Vertical Coordinate Formulations for Atmospheric Models

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Supplement to Lecture Notes
"Numerical Modeling of Weather and Climate"

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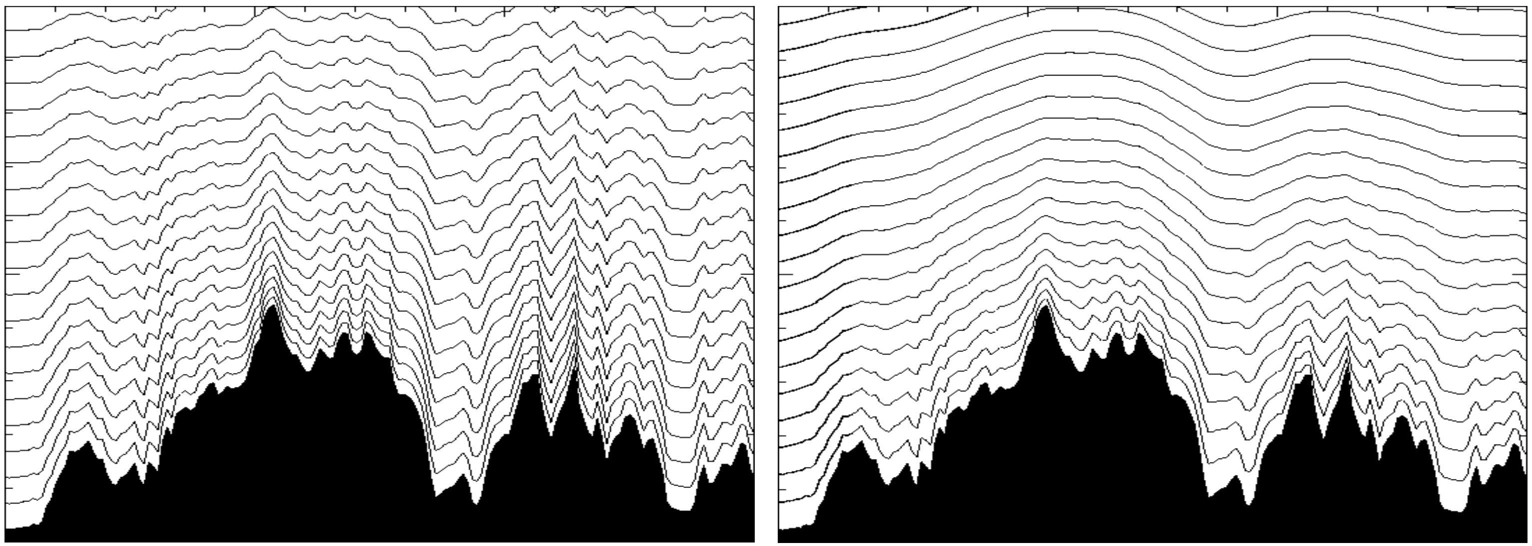


Outline

Introduction to terrain-following coordinates

Coordinate transformations

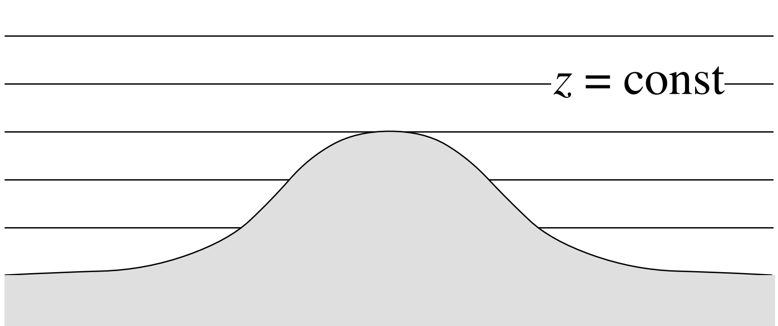
Pressure coordinates

Isentropic coordinates

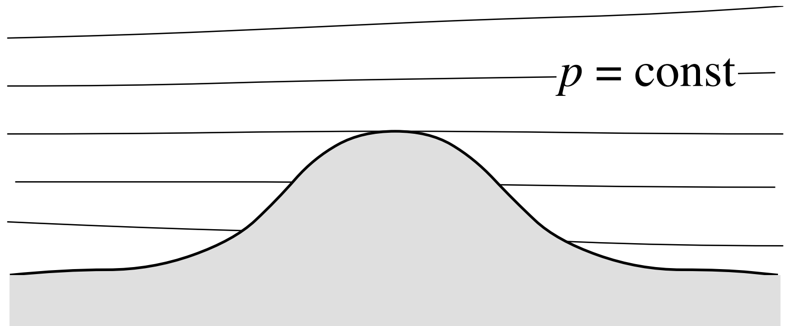
Sigma coordinates

Smooth terrain-following coordinates

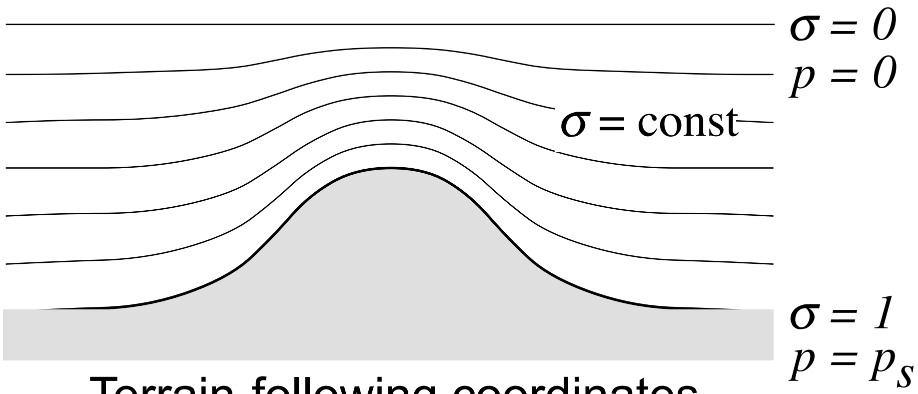
Vertical coordinates



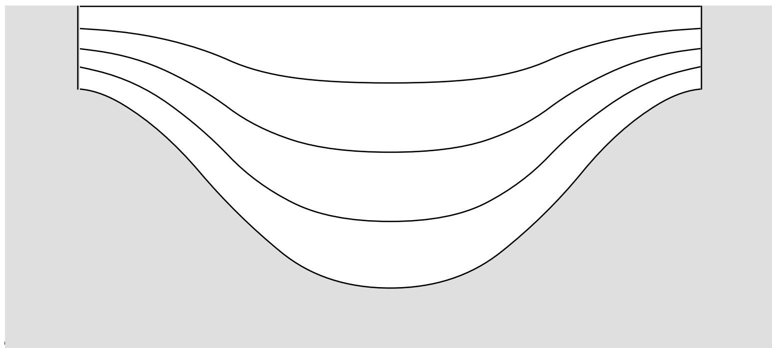
Height coordinates



Pressure coordinates



Terrain-following coordinates
(σ-coordinates)



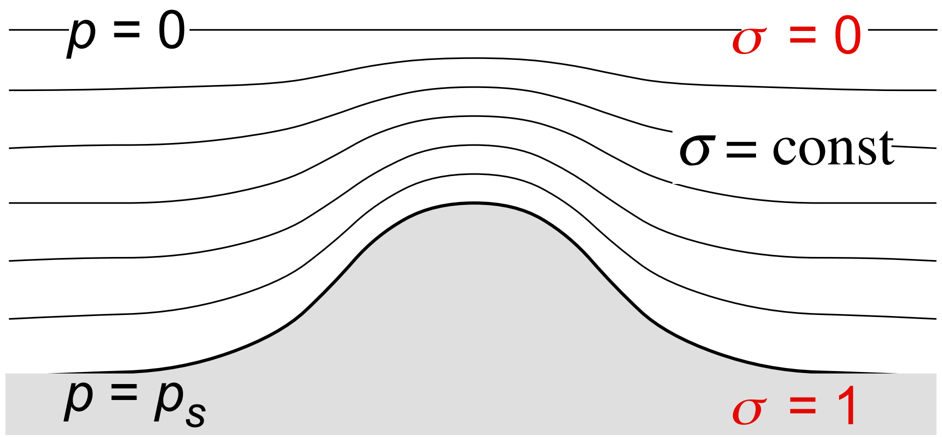
Terrain-following coordinates
in ocean models

Terrain-following Sigma coordinates

Special case: upper boundary at $p_t = 0$

Surface pressure: $p_s = p_s(x,y,t)$

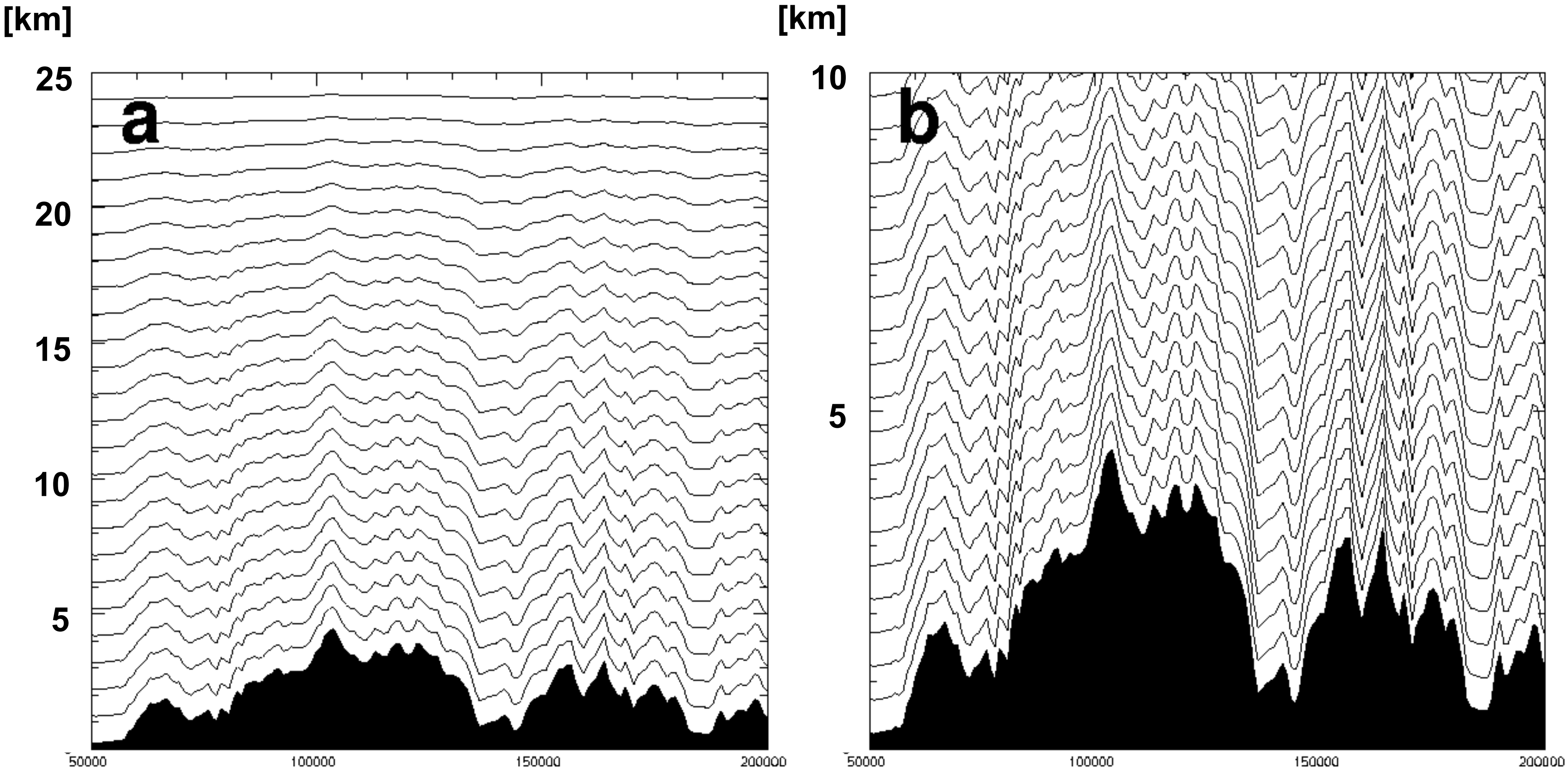
σ-coordinate: $\sigma \coloneqq \dfrac{p}{p_s}$



General case: upper boundary at $p_t = \text{const}$

σ-coordinate: $\sigma \coloneqq \dfrac{p - p_t}{p_s - p_t}$

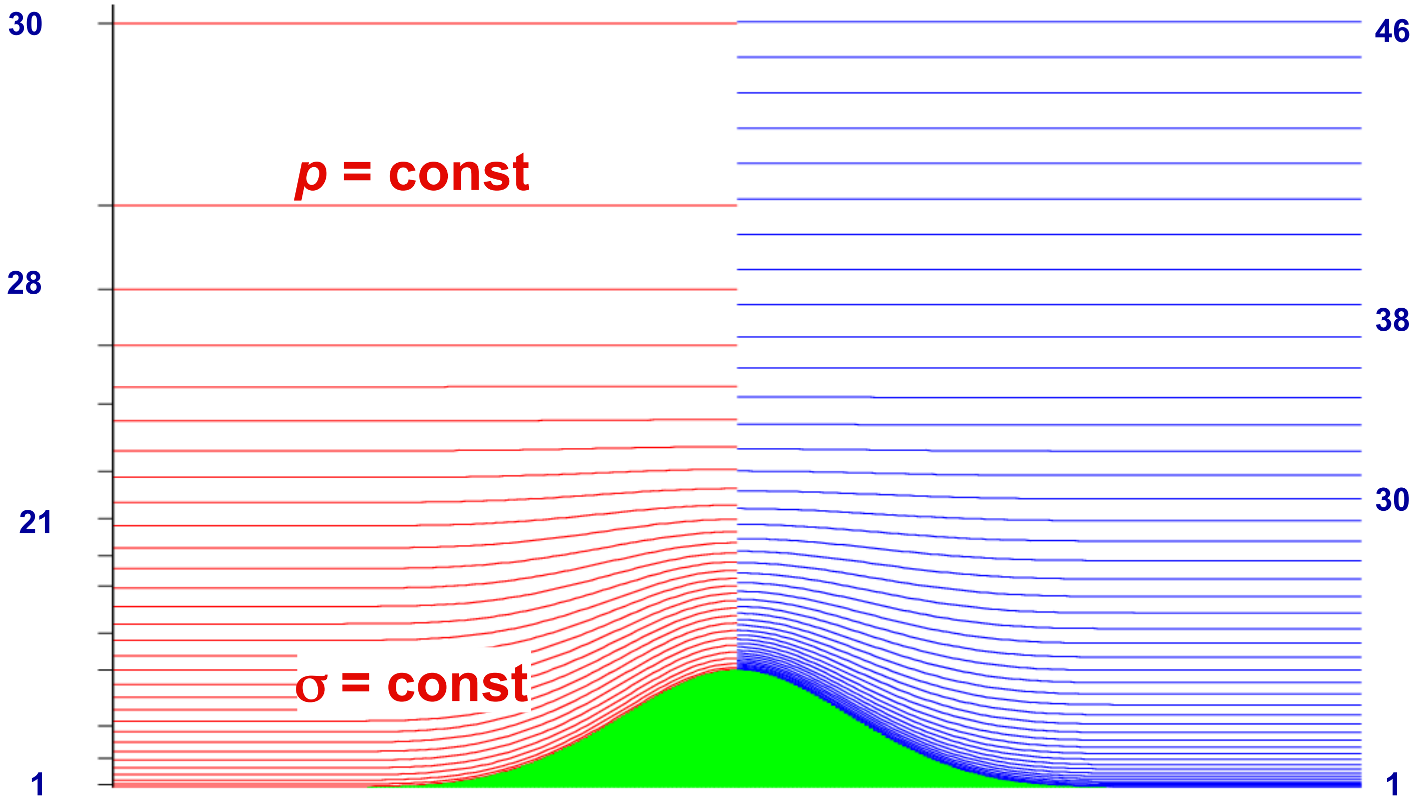
Terrain-following Sigma coordinates



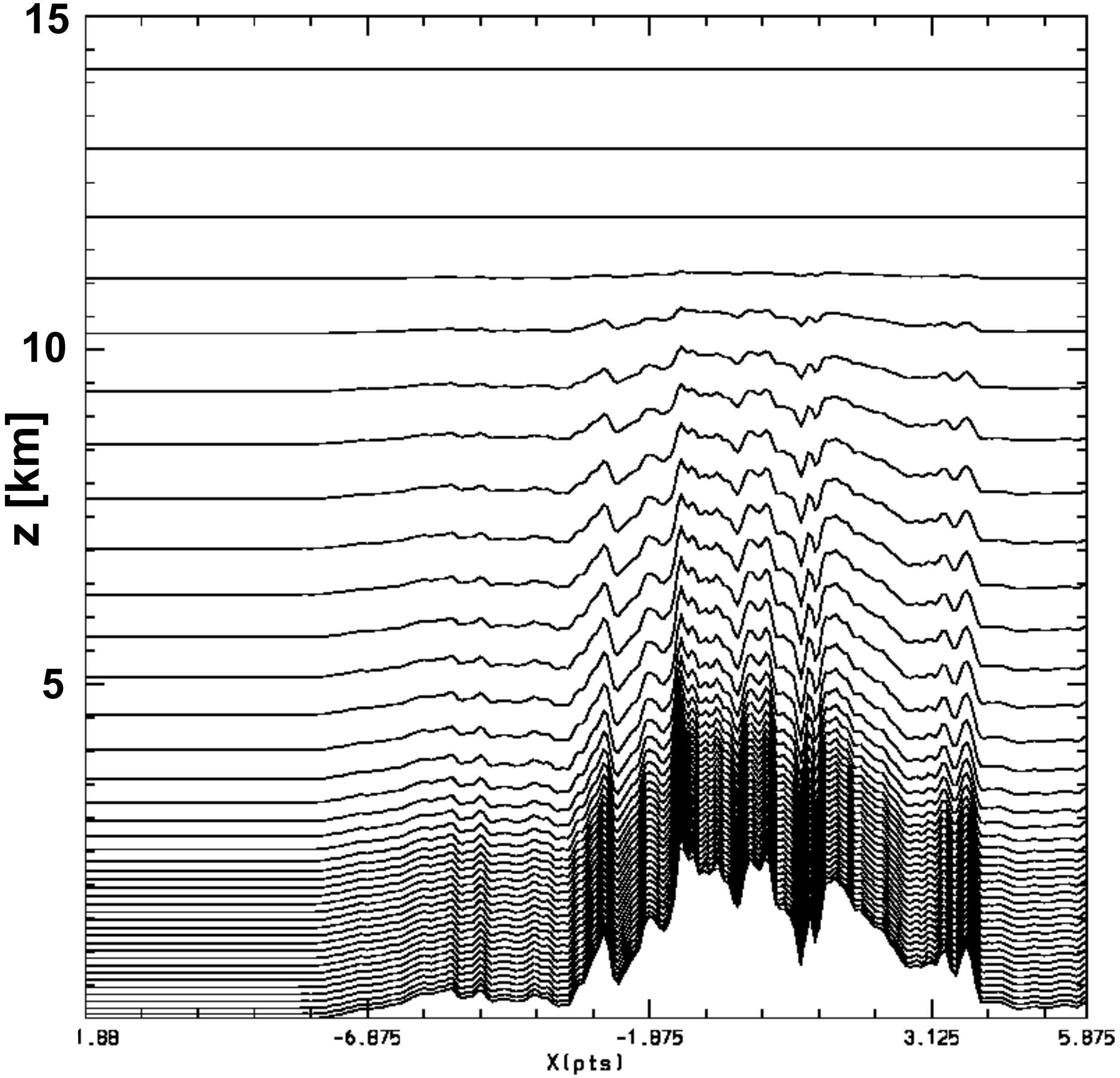
Terrain-following hybrid coordinates

Hybrid coordinates:

- transition between two different coordinate formulations, here: σ-coordinates at lower levels, p-coordinates at upper levels
- finer spacing at lower levels



Example of hybrid coordinate



Standard coordinate setting in the COSMO model of MeteoSwiss and DWD

σ-based hybrid coordinate

Outline

Introduction to terrain-following coordinates

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Smooth terrain-following coordinates

Basic considerations and objectives

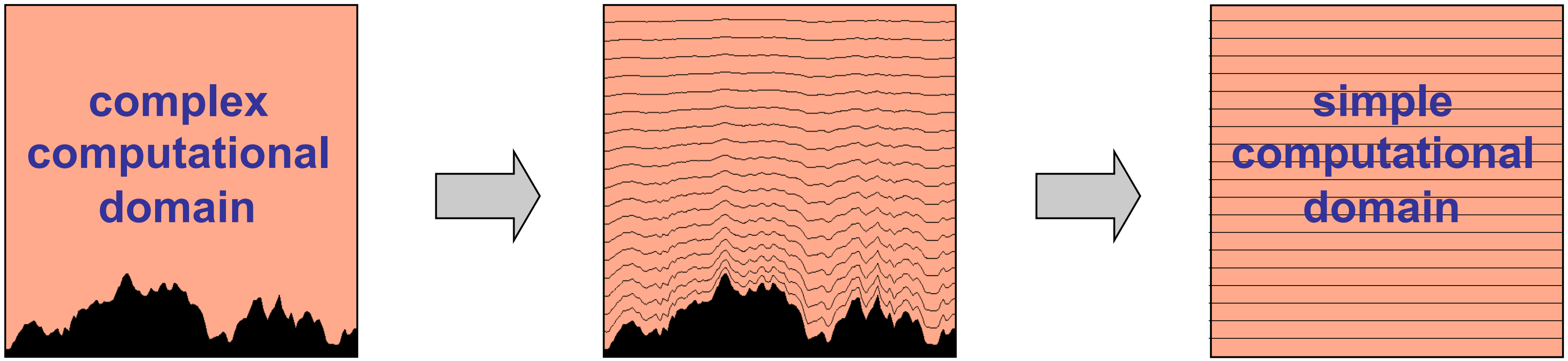
(1) Transformation should simplify structure of governing equations:

Example: continuity equation

height-coordinates: $$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$ (nonlinear, prognostic)

pressure-coordinates: $$\left(\frac{\partial u}{\partial x}\right)_p+\left(\frac{\partial v}{\partial y}\right)_p+\frac{\partial \omega}{\partial p}=0$$ (linear, diagnostic)

(2) Transformation should map computational domain to a rectangular grid:



physical space | physical space | computational space

Basic considerations and objectives

(3) Transformation should simplify boundary conditions:

Example: lower boundary condition at the height of the topography $h=h(x,y)$

height-coordinates:

$$w(z=h) = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

wind is tangential to lower boundary

sigma-coordinates:

$$\dot{\sigma}(z=h) = 0$$

$$\dot{\sigma} = \frac{D\sigma}{Dt}$$ "sigma dot"

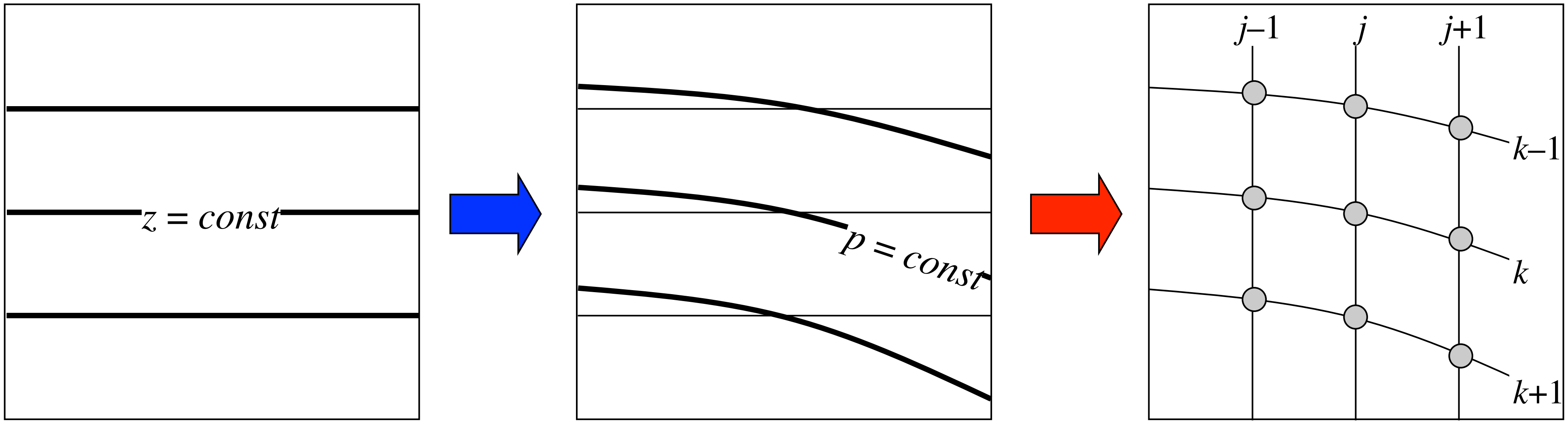
vertical wind in sigma coordinates disappears at lower and upper boundary!

Implementation of coordinate transformations

Example: two-dimensional continuity equation

Step 1:
Transform equations

Step 2:
Descretize equations



height-coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho w)}{\partial z} = 0$$

pressure-coordinates

$$\left(\frac{\partial u}{\partial x}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

descretized *p*-coordinates

$$\frac{u_{j+1,k} - u_{j-1,k}}{2\Delta x} + \frac{\omega_{j,k+1} - \omega_{j,k-1}}{2\Delta p} = 0$$

Definition of wind in general coordinates

Wind vector in (x,y,z)-coordinates

$$u \coloneqq \frac{Dx}{Dt}, \quad v \coloneqq \frac{Dy}{Dt}, \quad w \coloneqq \frac{Dz}{Dt}$$

Wind vector in generalized coordinates $(\tilde{x},\tilde{y},\tilde{z})$

$$\tilde{u} \coloneqq \frac{D\tilde{x}}{Dt}, \quad \tilde{v} \coloneqq \frac{D\tilde{y}}{Dt}, \quad \tilde{w} \coloneqq \frac{D\tilde{z}}{Dt}$$

Vertical wind in (x,y,p)-coordinates

$$\omega \coloneqq \frac{Dp}{Dt}$$

[hPa/s] Note: ascending ω < 0, descending ω > 0

Vertical wind in (x,y,θ)-coordinates

$$\dot{\theta} \coloneqq \frac{D\theta}{Dt}$$

[K/s] Note: $D\theta/Dt=0$ for adiabatic flows

Transformation of advection operator

Advection in (x,y,z)-coordinates

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underbrace{\frac{Dx}{Dt}}_{u}\left(\frac{\partial}{\partial x}\right)_{y,z} + \underbrace{\frac{Dy}{Dt}}_{v}\left(\frac{\partial}{\partial y}\right)_{x,z} + \underbrace{\frac{Dz}{Dt}}_{w}\left(\frac{\partial}{\partial z}\right)_{x,y}$$

The advection operator is defined as the total derivative

Careful! The partial derivatives are taken with respect to the coordinate considered, i.e. have different meanings in different coordinate systems!

Advection in generalized coordinates $(\tilde{x},\tilde{y},\tilde{z})$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underbrace{\frac{D\tilde{x}}{Dt}}_{\tilde{u}}\left(\frac{\partial}{\partial \tilde{x}}\right)_{\tilde{y},\tilde{z}} + \underbrace{\frac{D\tilde{y}}{Dt}}_{\tilde{v}}\left(\frac{\partial}{\partial \tilde{y}}\right)_{\tilde{x},\tilde{z}} + \underbrace{\frac{D\tilde{z}}{Dt}}_{\tilde{w}}\left(\frac{\partial}{\partial \tilde{z}}\right)_{\tilde{x},\tilde{y}}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \tilde{u}\left(\frac{\partial}{\partial \tilde{x}}\right)_{\tilde{y},\tilde{z}} + \tilde{v}\left(\frac{\partial}{\partial \tilde{y}}\right)_{\tilde{x},\tilde{z}} + \tilde{w}\left(\frac{\partial}{\partial \tilde{z}}\right)_{\tilde{x},\tilde{y}}$$

The transformed advection operator has the same formal structure as its Cartesian counterpart

Transformation of partial derivatives

Transformation of z-coordinates to s-coordinates:

Assume a new general coordinate defined by:

$s=s(x,y,z)$ or $z=z(x,y,s)$

Invertibility implies that $s=s(x,y,z)$ must be a strictly monotone function at each point (x,y)

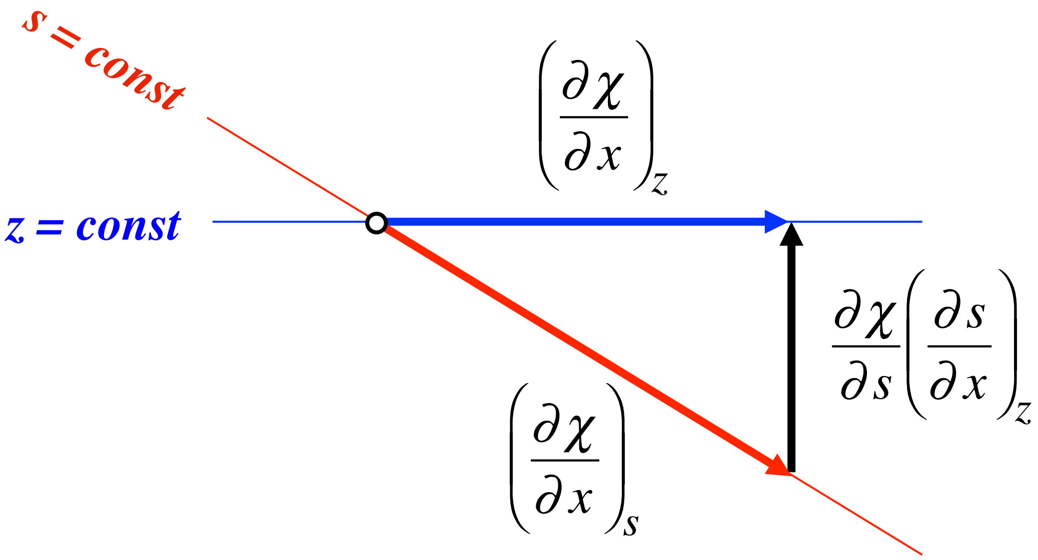
Transformation of $\partial\chi/\partial z$

$$\frac{\partial \chi}{\partial z} = \frac{\partial \chi}{\partial s}\frac{\partial s}{\partial z}$$

Note: Terms like $\partial s/\partial z$ and $(\partial s/\partial x)_z$ are known from coordinate transformation

Transformation of $(\partial\chi/\partial x)_z$

$$\left(\frac{\partial \chi}{\partial x}\right)_z = \left(\frac{\partial \chi}{\partial x}\right)_s + \frac{\partial \chi}{\partial s}\left(\frac{\partial s}{\partial x}\right)_z$$



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Pressure coordinates

In pressure coordinates, all variables are expressed as $\chi=\chi(x,y,p,t)$. For instance, the pressure field $p(x,y,z,t)$ is represented by

	$z\,(x,y,p,t)$	height [m]
or	$\phi\,(x,y,p,t) \approx g_o z$	geopotential [m^2/s^2]
or	$z_g\,(x,y,p,t)$	geopotential height [m]

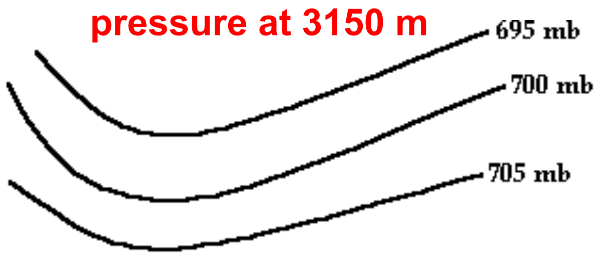
The geopotential height is a gravity-adjusted height accounting for variations of $g=g(x,y,z)$.

$$z_g = \frac{\phi}{g_o} = \frac{1}{g_o} \int_{z=0}^{z} g(z)\,dz \qquad g_o = 9.81\ \mathrm{m/s^2}$$

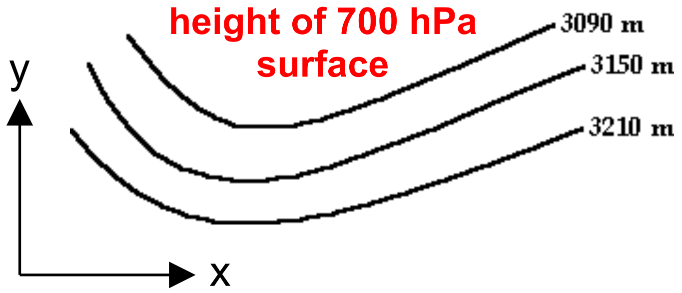
The geopotential is the potential of gravity:

$$g = \partial\phi/\partial z$$

***z*-coordinates**



***p*-coordinates**



Hydrostatic system in Cartesian coordinates

Horizontal momentum equations
$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F^x \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F^y$$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

Hydrostatic equation
$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Equation of state
$$p = \rho R T$$

Thermodynamic equation
$$\frac{DT}{Dt} - \frac{1}{c_p \rho}\frac{Dp}{Dt} = H$$

Continuity equation
$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} + \frac{\partial (w\rho)}{\partial z} = 0$$

F^x, F^y, H Externally specified force and heating rate

Hydrostatic system in pressure coordinates

Horizontal momentum equations $$\frac{Du}{Dt} - fv = -\left(\frac{\partial \phi}{\partial x}\right)_p + F^x \qquad \frac{Dv}{Dt} + fu = -\left(\frac{\partial \phi}{\partial y}\right)_p + F^y$$

with $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \omega\frac{\partial}{\partial p}$$ and $$\omega = \frac{Dp}{Dt}$$

Hydrostatic equation $$\frac{\partial \phi}{\partial p} = -\frac{1}{\rho}$$

Equation of state $$p = \rho R T$$

Thermodynamic equation $$\frac{DT}{Dt} = \frac{\omega}{\rho c_p} + H$$

Continuity equation $$\left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

F^x, F^y, H	Externally specified force and heating rate
$\phi \approx g_o z$	Geopotential

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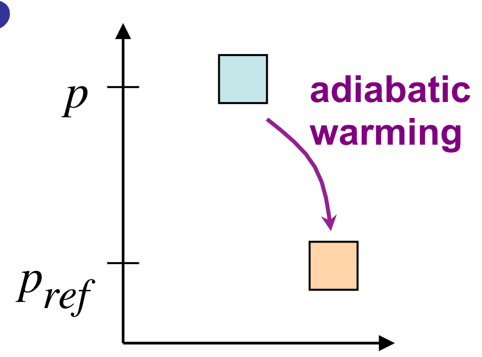
Isentropic coordinates



Definition: The potential (or isentropic) temperature θ of an air parcel at pressure p is the temperature that the parcel would acquire if adiabatically brought to some reference pressure p_{ref}.

$$\theta = T\left(\frac{p_{ref}}{p}\right)^{R/c_p}$$

p_{ref}	1000 hPa	reference pressure
R	287 J/(K kg)	gas constant for dry air
c_p	1004 J/(K kg)	specific heat of dry air at constant pressure

In isentropic coordinates, all variables are expressed as $\chi=\chi\,(x,y,\theta,t)$.

Advantage: Isentropic coordinates are particularly attractive for adiabatic flows, as the vertical velocity (=diabatic heating rate measured in K/s) vanishes, i.e.

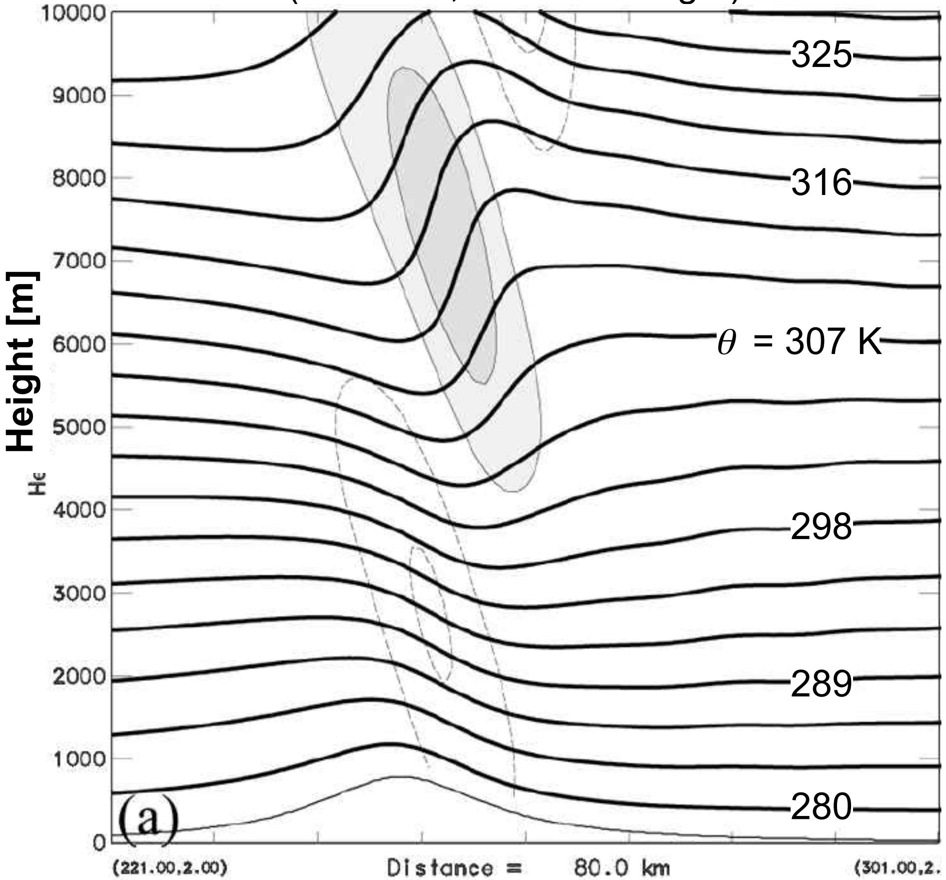
$$\dot{\theta} := \frac{D\theta}{Dt} = 0$$

Disdvantage: The invertibility condition (i.e., existence of an inverse coordinate transformation) implies that isentropic coordinates are restricted to flows with $\partial\theta/\partial z>0$.

Example 1: Flow past topography

Vertically propagating gravity wave

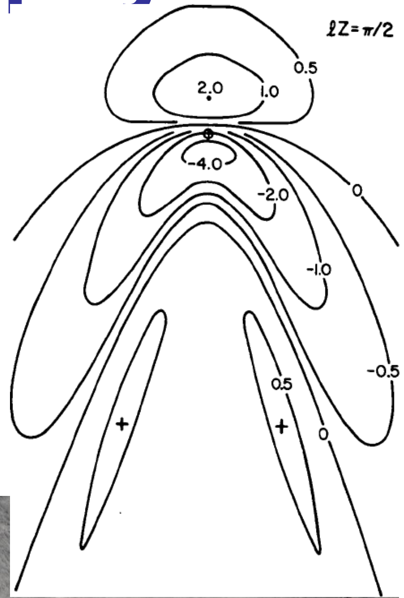
Idealized simulation showing θ and w
(U=20m/s, from left to right)

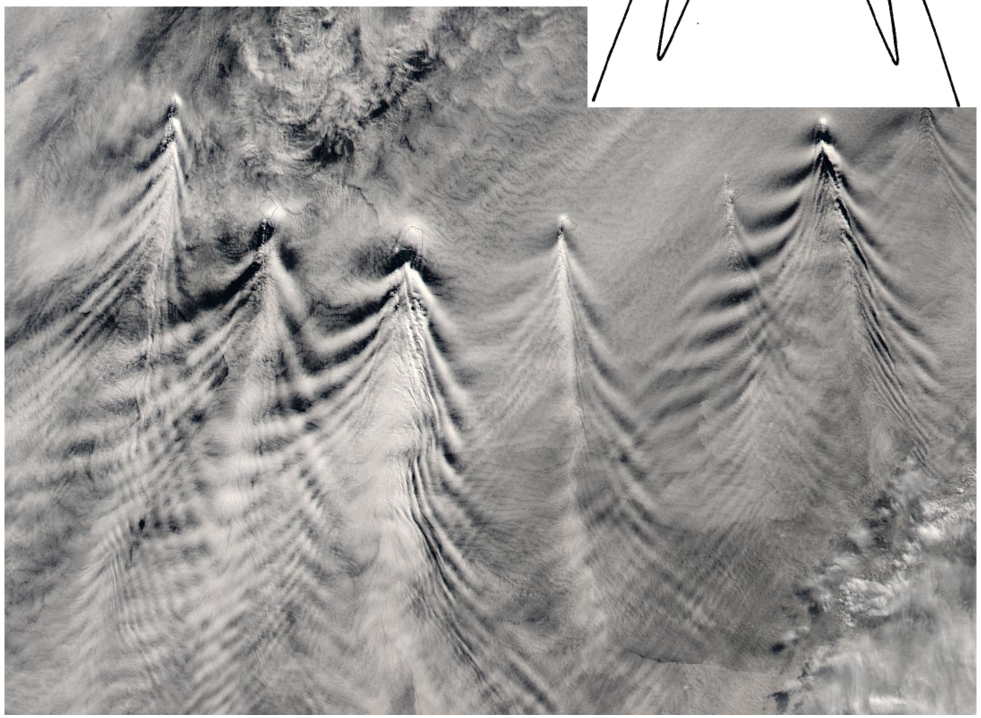


Steady inviscid adiabatic flows: isentropes represent streamlines

"Ship waves" above isolated mountains

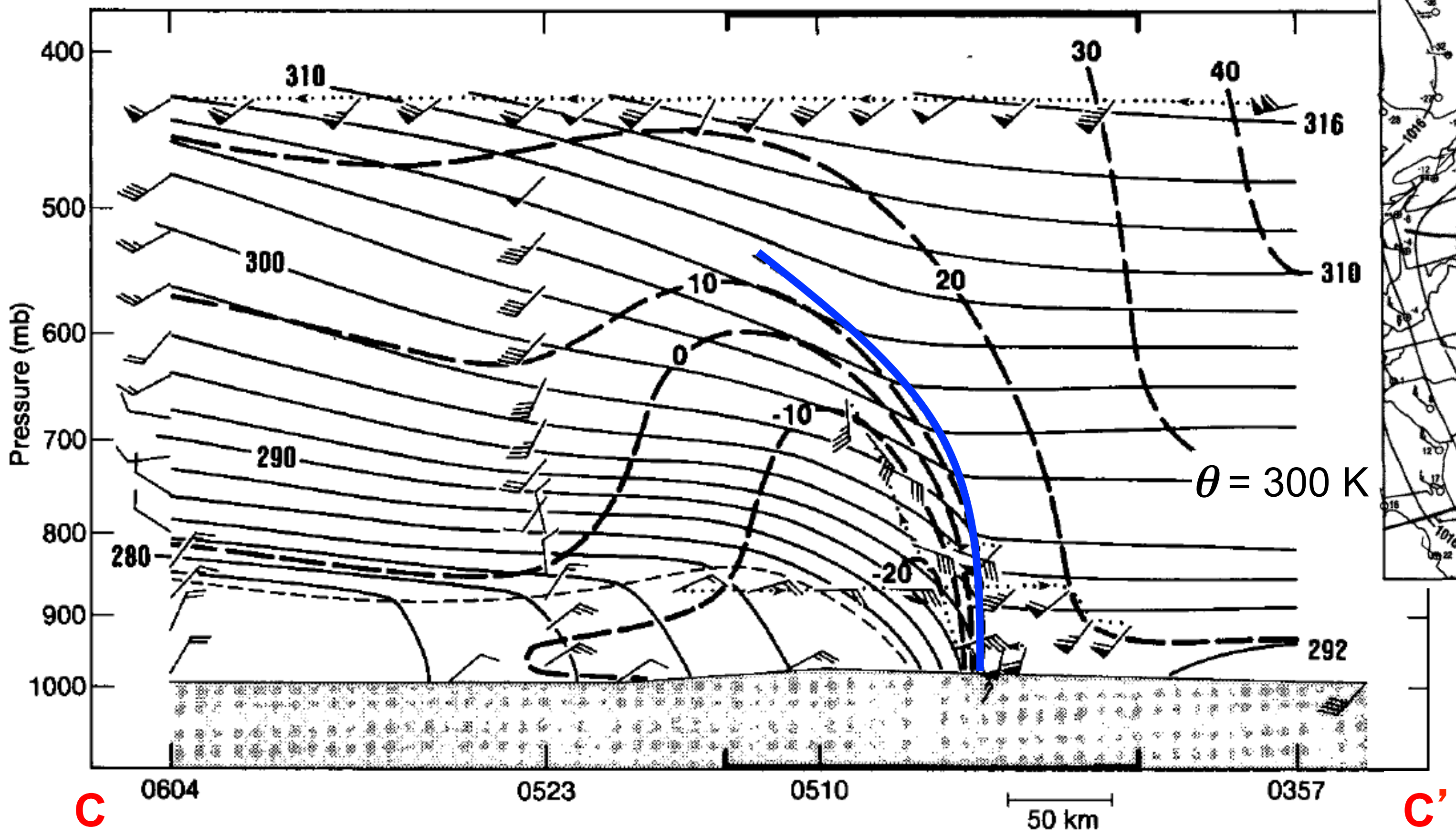
Idealized analysis and satellite picture at south-sandwich islands





(Doyle and Smith 2003, Smith 1979)

Example 2: Cold front



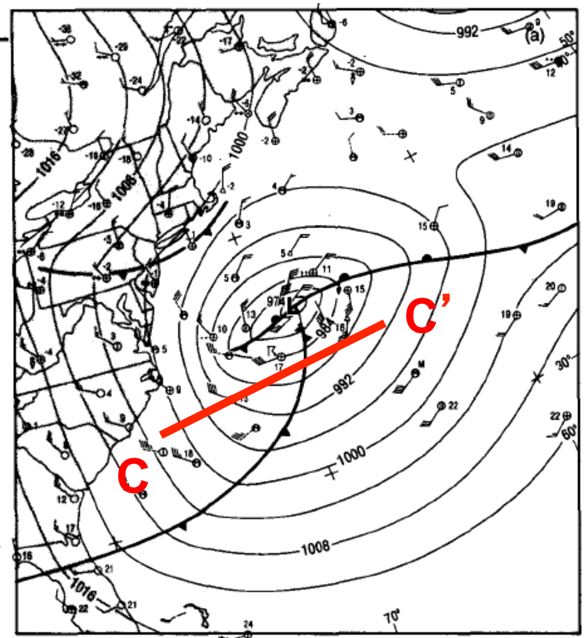
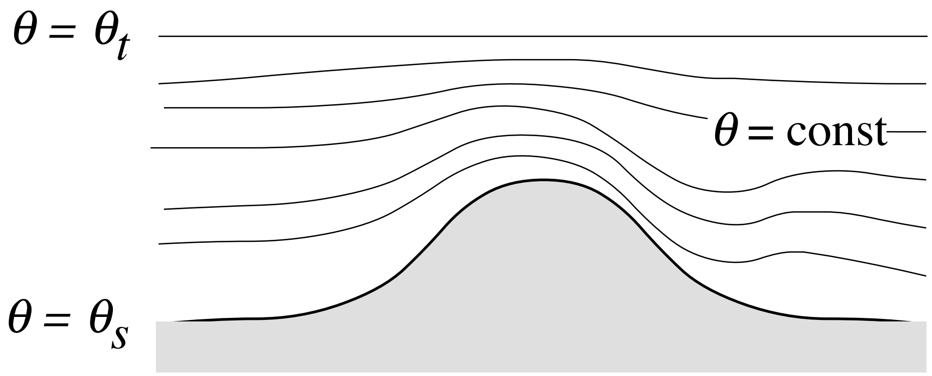


FIG. 11. Cross section of potential temperature (K, solid) and section-normal wind speed (m s^{-1}, bold dashed) along line CC' of Fig. 4 at 0600 UTC 4 January 1989. Thin-dashed lines are frontal boundaries. Dropwindsondes are labeled on line CC' with times of deployment. NOAA WP-3D flight tracks are depicted by small-dotted lines, with selected flight-level wind vectors plotted. Wind flags and barbs are as in Fig. 2. Bold lines on the top and bottom of the figure frame show the horizontal domain of the cross-sectional enlargements in Fig. 12.

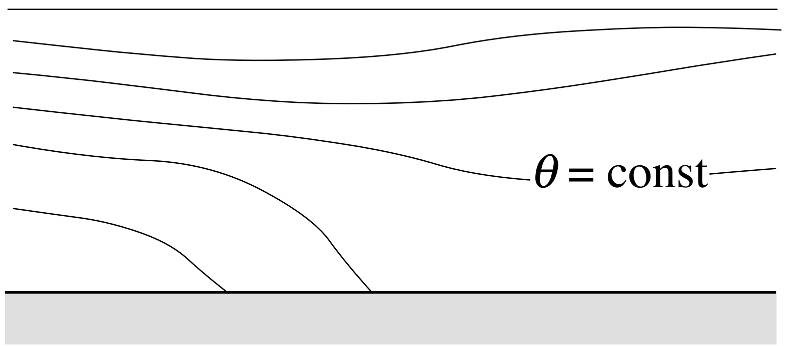
A cold front is evident from its horizontal temperature contrast (i.e. from sloping isentropes)

(Neiman, Shapiro, Fedor, 1993, MWR, 121)

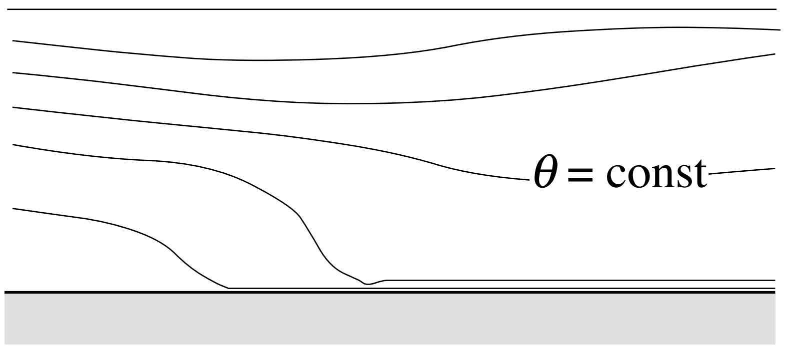
Isentropic coordinates



Isentropic coordinates have a simple lower boundary conditions if $\theta(z=h)$ is constant, as in this example.



In a more general case, the lower b.c. becomes very cumbersome.



Models can use a formulation that represent these circumstances with collapsed computational layers.

New variables occurring in the θ-system

Comparison of momentum and continuity equations (for f =0)

Pressure coordinates	Shallow-water system	Isentropic coordinates
Geopotential ϕ	Surface height $H+h$	Montgomery potential M
$\frac{Du}{Dt}=-\left(\frac{\partial\phi}{\partial x}\right)_p$	$\frac{Du}{Dt}=-\frac{\partial(H+h)}{\partial x}$	$\frac{Du}{Dt}=-\left(\frac{\partial M}{\partial x}\right)_\theta$
Density "1"	Fluid depth H	Isentropic mass density σ
$\nabla_p\cdot(u,v,\omega)=0$ $\frac{\partial 1}{\partial t}+\nabla_p\cdot[(u,v,\omega)\ 1]=0$	$\frac{\partial H}{\partial t}+\nabla_h\cdot[(u,v)\ \ H]=0$	$\frac{\partial\sigma}{\partial t}+\nabla_\theta\cdot\left[(u,v,\dot{\theta})\ \ \sigma\right]=0$

Note: Different σ than with σ-coordinates!

Hydrostatic system in isentropic coordinates

Horizontal momentum equations
$$\frac{Du}{Dt} - fv = -\left(\frac{\partial M}{\partial x}\right)_\theta + F^x \qquad \frac{Dv}{Dt} + fu = -\left(\frac{\partial M}{\partial y}\right)_\theta + F^y$$

with $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_\theta + v\left(\frac{\partial}{\partial y}\right)_\theta + \dot{\theta}\frac{\partial}{\partial \theta}$$

Hydrostatic equation
$$\frac{c_p T}{\theta} = \frac{\partial M}{\partial \theta}$$ with $\phi = gz$

Equation of state
$$p = \rho R T$$

Thermodynamic equation
$$\frac{D\theta}{Dt} = \dot{\theta}$$

Continuity equation
$$\frac{\partial \sigma}{\partial t} + \left(\frac{\partial(\sigma u)}{\partial x}\right)_\theta + \left(\frac{\partial(\sigma v)}{\partial y}\right)_\theta + \frac{\partial(\sigma \dot{\theta})}{\partial \theta} = 0 \qquad \sigma := -\frac{1}{g}\frac{\partial p}{\partial \theta}$$

F^x, F^y, H	Externally specified force and heating rate
$\phi = g_o z$	Geopotential
$M = \phi + c_p T$	Montgomery potential

Simplified system in isentropic coordinates

Horizontal momentum equations
$$\frac{Du}{Dt} - \cancel{fv} = -\left(\frac{\partial M}{\partial x}\right)_\theta + \cancel{F^x} \qquad \cancel{\frac{Dv}{Dt} + fu = -\left(\frac{\partial M}{\partial y}\right)_\theta + F^y}$$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_\theta + \cancel{v\left(\frac{\partial}{\partial y}\right)_\theta} + \cancel{\dot{\theta}\frac{\partial}{\partial \theta}}$

Hydrostatic equation
$$\frac{c_p T}{\theta} = \frac{\partial M}{\partial \theta} \qquad \text{with} \quad \phi = gz$$

Equation of state
$$p = \rho R T$$

Thermodynamic equation
$$\frac{D\theta}{Dt} = \cancel{\dot{\theta}}$$

Continuity equation
$$\frac{\partial \sigma}{\partial t} + \left(\frac{\partial(\sigma u)}{\partial x}\right)_\theta + \cancel{\left(\frac{\partial(\sigma v)}{\partial y}\right)_\theta} + \cancel{\frac{\partial(\sigma \dot{\theta})}{\partial \theta}} = 0 \qquad \sigma := -\frac{1}{g}\frac{\partial p}{\partial \theta}$$

Two-dimensional	$\partial/\partial y = 0$	Simplifications as in tutorials
Adiabatic	$D\theta/Dt = 0$	
Inviscid	$F = 0$	
Non-rotating	$f = 0$	

Simplified system in isentropic coordinates

Simplifications: $\partial/\partial y = 0,\ D\theta/Dt = 0,\ \mathbf{F} = 0,\ f = 0$

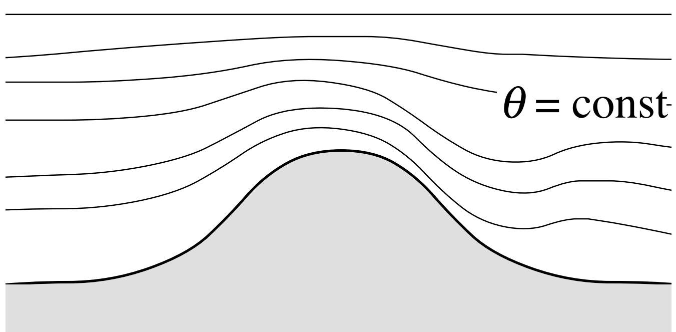
Momentum equations $\quad \dfrac{Du}{Dt} = -\left(\dfrac{\partial M}{\partial x}\right)_\theta \quad$ with $\quad \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\left(\dfrac{\partial}{\partial x}\right)_\theta$

Continuity equation $\quad \dfrac{\partial \sigma}{\partial t} + \left(\dfrac{\partial(\sigma u)}{\partial x}\right)_\theta = 0 \quad$ with $\quad \sigma = -\dfrac{1}{g}\dfrac{\partial p}{\partial \theta}$

Hydrostatic equation $\quad \pi = \dfrac{\partial M}{\partial \theta} \quad$ with $\quad \pi(p) = c_p \dfrac{T}{\theta} = c_p\left(\dfrac{p}{p_{ref}}\right)^{R/c_p}$ Exner function

Boundary conditions:

- upper boundary is horizontal isentropic surface
- lower boundary is isentropic surface



$\theta = \theta_t \qquad p_t = const$

$\theta = \theta_s \qquad M_s = c_p T + g z_s$

Schär, ETH Zürich

Grid structure for θ-coordinates in 2D

Primary model variables

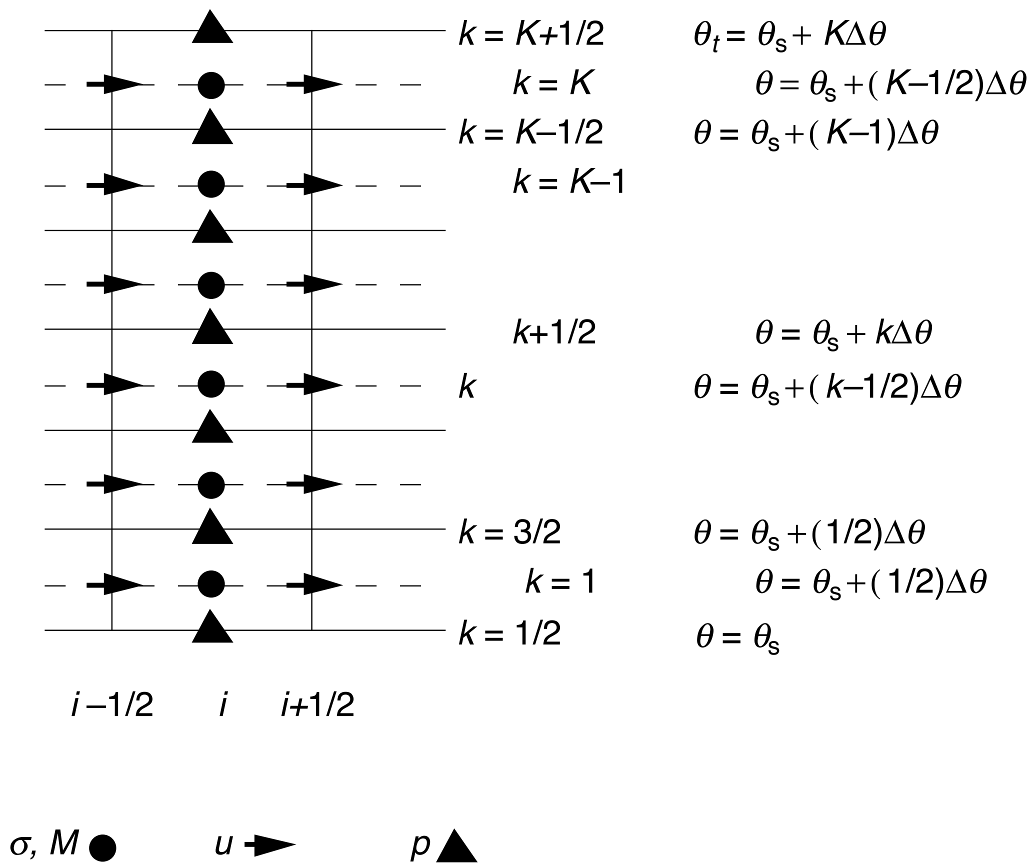
$\sigma_{i,k}$ $(k=1,..\ K)$ isentropic density

$M_{i,k}$ $(k=1,..\ K)$ Montgomery potential

$u_{i+1/2,k}$ $(k=1,..\ K)$ horizontal velocity

$p_{i,k+1/2}$ $(k=1/2,..\ K+1/2)$ pressure

vertical section



Simplified system in isentropic coordinates

Simplifications: $\partial/\partial y = 0,\ D\theta/Dt = 0,\ \mathbf{F} = 0,\ f = 0$

Momentum equations $$\frac{Du}{Dt} = -\left(\frac{\partial M}{\partial x}\right)_\theta$$ with $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_\theta$$

Continuity equation $$\frac{\partial \sigma}{\partial t} + \left(\frac{\partial(\sigma u)}{\partial x}\right)_\theta = 0$$ with $$\sigma = -\frac{1}{g}\frac{\partial p}{\partial \theta}$$

Hydrostatic equation $$\pi = \frac{\partial M}{\partial \theta}$$ with $$\pi(p) = c_p \frac{T}{\theta} = c_p \left(\frac{p}{p_{ref}}\right)^{R/c_p}$$ Exner function

Integration:

Step 1: Integration of momentum and continuity equations => u^{n+1}, σ^{n+1}

Step 2: Diagnosis of pressure and Exner function => p^{n+1}, π^{n+1}

Step 3: Diagnosis of Montgomery potential => M^{n+1}

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Isentropic coordinates

Sigma coordinates

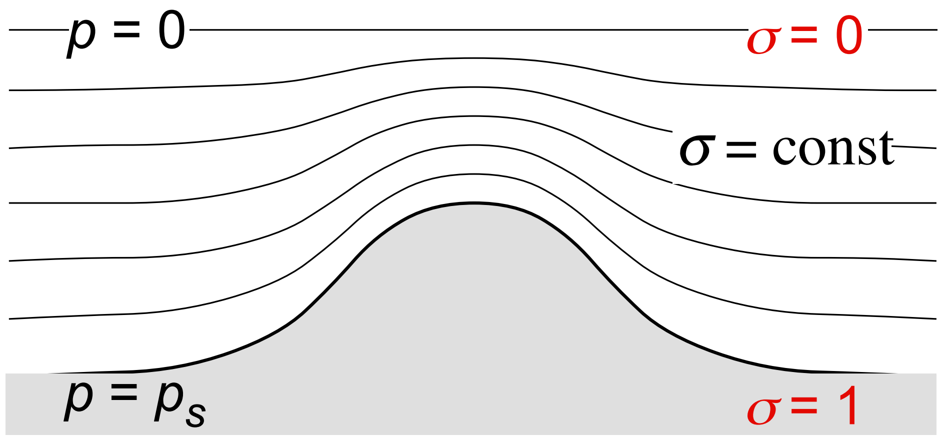
Smooth terrain-following coordinates

Terrain-following Sigma coordinates

Special case: upper boundary at $p_t = 0$

Surface pressure: $p_s = p_s(x, y, t)$

σ-coordinate: $\sigma := \dfrac{p}{p_s}$



General case: upper boundary at $p_t = \text{const}$

σ-coordinate: $\sigma := \dfrac{p - p_t}{p_s - p_t}$

Phillips (1957)

J. Meteorol., Vol 14, 184-185 (1957)

A COORDINATE SYSTEM HAVING SOME SPECIAL ADVANTAGES FOR NUMERICAL FORECASTING

By N. A. Phillips

Massachusetts Institute of Technology[1]

(Manuscript received 29 October 1956)

The coordinate system used to date in numerical forecasting schemes has been the x, y, p, t-system introduced by Sutcliffe and Godart [4] and also by Eliassen [3]. This system, in common with the ordinary x, y, z, t-system, has certain computational disadvantages in the vicinity of mountains, because the lower limit of the atmosphere is not a coordinate surface. The purpose of this brief note is to describe a modified coordinate system in which the ground is always a coordinate surface.

It is obtained by replacing the vertical coordinate p in the x, y, p, t-system by the independent variable $\sigma = p/\pi$, where $\pi = \pi(x, y, t)$ is the pressure at ground level. σ ranges monotonically from zero at the top of the atmosphere to unity at the ground. In describing the relation between this x, y, σ, t-system and the usual x, y, p, t-system, we will use a subscript p to indicate a derivative along a pressure surface. Differentiation in the new x, y, σ, t-system will have no subscripts.

The following relation holds, where ξ can be x, y, or t:

$$\left(\frac{\partial}{\partial \xi}\right)_p = \frac{\partial}{\partial \xi} - \frac{\sigma}{\pi}\frac{\partial \pi}{\partial \xi}\frac{\partial}{\partial \sigma}.$$

tude have been neglected. Equations (1) and (2) differ from those in the x, y, p, t-system only by the inclusion of the terms in $\partial\phi/\partial\sigma$.

The operator d/dt is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \dot{\sigma}\frac{\partial}{\partial \sigma}, \tag{3}$$

where $\dot{\sigma} = d\sigma/dt$.

The hydrostatic equation is obtained from the relation

$$\frac{\partial}{\partial p} = \frac{\partial \sigma}{\partial p}\frac{\partial}{\partial \sigma} = \frac{1}{\pi}\frac{\partial}{\partial \sigma},$$

and becomes, simply,

$$\partial\phi/\partial\sigma = -RT/\sigma. \tag{4}$$

Here R is the gas constant, and T is the absolute temperature. The coefficient $\pi^{-1}\sigma(\partial\phi/\partial\sigma)$ appearing in (1) and (2) can thus be replaced by $-RT/\pi$. Since ϕ is known at $\sigma = 1$ (at the ground), a knowledge of $T(\sigma)$ will give $\phi(\sigma)$ from (4) by integration.

The equation of continuity in the x, y, p, t-system is

$$\nabla_p \cdot v + \partial\omega/\partial p = 0,$$

Hydrostatic system in pressure coordinates

Horizontal momentum equations $\frac{Du}{Dt} - fv = -\left(\frac{\partial \phi}{\partial x}\right)_p + F^x$ $\frac{Dv}{Dt} + fu = -\left(\frac{\partial \phi}{\partial y}\right)_p + F^y$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \omega\frac{\partial}{\partial p}$ and $\omega = \frac{Dp}{Dt}$

Hydrostatic equation $\frac{\partial \phi}{\partial p} = -\frac{1}{\rho}$

Equation of state $p = \rho R T$

Thermodynamic equation $\frac{DT}{Dt} = \frac{\omega}{\rho c_p} + H$

Continuity equation $\left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$

Boundary conditions at top: $p = 0:$ $\omega = 0$

Boundary conditions at surface: $p = p_s:$ $\omega = \partial p_s / \partial t + \mathbf{v}_s \cdot \nabla_h p_s$ $\phi = g z_s$

Hydrostatic system in sigma coordinates

Horizontal momentum equations
$$\frac{Du}{Dt} - fv = -\left(\frac{\partial \phi}{\partial x}\right)_\sigma - \frac{RT}{p_s}\frac{\partial p_s}{\partial x} + F^x$$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_\sigma + v\left(\frac{\partial}{\partial y}\right)_\sigma + \dot{\sigma}\frac{\partial}{\partial \sigma}$ and $\dot{\sigma} = \frac{D\sigma}{Dt}$

Hydrostatic equation
$$\frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma}$$

Equation of state
$$\sigma\, p_s = \rho\, RT$$

Thermodynamic equation
$$\frac{DT}{Dt} = \frac{RT}{c_p \sigma p_s}\omega + H \qquad \omega = \sigma\left[\frac{\partial p_s}{\partial t} + u\frac{\partial p_s}{\partial x} + v\frac{\partial p_s}{\partial y}\right] + \dot{\sigma} p_s$$

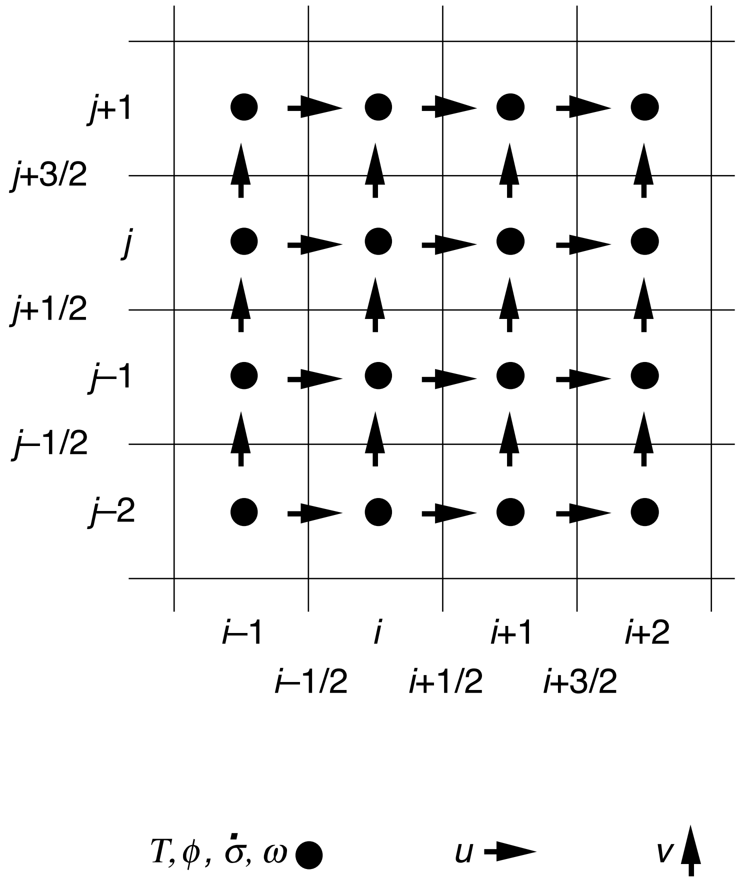
Continuity equation
$$\frac{\partial p_s}{\partial t} + \nabla_\sigma \cdot (p_s \mathbf{v}) + p_s \frac{\partial \dot{\sigma}}{\partial \sigma} = 0$$

Boundary conditions at top: $p = 0: \quad \sigma = 0, \quad \dot{\sigma} = 0$

Boundary conditions at surface: $p = p_s: \quad \sigma = 1, \quad \dot{\sigma} = 0 \qquad \phi = g z_s$

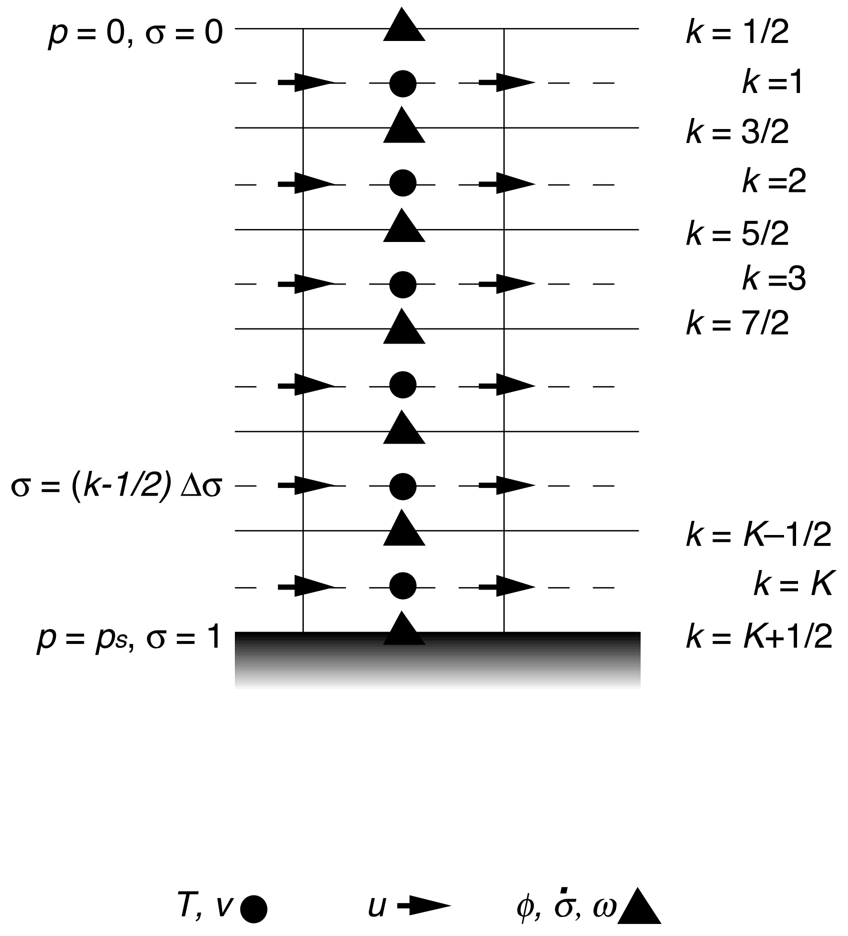
Grid structure in σ-coordinates

horizontal section



vertical section

$\sigma = p / p_s$



Vertical wind and surface pressure

If the variables u, v, T, p_s are known, the vertical wind $\dot{\sigma}$ may be diagnosed

Integrate continuity equation from $\sigma'=0$ to $\sigma'=\sigma$

$$\frac{\partial p_s}{\partial t}+\nabla_\sigma \cdot (p_s \mathbf{v})+p_s\frac{\partial \dot{\sigma}}{\partial \sigma}=0 \qquad \Rightarrow \qquad \sigma\frac{\partial p_s}{\partial t}+\int_0^\sigma \nabla_\sigma \cdot (p_s\mathbf{v})\,d\sigma'+p_s\dot{\sigma}\Big|_\sigma=0$$

Solve for vertical wind:
$$\dot{\sigma}(\sigma)=-\frac{\sigma}{p_s}\frac{\partial p_s}{\partial t}-\frac{1}{p_s}\int_0^\sigma \nabla_\sigma \cdot (p_s\mathbf{v})\,d\sigma'$$

Prognosis of surface pressure tendency: Integrate from $\sigma'=0$ to $\sigma'=1$

$$\frac{\partial p_s}{\partial t}=-\int_0^1 \nabla_\sigma \cdot (p_s\mathbf{v})\,d\sigma$$

Integration in sigma-coordinates

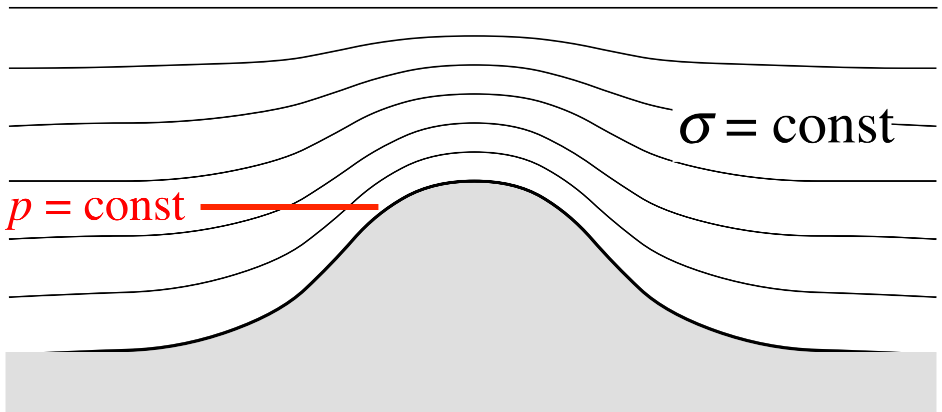
Variables u, v, T, p_s and ϕ known at time t

- Computation of surface pressure tendency $\partial p_s/\partial t$
- Diagnosis of vertical wind $\dot{\sigma}$
- Computation of tendencies $\partial u/\partial t$, $\partial v/\partial t$ and $\partial T/\partial t$
- Time step for u, v, T and p_s to time $t+\Delta t$
- Diagnosis of gepotential ϕ

Variables u, v, T, p_s and ϕ known at time $t+\Delta t$

The pressure-gradient problem

The main disadvantage of terrain-following coordinates are inaccuracies associated with the horizontal pressure gradient term.



In pressure coordinates

$$\frac{Du}{Dt} - fv = -\left(\frac{\partial \phi}{\partial x}\right)_p + F^x$$

In sigma coordinates

$$\frac{Du}{Dt} - fv = \underbrace{-\left(\frac{\partial \phi}{\partial x}\right)_\sigma - \frac{RT}{p_s}\frac{\partial p_s}{\partial x}}_{-\left(\frac{\partial \phi}{\partial x}\right)_p} + F^x$$

The horizontal pressure gradient is computed as the sum of two terms. Both of these may be very large, but the sum of the terms is very small. This is an error-prone constellation! It has important implications in the case of steep topography.

Outline

Introduction to terrain-following coordinates

Coordinate transformations

Pressure coordinates

Isentropic coordinates

Sigma coordinates

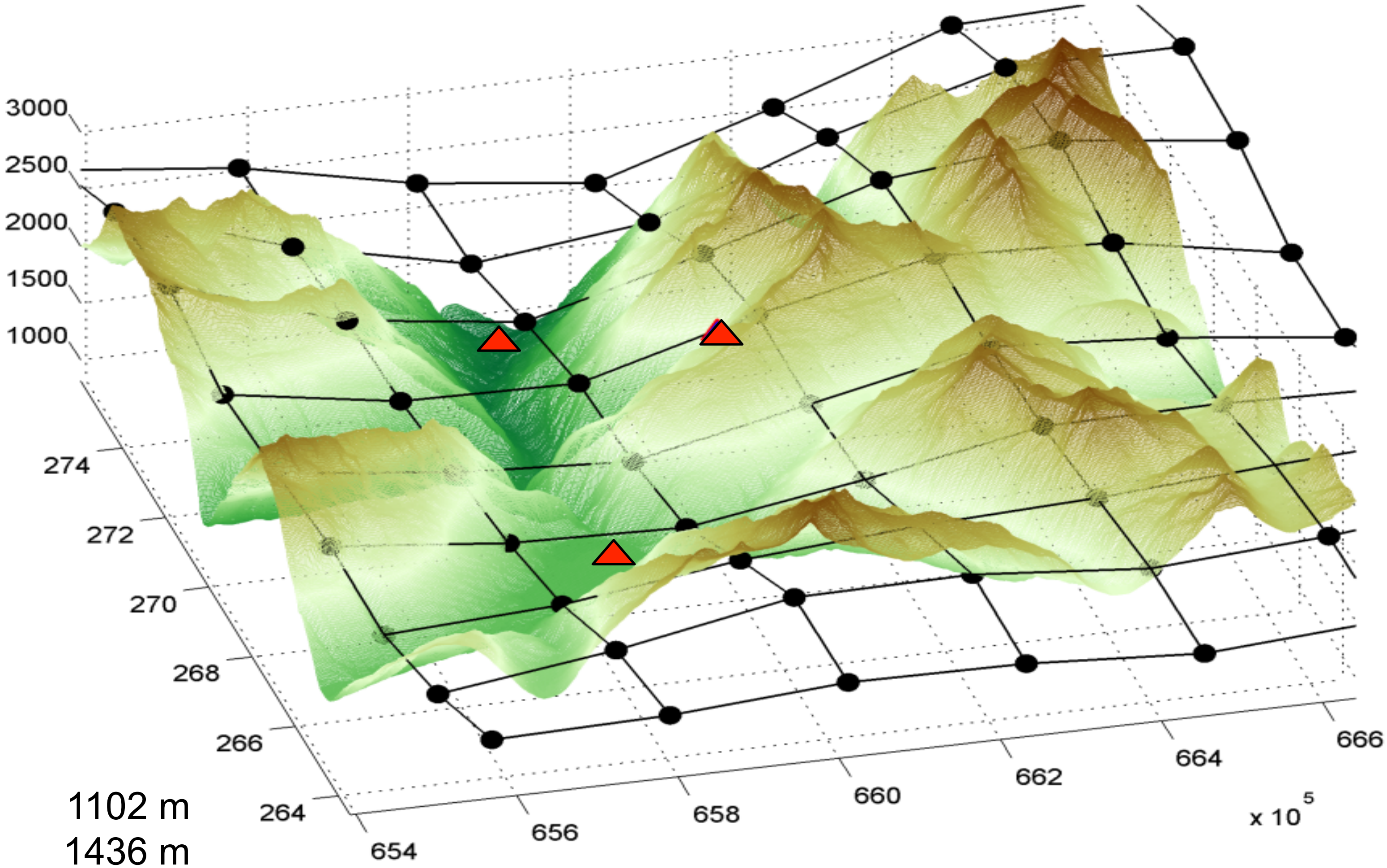
Smooth terrain-following coordinates

Real Topography is extremely complex

This implies awkward terrain-following grids.
How will this affect quality of numerical solutions?
=> Compare different coordinate formulations! <=

Even at Δ=2km only poorly captured

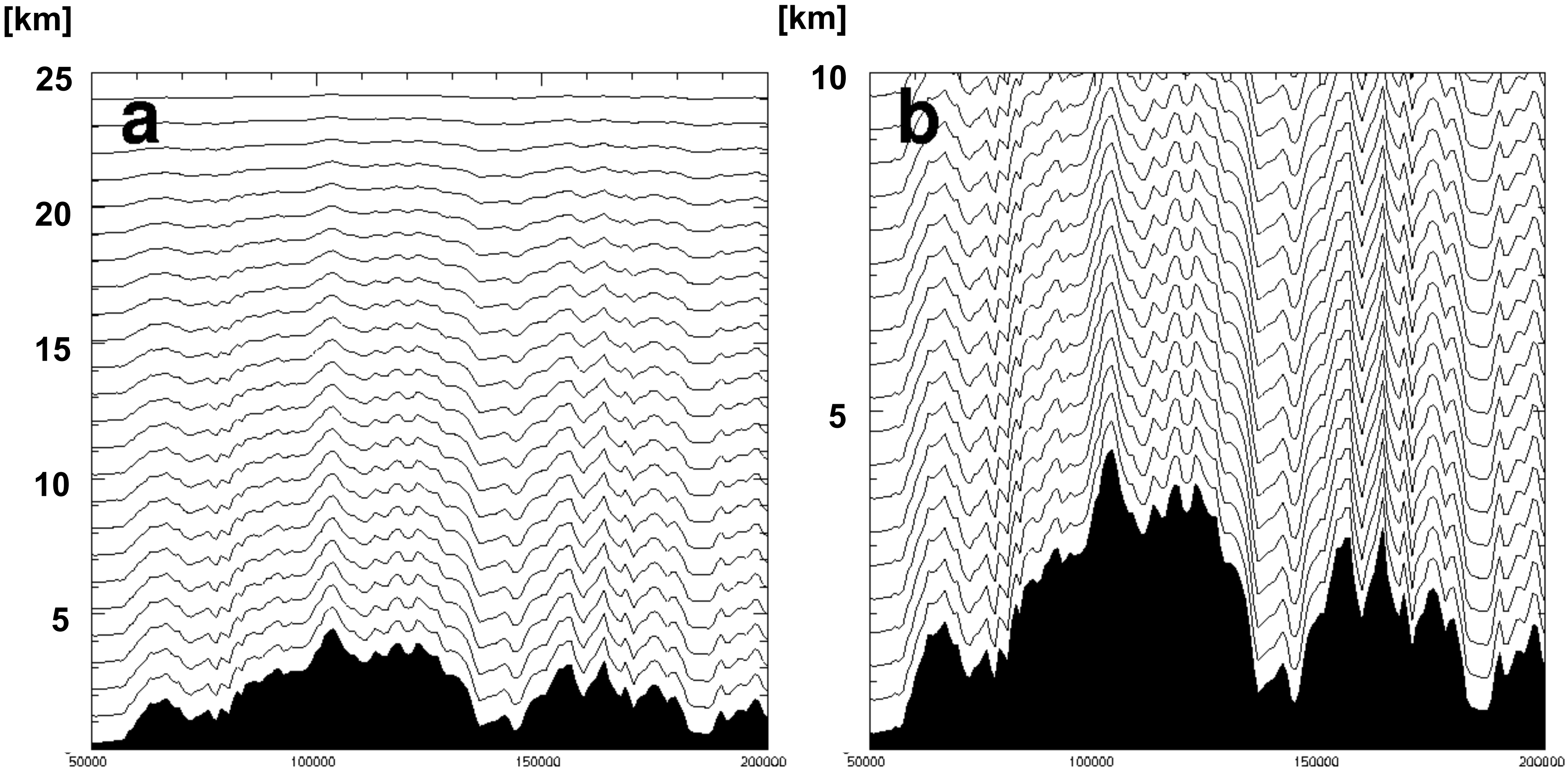
Topography with COSMO grid



Göschenen	1102 m
Andermatt	1436 m
Gütsch	2282 m

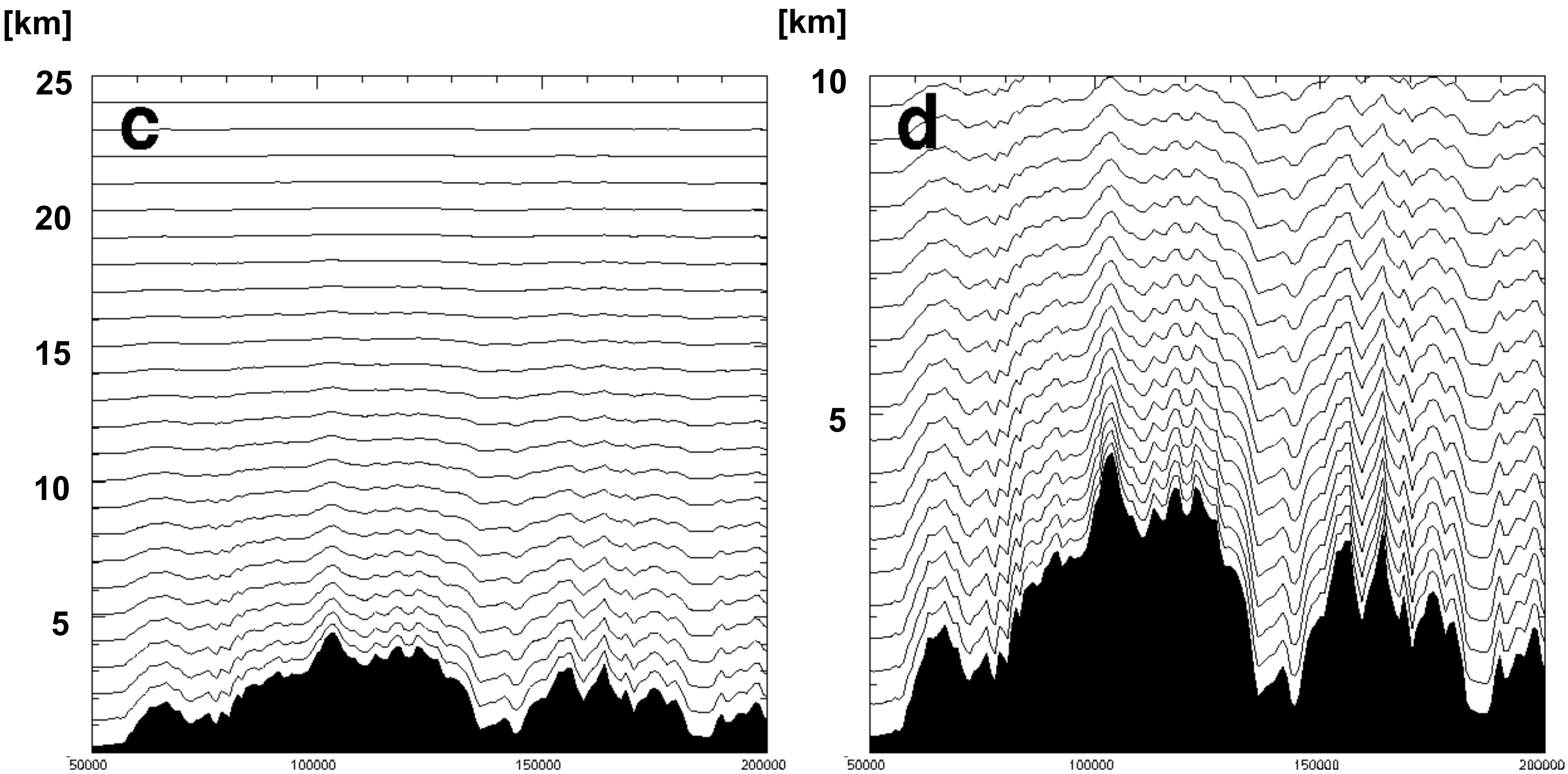
(Daniel Leuenberger, MeteoSwiss)

Sigma coordinates

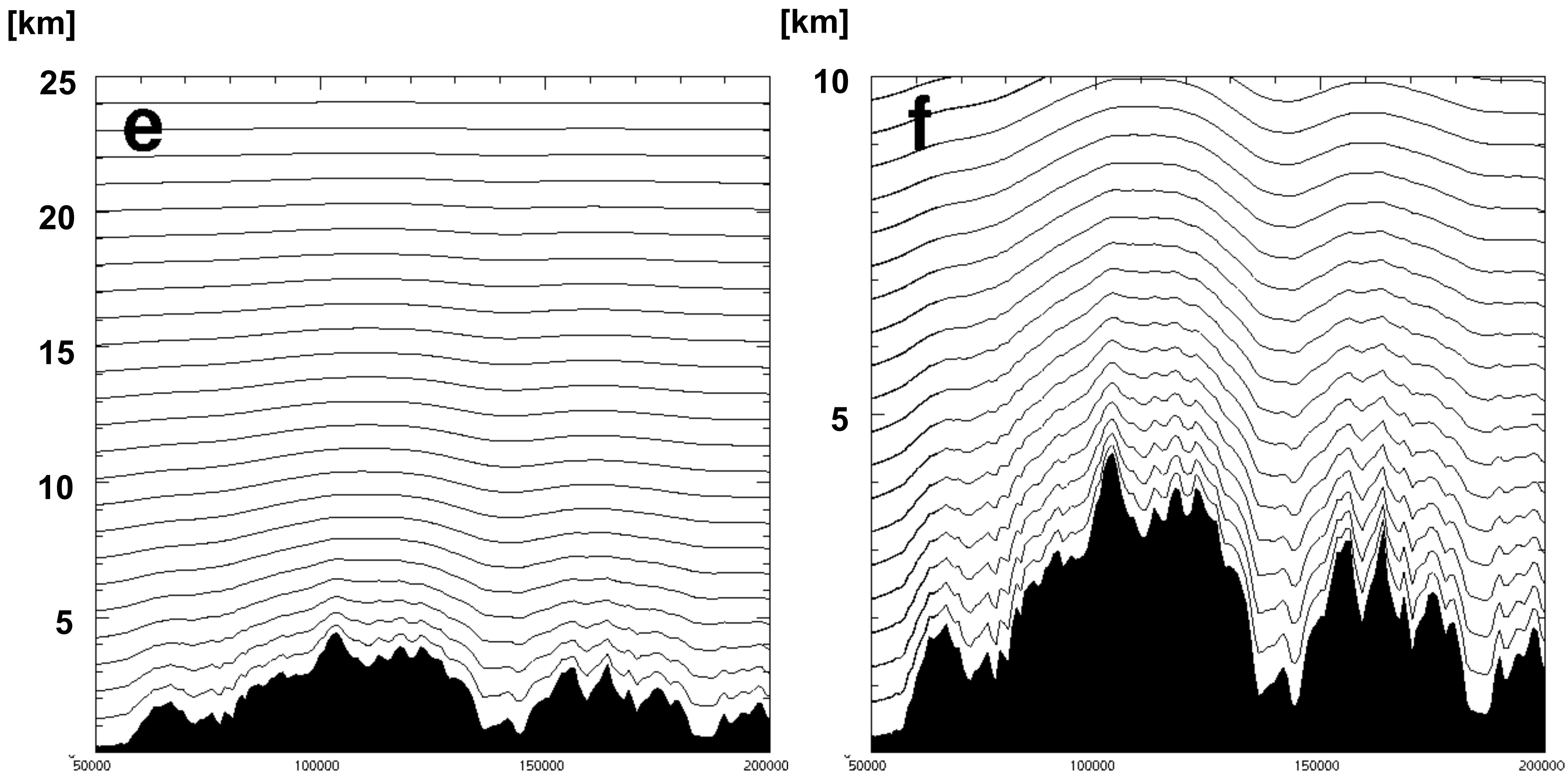


Schär, ETH Zürich

Hybrid coordinates



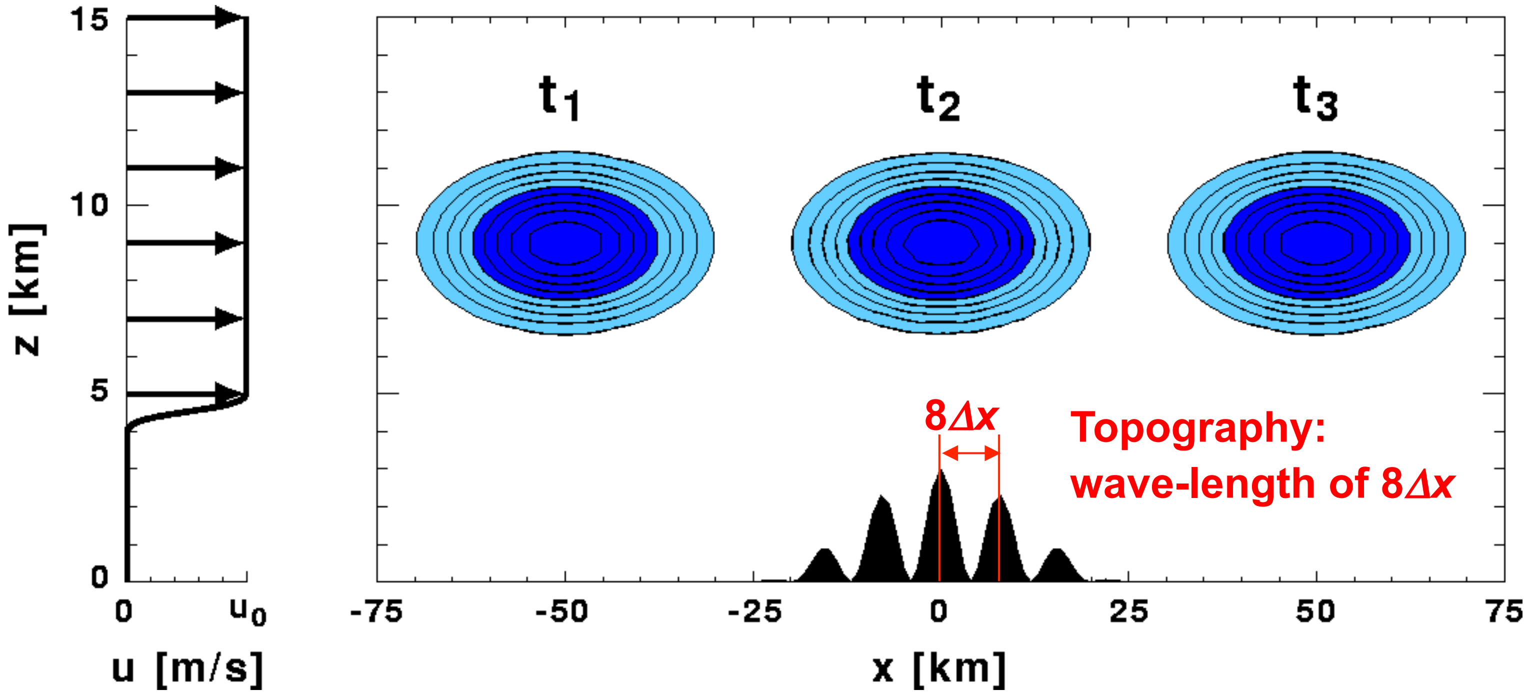
Smooth Level Vertical (SLEVE) coordinate



Scale-dependent decay of terrain-features with height

Schär et al. 2002, Leuenberger et al. 2010

Idealized Advection Test

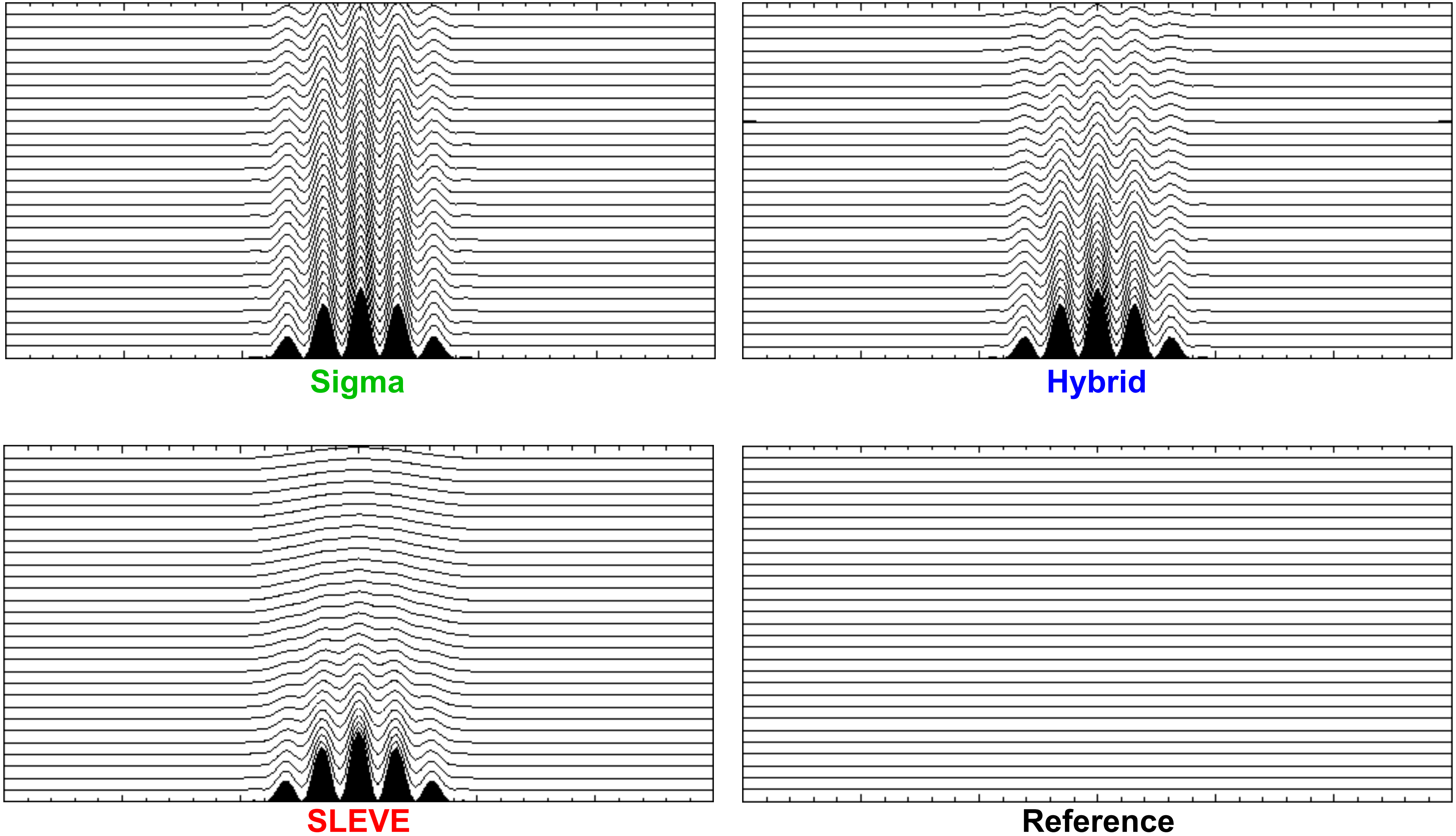


Tested coordinates:
- Sigma
- Hybrid
- SLEVE
- no topography (reference)

Tested numerical schemes:
- Centered in space and time, 2nd order
- Centered in space and time, 4th order
- Upstream, 1st order
- MPDATA (Smolarkiewicz), 2nd order

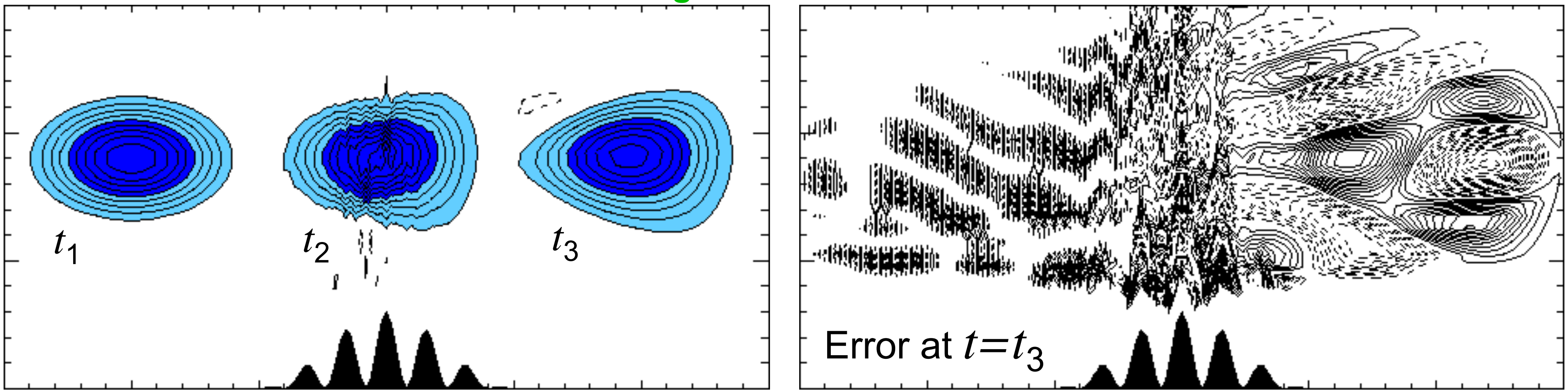
Numerics in conservative flux form

Coordinates

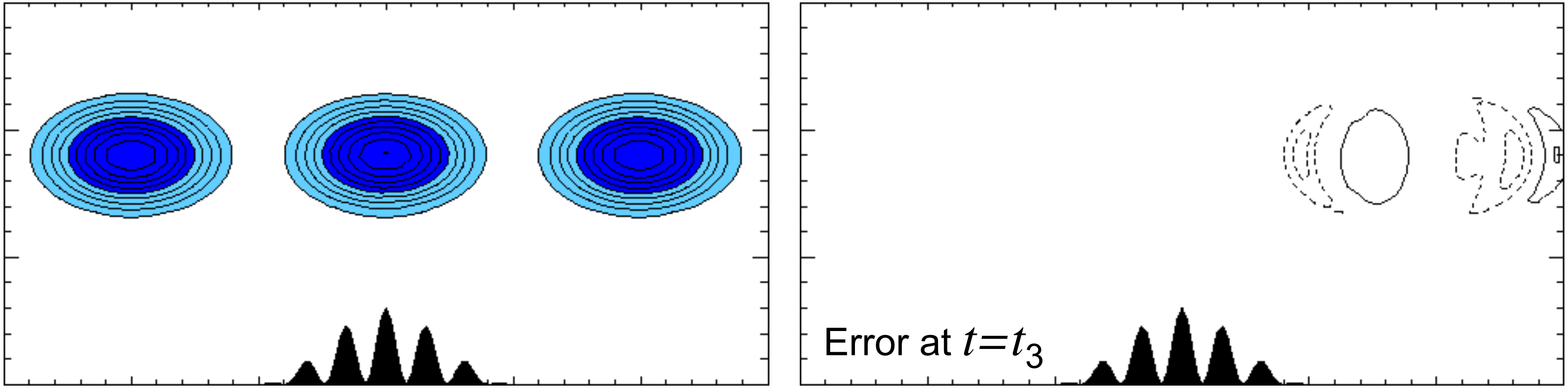


Results (Centered in Space and Time)

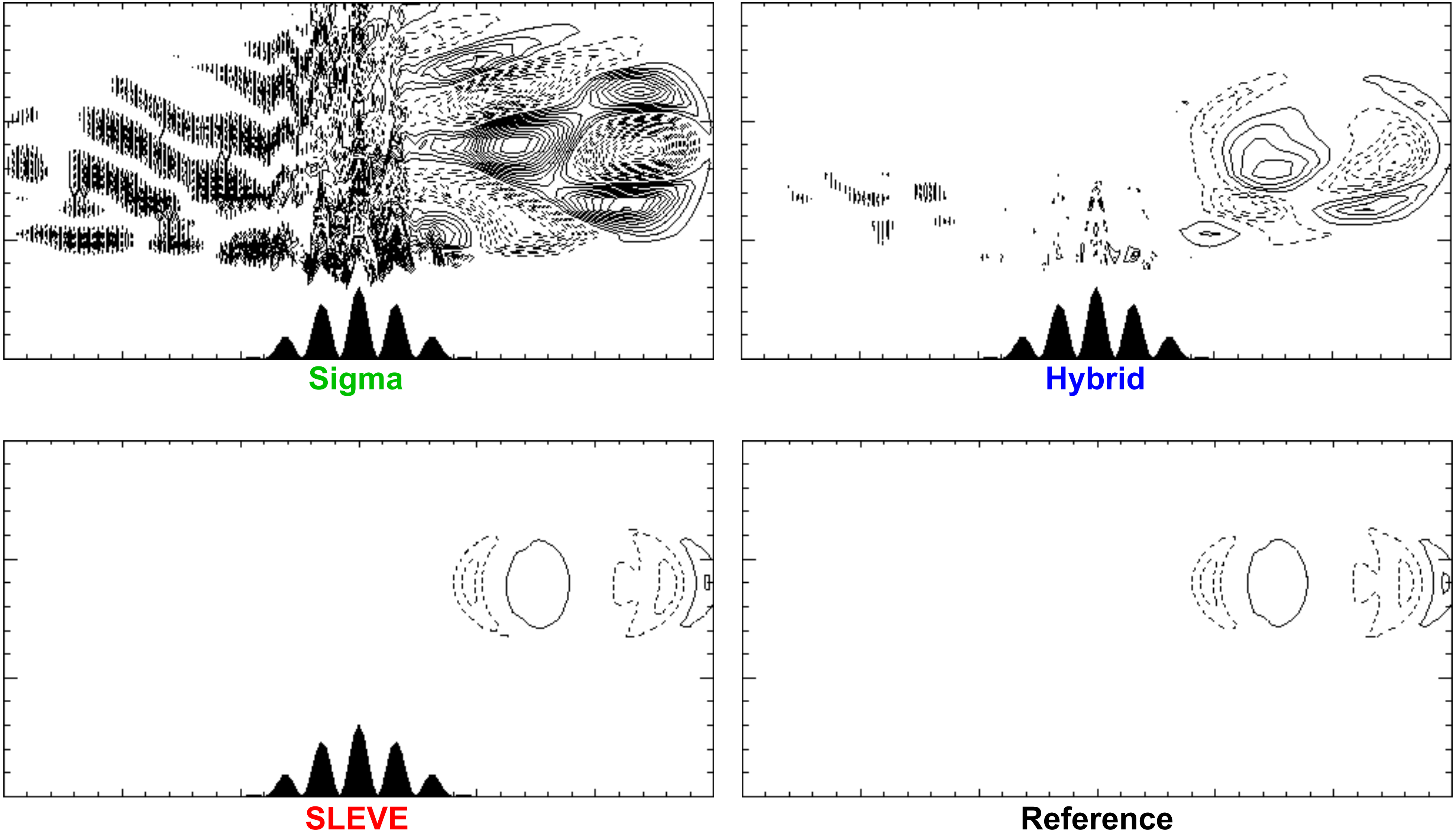
Sigma Coordinate



SLEVE Coordinate



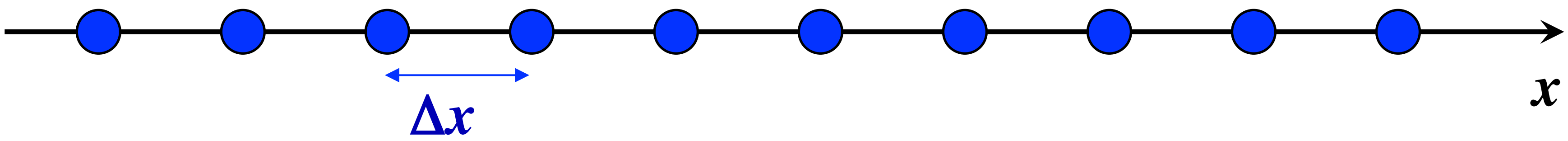
Error fields of idealized advection test



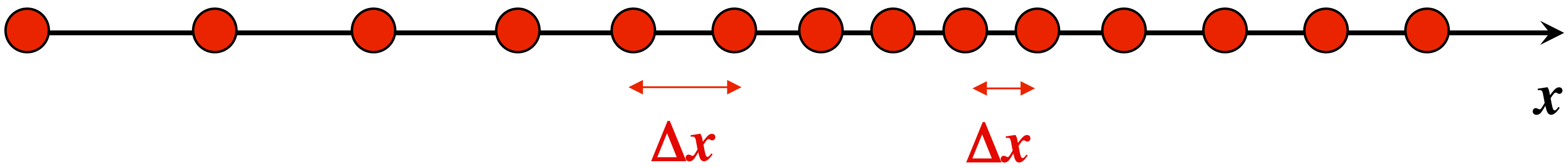
Interpretation (for 2nd-order centered scheme)

Truncation error for 1D linear advection equation: $$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} = 0$$

(a) regular grid:



(b) general grid:



Interpretation (for 2nd-order centered scheme)

Truncation error for 1D linear advection equation: $$\frac{\partial\rho}{\partial t}+\frac{\partial(u\rho)}{\partial x}=0$$

(a) regular grid:

Truncation error $$E_{reg} = \frac{\Delta x^2}{6}u\frac{\partial^3\rho}{\partial x^3}+O\left(\Delta x^3\right)$$

2nd-order scheme

(b) general grid:

Transformed equation $$\frac{\partial\rho}{\partial t}+J\frac{\partial(u\rho)}{\partial X}=0$$ with Jacobian $$J=\frac{\partial X}{\partial x}\approx\frac{\Delta X}{\Delta x}$$

have same leading order

Truncation error $$E_{trans} = \frac{\Delta x^2}{6}u\frac{\partial^3\rho}{\partial x^3} + J^2\frac{\Delta x^2}{12}u\left[\frac{\partial^2 J^{-2}}{\partial x^2}\frac{\partial\rho}{\partial x}+3\frac{\partial J^{-2}}{\partial x}\frac{\partial^2\rho}{\partial x^2}\right]+O\left(\Delta x^3\right)$$

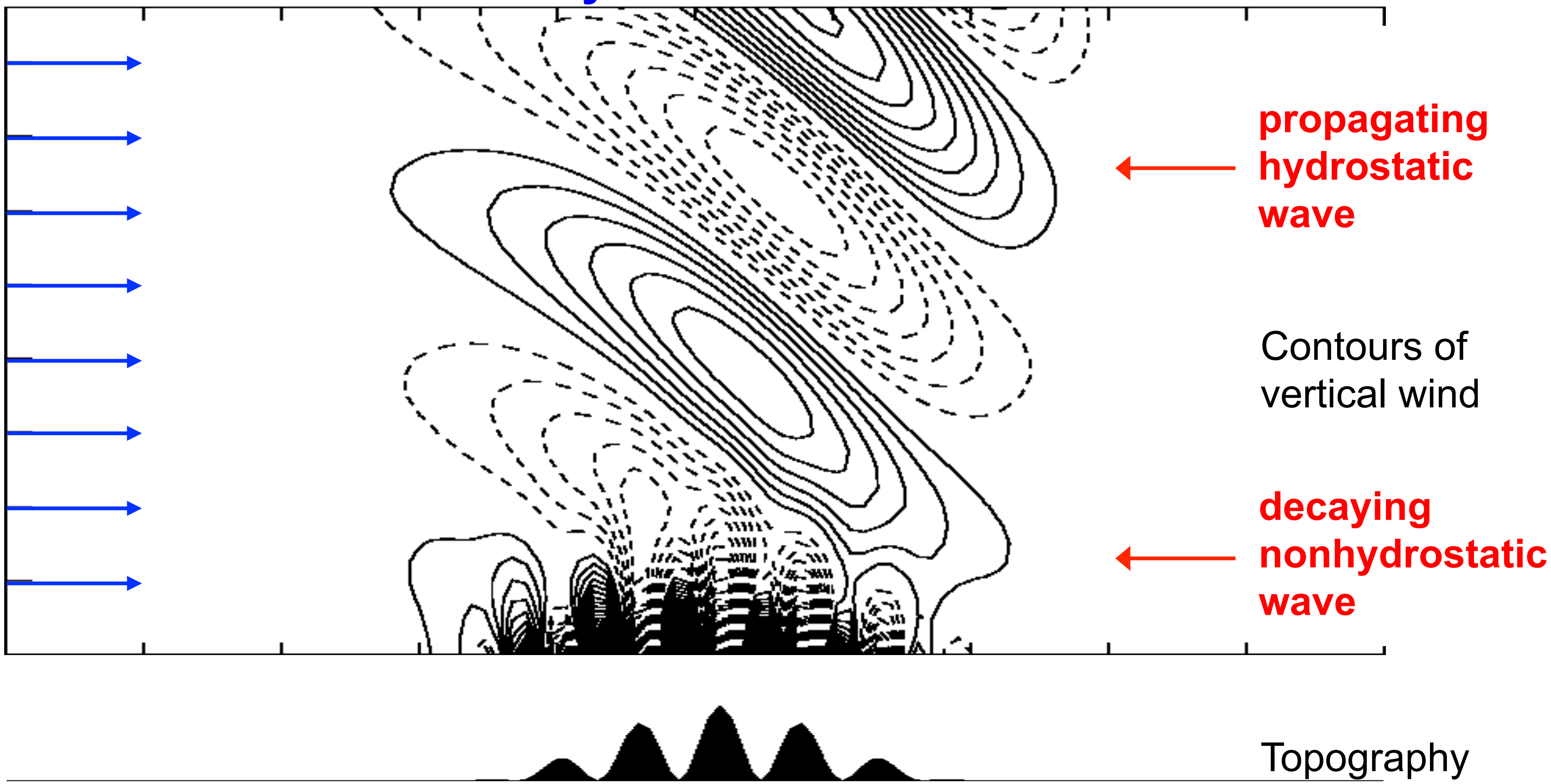
dominates in presence of small-scale anomalies

dominates in presence of small-scale grid deformations

Idealized Flow Test

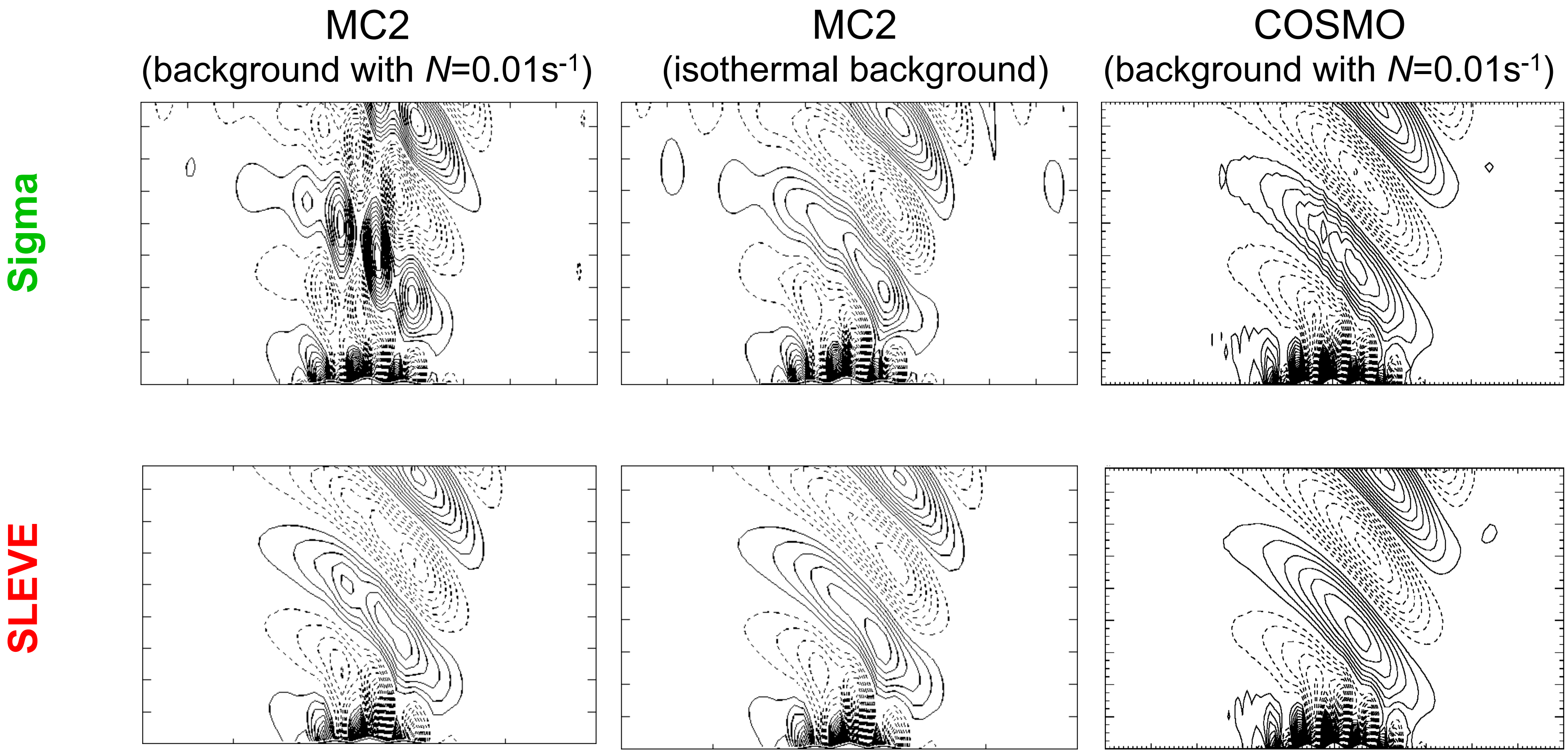
Nonhydrostatic flow past small-amplitude topography

Analytical solution



Schär et al. 2002

Idealized Flow Tests



Contours of vertical wind

Schär et al. 2002, Klemp et al. 2003

Summary

- **From a dynamical point of view, the best vertical coordinates for hydrostatic models are the <u>pressure</u> and <u>isentropic</u> coordinates.**

  The governing equations in these coordinates are actually simpler than in Cartesian coordinates, but the lower boundary conditions become very complicated.

  Use of these coordinates is thus restricted to theoretical studies and idealized numerical models.

- **The overwhelming majority of weather and climate models uses some version of <u>terrain-following</u> coordinates.**

  Particularly common are σ/p hybrid coordinates.

  Terrain-following coordinates have a simple lower boundary condition and allow for an excellent treatment of boundary-layer processes.

  The main disadvantage of these coordinates are truncation errors associated with horizontal derivatives (in particular the horizontal pressure-gradient term).