Review-2 and Practice problems

1. Introduction.
$$y' = f(x,y)$$
Constant solutions Solve $f(x,y)=0$
Increasing region Solve $f(x,y)>0$
Decreasing region Solve $f(x,y)<0$
Euler's Method
$$\begin{cases} y' = f(x,y) \\ y(a) = y_0 \end{cases}$$
To estimate solution at b, set $a = x_0 < x_1 < \cdots < x_n = b$, where $x_{i+1} - x_i = \Delta x = (b-a)/n$,
$$\begin{cases} x_{i+1} = x_i + \Delta x \\ y_{i+1} = y_i + f(x_i, y_i)\Delta x \end{cases}$$
then y_n is the approximation of $y(b)$.

2. Separable DE.
$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$
$$g(y)dy = f(x)dx$$
$$\int g(y)dy = \int f(x)dx$$

3. Applications of separable DE.
Tank's problem: Rate of change = Rate in − Rate out.
Exponential Grow/Decay Model:
P: population; t: time; k: relative growth rate.
$$\begin{cases} P' = kP \\ P(0) = P_0 \end{cases} \to P(t) = P_0 e^{kt}$$
Compound interest:
P: initial investment; t: time; r: annual interest rate; n: the interest is compounded n times per year.
Regular compound interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
Continuous compounded interest: $A = Pe^{rt}$
Newton's Law of heating and cooling:
T: the temperature of the object; t: time; k: positive constant; T_m: the temperature of surrounding environment.
$$\begin{cases} T' = k(T_m - T) \\ T(0) = T_0 \end{cases}$$

Logistic Model:
P: population; t: time; r: relative growth rate; K: maximum supportable population.

$$\begin{cases} P' = rP\left(1 - \dfrac{P}{K}\right) \\ P(0) = P_0 \end{cases} \to P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

4. Second order linear homogeneous DE

$$ay'' + by' + cy = 0$$

Characteristic Equation: $ar^2 + br + c = 0 \to r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Case-1 $b^2 - 4ac > 0$**,** $r_1 \neq r_2$: $\to y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
Case-2 $b^2 - 4ac = 0$**,** $r_1 = r_2 = r$: $\to y(x) = C_1 e^{rx} + C_2 x e^{rx}$
Case-3 $b^2 - 4ac < 0$**,** $r_{1,2} = \alpha \pm \beta i$: $\to y(x) = e^{\alpha x}\Big(C_1 \cos(\beta x) + C_2 \sin(\beta x)\Big)$

5. Second order linear non-homogeneous DE

$$ay'' + by' + cy = F(x)$$

Step-1: Solve $ay'' + by' + cy = 0 \to y_c(x)$
Step-2: Find 1 particular solution of $ay'' + by' + cy = F(x) \to y_p$

$F(x)$ **is a polynomial, ex.** $3x^2 + x + 1$	$\to y_p = Ax^2 + BX + C$
$F(x)$ **is a Exp. function, ex.** $10e^{2x}$	$\to y_p = Ae^{2x}$
$F(x)$ **is a Trig. function, ex.** $\sin(3x)$	$\to y_p = A\sin(3x) + B\cos(3x)$
$F(x)$ **is a polynomial×Exp., ex.** $x^2 e^{4x}$	$\to y_p = (Ax^2 + Bx + C)e^{4x}$
$F(x)$ **is** $F_1(x) + F_2(x)$**, ex.** $xe^x + \sin(x)$	$\to y_p = y_{p_1} + y_{p_2}$
$F_1(x) = xe^x$	$\to y_{p_1} = (Ax + B)e^x$
$F_2(x) = \sin(x)$	$\to y_{p_2} = A\sin(x) + B\cos(x)$

Remark: If $F(x)$ is part of by y_c, then the y_p need to be multiplied by x. If xy_p is still part of y_c, then use $x^2 y_p$.
Step-3: General solution $y(x) = y_c(x) + y_p(x)$

6. Applications of Second order linear DE.
Eclectic circuit
E: Electromotive force; R: Resistance; L: Inductor; C: Capacitor; $Q(t)$, the charge in the capacitor at time t; $I(t)$: the current at time t, $I(t) = Q'(t)$.

$$LQ'' + RQ' + \frac{1}{C}Q = E$$

Oscillate spring
k: Spring constant; m: the mass of the object; $x(t)$: location of the object (when the

spring is of its natural length, the object is at origin.); c Damping constant; F_e is the external force.
Without damping:

$$mx'' + kx = 0 \rightarrow x(t) = C_1 \cos(wt) + C_2 \sin(wt) = A \sin(wt + \phi)$$

where $w = \sqrt{\frac{k}{m}}$ is the angular frequency, $A = \sqrt{C_1^2 + C_2^2}$ is the amplitude and $\phi = arcsin\left(\frac{C_1}{A}\right)$ is the phase angle.
With damping: $t \rightarrow +\infty, \quad x(t) \rightarrow 0$

$$mx'' + cx' + kx = 0 \rightarrow r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$c^2 - 4mk > 0 \rightarrow$ over-damping;
$c^2 - 4mk = 0 \rightarrow$ critical-damping;
$c^2 - 4mk < 0 \rightarrow$ under-damping;
With damping and external force: $t \rightarrow +\infty, \quad x(t) \rightarrow x_p(t)$

$$mx'' + cx' + kx = F_e \rightarrow x(t) = x_c(t) + x_p(t)$$