Review and Practice problems

1. Arc length.

   (a) $y = f(x)$, $a \le x \le b$
   Formula: $L = \int_a^b \sqrt{1 + (f'(x))^2}dx$
   Example: $y = \frac{1}{4}x^2 - \frac{1}{2}\ ln(x)$, $x \in (1, 2)$. Find the exact value of arc length.
   Solution:

$$\begin{aligned} f'(x) &= \frac{x}{2} - \frac{1}{2x} \\ L &= \int_1^2 \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}} dx \\ &= \int_1^2 \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}} dx = \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2} dx \\ &= \int_1^2 \frac{x}{2} + \frac{1}{2x} dx = \left(\frac{1}{4}x^2 + \frac{1}{2}\ ln(x)\right)\Bigg|_{x=1}^{x=2} \\ &= \frac{3}{4} + \frac{1}{2}\ ln(2) \end{aligned}$$

   (b) $x = f(y)$, $a \le y \le b$
   Formula: $L = \int_a^b \sqrt{1 + (f'(y))^2}dy$

   (c) $x = x(t),\ y = y(t),\ a \le t \le b$
   Formula: $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$
   Example: $x = 3sin^2(t),\ y = sin^3(t)$, $t \in (0, \pi/2)$. Find the exact value of arc length.
   Solution:

$$\begin{aligned} x'(t) &= 6sin(t)cos(t) \\ y'(t) &= 3sin^2(t)cos(t) \\ L &= \int_0^{\frac{\pi}{2}} \sqrt{36sin^2(t)cos^2(t) + 9sin^4(t)cos^2(t)}dt \\ &= \int_0^{\frac{\pi}{2}} \sqrt{9sin^2(t)cos^2(t)(4 + sin^2(t))}dt \\ &= \int_0^{\frac{\pi}{2}} 3sin(t)cos(t)\sqrt{4 + sin^2(t)}dt \end{aligned}$$

$$\textbf{Let } u = 4 + sin^2(t)\textbf{, then } du = 2sin(t)cos(t)dt$$

$$= \int_{t=0}^{t=\frac{\pi}{2}} \frac{3}{2}\sqrt{u}du = \left(u^{\frac{3}{2}}\right)\Big|_{t=0}^{t=\frac{\pi}{2}}$$
$$= \left(u^{\frac{3}{2}}\right)\Big|_{u=4}^{u=5} = 5\sqrt{5} - 8$$

2. Average value of a function.
   Definition: $f(x)$ is a function defined on $[a, b]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

   Theorem: $f(x)$ is a continuous function on $[a, b]$, then there exists at least one value c in $[a, b]$, such that

$$f(c) = f_{ave}$$

   Example:$f(x) = (x-3)^2$, $x \in [2, 5]$. Find f_{ave} and c such that $f(c) = f_{ave}$
   Solution:

$$\begin{aligned} f_{ave} &= \frac{1}{5-2}\int_2^5 (x-3)^2 dx \\ &= \frac{1}{3}\left(\frac{1}{3}(x-3)^3\right)\Big|_2^5 \\ &= 1 \end{aligned}$$

$$\begin{aligned} &f(c) = f_{ave} \\ &(c-3)^2 = 1 \\ &c = 2 \quad or \quad c = 4 \end{aligned}$$

3. Work.
   Newton's 2nd Law: $Force = Mass * Acceleration$
   Definition of work: $Work = Force * Distance$
   Work by a variable force $f(x)$ acting on a particle as it moves from $x = a$ to $x = b$:

$$W = \int_a^b f(x)dx$$

   (a) Spring problem
      Hooke's law: The force to maintain a spring x units beyond its natural length ℓ_n is $f(x) = kx$, where k is the spring constant.
      Formula: The work done in stretching a spring from length ℓ_a to ℓ_b is

$$W = \int_{\ell_a - \ell_n}^{\ell_b - \ell_n} kx\ dx$$

Example: The force to stretch a spring to 0.12m is 0.4N. The work done in stretching this spring from 0.1m to 0.2m is 0.1J. How much work is done in stretching a spring from 0.14m to 0.16m?
Solution:
Step-1. Find ℓ_n and k
The force to stretch a spring to 0.12m is 0.4N

$$k(0.12 - \ell_n) = 0.4 \qquad (1)$$

The work done in stretching this spring from 0.1m to 0.2m is .

$$\int_{0.1-\ell_n}^{0.2-\ell_n} kxdx = 0.1 \qquad (2)$$

Simplify (2),

$$k(0.015 - 0.1\ell_n) = 0.1 \qquad (3)$$

$$\frac{(1)}{(3)} \to \frac{0.12 - \ell_n}{0.015 - 0.1\ell_n} = 4 \to \ell_n = 0.1$$

Let $\ell_n = 0.1$ in (1), then $k = 20$.
Step-2. Find W

$$W = \int_{0.04}^{0.06} 20x\ dx = 10x^2\Big|_{0.04}^{0.06} = 0.02J$$

(b) Cable problem
Example: A cable 100 feet long and weighing 400 pounds hangs vertically from the top of a building. If a 100-lb object is attached to the end of the cable, how much work is required to pull it to the top?
Solution:
This cable weighs 4 lb/ft. Let $f(x)$ be the force to lift the cable and the object, where x is the length of cable on the top of building. $f(x)$=total weight-the weight of cable on the top of the building, which is

$$f(x) = 500 - 4x$$

$$W = \int_0^{100} 500 - 4x\ dx = (500x - 2x^2)\Big|_0^{100} = 30000\texttt{ft-lb}$$

(c) Tank problem
Example: A tank has the shape of a hemisphere (circular part at the top) with a radius of 5 meters. If the height of water in the tank is 3 meters, find the work required to pump the water to the top of the tank until the water height is 1 meters. (The mass density of water is 1000 kg/m^3).
Solution:
For simplicity, we set the origin at the center of the circular top and y as the vertical axis. We want to pump the water between $y = -4$ and $y = -2$ to the top of the tank at $y = 0$.
Step-1, Split the water into n layers of water and estimate the work to pump each layer

of water to the top of the tank.
For water between y_i and y_{i+1},
The volume is

$$V_i = \pi(\sqrt{25 - y_i^2})^2 \Delta y = \pi(25 - y_i^2)\Delta y$$

The mass is

$$M_i = 1000V_i = 1000\pi(25 - y_i^2)\Delta y$$

The force of gravity is

$$F_i = 9.8M_i = 9800\pi(25 - y_i^2)\Delta y$$

The distance from this layer of water to the top of tank is

$$D_i = 0 - y_i = -y_i$$

The work done for this layer of water

$$W_i = F_i * D_i = 9800\pi(25 - y_i^2)(-y_i)\Delta y$$

The total work will be

$$W = \lim_{n\to\infty} \sum_i 9800\pi(25 - y_i^2)(-y_i)\Delta y = \int_{-4}^{-2} 9800\pi(25 - y^2)(-y)dy$$

Step-2, compute W.

$$\begin{aligned} W &= \int_{-4}^{-2} 9800\pi(25 - y^2)(-y)dy \\ &= 9800\pi\left(\frac{1}{4}y^4 - \frac{25}{2}y^2\right)\Big|_{-4}^{-2} \\ &= 9800\pi[(4 - 50) - (64 - 200)] \\ &= 882000\pi(J) \end{aligned}$$

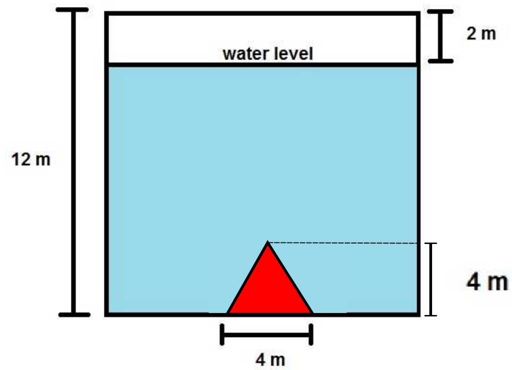
4. Force due to fluid pressure.
Definition: $Pressure = \dfrac{Force}{Area}$
Formula: Suppose region A is on one side of a tank and under water surface $y = h$, which is bounded by $y = a$ from below, $y = b$ from above, $x = f(y)$ from right and $x = g(y)$ from left. Then the force due to fluid pressure over region A is,

$$F = \int_a^b Density(\texttt{lb/ft}^3) * (h - y)(f(y) - g(y))dy \ (\texttt{lb})$$

or

$$F = \int_a^b Density(\texttt{kg/m}^3) * Accel.\ of\ gravity * (h - y)(f(y) - g(y))dy \ (\texttt{N})$$

Example: A vertical dam has a triangular gate as shown in the figure. Find the hydrostatic force against the gate.
Solution:



Set the origin at the top of the gate, then the water surface is at $y = 6$. The gate is bounded by $y = -4$, $y = 0$, $x = -\frac{1}{2}y$(from right) and $x = \frac{1}{2}y$(from left).

$$F = \int_{-4}^{0} 1000 * 9.8 * (6-y)(-\frac{1}{2}y - \frac{1}{2}y)dy = 9800\left(\frac{1}{3}y^3 - 3y^2\right)\Big|_{-4}^{0} = \frac{2038400}{3}(\mathtt{N})$$

5. Moments and Center of mass.
Definition and Formula:
1-D case: n particles with masses $m_1,\ m_2, \cdots, m_n$ locate at $x_1,\ x_2, \cdots, x_n$.
The moment of m_i with respect to the origin is $m_i x_i$.
The total moment: $M = \sum m_i x_i$
The total mass: $m = \sum m_i$
The center of mass: $\bar{x} = \dfrac{M}{m}$

2-D case: n particles with masses $m_1,\ m_2, \cdots, m_n$ locate at $(x_1, y_1),\ (x_2, y_2), \cdots, (x_n, y_n)$.
The moment of m_i with respect to the y-axis is $m_i x_i$.
The moment of m_i with respect to the x-axis is $m_i y_i$.
The total y-moment: $M_y = \sum m_i x_i$
The total x-moment: $M_x = \sum m_i y_i$
The total mass: $m = \sum m_i$
The center of mass: $\bar{x} = \dfrac{M_y}{m},\quad \bar{y} = \dfrac{M_x}{m}$

Lamina case: A plane lamina with a constant density ρ is bounded by $x = a$ from left, $x = b$ from right, $y = 0$ from below and $y = f(x)$ from above.
The total y-moment: $M_y = \rho \displaystyle\int_a^b x f(x)dx$
The total x-moment: $M_x = \rho \displaystyle\int_a^b \frac{1}{2}[f(x)]^2 dx$
The total mass: $m = \rho \displaystyle\int_a^b f(x)dx$

The center of mass: $\bar{x} = \dfrac{M_y}{m} = \dfrac{\int_a^b xf(x)dx}{\int_a^b f(x)dx}$, $\bar{y} = \dfrac{M_x}{m} = \dfrac{\int_a^b \frac{1}{2}[f(x)]^2dx}{\int_a^b f(x)dx}$
Example: A plane lamina with a constant density $\rho = 10$ is enclosed by $x = 0$ from left, $x = \pi$ from right, $y = 0$ from below and $y = sin(x)$ from above. Find M_x, M_y and $(\bar{x}, \bar{y})$
Solution:

$$M_y = 10\int_0^{\pi} xsin(x)dx = 10([-xcos(x)]_0^{\pi} - \int_0^{\pi}(-cos(x))dx) = 10(\pi + sin(x)\Big|_0^{\pi}) = 10\pi$$

$$M_x = 10\int_0^{\pi} \frac{1}{2}sin^2(x)dx = \frac{10}{4}\int_0^{\pi}\big(1 - cos(2x)\big)dx = \frac{10}{4}\left(x - \frac{1}{2}sin(2x)\right)\Bigg|_0^{\pi} = \frac{10\pi}{4}$$

$$m = 10\int_0^{\pi} sin(x)dx = 10(-cos(x))\Big|_0^{\pi} = 20$$

$$\bar{x} = \frac{M_y}{m} = \frac{\pi}{2}, \qquad \bar{y} = \frac{M_x}{m} = \frac{\pi}{8}$$