Key concepts & formulas

1 3-D coordinate System

- Distance formula: point $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$

$$d(A, B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

- Sphere with center $C(x_c, y_c, z_c)$ and radius r:

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

2 Vectors

- vector addition: $\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
- scalar multiplication: $k\langle a_1, a_2, a_3 \rangle = \langle ka_1, ka_2, ka_3 \rangle$
- length of a vector: $\|\langle x, y, z \rangle\| = \sqrt{x^2 + y^2 + z^2}$
  - zero vector: $\overrightarrow{0} = \langle 0, 0, 0 \rangle$ and its magnitude is $\|\overrightarrow{0}\| = 0$
  - unit vector: $\widehat{v}$ with magnitude $\|\widehat{v}\| = 1$
- basis vectors: $\widehat{i} = \langle 1, 0, 0 \rangle, \widehat{j} = \langle 0, 1, 0 \rangle, \widehat{k} = \langle 0, 0, 1 \rangle$
  - For any vector $\overrightarrow{a} = \langle x, y, z \rangle$, we have $\overrightarrow{a} = x\widehat{i} + y\widehat{j} + z\widehat{k}$
- properties: vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$; real numbers k and ℓ
  - $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$ $\qquad$ $\overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}) = (\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c}$
  - $k(\overrightarrow{a} + \overrightarrow{b}) = k\overrightarrow{a} + k\overrightarrow{b}$ $\qquad$ $(k + \ell)\overrightarrow{a} = k\overrightarrow{a} + \ell\overrightarrow{a}$
  - $\overrightarrow{a} + \overrightarrow{0} = \overrightarrow{a}$ $\qquad$ $\overrightarrow{a} + (-\overrightarrow{a}) = \overrightarrow{0}$ $\qquad$ $1\overrightarrow{a} = \overrightarrow{a}$

3 Dot product

- definition $\overrightarrow{a} \cdot \overrightarrow{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1b_1 + a_2b_2 + a_3b_3$
- theorem: Suppose the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is θ, then $\overrightarrow{a} \cdot \overrightarrow{b} = \|\overrightarrow{a}\|\|\overrightarrow{b}\|\cos\theta$
- corollary: $\cos\theta = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|\overrightarrow{a}\|\|\overrightarrow{b}\|}$
  - if $\overrightarrow{a} \cdot \overrightarrow{b} = 0$, then $\overrightarrow{a} \perp \overrightarrow{b}$.
- projections:

- vector projection $proj_{\overrightarrow{b}}\overrightarrow{a} = \dfrac{\overrightarrow{a}\cdot\overrightarrow{b}}{\|\overrightarrow{b}\|^2}\overrightarrow{b}$
- scalar projection $\dfrac{\overrightarrow{a}\cdot\overrightarrow{b}}{\|\overrightarrow{b}\|}$

- properties: vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$; real number k
  - $\overrightarrow{a}\cdot\overrightarrow{b} = \overrightarrow{b}\cdot\overrightarrow{a}$ $\qquad$ $\overrightarrow{a}\cdot\overrightarrow{a} = \|\overrightarrow{a}\|^2$
  - $\overrightarrow{a}\cdot(\overrightarrow{b}+\overrightarrow{c}) = \overrightarrow{a}\cdot\overrightarrow{b}+\overrightarrow{a}\cdot\overrightarrow{c}$
  - $\overrightarrow{a}\cdot\overrightarrow{0} = 0$ $\qquad$ $(k\overrightarrow{a})\cdot\overrightarrow{b} = k(\overrightarrow{a}\cdot\overrightarrow{b}) = \overrightarrow{a}\cdot(k\overrightarrow{b})$
- application: work done by a force $\overrightarrow{F}$ moving an object with displacement vector $\overrightarrow{d}$:

$$W = \overrightarrow{F}\cdot\overrightarrow{d}$$

4 Cross Product

- definition: Suppose the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is θ, then $\|\overrightarrow{a}\times\overrightarrow{b}\| = |\overrightarrow{a}||\overrightarrow{b}|\sin\theta$ and $\overrightarrow{a}\times\overrightarrow{b}$ is perpendicular to $\overrightarrow{a}$ and $\overrightarrow{b}$ and the direction is determined by right-hand rule.

$$\overrightarrow{a}\times\overrightarrow{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

$$\overrightarrow{a}\times\overrightarrow{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$$

- properties: vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$; real number k
  - $\overrightarrow{a}\times\overrightarrow{b} = -\overrightarrow{b}\times\overrightarrow{a}$
  - $(k\overrightarrow{a})\times\overrightarrow{b} = k(\overrightarrow{a}\times\overrightarrow{b}) = \overrightarrow{a}\times(k\overrightarrow{b})$
  - $\overrightarrow{a}\times(\overrightarrow{b}+\overrightarrow{c}) = \overrightarrow{a}\times\overrightarrow{b}+\overrightarrow{a}\times\overrightarrow{c}$
  - $(\overrightarrow{a}+\overrightarrow{b})\times\overrightarrow{c} = \overrightarrow{a}\times\overrightarrow{c}+\overrightarrow{b}\times\overrightarrow{c}$
  - $\overrightarrow{a}\cdot(\overrightarrow{b}\times\overrightarrow{c}) = (\overrightarrow{a}\times\overrightarrow{b})\cdot\overrightarrow{c}$
  - $\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c}) = (\overrightarrow{a}\cdot\overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a}\cdot\overrightarrow{b})\overrightarrow{c}$
  - $\overrightarrow{a}\times\overrightarrow{a} = \overrightarrow{0}$
- applications:
  - volume of a parallelpiped determined by vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$:

$$V = \overrightarrow{a}\cdot(\overrightarrow{b}\times\overrightarrow{c})$$

  - torque: force $\overrightarrow{F}$ on a wrench $\overrightarrow{r}$, the torque $\overrightarrow{\tau} = \overrightarrow{r}\times\overrightarrow{F}$, measures the tendency of rotation.

5 Equations of lines and planes

- Equations of a line passing through $P(x_0, y_0, z_0)$ with direction vector $\overrightarrow{v} = \langle a, b, c\rangle$
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$
- Equations of a plane contains $P(x_0, y_0, z_0)$ with normal vector $\overrightarrow{n} = \langle a, b, c\rangle$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
or
$$ax + by + cz = d$$
where $d = ax_0 + by_0 + cz_0$
- angle φ between 2 lines (ℓ_1 and ℓ_2 with direction vectors $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$):
  - step-1: find the angle θ between $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$;
  - step-2: $\varphi = \begin{cases} \theta & \text{if } \theta \leq \frac{\pi}{2} \\ \pi - \theta & \text{if } \theta > \frac{\pi}{2} \end{cases}.$
- angle φ between 2 planes (S_1 and S_2 with normal vectors $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$):
  - step-1: find the angle θ between $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$;
  - step-2: $\varphi = \begin{cases} \theta & \text{if } \theta \leq \frac{\pi}{2} \\ \pi - \theta & \text{if } \theta > \frac{\pi}{2} \end{cases}.$
- angle φ between line ℓ and plane S (ℓ with direction vector $\overrightarrow{v}$; S with normal vectors $\overrightarrow{n}$):
  - step-1: find the angle θ between $\overrightarrow{v}$ and $\overrightarrow{n}$;
  - step-2: $\varphi = \frac{\pi}{2} - \theta$.
- distance formulas:
  - point P_0 to line ℓ (ℓ contains P with direction $\overrightarrow{v}$): $d(P_0, \ell) = \left\| \overrightarrow{PP_0} - proj_{\overrightarrow{v}} \overrightarrow{PP_0} \right\|$
  - point P_0 to plane S (S contains P with normal $\overrightarrow{n}$): $d(P_0, S) = \left\| proj_{\overrightarrow{n}} \overrightarrow{PP_0} \right\|$
  - line ℓ_1 (determined by P_1 and $\overrightarrow{v}_1$) to line ℓ_2 (determined by P_2 and $\overrightarrow{v}_2$):
    * $\ell_1 /\!/ \ell_2$: $d(\ell_1, \ell_2) = \dfrac{\|\overrightarrow{P_1P_2} \times \overrightarrow{v}_1\|}{\|\overrightarrow{v}_1\|} = \dfrac{\|\overrightarrow{P_1P_2} \times \overrightarrow{v}_2\|}{\|\overrightarrow{v}_2\|}$
    * $\ell_1 \not/\!/ \ell_2$: $d(\ell_1, \ell_2) = \left\| proj_{\overrightarrow{n}} \overrightarrow{P_1P_2} \right\|$, where $\overrightarrow{n} = \overrightarrow{v}_1 \times \overrightarrow{v}_2$
  - plane S_1 ($ax + by + cz = d_1$) to plane S_2 ($ax + by + cz = d_2$):
$$d(S_1, S_2) = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

6 3D curves

- vector function: $\overrightarrow{r}(t) = \langle f(t), g(t), h(t) \rangle$
- limit: $\lim\limits_{t \to a} \overrightarrow{r}(t) = \langle \lim\limits_{t \to a} f(t), \lim\limits_{t \to a} g(t), \lim\limits_{t \to a} h(t) \rangle$
- continuity of $\overrightarrow{r}(t)$: $\overrightarrow{r}(t)$ is continuous at t_0 if $\lim\limits_{t \to t_0} \overrightarrow{r}(t) = \overrightarrow{r}(t_0)$
- derivative:
  - definition: $\overrightarrow{r}'(t) = \lim\limits_{h \to 0} \dfrac{\overrightarrow{r}(t+h) - \overrightarrow{r}(t)}{h}$
  - theorem: $\overrightarrow{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$
  - rules:
    * $(\overrightarrow{u}(t) + \overrightarrow{v}(t))' = \overrightarrow{u}'(t) + \overrightarrow{v}'(t)$ $\qquad (k\overrightarrow{u}(t))' = k\overrightarrow{u}'(t)$
    * $(\overrightarrow{u}(t) \cdot \overrightarrow{v}(t))' = \overrightarrow{u}'(t) \cdot \overrightarrow{v}(t) + \overrightarrow{u}(t) \cdot \overrightarrow{v}'(t)$
    * $(\overrightarrow{u}(t) \times \overrightarrow{v}(t))' = \overrightarrow{u}'(t) \times \overrightarrow{v}(t) + \overrightarrow{u}(t) \times \overrightarrow{v}'(t)$
    * $(f(t)\overrightarrow{u}(t))' = f'(t)\overrightarrow{u}(t) + f(t)\overrightarrow{u}'(t)$
    * $(\overrightarrow{u}(f(t)))' = \overrightarrow{u}'(f(t))f'(t)$
- integral: $\displaystyle\int_a^b \overrightarrow{r}(t)\ dt = \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt, \int_a^b h(t)\ dt \right\rangle$
- arc-length of curve $\overrightarrow{r}(t)$, $t \in [a, b]$
  - formula: $L = \displaystyle\int_a^b \|\overrightarrow{r}'(t)\|\ dt$
  - arc-length function: $s(t) = \displaystyle\int_a^t \|\overrightarrow{r}'(u)\|\ du$
  - parametrize the curve with respect to arc-length:
    * step-1: find arc-length function $s(t)$;
    * step-2: rewrite $s = s(t)$ as $t = t(s)$;
    * step-3: replace t in $\overrightarrow{r}(t)$ by $t(s)$.
- curvature
  - definition: $\kappa = \left\| \dfrac{d\widehat{T}}{ds} \right\|$, measures how curved a curve is.
  - formula: $\kappa = \dfrac{\|\widehat{T}'(t)\|}{\|\overrightarrow{r}'(t)\|} = \dfrac{\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\|}{\|\overrightarrow{r}'(t)\|^3}$
- special vectors and planes
  - unit tangent vector: $\widehat{T} = \dfrac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}$

- unit normal vector: $\widehat{N} = \dfrac{\widehat{T}'(t)}{\|\widehat{T}'(t)\|}$
- unit binormal vector: $\widehat{B} = \widehat{T} \times \widehat{N}$
- normal plane S_n at point $\overrightarrow{r}(t_0)$: its normal vector is $\widehat{T}(t_0)$.
- osculating plane S_o at point $\overrightarrow{r}(t_0)$: its normal vector is $\widehat{B}(t_0)$.
- osculating circle C_o at point $\overrightarrow{r}(t_0)$: it is on the osculating plane, at the concave side of the curve $\overrightarrow{r}(t)$, contains point $\overrightarrow{r}(t_0)$ and has the same tangent line and curvature as curve $\overrightarrow{r}(t)$ at point $\overrightarrow{r}(t_0)$.

• applications

- motion in space: $\overrightarrow{F} \to \overrightarrow{a} \to \overrightarrow{v} \to \overrightarrow{r}$
  * location vector at time t: $\overrightarrow{r}(t)$
  * velocity vector at time t: $\overrightarrow{v}(t) = \overrightarrow{r}'(t)$
  * acceleration vector at time t: $\overrightarrow{a}(t) = \overrightarrow{v}'(t) = \overrightarrow{r}''(t)$
  * newton's second law: force $\overrightarrow{F}(t) = m\overrightarrow{a}(t)$
- tangential and normal component components of acceleration (a_T and a_N)
  * Definition: $\overrightarrow{a} = a_T\widehat{T} + a_N\widehat{N}$
  * $a_T = \|v(t)\|' = \dfrac{\overrightarrow{r}'(t) \cdot \overrightarrow{r}''(t)}{\|\overrightarrow{r}'(t)\|}$
  * $a_N = k\|v(t)\|^2 = \dfrac{\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\|}{\|\overrightarrow{r}'(t)\|}$
- torsion
  * Definition: $\tau = -\widehat{N} \cdot \dfrac{d\widehat{B}}{ds}$
  * Curve C $\overrightarrow{r}(t)$: $\kappa \equiv 0 \iff C$ is a line.
  * Curve C $\overrightarrow{r}(t)$: $\kappa > 0$ and $\tau \equiv 0 \iff C$ is on a unique plane.