Homotopy Coherence Problem and ∞-categories

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June 12, 2021

Motivation

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Why homotopy is important?

Theorem (Representability of ordinary cohomology)

Let π be an abelian group, $K(\pi, n)$ be the Eilenberg-Maclane space, and X is any topological space, we have a canonical isomorphism

$$[X, K(\pi, n)] \cong H^n(X; \pi)$$

Theorem (Classification of principal G-bundles)

Let G be a topological group and BG be the classifying space of G, then there is a 1-1 correspondence

$$[X, BG] \cong \{\textit{Equivalent classes principal } G\textit{-bundles on } X\}$$

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Homotopies are paths in mapping spaces

Let $\mathcal{S}\mathrm{pace}$ be the category of spaces (spaces means CW-complexes or compactly-generated and weak Hausdorff spaces), where morphisms are continuous maps

Definition (Mapping space)

For any $x, y \in \mathcal{S}\mathrm{pace}$, $\mathrm{Hom}_{\mathcal{S}\mathrm{pace}}(x, y)$ can be endowed with compact-open topology to be a space, we call it mapping space and denote it by $\mathrm{Maps}(x, y)$.

Proposition

There is a canonical correspondence

$$\mathrm{Maps}(x, \mathrm{Maps}(y, z)) \cong \mathrm{Maps}(x \times y, z)$$

In this way, let $f, g \in \mathrm{Maps}(x, y)$, a homotopy from f to g is a path in $\mathrm{Maps}(x, y)$, vice versa.

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Definition

The homotopy category of spaces $\mathrm{h}\mathcal{S}\mathrm{pace}$ *has the same objects as* $\mathcal{S}\mathrm{pace}$*, the morphisms are homotopy classes, i.e.*

$$\mathrm{Hom}_{\mathrm{h}\mathcal{S}\mathrm{pace}}(x, y) = \pi_0 \mathrm{Maps}(x, y)$$

We will see that most of functors which we use frequently in topology is representable in the homotopy category. The representability allows us to study spaces by studying morphisms. **Moreover, we get information of a space from certain diagrams in** $\mathrm{h}\mathcal{S}\mathrm{pace}$**.**

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The blindness of homotopy category

For a space X, X has much more information than $\pi_0(X)$ clearly, hence $\mathrm{h}\mathcal{S}\mathrm{pace}$ has less information than $\mathcal{S}\mathrm{pace}$ by modulo homotopy.

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The blindness of homotopy category

To measure the blindness more concretely, we need the following definition.

Definition

Let A be a small category, a commutative diagram (of A-shape) is a functor $F\colon A \to \mathcal{S}\mathrm{pace}$; a homotopy commutative diagram is a functor $G\colon A \to \mathrm{h}\mathcal{S}\mathrm{pace}$.

Now we may describe the blindness more specifically,

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Example: (co)limits in homotopy category

An effective way to see the difference is to consider (co)limits of the diagrams.

Example

Let's consider two diagrams

$$* \longleftarrow S^n \xrightarrow{i} D^{n+1} \tag{0.1}$$

$$* \longleftarrow S^n \longrightarrow * \tag{0.2}$$

where i is the inclusion of the boundary. Since D^{n+1} and $*$ are isomorphic in $\mathrm{h}\mathcal{S}\mathrm{pace}$, these two diagrams are equivalent in $\mathrm{h}\mathcal{S}\mathrm{pace}$. However, the colimit of Diagram 0.1 in $\mathcal{S}\mathrm{pace}$ is $D^{n+1}/S^n \cong S^{n+1}$ while the colimit of Diagram 0.2 is just a single point $*$, which shows the difference.

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We have shown there is a loss of information when passing to homotopy category, now the question is how can we measure the deviation?

Question (The realization problem)

Given a homotopy commutative diagram $F\colon A \to \mathrm{h}\mathcal{S}\mathrm{pace}$, can we lift the functor to $\mathcal{S}\mathrm{pace}$? Namely, there is a functor $G\colon A \to \mathcal{S}\mathrm{pace}$ such that the composition $\pi \circ G\colon A \to \mathrm{h}\mathcal{S}\mathrm{pace}$ is natural isomorphic to F.

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Example: G-spaces and homotopy G-spaces

Definition

Suppose G is a group, the associated groupoid BG is a category with one object $*$, and $\mathrm{Hom}(x, x) := G$ where the composition rule is given by the group multiplication.

Definition

A G-space is a functor $BG \to \mathcal{S}\mathrm{pace}$. We may also say a space X is a G-space if there is a functor $BG \to \mathcal{S}\mathrm{pace}$ such that X is the image of $*$. Similarly, a homotopy G-space is a functor $BG \to \mathrm{h}\mathcal{S}\mathrm{pace}$.

Let X be a G-space, if $f : Y \to X$ is a homotopy equivalence, then Y is a homotopy G-space. We may say X is a realization of Y.

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A homotopy G-space can be realized by a G-space X if and only if the lifting problem 0.3 has a solution.

$$\begin{tikzcd} & B\mathrm{Aut}(Y) \arrow[d, "B\tau"] \\ BG \arrow[r, "B\alpha"] \arrow[ur, dashed] & B\mathrm{Aut}_0(Y) \end{tikzcd} \tag{0.3}$$

where $\mathrm{Aut}(Y)$ be the group of automorphisms of Y in $\mathcal{S}\mathrm{pace}$, $\mathrm{Aut}_0(Y)$ be the group of automorphisms of Y in $\mathrm{h}\mathcal{S}\mathrm{pace}$ and $\alpha\colon G \to \mathrm{Aut}_0(Y)$ is determined by the homotopy group action. B is the functor of classifying space.

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Example: cup products and Steenrod squares

The Alexander-Whitney approximation D_0 of the diagonal map $D\colon X \to X \times X$ determines the cup product on X.
Let $X \times X$ be a $\mathbb{Z}/2$-space given by $T\colon (x, y) \mapsto (y, x)$.

Problem

$D_0 \simeq D$, but D is T-invariant while D_0 is not! The following diagram is homotopy commutative but not strictly commutative!

$$\begin{tikzcd} & X \times X \arrow[dd, "T"] \\ X \arrow[ur, "D_0"] \arrow[dr, "D_0"'] & \\ & X \times X \end{tikzcd} \tag{0.4}$$

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If we let $\mathbb{Z}/2$ act on X trivially, then D is a T-equivariant map while D_0 is not! We may say D_0 is a homotopy T-invariant.

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Can we realize Diagram 0.4 by cellular map or simplicial map?
Namely, can we make it be a T-equivariant diagram?
If we can, what are the benefits of the realization?

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$$\begin{tikzcd} S^\infty \times X \arrow[r, "\phi"] \arrow[d, "T\times\mathrm{id}"'] & X \times X \arrow[d, "T"] \\ S^\infty \times X \arrow[r, "\phi"] & X \times X \end{tikzcd} \tag{0.5}$$

where T acts on S^∞ by reflection. Note that the diagram is **strictly commutative**; *S^∞ is contractible, hence $S^\infty \times X$* **and** X **are homotopy equivalence.**

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Since $\phi\colon S^\infty \times X \to X \times X$ is T-equivariant, then by quotient the group action, we have

$$\bar{\phi}\colon \mathbb{RP}^\infty \times_{\mathbb{Z}/2} X \to X$$

When passing to $\mathbb{Z}/2$-cohomology, we have

$$\begin{array}{rrcl} \bar{\phi}^* = Sq: & H^*(X;\mathbb{Z}/2) & \longrightarrow & H^*(X;\mathbb{Z}/2)[t] \\ & x & \longmapsto & \sum Sq^i(x)t^i \end{array}$$

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Actually, the map $\phi\colon S^\infty \times X \to X \times X$ carries much more information than $D_0\colon X \to X \times X$.

D_0 makes $H^*(X)$ into a ring, while ϕ makes the singular cochain complex $C^\bullet(X)$ into an E_∞-algebra.

E_∞-algebra structure carries much more information than cohomology ring!

Theorem (Mandell)

Suppose X, Y are simply connected spaces, a continuous map $f\colon X \to Y$ induces a quasi-isomorphism between $C^*(Y)$ and $C^*(X)$ as E_∞ algebra, if and only if f is a weak homotopy equivalence.

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Homotopy coherence structures

Theorem (Dwyer-Kan-Smith,1989)

A homotopy diagram has a realization of and only if it may be lifted to a **homotopy coherent** *diagram.*

Example (Homotopy coherent structure on cup products)

1. *There exists a homotopy* D_1 *from* D_0 *to* TD_0. *In particular,* TD_1 *is a homotopy from* TD_0 *to* D_1;
2. *There exists a homotopy* D_2 *from* $D_1 + TD_1$ *to the constant homotopy of* D_1;
3. $D_2 + TD_2$ *is a homotopy from* $D_1 + TD_1$ *to itself and it is also homotopy to the constant homotopy via* D_3;
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Finally, we have $\{D_n\}_{n\geq 0} \implies S^\infty \times X \to X \times X$.

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3. $D_2 + TD_2$ *is a homotopy from* $D_1 + TD_1$ *to itself and it is also homotopy to the constant homotopy via* D_3*;*
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Finally, we have $\{D_n\}_{n\geq 0} \implies S^\infty \times X \to X \times X$.

Homotopy coherence structures

Theorem (Dwyer-Kan-Smith,1989)

A homotopy diagram has a realization of and only if it may be lifted to a **homotopy coherent** *diagram.*

Example (Homotopy coherent structure on cup products)

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Example

Let's consider a diagram

$$\omega := 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

An ω-shaped diagram in $\mathcal{S}\mathrm{pace}$ consists of space X_k for $k \in \omega$ and morphisms $f_{i,k} \colon X_i \to X_k$ for $i < k$.
If it is a homotopy commutative diagram, then for any $i < j < k$, there is a homotopy $h_{i,j,k} \colon f_{i,k} \simeq f_{j,k} \circ f_{i,j}$.
This process specifies a path in $\mathrm{Maps}(X_i, X_k)$ from vertex $f_{i,k}$ to $f_{j,k} \circ f_{i,j}$.

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Example

Take the same ω-shaped diagram. If it is homotopy coherent. Then for any $i < j < k < l$, the chosen homotopies provides four paths in $\mathrm{Maps}(X_i, X_l)$:

$$\begin{tikzcd} f_{i,l} \arrow[r, dash, "h_{i,k,l}"] \arrow[d, dash, "h_{i,j,l}"'] & f_{k,l} \circ f_{i,k} \arrow[d, dash, "f_{k,l} \circ h_{i,j,k}"] \\ f_{j,l} \circ f_{i,j} \arrow[r, dash, "h_{j,k,l} \circ f_{i,j}"] & f_{k,l} \circ f_{j,k} \circ f_{i,j} \end{tikzcd} \tag{0.6}$$

there is a 2-homotopy to filling the square in $\mathrm{Maps}(X_i, X_l)$. *Similarly, for $i < j < k < l < m$, there are twelve paths and six 2-squares in $\mathrm{Maps}(X_i, X_m)$ and then we specify a 3-homotopy to filling in this cube.*

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Homotopy coherent structures

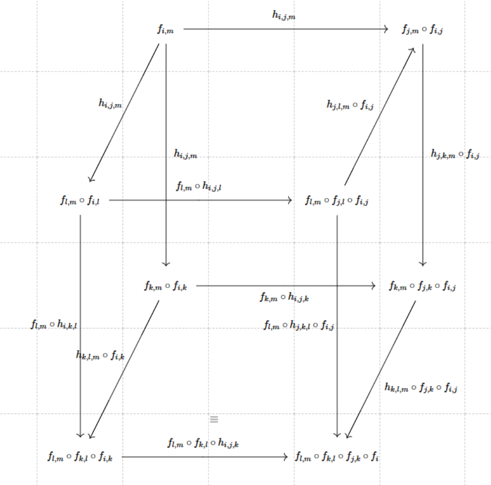

Figure: 3-homotopy filling the cube

Homotopy coherent structures

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Proceeding the procedure, homotopy coherence means that all such n-homotopies exits! In other words, any such n-cubes in the mapping spaces can be filled by higher homotopies.

Even in this simple case of ω, the data of homotopy coherence is much richer than the data of homotopy commutivity. The existence of higher homotopies carries a lot of data.

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Example: non-homotopy coherent diagram

Example (A homotopy commutative diagram that is not homotopy coherent)

Let p be the Hopf fibration, i be inclusion of fiber at the based point and n is a degree map $e^{i\theta} \mapsto e^{in\theta}$:

$$\begin{tikzcd} & S^1 \arrow[dl, "n"'] \arrow[dr, "i"] \arrow[dd] & \\ S^1 \arrow[rr, "i" near start, "*" near end] \arrow[dr, "*"'] & & S^3 \arrow[dl, "p"] \\ & S^2 & \end{tikzcd} \tag{0.7}$$

Since $\pi_1(S^3)$ is trivial, let $\alpha\colon i \simeq i \circ n$ be the homotopy. However, $p \circ \alpha$ is not 2-homotopic to the constant homotopy $*$.

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Why we need ∞-categories?

Motivation

If we just modulo homotopy directly, we will lose the data of homotopy coherence i.e. the higher homotopies. It is very complicated to describe the phenomenon by ordinary category and homotopy category. **∞-categories provides a new framework to describe the homotopy coherence!**

Slogan: the significance of ∞-categories is that they carries higher homotopies information, just like higher homotopy groups carry more information than fundamental groups!

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Example: fundamental groupoid

Definition

Let X be a topological space, the fundamental groupoid $\pi_{\leq 1}(X)$ associated to X is a category whose object are points in X, morphisms between $x, y \in X$ are **homotopy classes** *of paths from x to y and* **the composition rule is given by path multiplications**.

Remark

For each x, the automorphism group $\mathrm{Aut}_{\pi_{\leq 1}(X)}(x)$ is $\pi_1(X, x)$.

Note that all the morphisms in the fundamental groupoid are invertible, since every path admits an inverse up to homotopy. This fundamental groupoid **only depends on the 1-type of X and hence discards a lot of information.**

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Notice that when defining the fundamental groupoid, we modulo the homotopy relations of paths, which leads to loss of information. **What will happen if we do not modulo the homotopy relation?**

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If we do not modulo the homotopy relation, namely, let objects be points of X, morphisms be paths in X and composition rule be path multiplication, **that will be NOT a category! The reason is the failure of associativity law! Though it cannot be a category, it is can be an ∞-category!**

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∞-groupoid and Grothendieck homotopy hypothesis

Definition (Informal definition of ∞-groupoids)

The ∞-groupoid $\pi_{\leq\infty}(X)$ has the following data:

1. *objects are points;*
2. *morphisms are paths;*
3. *2-morphisms are homotopies;*
4. *3-morphisms are 2-homotopies;*
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Grothendieck homotopy hypothesis: spaces and ∞-groupoids should be the same!

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