Key concepts & formulas

1 double integrals

- definition: $\iint_D f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{k=1}^{m} \sum_{j=1}^{n} f(x_j, y_k)\Delta A$
- fubini's theorem $D = \{(x,y)|a \le x \le b, c \le y \le d\}$,

$$\iint_D f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$$

- general region D:
  - $D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

$$\iint_D f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$

  - $D = \{(x,y)|c \le y \le d, g_1(y) \le x \le g_2(y)\}$,

$$\iint_D f(x,y)\ dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\ dxdy$$

- applications:
  - area of a region $D = \iint_D 1\ dA$
  - average value of a function $f(x,y)$ over a region D: $f_{avg} = \dfrac{\iint_D f(x,y)\ dA}{\iint_D 1\ dA}$
  - volume of a solid between 2 surfaces $f(x,y)$ and $g(x,y)$ ($f \ge g$) over a region D,

$$V = \iint_D f(x,y) - g(x,y)\ dA$$

  - center of mass: a plane lamina D, with density $f(x,y)$, then the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\iint_D xf(x,y)\ dA}{\iint_D f(x,y)\ dA}, \frac{\iint_D yf(x,y)\ dA}{\iint_D f(x,y)\ dA}\right)$$

2 triple integrals

- definition: $\iiint_D f(x,y,z)\ dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i,y_j,z_k)\Delta V$

- fubini's theorem $D = \{(x,y,z)|a \le x \le b, c \le y \le d, s \le z \le t\}$,

$$\iiint_D f(x,y)\ dA = \int_s^t \int_a^b \int_c^d f(x,y)\ dydxdz = \cdots \text{(totally 6 different ways)}$$

- general region E:
  - $E = \{(x,y,z)|(x,y) \in D, g_1(x,y) \le z \le g_2(x,y)\}$,

$$\iiint_E f(x,y,x)\ dA = \iint_D \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\ dz\right] dA$$

  - $E = \{(x,y,z)|(x,z) \in D, g_1(x,z) \le y \le g_2(x,z)\}$,

$$\iiint_E f(x,y,x)\ dA = \iint_D \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\ dy\right] dA$$

  - $E = \{(x,y,z)|(y,z) \in D, g_1(y,z) \le x \le g_2(y,z)\}$,

$$\iiint_E f(x,y,x)\ dA = \iint_D \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\ dx\right] dA$$

- applications:
  - volume of a region $E = \iiint_E 1\ dV$
  - average value of a function $f(x,y,z)$ over a region E: $f_{avg} = \dfrac{\iiint_E f(x,y,z)\ dV}{\iiint_E 1\ dV}$
  - center of mass: a solid E, with density $f(x,y,z)$, then the center of mass is

$$(\overline{x},\overline{y},\overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right) = \left(\frac{\iiint_E xf\ dV}{\iiint_E f\ dV}, \frac{\iiint_E yf\ dV}{\iiint_E f\ dV}, \frac{\iiint_E zf\ dV}{\iiint_E f\ dV}\right)$$

3 curvilinear coordinates

- polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \Rightarrow dA = r dr d\theta$$

- cylindrical coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases} \Rightarrow dV = r dr d\theta dz$$

- spherical coordinates:

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases} \Rightarrow dV = \rho^2\sin(\phi) d\rho d\phi d\theta$$

4 line integrals

- line integral of a function: curve C: $\overrightarrow{r}(t)$, $t \in [a, b]$

$$\int_C f\ ds = \int_a^b f\|\overrightarrow{r}'\|\ dt$$

- line integral of a vector field: curve C: $\overrightarrow{r}(t)$, $t \in [a, b]$

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_a^b \overrightarrow{F} \cdot \overrightarrow{r}'\ dt$$

- Fundamental theorem of line integral curve C: $\overrightarrow{r}(t)$, $t \in [a, b]$

$$\int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a))$$

  How to use this theorem to find $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$
  2-D case: $\overrightarrow{F} = \langle P(x, y), Q(x, y)\rangle$

  - step-1: check $P_y = Q_x$, if true, go to step-2;
  - step-2: find f such that $\nabla f = F$;
  - step-3: $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a))$.

  3-D case: $\overrightarrow{F} = \langle P(x, y, z),\ Q(x, y, z),\ R(x, y, z)\rangle$

  - definition: $curl(\overrightarrow{F}) = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle$

- step-1: check $curl(\overrightarrow{F}) = \overrightarrow{0}$, if true, go to step-2;
- step-2: find f such that $\nabla f = F$;
- step-3: $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a))$.

- Green's theorem: C is a positively oriented, piece-wise smooth, simple close curve; D is the region enclosed by C; $\overrightarrow{F} = \langle P(x,y), Q(x,y)\rangle$ is a vector field, whose components have continuous partial derivatives in the open region which contains D, then

$$\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = \oint_C Pdx + Qdy = \iint_D (Q_x - P_y)\, dA$$