The Shape of Space [1]

Topology vs Geometry

Topology of a surface: the aspect of a surface's nature that is unaffected by deformation.
Geometry of a surface: consists of the properties that **DO** change when the surface is deformed.
Geometrical properties: curvature (most important), areas, distances, angles...

Intrinsic vs Extrinsic properties

Intrinsic topology: same intrinsic topology if inside the surface one cannot tell them apart.
Extrinsic topology: same extrinsic topology if one can be deformed within a higher-dimensional space to look like the other.
Intrinsic geometry: properties of the surface.
Extrinsic geometry: only to be appreciated from higher dimensions.
Geodesic: intrinsically straight line.

Local vs Global properties

Local properties are those observable within a small region of the manifold.
Global properties require consideration of the manifold as a whole.
Homogeneous manifold: one whose local geometry is the same at all points.
Remark: most often used in "local geometry" and "global topology".
Examples:

- a two-dimensional manifold (surface) is a space with local topology of a plane. All two-manifolds have the same local topology.
- a three-dimensional manifold is a space with local topology of "ordinary" 3D space; they all have the same local topology.

Close vs Open

Intuitively, closed means finite and open means infinite.
Remark 1: Edges: anything with edges is NOT EVEN a manifold (manifold-with-boundary; terms closed and open imply manifold has NO edges).
Remark 2: Area: there are surfaces that are infinitely long, yet only have a finite area (cusp).
— By convention, a surface is classified as closed or open accordingly to its distance across rather than its area (cusp = open).

Orientability

Manifolds that do not contain orientation-reversing paths are called orientable.

	orientable	**non-orientable**
curved local geometry	sphere	projective plane
flat local geometry	torus	klein-bottle

Common Manifolds

Most simple manifolds have shorthand names:

one-dim	**surfaces**	**three-manifolds**
$\mathbf{E}^1$ The line	$\mathbf{E}^2$ Euclidean plane	$\mathbf{E}^3$ Euclidean space
$\mathbf{S}^1$ The circle	$\mathbf{S}^2$ The sphere	
$\mathbf{I}$ The interval	$\mathbf{T}^2$ The torus	$\mathbf{T}^3$ The three-torus
	$\mathbf{K}^2$ Klein bottle	
	$\mathbf{P}^2$ Projective plane	$\mathbf{P}^3$ Projective 3-space
	$\mathbf{D}^2$ The disk	$\mathbf{D}^3$ A solid ball

Connected Sums

— # (connect sum) operation
Remark: every conceivable surface is a connected sum of tori and/or projective planes!
— a sphere is a connected sum of zero tori and zero projective planes.
Remark: every surface is a connected sum of either tori **only** or projective planes **only**!
— a surface written as a connected sum of both tori and projective planes can ALWAYS be written as a connected sum of projective planes only.
— without changing the surface's global topology, tori can be converted to Klein bottles, and Klein bottles to projective planes.

Products

— $\times$ (cross) operation
Remark: the torus $\mathbf{T}^2$ is the only closed surface (with no edges) that is a product.
— $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ is the only two-dimensional product having neither an edge nor an infinite area.
Geometrical product must satisfy the following 3 conditions:

1. All elements of first term are the same size
2. All elements of second term are the same size
3. Each element of first term is perpendicular to each element of second term

Isotropic

An **isotropic** manifold is one in which the geometry is the same in all directions.

References

[1] J. R. Weeks, *The shape of space.* CRC press, 2001.