembling the pattern in $H\mathbb{Z}/p$.

Power Operations in MU^* (Homotopical Construction)

We introduce the following conventions

$$\Gamma_n^p(X) := (S^{2n-1} \times X^p)/p$$

$$\Gamma_n^{p+}(X) := (S^{2n-1} \wedge X^{\wedge p})/p$$

Let ξ be a complex vector bundle on X and $\pi\colon S^{2n-1} \times X^p \to X^p$ be the natural projection. we define a vector bundle $\xi_n(p) : \pi^*(\xi^p)/p \to \Gamma_n^p(X)$.

Lemma

By taking Thom spaces, we have

$$\mathrm{Th}(\xi_n(p)) \cong \Gamma_n^{p+}(\mathrm{Th}(\xi))$$

Power Operations in MU^* (Homotopical Construction)

Definition

Given integer r, n and prime p, the *external power operation*

$$EP^{2r}_{n,p} \colon \widetilde{MU}^{2r}(X) \to \widetilde{MU}^{2pr}(\Gamma^{p+}_n(X))$$

is defined to be: for any $\alpha \in \widetilde{MU}^{2r}(X)$ that can be represented by $f \colon \Sigma^{2l} X \to MU_{2r+2l}$, we have

$$\Gamma f \colon \Gamma^{p+}_n \Sigma^{2l} X \to \Gamma^{p+}_n MU_{2r+2l}$$

Note that the Thom class of $\eta_{r+l}(p)$ denoted by $\Phi_{\eta_{r+l}(p)}$ is in $MU^{2p(r+l)}(\Gamma^{p+}_n MU_{2r+2l})$, and we define

$$EP^{2r}_{n,p}(\alpha) := \Gamma f^*(\Phi_{\eta_{r+l}(p)}) \in \widetilde{MU}^{2p(r+l)}(\Gamma^{p+}_n \Sigma^{2l} X) \cong \widetilde{MU}^{2pr}(\Gamma^{p+}_n X)$$

Power Operations in MU^* (Homotopical Construction)

Definition

Given a positive integer n and a prime p, let $\Delta\colon X \to X^p$ be the diagonal map. Then we have

$$\Delta\colon L^n(p)_+ \wedge X \to \Gamma_n^{p+} X$$

The *mod-p total power operation of degree n* is defined to be

$$\begin{array}{rccc} \mathcal{P}^{2r}_{n,p}: & \widetilde{MU}^{2r}(X) & \longrightarrow & \widetilde{MU}^{2pr}(L^n(p)_+ \wedge X) \\ & \alpha & \longmapsto & \Delta^* EP^{2r}_{n,p}(\alpha) \end{array}$$

Let $\mathcal{P}^{2r}_p = \mathcal{P}^{2r}_{\infty,p}$ and $X = Y_+$ for some space Y. Then we have

$$\mathcal{P}^{2r}_p\colon MU^{2r}(Y) \to MU^{2pr}(B\mathbb{Z}/p \times Y)$$

The homotopical construction and the geometric construction are equivalent.

Gysin Morphisms and Power Operations

Remark

In the homotopical construction of power operations, the essential structure is

$$\Gamma_n^{p+} MU_{2r+2l} \to MU_{2p(r+l)}$$

which is offered by a Thom class of a certain well-designed bundle. Furthermore, these structure maps can be refined as H_∞-structures.

Gysin Morphisms and Power Operations

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Remark

Informally speaking, the following diagram illustrates how push-forward setting (Gysin morphisms) help us encode coherence data.

$$\boxed{\textit{Gysin morphisms}} \longrightarrow \boxed{\textit{Thom classes}} \longrightarrow \boxed{H_\infty\textit{-structures on } MU}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\textit{geometric perspective}} \quad \underbrace{\qquad\qquad\qquad\qquad}_{\textit{homotopical perspective}}$$

Outline

Auxiliary Classes

Given a complex G-representation τ and a trivial G-space X, denote X^τ the G-equivariant bundle $X \times \xi \to X$.

Let $V = \{(z_1, \ldots, z_p) \in \mathbb{C}^p \mid \sum_{i=1}^p z_i = 0\}$ and (ρ, V) be a representation of $\mathbb{Z}/p$ where $\mathbb{Z}/p$ acts on V by permuting factors cyclically.

Let σ be a 1-dimensional representation of $\mathbb{Z}/p$ sending n to $\exp(2n\pi i/p)$.

Fix a principal $\mathbb{Z}/p$-bundle $\xi \colon Q \to B$, define

$$v = e_{MU}(Q \times_{\mathbb{Z}/p} B^\sigma \to B)$$

$$w = e_{MU}(Q \times_{\mathbb{Z}/p} B^\rho \to B)$$

Technical Lemma I

Lemma

Given an positive integer q, there exists an integer n such that the p-th power operation associated to a principle $\mathbb{Z}/p$-bundle $\xi\colon Q \to B$ is related to the Landweber-Novikov operations by the formula

$$w^{n+q} P_\xi x = \sum_{l(\alpha)\leq n} w^{n-l(\alpha)} \Big(\prod_{j\geq 1} a_j(v)^{\alpha_j}\Big) s_\alpha(x)$$

where a_j is a power series in $C[[t]]$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ with finitely many non-zero items, and $l(\alpha) = \sum \alpha_j$.

The Insight of Technical Lemma I

Riemann-Roch type theorem

$\downarrow$

localization at the fix point set

$\downarrow$

$$\boxed{\text{cobordism power operations}} \xleftrightarrow[\text{Technical Lemma I}]{} \boxed{\text{Landweber-Novikov operations}}$$

Technical Lemma II

Let $L^n(p)$ be the mod-p lens space S^{2n-1}/p.
Let $\zeta_{n,p}$ be the complex line bundle $S^{2n-1} \times_{\mathbb{Z}/p} \mathbb{C} \to L^n(p)$.
Let $z_n = e_{MU}(\zeta_{n,p})$.
Let $\theta_p(t) = [p]_{F_{MU}}(t)/t$.
Let $j_n\colon X \times L^{n-1}(p) \to X \times L^n(p)$ be the map induced by the natural inclusion $i_n\colon L^{n-1}(p) \hookrightarrow L^n(p)$.

Lemma

If $x \in MU^q(X \times L^n(p))$ such that $x \cdot z_n = 0$, then there exists $y \in MU^q(X)$ such that $y \cdot \theta_p(z_{n-1}) = j_n^*(x)$.

The Insight of Technical Lemma II

Theorem (Landweber)

For finite complex X, we have

$$MU^*(B\mathbb{Z}/n \times X) \cong MU^*(X)[[z]]/[n]_{F_{MU}}(z)$$

where z is the Euler class of the complex line bundle $B\mathbb{Z}/n \times_{\mathbb{Z}/n} \mathbb{C}$ with $\mathbb{Z}/n$ acting on $\mathbb{C}$ by multiplying $\exp(2\pi i/n)$. In particular, $MU^*(B\mathbb{Z}/p) = MU^*(pt)[[z]]/[p]_{F_{MU}}(z)$.

This theorem indicates the connection between power operations and the formal group law.

Preliminaries on U^*

For a based space (X, x_0), the reduced cobordism theory is defined to be

$$\widetilde{U}^*(X) := \ker(U^*(X) \to U^*(x_0))$$

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4. $U^*(X) \cong \widetilde{U}^*(X) \oplus U^*(x_0)$

Lemma

$\widetilde{U}^0(X)$ is a nilpotent ideal of $U^0(X)$.

The Structure Theorem

Let $F_{MU}(x,y) = \sum c_{ij} x^i y^j$ be the formal group law on MU, where $c_{ij} \in MU^{2-2i-2j}$. Let $C \subset MU^*$ be the subring of MU^* generated by $\{c_{ij}\}$.

Theorem

If X is of the homotopy type of a compact smooth manifold, then

$$U^*(X) = C \cdot \sum_{q \geq 0} U^q(X)$$

$$\widetilde{U}^*(X) = C \cdot \sum_{q > 0} U^q(X)$$

Outline of the Proof of the Structure theorem

Sketch proof.

We give the outline of the proof here.

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$$\widetilde{U}^{2*}(X) = C \cdot \bigoplus_{q>0} U^{2q}(X)$$

2. Now we set

$$R = C \cdot \bigoplus_{q>0} U^{2q}(X)$$

and we need to show $U^{2*}(X) = R$.

$\square$

The Inductive Argument

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4. (**Key Step**) Proceed the induction by using power operations in cobordism theory. The rough idea is to do operations on $x \in U^{-2q}(X)$ such that x can be decomposed to be a sum of elements in R, where we use Technical Lemma I II.

Outline of the Key Step

1. Let ε_m be the principal $\mathbb{Z}/p$-bundle $S^{2m-1} \to L^m(p) = S^{2m-1}/p$. **Technical Lemma I** will help us deduce that there exists an integer m such that there indeed exists some formal power series $f(t) \in R_{(p)}[[t]]$ such that

$$v^m(w^q P_{\varepsilon_m} x - x) = f(v) \in U^*(L^m(p) \times X)_{(p)}$$

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2. **Technical Lemma II** will help us deduce that the minimal choice of such positive integer m is 1, which means that we eventually have

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3. Restricting Equation 1 on X, we will deduce that

$$\widetilde{U}^{-2q}(X)_{(p)} \subset R^{-2q}_{(p)} + p\widetilde{U}^{-2q}(X)_{(p)}$$

Details in the Key Step

According to Technical Lemma I, for some $x \in \widetilde{U}^{-2q}(X)$ and some large n, we have

$$w^{n+q}P_{\xi}x = \sum_{l(\alpha)\leq n} w^{n-l(\alpha)}a(v)^{\alpha}s_{\alpha}(x) = w^{n}x + \sum_{0<l(\alpha)\leq n} w^{n-l(\alpha)}a(v)^{\alpha}s_{\alpha}(x) \tag{2}$$

where $a(v)^{\alpha} = \prod a_j(v)^{\alpha_j}$.

Since $\bigoplus_{k=1}^{p-1} \sigma^{\otimes k} \cong \rho$,we have

$$w = \prod_{k=1}^{p-1}[k]_{F_{MU}}(v) = (p-1)!v^{p-1} + \sum_{j\geq p} d_j v^j \tag{3}$$

where $d_j \in C$ for all j.

Details in the Key Step

By localization on p and Equation 3, we have

$$v^{p-1} = w \cdot \theta(v)$$

where θ is a power series with the coefficients in $C_{(p)}$ such that $\theta^{-1}(x) = (p-1)! + \sum_{j \geq 1} d_j x^{j-p+1}$.
Now we let ε_m be principal $\mathbb{Z}/p$-bundle $S^{2m-1} \to L^m(p) = S^{2m-1}/p$.
Then we modify Equation 2 into

$$w^n(w^q P_{\varepsilon_m} x - x) = \sum_{0 < l(\alpha) \leq n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x) \tag{4}$$

$$(v^{p-1}\theta^{-1}(v))^n(w^q P_{\varepsilon_m} x - x) = \sum_{0 < l(\alpha) \leq n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x) \tag{5}$$

$$(v^{p-1})^n(w^q P_{\varepsilon_m} x - x) = \psi(v) \tag{6}$$

where $\psi(t) \in R_{(p)}[[t]]$, since $s_\alpha(x) \in R_p$ according to the inductive hypothesis.

Details in the Key Step

Let $m = n(p-1)$ and $r > 0$. Then we have

$$v^m(w^q P_{\varepsilon_m} x - x) = \psi(v) \in U^*(L^m(p) \times X)_{(p)}$$

We may assume m **is the minimal positive integer such that there indeed exists some formal power series** $f(t) \in R_{(p)}[[t]]$ **such that** $v^m(w^q P_{\varepsilon_m} x - x) = f(v)$. **Our goal is to show that the** $m = 1$.

Let $m = n(p-1)$ and $r > 0$. Then we have

$$v^{m}(w^{q}P_{\varepsilon_{m}}x - x) = \psi(v) \in U^{*}(L^{m}(p) \times X)_{(p)}$$

We may assume m is the minimal positive integer such that there indeed exists some formal power series $f(t) \in R_{(p)}[[t]]$ such that $v^{m}(w^{q}P_{\varepsilon_{m}}x - x) = f(v)$. Our goal is to show that the $m = 1$.

Details in the Key Step

Let $i \colon X \to L^m(p) \times X$ be an inclusion for some point at $L^m(p)$ and $i^* v = 0$ because $i^*\sigma$ is a trivial bundle over X.

Note that $i^*(\psi(v)) = \psi(0)$ and $\psi(0) = 0$ by previous equation. Therefore, $t \mid \psi(t)$ and we let $t\psi_1(t) = \psi(t)$:

$$v(v^{m-1}(w^q P_{\varepsilon_m} x - x) - \psi_1(v)) = 0$$

Note that v is exactly the Euler class of $S^{2m-1} \times_{\mathbb{Z}/p} \mathbb{C} \to L^m(p)$. **This remind us of Technical Lemma II**

Details in the Key Step

By Technical Lemma II, there exists $y \in U^{2(m-1)-2q}(X)$ such that

$$j_m^*(v^{m-1}(w^q P_{\varepsilon_m} x - x) - \psi_1(v)) = y\theta_p(v) \in U^*(L^{m-1}(p) \times X)_{(p)} \quad (7)$$

$$v^{m-1}(w^q P_{\varepsilon_{m-1}} x - x) = \psi_1(v) + y\theta_p(v) \quad (8)$$

(Warning: there exists abuse of notations. The definitions of v and w should adjust to the chosen principal bundle automatically.)

We may identify $y \in \widetilde{U}^{2(m-1)-2q}(X)_{(p)}$ by modulo the part on the base point.

Here m must be 1, otherwise it against the minimality of m because $y \in R^{2(m-1)-2q}$ according to the inductive hypothesis.

For $m = 1$, we further have

$$w^q P_{\varepsilon_1} x - x = \psi_1(v) + y\theta_p(v) \in U^*(S^1/p \times X)_{(p)} \tag{9}$$

Let $i\colon X \to S^1/p \times X$ be a natural inclusion as we did it before and apply it to Equation 9. Then we have

$$-x = \psi_1(0) + py \qquad q > 0 \tag{10}$$

$$x^p - x = \psi_1(0) + py \qquad q = 0 \tag{11}$$

For the case $q > 0$: Since x is arbitrary, we have

$$\widetilde{U}^{-2q}(X) \subset R_{(p)}^{-2q} + p\widetilde{U}^{-2q}(X)_{(p)}$$

Then we have

$$\widetilde{U}^{-2q}(X)_{(p)} \subset R_{(p)}^{-2q} + p^n\widetilde{U}^{-2q}(X)_{(p)}$$

for any n. Since $\widetilde{U}^{-2q}(X)$ is a finitely generated abelian group, we have $\widetilde{U}^{-2q}(X)_{(p)} \subset R_{(p)}^{-2q}$.

Details in the Key Step

For the case $q = 0$: Note that $x^p - x \in p\widetilde{U}^0(X) + R^0$. Let

$$\begin{array}{rccc} \gamma: & \widetilde{U}^0(X)/(p\widetilde{U}^0(X) + R^0) & \longrightarrow & \widetilde{U}^0(X)/(p\widetilde{U}^0(X) + R^0) \\ & z & \longmapsto & z^p \end{array}$$

be an endomorphism. Then $x \in \widetilde{U}^0(X)$ is a fixed point for γ. Since $\widetilde{U}^0(X)$ is a nilpotent ideal in $U^0(X)$, we conclude that $x \in p\widetilde{U}^0(X) + R^0$. Then we by using the techniques in the case $q > 0$, we deduce that $\widetilde{U}^0(X) = R^0$.

Outline

The Proof of Injectivity

Theorem

The induced map $L \to MU^*$ is bijective.

It remains to show the injective part. The rough idea is to use Landweber-Novikov operations to build a ring map $MU^* \to R$ for a simpler ring R with simple formal group law and show the composition $L \to MU^* \to R$ is injective.

Let $\phi\colon MU \to H\mathbb{Z}$ be an orientation of $H\mathbb{Z}$. ϕ preserves Euler classes of line bundles and Chern classes $c_\alpha(E)$ for vector bundle E. We define

$$\beta\colon U^*(X) \xrightarrow{s_t} U^*(X)\,[t_1, t_2, t_3, \dots] \xrightarrow{\phi} H^*(X)\,[t_1, t_2, t_3, \dots] \tag{12}$$

where s_t is the total Landweber-Novikov operation.

Proposition

If L is a complex line bundle, then

$$\beta(e_U(L)) = \sum_{j\geq 0} t_j(e_H(L))^{j+1}$$

where $t_0 = 1$.

The Proof of Injectivity

Proposition

The Lazard ring L is a polynomial ring over $\mathbb{Z}$ with a generator in degree q for each $q > 0$.

Let $\theta(x) = \sum_{j \geq 0} t_j x^{j+1}$. Then we have

$$\begin{aligned}
\beta F_{MU}(\theta(e_H(L_1)), \theta(e_H(L_2))) &= \sum_{i,j} \beta(c_{ij}) \theta(e_H(L_1)) \theta(e_H(L_2)) \\
&= \beta F(e_U(L_1), e_U(L_2)) \\
&= \beta(e_U(L_1 \otimes L_2)) \\
&= \theta(e_H(L_1 \otimes L_2)) \\
&= \theta(e_H(L_1) + e_H(L_2))
\end{aligned}$$

Thus we have $(\beta F_{MU})(\theta(x), \theta(y)) = \theta(x + y)$.

The Proof of Injectivity

Note that $\theta(x) = x +$ higher terms, there exists a power series $\theta^{-1}(x)$ such that $\theta \circ \theta^{-1}(x) = x$. Then we consider the following map

$$
\begin{array}{ccccc}
L & \xrightarrow{f} & U^* & \xrightarrow{\beta} & H^*[t_1, t_2, \ldots] \cong \mathbb{Z}[t_1, t_2, \ldots] \\
F_{Univ} & \longmapsto & F_{MU} & \longmapsto & \theta^{-1*} G_a(x, y)
\end{array}
$$

where $G_a(x, y) = x + y$ the additive formal group law and θ^{-1*} means conjugation action of invertible power series on formal group law.

The Proof of Injectivity

Since L is torsion free, we just need to should that $\mathbb{Q} \otimes \beta \circ f$ is injective. Consider the natural transformation

$$\mathrm{Hom}_{\textbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], -) \xrightarrow{(\beta \circ f)^*} \mathrm{Hom}_{\textbf{Cring}}(L, -)$$

There is an evident bijection between $\mathrm{Hom}_{\textbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], R)$ and the set of power series in $R[[x]]$ divided by x setting

$$u \in \mathrm{Hom}_{\textbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], R) \mapsto \theta_u(x) := \sum u(t_j) x^{j+1}$$

For $\mathbb{Q}$-algebra R, we have

$$\mathrm{Hom}_{\textbf{Cring}}(\mathbb{Z}[t_1, t_2, \dots], R) \cong \mathrm{Hom}_{\mathbb{Q}-\mathcal{A}\mathrm{lg}}(\mathbb{Q}[t_1, t_2, \dots], R)$$

The Proof of Injectivity

According to our convention, we have $(\beta \circ f)^*(\theta_u) = (\theta_u^{-1})^* G_a(x, y)$.

Proposition

For each formal group law G over a $\mathbb{Q}$-algebra R, there exists a unique power series $\log_G(x)$ over G such that $G = \log_G^* G_a = G$.

Therefore, we have an isomorphism of functors

$$\mathrm{Hom}_{\mathbb{Q}-\mathcal{A}\mathrm{lg}}(\mathbb{Q}[t_1, t_2, \ldots], -) \to \mathrm{Hom}_{\mathbb{Q}-\mathcal{A}\mathrm{lg}}(\mathbb{Q} \otimes L, -)$$

According to Yoneda lemma, $\mathbb{Q} \otimes (\beta \circ f)$ is an isomorphism and thus injective.

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Levine-Morel's Algebraic Cobordism

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Motivation

*Enlightened by Quillen's work, Levine and Morel extend Quillen's notion of oriented cohomology to the category Sm_k of smooth quasi-projective k-schemes, and further construct the universal oriented cohomology Ω^* on Sm_k, which is called "***algebraic cobordism***".*

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Remark (Idea of construction)

For a finite type k-scheme X, let $\mathcal{M}(X)$ be the set of isomorphic classes of projective morphisms $f: Y \to X$, with $Y \in \mathrm{Sm}_k$, where $\deg[f: Y \to X] := \dim_k(Y)$. Let $\mathcal{M}(X)^+$ be the group completion. $\Omega^*(X)$ is constructed as a quotient of $\mathcal{M}(X)^+$.

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Algebraic Cobordism and Hopkins-Morel Problem

In analogy to stable homotopy category, Ω^* is represented by motivic Thom spectrum **MGL** in the following sense.

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For any $X \in \mathrm{Sm}_k$ and $n \in \mathbb{Z}$,

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Theorem (Hoyois, Hopkins-Morel)

Let k *be a field of characteristic exponent* c. *Let* L *be the Lazard ring. The canonical map*

$$\theta\colon L[\frac{1}{c}] \to \mathbf{MGL}_{2*,*}[\frac{1}{c}]$$

is an isomorphism.