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Poisson distribution: $\lambda \in \mathbb{R}_+ , \; k \in \mathbb{N} \quad P(\lambda): \; p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

② $x \sim P(\lambda)$

$$\mathbb{E}[x] = \sum_{x \in \mathbb{N}} x \frac{\lambda^x}{x!} e^{-\lambda} = \sum \frac{x \lambda \lambda^{x-1}}{x\,(x-1)!} e^{-\lambda}$$

$$= \lambda \underbrace{\sum \underbrace{\frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda}}_{p(x-1)}}_{1 \text{ (Definition of a distribution)}} \quad \underline{\underline{= \lambda}}$$

$$Var(x) = \mathbb{E}[x^2] - (\mathbb{E}[x])^2 = \Big(\sum_{x \in \mathbb{N}} x^2 \frac{\lambda^x}{x!} e^{-\lambda} \Big) - \lambda^2 = \Big(\lambda \sum x \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda} \Big) - \lambda^2$$

$$= \Big(\lambda \sum (1 + x - 1) \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda} \Big) - \lambda^2$$

$$= \lambda \Big(\underbrace{\sum (x-1) \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda}}_{\lambda \text{ (see above)}} + \underbrace{\sum \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda}}_{1} \Big) - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}$$

③

$$P(k = X + Y) = \sum_{i=0}^{k} P(X = i, Y = k - i) = \sum P(X = i) P(Y = k - i)$$

$$= \sum \frac{\lambda^i}{i!} e^{-\lambda} \frac{\mu^{k-i}}{(k-i)!} e^{-\mu}$$

$$= e^{-(\lambda+\mu)} \sum \frac{\lambda^i}{i!} \frac{\mu^{k-i}}{(k-i)!}$$

binomial theorem

$$= e^{-(\lambda+\mu)} \frac{1}{k!} \sum \underbrace{\frac{k!}{i!(k-i)!}}_{\binom{k}{i}} \lambda^i \mu^{k-i} \quad \Big| \; (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^k}{k!} \quad \square$$

Poisson : $P(\lambda)$: $p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$$\sum_k k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \sum \lambda \underbrace{\frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}}_{p(k-1)} = \lambda \cdot 1 = \lambda$$

$$Var[X] = E[x^2] - E[x]^2 = \sum k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \sum k \cdot \lambda \cdot \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}$$

$$= \sum (1 + k-1) \cdot \lambda \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}$$

$$= \lambda \underbrace{\sum}_{1} + \lambda^2 \underbrace{\sum}_{1} \left(- \lambda^2 \right) = \lambda$$

$$\overset{!}{=} \frac{(\lambda+\mu)^k}{k!} e^{-(\lambda+\mu)}$$

$$P(k = x+Y) = \sum_{i=0}^{k} P(X=i, Y=k-i) = \sum P_c(i; \lambda) P_c(k-i, \mu) = \sum_{i=0}^{k} \frac{\lambda^i}{i!} e^{-\lambda} \frac{\mu^{k-i}}{(k-i)!} e^{-\mu}$$

$$= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\lambda^i}{i!} \frac{\mu^{k-i}}{(k-i)!}$$