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Abstract

In this paper, a solution of difference equations is given explicitly and then using obtained result, the dual bases for Bézier surface defined on a rectangular domain or a simplex in R^m *are given.*

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1. Introduction

Difference equations are often appeared in spline interpolation, numerical solution of differential equations and also in CAGD. In this paper, we give an explicit solution of difference equations, then we construct the dual bases for Bézier surface defined on a rectangular cube or a simplex in R^m using obtained result.

Let us introduce some notation used in this paper. For any multi-index $\alpha=(\alpha_1,...,\alpha_m),\beta=(\beta_1,\ldots,\beta_m)\in Z_+^m$ and $x=(x_1,\ldots,x_m)$,we define $|\alpha|,\alpha!,x^\alpha$ and $\binom{\alpha}{\beta}$ as usual.

Let $d=(d_1,\ldots d_m)\in Z_+^m$ and $\{a_\alpha:\alpha\le d\}$, $\{b_\alpha:\alpha\le d\}$ are two sets of points in R^m. We introduce the symbolic shift operator [2], $E:=(E_1,\ldots,E_m)$ defined as following:

$$E_j a_\alpha := a_{\alpha+e_j},\quad j=1,2,\ldots m,$$

here $e_j :=$ the unit vector with 1 for its j-th component and 0 for other components.

Let $\Delta:=(\Delta_1,\ldots,\Delta_m)$ with $\Delta_j:=E_j-I$ be forward difference operator and $\Delta_j^k:=\Delta_j(\Delta^{k-1})$, $j=1,\ldots,m$, $k=2,\ldots m$. For simplicity we also use 0 denote zero vector in R^m and define $E\pm I:=(E_1\pm I,\ldots,E_m\pm I)$.

Then we have

Theorem 1 *Let* $\{a_\alpha:\alpha\le d\}$, $\{b_\alpha:\alpha\le d\}$ *are two sets of points in* R^m*, then*

$$a_\alpha=\Delta^\alpha b_0=(E-I)^\alpha b_0 \qquad (1)$$

if and only if $b_\alpha=(E+I)^\alpha a_0$ (2)

In section 2, a proof of this theorem is given. And in section 3, using this result, the dual bases for Bézier surface defined on a cube and a simplex in R^m are obtained.

2. The proof of Theorem 1

First we prove the necessity of the theorem. From $a_\alpha=(E-I)^\alpha b_0$, for all $\alpha\le d$,
it is ease to verity that $a_0=b_0$ and for $\alpha=e_j$, $j=1,2,\ldots m$ from

$$a_{e_j}=(E-I)^{e_j}b_0=b_{e_j}-b_0,$$

we have $b_{e_j}=(E+I)^{e_j}a_0$.

Now we suppose $b_{re_j}=(E+I)^{re_j}a_0$, for $r\le d_j-1$, $j=1,2,\ldots m$, then from

$$a_{(r+1)e_j}=(E_j-I)^{r+1}b_0$$
$$=b_{(r+1)e_j}+a_{(r+1)e_j}-(E_j+I)^{r+1}a_0,$$

we have $b_{(r+1)e_j}=(E+I)^{(r+1)e_j}a_0$.

So, we have proved formular (2) for $\alpha=re_j$, $j=1,2,\ldots m$, $r=1,2,\ldots d_j$.

Suppose the theorem is valid for $m\le j$, $j=1,2,\ldots m-1$, then for $m=j+1$, by assumption of induction, we may assume

$$b_{\alpha-e_i}=(E+I)^{\alpha-e_i}a_0,\quad i=1,2,\ldots j+1,$$

then $a_\alpha=(E-I)^\alpha b_0=\sum_{\sigma\le\alpha}(-)^{|\alpha-\sigma|}\binom{\alpha}{\sigma}b_\sigma$

$$=b_\alpha+a_\alpha-(E+I)^\alpha a_0.$$

Finally, we get $b_\alpha=(E+I)^\alpha a_0$.

The necessity of the theorem is proved, and the proof of the sufficiency is similar, we omit it. The theorem is confirmed.

In practice, $m=2,3$ are most important. We write these special case in the following corollary.

Corollary 1.1 *Let* $\{a_{ij}\}_{i=1}^{d_1}{}_{j=1}^{d_2}$, $\{b_{ij}\}_{i=1}^{d_1}{}_{j=1}^{d_2}$ *are two sets of points in* R^2, then

$$a_{ij}=\Delta_1^i\Delta_2^j b_{0,0} \qquad (3)$$

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if and only if

$$b_{ij} = (E_1 + I)^i (E_2 + I)^j a_{0,0}. \quad (4)$$

Corollary 1.2 *Let* $\{a_{ijk}\}_{i=1}^{d_1}\,_{j=1}^{d_2}\,_{k=1}^{d_3}$, $\{b_{ijk}\}_{i=1}^{d_1}\,_{j=1}^{d_2}\,_{k=1}^{d_3}$ *are two sets of points in* R^3, *then*

$$a_{ijk} = \Delta_1^i \Delta_2^j \Delta_3^k b_{0,0,0} \quad (5)$$

if and only if

$$b_{ijk} = (E_1 + I)^i (E_2 + I)^j (E_3 + I)^k a_{0,0,0}. \quad (6)$$

3. The Application in CAGD

As we known, the dual bases of univariate B-splines constructed by de Boor played a very important role in the research of univariate splines and how to express a spline function by a combination of B-splines [1]. Then, many authors work on this subject and obtain dual bases in different form from different approach. For examples, dual bases of multivariate Bernstein-Bézier polynomials are given in differential form by a straight forward calculation [4]. From blossoming approach, the dual bases in blossoming form explicitly are given in [3]. In this section, we will construct the dual bases using the solution of difference equations.

Let $P(x) = \sum_{\alpha \le d} P_\alpha b_\alpha^d(x) := \prod_{i=1}^{m} (x_i \Delta_i + I)^{d_i} P_0$ (7)

Then we have following

Theorem 2 *Let a Bézier surface* $P(x)$ *on a rectangular cube in* R^m, *defined as in (7). Then the Bézier coordinates* $P_\alpha, \alpha \le d$ *can be expressed as:*

$$P_\alpha = (E + I)^\alpha a_0 = \sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} \frac{(d - \sigma)!}{d!} \frac{\partial^{|\sigma|} P(x)}{\partial x^\sigma} \bigg|_{x=0}$$

here $a_\alpha := (E - I)^\alpha a_0$ *for* $\alpha \le d$.

For the case of $m = 2$, we have the following

Corollary 2.1 *For the Bézier surface*

$$P(u,v) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} P_{ij} b_i^{n_1}(u) b_j^{n_2}(v)$$
$$= (u\Delta_1 + I)^{n_1} (v\Delta_2 + I)^{n_2} P_{0,0}$$

defined on a rectangular in the plane, the dual basis of this surface are

$$\Lambda_{ij} := \sum_{r=0}^{i} \sum_{s=0}^{j} \frac{(n_1 - r)!}{n_1!} \frac{(n_2 - s)!}{n_2!} \binom{i}{r} \binom{j}{s} \frac{\partial^{r+s}}{\partial u^r \partial v^s} \bigg|_{(0,0)}.$$

The second application is for constructing dual bases of Bézier surface defined on a simplex in R^m.

Let $V_m := \langle V^{(1)}, \ldots, V^{(m+1)} \rangle$

$$:= \left\{ \sum_{i=1}^{m+1} \lambda_i V^{(i)} : 0 \le |\lambda| \le 1, \lambda_{m+1} := 1 - |\lambda| \right\}$$

be a non-degenerate simplex in R^m and

$$b_n(\lambda) = \sum_{|\alpha| \le n} C_\alpha B_\alpha^n(\lambda)$$
$$= \left(\lambda_1 E_1 + \cdots \lambda_m E_m + (1 - |\lambda|) I\right)^n C_0$$

be a Bézier surface defined on the simplex V_m.

Then we have

Theorem 3 *Let* $b_n(\lambda)$ *be the Bézier surface on the simplex* V_m *in* R^m, *and linear functional*

$$\Lambda_\alpha f := \sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} \frac{(n - |\sigma|)!}{n!} \frac{\partial^{|\sigma|}}{\partial \lambda^\sigma} f \bigg|_{\lambda=0}, \quad \text{for } |\alpha| \le n.$$

Then, $\{\Lambda_\alpha : |\alpha| \le n\}$ *is a dual bases of Bézier surface* $b_n(\lambda)$, *i.e.*

$$\Lambda_\alpha b_n(\lambda) = C_\alpha, \quad \text{for } |\alpha| \le n.$$

For the special case of $m = 2$, we have following

Corollary 3.1 *Let* $b_n(\lambda) = \sum_{\alpha_1 + \alpha_2 \le n} C_{\alpha_1 \alpha_2}$

$$\frac{n!}{\alpha_1! \alpha_2! (n - \alpha_1 - \alpha_2)!} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} (1 - \lambda_1 - \lambda_2)^{n - \alpha_1 - \alpha_2}$$

be a Bézier surface of degree n *defined a triangular*

$$V_2 := \langle V^{(1)}, V^{(2)}, V^{(3)} \rangle := \{\lambda_1 V^{(1)} + \lambda_2 V^{(2)} +$$
$$(1 - \lambda_1 - \lambda_2) V^{(3)} : \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_1 + \lambda_2 \le 1\}$$

and the linear functional is defined as

$$\Lambda_{\alpha_1 \alpha_2} f := \sum_{i=0}^{\alpha_1} \sum_{j=0}^{\alpha_2} \binom{\alpha_1}{i} \binom{\alpha_2}{j} \frac{(n - i - j)!}{n!}$$
$$\frac{\partial^{i+j}}{\partial \lambda_1^i \partial \lambda_2^j} f \bigg|_{(0,0)}, \alpha_1 + \alpha_2 \le n$$

Then

$$\Lambda_{\alpha_1 \alpha_2} b_n(\lambda) = C_{\alpha_1 \alpha_2}, \quad \text{for all } \alpha_1 + \alpha_2 \le n.$$

4. Conclusion

In this paper, we give a solution of difference equations explicitly, it seems that this result is meaningful, at least we easily obtained the dual bases for Bézier surface defined on a high dimension rectangular cube or a simplex. It is hopeful that this result has more applications in CAGD or other fields.

5. References

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