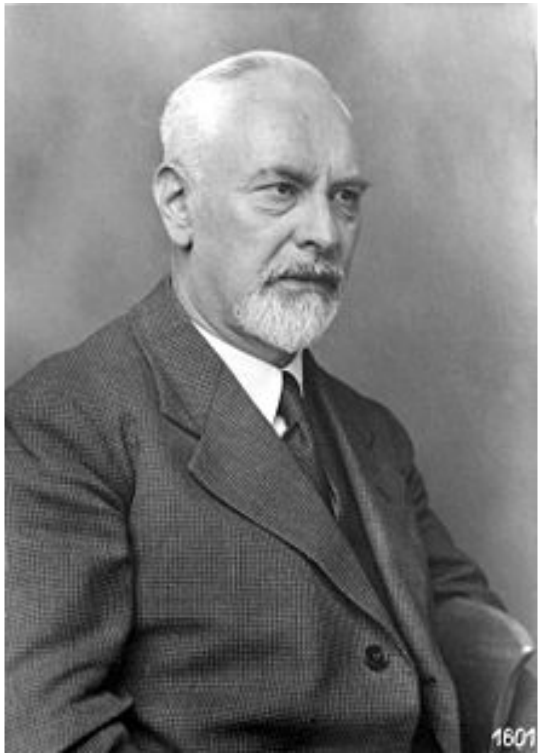
Numerical Modelling in FORTRAN day 9

Paul Tackley, 2014

Today's Goals

1. **Programming**: Low Prandtl number convection (i.e., almost any fluid)



Ludwig Prandtl (1875-1953)

Project
(optional, 1 KP)

1. Chosen topic, agreed upon with me (suggestions given, also ask the advisor of your MSc or PhD project).
   - Due end of Semesterprüfung (Friday 13th Feb 2015)
   - Start planning soon!

Project: general guidelines

- Choose something either
  - related to your research project and/or
  - that you are interested in
- Effort: 1 KP => 30 hours. About 4 days' work.
- I can supply information about needed equations and numerical methods that we have not covered

Some ideas for a project

- Involving solving partial differential equations on a grid (like the convection program)
  - Wave propagation
  - Porous flow (groundwater or partial melt)
  - Variable-viscosity Stokes flow
  - Shallow-water equations
  - 3-D version of convection code
- Involving other techniques
  - Spectral analysis and/or filtering
  - Principle component analysis (multivariate data)
  - Inversion of data for model parameters
  - N-body gravitational interaction (orbits, formation of solar system, ...)
  - Interpolation of irregularly-sampled data onto a regular grid

Values of the Prandt number Pr

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

Viscous diffusivity

Thermal diffusivity

Momentum compared heat

- Liquid metals: 0.004-0.03
- Air: 0.7
- Water: 1.7-12
- Rock: ~10^{24} !!! (effectively infinite)

can be regarded as infinity

Finite-Prandtl number convection

- Existing code assumes infinite Prandtl number
  - also known as Stokes flow
  - appropriate for highly-viscous fluids like rock, honey etc.
- Fluids like water, air, liquid metal have a lower Prandtl number so equations must be modified

Applications for finite Pr

- Outer core (geodynamo)
- Atmosphere
- Ocean
- Anything that's not solid like the mantle

Equations

- Conservation of mass (= ‘continuity’)
- Conservation of momentum (‘Navier-Stokes’ equation: F=ma for a fluid)
- Conservation of energy

Claude Navier
(1785-1836)

Sir George Stokes
(1819-1903)

Finite Pr Equations

Navier-Stokes equation: F=ma for a fluid

Coriolis force

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=-\nabla P+\rho\nu\nabla^2\vec{v}+2\rho\vec{\Omega}\times\vec{v}+g\rho\alpha T\hat{y}$$

advection of velocity

viscosity term

when considering ocean and atmosphere

unit vector

"ma"

Valid for constant viscosity only

continuity and energy equations same as before

$$\frac{\partial T}{\partial t}+\vec{v}\cdot\nabla T=\kappa\nabla^2 T+Q \qquad \nabla\cdot\vec{v}=0$$

modelling mental needs Q

ρ=density, ν=kinematic viscosity, g=gravity,
α=thermal expansivity

Non-dimensionalise the equations

- Reduces the number of parameters
- Makes it easier to identify the dynamical regime
- Facilitates comparison of systems with different scales but similar dynamics (e.g., analogue laboratory experiments compared to core or mantle)

Non-dimensionalise to thermal diffusion scales

- Lengthscale D (depth of domain)
- Temperature scale (T drop over domain)
- Time to D^2/κ thermal diffusion time, kappa=[m^2/s]
- Velocity to κ/D
- Stress to $\rho\nu\kappa/D^2$

viscosity

Nondimensional equations

$$\nabla \cdot \vec{v} = 0 \qquad \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \nabla^2 T$$

Lower Ek, the more important of this term

$$\frac{1}{\Pr}\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = -\nabla P + \nabla^2 \vec{v} + \frac{1}{Ek}\vec{\Omega} \times \vec{v} + Ra.T\hat{y}$$

$$\Pr = \frac{\nu}{\kappa} \qquad Ek = \frac{\nu}{2\Omega D^2} \qquad Ra = \frac{g\alpha \nabla T D^3}{\nu\kappa}$$

Prandtl number — Ekman number — Rayleigh number

As before, use streamfunction

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

Also simplify by assuming 1/Ek=0

Eliminating pressure

- Take curl of 2D momentum equation: curl of grad=0, so pressure disappears
- Replace velocity by vorticity: $\vec{\omega} = \nabla \times \vec{v}$
  旋涡
- in 2D only one component of vorticity is needed (the one perpendicular to the 2D plane), $\nabla^2 \psi = \omega_z$

$$\frac{1}{\Pr}\left(\frac{\partial \omega}{\partial t} + v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y}\right) = \nabla^2 \omega - Ra \frac{\partial T}{\partial x}$$

=> the streamfunction-vorticity formulation

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial\omega}{\partial t}+v_x\frac{\partial\omega}{\partial x}+v_y\frac{\partial\omega}{\partial y}\right)=\nabla^2\omega-Ra\frac{\partial T}{\partial x}$$

$$\nabla^2\psi=-\omega \qquad \left(v_x,v_y\right)=\left(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x}\right)$$

$$\frac{\partial T}{\partial t}+\vec{v}\cdot\nabla T=\nabla^2 T+Q$$

Note: Effect of high Pr

$$\frac{1}{\Pr}\left(\frac{\partial \omega}{\partial t}+v_x\frac{\partial \omega}{\partial x}+v_y\frac{\partial \omega}{\partial y}\right)=\nabla^2\omega - Ra\frac{\partial T}{\partial x}$$

If Pr->infinity, left-hand-side=>0 so equation becomes Poisson like before:

$$\nabla^2\omega = Ra\frac{\partial T}{\partial x}$$

Taking a timestep

(i) Calculate ψ from ω using: $\nabla^2\psi = \omega$

(ii) Calculate v from ψ $\left(v_x, v_y\right) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$

(iii) Time-step ω and T using explicit finite differences:

$$\frac{\partial T}{\partial t} = -v_x \frac{\partial T}{\partial x} - v_y \frac{\partial T}{\partial y} + \nabla^2 T$$

$$\frac{\partial \omega}{\partial t} = -v_x \frac{\partial \omega}{\partial x} - v_y \frac{\partial \omega}{\partial y} + \mathrm{Pr}\,\nabla^2\omega - Ra\,\mathrm{Pr}\frac{\partial T}{\partial x}$$

浮力因子

T time step is the same as before

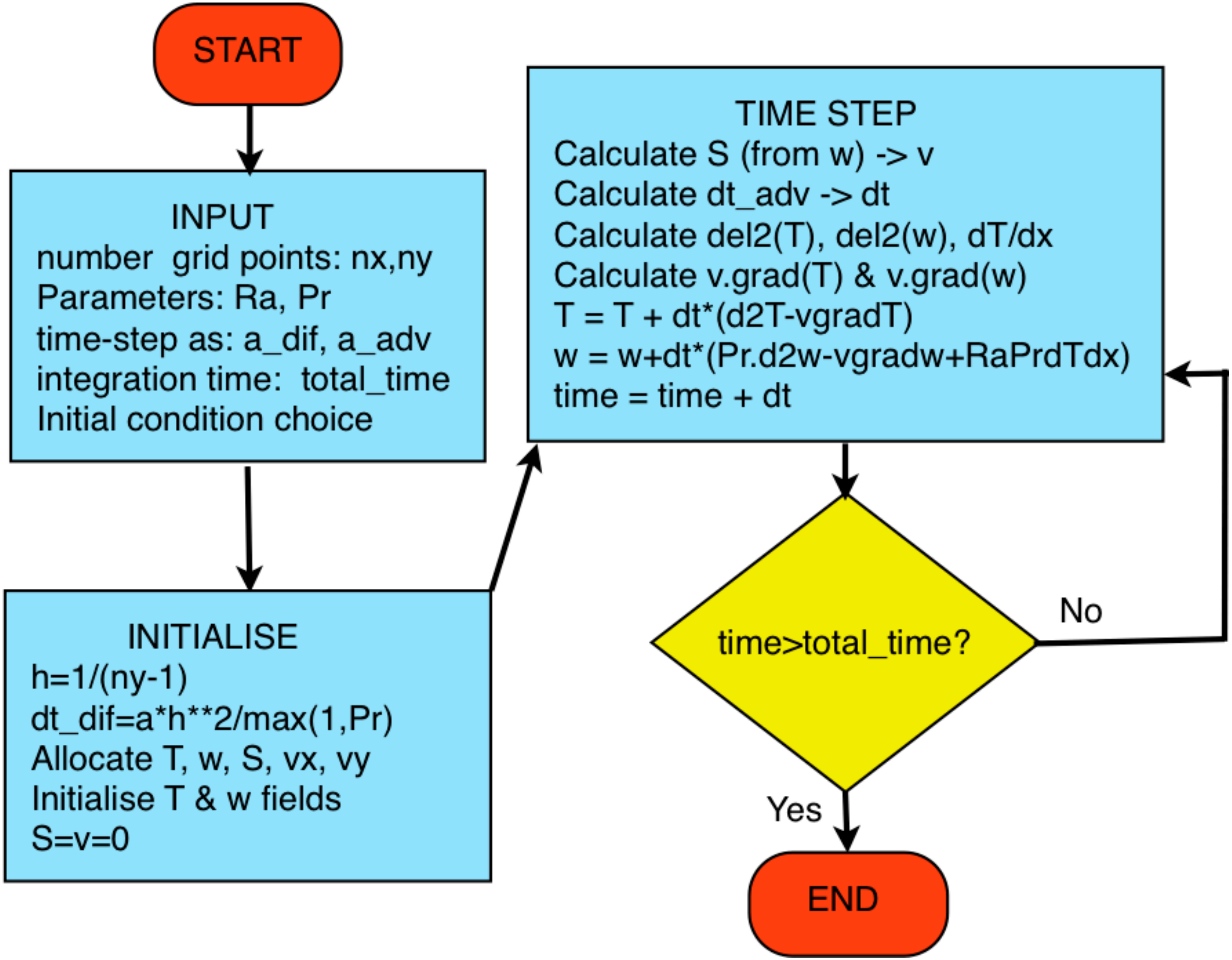
$$\frac{T_{new} - T_{old}}{\Delta t} = -v_x \frac{\partial T_{old}}{\partial x} - v_y \frac{\partial T_{old}}{\partial y} + \nabla^2 T_{old}$$

$$T_{new} = T_{old} + \Delta t \left(\nabla^2 T_{old} - v_x \frac{\partial T_{old}}{\partial x} - v_y \frac{\partial T_{old}}{\partial y} \right)$$

w must now be time stepped in a similar way

$$\frac{\omega_{new} - \omega_{old}}{\Delta t} = -v_x \frac{\partial \omega_{old}}{\partial x} - v_y \frac{\partial \omega_{old}}{\partial y} + \Pr \nabla^2 \omega_{old} - Ra \Pr \frac{\partial T_{old}}{\partial x}$$

$$\omega_{new} = \omega_{old} + \Delta t \left(\Pr \nabla^2 \omega_{old} - v_x \frac{\partial \omega_{old}}{\partial x} - v_y \frac{\partial \omega_{old}}{\partial y} - Ra \Pr \frac{\partial T_{old}}{\partial x} \right)$$


START
INPUT
number grid points: nx,ny
Parameters: Ra, Pr
time-step as: a_dif, a_adv
integration time: total_time
Initial condition choice
INITIALISE
h=1/(ny-1)
dt_dif=a*h**2/max(1,Pr)
Allocate T, w, S, vx, vy
Initialise T & w fields
S=v=0
TIME STEP
Calculate S (from w) -> v
Calculate dt_adv -> dt
Calculate del2(T), del2(w), dT/dx
Calculate v.grad(T) & v.grad(w)
T = T + dt*(d2T-vgradT)
w = w+dt*(Pr.d2w-vgradw+RaPrdTdx)
time = time + dt
time>total_time?
No
Yes
END

Stability condition

Diffusion: $$dt_{diff} = a_{diff} \frac{h^2}{\max(\Pr,1)}$$

Advection: $$dt_{adv} = a_{adv} \min\left(\frac{h}{\max val(abs(vx))}, \frac{h}{\max val(abs(vy))} \right)$$

Combined: $$dt = \min(dt_{diff}, dt_{adv})$$

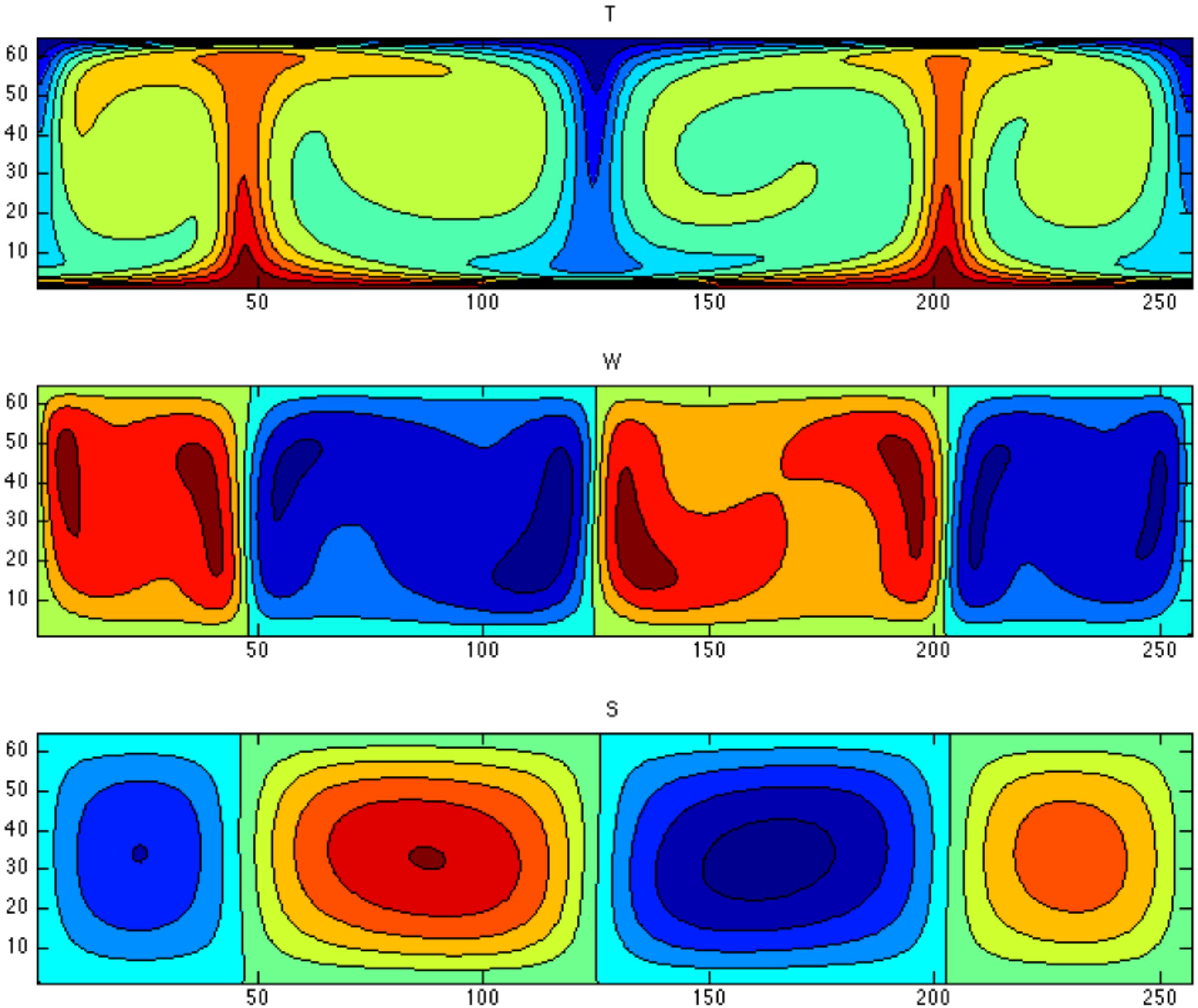
Modification of previous convection program

- Replace Poisson calculation of w with time-step, done at the same time as T time-step
- Get a compiling code!
- Make sure it is stable and convergent for values of Pr between 0.01 and 1e2
- Hand in your code, and your solutions to the test cases in the following slides
- Due date: 4 December (2 weeks)

Test cases

- All have nx=257, ny=65, Ra=1e5, total_time=0.1, and random initial T and w fields, unless otherwise stated

Pr=10



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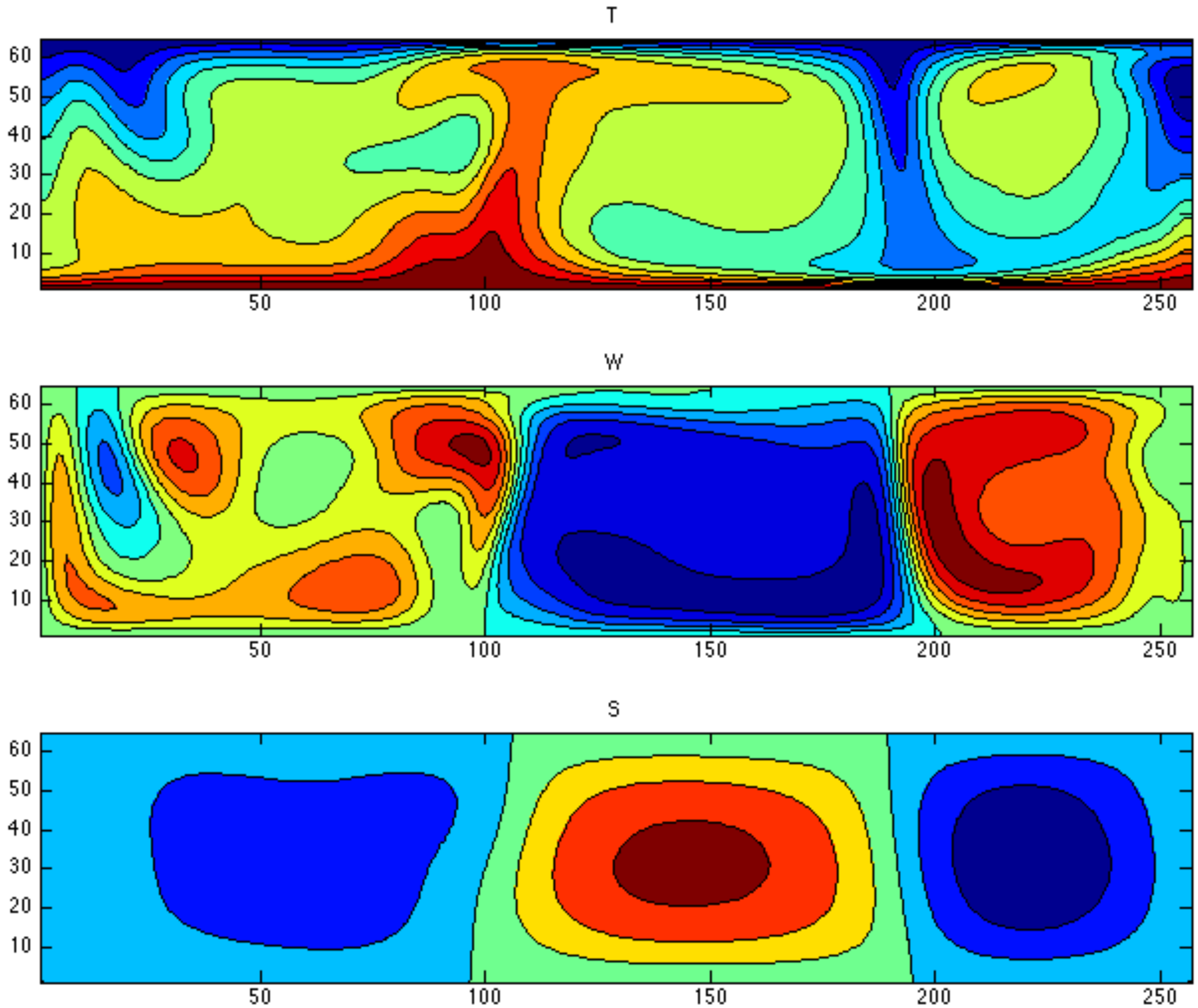
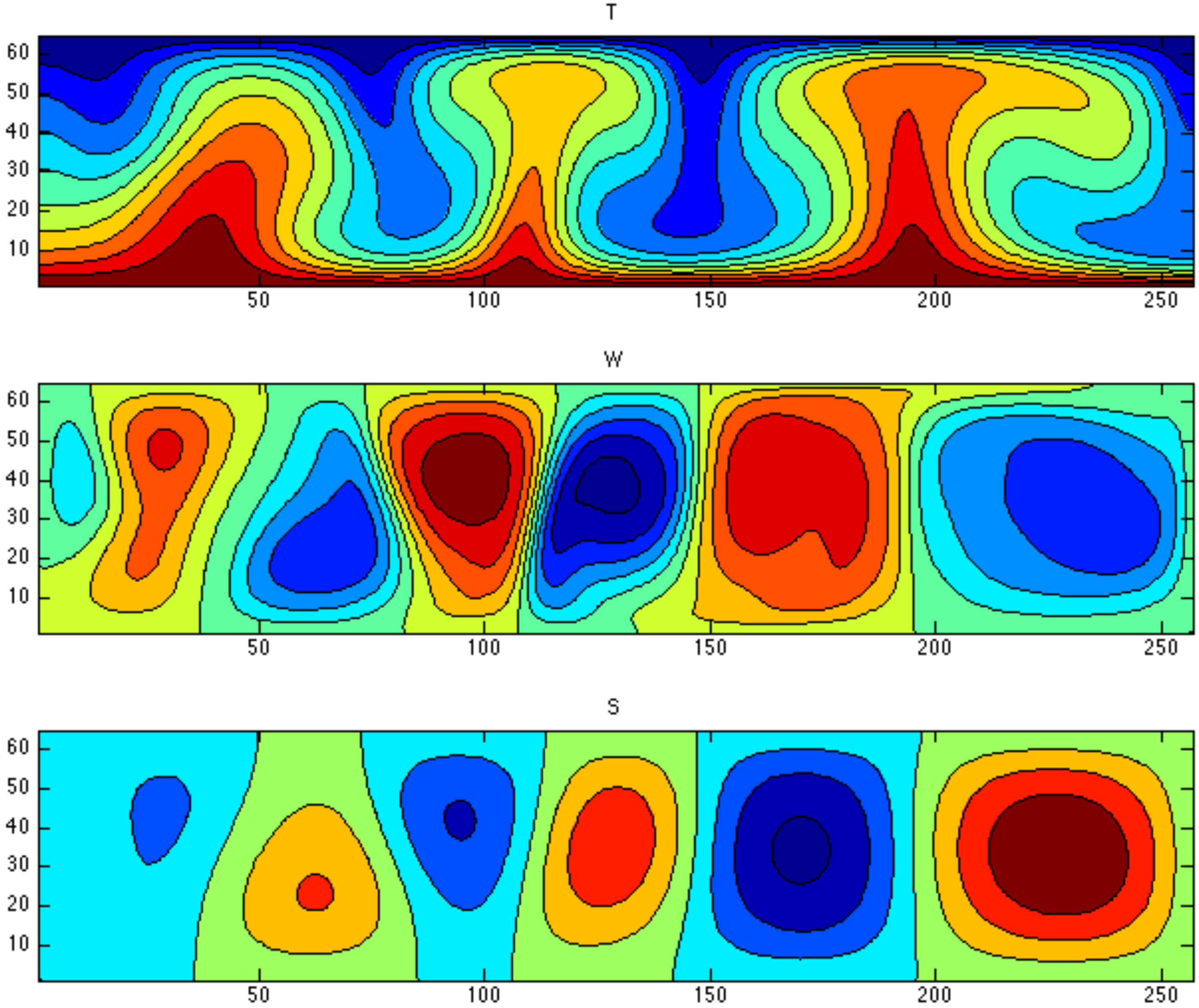
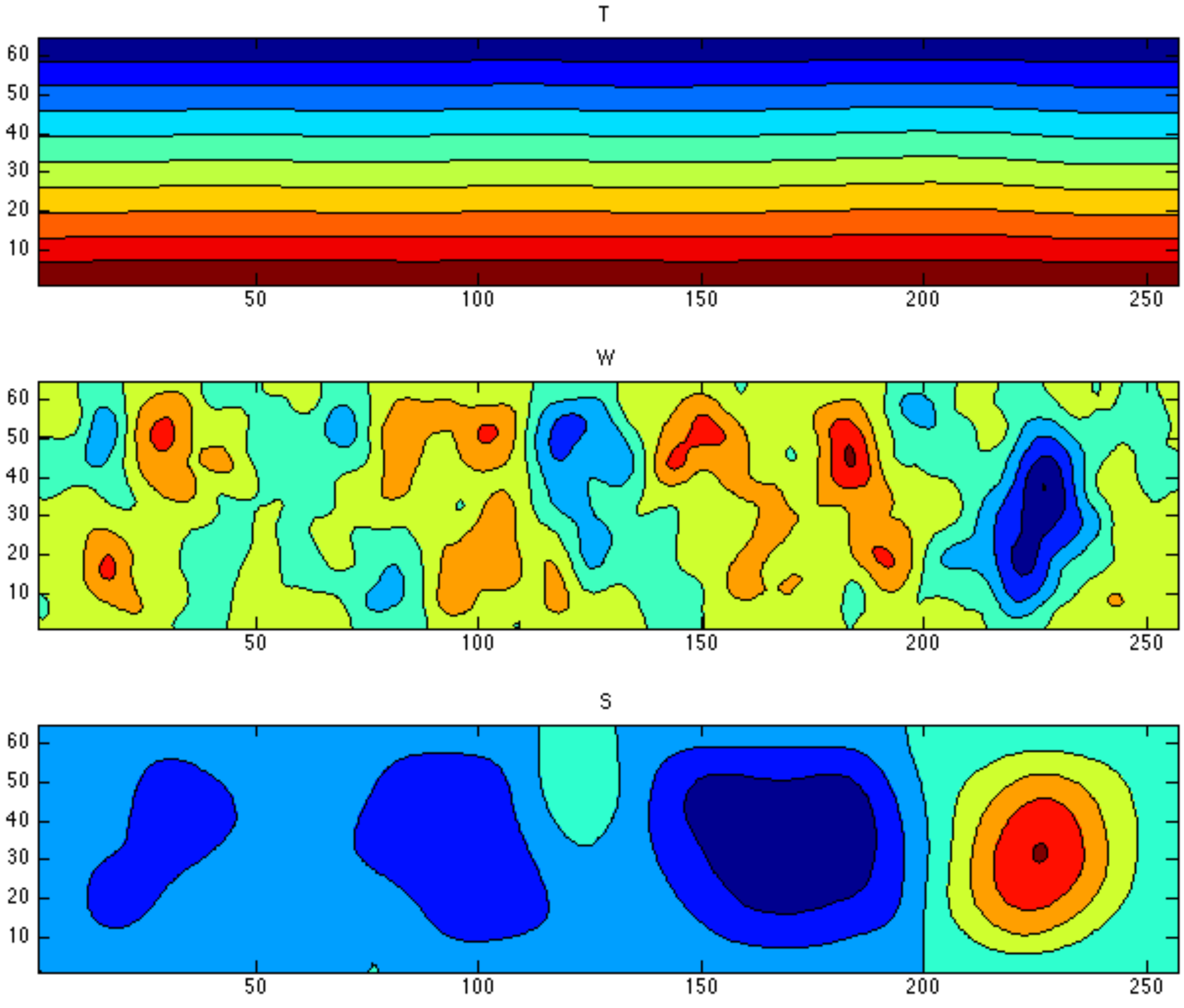
Pr=1



Pr=0.1



Pr=0.01


T
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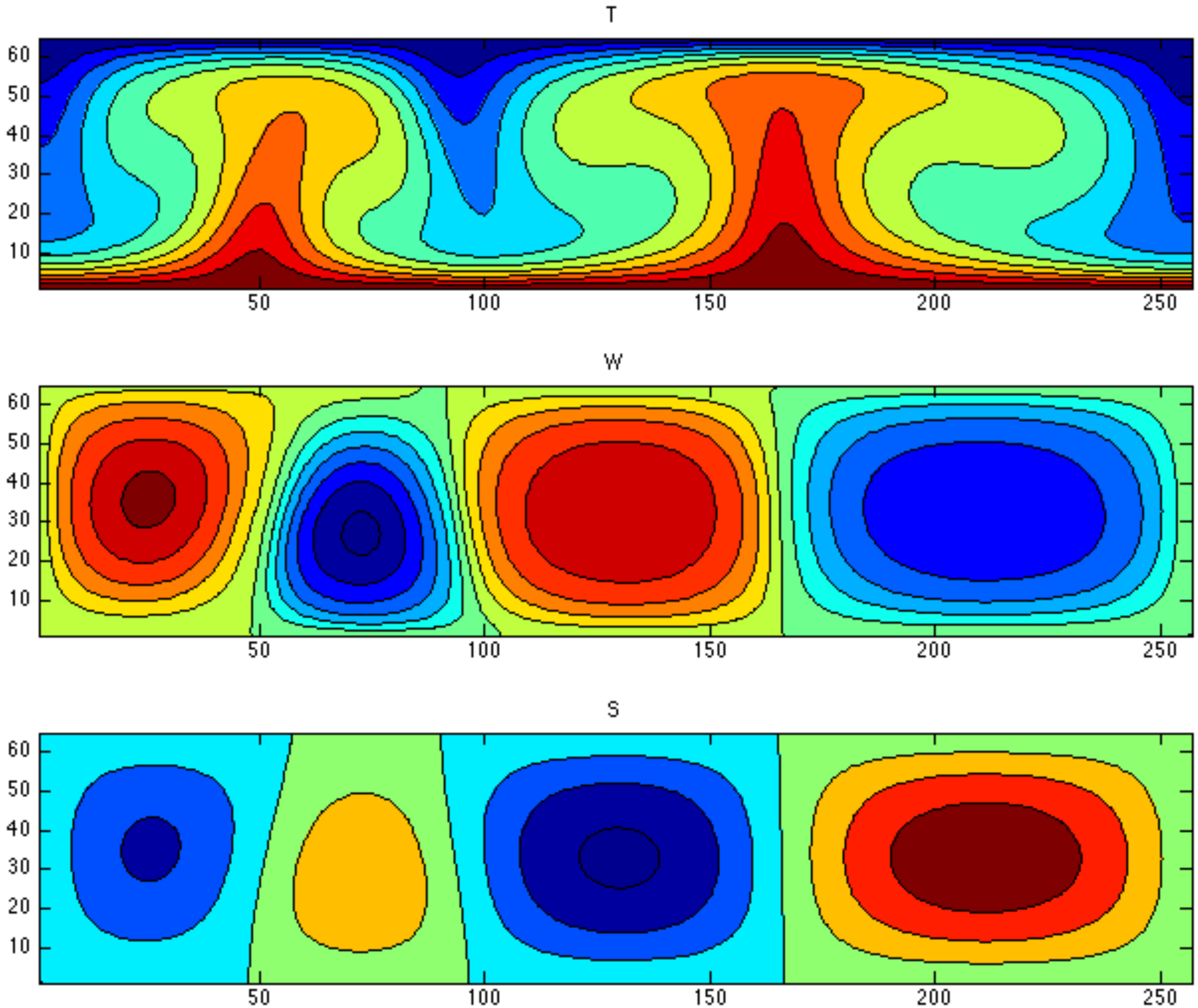
Pr=0.01, time=1.0



Pr=0.1, Ra=1e7