Numerical Modeling of Weather and Climate

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Lecture Notes Chapter 2:
Repetition of finite difference methods

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Further information:

Schär, C, 2011: Skript der Vorlesung "Numerische Methoden in der Umweltphysik". Verfügbar via Internet unter http://www.iac.ethz.ch/people/schaer/

Durran, D. R., 1998: Numerical methods for wave equations in geophysical fluid dynamics. *Text in Applied Mathematics*, **32**, Springer

Derivatives with finite differencing

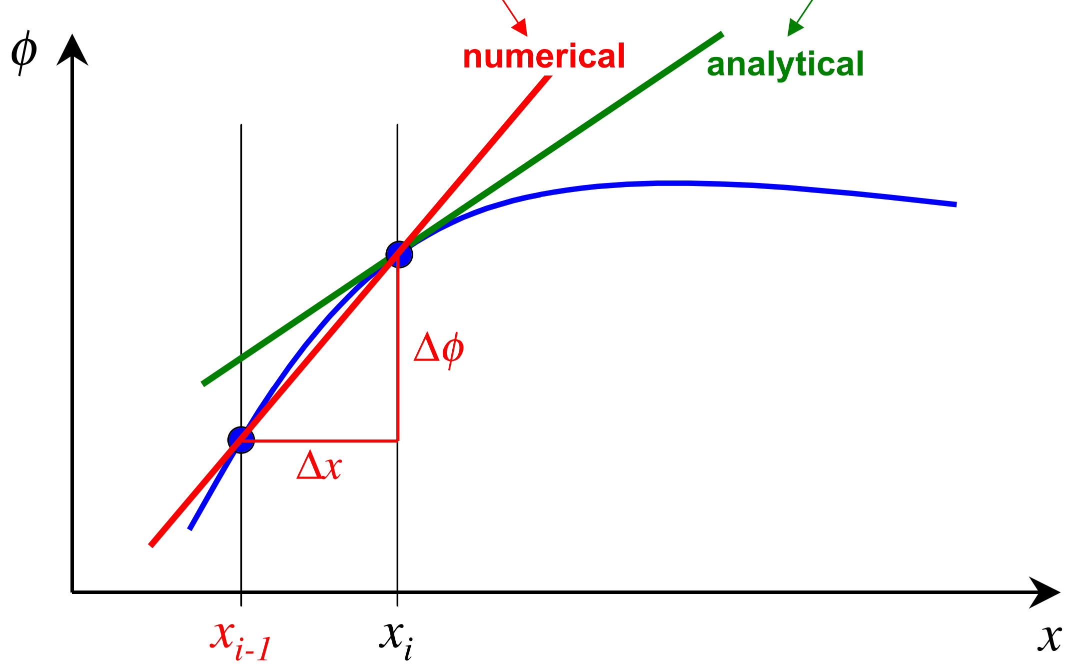
Discretization in space

$$x_i = i\Delta x$$

$$\phi_i = \phi(x_i)$$

Spatial derivative using finite differences

$$\boxed{\left.\frac{\partial \phi}{\partial x}\right|_{x_i} := \frac{\phi_i - \phi_{i-1}}{\Delta x}} \qquad \boxed{\left.\frac{\partial \phi}{\partial x}\right|_{x_i} := \lim_{\Delta x \to 0} \frac{\phi_i - \phi_{i-1}}{\Delta x}}$$



Leapfrog Scheme

One-dimensional linear advection (or transport) equation

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0 \qquad (u\text{=const})$$

Discretization in space and time

$$x_i = i\,\Delta x, \quad t^n = n\Delta t$$

$$\phi_i^n = \phi(x_i, t^n)$$

Centered differences in space and time

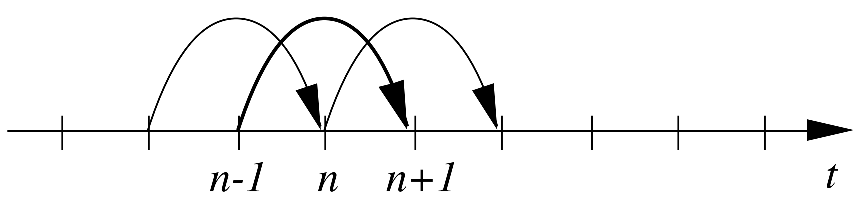
$$\left(\frac{\partial \phi}{\partial t}\right)_i^n \approx \frac{\phi_i^{n+1} - \phi_i^{n-1}}{2\,\Delta t}$$

$$\left(\frac{\partial \phi}{\partial x}\right)_i^n \approx \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\,\Delta x}$$

Solving for ϕ_i^{n+1} yields the *Leapfrog scheme*

$$\phi_i^{n+1} = \phi_i^{n-1} - \alpha\left(\phi_{i+1}^n - \phi_{i-1}^n\right) \quad \text{with} \quad \alpha = \frac{u\Delta t}{\Delta x} = \text{Courant number}$$

This is a three-level time-stepping scheme:



Upstream Scheme

Forward discretization in time (*Euler or forward time stepping*)

$$\left(\frac{\partial \phi}{\partial t}\right)_i^n \approx \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t}$$

Upstream discretization in space

$$\left(\frac{\partial \phi}{\partial x}\right)_i^n \approx \frac{\phi_i^n - \phi_{i-1}^n}{\Delta x} \qquad \text{(for } u \geq 0\text{)}.$$

Solving for ϕ_i^{n+1} yields the *Upstream scheme*

$$\phi_i^{n+1} = \phi_i^n - \alpha\left(\phi_i^n - \phi_{i-1}^n\right) \quad \text{with} \quad \alpha = \frac{u\Delta t}{\Delta x}$$

where α is the Courant number.

This is a two-level time-stepping scheme.

Courant-Friedrichs-Levy stability criterion

Stability criterion for a large class of schemes:

$$\left|\frac{u\ \Delta t}{\Delta x}\right| \le 1$$

For the advection equation: u refers to the advection velocity

=> Interpretation: the physical domain of dependence of ϕ_j^{n+1} must be included in the numerical domain of dependence.

=> For a large class of schemes, this is a necessary but not sufficient stability criterion!

In the general case, u is the largest speed at which information may propagate in the system under consideration (propagation speed). Information can propagate by advective processes (with velocity $\mathbf{v}$), or by wave propagation (with group velocity $\mathbf{c}_g$).

Thus, in the general case u represents:

$$u = \max\left|\mathbf{v} + \mathbf{c}_g\right|$$

Von Neumann analysis

Ansatz: wave-like perturbation of the form

$$\phi(x,t) = e^{i\,(k\,x-\omega\,t)}$$

For linear advection we have $\omega = uk$.

Conversion to computational grid $(x,t) = (j\Delta x, n\Delta t)$

$$\phi_j^n = e^{i\,k\,(j\,\Delta x - u\,n\,\Delta t)} = e^{i\,k\,j\,\Delta x}\,\underbrace{(e^{-i\,k\,u\,\Delta t})}_{\lambda}{}^n = e^{i\,k\,j\,\Delta x}\,\lambda^n$$

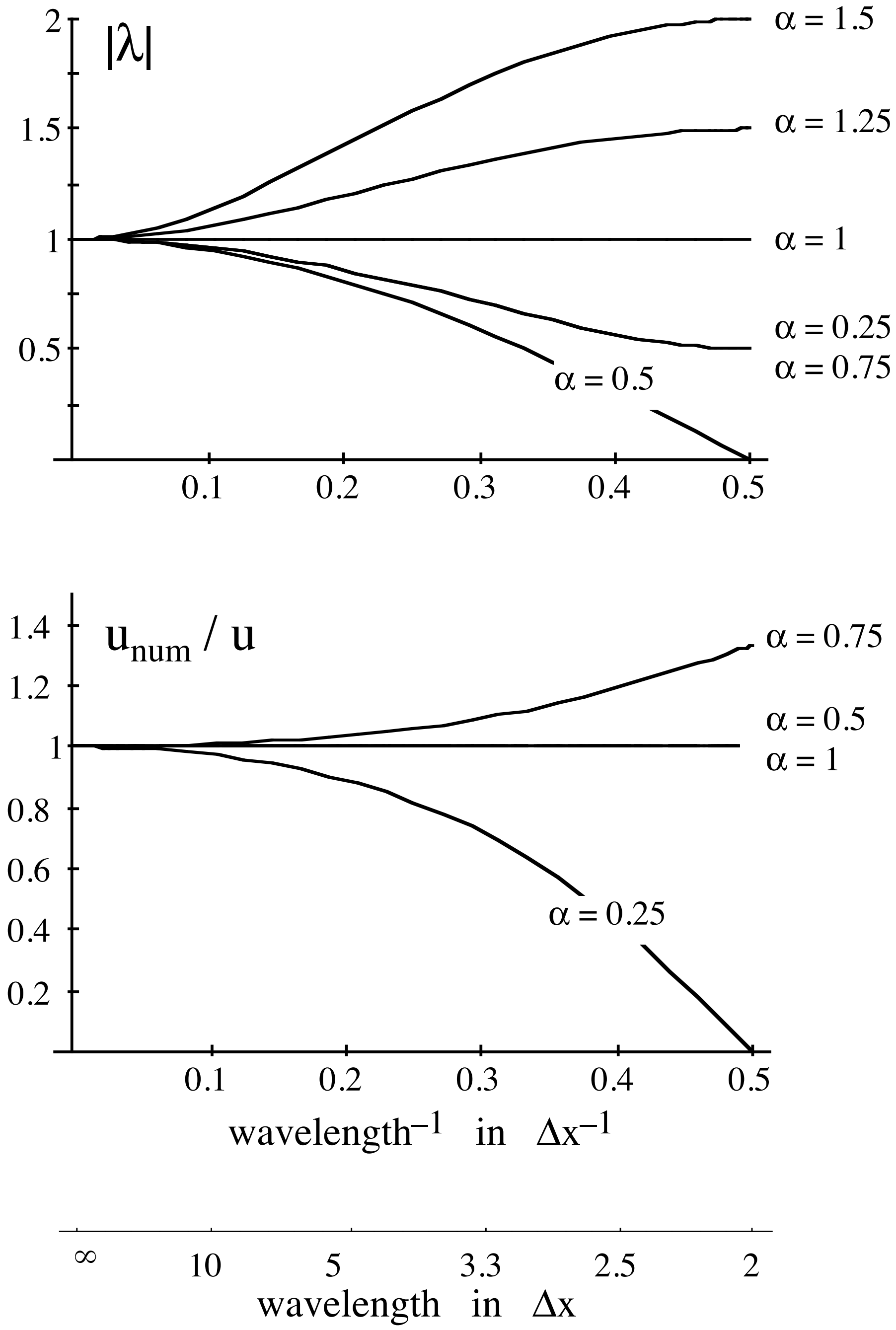
The complex number $\lambda = \lambda(k)$ determines time evolution.

Plug Ansatz into numerical scheme to derive $\lambda = \lambda(k)$.
Then, λ determines stability of scheme:

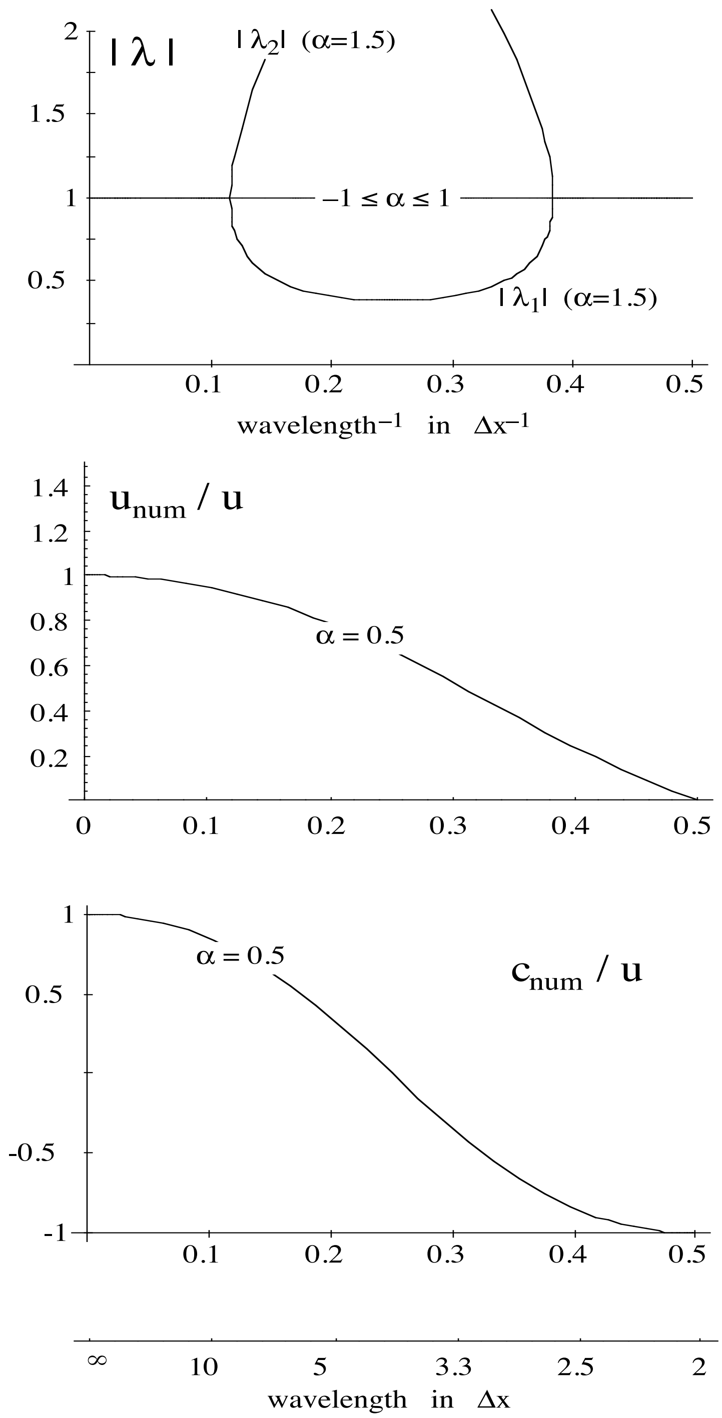
- $|\lambda(k)| > 1$ for one wave number k: unstable
- $|\lambda(k)| = 1$ for all wave numbers k: stable, neutral
- $|\lambda(k)| < 1$ for all wave numbers k: stable

In addition, $\lambda(k)$ contains detailed information about the amplitude and phase error of a scheme, as a function of wave number k.

Example: Upstream scheme



Example: Leapfrog scheme



Classification of time stepping algorithms

Consider an ordinary differential equation

$$\frac{\partial \phi}{\partial t} = F(\phi)$$

Explicit schemes: the discretized form of the equation does not contain the time level *n+1* on the right-hand side.

Example: Forward (Euler) step

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = F\left(\phi^n\right)$$

Explicit schemes can be solved for ϕ^{n+1}, i.e. there is an explicit form of the time step.

Implicit schemes: $F(\phi)$ is evaluated at time level *n+1* (or some time level > n).

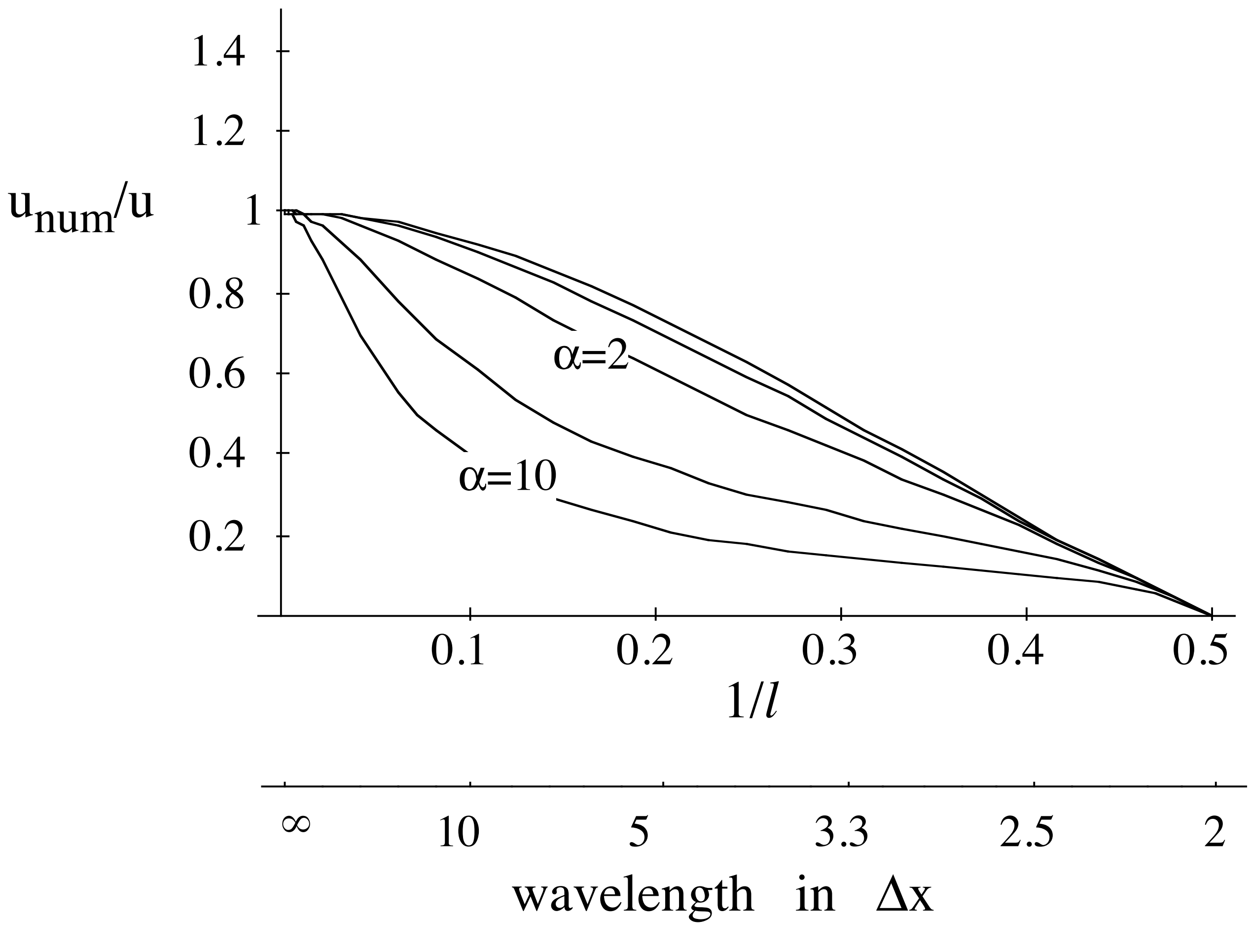
Example: Trapezoidal rule

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{1}{2}\left[F\left(\phi^{n+1}\right) + F\left(\phi^n\right)\right]$$

In general (depending on *F*) such an equation cannot explicitly be solved for ϕ^{n+1}.

Phase error of implicit advection

Implicit schemes have attractive stability properties, and may be stable irrespective of the time-step selected. In general, however, the accuracy will dramatically decrease with increasing time step. Below an example for the implicit advection, which is unconditionally stable but has large phase errors for large Courant numbers.



Nonlinear instability

The von Neumann method is restricted to linear schemes and linear (or linearized) governing equations.

However, instability can also result from nonlinear interactions, even if linear stability criteria (e.g. the CFL condition) are met.

Nonlinear instability often derives in the context of strong scale interactions, for instance when the governing equations attempt to create a scale collapse. As the scale collapse cannot fully be represented on the computational grid (as a result of limited spatial resolution), the nonlinear amplification of small-scale wave may result in instability.

Order of schemes

Example: Upstream scheme

Using Taylor series expansion, one can estimate the error of a discretization. In the case of the upstream finite differencing

$$\phi_i^{'} = \frac{\phi_i - \phi_{i-1}}{\Delta x}$$

$$+ \phi_i^{''} \frac{\Delta x}{2} - \phi_i^{'''} \frac{\Delta x^2}{3!} + \phi_i^{(iv)} \frac{\Delta x^3}{4!} + \mathrm{O}\left(\Delta x^4\right)$$

or

$$\phi_i^{'} = \left(\phi_i^{'}\right)^{upstream} + F$$

$$\text{mit} \quad F = \phi_i^{''} \frac{\Delta x}{2} - \phi_i^{'''} \frac{\Delta x^2}{3!} + \mathrm{O}\left(\Delta x^3\right)$$

The leading term of the error *F* is $\mathrm{O}\left(\Delta x^1\right)$ => 1^{st} order in space

A scheme of order *n* has leading error terms of (at least) order *n* in space and time.

Example: Leapfrog scheme

Leading term of the error *F* is $\mathrm{O}\left(\Delta x^2\right)$ => 2^{nd} order in space

4^{th}-order centered differencing

2^{nd}-order centered spatial differencing

$$\phi'_j = \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}$$

4^{th}-order centered spatial differencing

$$\phi'_j = \beta \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} + \gamma \frac{\phi_{j+2} - \phi_{j-2}}{4\Delta x}$$

With $\beta = 4/3$ and $\gamma = -1/3$, the leading error term is $O\left(\Delta x^4\right)$

=> fourth order in space

Convergence: Lax-Richtmeyer theorem

A numerical scheme is *convergent*, if the integration (over a finite time interval *T*) for $\Delta x \to 0$ and $\Delta t \to 0$ yields the exact solution.

With $\Delta x \to 0$, stability and accuracy considerations imply that also $\Delta t \to 0$, and thus the number of time steps increases as $N \to \infty$. Addressing convergence thus is a difficult topic, as it requires consideration of an infinite number of time steps.

Lax-Richtmeyer theorem (Lax and Richtmeyer 1956):

> A finite difference numerical scheme is *convergent*, if and only if it is *consistent* and *stable*.

A scheme is referred to as *consistent*, if the discretization error *per time step* disappears with $\Delta x \to 0$ and $\Delta t \to 0$. Thus, any scheme that is at least 1^{st} order accurate in space and time is consistent.

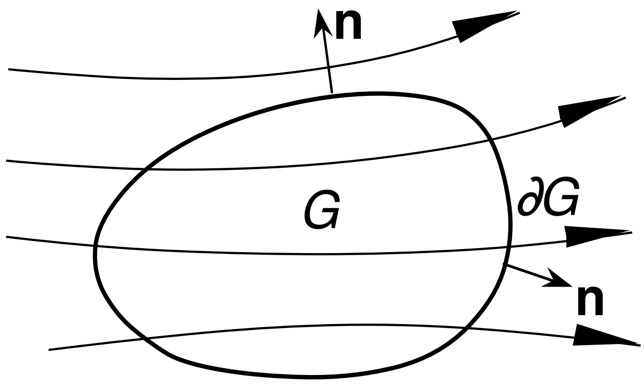
The Lax-Richtmeyer theorem thus allows addressing convergence (which involves an infinite number of time steps) by considering consistency and stability (which may often be framed in terms of single time steps).

Conservative schemes

Equations in divergence form, e.g. the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) = 0$$

are <u>*conservative*</u>. This expresses that the integrated mass in a domain G can only change in response to mass fluxes at the surface ∂G.



At the level of numerical schemes, conservation may be guaranteed by using the flux form. For the one-dimensional continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(u \rho) = 0$$

this is

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}}{\Delta x} = 0$$

where F is an approximation of the mass flux $u\rho$.