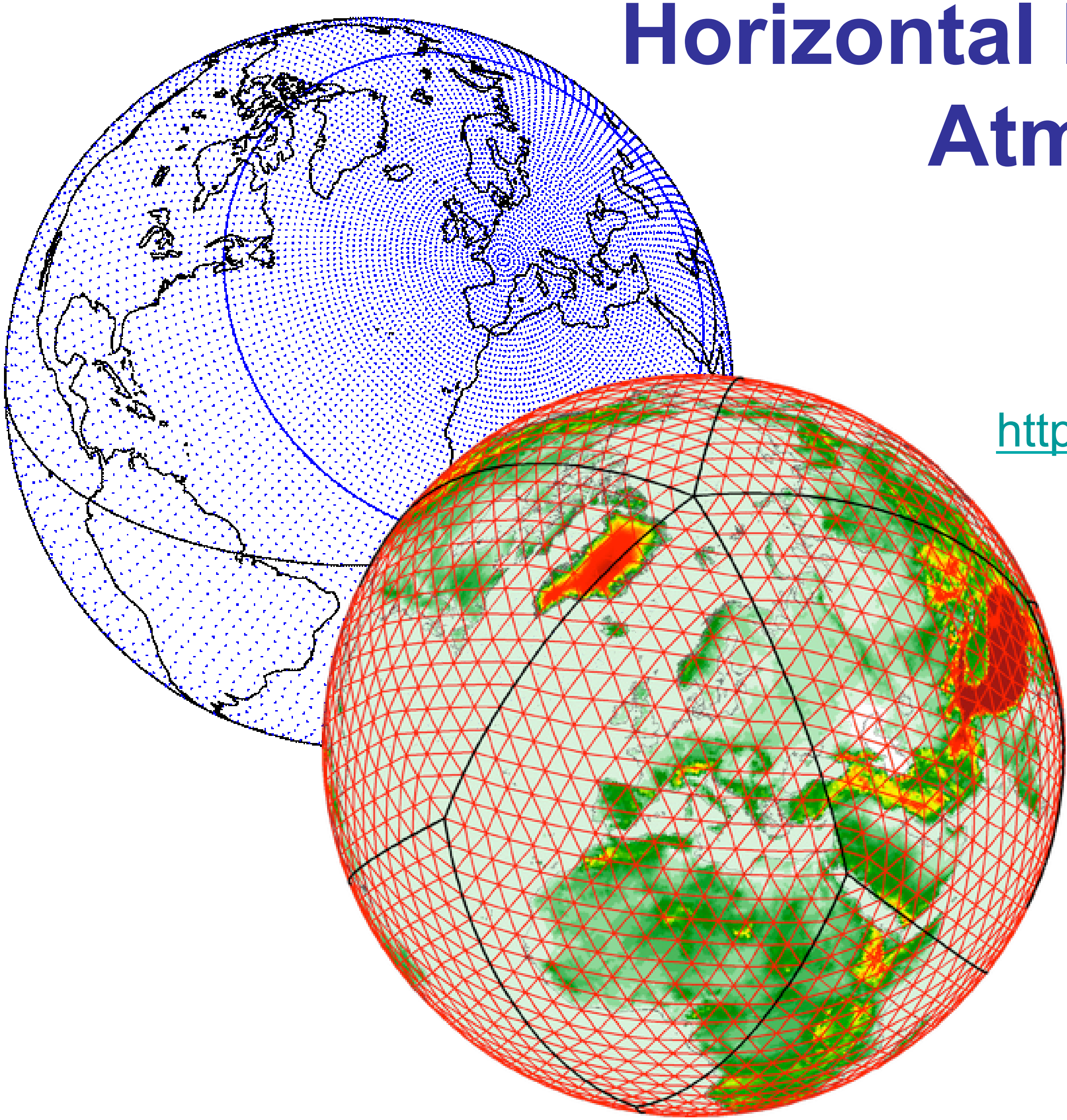
Horizontal Discretizations in Atmospheric Models



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Supplement to Lecture Notes
"Numerical Modeling of
Weather and Climate"

March, 2015

The pole problem

The simplest model grid on the sphere is a regular latitude / longitude grid.

This grid suffers from the "pole problem."

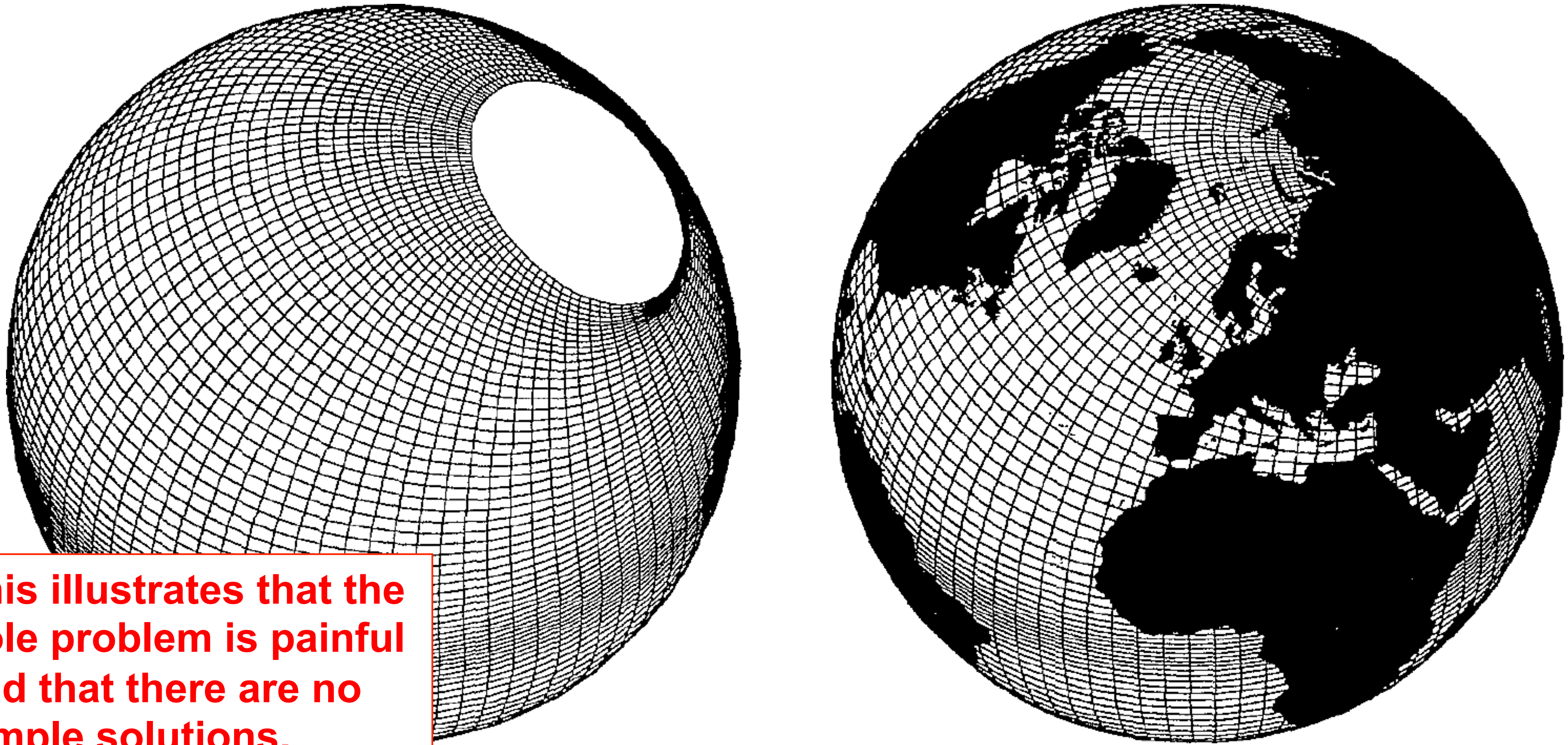
The poles represent singularities in latitude / longitude grids.

As the grid spacing approaches zero near the poles, there are difficult stability issues.

CFL criterion: $\left|\frac{u \ \Delta t}{\Delta x}\right| \leq 1$

The pole problem in a global ocean model

In ocean models, the Earth's axes may be bent to place the North Pole over Asia.



This illustrates that the pole problem is painful and that there are no simple solutions.

Madec and Imbard, 1996, Clim. Dyn., 12 (6), 381-388
A global ocean mesh to overcome the North Pole singularity
(used in the MICOM ocean model and the Bergen climate model)

Outline

The pole problem

Regional models

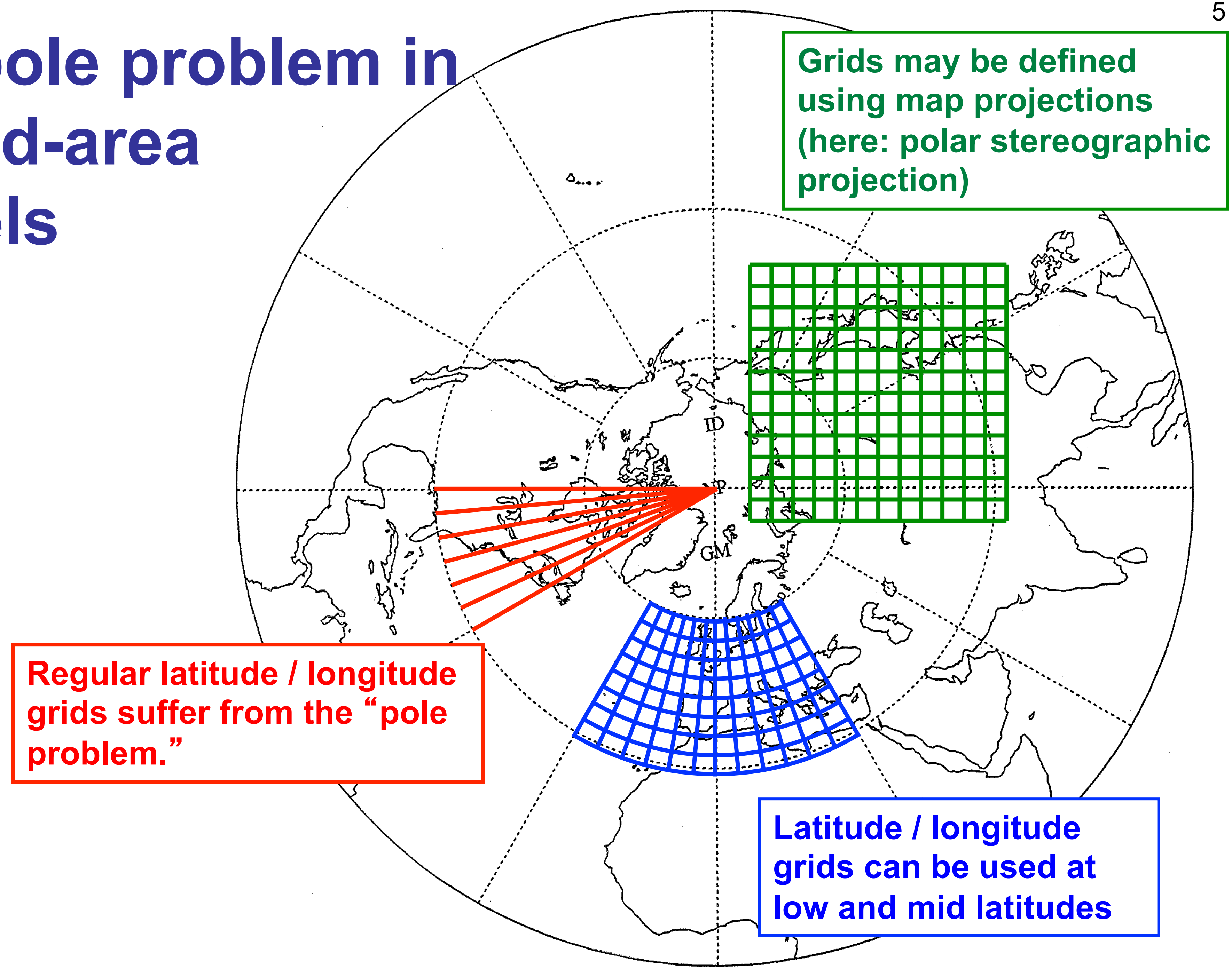
Introduction to spectral methods

Global spectral models

Other global grid approaches

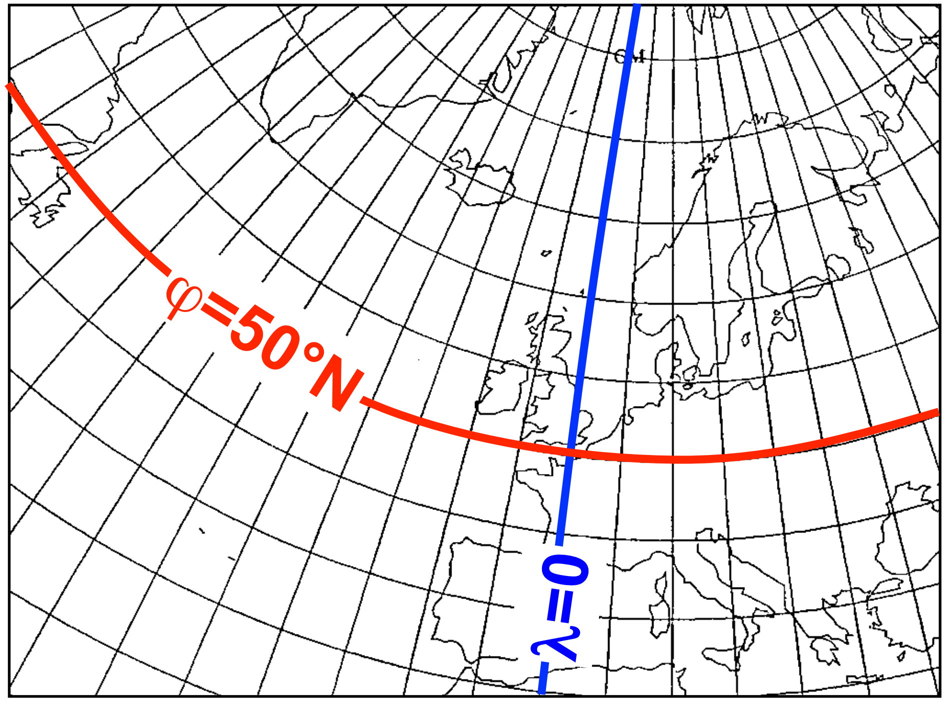
Linking “dynamics” and “physics”

The pole problem in limited-area models

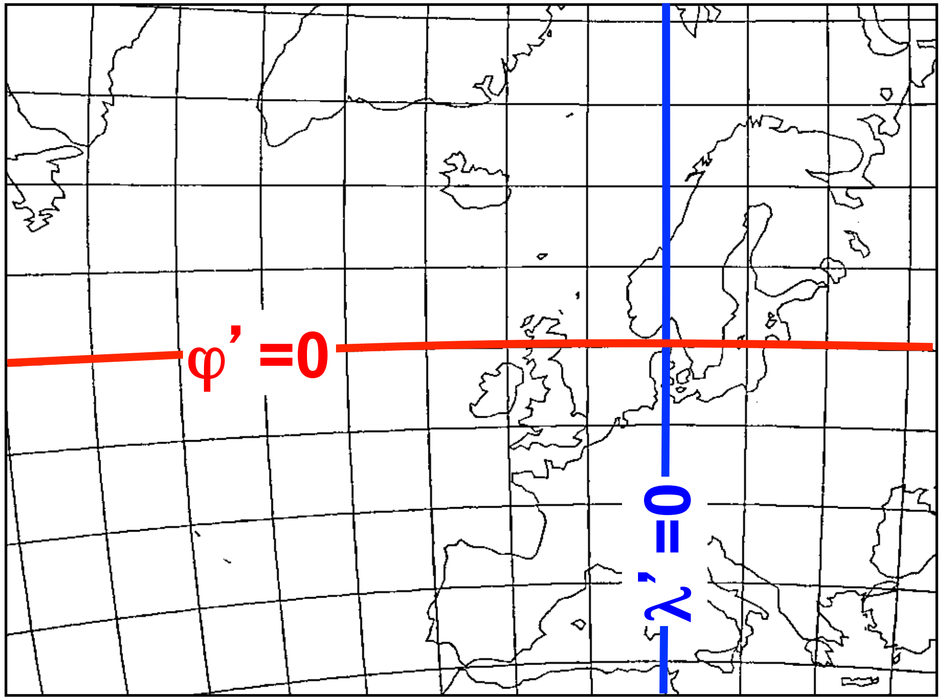


Rotated latitude / longitude grids

Regular latitude / longitude grid



Rotated latitude / longitude grid



This is a regular latitude/longitude grid based on rotated spherical coordinates.
It has a much more isotropic structure than the unrotated grid!

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Spatial discretizations in one dimension

Local value of function (finite differences)

$$\boxed{\phi_i(t)} = \phi(i\Delta x, t)$$

Mean over grid box (finite volume, finite differences)

$$\boxed{\phi_i(t)} = \frac{1}{\Delta x} \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} \phi(x,t) \; dx$$

Spectral representation (spectral method)

$$\phi(x,t) = \sum_m \boxed{\psi_m(t)} \, e^{i k_m x}$$

Linear advection with the spectral method

Governing equation

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

Discretization

$$\phi(x,t) = \sum_m \psi_m(t)\, e^{i\, k_m\, x}$$

Discretized equation

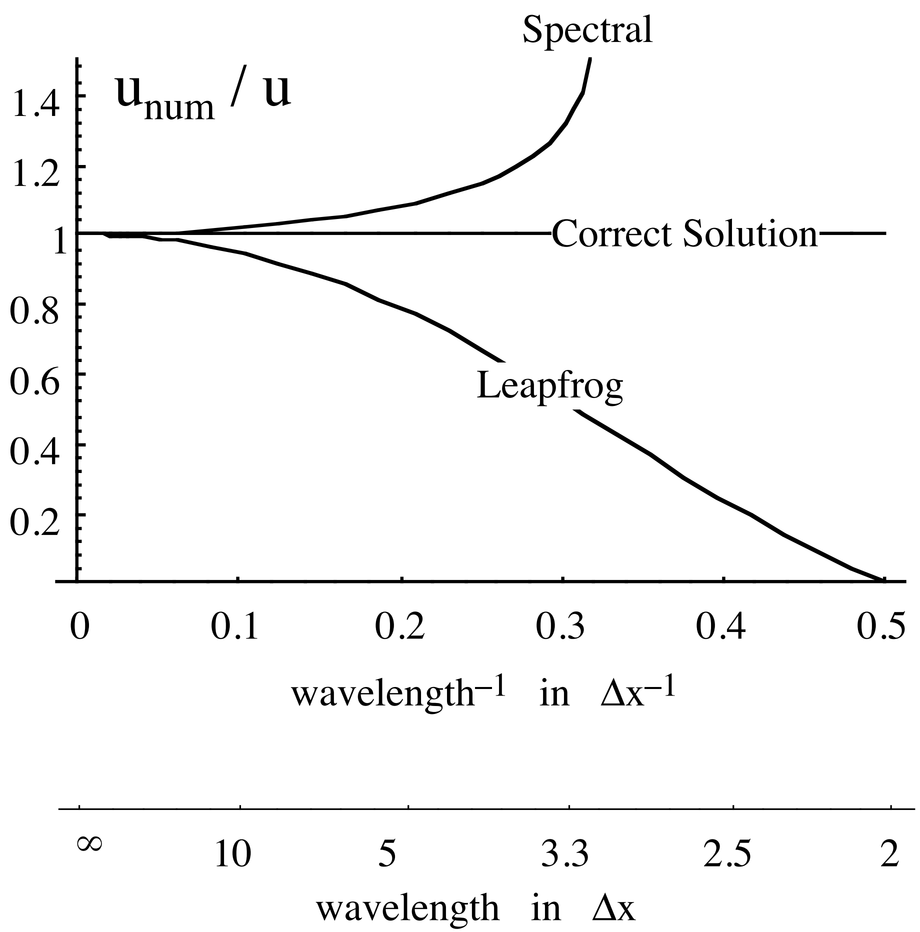
$$\sum_{m=-M}^{M} \left[\frac{\partial \psi_m}{\partial t} + i\, u\, k_m\, \psi_m \right] e^{i\, k_m\, x} = 0$$

$$\frac{\partial \psi_m}{\partial t} = -i\, u\, k_m \psi_m \qquad (m = -M \dots M)$$

With Leapfrog time step

$$\psi_m^{n+1} = \psi_m^{n-1} - 2\, i\, \Delta t\, u\, k_m \psi_m^n$$

Comparison against centered finite differences



Phase error of the spectral method and leapfrog scheme for the linear advection equation with Courant number α=0.5, in comparison with centered differencing

Nonlinear advection with the spectral method

Governing equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Discretization

$$u(x,t) = \sum_m \psi_m(t) e^{i k_m x}$$

Discretized equation

$$\sum_{m=-M}^{M} \frac{\partial \psi_m}{\partial t} e^{i k_m x} + \left[\sum_{j=-M}^{M} \psi_j e^{i k_j x}\right]\left[\sum_{l=-M}^{M} i\, k_l\, \psi_l e^{i k_l x}\right] = 0$$

$$\sum_{m=-M}^{M} \frac{\partial \psi_m}{\partial t} e^{i k_m x} + \sum_{j,l=-M}^{M} \psi_j\, i k_l\, \psi_l\, e^{i k_{j+l} x} = 0$$

Using interaction coefficients

$$\frac{\partial \psi_p}{\partial t} = -\sum_{j,l=-M}^{M} I_{j,l,p} \psi_j \psi_l \qquad (p = -M \ldots M)$$

with

$$I_{j,l,p} = \frac{1}{L}\int_0^L i k_l\, e^{i k_{j+l-p}}\, dx$$

The interaction matrix $I_{j,l,p}$ has size $(2M+1)^3$ and is prohibitively large!

Spectral approach for linear versus nonlinear equations

- **Linear advection** $$\frac{\partial \psi_p}{\partial t} = iuk_p \psi_p$$

  Approach is simple and provides high accuracy

- **Nonlinear advection** $$\frac{\partial \psi_p}{\partial t} = -\sum_{j,l=-M}^{M} I_{j,l,p} \psi_j \psi_l \qquad (p = -M \ldots M)$$

  Storage of interaction tensor is very costly, has dimension $(2M+1)^3$.

  For weather and climate, approach is prohibitively expensive.

- **Alternative: pseudo-spectral approach.**

  Combine
  - computations in spectral space (for linear terms) with
  - computations in physical space (for non-linear terms).

  Approach is suited for nonlinear systems, but implies frequent use of transformations between spectral and physical space.

Pseudospectral method with $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

Spectral space

$$u(x,t) = \sum_m \psi_m^n e^{i k_m x}$$

Computation of derivatives

$$\partial/\partial x: \quad \psi_m^n \rightarrow ik_m\psi_m^n$$

Time step

$$\psi_m^{n+1} = \psi_m^{n-1} + 2\Delta t\,\tilde{\psi}_m^n$$

Transformations

$$\psi_m^n \xrightarrow[FT^{-1}]{} u_j^n$$

$$i k_m \psi_m^n \xrightarrow[FT^{-1}]{} \left(\frac{\partial u}{\partial x}\right)_j^n$$

$$\tilde{\psi}_m^n \xleftarrow[FT]{} \tilde{u}_j^n$$

Physical space

$$u_j^n = u\left(x_j, t^n\right)$$

Computation of tendencies

$$\tilde{u}_j^n := \left(\frac{\partial u}{\partial t}\right)_j^n = -u_j^n\left(\frac{\partial u}{\partial x}\right)_j^n$$

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Spectral methods in different geometries

Geometry	Eigenvalue problem	Eigenvalue	Eigenfunction (base functions)
1D: $0 \le x \le L_x$	$\frac{\partial^2 f}{\partial x^2} = -\mu f$	$\mu = \left(n \frac{2\pi}{L_x}\right)^2$	$f(x) = e^{ik_n x}$ with $k_n = n 2\pi / L_x$
2D: $0 \le x \le L_x$ $0 \le y \le L_y$	$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = -\mu f$	$\mu = \left(n \frac{2\pi}{L_x}\right)^2 + \left(m \frac{2\pi}{L_y}\right)^2$	$f(x,y) = e^{i(k_n x + l_m y)} = e^{ik_n x} e^{il_m y}$
Sphere: Radius a	$\nabla^2 f = -\mu f$	$\mu = \frac{n(n+1)}{a^2}$	$f(\lambda,\phi) = Y_n^m(\lambda,\phi) = e^{im\lambda} P_n^m[\sin(\phi)]$

Spectral representation on the sphere

Two-dimensional fields ϕ on the sphere are represented as an expansion using spherical harmonics $Y_n^m(\lambda,\varphi)$:

$$\phi(\lambda,\varphi,t) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \psi_n^m(t)\ Y_n^m(\lambda,\varphi)$$

Time-dependent coefficients
(stored to represent ϕ)

Spherical harmonics: $$Y_n^m(\lambda,\varphi) = e^{i\,m\,\lambda}\ P_n^m[\sin(\varphi)]$$

sin and cos-functions
(as in Fourier series),
determine variations in longitude λ

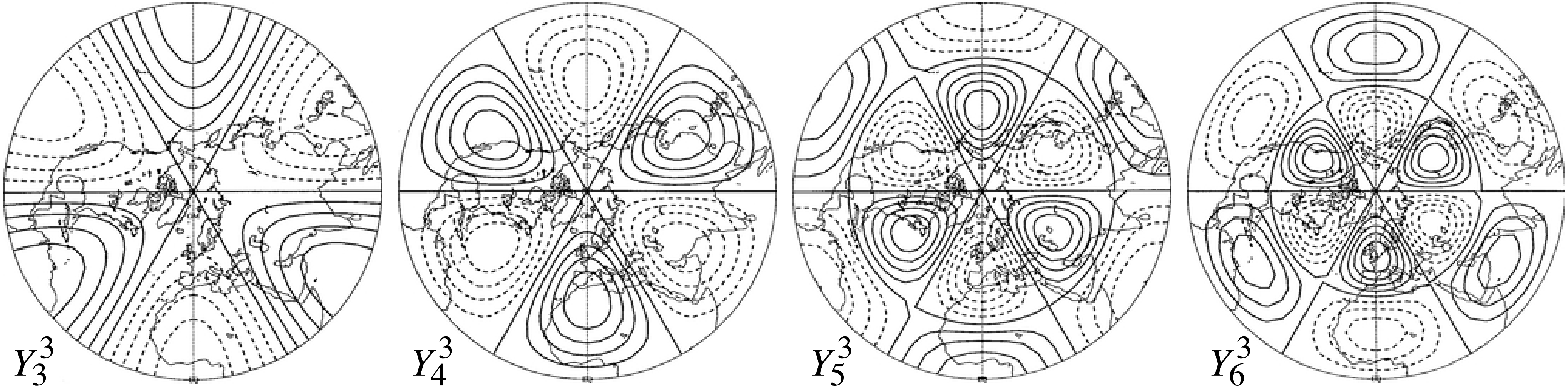
Associated Legendre polynomials,
defined for $n \geq |m|$,
determine variations in latitude φ

Examples:

$$P_2^0(x) = \frac{1}{2}\left(3x^2 - 1\right) \qquad P_2^1(x) = -3x\sqrt{1-x^2}$$

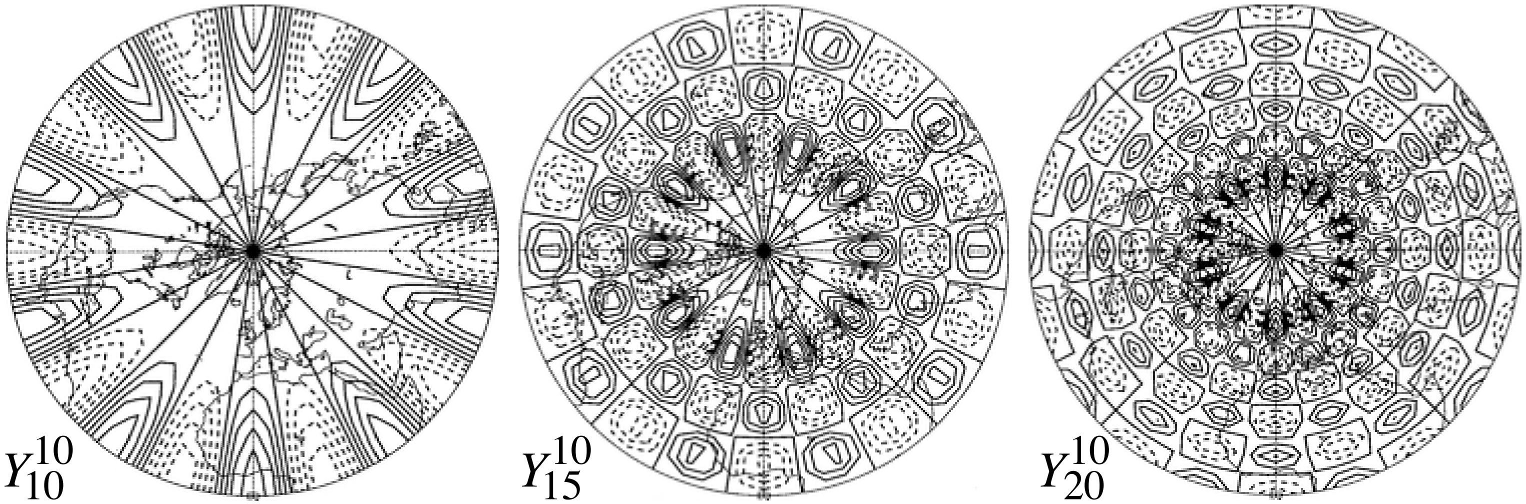
Spherical harmonics

Family Y^3 (order m=3)



Y_3^3 Y_4^3 Y_5^3 Y_6^3

Family Y^{10} (order m=10)



Y_{10}^{10} Y_{15}^{10} Y_{20}^{10}

Schär, ETH Zürich

Properties of spherical harmonics

Symmetry

$$Y_n^{-m} = (-1)^m \overline{Y_n^m}$$

Orthogonality

$$\frac{1}{2\pi} \iint Y_n^m(\lambda,\varphi) \overline{Y_r^s(\lambda,\varphi)} \, d\lambda \, \cos(\varphi) \, d\varphi = \delta_{nr} \, \delta_{ms}$$

Operations

$$\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2} Y_n^m$$

$$\frac{\partial}{\partial \lambda} Y_n^m = i m Y_n^m$$

$$\frac{\partial}{\partial \varphi} Y_n^m = -n \varepsilon_{n+1}^m Y_{n+1}^m + (n+1) \varepsilon_n^m Y_{n-1}^m \qquad \text{with } \varepsilon_n^m = \sqrt{\frac{n^2 - |m|^2}{4n^2 - 1}}$$

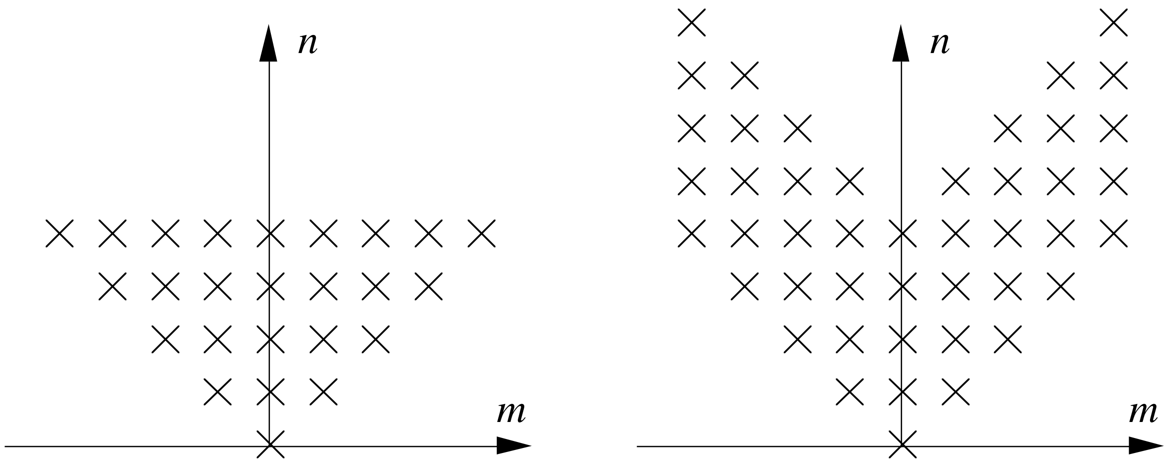
Truncation of series

Two-dimensional fields ϕ on the sphere are represented as an expansion using spherical harmonics $Y_n^m(\lambda,\varphi)$:

$$\phi(\lambda,\varphi,t) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \psi_n^m(t)\ Y_n^m(\lambda,\varphi) \qquad \text{(summation over } n \geq |m|\text{)}$$

Order m determines variations in λ-direction, degree n variations in φ-direction.

Truncation of series (example M=4)



triangular truncation
$n \leq M$

rhomboidal truncation
$n \leq M + |m|$

Resolution in spectral spherical models

Discretization:

$$\phi(\lambda,\varphi,t) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \psi_n^m(t)\; Y_n^m(\lambda,\varphi) \qquad\qquad Y_n^m(\lambda,\varphi) = e^{im\lambda}\; P_n^m[\sin(\varphi)]$$

Notation (convention): Examples

- Spectral model with resolution R25: Rhomboidal truncation with M=25
- Spectral model with resolution T106: Triangular truncation with M=106

Equivalent grid spacing: Example T106

Represents 106 waves at the equator

4 grid points needed per wave length.

$$\left.\begin{array}{l}\text{Resolved}\\ \text{wavelength}\end{array}\right\} = \frac{40000\ \text{km}}{106} = 380\ \text{km}$$

$$\left.\begin{array}{l}\text{Equivalent}\\ \text{grid spacing}\end{array}\right\} \approx \frac{380\ \text{km}}{4} \approx 100\ \text{km}$$

Computational aspects

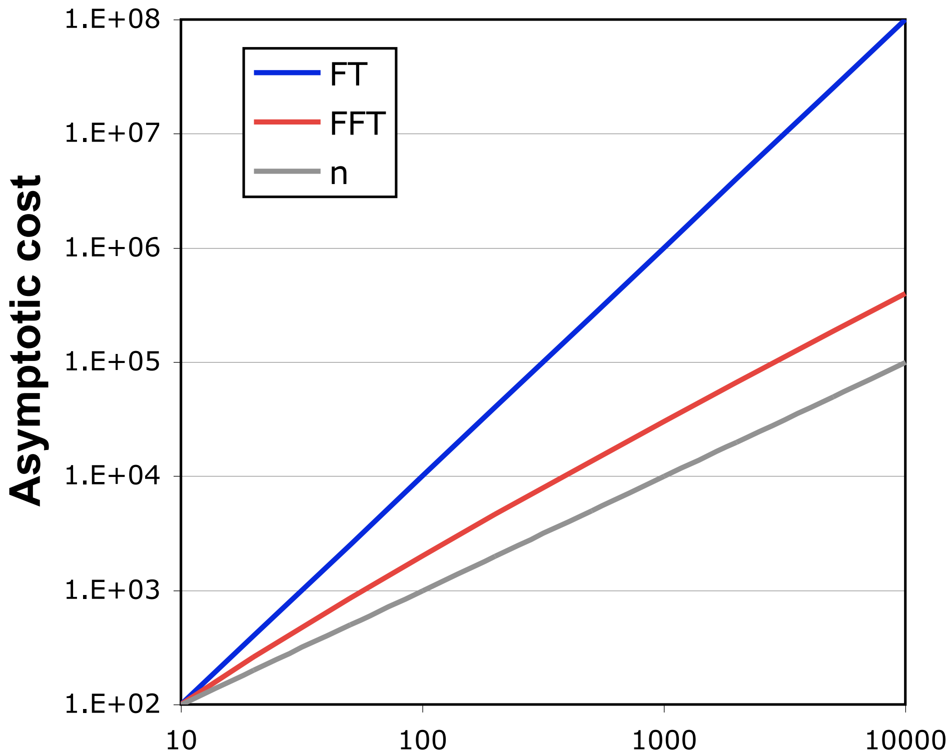
Spectral and pseudo-spectral approaches require frequent Fourier transforms.

Computational costs for a simple-minded one-dimensional FT:

O(N^2) arithmetic operations increases with square of N

Using the Fast Fourier Transform (FFT, Cooley and Tuckey 1965)

O(N log N) arithmetic operations increases much slower



The feasibility of the spectral and pseudo-spectral approach rests upon FFT.

Even with FFT, the computational costs increases faster than the number of grid points. At very high resolution, grid point models will become computationally advantageous.

On the sphere, both Fourier and Legendre transformations are needed. Recently, fast Legendre transforms (FLT) have been developed (Tygert 2008, 2010), and successfully tested at ECMWF (Wedi 2012).

Global spectral model at T_L7999

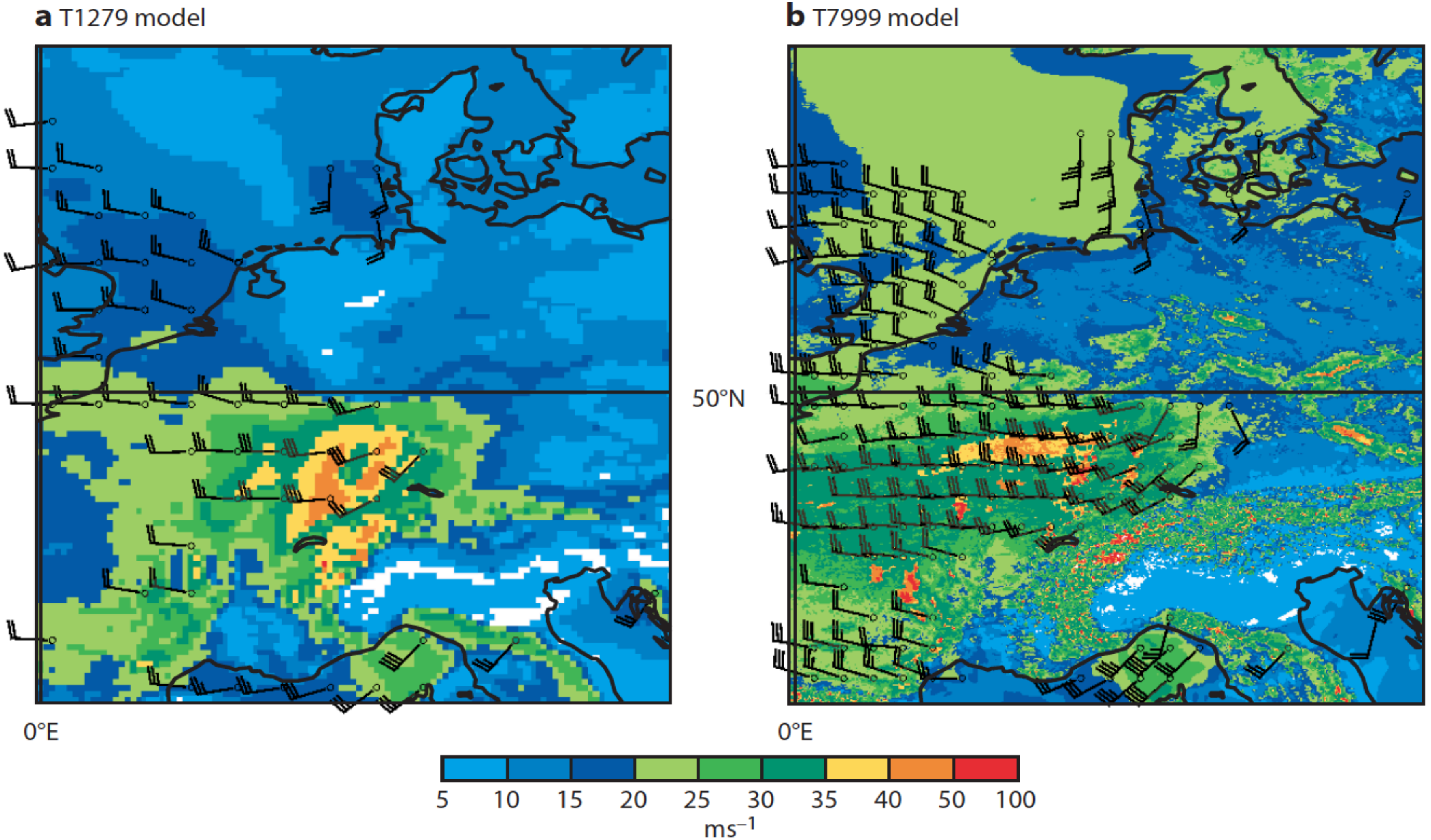


Figure 4 11-hour forecast of wind gusts and 10-metre surface winds for the Christmas storm 'Lothar' on 26 December 1999 from: (a) T1279 (~16 km) model and (b) T7999 (~2.5 km) model. During the storm some of the highest wind speeds ever recorded in Europe were observed (69 ms^{-1} on Jungfraujoch, Switzerland; 75 ms^{-1} on Hohentwiel, Singen, Germany). The trail of 'destruction' is clearly marked by the wind gust data. Note that the lower mountain ranges now prominently feature in the wind gust data for T7999.

(Wedi 2012, ECMWF NL)

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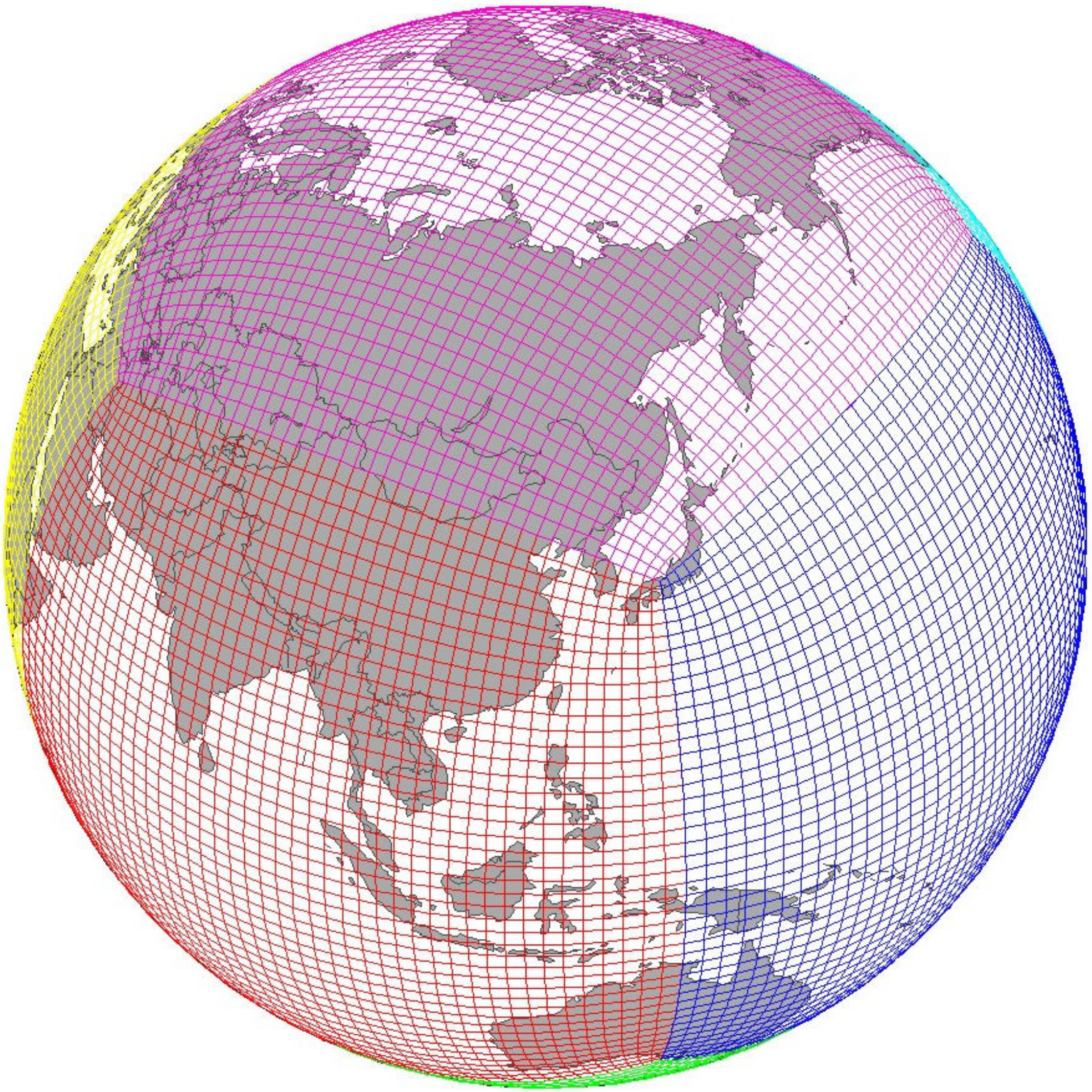
Spherical cube grid

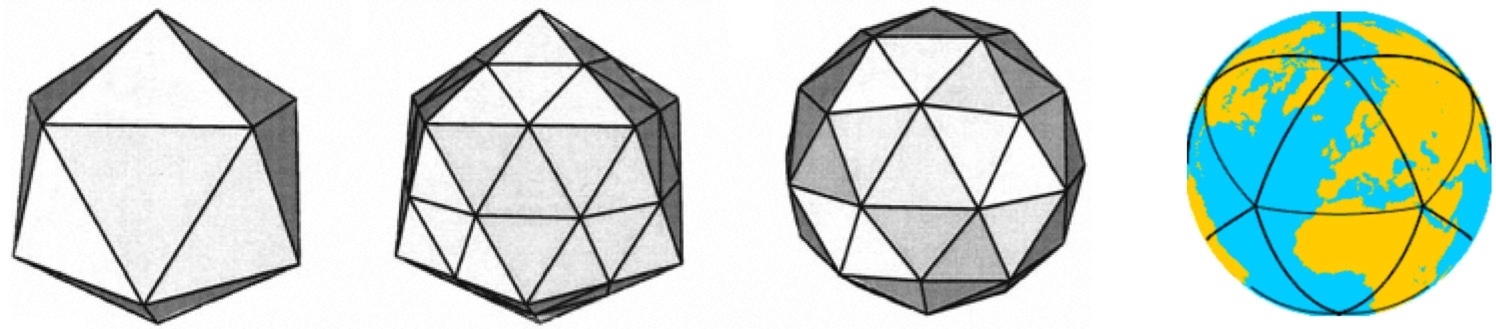


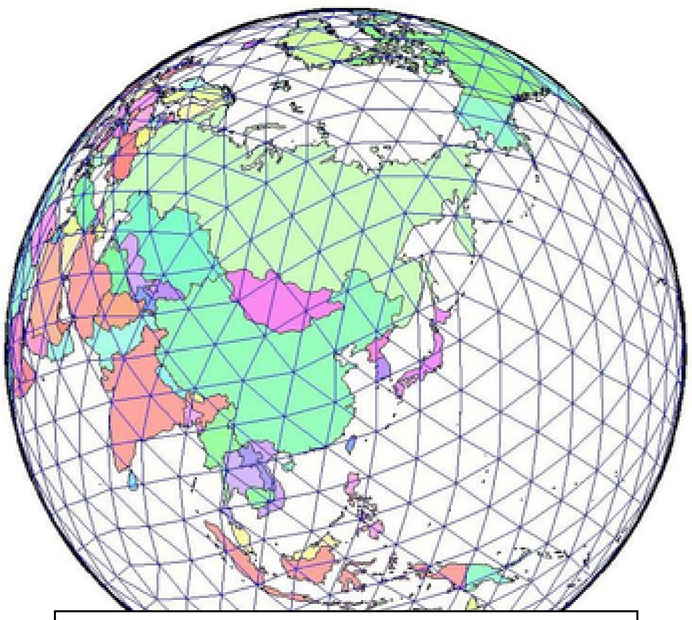
The spherical cube grid is formed by projecting a cube onto the sphere. There are 8 non-orthogonal grid points corresponding to the corners of the original cube.

Adcroft et al. 2004, MWR, 132 (12): 2845-2863; William Sawyer, PhD ETH

Geodesic grids

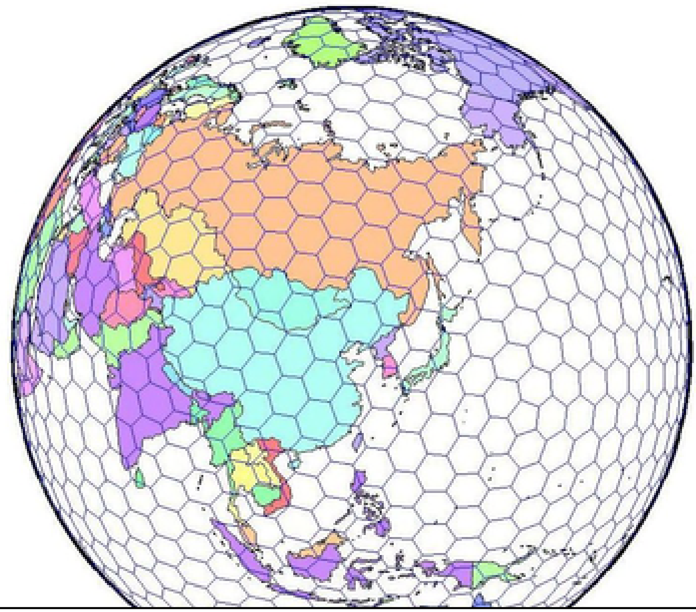
Geodesic grids are formed from an icosahedron by iteratively bisecting the edges and projecting the new points onto the sphere.





Delaunay (triangular) grid

Resulting grids



Voronoi (hexagonal–pentagonal) grid

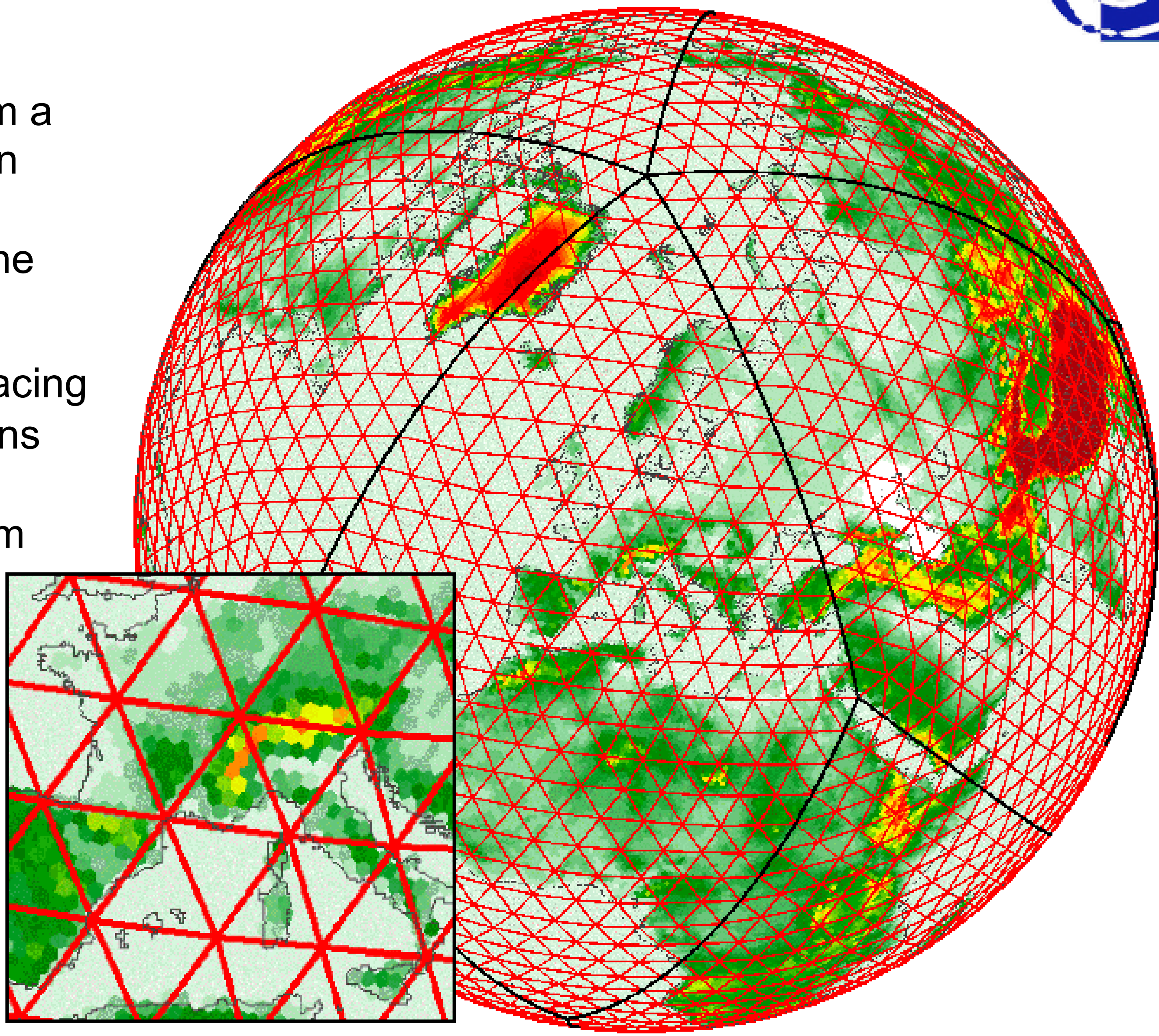
William Sawyer, PhD ETH
Bonaventura and Ringler 2005, MWR 133; Ringler et al. 2000, MWR, 128

Icosahedral Global Mesh

This grid is constructed from a projection of an icosahedron onto the sphere, and subsequent refinement of the 20 triangles.

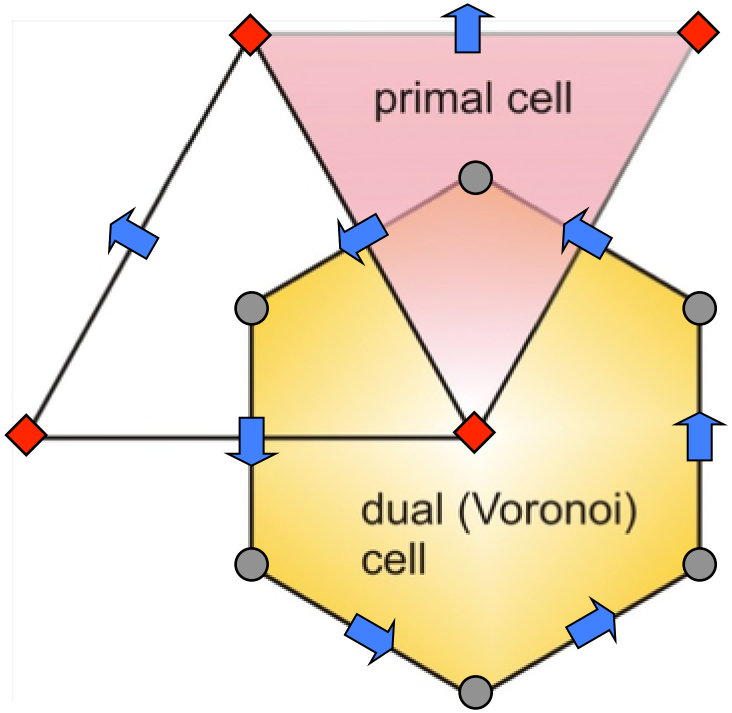
Current operational grid spacing as used for NWP applications by the German Weather Service (DWD): about 60 km

The current German weather and climate models (GME and ECHAM) will be replaced around 2014 by a new model ICON: **ico**sahedral grid with **n**onhydrostatic dynamics.

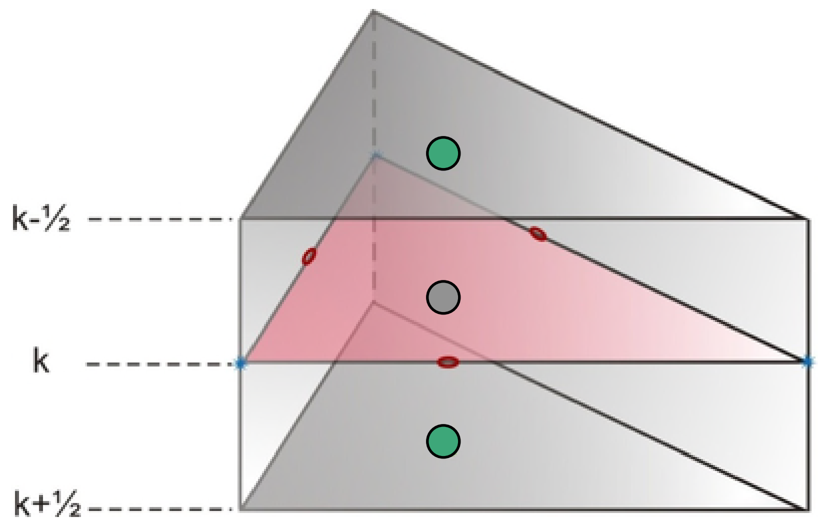


(German Weather Service, DWD; Max Planck Institute, MPI Hamburg)

Staggering in ICON



horizontal



vertical

C-type staggering

T, q, p, Φ

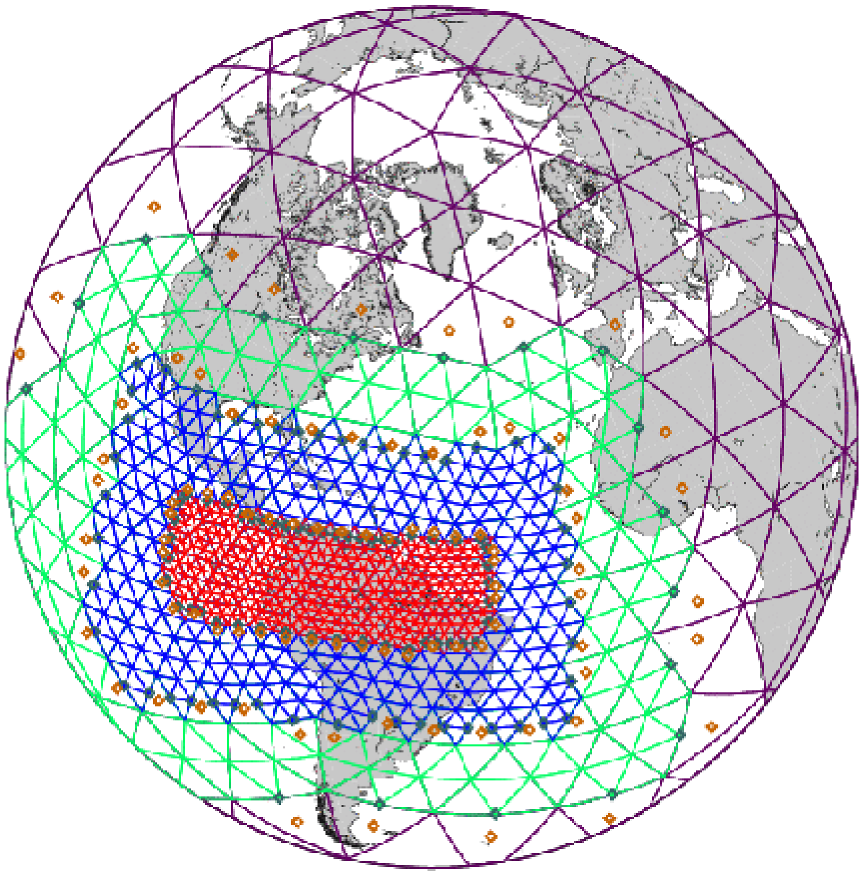
v_n

$\vec{k} \bullet (\vec{\nabla} \times \vec{v})$

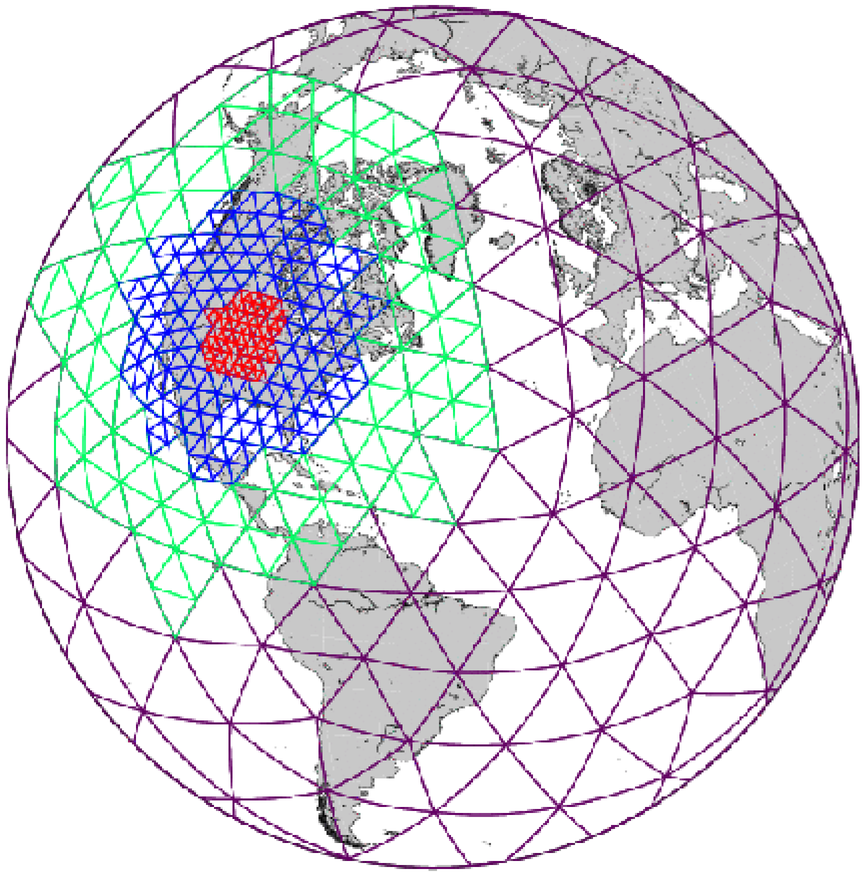
$\dot{\eta} \frac{\partial p}{\partial \eta}, \Phi, p$

(Detlev Majewski, DWD)

Local grid refinement in ICON



Latitude-longitude window



Circular window

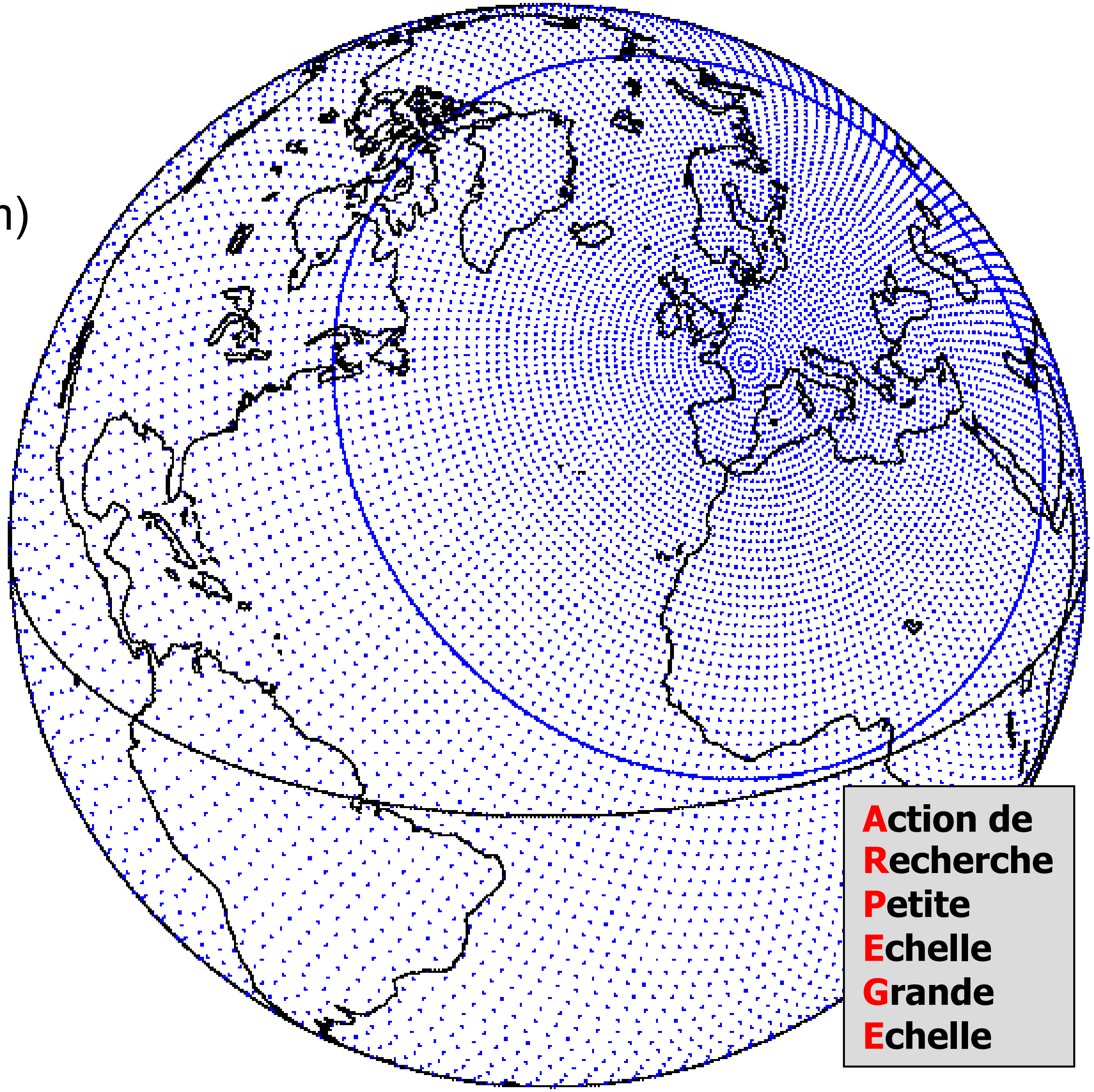
(Detlev Majewski, DWD)

Global Models with variable resolution

ARPEGE (Meteo France)

Representation on the globe with the spectral approach, but using
- grid stretching (Schmidt transform)
- pole rotation

to enhance resolution over the region of interest



Current resolution in NWP applications:

spectral resolution: T_L358

stretching factor: C=2.4

Resulting grid:

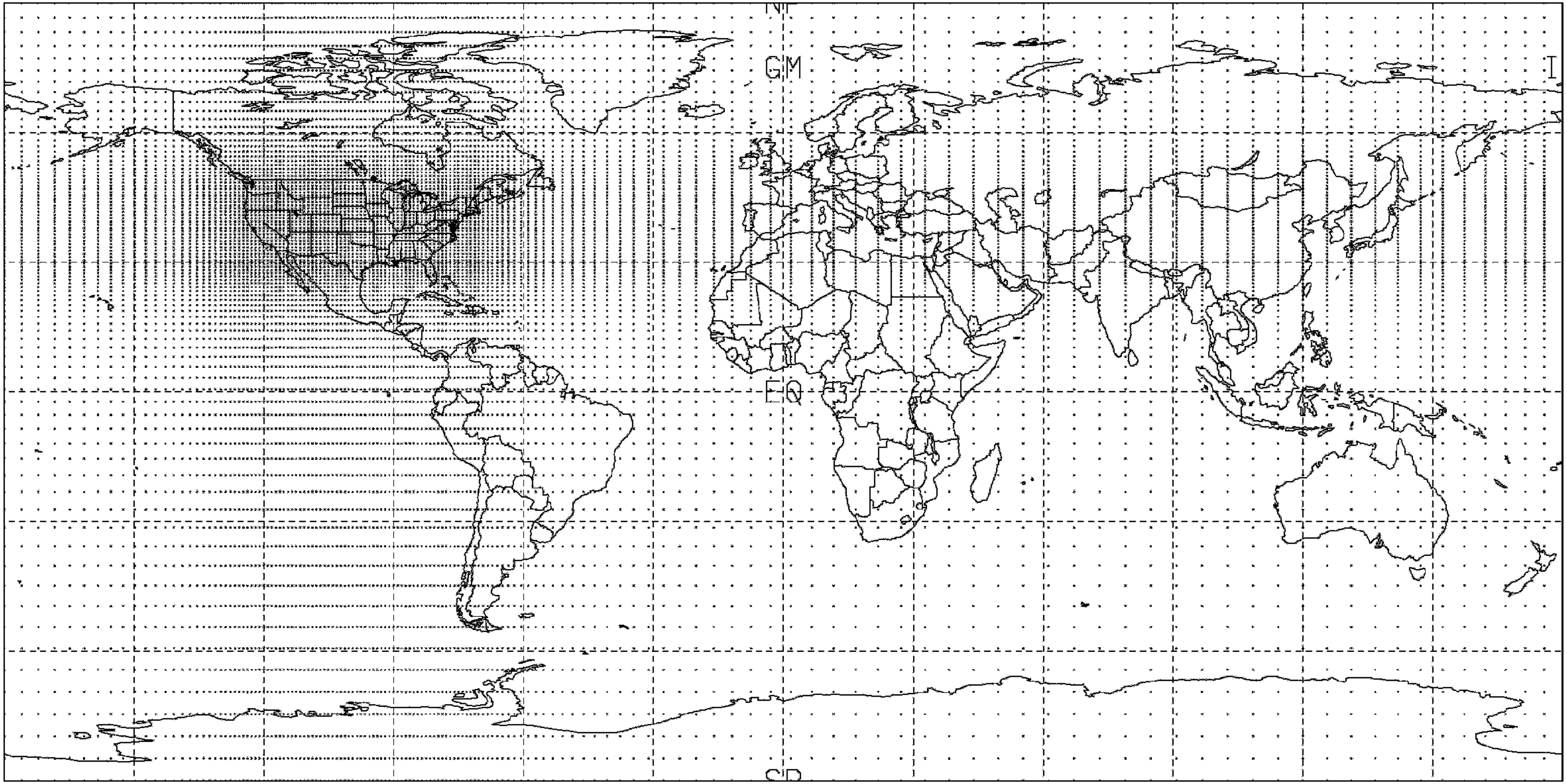
Δ = 23 km (France)

Δ = 133 km (antipodes)

Action de
Recherche
Petite
Echelle
Grande
Echelle

(CAS 2003, Annecy; transformation based on Schmidt 1977, 1982, Beiträge Phys. Atmosph.)

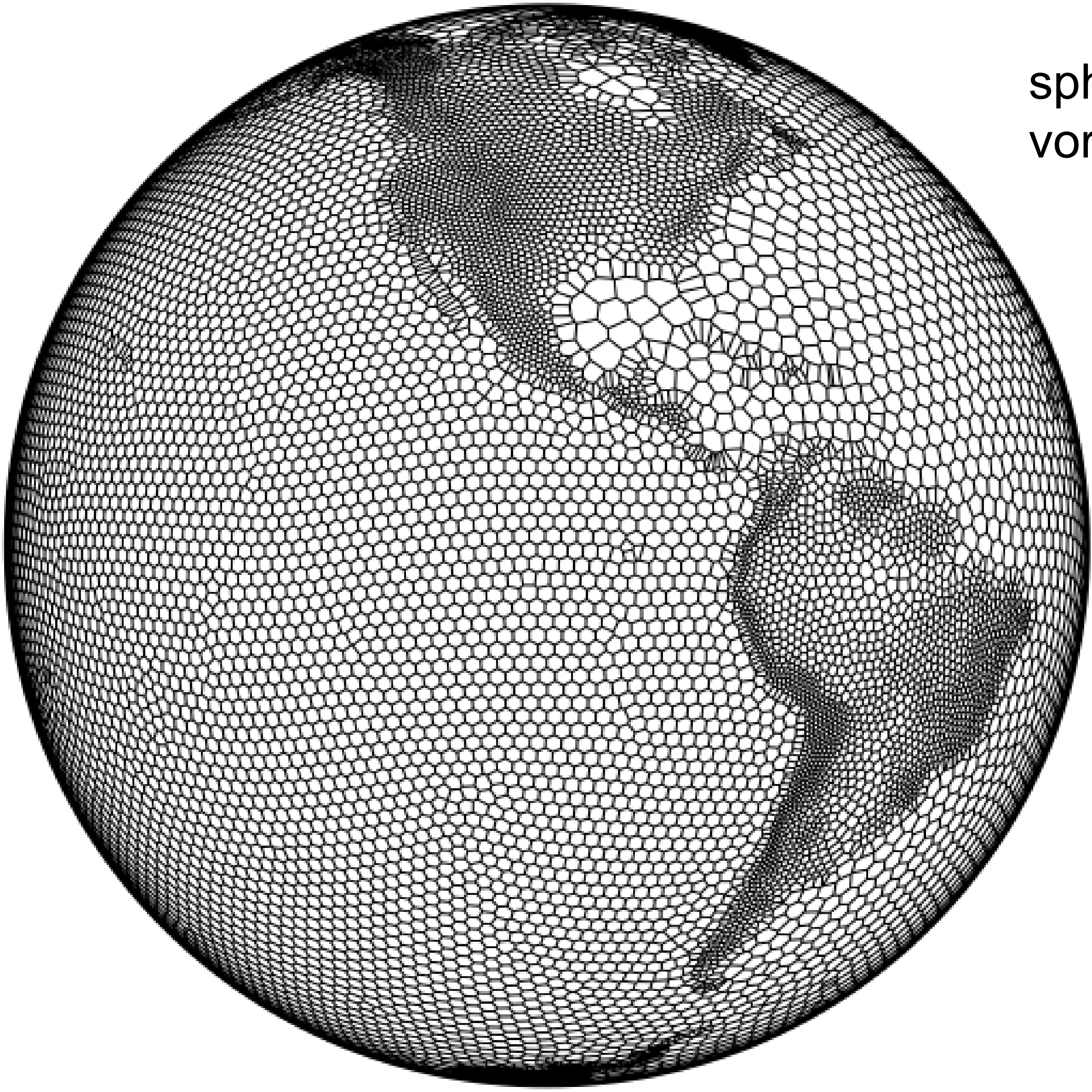
Latitude / longitude mesh with variable resolution



Grid has spacing between 55 and 250 km
(GEOS Model, NASA/GSFC)

(Fox-Rabinowitz 2005, 2008)

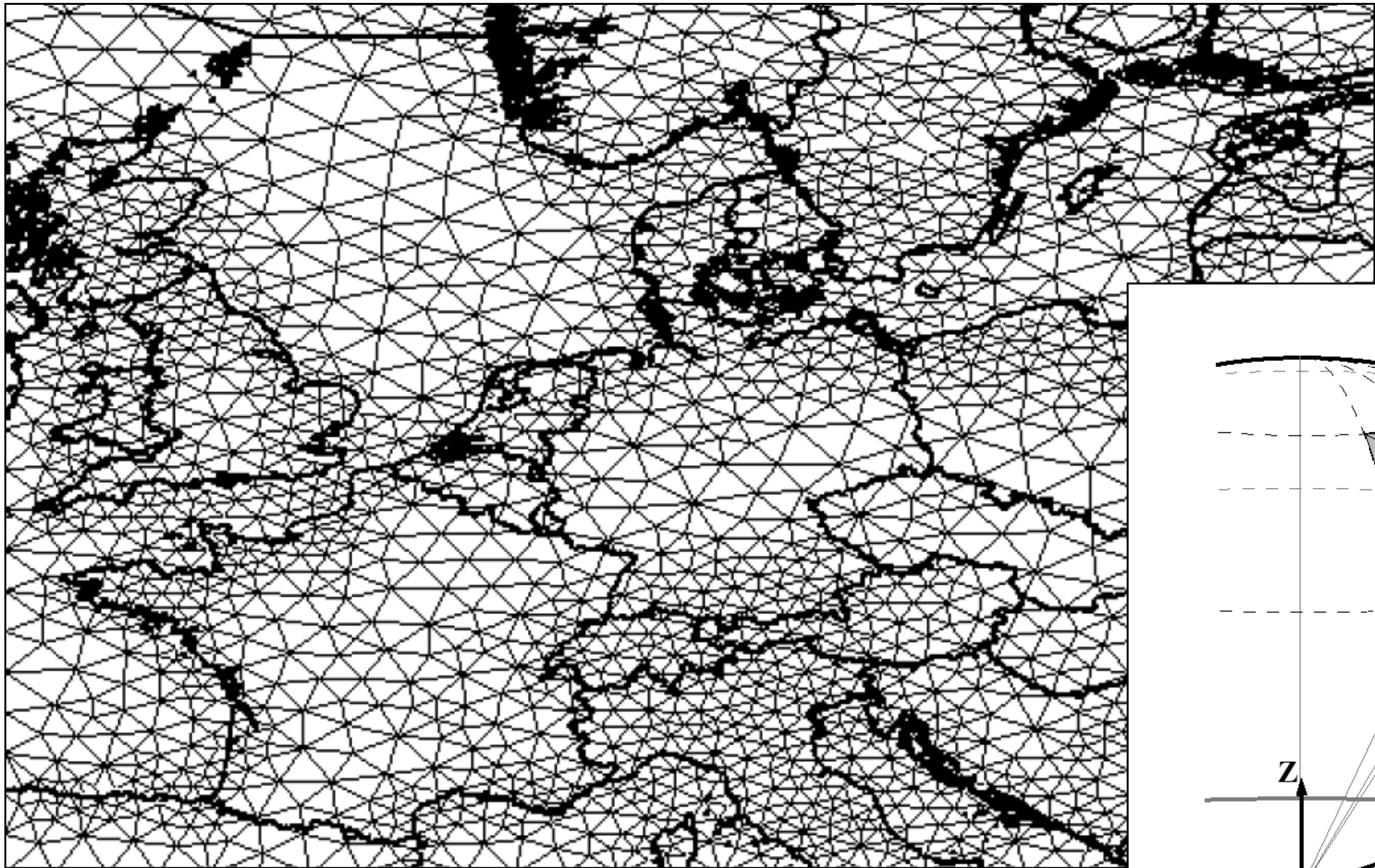
Mesh refinement based on topograhpy

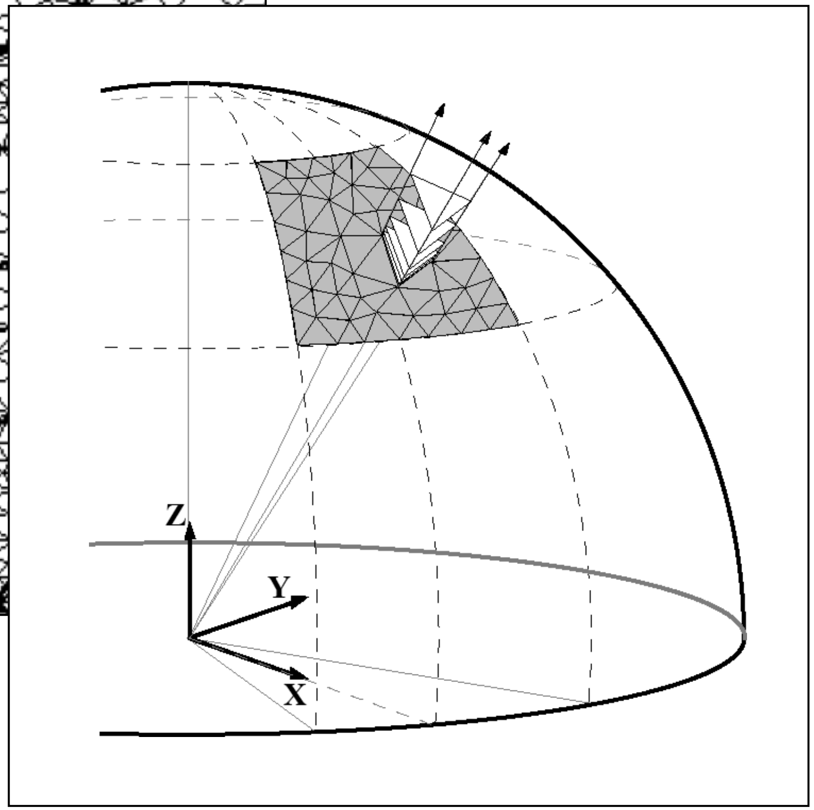


spherical centroidal
voronoi tessellation

(Michael Duda, NCAR, MMM)

Unstructured grids

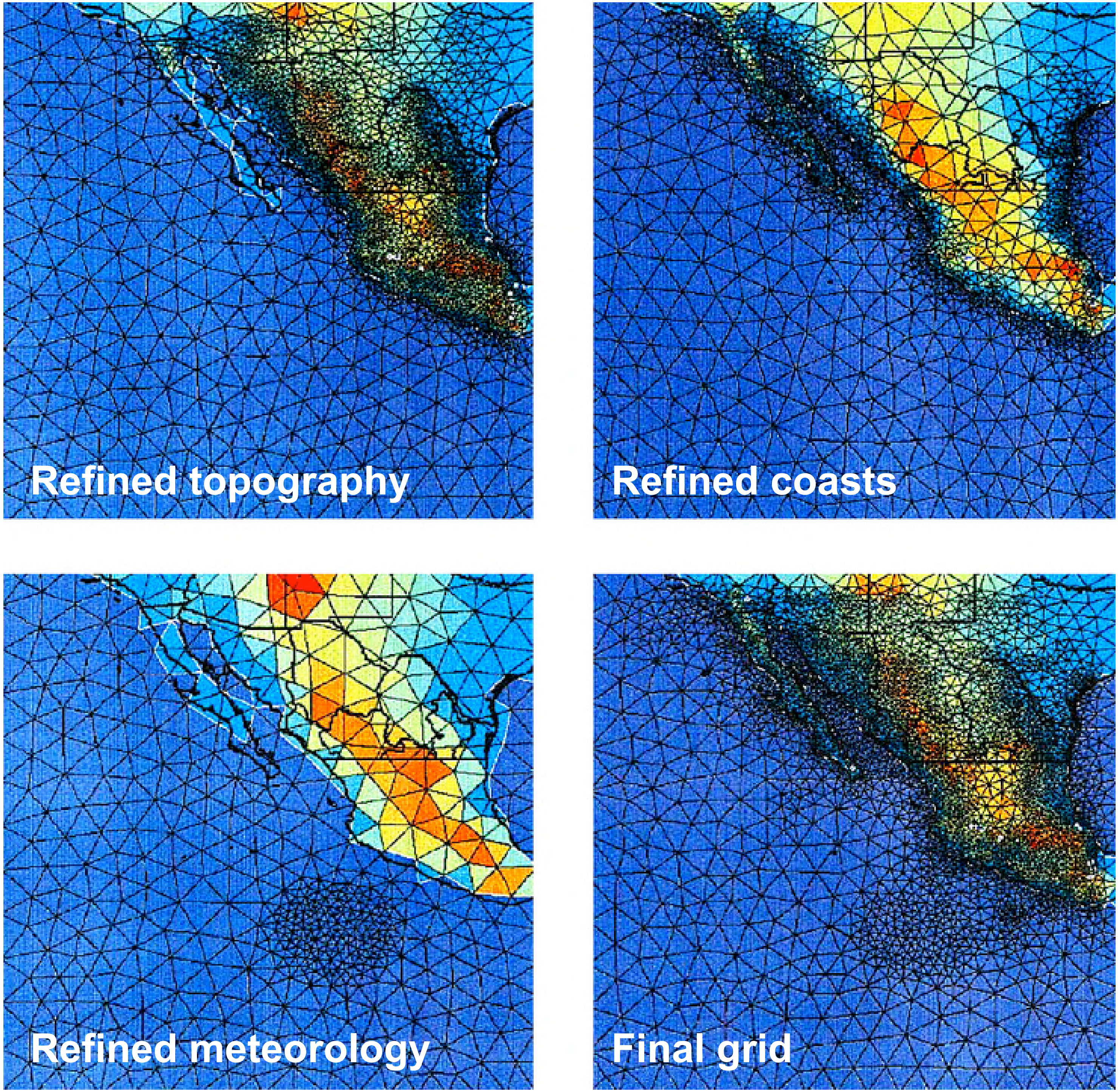




Unstructured grids allow for the enhancement of coast-lines, topography, etc. They require the explicit definition of each computational cell.

http://vortex.atgteam.com/

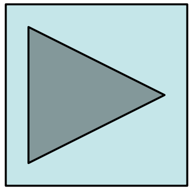
Adaptive grids

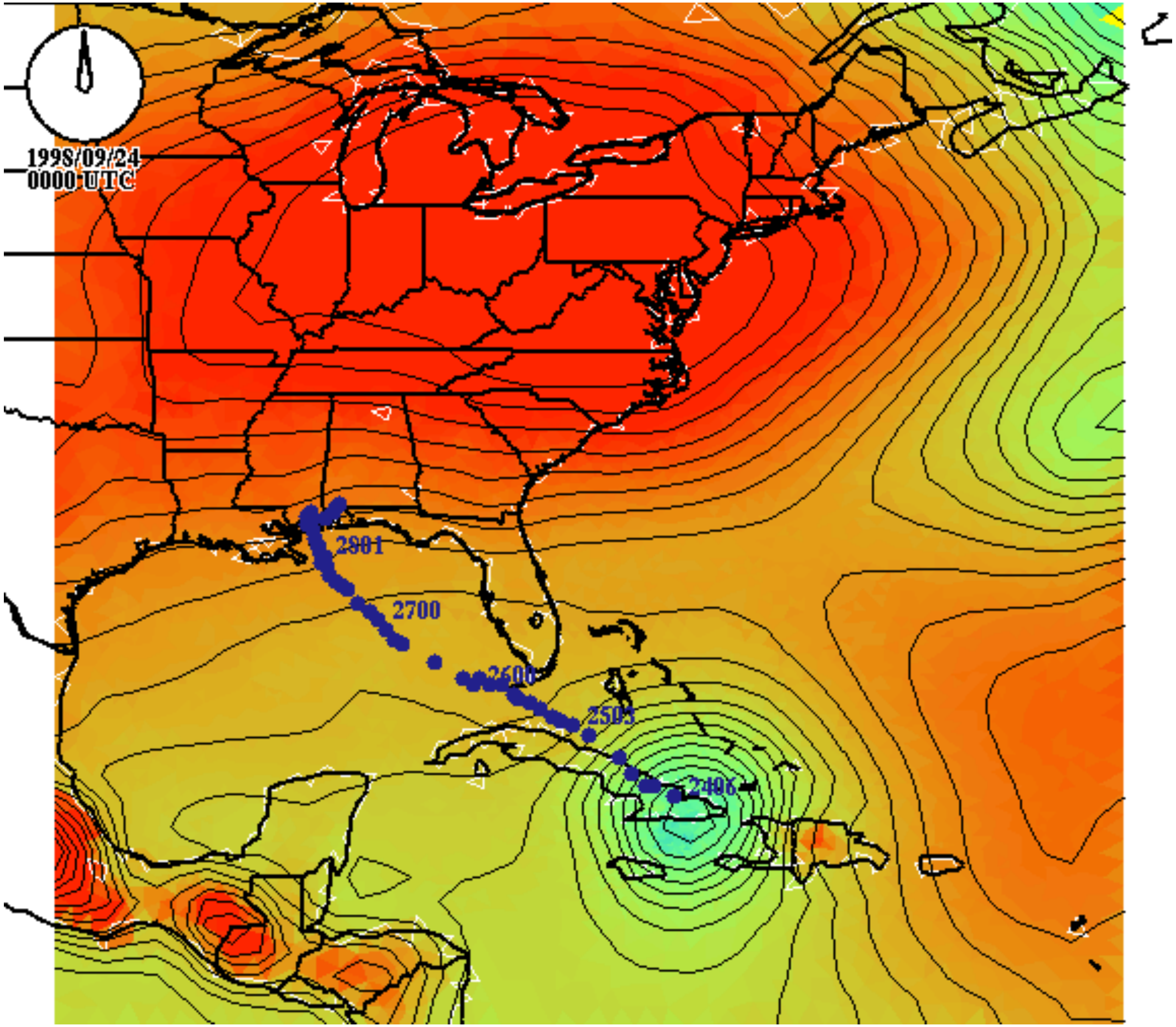


Adaptive grids dynamically adapt to the (meteorological) situation under consideration.

D.P. Bacon et al., 2000, MWR 128, 2044-2076: Operational Multiscale Environment Model with Grid Adaptivity (OMEGA). http://vortex.atgteam.com/

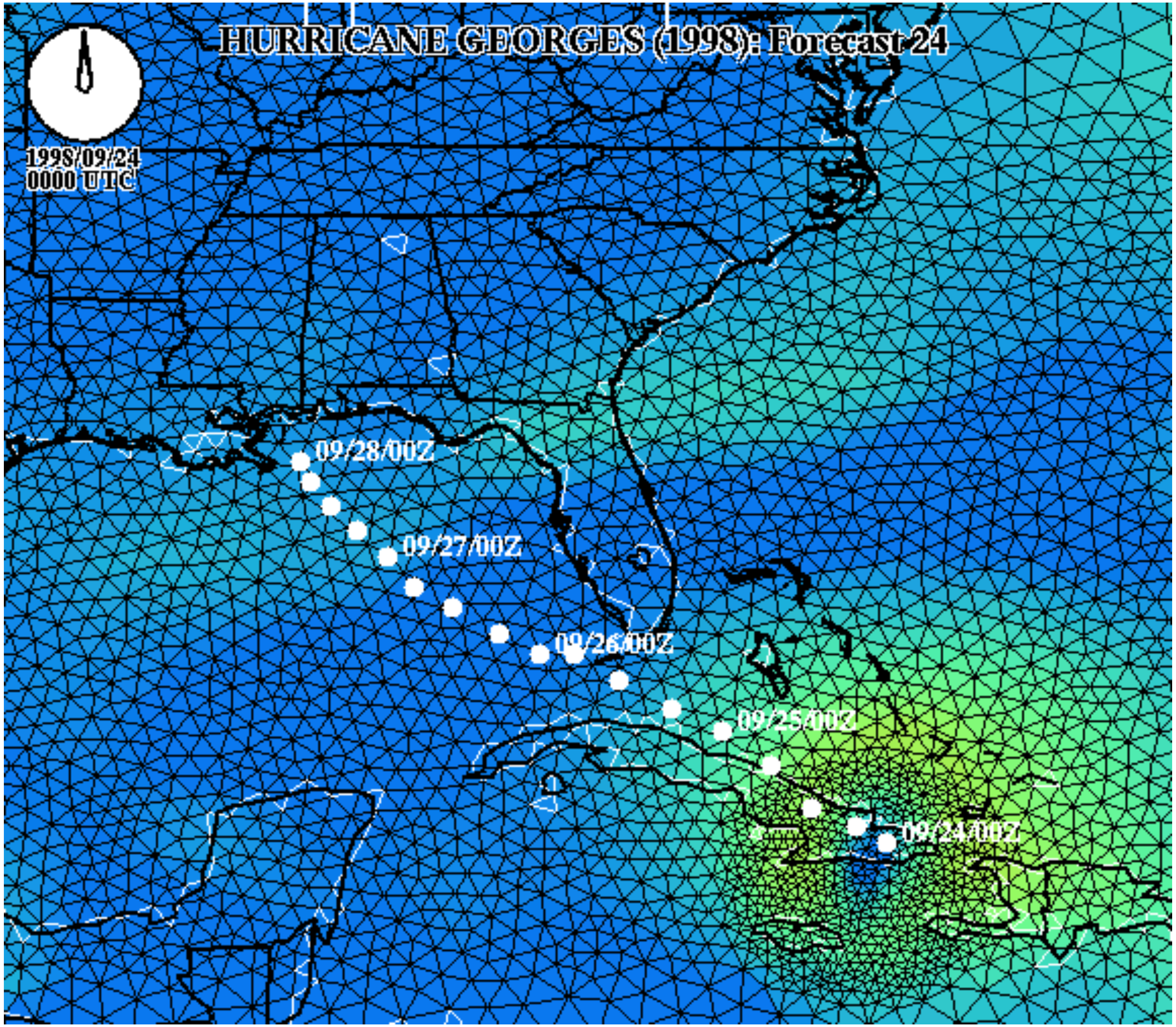
Hurricane Georges (1998) with an adaptive mesh





Surface pressure

Hurricane Georges (1998) with an adaptive mesh



Mesh

Wind speed

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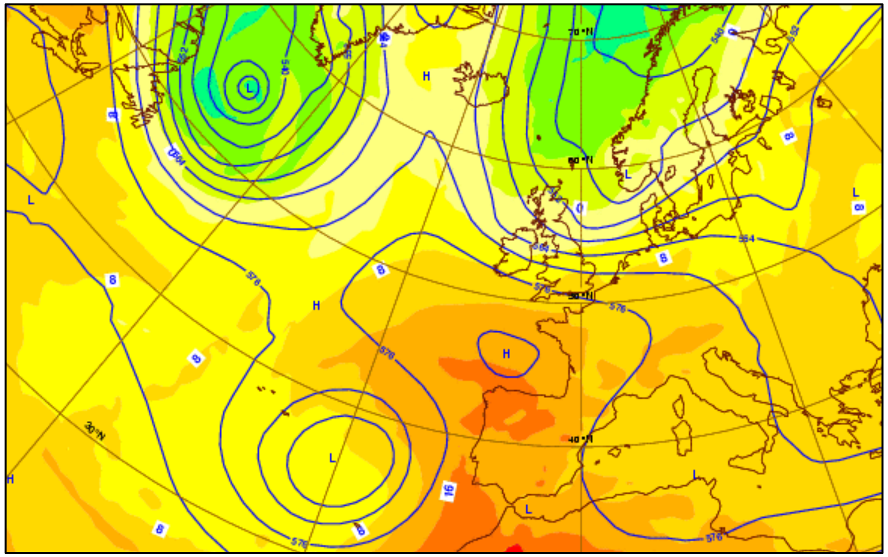
Linking “dynamics” and “physics”

„Dynamics“

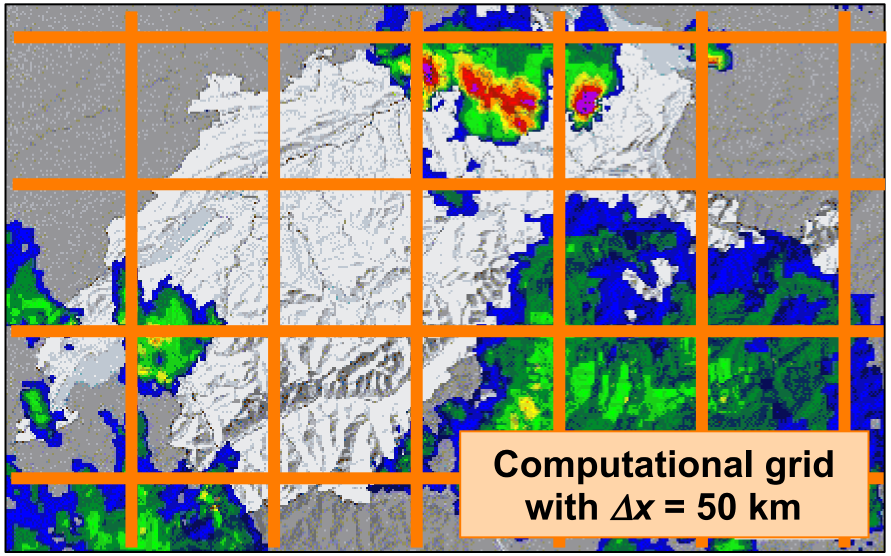


Solution of the governing equations on a computational grid.

Addresses resolved part of atmospheric dynamics and thermodynamics.

Examples of resolved structures:
- general circulation of atmosphere
- low and high pressure systems
- mountain flows, etc.

„Physics“



Representation of unresolved scales by parameterizations.

Addresses unresolved part of atmospheric dynamics and physical processes.

Examples of parameterized processes:
- boundary layer
- short-wave and long-wave radiation
- convection, precipitation and clouds

Parameterization

Resolved variables: **Variables that are represented explicitly in a model, using a governing set of equations.**

In pressure coordinates this includes:

$u, v, \omega, T, \phi, p_{srf}$ **and derived quantities (e.g. ρ).**

Parameterization: **Representation of unresolved processes by means of resolved variables. For instance:**

Frictional force $\boldsymbol{F}$ due to boundary layer processes:

$$\boldsymbol{F} = \boldsymbol{F}\,(u, v, \omega, T, \ \ldots)$$

Diabatic heating H due to condensation of vapor:

$$H = H\,(u, v, \omega, T, \ \ldots)$$

Additional resolved equations:

In general, the approach requires the introduction of additional resolved quantities, e.g.:

$$\frac{Dq_{vap}}{Dt} = S_{vap} \qquad \frac{Dq_{cld}}{Dt} = S_{cld}$$

Equations for specific water vapour and cloud water content

Linking "Dynamics" and "Physics"

Governing equations in pressure coordinates

Horizontal momentum equations $\frac{Du}{Dt} - fv = -\left(\frac{\partial \phi}{\partial x}\right)_p + F^x \qquad \frac{Dv}{Dt} + fu = -\left(\frac{\partial \phi}{\partial y}\right)_p + F^y$

Hydrostatic equation $\frac{\partial \phi}{\partial p} = -\frac{1}{\rho}$

Equation of state $p = \rho R T$

Thermodynamic equation $\frac{DT}{Dt} = \frac{\omega}{\rho c_p} + H$

Continuity equation $\left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{dy}\right)_p + \omega\frac{\partial}{\partial p}$

and $\omega = \frac{Dp}{Dt}$ and $\phi \approx g_o z$

Parameterized terms	$(F_x, F_y)/\rho$	**non-conservative forces**
	H/c_p	**diabatic heating rate**

Summary

- **For global models, the pole problem implies the use of advanced numerical methods**

- **The pseudo-spectral approach is up-to-now the preferred method in global weather and climate models.**

  At low resolution, it combines computational accuracy with a simple and fully isotropic representation of the data.

  At very high resolution, it suffers from the computational load implied by the spectral (Legendre) transforms.

- **Currently, several advanced dynamical cores are under development, testing and early use. New approaches include:**

  almost isotropic grid-point models (e.g. spherical cube, geodesic grids)

  regional grid refinement

  unstructured and adaptive grids

  global non-hydrostatic models