Review-2 and Practice problems

1. Trigonometric Integrals.

   (a) $\displaystyle\int \sin^m(x)\cos^n(x)dx$

   Case-1: m is odd $\rightarrow$ let $u = \cos(x)$

   Example: $\displaystyle\int \sin^3(x)\cos^2(x)dx$

   Solution:

$$
\begin{aligned}
\int \sin^3(x)\cos^2(x)dx &= \int \sin^2(x)\cos^2(x)(\sin(x)dx) \\
&= \int (1-\cos^2(x))\cos^2(x)(\sin(x)dx) \\
&\quad \texttt{let } u = \cos(x),\ du = -\sin(x)dx \\
&= \int (1-u^2)u^2(-du) \\
&= \int u^4 - u^2 du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C
\end{aligned}
$$

   Case-2: n is odd $\rightarrow$ let $u = \sin(x)$

   Example: $\displaystyle\int \cos^5(2x)dx$

   Solution:

$$
\begin{aligned}
\int \cos^5(2x)dx &= \int (\cos^2(2x))^2(\cos(2x)dx) \\
&= \int (1-\sin^2(2x))^2(\cos(2x)dx) \\
&\quad \texttt{let } u = \sin(2x),\ du = 2\cos(2x)dx \\
&= \int (1-u^2)^2(\frac{1}{2}du) \\
&= \frac{1}{2}\int u^4 - 2u^2 + 1du \\
&= \frac{1}{10}u^5 - \frac{1}{3}u^3 + \frac{1}{2}u + C \\
&= \frac{1}{10}\sin^5(2x) - \frac{1}{3}\sin^3(2x) + \frac{1}{2}\sin(2x) + C
\end{aligned}
$$

   Case-3: m and n are even $\rightarrow$ use $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ and $\sin^{(} x) = \frac{1}{2}(1-\cos(2x))$

   Example: $\displaystyle\int 16\sin^4(x)\cos^4(x)dx$

Solution:

$$\begin{aligned}
\int 16\sin^4(x)\cos^4(x)dx &= \int 16\left(\frac{1}{2}(1+\cos(2x))\right)^2\left(\frac{1}{2}(1-\cos(2x))\right)^2 dx\\
&= \int \left(1-2\cos^2(2x)+\cos^4(2x)\right)dx\\
&= \int \left(1-2\left(\frac{1}{2}(1+\cos(4x))\right)+\left(\frac{1}{2}(1+\cos(4x))\right)^2\right)dx\\
&= \int \left(-\cos(4x)+\frac{1}{4}\left(1+2\cos(4x)+\cos^2(4x)\right)\right)dx\\
&= \int \left(\frac{1}{4}-\frac{1}{2}\cos(4x)+\frac{1}{8}\left(1+\cos(8x)\right)\right)dx\\
&= \int \left(\frac{3}{8}-\frac{1}{2}\cos(4x)+\frac{1}{8}\cos(8x)\right)dx\\
&= \frac{3}{8}x-\frac{1}{8}\sin(4x)+\frac{1}{64}\sin(8x)+C
\end{aligned}$$

(b) $\int \tan^m(x)\sec^n(x)dx$

Case-1: n is even $\rightarrow$ let $u=\tan(x)$

Example: $\int \tan^2(x)\sec^4(x)dx$

Solution:

$$\begin{aligned}
\int \tan^2(x)\sec^4(x)dx &= \int \tan^2(x)\sec^2(x)(\sec^2(x)dx)\\
&= \int \tan^2(x)(tan^2(x)+1)(\sec^2(x)dx)\\
&\quad \texttt{let } u=\tan(x),\ du=\sec^2(x)dx\\
&= \int u^2(u^2+1)(du)\\
&= \int u^4+u^2du\\
&= \frac{1}{5}u^5+\frac{1}{3}u^3+C\\
&= \frac{1}{5}\tan^5(x)+\frac{1}{3}\tan^3(x)+C
\end{aligned}$$

Case-2: m is odd $\rightarrow$ let $u=\sec(x)$

Example: $\int \tan^3(2x)\sec(2x)dx$

Solution:

$$\begin{aligned}
\int \tan^3(2x)sec(2x)dx =& \int (\tan^2(2x))(\tan(2x)\sec(2x)dx) \\
=& \int (\sec^2(2x)-1)(\tan(2x)\sec(2x)dx) \\
& \texttt{let } u=\sec(2x),\ du = 2\tan(2x)\sec(2x)dx \\
=& \int (u^2-1)(\frac{1}{2}du) \\
=& \frac{1}{2}\int u^2 - 1du \\
=& \frac{1}{6}u^3 - \frac{1}{2}u + C \\
=& \frac{1}{6}\sec^3(2x) - \frac{1}{2}\sec(2x) + C
\end{aligned}$$

Case-3: n is odd and n is even $\rightarrow$ No general solution

Example: $\int \sec(2x)dx$

Solution:

$$\begin{aligned}
\int \sec(2x)dx =& \int \frac{\sec(2x)(\sec(2x)+\tan(2x))}{\sec(2x)+\tan(2x)}dx \\
=& \int \frac{(\sec^2(2x)+\sec(2x)\tan(2x))}{\sec(2x)+\tan(2x)}dx \\
& \texttt{let } u=\sec(2x)+\tan(2x),\ du = 2(\sec^2(2x)+\sec(2x)\tan(2x))dx \\
=& \int \frac{1}{u}(\frac{1}{2}du) = \frac{1}{2}ln(u) \\
=& \frac{1}{2}ln(\sec(2x)+\tan(2x))
\end{aligned}$$

2. Trigonometric substitution.
   Case-1: $\sqrt{x^2-a^2} \rightarrow$ let $x = a\sec(x)$
   Example: $\int \frac{1}{\sqrt{x^2-6x}}dx$
   Solution:

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-6x}}dx =& \int \frac{1}{\sqrt{x^2-6x+9-9}}dx \\
=& \int \frac{1}{\sqrt{(x-3)^2-3^2}}dx \\
& \texttt{let } u = x-3,\ du = dx \\
=& \int \frac{1}{\sqrt{u^2-3^2}}du \\
& \texttt{let } u = 3\sec(\theta),\ du = 3\sec(\theta)\tan(\theta)d\theta \\
=& \int \frac{1}{3\tan(\theta)}3\sec(\theta)\tan(\theta)d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int \sec(\theta) d\theta \\
&= \ln|\sec(\theta) + \tan(\theta)| + C \\
&= \ln\left|\frac{u}{3} + \frac{\sqrt{u^2 - 3^2}}{3}\right| + C \\
&= \ln\left|\frac{x - 3 + \sqrt{x^2 - 6x}}{3}\right| + C \\
&= \ln\left|x - 3 + \sqrt{x^2 - 6x}\right| + C
\end{aligned}
$$

Case-2: $\sqrt{a^2 - x^2} \to$ let $x = a\sin(x)$
Example: $\int \sqrt{4x - x^2} dx$
Solution: **Solution**:

$$
\begin{aligned}
\int \sqrt{4x - x^2} dx &= \int \sqrt{-(x^2 - 4x)} dx \\
&= \int \sqrt{-(x^2 - 4x + 4 - 4)} dx \\
&= \int \sqrt{-((x-2)^2 - 2^2)} dx \\
&= \int \sqrt{2^2 - (x-2)^2} dx \\
&\quad \texttt{let } u = x - 2,\ du = dx \\
&= \int \sqrt{2^2 - u^2} du \\
&\quad \texttt{let } u = 2\sin(\theta),\ du = 2\cos(\theta) d\theta \\
&= \int 2\cos(\theta) \cdot 2\cos(\theta) d\theta \\
&= 4 \int \cos^2(\theta) d\theta \\
&= 4 \int \frac{1}{2}(\cos(2\theta) + 1) d\theta \\
&= 4\left(\frac{1}{2}\left(\frac{1}{2}\sin(2\theta) + \theta\right)\right) + C \\
&= \sin(2\theta) + 2\theta + C \\
&= 2\sin(\theta)\cos(\theta) + 2\theta + C \\
&= 2\frac{u}{2}\frac{\sqrt{2^2 - u^2}}{2} + 2arcsin\left(\frac{u}{2}\right) + C \\
&= \frac{(x-2)\sqrt{4x - x^2}}{2} + 2arcsin\left(\frac{x-2}{2}\right) + C
\end{aligned}
$$

Case-3: $\sqrt{x^2+a^2} \rightarrow$ let $x = a\tan(x)$

Example: $\int \frac{1}{\sqrt{4x^2+9}}dx$

Solution:

$$
\begin{aligned}
\int \frac{1}{\sqrt{4x^2+9}}dx =& \int \frac{1}{\sqrt{(2x)^2+3^2}}dx \\
& \texttt{let } u = 2x,\ du = 2dx \\
=& \int \frac{1}{2\sqrt{u^2+3^2}}du \\
& \texttt{let } u = 3\tan(\theta),\ du = 3\sec^2(\theta)d\theta \\
=& \int \frac{1}{6\sec(\theta)}3\sec^2(\theta)d\theta \\
=& \frac{1}{2}\int \sec(\theta)d\theta \\
=& \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C \\
=& \frac{1}{2}\ln\left|\frac{\sqrt{u^2+3^2}}{3} + \frac{u}{3}\right| + C \\
=& \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}+2x}{3}\right| + C \\
=& \frac{1}{2}\ln\left|\sqrt{4x^2+9}+2x\right| + C
\end{aligned}
$$

3. Partial fraction.

Question: $\int f(x)\ dx = \int \frac{N(x)}{D(x)}\ dx$, where $N(x)$ and $D(x)$ are polynomials.

If degree of $N(x) \geq$ degree of $D(x) \rightarrow$ Long division $\rightarrow$ Partial fraction

If degree of $N(x) <$ degree of $D(x) \rightarrow$ Partial fraction;

Example-1: $\int \frac{2}{(x-1)(x+1)}dx$

Solution:

$$
\begin{aligned}
\frac{2}{(x-1)(x+1)} =& \frac{A}{x-1} + \frac{B}{x+1} \\
=& \frac{Ax+A+Bx-B}{(x-1)(x+1)} \\
=& \frac{(A+B)x+(A-B)}{(x-1)(x+1)}
\end{aligned}
$$

Compare the coefficients, $A+B=0$ and $A-B=2$, which leads to $A=1$ and $B=-1$.

$$
\int \frac{2}{(x-1)(x+1)}dx = \int \frac{1}{x-1} + \frac{-1}{x+1}dx
$$

$$
\begin{aligned}
&= \ln|x-1| - \ln|x+1| + C \\
&= \ln\left|\frac{x-1}{x+1}\right| + C
\end{aligned}
$$

Example-2: $\displaystyle\int \frac{2x^2+3x-1}{(x-1)(x+1)^2}dx$
Solution:

$$
\begin{aligned}
\frac{2x^2+3x-1}{(x-1)(x+1)^2} =& \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\
=& \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}
\end{aligned}
$$

To ensure

$$2x^2+3x-1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Take $x=1$, $4=4A \rightarrow A=1$;
Take $x=-1$, $-2=-2C \rightarrow C=1$;
Take $x=0$, $-1=A-B-C \rightarrow B=1$;

$$
\begin{aligned}
\int \frac{2x^2+3x-1}{(x-1)(x+1)}dx =& \int \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{(x+1)^2}dx \\
=& \ln|x-1| + \ln|x+1| + (-\frac{1}{x+1}) + C \\
=& \ln|(x-1)(x+1)| - \frac{1}{x+1} + C
\end{aligned}
$$

Example-3: $\displaystyle\int \frac{2x^2+2x+2}{(x+1)(x^2+1)}dx$
Solution:

$$
\begin{aligned}
\frac{2x^2+2x+2}{(x+1)(x^2+1)} =& \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
=& \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)}
\end{aligned}
$$

To ensure

$$2x^2+2x+2 = A(x^2+1) + (Bx+C)(x+1)$$

Take $x=-1$, $2=2A \rightarrow A=1$;
Take $x=0$, $2=A+C \rightarrow C=1$;

Take $x = 1$, $6 = 2A + 2B + 2C \to B = 1$;

$$\begin{aligned}\int \frac{2x^2+2x+2}{(x+1)(x^2+1)}dx =& \int \frac{1}{x+1} + \frac{x+1}{x^2+1}dx \\ =& \ln|x+1| + \int \frac{x}{x^2+1} + \frac{1}{x^2+1}dx \\ =& \ln|x+1| + \frac{1}{2}\ln(x^2+1) + arctan(x) + C\end{aligned}$$

Example-3: $\int \frac{x^4+x^3+3x^2+x+1}{x(x^2+1)^2}dx$

Solution:

$$\begin{aligned}\frac{x^4+x^3+3x^2+x+1}{x(x^2+1)^2} =& \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\ =& \frac{A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x}{(x+1)(x^2+1)^2} \\ =& \frac{(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A}{(x+1)(x^2+1)^2}\end{aligned}$$

Constant term, $A = 1$;
x^3 term, $C = 1$;
x term, $C + E = 1 \to E = 0$;
x^4 term, $A + B = 1 \to B = 0$;
x^2 term, $2A + B + D = 3 \to D = 1$;

$$\begin{aligned}\int \frac{x^4+x^3+3x^2+x+1}{x^2-1}dx =& \int \frac{1}{x} + \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}dx \\ =& \ln|x| + arctan(x) - \frac{1}{2(x^2+1)} + C\end{aligned}$$

Example-4: $\int \frac{x^4+x^2}{x^2-1}dx$

Solution:
Long division:

$$\begin{array}{r} x^2 \quad + \quad 2 \\ x^2-1 \overline{) x^4 + 0x^3 + x^2 + 0x + 0} \\ \underline{x^4 + 0x^3 - x^2 } \\ 2x^2 + 0x + 0 \\ \underline{2x^2 + 0x - 2} \\ 2 \end{array}$$

$$\int \frac{x^4+x^2}{x^2-1}dx = \int x^2+2+\frac{2}{x^2-1}dx$$
$$=\frac{1}{3}x^3+2x+\int \frac{2}{(x-1)(x+1)}dx$$
follow Example-1
$$=\frac{1}{3}x^3+2x+\ln\left|\frac{x-1}{x+1}\right|+C$$

4. Using integral table. **(No general procedures for problems in this section. Read lecture note of section 2.4 and try to do exercise problems in textbook.)**

5. Numerical Integration.
   Trapezoidal Rule: $a = x_0 < x_1 < \cdots < x_n = b$, $x_i - x_{i-1} = h = \frac{b-a}{n}$

$$I_T = \frac{1}{2}h\Big(f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\Big)$$

   Error Estimate for Trapezoidal Rule: Define $E_T = \displaystyle\int_a^b f(x)\ dx - I_T$,

$$|E_T| \le \frac{K(b-a)^3}{12n^2},$$

   where $K > 0$ and $|f''(x)| \le K$ for all x in (a,b).
   Simpson's Rule: $a = x_0 < x_1 < \cdots < x_n = b$, $x_i - x_{i-1} = h = \frac{b-a}{n}$

$$I_S = \frac{1}{3}h\Big(f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big)$$

   Error Estimate for Simpson's Rule: Define $E_S = \displaystyle\int_a^b f(x)\ dx - I_S$,

$$|E_S| \le \frac{K(b-a)^5}{180n^4},$$

   where $K > 0$ and $|f''''(x)| \le K$ for all x in (a,b).
   Example: Estimate $\displaystyle\int_{-2}^{2} 5x^4 dx$ using Trapezoidal rule and Simpson's rule and estimate their errors.
   Solution:

$$h = \frac{2-(-2)}{4} = 1$$

   Trapezoidal Rule:

$$I_T = \frac{1}{2}\Big(f(-2)+2f(-1)+2f(0))+2f(1)+f(2)\Big) = 90$$

Error Estimate for Trapezoidal Rule: $f''(x) = 60x^2 \to |f''(x)| < f''(2) = 240 = K$,

$$|E_T| \leq \frac{240(2-(-2))^3}{12 * 4^2} = 80.$$

Simpson's Rule:

$$I_S = \frac{1}{3}\bigg(f(-2) + 4f(-1) + 2f(0)) + 4f(1) + f(2)\bigg) = \frac{200}{3} \approx 66.667$$

Error Estimate for Simpson's Rule: $f''''(x) = 120 = K$

$$|E_S| \leq \frac{120(2-(-2))^5}{180 * 4^4} = 8/3 \approx 2.667.$$

6. Improper integrals.

Case-1.1: $\displaystyle\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_a^t f(x)dx$

Example: $\displaystyle\int_0^{\infty} xe^{-x}dx$

Solution:

$$\int_0^{+\infty} xe^{-x}dx = \lim_{t\to+\infty}\int_0^t xe^{-x}dx$$

$$\textbf{Integration by parts}$$

$$\begin{aligned}
&= \lim_{t\to+\infty}(-xe^{-x} - e^{-x})\Big|_0^t \\
&= \lim_{t\to+\infty}[(-te^{-t} - e^{-t}) - (-0e^{-0} - e^{-0})] \\
&= [(0-0) - (0-1)] = 1
\end{aligned}$$

Case-1.2: $\displaystyle\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx$

Example: $\displaystyle\int_{-\infty}^{0} \frac{2x}{x^2+1}dx$

Solution:

$$\int_{-\infty}^{0} \frac{2x}{x^2+1}dx = \lim_{t\to-\infty}\int_t^0 \frac{2x}{x^2+1}dx$$

$$\textbf{let } u = x^2 + 1\textbf{, } du = 2xdx$$

$$\begin{aligned}
&= \lim_{t\to-\infty}\int_{x=t}^{x=0} \frac{1}{u}du \\
&= \lim_{t\to-\infty}[\ln(u)]\Big|_{x=t}^{x=0} \\
&= \lim_{t\to-\infty}[\ln(x^2+1)]\Big|_{x=t}^{x=0} \\
&= \lim_{t\to-\infty}[\ln(1) - \ln(t^2+1)] \\
&= \infty
\end{aligned}$$

$$\int_{-\infty}^{0} \frac{2x}{x^2+1}dx \textbf{ is divergent.}$$

Case-1.3: $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{+\infty} f(x)dx$

Example: $\int_{-\infty}^{+\infty} 2|x|e^{-x^2}dx$

Solution:

$$\begin{aligned}
\int_{-\infty}^{+\infty} 2|x|e^{-x^2}dx &= \int_{-\infty}^{0} -2xe^{-x^2}dx + \int_{0}^{+\infty} 2xe^{-x^2}dx \\
&= \lim_{t\to-\infty}\int_{t}^{0} -2xe^{-x^2}dx + \lim_{t\to+\infty}\int_{0}^{t} 2xe^{-x^2}dx \\
&\quad \textbf{let } u = x^2\textbf{,}\ du = 2xdx \\
&= \lim_{t\to-\infty}\int_{x=t}^{x=0} -e^{-u}du + \lim_{t\to+\infty}\int_{x=0}^{x=t} e^{-u}du \\
&= \lim_{t\to-\infty}(e^{-u})\Big|_{x=t}^{x=0} + \lim_{t\to+\infty}(-e^{-u})\Big|_{x=0}^{x=t} \\
&= \lim_{t\to-\infty}(e^{-x^2})\Big|_{x=t}^{x=0} + \lim_{t\to+\infty}(-e^{-x^2})\Big|_{x=0}^{x=t} \\
&= \lim_{t\to-\infty}(1-e^{-t^2}) + \lim_{t\to+\infty}(-e^{-t^2}+1) \\
&=(1-0)+(-0+1) \\
&=2
\end{aligned}$$

Case-2.1: If $f(x)$ is continuous in $[a, b)$ but discontinuous at b,

$$\int_{a}^{b} f(x)dx = \lim_{t\to b^-}\int_{a}^{t} f(x)dx$$

Example: $\int_{0}^{1} \frac{1}{\sqrt{1-x}}dx$

Solution:

$$\frac{1}{\sqrt{1-x}} \to +\infty \ \ \textbf{as} \ \ x \to 1^-$$

$$\begin{aligned}
\int_{0}^{1} \frac{1}{\sqrt{1-x}}dx &= \lim_{t\to1^-}\int_{0}^{t} \frac{1}{\sqrt{1-x}}dx \\
&= \lim_{t\to1^-}\left(-2(1-x)^{\frac{1}{2}}\right)\Big|_{0}^{t} \\
&= \lim_{t\to1^-}[-2(1-t)^{\frac{1}{2}} - (-2(1-0)^{\frac{1}{2}})] \\
&=[0+2(1)] = 2
\end{aligned}$$

Case-2.2: If $f(x)$ is continuous in $(a, b]$ but discontinuous at a,

$$\int_{a}^{b} f(x)dx = \lim_{t\to a^+}\int_{t}^{b} f(x)dx$$

Example: $\displaystyle\int_{-3}^{0} \frac{1}{\sqrt{9-x^2}}dx$

Solution:

$$\frac{1}{\sqrt{9-x^2}} \to +\infty \quad \textbf{as} \quad x \to -3^+$$

$$\begin{aligned}
\int_{-3}^{0} \frac{1}{\sqrt{9-x^2}}dx &= \lim_{t\to -3^+} \int_{-3}^{0} \frac{1}{\sqrt{9-x^2}}dx \\
&\textbf{let } x = 3\sin(\theta)\textbf{,}\ dx = 3\cos(\theta)d\theta \\
&= \lim_{t\to -3^+} \int_{-3}^{0} \frac{1}{3\cos(\theta)} 3\cos(\theta)d\theta \\
&= \lim_{t\to -3^+} \int_{-3}^{0} 1 d\theta \\
&= \lim_{t\to -3^+} \theta \Big|_{x=t}^{x=0} \\
&= \lim_{t\to -3^+} \left[arcsin\left(\frac{x}{3}\right)\right] \Big|_{x=t}^{x=0} \\
&= \lim_{t\to -3^+} [arcsin(0) - arcsin(t/3)] \\
&=[arcsin(0) - arcsin(-1)] \\
&=0 - (-\frac{\pi}{2}) \\
&=\frac{\pi}{2}
\end{aligned}$$

Case-2.3: If $f(x)$ is continuous in $[a, c)$ and $(c, b]$ but discontinuous at c,

$$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$$

Example: $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{|x|}}dx$

Solution:

$$\frac{1}{\sqrt{|x|}} \to +\infty \quad \textbf{as} \quad x \to 0$$

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{|x|}}dx &= \int_{-1}^{0} \frac{1}{\sqrt{-x}}dx + \int_{0}^{1} \frac{1}{\sqrt{x}}dx \\
&= \lim_{t\to -1^+} \int_{t}^{0} \frac{1}{\sqrt{-x}}dx + \lim_{t\to 1^-} \int_{0}^{t} \frac{1}{\sqrt{x}}dx \\
&= \lim_{t\to -1^+} (-2(-x)^{\frac{1}{2}})\Big|_{x=t}^{x=0} + \lim_{t\to 1^-} (2x^{\frac{1}{2}})\Big|_{x=0}^{x=t} \\
&= \lim_{t\to -1^+} (0 - (-2(-t)^{\frac{1}{2}})) + \lim_{t\to 1^-} (2t^{\frac{1}{2}} - 0)
\end{aligned}$$

$$
\begin{aligned}
&= (0 - (-2)) + (2 - 0) \\
&= 4
\end{aligned}
$$

Case-2.4: If $f(x)$ is continuous in (a, b) but discontinuous at a and b,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

where c could be any point in (a, b).

Example: $\displaystyle\int_{-2}^{2} \frac{1}{\sqrt{4 - x^2}} dx$

Solution:

$$\frac{1}{\sqrt{4 - x^2}} \to +\infty \quad \textbf{as} \quad x \to 2 \textbf{ or } -2$$

$$
\begin{aligned}
\int_{-2}^{2} \frac{1}{\sqrt{4 - x^2}} dx &= \int_{-2}^{0} \frac{1}{\sqrt{4 - x^2}} dx + \int_{0}^{2} \frac{1}{\sqrt{4 - x^2}} dx \\
&\quad \textbf{let } x = 2\sin(\theta)\textbf{,}\ dx = 2\cos(\theta)d\theta \\
&= \int_{-2}^{0} \frac{1}{2\cos(\theta)} 2\cos(\theta)d\theta + \int_{0}^{2} \frac{1}{2\cos(\theta)} 2\cos(\theta)d\theta \\
&= \int_{-2}^{0} 1d\theta + \int_{0}^{2} 1d\theta \\
&= \lim_{t \to -2^+} (\theta)\Big|_{x=t}^{x=0} + \lim_{t \to 2^-} (\theta)\Big|_{x=0}^{x=t} \\
&= \lim_{t \to -2^+} \left(arcsin\left(\frac{x}{2}\right)\right)\Big|_{x=t}^{x=0} + \lim_{t \to 2^-} \left(arcsin\left(\frac{x}{2}\right)\right)\Big|_{x=0}^{x=t} \\
&= \lim_{t \to -2^+} \left(arcsin\,(0) - arcsin\left(\frac{t}{2}\right)\right) + \lim_{t \to 2^-} \left(arcsin\left(\frac{t}{2}\right) - arcsin\,(0)\right) \\
&= (arcsin\,(0) - arcsin\,(-1)) + (arcsin\,(1) - arcsin\,(0)) \\
&= (0 - (-\frac{\pi}{2})) + (\frac{\pi}{2} - 0) \\
&= \pi
\end{aligned}
$$