Numerical Modeling of Weather and Climate

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Chapter 3: Adiabatic Formulation of Atmospheric Models

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3. Adiabatic formulation of atmospheric models

3.1 Shallow water dynamics

3.1.1 Background

The shallow water equations represent the simplest model problem for stratified fluids (a fluid with vertically decreasing density). The system of equations is employed in atmospheric and oceanic sciences to investigate elementary properties of stratified fluid dynamics.

The shallow water equations represent the dynamics of a homogeneous layer of incompressible fluid of constant density, which is confined above by a free surface. The free surface represents a discontinuous density interface, and it implies stratification effects. The one-dimensional shallow-water system is sketched in Fig.3.1.1. We make two important assumptions: First, the interaction with the overlying layer of fluid (e.g. air) is neglected. Second, it is assumed that the horizontal velocity is independent of height, i.e. $\partial u/\partial z=0$. The latter assumption restricts the validity of the system to shallow fluid layers.

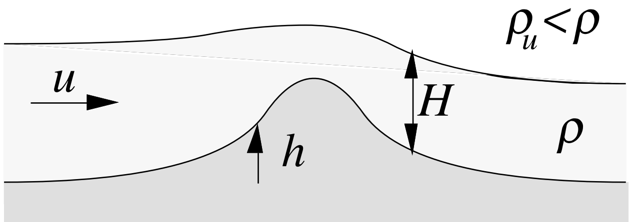


Fig.3.1.1: Variables in the one-dimensional shallow water equations: horizontal velocity u, layer depth H, height of topography h.

In the one-dimensional case and in the absence of background rotation, the shallow-water equations encompass a horizontal momentum equation

$$\frac{Du}{Dt}+g^*\frac{\partial(h+H)}{\partial x}=0 \quad \text{with} \quad \frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x} \ , \tag{3.1.1}$$

and a continuity equation

$$\frac{\partial H}{\partial t}+\frac{\partial(uH)}{\partial x}=0 \ . \tag{3.1.2}$$

Here $g^* = g\Delta\rho/\rho$ with $\Delta\rho=\rho-\rho_u$ denotes the reduced gravity. For a lake or river (where ρ and ρ_u represent water and air, respectively), we have $g^*\approx g$. However, the shallow water equations may be used to represent a wide range of other two-layer systems. For instance, an atmospheric two-layer structure may be represented as $g^* = g\Delta\theta/\theta$, where $\Delta\theta$ denotes the potential temperature contrast across the interface between the two layers.

The shallow water equations support propagating shallow-water waves. In the approximation considered, they are non-dispersive and have phase (and group) velocities given by

$$c = \sqrt{g^* H} \tag{3.1.3}$$

An associated non-dimensional parameter is the *Froude number*

$$Fr = \frac{|u|}{\sqrt{g^* H}} \ , \tag{3.1.4}$$

which is defined as the ratio between advective and phase velocities. The Froude number is used to distinguish between two major flow regimes. Flows with $Fr < 1$ are referred to as *subcritical*. In these flows, the group velocity exceeds the advective velocity. Thus, waves are able to propagate against the mean flow. Flows with $Fr > 1$ are referred to as *supercritical*. In supercritical flow, upstream propagating waves are flushed downstream by the advective velocity. Thus, there is a kind of communication problem, in the sense that the presence of an obstacle in a supercritical flow cannot be communicated into the upstream direction. Supercritical flows can thus lead to the formation of a *hydraulic jump* (or *shock*). In analytical theory, these are discontinuities in layer depth and flow velocity. Examples of such features in the shallow water systems include hydraulic jumps in rivers and kitchen sinks (kayak sportsmen and cooks should know about these!). Related phenomena occur in supersonic booms (aviation), foehn winds and downslope windstorms (meteorology), or traffic jams (transportation). In all these examples, the key issue is the inability of wave propagation against the mean flow.

For further analysis, we cast (3.1.1-2) into a dimensionless form by choosing the following scales:

- a horizontal scale L (e.g. defined by the half width of an obstacle),
- a vertical scale H_o (usually defined by the mean water depth), and
- a velocity scale (defined by the phase speed $\sqrt{g^* H_o}$).

The resulting dimensionless system reads

$$\frac{Du}{Dt} + \frac{\partial (h+H)}{\partial x} = 0 \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \ , \tag{3.1.5}$$

$$\frac{\partial H}{\partial t} + \frac{\partial (uH)}{\partial x} = 0 \ , \tag{3.1.6}$$

and the Froude number simplifies to $Fr = |u| / \sqrt{H}$.

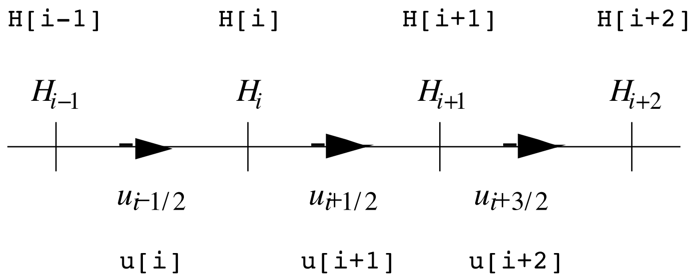
3.1.2 Integration in advective form using centered differences

A simple scheme to integrate the shallow-water equations uses centered differences in space and time. The accuracy of the scheme can be improved by using a *staggered grid* defined by

$$H_j^n = H(x = j\Delta x,\ t = n\Delta t)\ , \tag{3.1.10}$$

$$u_{j+1/2}^n = u(x = (j+1/2)\ \Delta x,\ t = n\,\Delta t)\ . \tag{3.1.11}$$

Thus, the grid has the following structure:



In this staggered grid, the location of the u-grid is shifted by half a grid increment in the x-direction. The two grids points are commonly referred to as mass and velocity grids, respectively. When implementing the code on a staggered grid, care is required regarding the staggered indices. In analytical terminology, we use the notation with half integers ($u_{i+1/2}$), while in the code itself only integer array indices are allowed (`u[i+1]`$=u_{i+1/2}$).

The discretization of (3.1.5-6) on this grid, using centered differencing, yields

$$\begin{aligned} \frac{1}{2\Delta t}\left[u_{j+1/2}^{n+1} - u_{j+1/2}^{n-1}\right] \ + \ \frac{u_{j+1/2}^n}{2\Delta x}\left[u_{j+3/2}^n - u_{j-1/2}^n\right] & \\ + \ \frac{1}{\Delta x}\left[\left(H_{j+1}^n + h_{j+1}\right) - \left(H_j^n + h_j\right)\right] \ &= \ 0 \end{aligned}\ , \tag{3.1.12}$$

$$\begin{aligned} \frac{1}{2\Delta t}\left[H_j^{n+1} - H_j^{n-1}\right] \ + \ \frac{1}{2\Delta x}\left[u_{j+1}^n H_{j+1}^n - u_{j-1}^n H_{j-1}^n\right] = 0 & \\ \text{with} \quad u_j^n = \frac{1}{2}\left(u_{j-1/2}^n + u_{j+1/2}^n\right) & \end{aligned}\ , \tag{3.1.13}$$

This is an explicit scheme, i.e. (3.1.12-13) can be solved for u^{n+1} and H^{n+1}.

The scheme (3.1.12-13) is stable, but in complex circumstances (e.g. hydraulic jumps), additional steps to preserve stability are needed. This may include an Asselin filter to suppress the computational mode of centered time stepping schemes, or some digital diffusion to suppress nonlinear instability (see Schär, 2006, chapters 4.3 and 5).

3.1.3 Integration in flux form using the Euler time step

The numerical integration (3.1.12-13) is well suited for a range of purposes (e.g. wave propagation) but it has some serious deficiencies that become important in nonlinear flows. In particular, the scheme conserves mass, but the momentum equation is not in conservative flux form, and this implies that momentum can unphysically be created or destroyed.

To suppress such artifacts, we convert (3.1.5) into its conservative flux form. To this end, we start with the product rule from differential calculus, i.e. $\partial(Hu)/\partial t = H(\partial u/\partial t) + u(\partial H/\partial t)$, and substitute from (3.1.5-6) to obtain

$$\frac{\partial(Hu)}{\partial t} + \frac{\partial[u(Hu)]}{\partial x} + H\frac{\partial(h+H)}{\partial x} = 0 . \tag{3.1.20}$$

In absence of topography ($h \equiv 0$), the third term may be written as $\partial(H^2/2)/\partial x$, and this shows that (3.1.20) is in momentum conservative flux form. In particular, integration of (3.1.20) over an interval [*a,b*] yields

$$\frac{\partial}{\partial t}\int_a^b Hudx = -\left[Hu^2 + H^2/2\right]\Big|_a^b , \tag{3.1.21}$$

This shows that the total momentum in this interval can only change in response to momentum fluxes at the boundaries. In the case of topography ($h \neq 0$) however, there is some momentum exchange with the underlying surface by pressure forces. This momentum exchange is the shallow-water analog of the mountain drag (or orographic drag), which is an important term in the atmospheric momentum balance. However, even in this case, the conservative flux form (3.1.20) is preferable, as it describes the physics of the system more realistically than its advective counterpart (3.1.5).

INTEGRATION OF THE CONTINUITY EQUATION

The continuity equation (3.1.6) is integrated using the first-order flux form of the upstream scheme (see Schär 2006, chapter 6.3).

$$\frac{H_i^{n+1} - H_i^n}{\Delta t} + \frac{F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2}}{\Delta x} = 0 \tag{3.1.23}$$

with the upstream fluxes estimated as

$$F_{i+1/2}^{n+1/2} = \begin{cases} u_{i+1/2}^n H_i^n & \text{for} \quad u_{i+1/2}^n \geq 0 \\ u_{i+1/2}^n H_{i+1}^n & \text{for} \quad u_{i+1/2}^n \leq 0 \end{cases} . \tag{3.1.24}$$

On the computer, this is implemented as

$$F_{i+1/2}^{n+1/2} = Max\left(0, u_{i+1/2}^n\right) H_i^n + Min\left(0, u_{i+1/2}^n\right) H_{i+1}^n \tag{3.1.25}$$

in order to avoid a case distinction (if statement). The main advantage of the staggered grid (3.1.10-11) becomes evident: it allows one to employ horizontal finite differences over Δx (rather than $2\Delta x$) in equation (3.1.23).

INTEGRATION OF THE MOMENTUM EQUATION

Upon defining the momentum variable q:=Hu, (3.1.20) can be written as

$$\frac{\partial q}{\partial t} + \frac{\partial(uq)}{\partial x} + H\frac{\partial(h+H)}{\partial x} = 0. \tag{3.1.26}$$

As a first step, we need to diagnose the momentum q at the mass points from the other variables as

$$q_i^n := H_i^n \frac{u_{i-1/2}^n + u_{i+1/2}^n}{2} \tag{3.1.27}$$

The simplest discretization of (3.1.26) using the flux-form of the upstream scheme then reads

$$\frac{q_i^{n+1} - q_i^n}{\Delta t} + \frac{Q_{i+1/2}^{n+1/2} - Q_{i-1/2}^{n+1/2}}{\Delta x} = H_i^n \frac{\left(h_{i+1} - h_{i-1}\right) + \left(H_{i+1}^n - H_{i-1}^n\right)}{2\Delta x} \tag{3.1.28}$$

where, analogous to (3.1.25), the upstream fluxes

$$Q_{i+1/2}^{n+1/2} = Max\left(0, u_{i+1/2}^n\right) q_i^n + Min\left(0, u_{i+1/2}^n\right) q_{i+1}^n \tag{3.1.29}$$

are employed. Once q_i^{n+1} has been computed with (3.1.28), one diagnoses the “advective velocities” on the staggered grid as

$$u_{i+1/2}^{n+1} = \frac{q_i^{n+1} + q_{i+1}^{n+1}}{H_i^{n+1} + H_{i+1}^{n+1}}. \tag{3.1.30}$$

This procedure is referred to as “momentum averaging”. It has some advantages over “velocity averaging”, which would average the diagnosed velocities at gridpoints i and i+1.

The procedure outlined above has some attractive properties. In particular, we have employed the conservative flux form, which is well suited to treat hydraulic jumps. However, the procedure also has some disadvantages. First, the use of the upstream fluxes yields a shallow-water scheme merely of first-order accuracy. Second, the treatment of the pressure gradient term is not optimal. As it employs differences over $2\Delta x$, spurious small-scale oscillations may result, in particular near hydraulic jumps. Both these disadvantages can be addressed using more sophisticated approaches.