SETA: supersingular encryption from torsion attacks

Antonin Leroux, joint work with L. De Feo, C. Delpech de Saint Guilhem, T. B. Fouotsa, P. Kutas, C. Petit, J. Silva, B. Wesolowski

DGA, Ecole Polytechnique, Institut Polytechnique de Paris, Inria Saclay

The State of Post-Quantum Cryptography

Six families still in Round 3 NIST post-quantum competition (Finalists + Alternate Candidates):

Lattices	4 encryption	2 signature
Codes	3 encryption	
Multivariate		2 signature
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Other encryption schemes?

Contributions

SETA: a *public key encryption* based on a *trapdoor one way function* inspired by the **torsion points attacks** from Petit, 2017.

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A concrete set of parameters for SETA and a first implementation.

A new "uber"-isogeny assumption to encompass all isogeny-based assumption.

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Introduction to isogeny-based cryptograpy

Elliptic curve and Isogeny notations

Elliptic Curve over $\mathbb{F}_q$:

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$$\varphi : E \to F, \text{ uniquely defined by } \ker \varphi$$

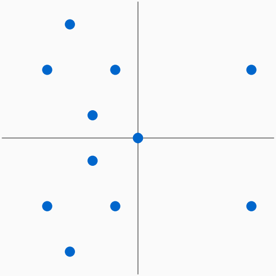
The **degree** is $\deg(\varphi) = \# \ker(\varphi)$.

The **dual** isogeny $\hat{\varphi} : F \to E$

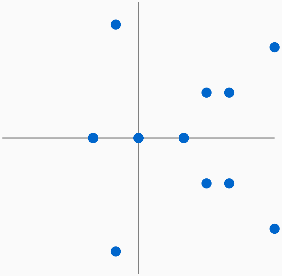
$$\hat{\varphi} \circ \varphi = [\deg(\varphi)]_E.$$

Isogenies: an example over $\mathbb{F}_{11}$

$E\ :\ y^2 = x^3 + x$



$F\ :\ y^2 = x^3 - 4x$



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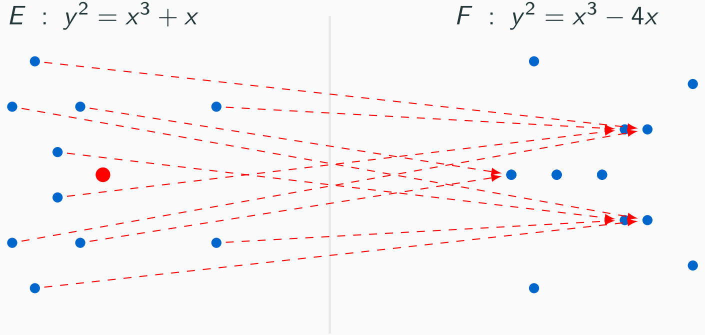

$$\varphi(x, y) = \left(\frac{x^2+1}{x},\quad y\frac{x^2-1}{x^2}\right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Endomorphism ring

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This talk $\to$ supersingular curves.

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PK_A (resp. for PK_B) include $\varphi_A(P_B), \varphi_A(Q_B)$ (resp. $\varphi_B(P_A), \varphi_B(Q_A)$)
with $E_0[N_B] = \langle P_B, Q_B\rangle$ and $E_0[N_A] = \langle P_A, Q_A\rangle$.

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Key recovery Problem

The **Computational Supersingular Isogeny (CSSI)** problem:

CSSI: Find $\varphi_A : E_0 \to E_A$ of degree N_A given E_0, E_A.

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Endomorphism Ring problem: Given a curve E, find $\mathrm{End}(E)$.

Torsion point attacks and applications

Starting curve with known endomorphism ring

Target **CSSI-T** when $\mathrm{End}(E_0)$ is known.

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$\varphi_A : E_0 \to E_A$ imply

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When $\psi = [d] + \varphi_A \circ \theta \circ \hat{\varphi}_A$ has **degree** N_B^2, $\ker \psi$ computed from $\varphi_A(P_B), \varphi_A(Q_B)$.

The torsion point attack

Once $\psi = [d] + \varphi_A \circ \theta \circ \hat{\varphi}_A$ is known:

$$\ker \hat{\varphi}_A =^1 \ker(\psi - [d]) \cap E_2[N_A]$$

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Known solutions when $N_B > pN_A$. SIDH : $N_A \approx N_B \approx \sqrt{p}$. Still Secure !

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We have a *trapdoor mechanism*!

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Let d, n, N_A, N_B be a solution of $d^2 + {N_A}^2 n = N_B^2$.

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Inversion pb is the **CSSI-T** + $\mathbb{Z}[\sqrt{-n}] \hookrightarrow \mathrm{End}(E_0)$.

Implementation and parameters

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1. Find θ of norm n and trace 0 inside *quaternion algebra*.
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Bottleneck is Step 4.

Choice of parameters

Need smooth deg and have kernel def. over $\mathbb{F}_{p^k}$ for small k (deg $|p^k - 1$).

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Torsion requirement:

- N_A, N_B with $N_B > N_A^2$ and $\gcd(N_A, N_B) = 1$ (for enc/dec).
- T, ℓ^e with $T > p^{3/2}$ and $\gcd(\ell, T) = 1$ (for key generation).

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- $T\ell^e \approx N_A N_B$ def. over $\mathbb{F}_{p^2}$: reasonnable key gen, slow enc/dec

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N_A &= 43^{12} \cdot 84719^{11}, \\
N_B &= 3^{21} \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 73 \cdot 257^{12} \cdot 313 \cdot 1009 \cdot 2857 \cdot 3733 \cdot 5519 \cdot 696 \\
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Implementation result

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Keygen	Encryption	Decryption
10h	4.6s	10.6m

Table 1: SETA performances

Uber-isogeny assumption

CSIDH and group actions

Quadratic order $\mathfrak{O}$,

$$\mathcal{F}_{\mathfrak{O}} = \{(E,\iota) \mid \iota : \mathfrak{O} \hookrightarrow \mathrm{End}(E)\}, \ \ \mathcal{E}_{\mathfrak{O}} = \{E \mid \exists \iota, (E,\iota) \in \mathcal{F}_{\mathfrak{O}}\}.$$

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Castryck et al., "CSIDH: an efficient post-quantum commutative group action," 2018

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Uber isogeny problem

The $\mathfrak{O}$-**Uber-isogeny problem ($\mathfrak{O}$-UIP)**

$\mathfrak{O}$-**UIP**: Given $(E_0, \iota_0) \in \mathcal{F}_{\mathfrak{O}}$ and $E \in \mathcal{E}_{\mathfrak{O}}$. Find $\mathfrak{a}$ such that
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If ι is given, there is a subexponential algorithm in disc $\mathfrak{O}$. This is the case for **CSIDH** where ι is trivial from Frobenius.

Link with isogeny-based crypto

CSIDH: $\mathfrak{O} = \mathbb{Z}[\sqrt{-p}]$.

$\mathbb{Z}[\sqrt{-p}]$-UIP $\Leftrightarrow$ **CSIDH** key recovery.

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CSSI: Exists $\mathfrak{O}$ such that every curve $E \in \mathcal{E}_{\mathfrak{O}}$.

$\mathfrak{O}$-UIP $\Rightarrow$ **CSSI**.

Conclusion and Open Problems

We presented a new post-quantum encryption scheme inspired by torsion attacks on SIDH. Efficiency is still order of magnitudes below SIDH but security relies on new problems. We have introduced the $\mathfrak{O}$-UIP, a new generic assumption for isogeny-based cryptography.

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https://eprint.iacr.org/2019/1291