



Low-rank parameterization of planar domains for isogeometric analysis[☆]



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ABSTRACT

Construction of spline surfaces from given boundary curves is one of the classical problems in computer aided geometric design, which regains much attention in isogeometric analysis in recent years and is called domain parameterization. However, for most of the state-of-the-art parameterization methods, the rank of the spline parameterization is usually large, which results in higher computational cost in solving numerical PDEs. In this paper, we propose a low-rank representation for the spline parameterization of planar domains using low-rank tensor approximation technique, and apply quasi-conformal map as the framework of the spline parameterization. Under given correspondence of boundary curves, a quasi-conformal map with low rank and low distortion between a unit square and the computational domain can be modeled as a non-linear optimization problem. We propose an efficient algorithm to compute the quasi-conformal map by solving two convex optimization problems alternatively. Experimental results show that our approach outperforms previous approaches in producing bijective and low-rank parametric spline representations of planar domains.

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1. Introduction

Given the boundary curves in 2D or 3D, constructing a parametric spline representation to interpolate the given boundary is a fundamental problem in Computer Aided Geometric Design, and Coons surfaces are a classic tool to solve the problem (Farin and Hansford, 1999). This problem has been revived in recent years due to its applications in isogeometric analysis (IGA), and it is called *domain parameterization*. Domain parameterization has a great effect on the accuracy and efficiency in subsequent analysis (Cohen et al., 2010; Xu et al., 2011a; Pilgerstorfer and Jüttler, 2014). It is a common requirement that the parameterization should be injective, i.e., the mapping from the parametric domain (generally a unit square) to the computational domain is self-intersection free. In addition, the distortion of the map should be as small as possible, i.e., the areas and angles after mapping should be preserved as much as possible. So far many approaches have been proposed to solve the parameterization problem, e.g. the discrete Coons interpolation (Farin and Hansford, 1999), the harmonic mapping (Martin et al., 2009; Nguyen and Jüttler, 2010; Xu et al., 2011b), the spring model (Gravesen et al., 2012), the nonlinear optimization method (Xu et al., 2011a), parameterization with non-standard B-splines (e.g., T-splines (Zhang et al., 2012, 2013), THB-splines (Falini et al., 2015)), the method based on Teichmüller mapping (Nian and Chen, 2016) and so on. While all these methods focus on low distortion and bijectivity of the parameterization, the problem of low-rank

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parameterization is not discussed. In fact, the rank of the parametric spline representation by these methods is usually large, which results in higher computational cost in subsequent isogeometric analysis (Mantzaflaris et al., 2014; Jüttler and Mokriš, 2017). Recently, Juettner and his collaborators have observed that reducing the rank of a parameterization can lead to substantial improvements of the overall efficiency of the numerical simulation (Mantzaflaris et al., 2014, 2017; Jüttler and Mokriš, 2017). This observation motivates us to explore parameterization techniques which are able to generate low-rank spline representations.

In this paper, by using low-rank tensor approximation technique, we propose a low-rank representation for planar domain parameterization based on quasi-conformal mapping. Quasi-conformal mapping is a natural extension of conformal mapping which preserves angles (Lehto et al., 1973). The angular distortion and the bijectivity of a quasi-conformal map can be characterized by a complex function called the *Beltrami coefficient* (Lui et al., 2010, 2012, 2013). By optimizing the norm of the Beltrami coefficient and the rank of the spline representation, we are able to find a planar domain parameterization with low rank and low distortion as much as possible.

The remainder of this paper is organized as follows. Section 2 reviews some related work about domain parameterization and the applications of low-rank tensors in science and engineering. Section 3 presents some preliminary knowledge about quasi-conformal mapping and low-rank tensor approximation. In Section 4, we propose a mathematical model followed by an algorithm to compute a low-rank quasi-conformal map for domain parameterization. Section 5 demonstrates some experimental results of our algorithm. Comparisons with the state-of-the-art methods are also provided. Finally, we conclude the paper with a summary and future work in Section 6.

2. Related work

2.1. Domain parameterization

Domain parameterization is one essential step in isogeometric analysis (Hughes et al., 2005). The quality of the parameterization greatly influences the numerical accuracy and efficiency of the numerical simulations (Cohen et al., 2010; Xu et al., 2013c; Pilgerstorfer and Jüttler, 2014; Xu et al., 2017). Over the past decade, many approaches have been proposed to solve the parameterization problem. A simple way for domain parameterization is based on discrete Coons patches proposed by Farin and Hansford (1999). A spring model was suggested by Gravesen et al. to solve the problem (Gravesen et al., 2012). The harmonic functions have many good properties and they were used in Martin et al. (2009), Nguyen and Jüttler (2010), Xu et al. (2011b) to construct domain parameterizations. These methods are generally computational inexpensive but the resulting parameterization may not be injective—a deficiency that should be avoided in such type of applications. Xu et al. (2011a) presented a sophisticated nonlinear optimization technique with the injectivity and the quality of the parameterization as an objective. In Falini et al. (2015), THB-splines is used for planar domain parameterization with varying levels of computational complexity. Recently, Nian and Chen (2016) proposed an approach for planar domain parameterization based on Teichmüller mapping, which guarantees a bijective and high-quality parameterization. For 3D domains, a framework was developed in Martin et al. (2009) to model a single trivariate B-spline from input boundary triangle meshes with genus-zero topology. Aigner et al. (2009) presented a variational framework for generating NURBS parametrizations of swept volumes, in which the control points can be obtained by solving an optimization problem. Escobar et al. (2011) proposed a solid T-spline modeling algorithm from a surface triangular mesh. Zhang et al. (2012) developed a mapping-based method to construct rational trivariate solid T-splines for genus-zero geometry from the boundary triangulations. For meshes with more general geometry, they (Wang et al., 2013) further used the mapping, subdivision and pillowing techniques to generate high quality T-spline representations. In Xu et al. (2013b), the authors proposed a variational harmonic method to construct analysis-suitable parameterization of a computational domain from given CAD boundary information. For models topologically equivalent to a set of cubes and bounded by B-spline surfaces, they (Xu et al., 2013a) further studied the volume parameterization of the multi-block computational domain using the nonlinear optimization method proposed in Xu et al. (2011a). When dealing with more complex geometric shapes, however, single-patch representations do not provide sufficient flexibility. Multi-patch structures are generally constructed to fulfill the task of low distortion parameterization (Xu et al., 2015a; Buchegger and Jüttler, 2017; Xu et al., 2018; Xiao et al., 2018). In this paper, we focus on 2D domain parameterization.

2.2. Applications of low-rank tensor approximation

Low-rank approximation is very helpful for dimension reduction and data compression, and has been successfully applied in many fields like signal processing, computer vision, pattern recognition, computer graphics, etc. A thorough survey on this topic is out of the scope of this paper, and we refer the reader to Markovsky (2011), Ma et al. (2012) and references therein. Tensors, as a generalization of matrices in higher dimensions have important applications in science and engineering, e.g., psychometrics and data mining (Kolda and Bader, 2009). The details of low-rank tensor approximation and its applications have been discussed in depth in Grasedyck et al. (2013). Recently, the low-rank tensor optimization has been applied in graphics and geometric modeling community, e.g., in finding the upright orientation of 3D shapes (Wang et al., 2014) and in compact implicit surface reconstructions (Pan et al., 2016). For other applications of low-rank tensors in geometric modeling and processing, please refer to Xu et al. (2015b) and references therein. Juettner and his collaborators recently addressed the

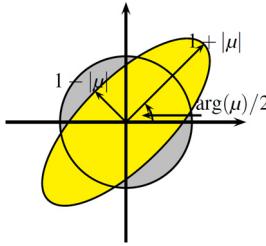


Fig. 1. Geometric meaning of the Beltrami coefficient.

problem of low-rank approximation for isogeometric analysis applications. Mantzaflaris et al. (2014) applied low-rank matrix approximation for accelerating the assembly process of stiffness matrices in isogeometric analysis. They further extended their work to 3D case and employed the tensor decomposition technique for Galerkin-based isogeometric analysis, which can reduce the computation time and storage requirements dramatically (Mantzaflaris et al., 2017). A construction for low-rank tensor-product spline surfaces from given boundary curves is also proposed by Jüttler and Mokriš (2017).

3. Preliminaries

In this section, we give some preliminary knowledge about quasi-conformal mapping and low-rank tensor approximation followed by the definition of rank- R spline functions.

3.1. Quasi-conformal mapping

The most convenient way to explain quasi-conformal mapping is in complex setting. Let $z = x + \sqrt{-1}y$ be a complex variable with x and y being the real and imaginary part of z respectively, and $\bar{z} = x - \sqrt{-1}y$ be the conjugate of z . For a differentiable complex function $f(z)$, its complex derivatives are defined as $f_z = \frac{1}{2}(f_x - \sqrt{-1}f_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + \sqrt{-1}f_y)$. A complex function defines a map from a complex plane to a complex plane. When $f_{\bar{z}} = 0$, f defines a *conformal map* which preserves angles and maps an infinitesimal circle to an infinitesimal circle. A quasi-conformal map is a generalization of a conformal map which maps an infinitesimal circle to an infinitesimal ellipse.

Definition 1. Suppose $f : \hat{\Omega} \rightarrow \Omega$ is a complex function, where $\hat{\Omega}$ and Ω are two domains in \mathbb{C} . If f is assumed to have continuous partial derivatives, then f is quasi-conformal provided it satisfies the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} \quad (1)$$

for some complex valued Lebesgue measurable μ satisfying $\sup |\mu| < 1$. μ is called the Beltrami coefficient of the map f .

The Beltrami coefficient μ determines the angular deviation from conformality. When $\mu = 0$, the quasi-conformal map becomes conformal. Define the dilatation of f at the point z by

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$

Then a quasi-conformal map takes infinitesimal circles to infinitesimal ellipses with bounded eccentricity given by the dilatation $K = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$ and the orientation of axis rotates an $\arg(\mu)/2$, as shown in Fig. 1. Furthermore, $f(z)$ is orientation preserving and bijective provided $\|\mu\|_\infty < 1$ and $f_z \neq 0$.

Besides angular deviation, another quantity that characterizes the area distortion of a map f is the Jacobian $J(f)$ of the map. In order to eliminate the area difference between the parametric domain (a unit square) and the computational domain, usually the scaled Jacobian $J_s(f)$ is employed:

$$J_s(f) = \frac{J(f)}{A_\Omega}$$

where A_Ω is the area of the computational domain Ω .

3.2. Low-rank tensor approximation

A *tensor* is a multidimensional array. More formally, an n th-order or n -way tensor is an element of the tensor product of n vector spaces, each of which has its own coordinate system (Kolda and Bader, 2009). An n th-order tensor is usually

denoted by boldface Euler script letters, e.g., $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \dots \times I_n}$. A first-order tensor is a vector, a second-order tensor is a matrix, and tensors of order three or higher are called higher-order tensors.

An n th-order tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \dots \times I_n}$ is of *rank one* if it can be written as the outer product of n vectors, i.e.,

$$\mathcal{X} = \mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \dots \circ \mathbf{a}^{(n)},$$

where \circ denotes the outer product, and $\mathbf{a}^{(i)} \in \mathbb{C}^{I_i}$.

CP decomposition Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \dots \times I_n}$ be an n th-order tensor, the *CP decomposition* factorizes \mathcal{X} into a sum of component rank-one tensors as follows:

$$\mathcal{X} = \sum_{r=1}^R \lambda_r \mathbf{a}_r^{(1)} \circ \mathbf{a}_r^{(2)} \circ \dots \circ \mathbf{a}_r^{(n)}, \quad (2)$$

where R is a positive integer and $\lambda_r > 0$, $\mathbf{a}_r^{(i)} \in \mathbb{C}^{I_i}$ for $r = 1, \dots, R$. It's often useful to assume that $\mathbf{a}_r^{(i)}$ are normalized to length one with the weights absorbed into λ_r .

The *rank* of a tensor \mathcal{X} , denoted as $\text{rank}(\mathcal{X})$, is the smallest number of components in the above expression (2). The CP decomposition can be considered to be a higher-order generalization of the matrix singular value decomposition (SVD) which can be described as follows:

Singular value decomposition Let $X \in \mathbb{C}^{I_1 \times I_2}$ be a matrix, the SVD factorizes X into a sum of component rank-one matrices as follows:

$$X = U \Sigma V^H = \sum_{r=1}^R \lambda_r \mathbf{a}_r \circ \mathbf{b}_r \quad (3)$$

where U is an $I_1 \times I_1$ complex unitary matrix, Σ is an $I_1 \times I_2$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and V is an $I_2 \times I_2$ complex unitary matrix. The diagonal entries of Σ are known as the singular values of X and λ_r are the nonzero singular values. $\mathbf{a}_r, \mathbf{b}_r$ are the left R column vectors of U and V respectively.

Theorem 1. (Eckart and Young, 1936) *The best rank- k approximation of X is given by a truncated SVD of X , that is*

$$\hat{X} = U \hat{\Sigma} V^H = \sum_{r=1}^k \lambda_r \mathbf{a}_r \circ \mathbf{b}_r \quad (4)$$

where \hat{X} has a specific rank k , and $\hat{\Sigma}$ is the same matrix as Σ except that it contains only the k largest singular values (the other singular values are replaced by zero). \hat{X} is called rank- k approximation of X .

From Theorem 1, we can see that the *rank* of a matrix X , denoted as $\text{rank}(X)$, is equal to the number of nonzero singular values in SVD of X . However, $\text{rank}(X)$ is a nonconvex function, and solving a rank-constrained problem is generally NP-hard. Recently several works (Recht et al., 2010; Cai et al., 2010; Liu et al., 2013) use the nuclear norm of a matrix to approximately calculate the rank, which leads to a convex optimization problem. The nuclear norm of X is defined as follows

$$\|X\|_* := \sum_i \sigma_i(X) \quad (5)$$

where $\sigma_i(X)$ is the i th largest singular value of X .

3.3. Rank- R spline functions

A multivariate function $f(x_1, \dots, x_n)$ is said to have *rank R* if it can be represented as a sum of separable functions

$$f(x_1, \dots, x_n) = \sum_{r=1}^R \prod_{k=1}^n f_r^{(k)}(x_k) \quad (6)$$

where $f_r^{(k)}(x_k)$ are univariate functions.

Let $g(x_1, \dots, x_n)$ be an n -variate tensor product spline function of n -degree (d_1, \dots, d_n) defined over an n -dimensional domain Ω :

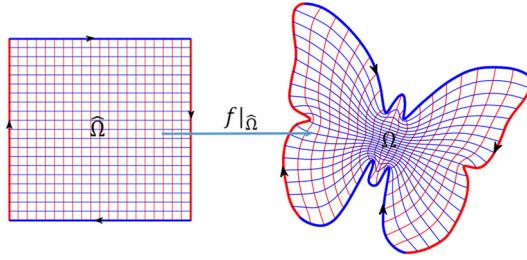


Fig. 2. Parameterization—a map f from a unit square $\hat{\Omega}$ to a computational domain Ω .

$$g(x_1, \dots, x_n) = \sum_{i_1, i_2, \dots, i_n} c_{i_1, i_2, \dots, i_n} \prod_{k=1}^n \beta_{i_k}^{(k)}(x_k) \quad (7)$$

where $\{c_{i_1, i_2, \dots, i_n}\}$ are the control coefficients, $\{\beta_{i_k}^{(k)}(x_k)\}$ are B-spline basis functions in n possibly different univariate spline spaces $S_{\tau_k}^{d_k}, k = 1, \dots, n$, and each space is defined by a knot vector τ_k and a degree d_k . We collect the basis functions in the knot vector τ_k

$$\boldsymbol{\beta}^{(k)}(x_k) = [\beta_{i_k}^{(k)}(x_k)], \quad k = 1, \dots, n.$$

Let \mathcal{C} be the n -order coefficient tensor associated with the coefficients $\{c_{i_1, i_2, \dots, i_n}\}$ defined in (7). If the rank of \mathcal{C} is R and we perform the CP decomposition of \mathcal{C} as

$$\mathcal{C} \stackrel{(2)}{=} \sum_{r=1}^R \mathbf{c}_r^{(1)} \circ \mathbf{c}_r^{(2)} \circ \dots \circ \mathbf{c}_r^{(n)}, \quad (8)$$

then g can be expressed in a sum of R products

$$g(x_1, \dots, x_n) = \sum_{r=1}^R \prod_{k=1}^n g_r^{(k)}(x_k) \quad (9)$$

of the univariate spline functions

$$g_r^{(k)}(x_k) = \mathbf{c}_r^{(k)} \cdot \boldsymbol{\beta}^{(k)}(x_k).$$

Thus g also has rank R , and we say that g is a *rank- R spline function*.

4. Parameterization of computational domains via low-rank tensor approximation

4.1. Representation of parameterization

Suppose we are given the B-spline representations of the four boundary curves of a computational domain Ω . Our aim is to compute a B-spline representation for the parameterization domain Ω , that is, a map from the unit square $\hat{\Omega} = [0, 1]^2$ to Ω which is bijective, low distortion and low rank. An example is illustrated in Fig. 2.

Assume the parameterization of the computation domain Ω is expressed by a tensor product B-spline function

$$\mathbf{P}(x, y) := \sum_{i=0}^m \sum_{j=0}^n \mathbf{P}_{ij} M_i^p(x) N_j^q(y) \quad (10)$$

where $\mathbf{P}_{ij} = (x_{ij}, y_{ij}) \in \mathbb{R}^2$ are the control points, $M_i^p(x)$ and $N_j^q(y)$ are the B-spline basis functions of degree p and q w.r.t. the knot sequences U and V in $[0, 1]$ respectively. Since we are working in complex settings, we rewrite the parameterization by a complex function

$$f(z) := \sum_{i=0}^m \sum_{j=0}^n c_{ij} M_i^p(x) N_j^q(y), \quad (11)$$

where $c_{ij} = x_{ij} + \sqrt{-1}y_{ij}$, $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. Since the boundary curves of the domain Ω are given, c_{ij} is known for $i = 0, 1, \dots, m$, $j = 0, n$ and $j = 0, 1, \dots, n$, $i = 0, m$.

From the equation (1), the Beltrami coefficient of f can be computed as

$$\mu(f) = \frac{(a - b) + \sqrt{-1}(c + d)}{(a + b) + \sqrt{-1}(c - d)} \quad (12)$$

where

$$\begin{aligned} a &= \sum_{i=0}^m \sum_{j=0}^n x_{ij} \frac{\partial M_i^p(x)}{\partial x} N_j^q(y), & b &= \sum_{i=0}^m \sum_{j=0}^n y_{ij} M_i^p(x) \frac{\partial N_j^q(y)}{\partial y}, \\ c &= \sum_{i=0}^m \sum_{j=0}^n y_{ij} \frac{\partial M_i^p(x)}{\partial x} N_j^q(y), & d &= \sum_{i=0}^m \sum_{j=0}^n x_{ij} M_i^p(x) \frac{\partial N_j^q(y)}{\partial y} \end{aligned}$$

4.2. Low-rank parameterization model

As explained in Section 3.1, the distortion of a quasi-conformal map f is determined by its Beltrami coefficient $\mu(f)$, thus we formulate the parameterization problem as the following model

$$\begin{aligned} \arg \min_f \quad & \int_{\hat{\Omega}} |\mu(f)|^2 dz + \omega_1 \int_{\hat{\Omega}} |\nabla \mu(f)|^2 dz + \omega_2 \text{rank}(C_f) \\ \text{s.t.} \quad & \|\mu(f)\|_{\infty} < 1 \\ & c_{i0}, c_{in}, c_{0j}, c_{mj} \quad (i = 0, 1, \dots, m, j = 0, 1, \dots, n) \text{ are given} \end{aligned} \quad (13)$$

where $C_f = (c_{ij})_{(m+1) \times (n+1)}$ is a complex matrix whose elements are the coefficients of f defined in (11), ω_1 and ω_2 are non-negative weights. The first term of the objective function aims to minimize the conformality distortion of f , the second term measures the smoothness of f and the third term is the low-rank regularization term which tries to reduce the rank of f . In terms of the constraints, the first one guarantees that f is locally bijective and the second one is the boundary conditions.

4.3. Numerical algorithm

Solving the optimization problem (13) for f is challenging since it is highly nonlinear and nonconvex. Instead, we set $v := \mu(f)$ as the auxiliary variable and replace the function $\text{rank}(\cdot)$ with the nuclear norm $\|\cdot\|_*$ introduced in Section 3.2. Thus we obtain the following optimization problem

$$\begin{aligned} \arg \min_{f, v} \quad & \int_{\hat{\Omega}} |v|^2 dz + \omega_1 \int_{\hat{\Omega}} |\nabla v|^2 dz + \omega_2 \|C_f\|_* \\ \text{s.t.} \quad & v = \mu(f) \\ & \|v\|_{\infty} < 1 \\ & c_{i0}, c_{in}, c_{0j}, c_{mj} \quad (i = 0, 1, \dots, m, j = 0, 1, \dots, n) \text{ are given} \end{aligned} \quad (14)$$

The above problem is relaxed as

$$\begin{aligned} \arg \min_{f, v} \quad & \int_{\hat{\Omega}} |v|^2 dz + \omega_1 \int_{\hat{\Omega}} |\nabla v|^2 dz + \omega_2 \|C_f\|_* + \omega_3 \int_{\hat{\Omega}} |v - \mu(f)|^2 dz \\ \text{s.t.} \quad & \|v\|_{\infty} < 1 \\ & c_{i0}, c_{in}, c_{0j}, c_{mj} \quad (i = 0, 1, \dots, m, j = 0, 1, \dots, n) \text{ are given} \end{aligned} \quad (15)$$

For large enough weight ω_3 , the optimal solution of the model (15) approximates the solution of (14), where v is close enough to $\mu(f)$. To efficiently solve (15), we solve two sub-problems alternatively. More specifically, we set $v_0 = 0$ initially. Suppose v_n is obtained at the n th iteration. Fixing $v = v_n$, we first minimize (15) for f to obtain f_n . Then by fixing $f = f_n$, we obtain v_{n+1} by minimizing (15) for v . The procedure runs until $\|v_{n+1} - v_n\|_{\infty} < \epsilon$ for a user-specified ϵ . In the following, we will discuss the two sub-problems in detail.

Problem 1 Given ν , find f such that the following objective function is minimized

$$\arg \min_f \omega_2 \|C_f\|_* + \omega_3 \int_{\hat{\Omega}} |\nu - \mu(f)|^2 dz \quad (16)$$

s.t. $c_{i0}, c_{in}, c_{0j}, c_{mj}$ ($i = 0, 1, \dots, m, j = 0, 1, \dots, n$) are given

Problem (16) is similar to the complex matrix completion problem (Cai et al., 2010). However, since $\mu(f) = f_{\bar{z}}/f_z$ is a rational B-spline function, the problem is still hard to solve. Instead we solve the following relaxed model

$$\arg \min_f \omega_2 \|C_f\|_* + \omega_3 \int_{\hat{\Omega}} |f_{\bar{z}} - \nu f_z|^2 dz + \lambda \|Pr(C_f) - y\|^2 \quad (17)$$

where λ is a large positive weight, $y \in \mathbb{C}^{2(m+n)}$ is the vector whose elements are comprised of $c_{i0}, c_{in}, c_{0j}, c_{mj}$ ($i = 0, 1, \dots, m, j = 0, 1, \dots, n$), and $Pr : \mathbb{C}^{(m+1) \times (n+1)} \rightarrow \mathbb{C}^{2(m+n)}$ is a linear operator that shapes the boundary elements of C_f into a vector, i.e., $Pr(C_f) = (c_{00}, \dots, c_{m0}, c_{m1}, \dots, c_{mn}, c_{m-1,n}, \dots, c_{0n}, c_{0,n-1}, \dots, c_{01})^T$. Now (17) is a convex optimization problem which can be solved by the alternating direction method of multipliers (ADMM) efficiently. The ADMM can be viewed as an attempt to blend the benefits of dual decomposition and augmented Lagrangian methods, and is used to solve constrained optimization problems with separable objective functions. The basic approach is outlined as follows.

Variable splitting Since the objective function in (17) is the sum of three functions and one of which is dependent on the others, using variable splitting technique leads to the following constrained optimization problem

$$\arg \min_f \omega_2 \|Z\|_* + \omega_3 \int_{\hat{\Omega}} |f_{\bar{z}} - \nu f_z|^2 dz + \lambda \|Pr(C_f) - y\|^2 \quad (18)$$

s.t. $c = z$

where c is the vectorization of C_f , i.e., $c = (c_{00}, \dots, c_{m0}, c_{01}, \dots, c_{m1}, \dots, c_{0n}, \dots, c_{mn})^T$, Z is an auxiliary matrix of the same size as C_f , and z is the vectorization of Z .

Augmented Lagrangian One typical way for solving (18) is to use the augmented Lagrangian scheme. In our problem, the augmented Lagrangian function is defined as

$$\begin{aligned} \mathcal{L}_\rho(c, Z, \eta) = & \omega_2 \|Z\|_* + \omega_3 \int_{\hat{\Omega}} |f_{\bar{z}} - \nu f_z|^2 dz + \lambda \|Pr(C_f) - y\|^2 \\ & + \mathbf{Re}(\eta^H(c - z)) + \frac{\rho}{2} \|c - z\|^2 \end{aligned} \quad (19)$$

where $\mathbf{Re}(z)$ is the real part of the complex number z , η is a vector of Lagrangian multiplier corresponding to the constraint $c = z$, and $\rho > 0$ is the penalty parameter. Now the ADMM algorithm can be outlined as follows

Algorithm 1 The ADMM algorithm.

Input:

$\omega_2, \omega_3, \lambda, \rho > 0$, and initial values for Z^0, η^0 .

Output:

an optimal c^*

1: $t \leftarrow 0$

2: **repeat**

3: $c^{t+1} = \arg \min_c \mathcal{L}_\rho(c, Z^t, \eta^t)$

4: $Z^{t+1} = \arg \min_Z \mathcal{L}_\rho(c^{t+1}, Z, \eta^t)$

5: $\eta^{t+1} = \eta^t + \rho(c^{t+1} - z^{t+1})$

6: $t \leftarrow t + 1$

7: **until** stopping criterion is satisfied.

c-subproblem The subproblem for c is

$$\arg \min_c \mathcal{L}_\rho(c, Z^t, \eta^t) = \omega_3 \int_{\hat{\Omega}} |f_{\bar{z}} - \nu f_z|^2 dz + \lambda \|Pr(C_f) - y\|^2 + \mathbf{Re}((\eta^t)^H(c - z^t)) + \frac{\rho}{2} \|c - z^t\|^2 \quad (20)$$

This is a quadratic optimization problem and the solution can be obtained by solving a sparse and symmetric linear system of equations. The preconditioned conjugate gradient method with incomplete Cholesky factorization is applied in our algorithm.

Z-subproblem The subproblem for Z is

$$\arg \min_Z \mathcal{L}_\rho(c^{t+1}, Z, \eta^t) = \omega_2 \|Z\|_* + \text{Re}((\eta^t)^H(c^{t+1} - z)) + \frac{\rho}{2} \|c^{t+1} - z\|^2 \quad (21)$$

which has the following closed form solution (Cai et al., 2010):

$$Z^{t+1} = \text{prox}_{\omega_2/\rho}^{\text{tr}}(c^{t+1} + \eta^t/\rho). \quad (22)$$

Note that the argument $c^{t+1} + \eta^t/\rho$ must be converted into a matrix of the same size as Z^t . Here the proximal operator $\text{prox}_{\omega_2/\rho}^{\text{tr}}$ can be considered as a shrinkage operation on the singular values and is defined as follows

$$\text{prox}_{\omega_2/\rho}^{\text{tr}}(Y) = U \max(S - \omega_2 I/\rho, 0) V^H, \quad (23)$$

where $Y = USV^H$ is the singular value decomposition (SVD) of Y , and the max operation is taken element-wise. Please refer to Cai et al. (2010) for the detailed derivation.

In our implementation, Z^0 and η^0 are set as zero, and the stopping criterion is that the value of c has small change or the maximum number of iterations reaches.

Problem 2 Given a mapping f from $\hat{\Omega}$ to Ω , $\mu(f)$ can be computed by (12), and the problem (15) reduces to the following model

$$\begin{aligned} \arg \min_v & \int_{\hat{\Omega}} |v|^2 dz + \omega_1 \int_{\hat{\Omega}} |\nabla v|^2 dz + \omega_3 \int_{\hat{\Omega}} |v - \mu(f)|^2 dz \\ \text{s.t. } & \|v\|_\infty < 1 \end{aligned} \quad (24)$$

Let

$$v = \sum_{i=0}^{\tilde{m}} \sum_{j=0}^{\tilde{n}} \tilde{c}_{ij} \tilde{M}_i^p(x) \tilde{N}_j^q(y), \quad (25)$$

where $\tilde{c}_{ij} = \tilde{x}_{ij} + \sqrt{-1}\tilde{y}_{ij}$, and $\tilde{M}_i^p(x)$ and $\tilde{N}_j^q(y)$ are the B-spline basis functions of degree p and q w.r.t. the knot sequences \tilde{U} and \tilde{V} in $[0, 1]$ respectively. For the simplicity of computation, the constraint $\|v\|_\infty < 1$ in the above optimization problem is replaced by

$$-\frac{\sqrt{2}}{2} < \tilde{x}_{ij} < \frac{\sqrt{2}}{2}, \quad -\frac{\sqrt{2}}{2} < \tilde{y}_{ij} < \frac{\sqrt{2}}{2}, \quad i = 0, 1, \dots, \tilde{m}, j = 0, 1, \dots, \tilde{n} \quad (26)$$

Then (24) becomes a quadratic optimization problem which can be easily solved.

4.4. Post-processing

The above algorithm iteratively solves two sub-problems to obtain two sequences of complex functions $\{v_k\}$ and $\{f_k\}$. However, in some cases, no matter how many steps the algorithm iterates for, $|\mu(f)|$ is very close to 1 and the error $|v - \mu(f)|$ is usually large at some concave regions (and hence the parameterization has a bad quality in these regions, see Fig. 3 for an example). In order to further improve the quality of parameterization and accelerate the convergence of the algorithm, we add a weight function into the second term of the problem (16) after t_0 iterations, where t_0 satisfies $\|\mu(f_{t_0})\|_\infty - 1 < \epsilon_0$ for a threshold ϵ_0 . At the k th ($k > t_0$) iteration, the problem (16) is reformulated as

$$\begin{aligned} \arg \min_f & \omega_2 \|C_f\|_* + \omega_3 \int_{\hat{\Omega}} \alpha |v - \mu(f)|^2 dz \\ \text{s.t. } & c_{i0}, c_{in}, c_{0j}, c_{mj} \quad (i = 0, 1, \dots, m, j = 0, 1, \dots, n) \text{ are given} \end{aligned} \quad (27)$$

where the weight function $\alpha = 1/((1 - |\mu(f_{k-1})|)^2 + \delta)$ and δ is a threshold which helps to avoid division by zero. The problem (27) can be solved in the same way as the problem (16). It's easy to see that the closer $|\mu(f_{k-1})|$ is to 1, the larger the weight is. Thus the added weight function α is mainly used to penalize the error $|v - \mu(f)|$ at those local regions

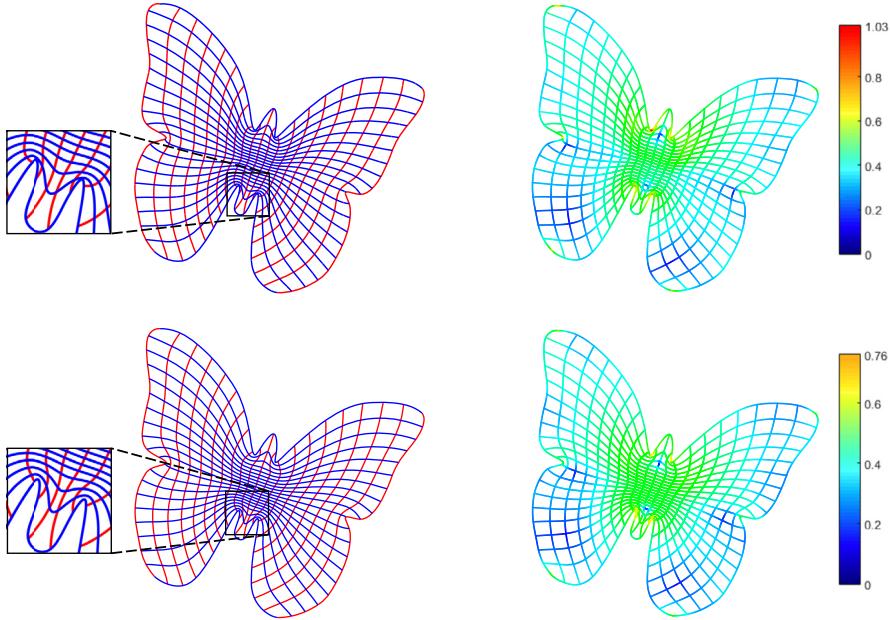


Fig. 3. Comparison of parameterization results with (bottom row) and without (top row) post-processing step. The left column shows the iso-parametric curves with some parts enlarged and the right one shows the colormaps of $|\mu(f)|$ (i.e., angular distortion). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

where $|\mu(f)|$ is very close to 1, which forces $|\mu(f)|$ not too close to 1 everywhere. From the numerical examples, we can see that this post-processing step is essential and effective, see Fig. 3 for a comparison result.

Now the overall algorithm of our parameterization method is summarized in Algorithm 2.

Algorithm 2 The low-rank parameterization algorithm.

Input:

B-spline representations of four boundary curves of the domain Ω , and the parameters $\omega_1, \omega_2, \omega_3, \lambda, \rho, \epsilon, \epsilon_0, \delta$.

Output:

```

A low-rank quasi-conformal map  $f$  from the unit square  $\hat{\Omega}$  to  $\Omega$ .
1:  $v_0 = 0, k = 0$ , compute an initial map  $f_0$  using the discrete Coons interpolation method proposed in Farin and Hansford (1999).
2: repeat
3:   if  $\|\mu(f_k)\|_\infty - 1 \geq \epsilon_0$  then
4:     Fix  $v_k$ , and solve the optimization problem (16) to obtain  $f_k$  using Algorithm 1;
5:   else
6:     Fix  $v_k$  and  $\alpha$ , and solve (27) to obtain  $f_k$  using Algorithm 1;
7:   end if
8:   Set  $\alpha = 1/((1 - |\mu(f_k)|)^2 + \delta)$ ;
9:   Fix  $f_k$ , and solve (24) to obtain  $v_{k+1}$ ;
10:   $k \leftarrow k + 1$ ;
11: until  $\|v_{k+1} - v_k\|_\infty < \epsilon$ .
```

Remark 1. Our mathematical model (13) and the registration model presented in Lam and Lui (2014) both obtain a diffeomorphism via quasi-conformal mapping. However, there are several differences between the two methods. Firstly, we not only want to find a quasi-conformal map with low distortion but also add a low-rank regularization term in (13) to make the rank of the map as low as possible. Secondly, in Lam and Lui (2014), the map f is represented in a discrete form, while it is expressed in a continuous form, i.e., tensor product B-splines in our work. Finally, the second constraint in the model (13) is ignored in Lam and Lui (2014), which can not guarantee the bijectivity of f .

5. Results and discussions

In this section, we demonstrate some examples to show the effectiveness of our parameterization method by comparing it with several state-of-the-art methods.

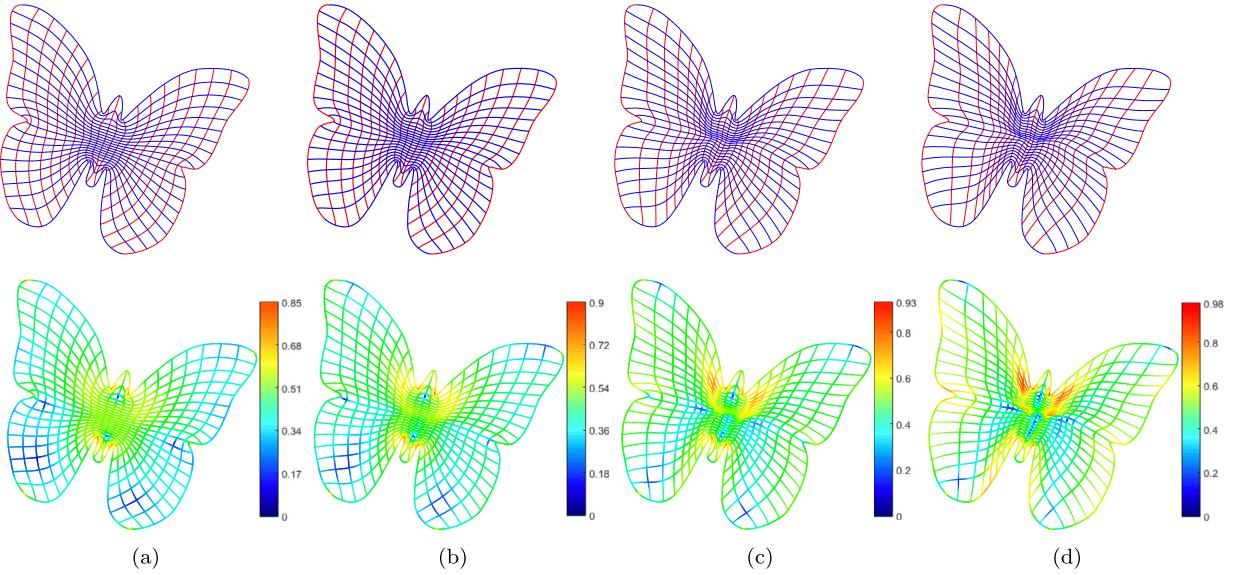


Fig. 4. Parameterization of the butterfly model using various weights ω_2 while leaving other parameters fixed. The top and bottom rows show the isoparametric curves and the colormaps of $|\mu(f)|$ respectively. (a) $\omega_2 = 1.5$, $\text{rank}(f) = 11$; (b) $\omega_2 = 5.5$, $\text{rank}(f) = 7$; (c) $\omega_2 = 7.5$, $\text{rank}(f) = 5$; (d) $\omega_2 = 10.0$, $\text{rank}(f) = 4$.

5.1. Implementation details

We implement our algorithm with a combination of C++ and MATLAB on a PC with a quad-core Intel i5 @3.1GHz and 8GB of RAM. Specifically, the overall framework of our algorithm is implemented in C++, the ADMM algorithm in **Problem 1** and the quadratic programming in **Problem 2** are implemented using MATLAB, and we use C++ to call MATLAB to fulfill the whole implementation. In addition, for the ADMM algorithm and quadratic programming, the MATLAB backslash operator and quadprog function are utilized respectively.

In our algorithm, there are several parameters for setting. Most of them are set as default values, e.g., the penalty parameter ρ is typically set to be 1, the threshold ϵ_0 and δ in Section 4.4 are set as 0.05 and 0.0001 respectively, and the weight λ in (17) is set to be 1000. We use bicubic uniform B-splines to represent the map f and the auxiliary variable v (i.e., $p = q = 3$ in (11) and (25)). Unless specified, the knot parameters in (11) and (25) are chosen as $m = n = \tilde{m} = \tilde{n} = 24$ in our examples, which is proven to work well.

There are three weights ω_1 , ω_2 and ω_3 in the mathematical model (15). The weight ω_1 controls the smoothness of f and we typically set $\omega_1 \in [3, 5.5]$. The penalty weight ω_3 is used to control the difference between v and $\mu(f)$ and is set to be 100 in practice. The weight ω_2 can be used to balance the rank of the map f and parameterization quality. Clearly, larger ω_2 can reduce the rank of f while smaller ω_2 leads to parameterization results of higher quality. We observe that $\omega_2 \in [4.5, 6]$ provides a good compromise between the rank and the quality. Fig. 4 provides an illustrating example.

5.2. Parameterization results

In the following, we present some examples to demonstrate the low rank and low distortion properties of our parameterization method.

5.2.1. Low rank

To demonstrate the superiority of our method in terms of the rank of the map, we provide a comparison with several state-of-the-art parameterization methods: nonlinear optimization method (Xu et al., 2011a), variational harmonic method (Xu et al., 2013b), the Teichmüller mapping method (T-Map) (Nian and Chen, 2016) and the low-rank spline interpolation method (low-rank spline) (Jüttler and Mokriš, 2017). To have a fair comparison, the number of knots in the B-spline representation (11) are chosen to be the same ($m = n = 24$) for these methods. The rank of the map f is numerically computed as the number of singular values of the complex matrix C_f which are greater than a user-specified threshold (10^{-5} in our experiment). Table 1 shows the statistics of our experiments. Besides the low-rank spline method which sacrifices the parameterization quality, our method significantly outperforms other state-of-the-art parameterization methods in terms of the rank. Some of the parameterization results are shown in Fig. 5. As described in Section 3.3, owing to the low-rank property of our method, the map f can be represented in a sum of the product of univariate spline functions with a small number of terms, which helps to speed up the assembly process in IGA without sacrificing the overall accuracy of the simulation.

Table 1

Comparisons of the distortions and ranks among variational harmonic method, the low-rank spline method, nonlinear optimization method, T-map method and our method. The distortion includes the angular distortion $|\mu(f)|$ and the area distortion $J_s(f)$. Here $\min(J_s(f))$, $\max(J_s(f))$ are the minimal and maximal area distortions respectively.

Model	Quantity	Variational harmonic	Low-rank spline	Nonlinear optimization	T-Map	Ours
Sheep (Fig. 5)	$\ \mu(f)\ _\infty$	45.40	26.92	0.99	1.03	0.76
	$\min(J_s(f))$	-1.52	-2.59	0.06	0.11	0.17
	$\max(J_s(f))$	4.38	4.67	10.28	4.24	3.33
	rank(f)	24	5	22	25	7
Bear (Fig. 5)	$\ \mu(f)\ _\infty$	75.91	80.36	0.98	0.96	0.89
	$\min(J_s(f))$	-1.25	-1.89	0.11	0.21	0.26
	$\max(J_s(f))$	5.22	3.86	11.49	4.85	3.52
	rank(f)	25	5	24	25	6
Dolphin (Fig. 8)	$\ \mu(f)\ _\infty$	2.51	5.86	0.99	0.98	0.92
	$\min(J_s(f))$	-3.63	-4.32	0.05	0.10	0.23
	$\max(J_s(f))$	5.26	4.93	11.57	6.07	5.88
	rank(f)	24	5	21	25	8
Rabbit (Fig. 6)	$\ \mu(f)\ _\infty$	20.76	62.67	1.00	1.45	0.86
	$\min(J_s(f))$	-2.95	-4.12	0.08	-0.94	0.19
	$\max(J_s(f))$	4.37	5.15	8.34	6.18	4.22
	rank(f)	21	5	24	25	8
Butterfly (Fig. 7)	$\ \mu(f)\ _\infty$	14.80	4.12	0.96	0.8	0.76
	$\min(J_s(f))$	-0.52	-0.66	0.04	0.02	0.15
	$\max(J_s(f))$	3.67	3.01	7.58	3.82	3.51
	rank(f)	25	5	23	25	7
Jigsaw (Fig. 5)	$\ \mu(f)\ _\infty$	76.75	484.25	0.91	0.8	0.82
	$\min(J_s(f))$	-1.92	-8.85	0.04	0.22	0.29
	$\max(J_s(f))$	21.58	31.24	25.75	8.84	8.02
	rank(f)	18	2	25	25	9

5.2.2. Local injectivity

Fig. 6, Fig. 7 and Fig. 8 depict the parameterization results of the rabbit, the butterfly and the dolphin by different methods respectively. We observe that the variational harmonic method and the low-rank spline method have many self-intersections in the concave regions, the nonlinear optimization method produces non-injective mapping in the butterfly model, the T-map method is not injective in some regions in the rabbit model, e.g. in the ear of the rabbit, while our method is always injective in these examples.

5.2.3. Distortion

Besides injectivity, the map distortion (including angular distortion described by $|\mu(f)|$ and the area distortion represented by $J_s(f)$) is an important criterion to measure the quality of the parameterization. In our experiments, to measure the area distortion of a map, we firstly uniformly subdivide the parametric domain $\hat{\Omega}$ into $M \times N$ sub-rectangles $\{\hat{\Omega}_{ij}\}$ ($i = 1, \dots, M$, $j = 1, \dots, N$), then the area distortion over the sub-rectangle $\hat{\Omega}_{ij}$, denoted as $J_s(f)|_{\hat{\Omega}_{ij}}$, is calculated as follows

$$J_s(f)|_{\hat{\Omega}_{ij}} = \frac{\int_{\hat{\Omega}_{ij}} J_s(f) dx dy}{A_{\hat{\Omega}_{ij}}} \quad (28)$$

where $A_{\hat{\Omega}_{ij}}$ is the area of $\hat{\Omega}_{ij}$.

From Table 1, we can see that the variational harmonic method and the low-rank spline method are significantly worse than our method in terms of distortion. In Fig. 9, Fig. 10, and Fig. 11, we compare our method with the nonlinear optimization method and T-map method by displaying the colormaps of the Beltrami coefficients $|\mu(f)|$ and the scaled Jacobian $J_s(f)$. It can be seen that our method produces smaller angular distortion in some concave regions, e.g. in the ear of the rabbit, in the root of the wing of the butterfly, and in the body of the dolphin than the other two methods, and at the same time, our approach achieves best results in terms of the area distortion among the three methods.

In summary, our method produces much better parameterization results than the other state-of-the-art methods in all these examples. The reason might be as follows. The variational harmonic method and the low-rank spline interpolation method can't guarantee the injectivity in theory. The Teichmüller mapping method computes a Teichmüller map by solving a nonlinear and non-convex optimization problem. But their method has no convergence guarantee, and thus it may not be able to find the real Teichmüller map in some cases. The nonlinear optimization method puts some strict constraints, which could result in no solutions for complex shapes.

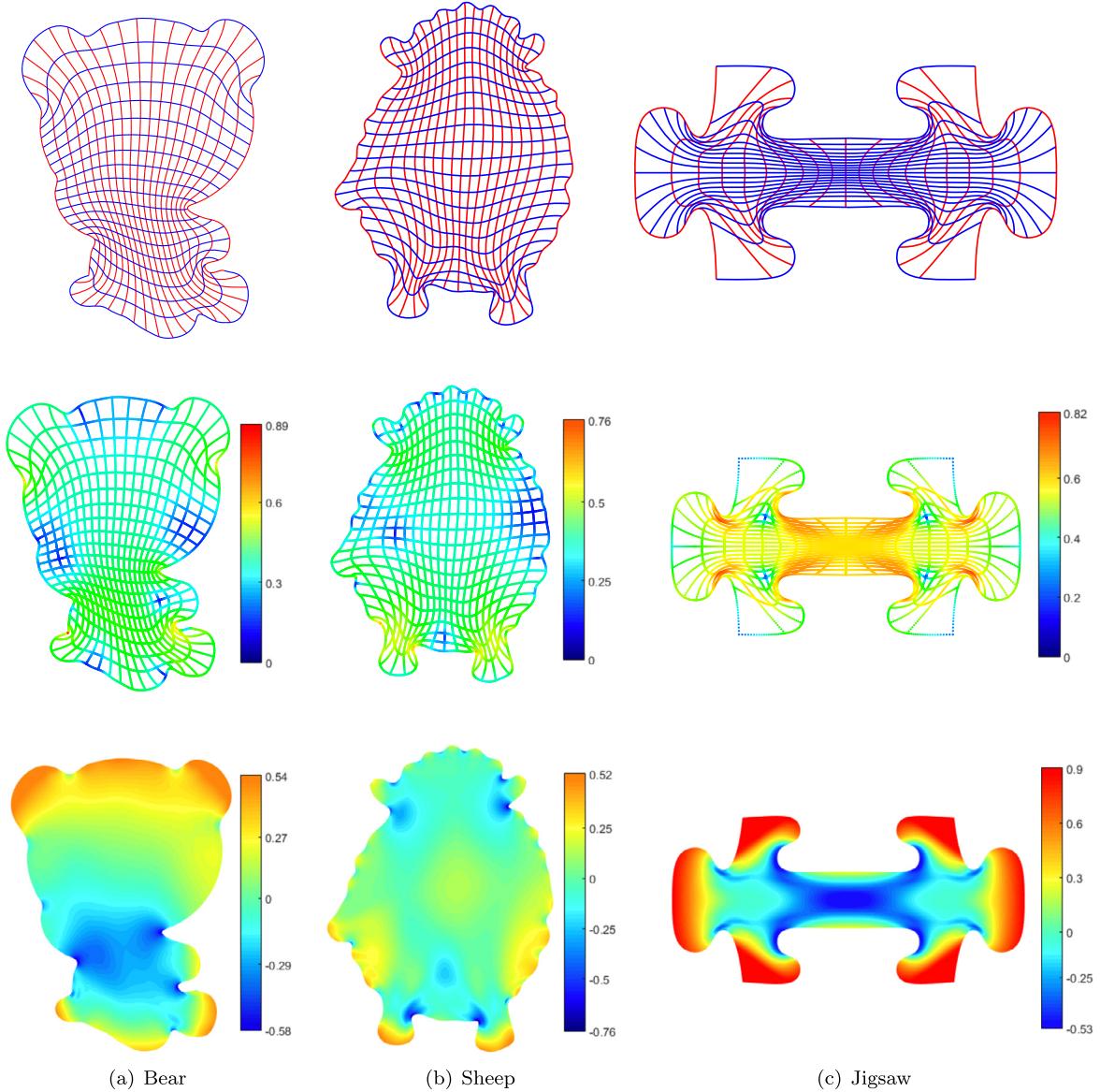


Fig. 5. Parameterization results of different models by our method. (a) the Bear model, (b) the Sheep model, (c) the Jigsaw model. The top row shows the iso-parametric curves, the middle and bottom rows show the colormaps of $|\mu(f)|$ and $\log_{10} J_s(f)$ respectively. Note that the optimal values of $|\mu(f)|$ and $\log_{10} J_s(f)$ are both 0.

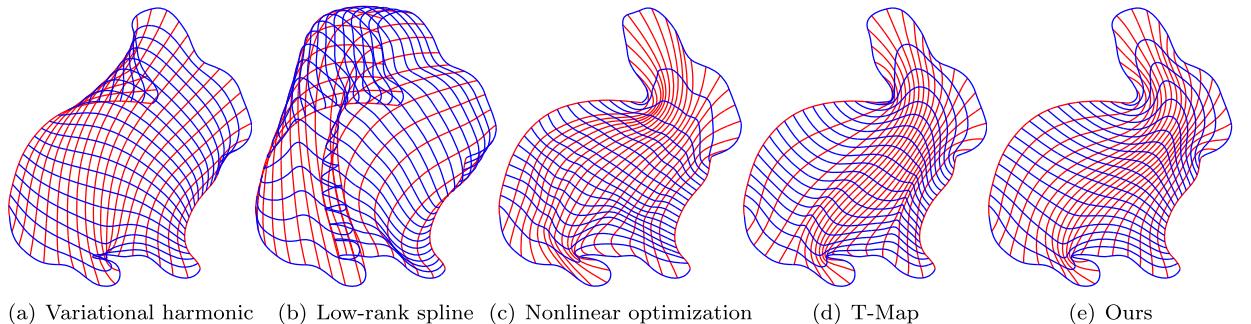


Fig. 6. Parameterization of the rabbit model using different methods: (a) variational harmonic, (b) low-rank spline, (c) nonlinear optimization, (d) T-map method and (e) our method.

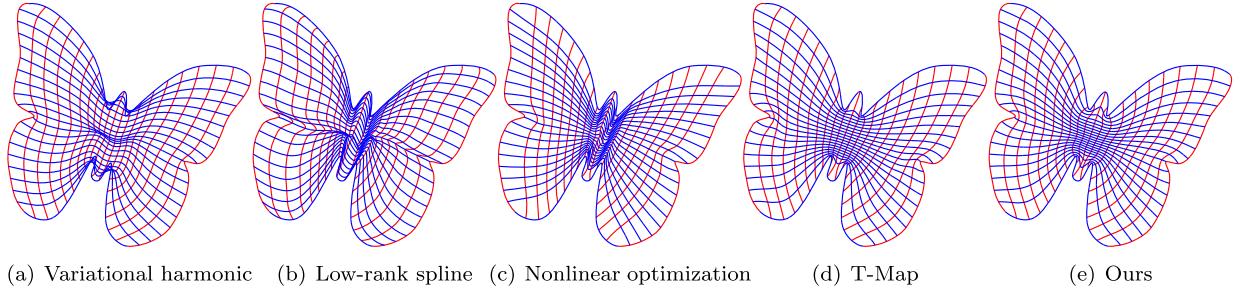


Fig. 7. Parameterization of the butterfly model using different methods: (a) variational harmonic, (b) low-rank spline, (c) nonlinear optimization, (d) T-map method and (e) our method.

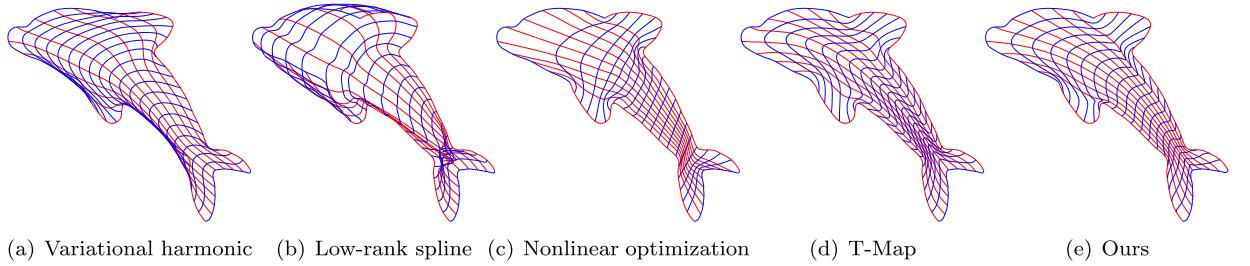


Fig. 8. Parameterization of the dolphin model using different methods: (a) variational harmonic, (b) low-rank spline, (c) nonlinear optimization, (d) T-map method and (e) our method.

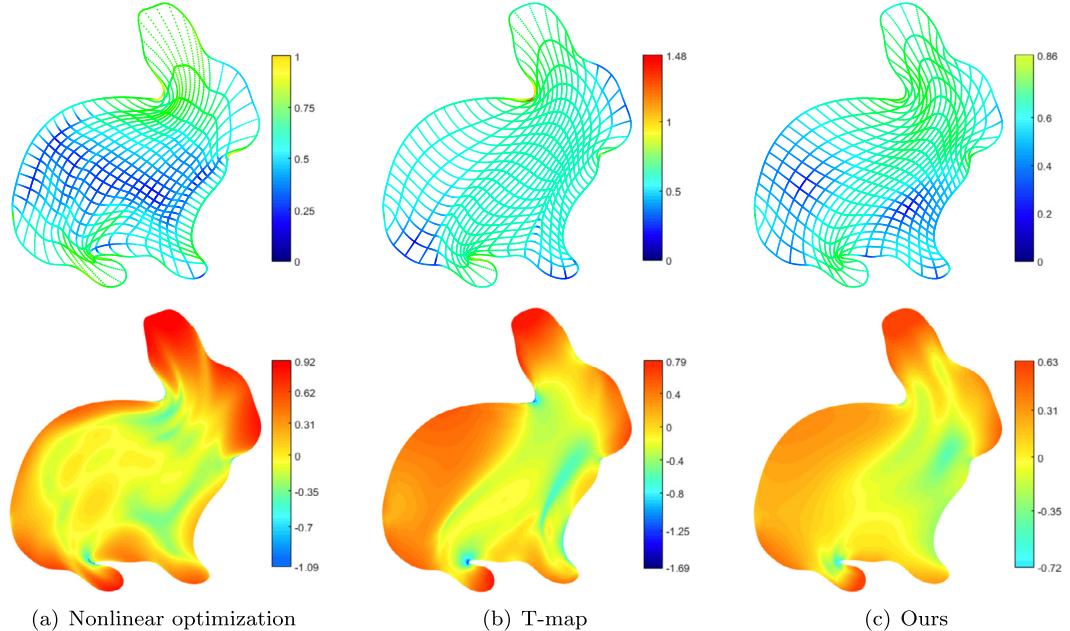


Fig. 9. Parameterization the rabbit model by three methods: (a) nonlinear optimization, (b) T-map method and (c) our method. The top and bottom row show the colormaps of $|\mu(f)|$ and $\log_{10} J_s(f)$ respectively.

5.3. Running time

As described in Section 4, our algorithm iteratively solves two sub-problems to obtain two sequences of complex functions. In each iteration, we need to use the ADMM algorithm to solve a convex optimization problem, which accounts for most of time in our method. Table 2 summarizes the running time of our method on a variety of models. Although the presented method is not the fastest compared with some of the state-of-the-art methods, it produces the best parameterization

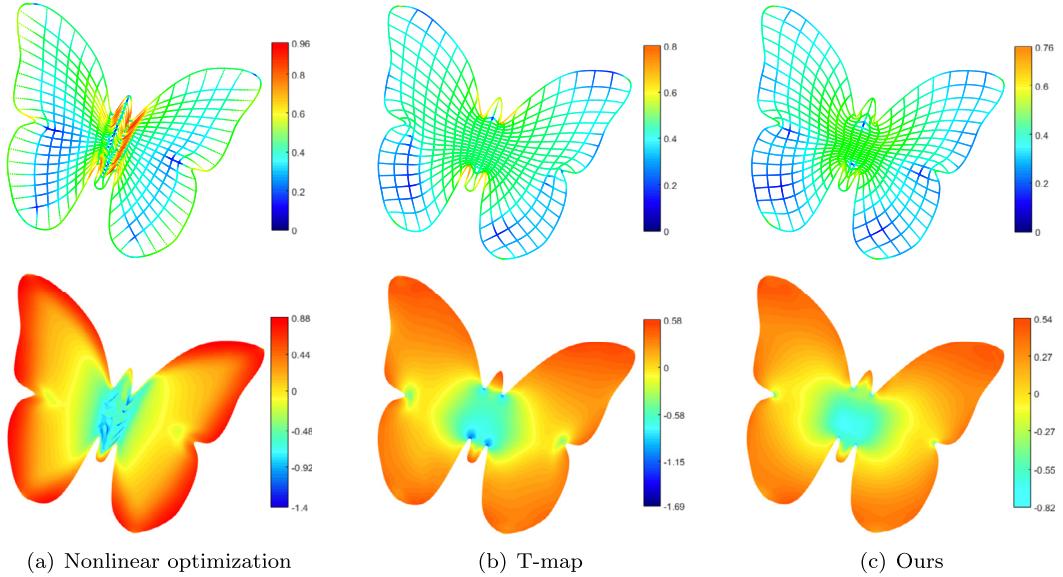


Fig. 10. Parameterization the butterfly model by three methods: (a) nonlinear optimization, (b) T-map method and (c) our method. The top and bottom row show the colormaps of $|\mu(f)|$ and $\log_{10} J_s(f)$ respectively.

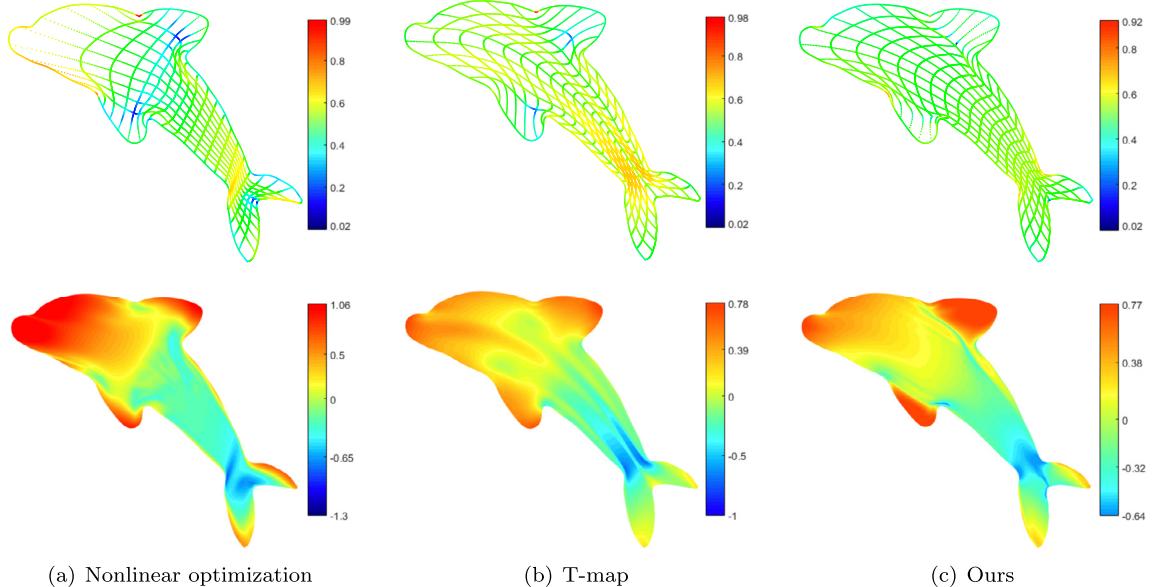


Fig. 11. Parameterization the dolphin model by three methods: (a) nonlinear optimization, (b) T-map method and (c) our method. The top and bottom row show the colormaps of $|\mu(f)|$ and $\log_{10} J_s(f)$ respectively.

Table 2
Runtime performance (in seconds) of our parameterization method on a variety of models.

Model	Sheep (Fig. 5)	Bear (Fig. 5)	Dolphin (Fig. 8)	Rabbit (Fig. 6)	Butterfly (Fig. 7)	Jigsaw (Fig. 5)
Running time (s)	141.23	130.49	128.39	160.72	162.40	105.56

quality among these methods, and thus it is worthwhile to sacrifice some speed. On the other hand, our algorithm can be accelerated by an implementation with C/C++.

6. Conclusions and future work

Parameterization of computational domains and efficiently assembling the mass and stiffness matrices are two essential steps in isogeometric analysis applications. In this paper, using low-rank tensor approximation technique, we propose a low-rank representation scheme for domain parameterization based on quasi-conformal mapping. The problem is formulated as a non-linear and non-convex optimization problem which minimizes the angular distortion and the rank of the map while ensuring the bijectivity of the map. The optimization problem is then converted into two quadratic optimization problems which are solved alternatively. Several experimental examples show that our approach can produce a low-rank and low-distortion parameterization which is superior to other state-of-the-art methods.

Regarding the future work, extending our work to three-dimensional volumetric parameterization is worthy of further research. However, the parameterization problem in three dimensional case is much harder since there is no analogous complex structure in three-dimensional space.

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