Key concepts & formulas

1 functions of two variables

(3-variable functions are similar.)

- notation: $z = f(x,y)$
- limit:
  - definition: we say $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, if for any positive real number $\varepsilon > 0$, there is a positive real number $\delta > 0$, such that if $0 < dist\big((x,y),(a,b)\big) < \delta$, then $|f(x,y) - L| < \varepsilon$
  - remark: if $f(x,y) \to L_1$, as $(x,y) \to (a,b)$ along a path C_1, and $f(x,y) \to L_2$, as $(x,y) \to (a,b)$ along another path C_2. if $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.
- continuity:
  - definition: we say function $f(x,y)$ is continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

2 partial derivatives of $z = f(x,y)$

(3-variable functions are similar.)

- definition: $\dfrac{\partial f}{\partial x}(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$, $\quad \dfrac{\partial f}{\partial y}(x,y) = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h}$
- notations: $\dfrac{\partial f}{\partial x} = f_x = \dfrac{\partial z}{\partial x} = z_x = D_x f$, $\quad \dfrac{\partial f}{\partial y} = f_y = \dfrac{\partial z}{\partial y} = z_y = D_y f$
- how to find partial derivatives of $z = f(x,y)$
  - to find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
  - to find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.
- second derivatives:
  - notations:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial^2 z}{\partial x^2} = z_{xx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = z_{xy}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = z_{yx}$$

  - theorem: if both of f_{xy} and f_{yx} exist and are continuous, then $f_{xy} = f_{yx}$

- linear approximation
  - tangent plane: for surface $z = f(x, y)$, at point $(x_0, y_0, f(x_0, y_0))$, its tangent plane (linear approximation) is $L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$
  - total differential: $dz = f_x dx + f_y dy$
- chain rule
  - case 1: $z = f(x, y)$, $x = x(t)$, $y = y(t) \Rightarrow z_t = f_x x_t + f_y y_t$
  - case 2: $z = f(x, y)$, $x = x(s, t)$, $y = y(s, t) \Rightarrow \begin{cases} z_s = f_x x_s + f_y y_s \\ z_t = f_x x_t + f_y y_t \end{cases}$
- implicit differentiation
  - $y = y(x)$ is implicitly defined by $F(x, y) = k$, then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$
  - $z = z(x, y)$ is implicitly defined by $F(x, y, z) = k$, then $\begin{cases} z_x = -\dfrac{F_x}{F_z} \\ z_y = -\dfrac{F_y}{F_z} \end{cases}$
- gradient vector: $\nabla f = \langle f_x, f_y \rangle$

3 directional derivatives

- definition $D_{\widehat{u}} f = \lim\limits_{h \to 0} \dfrac{f(x + ah, y + bh) - f(x, y)}{h}$, where $\widehat{u} = \langle a, b \rangle$ is a unit vector.
- formula: $D_{\widehat{u}} f = \widehat{u} \cdot \nabla f = a f_x + b f_y$
- theorem: for function $f(x, y)$, if $\nabla f \neq \overrightarrow{0}$, then
  - the maximum rate of increasing is $\|\nabla f\|$, in the direction of $\widehat{u} = \dfrac{\nabla f}{\|\nabla f\|}$
  - the maximum rate of decreasing is $-\|\nabla f\|$, in the direction of $\widehat{u} = -\dfrac{\nabla f}{\|\nabla f\|}$
  - in the direction where $\widehat{u} \perp \nabla f$, the rate of change is 0.
- tangent lines/planes for level curves/surfaces
  - for curve $F(x, y) = k$, for one point (x_0, y_0) on the curve, the tangent line there is

$$F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) = 0$$

  - for surface $F(x, y, z) = k$, for one point (x_0, y_0, z_0) on the surface, the tangent plane there is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

4 maximum and minimum

- local max/min of $f(x,y)$
  - find critical points by solving $\nabla f = \overrightarrow{0}$
  - second derivative test:

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 \Rightarrow \begin{cases} D > 0 \text{ and } f_{xx} > 0 \Rightarrow \text{local minimum} \\ D > 0 \text{ and } f_{xx} < 0 \Rightarrow \text{local maximum} \\ D < 0 \Rightarrow \text{saddle points} \\ D = 0 \Rightarrow \text{no conclustion} \end{cases}$$

- absolute max/min of $f(x,y)$ in a region D
  - find critical points in D by solving $\nabla f = \overrightarrow{0}$ and evaluate the function values there
  - find the max/min of $f(x,y)$ on the boundary of D
  - the largest function value from the previous 2 steps is the absolute maximum; the smallest one is the absolute minimum.

- method of lagrange multiplier:
  - max/min of $f(x,y)$ given $g(x,y) = k$
    * solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases}$ for x, y and λ, and then evaluate the function values there
    * the largest function value from the previous step is the maximum; the smallest one is the minimum.
  - max/min of $f(x,y,z)$ given $g(x,y,z) = k$
    * solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = k \end{cases}$ for x, y, z and λ, and then evaluate the function values there
    * the largest function value from the previous step is the maximum; the smallest one is the minimum.
  - max/min of $f(x,y,z)$ given $g(x,y,z) = k$ and $h(x,y,z) = \ell$
    * solve $\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(x,y,z) = k \\ h(x,y,z) = \ell \end{cases}$ for x, y, z, λ and μ, and then evaluate the function values there
    * the largest function value from the previous step is the maximum; the smallest one is the minimum.

5 double integrals

- definition: $\displaystyle\iint_D f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{k=1}^{m} \sum_{j=1}^{n} f(x_j, y_k)\Delta A$

- fubini's theorem $D = \{(x,y) | a \le x \le b, c \le y \le d\}$,

$$\iint_D f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$$

- general region D:
  - $D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

$$\iint_D f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$

  - $D = \{(x,y) | c \le y \le d, g_1(y) \le x \le g_2(y)\}$,

$$\iint_D f(x,y)\ dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\ dxdy$$

- applications:
  - area of a region $D = \iint_D 1\ dA$
  - average value of a function $f(x,y)$ over a region D: $f_{avg} = \dfrac{\iint_D f(x,y)\ dA}{\iint_D 1\ dA}$
  - volume of a solid between 2 surfaces $f(x,y)$ and $g(x,y)$ $(f \ge g)$ over a region D,

$$V = \iint_D f(x,y) - g(x,y)\ dA$$

  - center of mass: a plane lamina D, with density $f(x,y)$, then the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\iint_D xf(x,y)\ dA}{\iint_D f(x,y)\ dA}, \frac{\iint_D yf(x,y)\ dA}{\iint_D f(x,y)\ dA}\right)$$