Key concepts & formulas

1 surface integrals

Surface $\mathcal{S}$: $\overrightarrow{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$

- tangent plane at (x_0, y_0, z_0)

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0,$$

  where $\langle a, b, c\rangle = \overrightarrow{r_u} \times \overrightarrow{r_v}$ at (x_0, y_0, z_0).
  special cases:

  - $\mathcal{S} : z = f(x,y) \Rightarrow z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$;
  - $\mathcal{S} : F(x,y,z) = k \Rightarrow F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$.

- surface area:

$$Area(\mathcal{S}) = \iint_D \|\overrightarrow{r_u} \times \overrightarrow{r_v}\| dA$$

  special case:

  - $\mathcal{S} : z = f(x,y),\ (x,y) \in D \Rightarrow Area(\mathcal{S}) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\ dA$;

- surface integral of a function $g(x,y,z)$

$$\iint_{\mathcal{S}} g(x,y,z)\ d\mathcal{S} = \iint_D g(\overrightarrow{r}(u,v))\|\overrightarrow{r_u} \times \overrightarrow{r_v}\| dA$$

  special case:

  - $\mathcal{S} : z = f(x,y),\ (x,y) \in D \Rightarrow \iint_{\mathcal{S}} g(x,y,z)\ d\mathcal{S} = \iint_D g(x,y,z)\sqrt{1 + f_x^2 + f_y^2}\ dA$;

- surface integral of a vector field $F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

$$\iint_{\mathcal{S}} \overrightarrow{F}(x,y,z) \cdot d\overrightarrow{\mathcal{S}} = \iint_{\mathcal{S}} \overrightarrow{F}(\overrightarrow{r}(u,v)) \cdot \widehat{n}\ d\mathcal{S} = \iint_D \overrightarrow{F}(\overrightarrow{r}(u,v)) \cdot (\overrightarrow{r}_u \times \overrightarrow{r_v})\ dA$$

  special case:

  - $\mathcal{S} : z = f(x,y),\ (x,y) \in D \Rightarrow \iint_{\mathcal{S}} \overrightarrow{F}(x,y,z) \cdot d\overrightarrow{\mathcal{S}} = \iint_D \Big(-f_x P - f_y Q + R\Big)\ dA$;

- Stoke's theorem:

$$\int_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_{\mathcal{S}} curl\, \overrightarrow{F} \cdot d\overrightarrow{\mathcal{S}}$$

  where $\mathcal{S}$ is an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $\mathcal{C}$ with positive orientation and $\overrightarrow{F}$ is a vector field whose components have continuous partial derivatives.

- Divergence's theorem:
$$\iint_{\mathcal{S}} \overrightarrow{F} \cdot d\overrightarrow{\mathcal{S}} = \iiint_{\mathbf{E}} div \overrightarrow{F}\ dV$$
where $\mathbf{E}$ is a simple solid region, $\mathcal{S}$ is the boundary surface of mE, given with positive (outward) orientation and $\overrightarrow{F}$ is a vector field whose component functions have continuous partial derivatives.

Remark 1.1 you may need to use curvilinear coordinates to find the parametric surfaces or compute integrals

- polar coordinates:
$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \Rightarrow dA = rdrd\theta$$
- cylindrical coordinates:
$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases} \Rightarrow dV = rdrd\theta dz$$
- spherical coordinates:
$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases} \Rightarrow dV = \rho^2\sin(\phi)d\rho d\phi d\theta$$