Computer Aided Geometric Design 87 (2021) 101991

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

A sufficient condition for 3D typical curves

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ARTICLE INFO

Article history:
Available online 15 April 2021

Keywords:
Typical curves
Class A Bézier curves
Curvature
Torsion
Monotonicity

ABSTRACT

2D Typical curves (Mineur et al., 1998) are a class of special Bézier curves with monotone curvature, which play a key role in designing aesthetically pleasing surfaces for the automotive industry. To deal with 3D typical curves, Farin (2006) introduces the more general concept of Class A Bézier curves. These curves are defined by so-called Class A matrix that oughts to satisfy some appropriate conditions for guaranteeing the monotonicity of curvature and torsion. In this paper, we first present new conditions for Class A Bézier curves which complete the proof in Farin (2006). Then using these conditions, we propose a new sufficient condition for 3D typical curves. More, we discover that Farin's claim (Farin, 2006) on 3D typical curves is incorrect. Numerical examples are provided to validate the correctness of our theorems.

1. Introduction

The construction of fair or aesthetically pleasing surfaces (Sapidis, 1994; Hoschek and Kaklis, 1996) is a fundamental problem arising from industrial or conceptual design, particularly from automotive design, aerospace design and ship design. The prevailing way is to create a surface from a set of feature curves, i.e., a boundary curves network, which provides an intuitive way of defining a surface. To obtain desirable or acceptable shapes, feature curves are often demanded to have relatively few regions of monotonically varying curvature and torsion (Birkhoff, 1933; Farin and Sapidis, 1989; Hoschek and Lasser, 1996; Farin, 2002). Consequently, the monotone curvature variation conditions for Bézier curves and B-spline curves have been investigated by several researchers, see, e.g., Sapidis and Frey (1992); Frey and Field (2000); Wang et al. (2004); Li et al. (2006) and references therein. However, these conditions frequently lead to very complex formulas or nonlinear optimization problems as pointed out by Wang et al. (2019) and limit their usefulness.

Another popular approach for constructing curves with monotone curvature is to use a family of specific curves, e.g., spirals (Walton and Meek, 1996a; Walton et al., 2003; Walton and Meek, 2012), spirals with Pythagorean hodograph (Walton and Meek, 1996b; Farouki, 1997; Walton and Meek, 2013), log-aesthetic spirals (Meek et al., 2012), superspirals (Ziatdinov, 2012), conics (Yang, 2004), typical curves (Higashi et al., 1983, 1988; Mineur et al., 1998) and so on. In 2006, inspired by the term "Class A surfaces" from the automotive industry which denotes those parts of a car with visually pleasing appearance, Farin (2006) introduces the concept of Class A Bézier curves as follows. An n-th degree Bézier curve is defined by

$$\mathbf{x}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t), \quad t \in [0, 1], \tag{1}$$

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https://doi.org/10.1016/j.cagd.2021.101991
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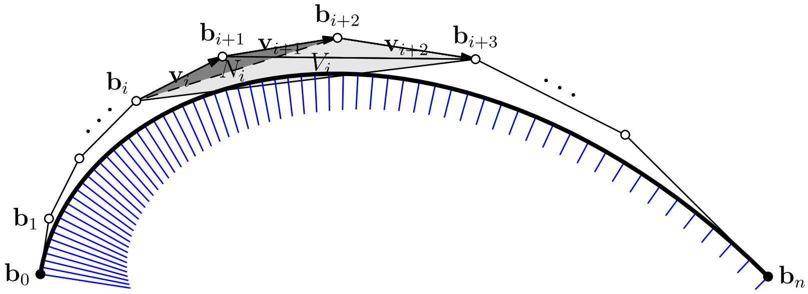

Fig. 1. A Class A Bézier curve with monotone curvature which is illustrated by the blue comb. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

where $\{\mathbf{b}_i\} \in \mathbb{R}^2$ or $\mathbb{R}^3$ are called control points, and $\{B_i^n(t)\}$ are the classical Bernstein polynomials given by

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, \dots, n.$$

The derivative or hodograph of the Bézier curve (1) can be expressed as

$$\dot{\mathbf{x}}(t) = n \sum_{i=0}^{n-1} \Delta \mathbf{b}_i B_i^{n-1}(t),$$

where $\Delta \mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i$, $i = 0, 1, \dots, n-1$. A class of Bézier curves, which have hodographs with control vectors

$$\Delta \mathbf{b}_i = \mathbf{v}_i = M^i \mathbf{v}, \quad \mathbf{v} = \mathbf{b}_1 - \mathbf{b}_0 \text{ and } i = 0, 1, \dots, n-1, \tag{2}$$

and have monotone curvature and torsion, are referred to as *Class A Bézier curves*, and M is called *Class A matrix*. For simplicity, the monotonicity of curvature and torsion in this paper is always assumed to be nonincreasing. Fig. 1 shows an example of Class A Bézier curve. Due to scale invariance of Bézier curves, we may assume without loss of generality that $\mathbf{v}$ is a unit vector.

In order to be Class A Bézier curves, Farin (2006) proposed that the matrix M should satisfy the following two conditions as follows

$$\|(1-t)\mathbf{v} + tM\mathbf{v}\| \geqslant \|\mathbf{v}\|, \quad \text{for all } t \in [0, 1] \text{ and } \|\mathbf{v}\| = 1, \tag{3}$$

and

$$\left(\min\{\sigma_i\}\right)^2 \geqslant \max\{\sigma_i\}, \tag{4}$$

where $\{\sigma_i\}$ are the singular values of M in decreasing order. However, Cao and Wang (2008) find that the proof in Farin (2006) is incomplete, and present some conditions on symmetric matrices for completing it. Furthermore, Wang and Zhao (2018) even show two counter examples to Farin's conditions. By relating the curvature at every point of the curve to the curvature at the endpoints in terms of the eigenvalues of the matrix M and the vector $\mathbf{v}$, Cantón et al. (2021) present new conditions for producing planar curves with monotonic curvature. They even provide a unified derivation of the existing results and give more general results in the planar case. However, the extension of their work to space curves seems not to be straightforward.

In this paper, we first introduce new conditions which guarantee matrices to be of Class A and complete the proof in Farin (2006) without the restriction of symmetric matrices. Then using these conditions, we propose a new sufficient condition for 3D typical curves. More, we discover that Farin's claim (Farin, 2006) on 3D typical curves is incorrect. Finally, numerical examples are provided to verify the correctness of our theorems. The rest of the paper is organized as follows: Section 2 introduces some notations and preliminary knowledge about Class A Bézier curves. Section 3 presents a useful lemma and new conditions for Class A Bézier curves. In Section 4, a new sufficient condition for 3D typical curves is proposed. Section 5 gives some numerical examples. Finally, we conclude the paper with proposals for future work in Section 6.

2. Preliminaries

Let $\mathbf{x}$ be a differentiable curve parameterized by $t \in I \subset \mathbb{R}$, the curvature and torsion (do Carmo, 2016) of $\mathbf{x}$ at t are given by

$$\kappa(t) = \frac{\|\mathbf{x}' \wedge \mathbf{x}''\|}{\|\mathbf{x}'\|^3}, \quad \tau(t) = \frac{|\det[\mathbf{x}', \mathbf{x}'', \mathbf{x}''']|}{\|\mathbf{x}' \wedge \mathbf{x}''\|^2},$$

where we assume $\kappa(t)$ and $\tau(t)$ are unsigned for simplicity. In particular, when $\mathbf{x}$ is a Bézier curve with control vectors (2), the curvatures of $\mathbf{x}$ at two endpoints can be written as

$$\kappa(0) = 2\frac{n-1}{n}k_0, \quad \kappa(1) = 2\frac{n-1}{n}\frac{\|\mathbf{v}_{n-2}\|^3}{\|\mathbf{v}_{n-1}\|^3}k_{n-2} \quad \text{with } k_i = \frac{|N_i|}{\|\mathbf{v}_i\|^3},\ i = 0, 1, \ldots, n-2, \tag{5}$$

where N_i is the triangle formed by vectors $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, and $|N_i|$ denotes the area of N_i, see Fig. 1. Similarly, the torsion of $\mathbf{x}$ at two endpoints can be written as

$$\tau(0) = \frac{3}{2}\frac{n-2}{n}q_0, \quad \tau(1) = \frac{3}{2}\frac{n-2}{n}\frac{|N_{n-3}|^2}{|N_{n-2}|^2}q_{n-3} \quad \text{with } q_i = \frac{|V_i|}{|N_i|^2},\ i = 0, 1, \ldots, n-3, \tag{6}$$

where V_i is the tetrahedron formed by vectors $\mathbf{v}_i$, $\mathbf{v}_{i+1}$, and $\mathbf{v}_{i+2}$, and $|V_i|$ denotes the volume of V_i. For details, please refer to Farin (2006); Cao and Wang (2008).

In order to prove the curvature and torsion monotonicity, we need to compare $\kappa(0)$ with $\kappa(1)$ in (5) and $\tau(0)$ with $\tau(1)$ in (6) respectively. Therefore, the relationship between $|N_i|$ and $|N_{i+1}|$, and the relationship between $|V_i|$ and $|V_{i+1}|$ play an important role in the proof. Observe that triangle N_{i+1} is the image of triangle N_i under a linear transformation represented by M. From the definitions of determinant and singular value (Strang, 2009), it is easy to see that

$$|N_{i+1}| = \sigma_1\sigma_2|N_i|, \tag{7}$$

where $\sigma_1 \geqslant \sigma_2 > 0$ are the singular values of $M \in \mathbb{R}^{2\times 2}$. Similarly, we have

$$|V_{i+1}| = \sigma_1\sigma_2\sigma_3|V_i|, \tag{8}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$ are the singular values of $M \in \mathbb{R}^{3\times 3}$. It should be noted that Eq. (7) is only correct in 2D space.

It is well known that if a Bézier curve (1) is subdivided at parameter t, then both resulting segments are Bézier curves whose control points can be computed by the de Casteljau algorithm. More, Farin (2006) claims that the segments generated by subdividing a Bézier curve with control vectors (2) also have their hodographs being the same form, with corresponding matrix becoming

$$T = (1-t)I + tM \text{ and } MT^{-1}.$$

Also, both T and MT^{-1} satisfy the condition (3). However, condition (4) may be violated after subdivision, see Cao and Wang (2008) for an example. To tackle this problem, Cao and Wang (2008) propose the following conditions for replacing condition (4) as follows:

Theorem 1. *A diagonal matrix $M \in \mathbb{R}^{2\times 2}$ with entries $\sigma_i,\ i = 1, 2$ is a Class A matrix if M satisfies*

$$\sigma_j \geqslant 1 \text{ and } 2\sigma_j \geqslant \sigma_k + 1, \quad (j,k) = (1,2), (2,1). \tag{9}$$

Theorem 2. *A diagonal matrix $M \in \mathbb{R}^{3\times 3}$ with entries $\sigma_i,\ i = 1, 2, 3$ is a Class A matrix if M satisfies*

$$\sigma_j \geqslant 1 \text{ and } 3\sigma_j \geqslant \sigma_k + \sigma_s + 1, \quad (j,k,s) = (1,2,3), (2,3,1), (3,1,2). \tag{10}$$

They prove that if a matrix M satisfies condition (9) or (10), then both T and MT^{-1} also satisfy these conditions, and consequently complete the proof presented in Farin (2006). Although the diagonal matrix M in Theorem 1 and 2 can be further generalized to symmetric matrices, this still limits the range of applications.

3. Conditions for Class A Bézier curves

In this section, we introduce new conditions for Class A Bézier curves. Before proceeding further, let us present a lemma on the areas of a triangle in 3D and its image under a linear transformation.

Lemma 1. *Let $M \in \mathbb{R}^{3\times 3}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{3\times 1}$, A is the triangle formed by $\mathbf{u}$ and $\mathbf{v}$, B is the image of A under a linear transformation represented by M (namely, the triangle formed by $M\mathbf{u}$ and $M\mathbf{v}$). Then we have*

$$\sigma_2\sigma_3|A| \leqslant |B| \leqslant \sigma_1\sigma_2|A|,$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$ are the singular values of M, and $|A|$, $|B|$ denote the areas of A and B respectively.

Proof. The areas of triangles A and B may be expressed as

$$|A|^2 = \frac{1}{4}\det\begin{bmatrix}\mathbf{u}^T\mathbf{u} & \mathbf{u}^T\mathbf{v}\\ \mathbf{v}^T\mathbf{u} & \mathbf{v}^T\mathbf{v}\end{bmatrix} = \frac{1}{4}\det\left[\begin{bmatrix}\mathbf{u}^T\\ \mathbf{v}^T\end{bmatrix}\begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}\right],\ |B|^2 = \frac{1}{4}\det\left[\begin{bmatrix}\mathbf{u}^T\\ \mathbf{v}^T\end{bmatrix}M^TM\begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}\right].$$

If the singular value decomposition of M is given by

$$M = U\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}V, \quad \text{with } U, V \in \mathbb{R}^{3\times 3} \text{ are orthogonal matrices,}$$

then we can rewrite $|A|$ and $|B|$ as

$$|A|^2 = \frac{1}{4}\det\left[\begin{bmatrix}\mathbf{u}^T\\ \mathbf{v}^T\end{bmatrix}V^TV\begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}\right],\ |B|^2 = \frac{1}{4}\det\left[\begin{bmatrix}\mathbf{u}^T\\ \mathbf{v}^T\end{bmatrix}V^T\begin{bmatrix}{\sigma_1}^2 & 0 & 0\\ 0 & {\sigma_2}^2 & 0\\ 0 & 0 & {\sigma_3}^2\end{bmatrix}V\begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}\right].$$

Assume that $V\mathbf{u} \triangleq (u_1, u_2, u_3)^T$ and $V\mathbf{v} \triangleq (v_1, v_2, v_3)^T$. According to the Cauchy-Binet formula, we have

$$|A|^2 = \frac{1}{4}\det\left[\begin{bmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix}\begin{bmatrix}u_1 & v_1\\ u_2 & v_2\\ u_3 & v_3\end{bmatrix}\right] = \frac{1}{4}\left(\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}^2 + \begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}^2 + \begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}^2\right).$$

Similarly, $|B|^2$ can be rewritten as

$$|B|^2 = \frac{1}{4}\det\left[\begin{bmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix}\begin{bmatrix}\sigma_1^2u_1 & \sigma_1^2v_1\\ \sigma_2^2u_2 & \sigma_2^2v_2\\ \sigma_3u_3 & \sigma_3v_3\end{bmatrix}\right] = \frac{1}{4}\left(\sigma_1^2\sigma_2^2\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}^2 + \sigma_1^2\sigma_3^2\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}^2 + \sigma_2^2\sigma_3^2\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}^2\right).$$

From $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$ it is easy to show that

$$\sigma_2^2\sigma_3^2|A|^2 \leqslant |B|^2 \leqslant \sigma_1^2\sigma_2^2|A|^2$$

completing the proof of the lemma. □

Applying Lemma 1 to N_i and N_{i+1}, we get

$$\sigma_2\sigma_3|N_i| \leqslant |N_{i+1}| \leqslant \sigma_1\sigma_2|N_i|, \tag{11}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$ are the singular values of $M \in \mathbb{R}^{3\times 3}$. Combining (11) and (8), we give a sufficient condition for the comparisons of curvature and torsion at the two endpoints of a Bézier curve.

Lemma 2. *Let $\mathbf{x}$ be a 3D Bézier curve with control vectors (2) and $M \in \mathbb{R}^{3\times 3}$ has the singular values $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$. If M satisfies condition (3) and the following conditions*

$$\sigma_3^3 \geqslant \sigma_1\sigma_2 \text{ and } \sigma_2\sigma_3 \geqslant \sigma_1, \tag{12}$$

then $\kappa(0) \geqslant \kappa(1)$ and $\tau(0) \geqslant \tau(1)$.

Proof. From (5) we have

$$\frac{\kappa(0)}{\kappa(1)} = \frac{k_0}{k_{n-2}}\frac{\|\mathbf{v}_{n-1}\|^3}{\|\mathbf{v}_{n-2}\|^3}.$$

Since $\mathbf{v}_{n-1} = M\mathbf{v}_{n-2}$ it follows

$$\sigma_3\|\mathbf{v}_{n-2}\| \leqslant \|\mathbf{v}_{n-1}\| \leqslant \sigma_1\|\mathbf{v}_{n-2}\|.$$

As pointed out in Farin (2006), if M satisfies condition (3), then both $M^T + M - 2I$ and $M^TM - I$ are positive definite matrices. This yields

$$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant 1.$$

So combining these two statements, we obtain

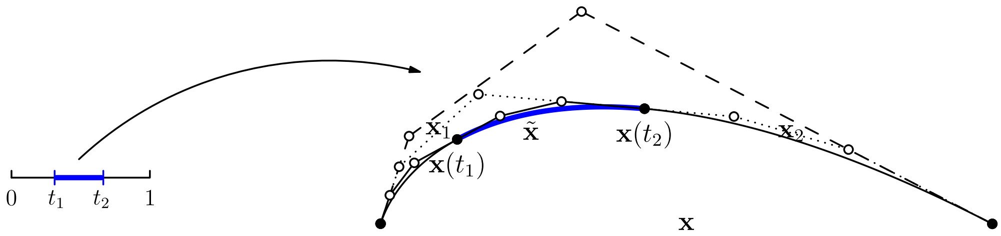

Fig. 2. Subdivide Bézier curve $\mathbf{x}$ twice and obtain $\tilde{\mathbf{x}}$ associated with the parameter interval $[t_1, t_2]$.

$$\frac{\kappa(0)}{\kappa(1)} = \frac{k_0}{k_{n-2}} \frac{\|\mathbf{v}_{n-1}\|^3}{\|\mathbf{v}_{n-2}\|^3} \geqslant \frac{k_0}{k_{n-2}} \sigma_3^3 \geqslant \frac{k_0}{k_{n-2}} = \frac{k_0}{k_1} \frac{k_1}{k_2} \cdots \frac{k_{n-3}}{k_{n-2}}.$$

Recalling from (5), (11) and (12), it is easy to show that

$$\frac{k_i}{k_{i+1}} = \frac{|N_i|}{|N_{i+1}|} \frac{\|\mathbf{v}_{i+1}\|^3}{\|\mathbf{v}_i\|^3} \geqslant \frac{\sigma_3^3}{\sigma_1 \sigma_2} \geqslant 1, \; i = 0, 1, \ldots, n-3.$$

From this, we immediately have $\kappa(0) \geqslant \kappa(1)$. Similarly, from (6) and (11) we have

$$\frac{\tau(0)}{\tau(1)} = \frac{q_0}{q_{n-3}} \frac{|N_{n-2}|^2}{|N_{n-3}|^2} \geqslant \frac{q_0}{q_{n-3}} \sigma_2^2 \sigma_3^2 \geqslant \frac{q_0}{q_{n-3}} = \frac{q_0}{q_1} \frac{q_1}{q_2} \cdots \frac{q_{n-4}}{q_{n-3}}.$$

Recalling from (6), (8) and (12), it is easy to show that

$$\frac{q_i}{q_{i+1}} = \frac{|V_i|}{|V_{i+1}|} \frac{|N_{i+1}|^2}{|N_i|^2} \geqslant \frac{\sigma_2 \sigma_3}{\sigma_1} \geqslant 1, \; i = 0, 1, \ldots, n-4.$$

This confirms $\tau(0) \geqslant \tau(1)$ and completes the proof of the lemma. $\square$

With the help of the last lemma, we can now prove the following theorem.

Theorem 3. *Let $\mathbf{x}$ be a 3D Bézier curve with control vectors satisfying (2) and $M \in \mathbb{R}^{3\times 3}$. Assume that*

$$T_1 = (1-t_2)I + t_2 M, \quad T_2 = (1-t_1)I + t_1 T_1, \quad \tilde{T} = T_1 T_2^{-1} \tag{13}$$

and $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$ are the singular values of $\tilde{T}$. If M satisfies condition (3) and $\tilde{T}$ satisfies condition (12) for any $t_1 < t_2 \in [0, 1]$, then $\mathbf{x}$ is a Class A Bézier curve, and M is a Class A matrix.

Proof. For any two points $\mathbf{p} = \mathbf{x}(t_1)$ and $\mathbf{q} = \mathbf{x}(t_2)$ on $\mathbf{x}$ with $t_1 < t_2 \in [0, 1]$, we may subdivide $\mathbf{x}$ at t_2 resulting in the two Bézier curve $\mathbf{x}_1$ and $\mathbf{x}_2$ associated with the parameter interval $[0, t_2]$ and $[t_2, 1]$ respectively. Again, we subdivide $\mathbf{x}_1$ at t_1 and obtain the Bézier curve $\tilde{\mathbf{x}}$ associated with the parameter interval $[t_1, t_2]$, see Fig. 2. As illustrated in Farin (2006), the corresponding matrix of $\tilde{\mathbf{x}}$ defined in (2) becomes $\tilde{T}$. Since Bézier curves are invariant under affine parameter transformations (Farin, 2002), we may reparametrize $\tilde{\mathbf{x}}(t)$ by introducing a new parameter $u = (t - t_1)/(t_2 - t_1) \in [0, 1]$. Applying Lemma 2 to $\tilde{\mathbf{x}}(u)$, we immediately see that $\kappa(t_1) \geqslant \kappa(t_2)$ and $\tau(t_1) \geqslant \tau(t_2)$. This completes the proof. $\square$

Remark 1. If M together with $T(= (1-t)I + tM)$ and MT^{-1} satisfy conditions (3) and (12) for any $t \in [0, 1]$, then $\mathbf{x}$ is a Class A Bézier curve and M is a Class A matrix. The proof is straightforward and so it is omitted. Clearly, conditions presented in Theorem 3 are weaker than the above conditions. These will be demonstrated in the next section.

In 2D, an argument similar to Theorem 3 leads to the following result.

Theorem 4. *Let $\mathbf{x}$ be a 2D Bézier curve with control vectors (2) and $M \in \mathbb{R}^{2\times 2}$. Assume that*

$$T_1 = (1-t_2)I + t_2 M, \quad T_2 = (1-t_1)I + t_1 T_1, \quad \tilde{T} = T_1 T_2^{-1}$$

and $\sigma_1 \geqslant \sigma_2 > 0$ are the singular values of $\tilde{T}$. If M satisfies condition (3) and $\tilde{T}$ satisfies the following condition

$$\sigma_2^2 \geqslant \sigma_1 \tag{14}$$

for any $t_1 < t_2 \in [0, 1]$, then $\mathbf{x}$ is a Class A Bézier curve, and M is a Class A matrix.

Remark 2. Cao and Wang (2008) show that when M is symmetric, condition (12) can be derived from condition (10). In fact, we can prove Theorem 1 and 2 using Theorem 3 and 4. Therefore, it is evident that our theorems hold in more general setting than Cao and Wang's theorems.

4. Typical curves

Typical curves (Mineur et al., 1998) are a class of Bézier curves which utilize control polygons generated by a constant rotation and scale of the previous leg and have monotonously increasing or decreasing curvature. Because of the fact that a composite transformation defined by a rotation and scaling may be represented by a single matrix M, typical curves fall into a category of Class A Bézier curves.

Definition 1. Let $\mathbf{x}$ be a 2D Bézier curve with control vectors satisfying (2) and assume

$$M = \sigma \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \sigma > 0. \tag{15}$$

If $\mathbf{x}$ has monotone curvature, then it is referred to as a 2D typical curve.

The early works of Higashi et al. (1983, 1988) investigate 2D typical curves of degree 3 and give some constraints of position and tangent direction at their endpoints for achieving curvature monotonicity. Later, Mineur et al. (1998) extend 2D typical curves to any degree $n \geqslant 3$ by employing complex numbers to describe rotations and scaling. They calculate the expression of curvature explicitly, and present a sufficient condition for guaranteeing the monotonicity of curvature as follows:

Theorem 5. *Let* $\mathbf{x}$ *be a 2D Bézier curve defined as (15). If M satisfies*

$$\sigma \cos\theta \geqslant 1, \tag{16}$$

then $\mathbf{x}$ *is a 2D typical curve.*

Since Mineur et al. (1998) use complex numbers to describe rotations and scaling, their method is limited to 2D curves. To overcome this, Farin (2006) introduces the matrix-based notation and presents a sufficient condition for 3D typical curves, i.e., $\sigma \cos\theta \geqslant 1$, but it is incorrect as illustrated in Section 5.

Definition 2. Let $\mathbf{x}$ be a 3D Bézier curve with control vectors satisfying (2) and assume without loss of generality

$$M = \sigma \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \sigma > 0. \tag{17}$$

If $\mathbf{x}$ has monotone curvature and torsion, then it is referred to as a 3D typical curve.

Using Theorem 3, we propose the following sufficient condition for 3D typical curves.

Theorem 6. *Let* $\mathbf{x}$ *be a 3D Bézier curve defined as (17). If M satisfies*

$$\sigma \cos\theta \geqslant \frac{\sigma + 1}{2}, \tag{18}$$

then $\mathbf{x}$ *is a 3D typical curve.*

Proof. From (18) and $\sigma > 0$, we know that $\sigma \geqslant 1$ and $\sigma \cos\theta \geqslant 1$. According to the assumption of M, it follows that

$$M + M^T - 2I = 2 \begin{bmatrix} \sigma\cos\theta - 1 & 0 & 0 \\ 0 & \sigma\cos\theta - 1 & 0 \\ 0 & 0 & \sigma - 1 \end{bmatrix} \text{ and } M^T M - I = (\sigma^2 - 1)I,$$

both of which are positive definite matrices. Therefore, M satisfies condition (3). Using the notation from Section 3 and conducting similar arguments described in Theorem 5, we also have

$$\tilde{T}\tilde{T}^T = \left[t_1^2 I + t_1(1 - t_1)(T_1^{-1} + {T_1^{-1}}^T) + (1 - t_1)^2 (T_1 T_1^T)^{-1} \right]^{-1}.$$

Similarly, from $T_1 = (1 - t_2)I + t_2 M$ we get

$$T_1 = \begin{bmatrix} (1-t_2)+t_2\sigma\cos\theta & -t_2\sigma\sin\theta & 0 \\ t_2\sigma\sin\theta & (1-t_2)+t_2\sigma\cos\theta & 0 \\ 0 & 0 & (1-t_2)+t_2\sigma \end{bmatrix}$$

and

$$T_1^{-1} = \frac{1}{\mu}\begin{bmatrix} (1-t_2)+t_2\sigma\cos\theta & t_2\sigma\sin\theta & 0 \\ -t_2\sigma\sin\theta & (1-t_2)+t_2\sigma\cos\theta & 0 \\ 0 & 0 & \dfrac{\mu}{(1-t_2)+t_2\sigma} \end{bmatrix},$$

where $\mu = (1-t_2)^2 + 2t_2(1-t_2)\sigma\cos\theta + t_2^2\sigma^2$. Furthermore,

$$(T_1T_1^T)^{-1} = \frac{1}{\mu}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\mu}{[(1-t_2)+t_2\sigma]^2} \end{bmatrix}$$

and

$$(T_1^{-1} + {T_1^{-1}}^T) = \frac{2}{\mu}\begin{bmatrix} (1-t_2)+t_2\sigma\cos\theta & 0 & 0 \\ 0 & (1-t_2)+t_2\sigma\cos\theta & 0 \\ 0 & 0 & \dfrac{\mu}{(1-t_2)+t_2\sigma} \end{bmatrix}.$$

Hence, we obtain

$$\tilde{T}\tilde{T}^T = \begin{bmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \eta \end{bmatrix}^{-1}, \quad \text{with } \nu = (1-t_1)^2\frac{1}{\mu} + 2t_1(1-t_1)\frac{(1-t_2)+t_2\sigma\cos\theta}{\mu} + t_1^2$$

and $\eta = (1-t_1)^2\dfrac{1}{[(1-t_2)+t_2\sigma]^2} + 2t_1(1-t_1)\dfrac{1}{(1-t_2)+t_2\sigma} + t_1^2 = \left[\dfrac{1-t_1}{(1-t_2)+t_2\sigma} + t_1\right]^2$. Clearly, we have

$$\frac{1}{\mu} = \frac{1}{(1-t_2)^2 + 2t_2(1-t_2)\sigma\cos\theta + t_2^2} \geqslant \frac{1}{[(1-t_2)+t_2\sigma]^2}, \quad \forall t_2 \in [0,1].$$

Since $\dfrac{(1-t_2)+t_2\sigma\cos\theta}{\mu} - \dfrac{1}{(1-t_2)+t_2\sigma} = \dfrac{t_2(t_2\sigma+t_2-1)(\sigma\cos\theta-\sigma)}{\mu[(1-t_2)+t_2\sigma]}$, it follows that

$$\frac{(1-t_2)+t_2\sigma\cos\theta}{\mu} \geqslant \frac{1}{(1-t_2)+t_2\sigma}, \quad \forall t_2 \in [0, \frac{1}{\sigma+1}],$$

and

$$\frac{(1-t_2)+t_2\sigma\cos\theta}{\mu} \leqslant \frac{1}{(1-t_2)+t_2\sigma}, \quad \forall t_2 \in [\frac{1}{\sigma+1}, 1].$$

By solving $\nu - \eta = 0$, we find

$$t_1 = \frac{1-t_2}{t_2\left[1+t_2(\sigma^2-1)\right]}.$$

Consequently, we have two cases:

$$\text{(I).}\ \nu \geqslant \eta, \quad \text{if } t_2 \in [0, \frac{1}{\sigma+1}] \text{ or } t_2 \in [\frac{1}{\sigma+1}, 1] \cap t_1 \in [0, \frac{1-t_2}{t_2\left[1+t_2(\sigma^2-1)\right]}],$$

and

$$\text{(II).}\ \nu \leqslant \eta, \quad \text{if } t_2 \in [\frac{1}{\sigma+1}, 1] \cap t_1 \in [\frac{1-t_2}{t_2\left[1+t_2(\sigma^2-1)\right]}, 1].$$

Next, we will prove that in both of these cases, the theorem holds.

Case (I): In this case, the singular values of $\tilde{T}$ are

$$\sigma_1(\tilde{T}) = \sqrt{\eta^{-1}} \geqslant \sigma_2(\tilde{T}) = \sigma_3(\tilde{T}) = \sqrt{\nu^{-1}}.$$

Thus condition (12) becomes

$$\sigma_3^2(\tilde{T}) \geqslant \sigma_1(\tilde{T}) \iff \sqrt{\eta} \geqslant \nu,$$

which is equivalent to

$$(1-t_1)\frac{1}{(1-t_2)+t_2\sigma} + t_1 \geqslant (1-t_1)^2\frac{1}{\mu} + 2t_1(1-t_1)\frac{(1-t_2)+t_2\sigma\cos\theta}{\mu} + t_1^2. \tag{19}$$

Using the Bernstein polynomials of the variable t_1, we can rewrite (19) as

$$\left[\frac{1}{(1-t_2)+t_2\sigma} - \frac{1}{\mu}\right]B_0^2(t_1) + \left[\frac{1}{2} + \frac{1}{2\left[(1-t_2)+t_2\sigma\right]} - \frac{(1-t_2)+t_2\sigma\cos\theta}{\mu}\right]B_1^2(t_1) \geqslant 0.$$

Thus, it suffices to prove

$$\frac{1}{(1-t_2)+t_2\sigma} - \frac{1}{\mu} \geqslant 0 \text{ and } \frac{1}{2} + \frac{1}{2\left[(1-t_2)+t_2\sigma\right]} - \frac{(1-t_2)+t_2\sigma\cos\theta}{\mu} \geqslant 0.$$

In fact, it can be easily verified that

$$\begin{aligned}
\frac{1}{(1-t_2)+t_2\sigma} - \frac{1}{\mu} \geqslant 0 &\iff (1-t_2)^2 + 2t_2(1-t_2)\sigma\cos\theta + t_2^2\sigma^2 \geqslant (1-t_2)+t_2\sigma \\
&\iff \left(\sigma\cos\theta - \frac{\sigma+1}{2}\right)B_1^2(t_2) + \sigma(\sigma-1)B_2^2(t_2) \geqslant 0 \\
&\Longleftarrow \sigma\cos\theta \geqslant \frac{\sigma+1}{2}.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
&\frac{1}{2} + \frac{1}{2\left[(1-t_2)+t_2\sigma\right]} - \frac{(1-t_2)+t_2\sigma\cos\theta}{\mu} \geqslant 0 \\
\iff &\left[(-1-\sigma+2\sigma\cos\theta) + 2(1-2\sigma\cos\theta+\sigma^2)t_2 + (\sigma-1)(1-2\sigma\cos\theta+\sigma^2)t_2^2\right]t_2 \geqslant 0 \\
\iff &(-1-\sigma+2\sigma\cos\theta)B_0^2(t_2) + \sigma(\sigma-1)B_1^2(t_2) + \sigma^2(1+\sigma-2\cos\theta)B_2^2(t_2) \geqslant 0 \\
\Longleftarrow &\ \sigma\cos\theta \geqslant \frac{\sigma+1}{2}.
\end{aligned}$$

Therefore, from (18) we arrive at the conclusion immediately.

Case (II): In this case, the singular values of $\tilde{T}$ are

$$\sigma_1(\tilde{T}) = \sigma_2(\tilde{T}) = \sqrt{\nu^{-1}} \geqslant \sigma_3(\tilde{T}) = \sqrt{\eta^{-1}}.$$

Thus condition (12) becomes

$$\sigma_3^3(\tilde{T}) \geqslant \sigma_1^2(\tilde{T}) \text{ and } \sigma_3(\tilde{T}) \geqslant 1 \iff \nu \geqslant \sqrt{\eta^3} \text{ and } \eta \leqslant 1,$$

which are equivalent to

$$(1-t_1)^2\frac{1}{\mu} + 2t_1(1-t_1)\frac{(1-t_2)+t_2\sigma\cos\theta}{\mu} + t_1^2 \geqslant \left[\frac{1-t_1}{(1-t_2)+t_2\sigma} + t_1\right]^3 \tag{20}$$

and

$$\frac{1-t_1}{(1-t_2)+t_2\sigma} + t_1 \leqslant 1. \tag{21}$$

For (21), it is straightforward to show that

$$\begin{aligned}
\frac{1-t_1}{(1-t_2)+t_2\sigma} + t_1 \leqslant 1 &\iff \frac{1}{(1-t_2)+t_2\sigma}B_0^1(t_1) + 1\cdot B_1^1(t_1) \leqslant 1 \\
&\iff (1-t_2)+t_2\sigma \geqslant 1 \\
&\iff \sigma \geqslant 1
\end{aligned}$$

and thus holds. Using the Bernstein polynomials of the variable t_1, we can rewrite (20) as

$$\left[\frac{1}{\mu}-\frac{1}{[(1-t_2)+t_2\sigma]^3}\right]B_0^3(t_1)+\left[\frac{1+2[(1-t_2)+t_2\sigma\cos\theta]}{3\mu}-\frac{1}{[(1-t_2)+t_2\sigma]^2}\right]B_1^3(t_1)$$
$$+\left[\frac{1}{3}-\frac{1}{[(1-t_2)+t_2\sigma]}+\frac{2[(1-t_2)+t_2\sigma\cos\theta]}{3\mu}\right]B_2^3(t_1)\geqslant 0.$$

Thus, it suffices to prove

$$\frac{1}{\mu}-\frac{1}{[(1-t_2)+t_2\sigma]^3}\geqslant 0,\quad \frac{1+2[(1-t_2)+t_2\sigma\cos\theta]}{3\mu}-\frac{1}{[(1-t_2)+t_2\sigma]^2}\geqslant 0\ \text{and}$$
$$\frac{1}{3}-\frac{1}{[(1-t_2)+t_2\sigma]}+\frac{2[(1-t_2)+t_2\sigma\cos\theta]}{3\mu}\geqslant 0.$$

In fact, it can be verified that

$$\begin{aligned}
\frac{1}{\mu}-\frac{1}{[(1-t_2)+t_2\sigma]^3}\geqslant 0 &\Longleftrightarrow [(1-t_2)+t_2\sigma]^3\geqslant(1-t_2)^2+2t_2(1-t_2)\sigma\cos\theta+t_2^2\sigma^2\\
&\Longleftrightarrow \left[(-1+3\sigma-2\sigma\cos\theta)+2\left(1-3\sigma+\sigma\cos\theta+\sigma^2\right)t_2+(\sigma-1)^3t_2^2\right]t_2\geqslant 0\\
&\Longleftrightarrow (-1+3\sigma-2\sigma\cos\theta)B_0^2(t_2)+\sigma(\sigma-\cos\theta)B_1^2(t_2)+\sigma^2(\sigma-1)B_2^2(t_2)\geqslant 0\\
&\Longleftarrow \sigma\geqslant 1.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
&\frac{1+2[(1-t_2)+t_2\sigma\cos\theta]}{3\mu}-\frac{1}{[(1-t_2)+t_2\sigma]^2}\geqslant 0\\
&\Longleftrightarrow [1+2[(1-t_2)+t_2\sigma\cos\theta]][(1-t_2)+t_2\sigma]^2\geqslant 3\left[(1-t_2)^2+2t_2(1-t_2)\sigma\cos\theta+t_2^2\sigma^2\right]\\
&\Longleftrightarrow 2\left[(-1+3\sigma-2\sigma\cos\theta)+[2-5\sigma+\sigma(1+2\sigma)\cos\theta]t_2+(\sigma-1)^2(\sigma\cos\theta-1)t_2^2\right]t_2\geqslant 0\\
&\Longleftrightarrow (-1+3\sigma-2\sigma\cos\theta)B_0^2(t_2)+\frac{\sigma(1-3\cos\theta+2\sigma\cos\theta)}{2}B_1^2(t_2)+\sigma^2(\sigma\cos\theta-1)B_2^2(t_2)\geqslant 0\\
&\Longleftrightarrow 2\sigma(1-\cos\theta)+(\sigma-1)\geqslant 0,\quad 2\sigma\cos\theta\geqslant 3\cos\theta-1,\quad \sigma\cos\theta\geqslant 1\\
&\Longleftarrow \sigma\cos\theta\geqslant\frac{\sigma+1}{2}.
\end{aligned}$$

Finally, we show that

$$\begin{aligned}
&\frac{1}{3}-\frac{1}{[(1-t_2)+t_2\sigma]}+\frac{2[(1-t_2)+t_2\sigma\cos\theta]}{3\mu}\geqslant 0\\
&\Longleftrightarrow 2[(1-t_2)+t_2\sigma\cos\theta][(1-t_2)+t_2\sigma]\geqslant\left[(1-t_2)^2+2t_2(1-t_2)\sigma\cos\theta+t_2^2\sigma^2\right][3-[(1-t_2)+t_2\sigma]]\\
&\Longleftrightarrow \left[(-1+3\sigma-2\sigma\cos\theta)+2(1-2\sigma-\sigma^2+2\sigma^2\cos\theta)t_2+(\sigma-1)(1-2\sigma\cos\theta+\sigma^2)t_2^2\right]t_2\geqslant 0\\
&\Longleftrightarrow (-1+3\sigma-2\sigma\cos\theta)B_0^2(t_2)+\sigma(\sigma-1)(2\cos\theta-1)B_1^2(t_2)+\sigma^2(-3+\sigma+2\cos\theta)B_2^2(t_2)\geqslant 0\\
&\Longleftrightarrow 2\sigma(1-\cos\theta)+(\sigma-1)\geqslant 0,\quad 2\cos\theta\geqslant 1,\quad 2\cos\theta\geqslant 3-\sigma\\
&\Longleftarrow \sigma\cos\theta\geqslant\frac{\sigma+1}{2},
\end{aligned}$$

which completes the proof. □

Remark 3. Note that even if M satisfies condition (18), we cannot guarantee that both $T(=(1-t)I+tM)$ and MT^{-1} also satisfy this condition. In fact, the resulting matrix T_1 after subdivision cannot be represented as a matrix of the form $\tilde{\sigma}\begin{bmatrix}\cos\tilde{\theta} & -\sin\tilde{\theta} & 0\\ \sin\tilde{\theta} & \cos\tilde{\theta} & 0\\ 0 & 0 & 1\end{bmatrix}$. On the other hand, we have just shown that M satisfies conditions (13). This confirms the statement made in Remark 1.

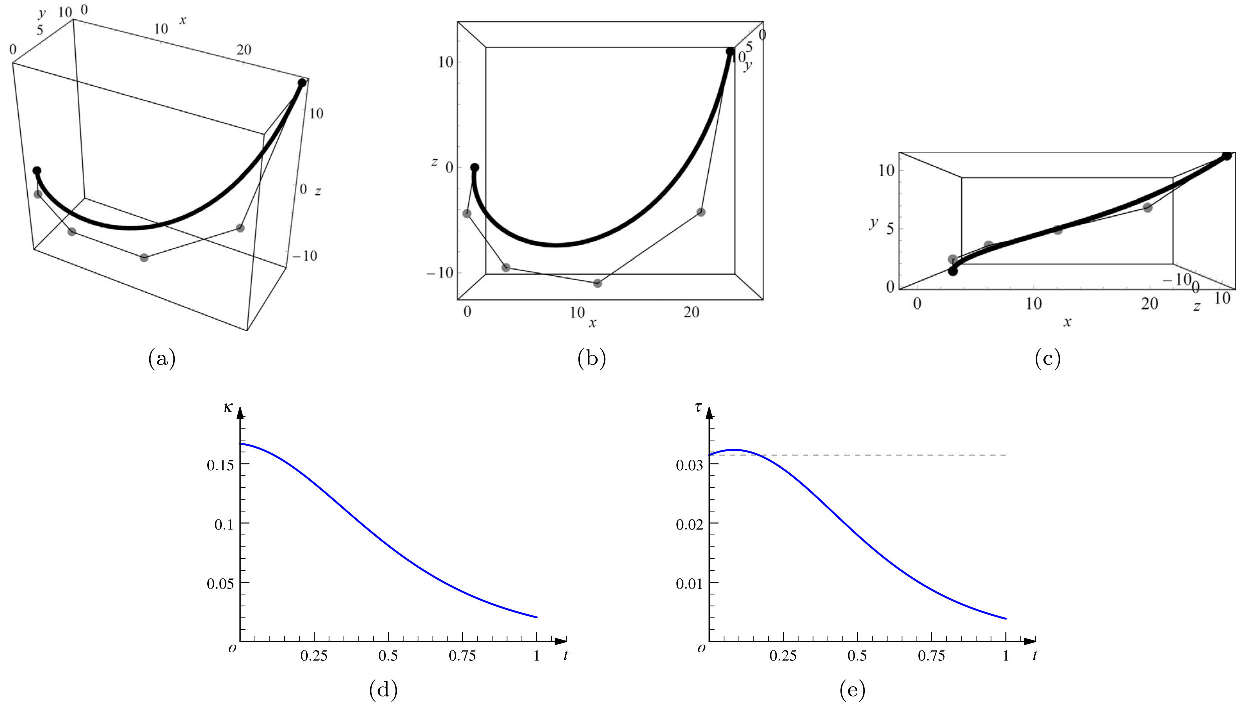

Fig. 3. A counter example to Farin's condition on 3D typical curve: (a)-(c) curve and control polygon from different views; (d) curvature plot; (e) torsion plot.

5. Examples

In this section, we show several numerical examples. The first one is a 3D Bézier curve with

$$n = 5, \quad \mathbf{v} = \begin{bmatrix} -1.00 \\ 1.00 \\ -4.50 \end{bmatrix}, \quad M = 1.42 \times \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix},$$

which is illustrated in Fig. 3(a)-(c). Here $\sigma = 1.42$ and $\theta = \frac{\pi}{4}$, $\sigma \cos\theta \approx 1.0041$, and hence it satisfies $\sigma \cos\theta \geqslant 1$, i.e., Farin's condition on 3D typical curves. However, from Fig. 3(d)-(e), we see that it is not a 3D typical curve.

The second example is a 3D Bézier curve using the same parameters as the first one except $\sigma = 2.42$, which is illustrated in Fig. 4(a)-(c). Now $\sigma = 2.42$ and $\sigma \cos\theta \approx 1.7112 \geqslant 1.71 = \frac{\sigma+1}{2}$, and hence condition (18) is satisfied. Fig. 4(d)-(e) confirms that it is a 3D typical curve.

The third one is a 3D Bézier curve with

$$n = 7, \quad \mathbf{v} = \begin{bmatrix} 0.3421 \\ -0.6551 \\ -0.6736 \end{bmatrix}, \quad M = 1.30 \times \begin{bmatrix} \cos\frac{5\pi}{36} & -\sin\frac{5\pi}{36} & 0 \\ \sin\frac{5\pi}{36} & \cos\frac{5\pi}{36} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is illustrated in Fig. 5(a)-(c). Here $\sigma = 1.30$ and $\theta = \frac{5\pi}{36}$, and condition (18) is satisfied. Thus it is a 3D typical curve, which is demonstrated by Fig. 5(d)-(e).

We now present two examples which satisfy our sufficient conditions for Class A Bézier curves but are not symmetric. The first one is a 2D Bézier curve with

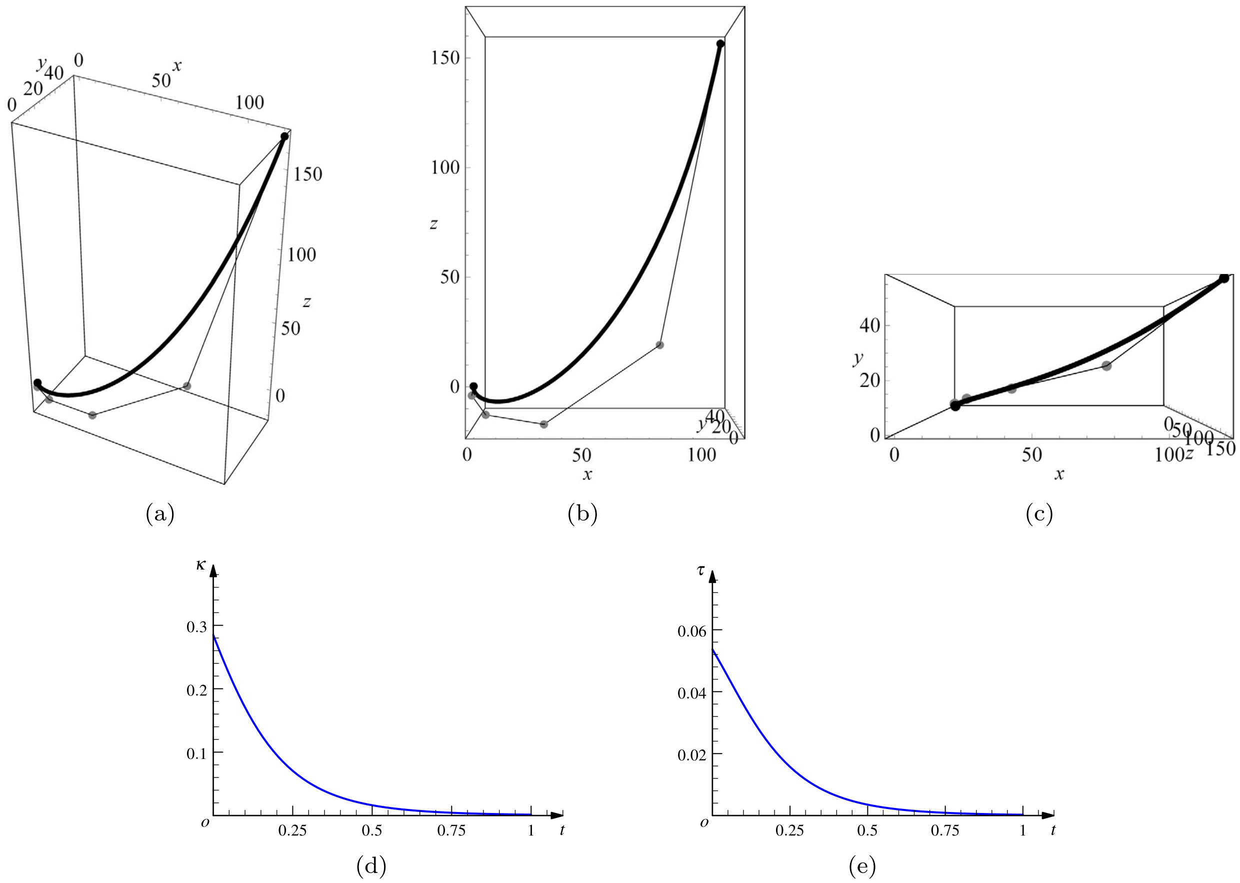

Fig. 4. A 3D typical curve: (a)-(c) curve and control polygon from different views; (d) curvature plot; (e) torsion plot.

$$n = 8, \quad \mathbf{v} = \begin{bmatrix} -0.9724 \\ 0.2333 \end{bmatrix}, \quad M = \begin{bmatrix} 1.1526 & -0.9097 \\ 0.9871 & 1.1079 \end{bmatrix},$$

which is illustrated in Fig. 6(a). Here M is a general 2×2 matrix, and the singular values of M are $\sigma_1 = 1.5201$ and $\sigma_2 = 1.4308$. A simple calculation shows that

$$M + M^T - 2I = \begin{bmatrix} 0.3052 & 0.0774 \\ 0.0774 & 0.2158 \end{bmatrix}, \quad MM^T - I = \begin{bmatrix} 1.1560 & 0.1299 \\ 0.1299 & 1.2018 \end{bmatrix},$$

which are both positive definite matrices. Therefore, M satisfies the condition (3). Using symbolic computation softwares, e.g., Mathematica or Maple, we plot the function $(\sigma_2^2 - \sigma_1)(\tilde{T})$ as shown in Fig. 6(b). Obviously, it meets the condition (14) and hence Theorem 4 holds. Fig. 6(c) demonstrates the monotonicity of curvature.

The second example is a 3D Bézier curve with

$$n = 5, \quad \mathbf{v} = \begin{bmatrix} -0.8066 \\ -0.3817 \\ -0.4513 \end{bmatrix}, \quad M = \begin{bmatrix} 2.5128 & -0.2562 & -0.0251 \\ 0.5947 & 2.1968 & -0.0157 \\ 0.3161 & 0.1003 & 2.6091 \end{bmatrix},$$

which is illustrated in Fig. 7(a). Here M is a general 3×3 matrix, and the singular values of M are $\sigma_1 = 2.7742$, $\sigma_2 = 2.4718$ and $\sigma_3 = 2.1614$. A simple calculation shows that

$$M + M^T - 2I = \begin{bmatrix} 3.0256 & 0.3385 & 0.2910 \\ 0.3385 & 2.3936 & 0.0846 \\ 0.2910 & 0.0846 & 3.2182 \end{bmatrix}, \quad MM^T - I = \begin{bmatrix} 5.3804 & 0.9319 & 0.7031 \\ 0.9319 & 4.1798 & 0.3674 \\ 0.7031 & 0.3674 & 5.9174 \end{bmatrix},$$

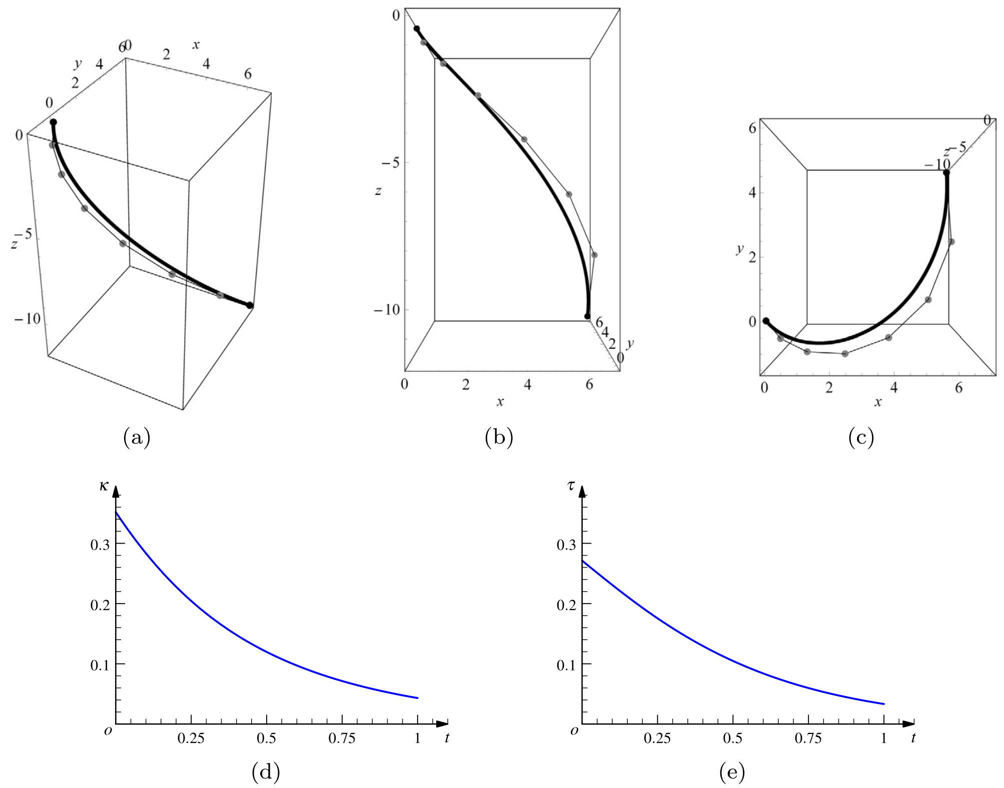

Fig. 5. A 3D typical curve: (a)-(c) curve and control polygon from different views; (d) curvature plot; (e) torsion plot.

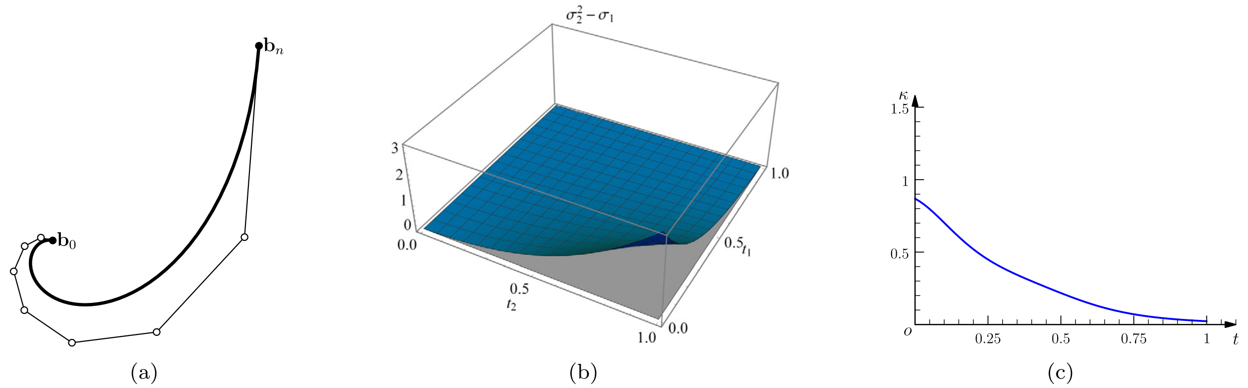

Fig. 6. A 2D Class A Bézier curve with general matrix: (a) curve and control polygon; (b) plot of function $\sigma_2^2 - \sigma_1$ with respect to t_1 and t_2; (c) curvature plot.

which are both positive definite matrices. Therefore, M satisfies the condition (3). Also, we plot the functions $(\sigma_3^2 - \sigma_1\sigma_2)(\tilde{T})$ and $(\sigma_2\sigma_3 - \sigma_1)(\tilde{T})$ as shown in Fig. 7(b) and 7(c) respectively. It is evident that M meets the condition (12) and hence Theorem 3 holds. The monotonicity of curvature and torsion is confirmed by Fig. 7(d) and 7(e).

6. Conclusions and future work

Typical curves and more general Class A curves play an important role in the shape design of aesthetically pleasing surfaces. In this work, we present two theorems which provide sufficient conditions for 2D and 3D Class A Bézier curves respectively. Compared with the existing conditions for Class A Bézier curves, our conditions are more general and useful. Based on them, we propose a new sufficient condition for 3D typical curves. More, we point out that Farin's claim on 3D typical curves is incorrect. Numerical examples demonstrate the correctness of our theorems.

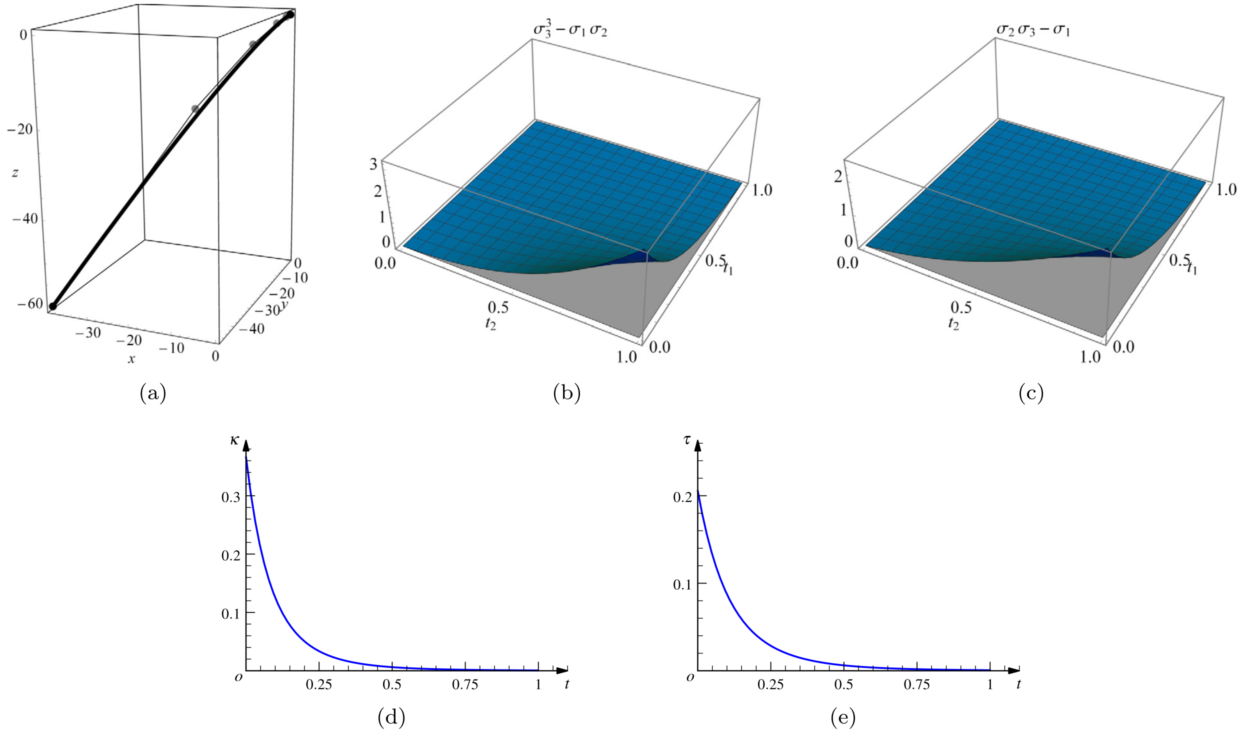

Fig. 7. A 3D Class A Bézier curve with general matrix: (a) curve and control polygon; (b) plot of function $\sigma_3^3-\sigma_1\sigma_2$; (c) plot of function $\sigma_2\sigma_3-\sigma_1$; (d) curvature plot; (e) torsion plot.

Currently our work is restricted to Class A Bézier curves. Future work may consider extending our theorems to B-spline curves or even NURBS curves. In addition, the topic of Class A surfaces is an interesting direction for further study. Since our conditions on general Class A matrices are not easy to be checked, an open question is how to simplify them. The derivation of the inequalities to bound the curvature and torsion distribution of typical curves is an interesting topic. It would be interesting to explore more applications of 3D typical curves, e.g., the design of ship hulls, aircraft, highway and railway routes, trajectories of mobile robots, tool path planning for CNC, etc. The constructive algorithms or fitting methods for 3D typical curves are also worthy of future research.

CRediT authorship contribution statement

Weihua Tong: Conceptualization, Methodology, Software, Supervision, Visualization, Writing – original draft, Writing – review & editing. **Ming Chen:** Formal analysis, Investigation, Software, Validation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We would like to thank the anonymous reviewers for their valuable comments and suggestions. Special thanks are due to Falai Chen for reading the draft of this paper and providing many insightful comments and suggestions. This work was supported by the National Natural Science Foundation of China (No. 61877056, 61972368), and the Anhui Provincial Natural Science Foundation, PR China (No. 1908085QA11).

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