The Courant-Friedrichs-Levy (CFL) Stability Criterion

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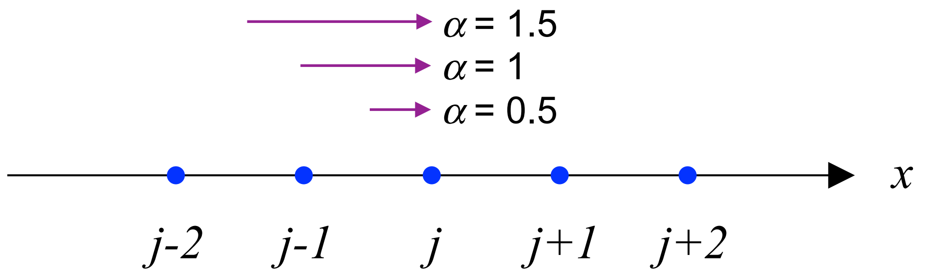
Interpretation of Courant number

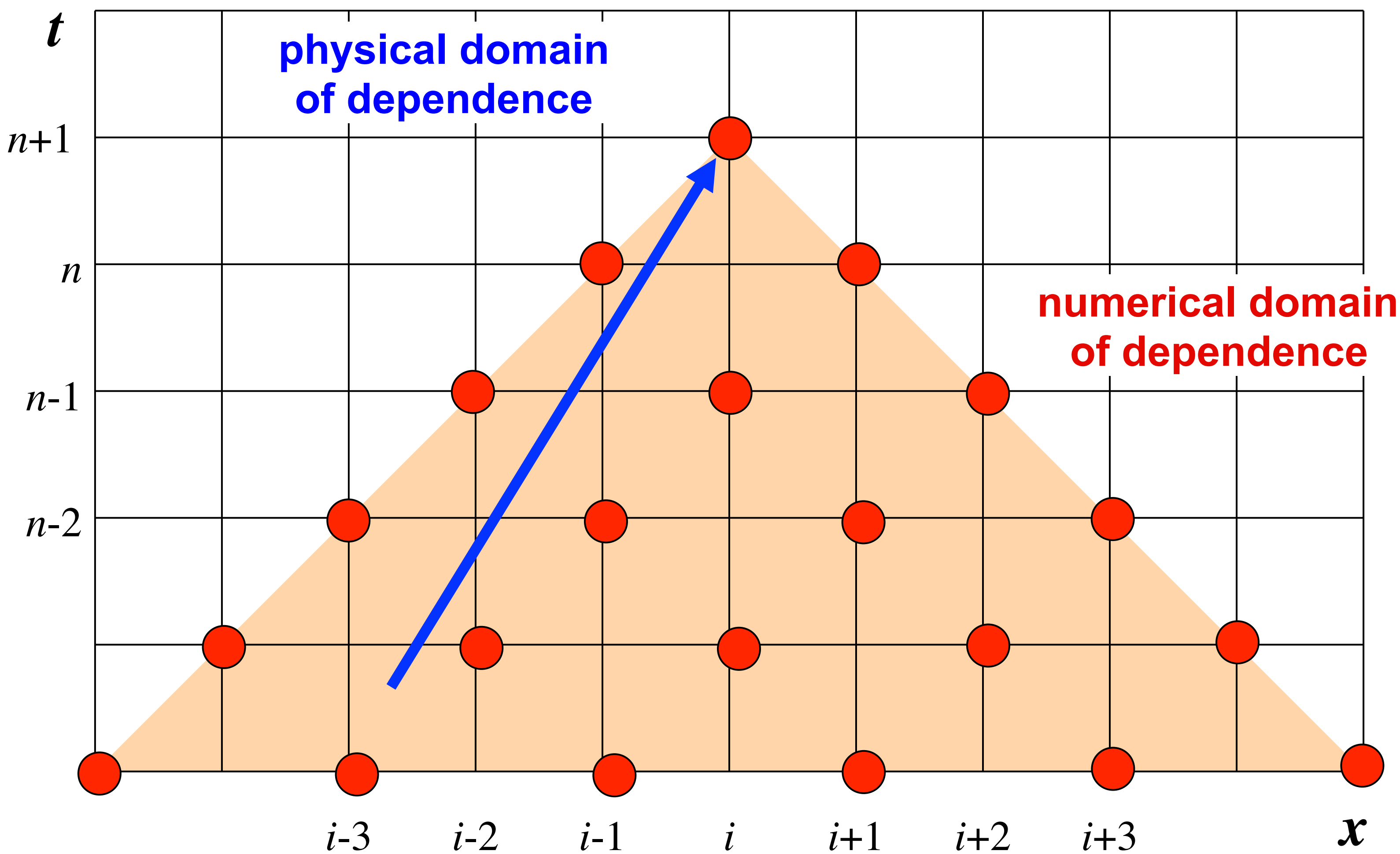
Courant number: $\alpha = u\Delta t / \Delta x$:



The Courant number is the dimensionless transport per time step (in units of Δx)

CFL criterion for Leapfrog scheme

Equation: $\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = 0$ Scheme: $\phi_i^{n+1} = \phi_i^{n-1} - \alpha\left(\phi_{i+1}^n - \phi_{i-1}^n\right)$ with $\alpha = \frac{u \Delta t}{\Delta x}$



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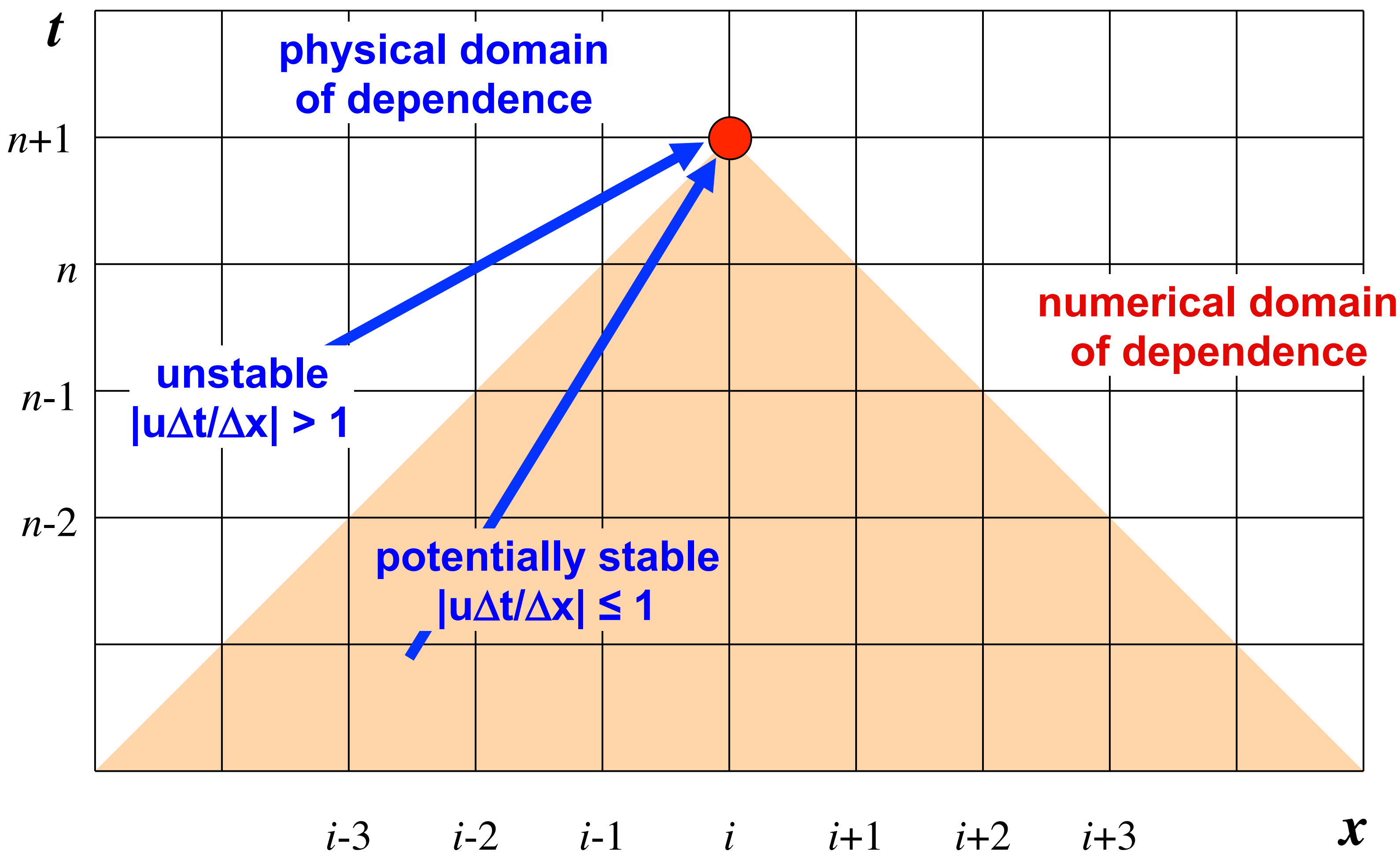

CFL criterion for Upstream scheme

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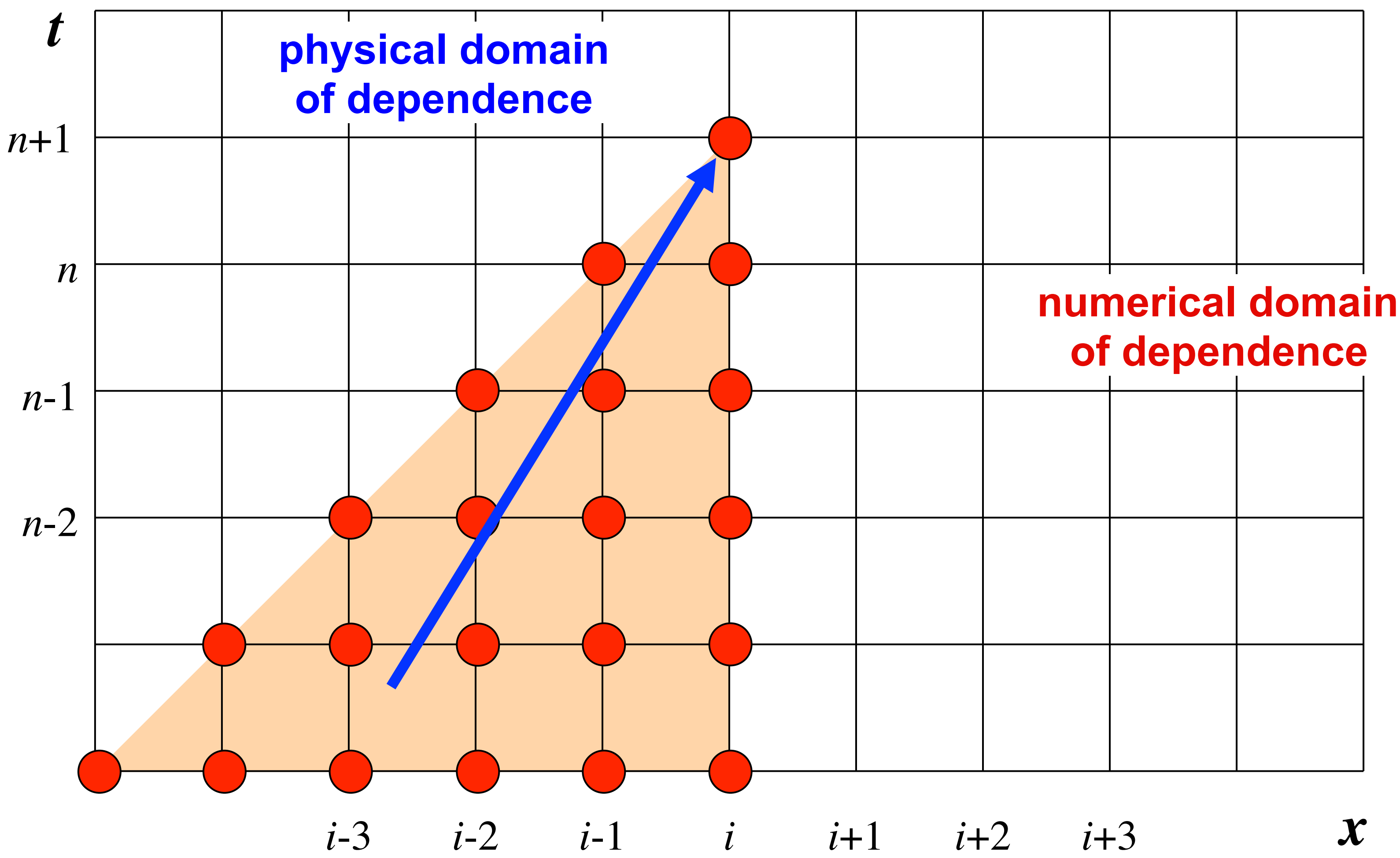

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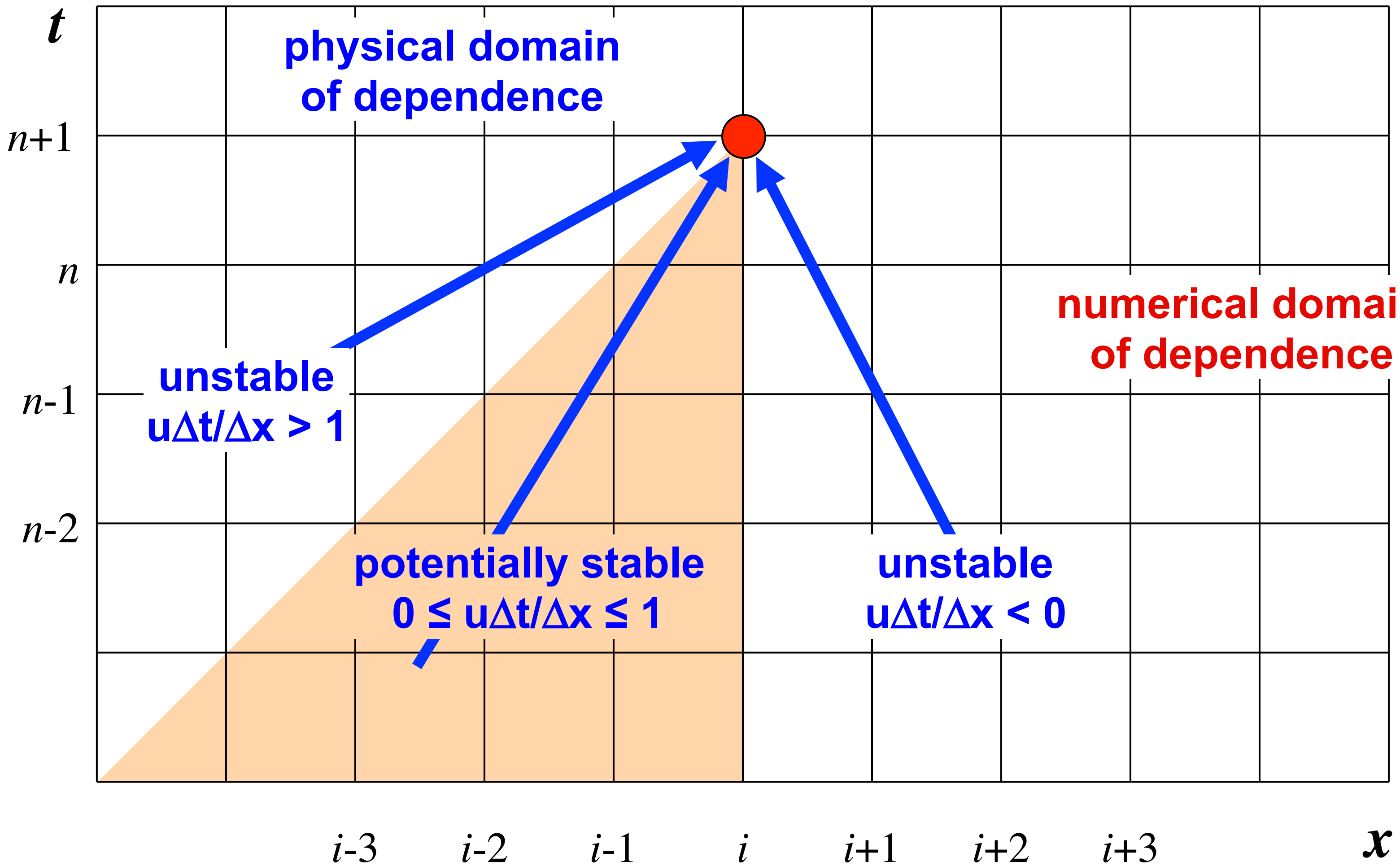

Summary

Courant-Friedrichs-Levy (CFL) principle:

The physical domain of dependence must be contained in the numerical domain of dependence!

Is a <u>necessary condition</u> for stability, but <u>not a sufficient condition!</u>

CFL criteria:

$$\left| \frac{u \ \Delta t}{\Delta x} \right| \leq 1$$

valid for a large category of schemes

Generalized form:

Velocity represents the <u>largest velocity in the system</u>, at which information can propagate. May include advective transport and wave propagation:

$$\frac{c \ \Delta t}{\Delta x} \leq 1 \qquad \text{with} \qquad c = \max\left(|u_{adv}| + |c_{group}| \right)$$

Example 1: shallow water system: gravity waves

$$c_{group} = \sqrt{g \ H}$$

Example 2: non-hydrostatic, compressible atmosphere: sound waves

$$c_{group} = 331 \text{ ms}^{-1} \sqrt{T/273.15 \text{ K}}$$