Review-4

1. Sequence.

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \cdots, a_n, \cdots\}$$

Limit of a Sequence: $\lim_{n\to\infty} a_n = L$.

- If $|L| < \infty$, a_n is convergent;
- If $L = \infty$ or does not exist, a_n is divergent.

Thm-1: Sequence $a_n = f(n)$ and $f(x)$ is defined on $[1, \infty)$. If $\lim_{x\to\infty} f(x) = L$, then

$$\lim_{n\to\infty} a_n = L.$$

Thm-2: If a_n and b_n are two convergent sequences with limits A and B, k is a real constant, then

- $\lim_{n\to\infty} k = k$ $\qquad \lim_{n\to\infty} ka_n = kA$
- $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$ $\qquad \lim_{n\to\infty} a_n b_n = AB$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B} \quad (if\ B \neq 0\ and\ b_n \neq 0)$ $\qquad \lim_{n\to\infty} a_n^k = A^k$

Thm-3: Squeeze Theorem
Let a_n, b_n and c_n be sequences such that $b_n \leq a_n \leq c_n$. If $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n$, then

$$\lim_{n\to\infty} a_n = L.$$

Corollary: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
Thm-4: Let $a_n = r^n$, then

$$\lim_{n\to\infty} a_n = \begin{cases} 0, & |r| < 1 \\ 1, & r = 1 \\ \infty, & r > 1 \\ DNE, & r < -1 \end{cases}$$

Definition:

- a_n is increasing if $a_{n+1} \geq a_n$ for all n;
- a_n is decreasing if $a_{n+1} \leq a_n$ for all n;
- a_n is monotonic if a_n is increasing or decreasing.
- a_n is bounded above if $a_n \leq M$ for all n;
- a_n is bounded below if $a_n \geq m$ for all n;
- a_n is bounded if $M \geq a_n \geq m$ for all n.

Thm-5: If a_n is a monotonic and bounded sequence then a_n is convergent.

2. Series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Definition: Define the partial sum sequence $\{S_n\}$ as

$$S_n = a_1 + a_2 + \cdots + a_n,$$

then

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L,$$

- If $|L| < \infty$, $\sum_{n=1}^{\infty} a_n$ is convergent;
- If $L = \infty$ or does not exist, $\sum_{n=1}^{\infty} a_n$ is divergent.

Geometric series: Let $a_n = ar^{n-1}$ and $a \neq 0$, then

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \dfrac{a}{1-r}, & |r| < 1 \\ \infty, & r \geq 1 \\ DNE, & r \leq -1 \end{cases}$$

Telescoping series: See examples in lecture notes and WebAssign.
Harmonic series: p-series with $p = 1$

Thm-1: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Thm-2: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Thm-3: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two convergent series with limits A and B, k is a real constant, then

$$\sum_{n=1}^{\infty} ka_n = kA \qquad \sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$$

3. Convergence test for positive series.
Integral test: If $a_n = f(n)$ and $f(x)$ is a real-valued, positive, decreasing continuous function for $x \geq 1$, then

- $\sum_{n=1}^{\infty} a_n$ is convergent if $\int_1^{\infty} f(x)dx$ is convergent;
- $\sum_{n=1}^{\infty} a_n$ is divergent if $\int_1^{\infty} f(x)dx$ is divergent.

p-series:

$$a_n = \frac{1}{n^p} \to \sum_{n=1}^{\infty} a_n = \begin{cases} convergent, & p > 1 \\ divergent, & p \leq 1 \end{cases}$$

Comparison Test:

Thm-1 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two positive series, then

- $\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} b_n$ is convergent and $b_n \geq a_n$;
- $\sum_{n=1}^{\infty} a_n$ is divergent if $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$.

Thm-2 Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two positive series. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent or both divergent.

4. Alternating series Test: $a_n \geq 0$

$$\sum_{n=1}^{\infty} (-1)^n a_n \qquad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

Thm-1 If $a_n \geq a_{n+1} \geq 0$ and $\lim_{n\to\infty} a_n = 0$, then both $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge.

Thm-2 Suppose $\sum_{n=1}^{\infty} (-1)^n a_n = S$ is convergent, where $a_n \geq a_{n+1} > 0$. Let $S_n = \sum_{i=1}^{n} (-1)^i a_i$ and $R_n = S - S_n$, then $|R_n| \leq a_{n+1}$.

5. Absolute convergence and Ratio Test
Definition:

- $\sum_{n=1}^{\infty} a_n$ is Absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent;
- $\sum_{n=1}^{\infty} a_n$ is Conditionally convergent if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Thm-1 If $\sum_{n=1}^{\infty} a_n$ is Absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Thm-2: Ratio Test Suppose $\sum_{n=1}^{\infty} a_n$ is a series, where $a_n \neq 0$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} L<1, & \textit{Absolutely convergent} \\ L>1, & \textit{Divergent} \\ L=1, & \textit{No conclusion} \end{cases}$$

6. Power series.

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

Interval of Convergence I:

Using ratio test to find the value of x such that $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges.

Thm-1 $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges in an interval with center $x=a$ and radius R,

- $R=0 \to I=[a,a]$;
- $R=\infty \to I=(-\infty,\infty)$;
- $0<R<\infty \to I=(a-R,a+R)$, $[a-R,a+R]$, $[a-R,a+R)$, or $(a-R,a+R]$.

7. Functions as power series.

Power series representation: Using $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x|<1$.

Thm-1 If $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ converges in an interval with center $x=a$ and radius $R>0$, then $f(x)$ is continuous and differentiable on $(a-R,a+R)$, then both

- $f'(x)=\sum_{n=1}^{\infty} c_n n(x-a)^{n-1}$,
- $\int f(x)dx = C+\sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}$,

have radius of convergence R about center $x=a$.

8. Taylor and Maclaurin series.

Thm-1 If $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ converges in an interval with center $x=a$ and radius $R>0$, then

$$c_n=\frac{f^{(n)}(a)}{n!}$$

Definition If $f(x)$ has derivatives of all orders converges at $x = a$, then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the Taylor series of $f(x)$ at $x = a$; when $a = 0$, it is called Maclaurin series.
Useful facts:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \left(\begin{array}{c} k \\ n \end{array} \right) x^n$$

Thm-2 If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the same interval I of convergence with radius $R > 0$, then

- $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ converges in an interval at least as large as I;
- $f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = a_0 b_0 + (a_1 b_0 + a0 b_1)x + (a_1 b_1 + a_0 b_2 + a_2 b_0)x^2 + \cdots$ converges in an interval at least as large as I;
- $\dfrac{f(x)}{g(x)} = \dfrac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}$ (Using long division) It is hard to find its interval of convergence.

9. Taylor and Maclaurin polynomials.
Definition If $f(x)$ has derivatives of all orders converges at $x = a$, then

$$T_n = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

is the n^{th} degree Taylor polynomial of $f(x)$ at $x = a$; when $a = 0$, it is called Maclaurin polynomial.
Thm-1 If $f(x)$ has a Taylor series about $x = a$ and let $R_n(x) = f(x) - T_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in [a - r, a + r]$ for some $r > 0$, then

$$f(x) = \lim_{n\to\infty} T_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $[a-r, a+r]$.

Thm-2: Taylor's Inequality

Let $M > 0$ and $r > 0$. If $|f^{(n+1)}| \leq M$ for all $x \in [a-r, a+r]$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

on $[a-r, a+r]$.