Dynamics and Numerics of Shallow Water Flows

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Supplement to Lecture Notes
"Numerical Modeling of Weather and Climate"

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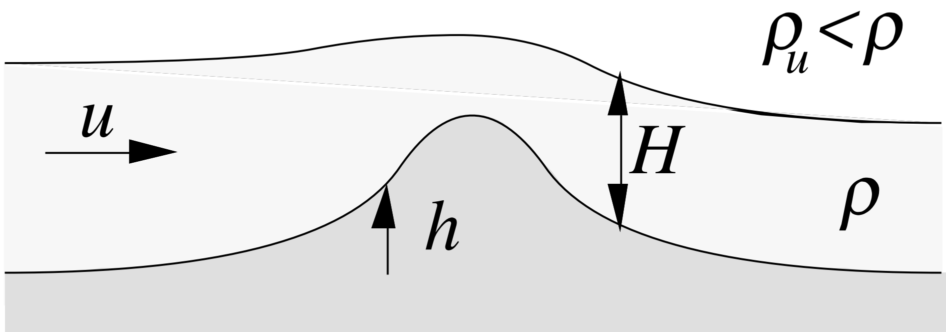
Outline:
- governing equations
- dimensionless parameters
- wave propagation, Tsunamis
- hydraulic jumps
- vortex shedding
- Seiche waves
- numerical implementation

Shallow water equations



Approximations

- Horizontal velocity u is independent of height, i.e. $u=u(x)$
- The influence of the overlaying layer of fluid (e.g. air) can be neglected.

System of equations

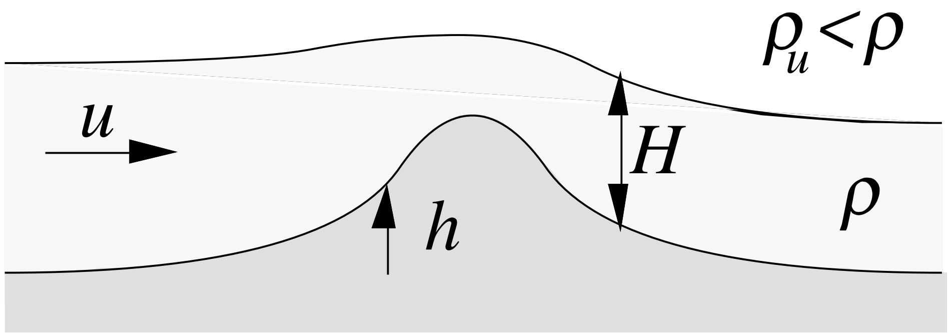
$$\frac{Du}{Dt} + g^* \frac{\partial(h+H)}{\partial x} = 0 \qquad \text{with} \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

$$\frac{\partial H}{\partial t} + \frac{\partial(uH)}{\partial x} = 0$$

Reduced gravity

$$g^* = g\frac{\Delta\rho}{\rho} = g\frac{\rho - \rho_u}{\rho}$$

Phase speed of shallow-water waves



Tsunami ($g^* = g = 10$ m/s^{-2})

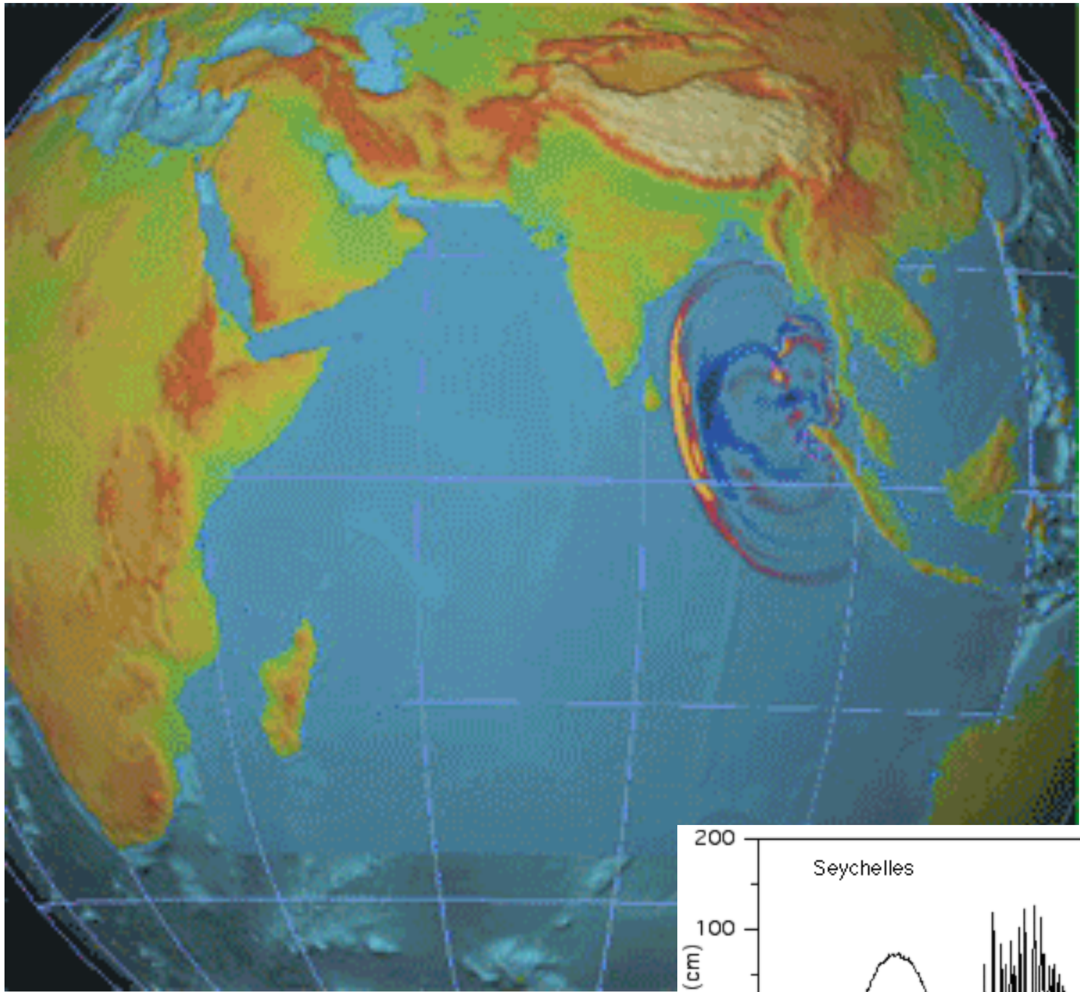
H [m]	c [m/s]	c [km/h]
1	3.2	12
10	10	36
100	32	115
1000	100	360
2000	140	504
4000	200	720
6000	245	882

Phase speed (non-rotating system, *f*=0)

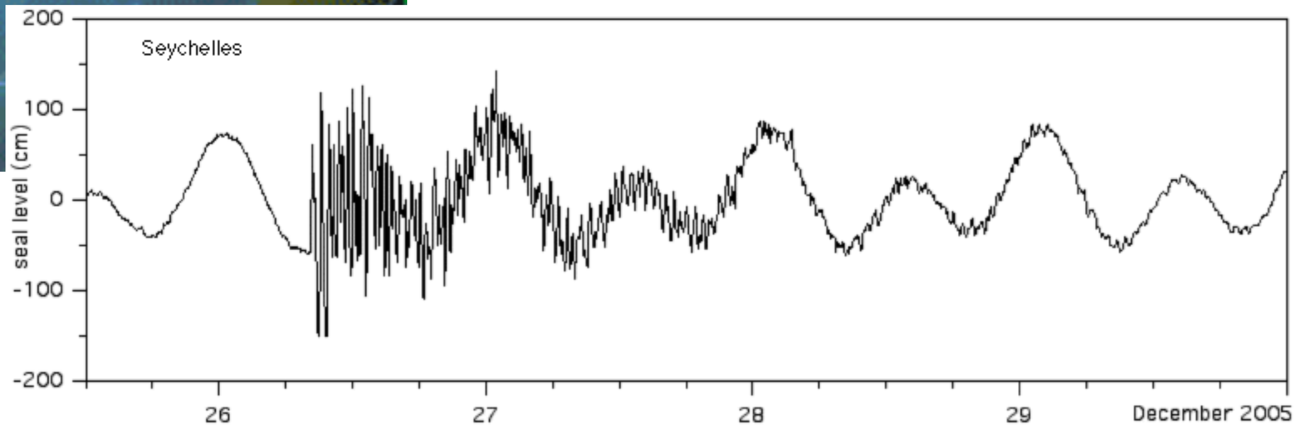
$$c = \sqrt{g^* H}$$ (non-dispersive)

$$g^* = g\frac{\Delta\rho}{\rho} = g\frac{\rho - \rho_u}{\rho}$$ (reduced gravity)

Tsunami of December 26, 2004

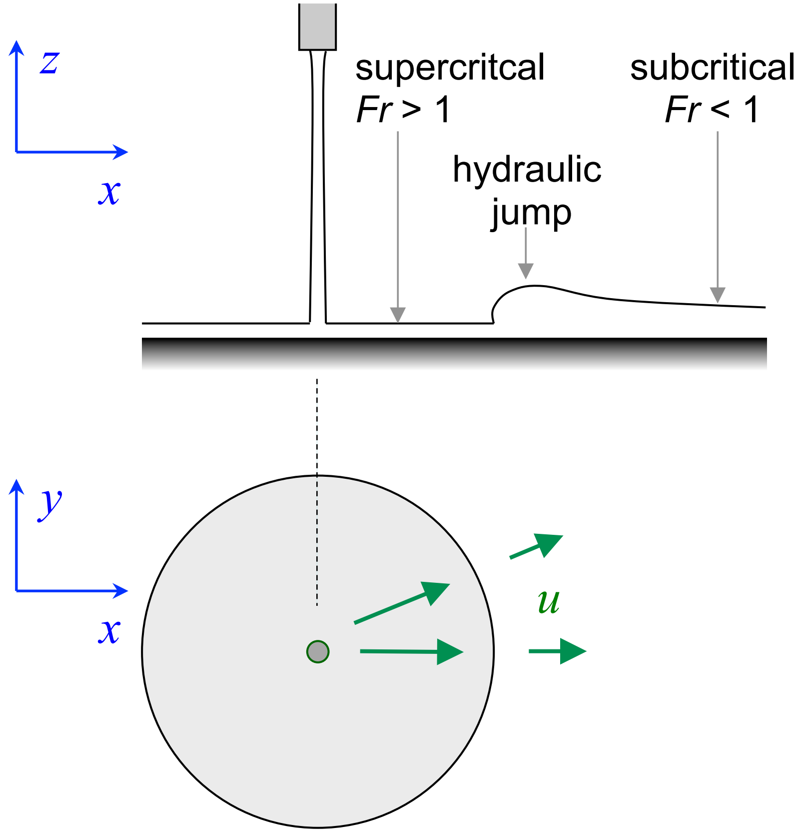


Earthquake
December 26, 2006, 00:59 UTC
Magnitude 9.1–9.3 Mw
Depth 30 km



Banda Aceh, Sumatra

Hydraulic jumps



Frounde number:

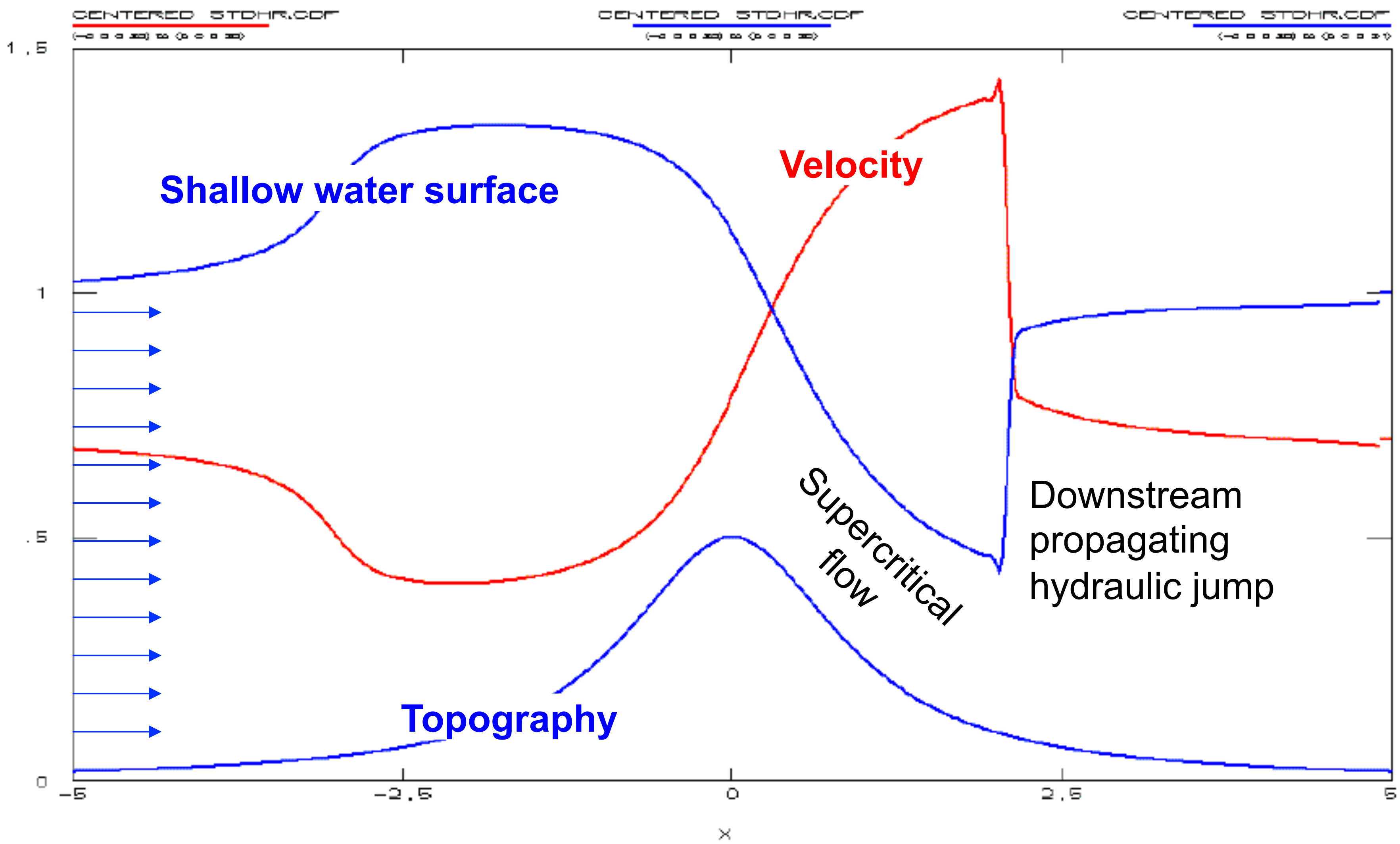
$$Fr = \frac{|u|}{c} = \frac{|u|}{\sqrt{g^* H}} = \frac{\text{advection velocity}}{\text{wave velocity}}$$

$Fr < 1$: subcritical (subsonic)

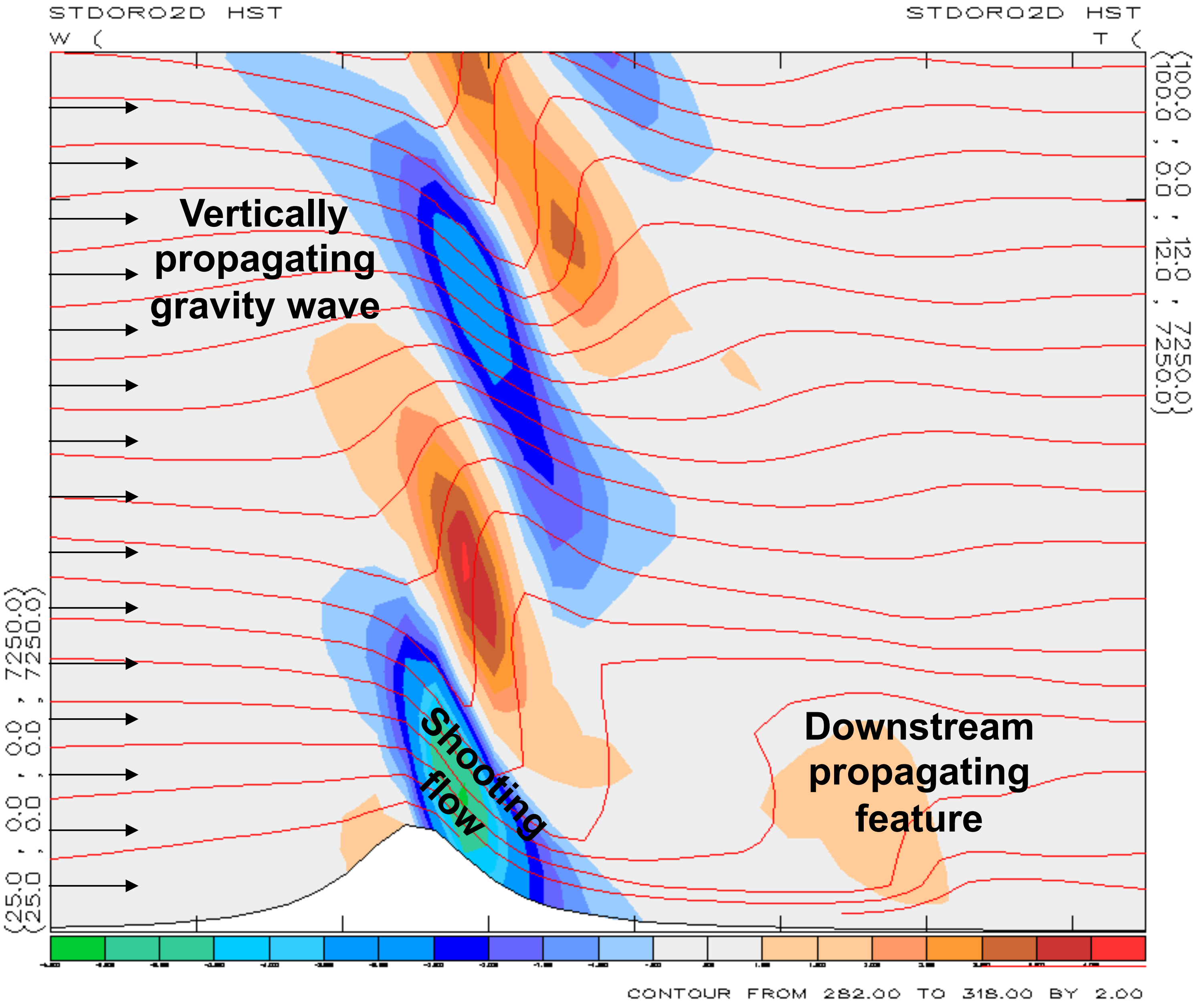
$Fr > 1$: supercritical (supersonic)

The velocity decreases with increasing distance from the point of impact (due to mass conservation). This provokes the transition from supercritical to subcritical conditions. The transition is accompanied by a hydraulic jump, dissipating some of the kinetic energy in turbulence. At the hydraulic jump, the velocity abruptly decreases, and fluid depth increases.

Shallow-water flow past a ridge



Atmospheric flow past a ridge



Contours: theta (density)

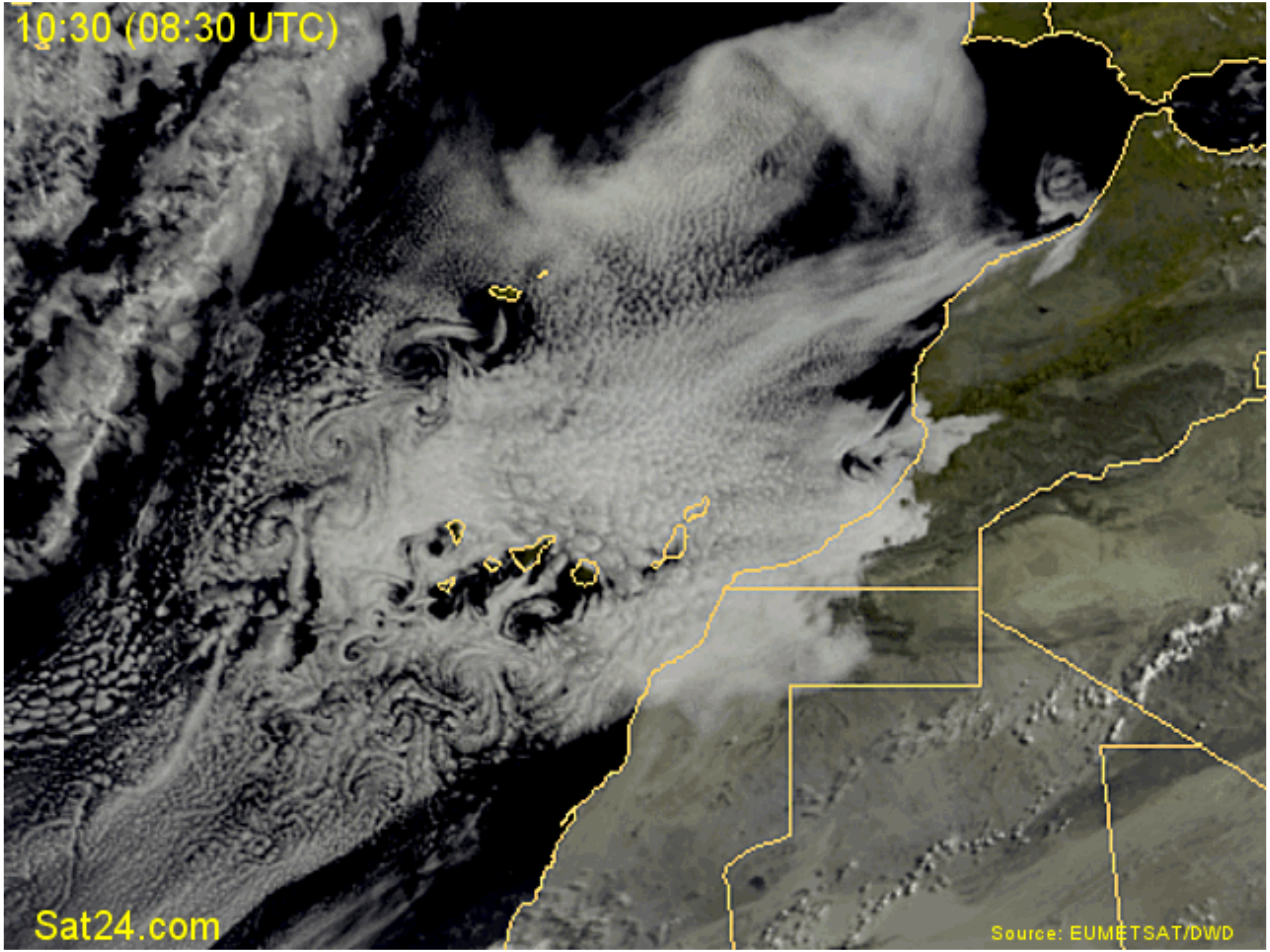
Color: vertical motion

Tidal Bore

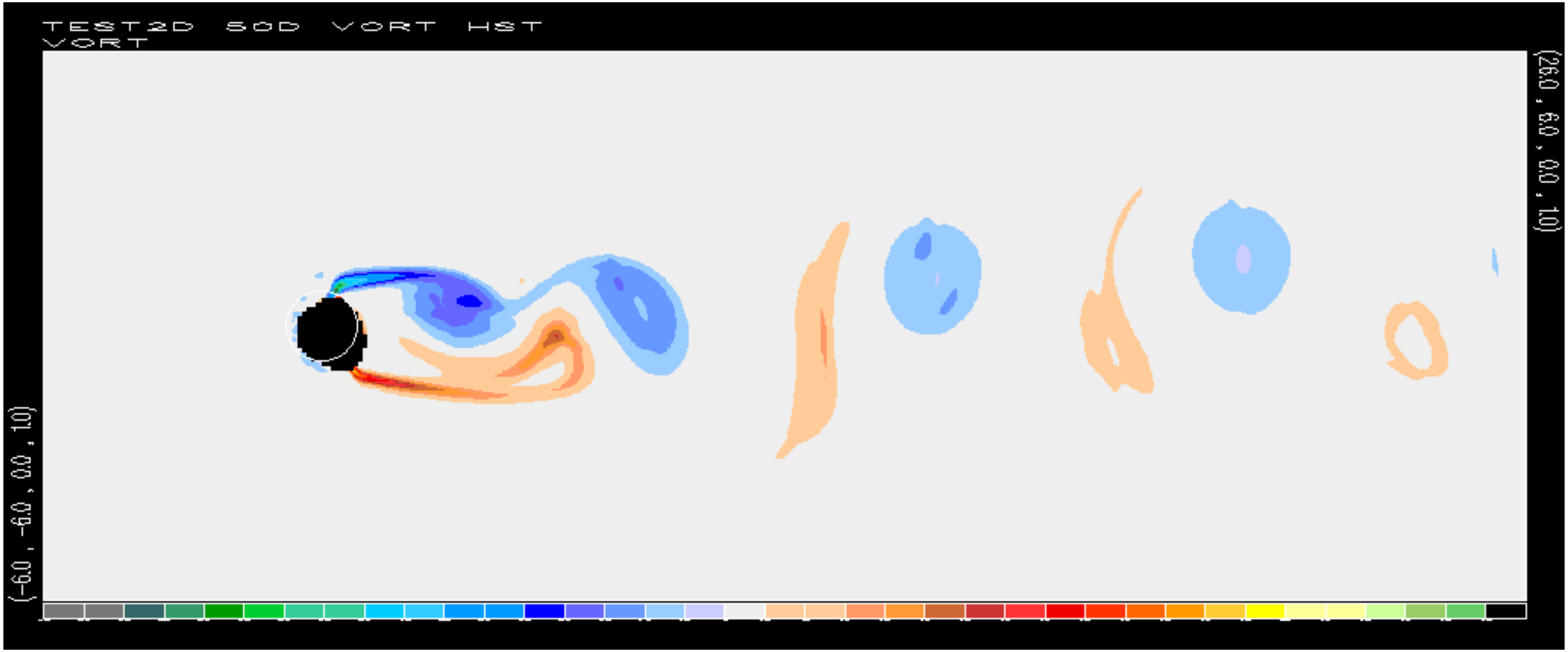

photo (C) Pierre-Yves Lagrée
Le Mascaret à Saint Pardon sur la Dordogne en Août 1997

Island Socorro (Mexiko)

Canary Islands, June 30, 2010



Shallow-water flow past an isolated mountain



Vorticity $\xi = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$\xi > 0:$

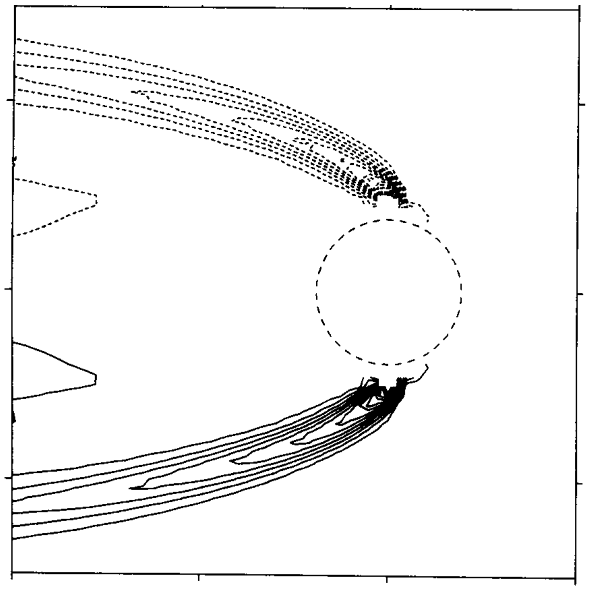
$\xi < 0:$

(Thomas Rösch, MSc ETH Zürich, Schär and Smith 1993)

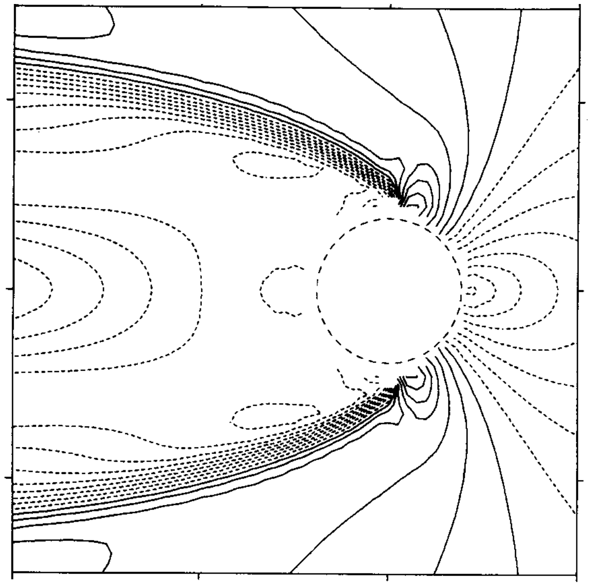
SW-Simulations of flow past isolated topography

Fr_∞=0.5, M=2

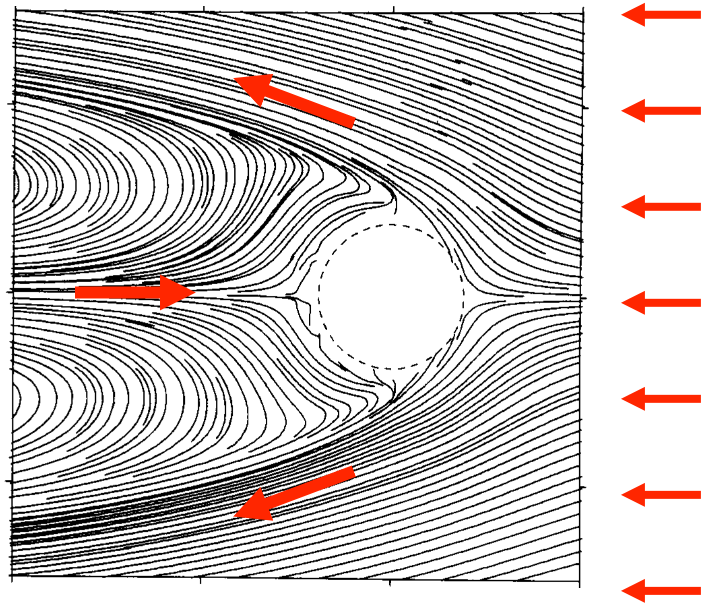
Vorticity



Windspeed

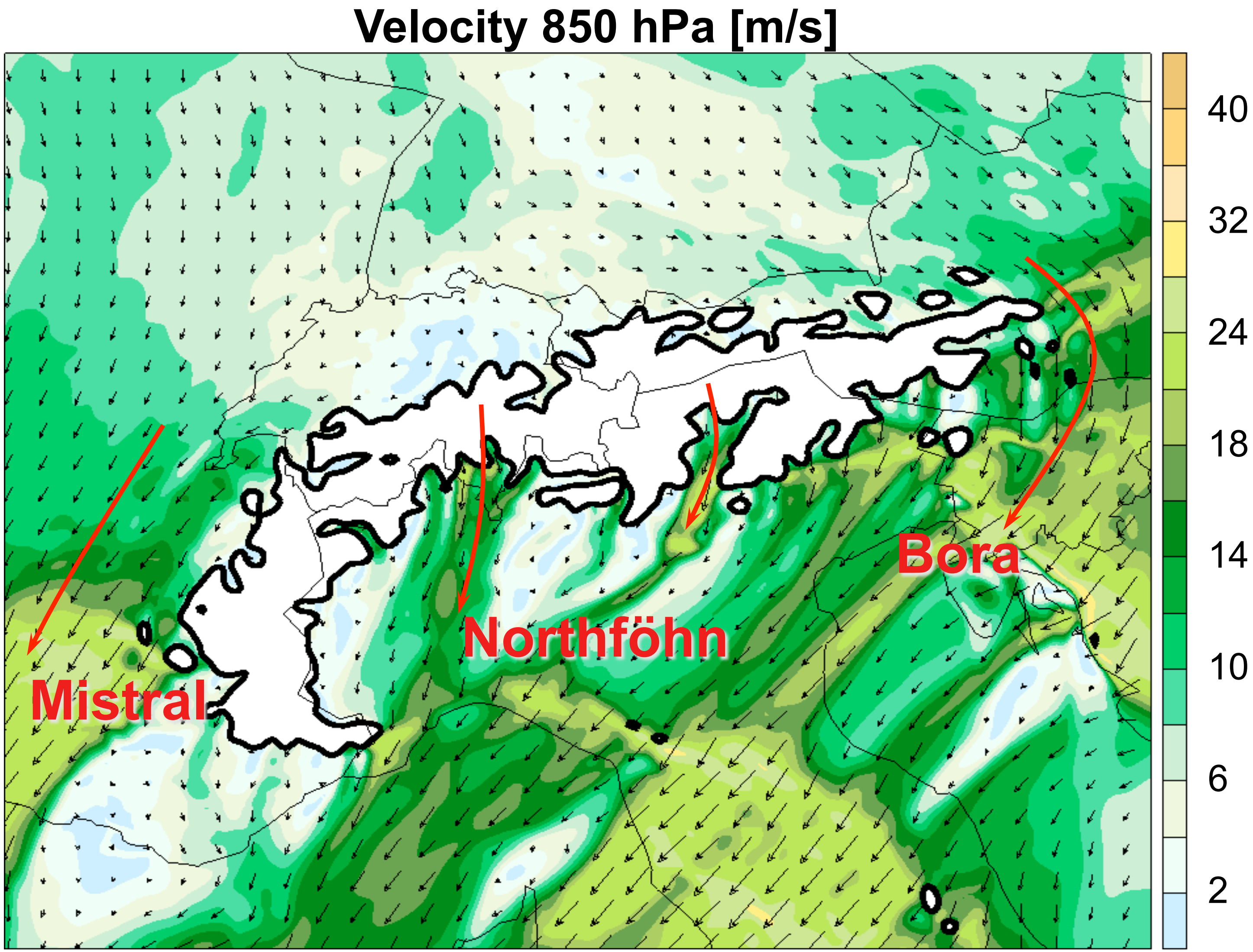


Streamlines



(Schär and Smith 1993)

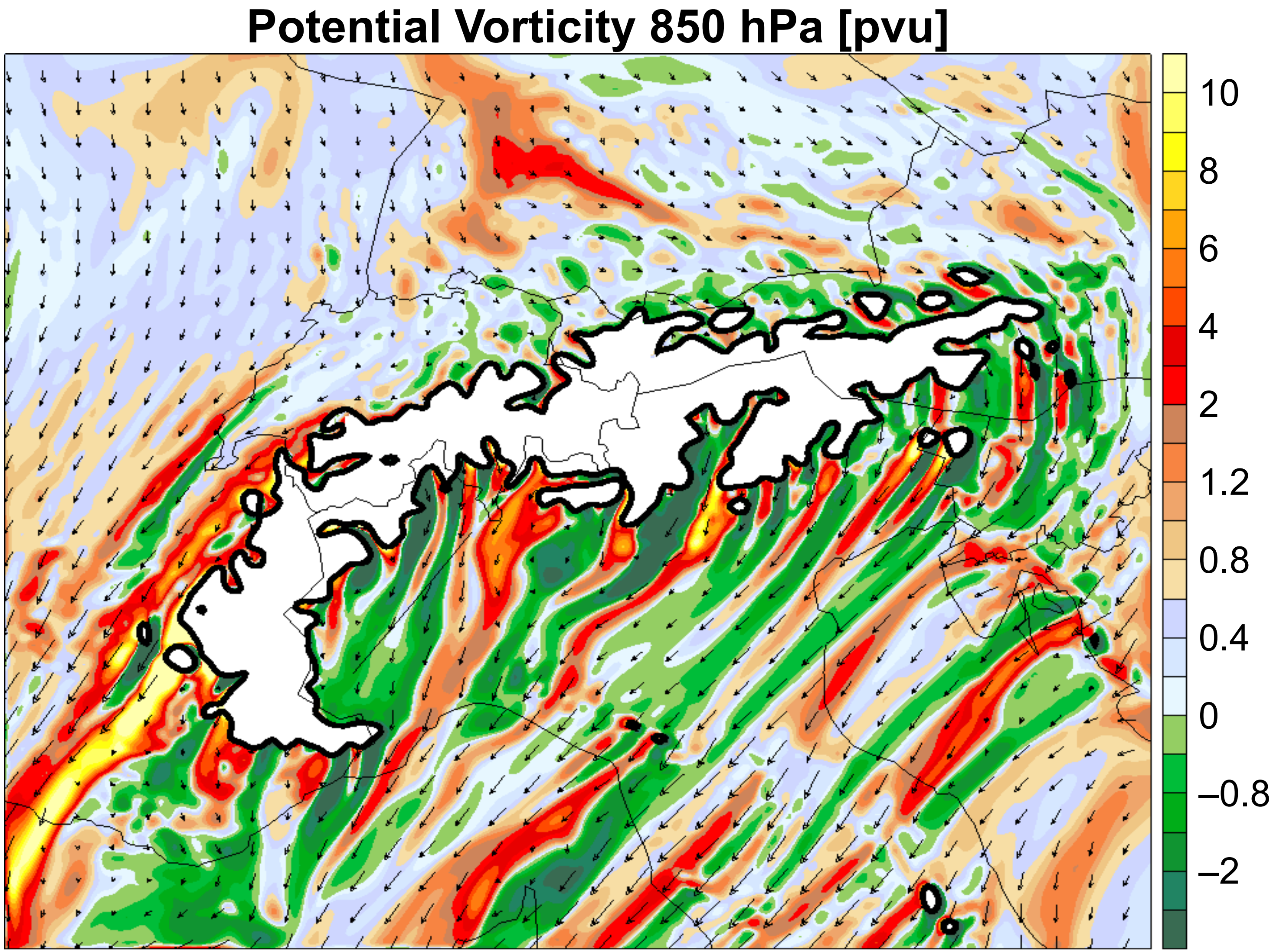
The Alpine wake …



IOP 15, November 8, 1999, 12 UTC (MC2-Model, Δ=3km)

(Benoit et al. 2002, Bull. AMS; Schär et al. 2003, QJ RMS)

… and its PV structure



IOP 15, November 8, 1999, 12 UTC (MC2-Model, Δ=3km)

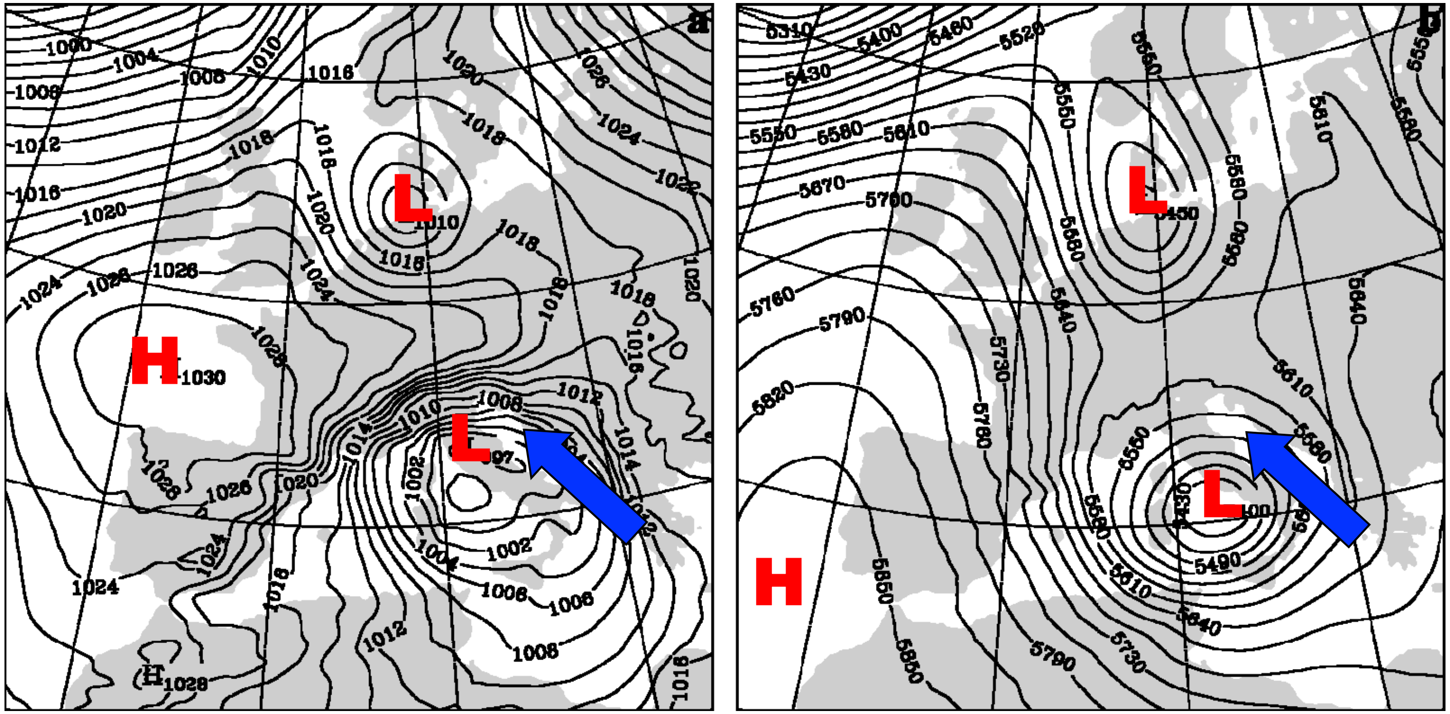
(Benoit et al. 2002, Bull. AMS; Schär et al. 2003, QJ RMS)

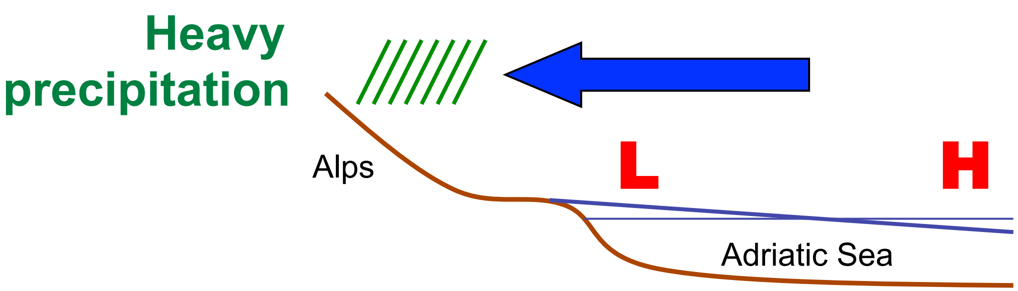
Acqua Alta, Venice

Typical synoptic situation for Acqua Alta

November 7, 1999, 06 UTC (Buzzi et al. 2003; MAP IOP 15)

Sea-Level Pressure | **Geopotential 500 hPa**

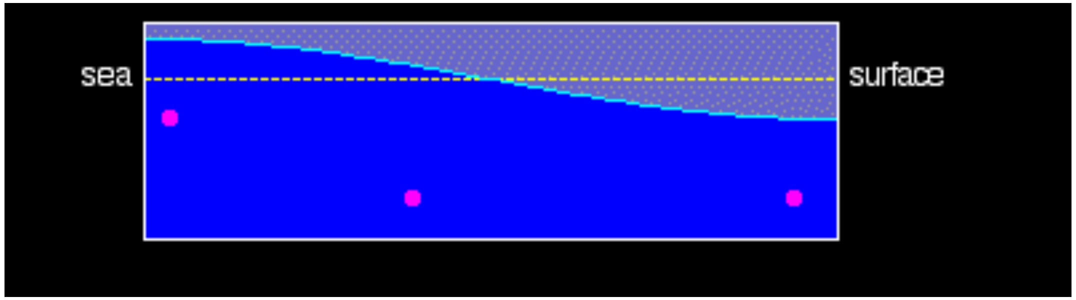




Seiche wave driven by:

- **pressure** (10 hPa = 10 cm)
- **wind**
- **runoff from precipitation**

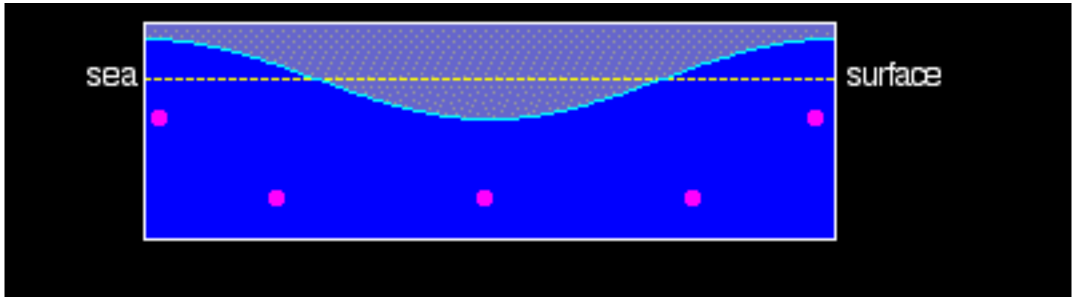
Seiche waves in SW dynamics



1st order Seiche wave

$$P = \frac{2L}{\sqrt{gH}}$$

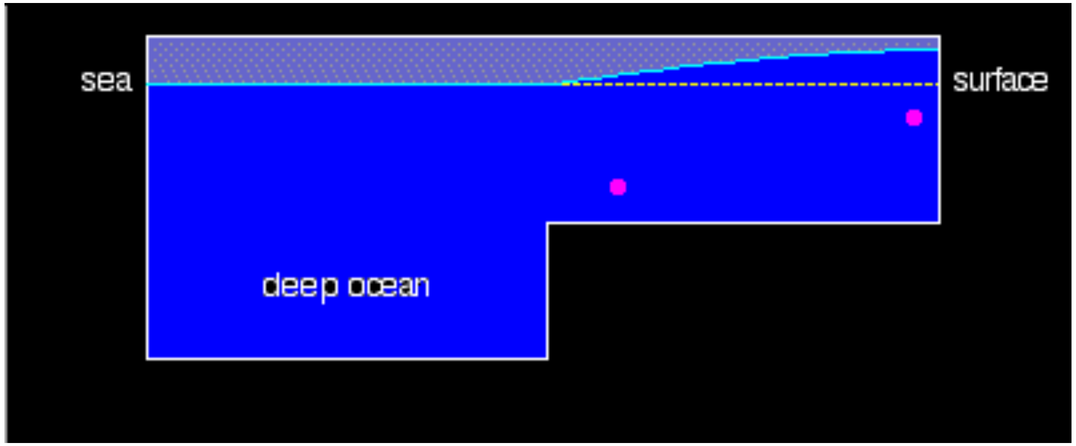
P=8.5h



2nd order Seiche wave

$$P = \frac{L}{\sqrt{gH}}$$

P=4.25h



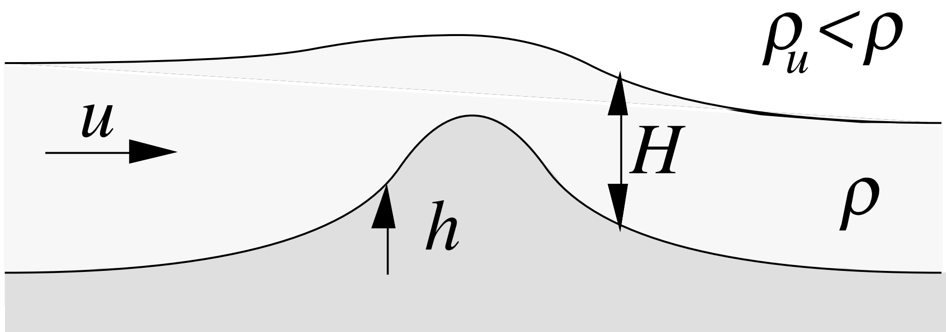
Open Seiche wave of 1st order

$$P = \frac{4L}{\sqrt{gH}}$$

P=17h

Adriatic sea:
L=700 km
H=200 m

Dimensionless formulation of shallow-water equations



Dimensional formulation

$$\frac{Du}{Dt} + g^* \frac{\partial(h+H)}{\partial x} = 0$$

$$\frac{\partial H}{\partial t} + \frac{\partial(uH)}{\partial x} = 0$$

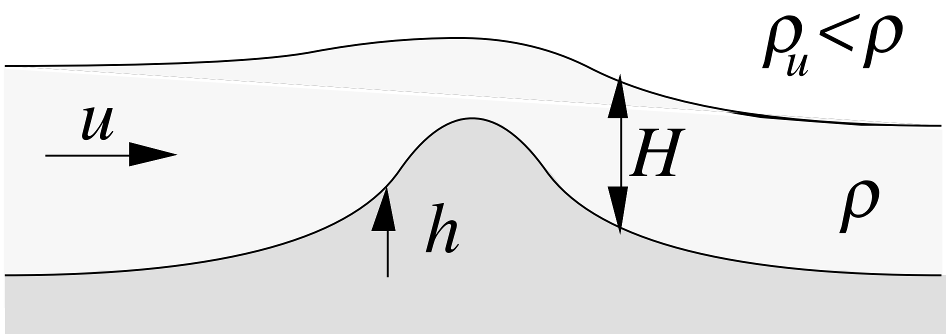
Dimensionless formulation

$$\frac{Du}{Dt} + \frac{\partial(h+H)}{\partial x} = 0$$

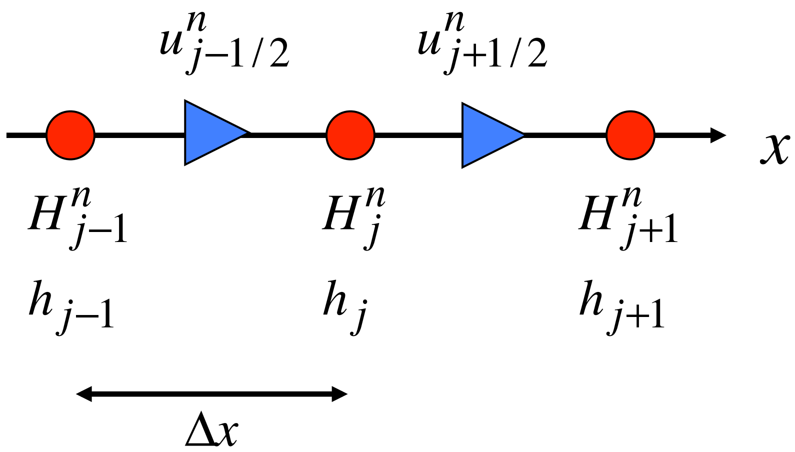
$$\frac{\partial H}{\partial t} + \frac{\partial(uH)}{\partial x} = 0$$

with $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

Numerical Implementation



Staggered grid



Array structure

u[j] u[j+1]

h[j-1] h[j] h[j+1]

Numerical Integration of Momentum Equation

Dimensionless formulation

$$\frac{Du}{Dt} + \frac{\partial(h+H)}{\partial x} = 0 \qquad \text{with} \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

Centered differences in space and time

$$\frac{1}{2\Delta t}\left[u_{j+1/2}^{n+1} - u_{j+1/2}^{n-1}\right] + \frac{u_{j+1/2}^{n}}{2\Delta x}\left[u_{j+3/2}^{n} - u_{j-1/2}^{n}\right] + \frac{1}{\Delta x}\left[\left(H_{j+1}^{n} + h_{j+1}\right) - \left(H_{j}^{n} + h_{j}\right)\right] = 0$$

Solve for time level *n*+1

$$u_{j+1/2}^{n+1} = u_{j+1/2}^{n-1} - \frac{u_{j+1/2}^{n}\ \Delta t}{\Delta x}\left[u_{j+3/2}^{n} - u_{j-1/2}^{n}\right] - \frac{2\Delta t}{\Delta x}\left[\left(H_{j+1}^{n} + h_{j+1}\right) - \left(H_{j}^{n} + h_{j}\right)\right]$$

Numerical Integration of Mass Equation

Dimensionless formulation

$$\frac{\partial H}{\partial t} + \frac{\partial (uH)}{\partial x} = 0$$

Centered differences in space and time

$$\frac{1}{2\Delta t}\left[H_j^{n+1} - H_j^{n-1}\right] \;+\; \frac{1}{2\Delta x}\left[u_{j+1}^n H_{j+1}^n - u_{j-1}^n H_{j-1}^n\right] = 0 \quad \text{with} \quad u_j^n = \frac{1}{2}\left(u_{j-1/2}^n + u_{j+1/2}^n\right)$$

Solve for time level *n*+1

$$H_j^{n+1} \;=\; H_j^{n-1} \;-\; \frac{\Delta t}{\Delta x}\left[u_{j+1}^n H_{j+1}^n - u_{j-1}^n H_{j-1}^n\right] \quad \text{with} \quad u_j^n = \frac{1}{2}\left(u_{j-1/2}^n + u_{j+1/2}^n\right)$$