The Shape of Space [1]

Topology vs Geometry

Topology of a surface: the aspect of a surface's nature that is unaffected by deformation.
Geometry of a surface: consists of the properties that **DO** change when the surface is deformed.
Geometrical properties: curvature (most important), areas, distances, angles...

Intrinsic vs Extrinsic properties

Intrinsic topology: same intrinsic topology if inside the surface one cannot tell them apart.
Extrinsic topology: same extrinsic topology if one can be deformed within a higher-dimensional space to look like the other.
Intrinsic geometry: properties of the surface.
Extrinsic geometry: only to be appreciated from higher dimensions.
Geodesic: intrinsically straight line.

Local vs Global properties

Local properties are those observable within a small region of the manifold.
Global properties require consideration of the manifold as a whole.
Homogeneous manifold: one whose local geometry is the same at all points.
Remark: most often used in "local geometry" and "global topology".
Examples:

- a two-dimensional manifold (surface) is a space with local topology of a plane. All two-manifolds have the same local topology.
- a three-dimensional manifold is a space with local topology of "ordinary" 3D space; they all have the same local topology.

Close vs Open

Intuitively, closed means finite and open means infinite.
Remark 1: Edges: anything with edges is NOT EVEN a manifold (manifold-with-boundary; terms closed and open imply manifold has NO edges).
Remark 2: Area: there are surfaces that are infinitely long, yet only have a finite area (cusp).
— By convention, a surface is classified as closed or open accordingly to its distance across rather than its area (cusp = open).

Orientability

Manifolds that do not contain orientation-reversing paths are called orientable.

	orientable	**non-orientable**
curved local geometry	sphere	projective plane
flat local geometry	torus	klein-bottle

Common Manifolds

Most simple manifolds have shorthand names:

one-dim	**surfaces**	**three-manifolds**
$\mathbf{E}^1$ The line	$\mathbf{E}^2$ Euclidean plane	$\mathbf{E}^3$ Euclidean space
$\mathbf{S}^1$ The circle	$\mathbf{S}^2$ The sphere	
I The interval	$\mathbf{T}^2$ The torus	$\mathbf{T}^3$ The three-torus
	$\mathbf{K}^2$ Klein bottle	
	$\mathbf{P}^2$ Projective plane	$\mathbf{P}^3$ Projective 3-space
	$\mathbf{D}^2$ The disk	$\mathbf{D}^3$ A solid ball

Connected Sums

— # (connect sum) operation
Remark: every conceivable surface is a connected sum of tori and/or projective planes!
— a sphere is a connected sum of zero tori and zero projective planes.
Remark: every surface is a connected sum of either tori **only** or projective planes **only**!
— a surface written as a connected sum of both tori and projective planes can ALWAYS be written as a connected sum of projective planes only.
— without changing the surface's global topology, tori can be converted to Klein bottles, and Klein bottles to projective planes.

Products

— × (cross) operation

References

[1] J. R. Weeks, *The shape of space*. CRC press, 2001.