Dynamics Cheat Sheet

Linear Momentum L

Definition: total force acting on a particle is equal to the time rate of change of its linear momentum.
Equation: $\mathbf{L} = m\mathbf{v}$
- for a system of particles (G: center of mass):

$$\mathbf{L} = \sum_i m_i \mathbf{v}_i = m\mathbf{v}_G$$

Time rate of change: $\dot{\mathbf{L}} = \mathbf{F}$
- for a system of particles (G: center of mass):

$$\dot{\mathbf{L}} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i = \underbrace{\mathbf{F}}_{\text{external forces}} = m\mathbf{a}_G$$

Angular Momentum H

Definition: "moment" of the particle's linear momentum $\mathbf{L}$ about a point O.
Equation: $\mathbf{H}_O = \mathbf{r} \times \mathbf{L} = \mathbf{r} \times m\mathbf{v}$
Equation: $\mathbf{H}_O = \mathbf{r}_G \times m\mathbf{v}_G + \mathbf{H}_G$
Equation: $\mathbf{H}_G = [\mathbf{I}_G]\boldsymbol{\omega}$
- for a system of particles (about fixed point O):

$$\mathbf{H}_O = \sum_i (\mathbf{r}_i \times m_i \mathbf{v}_i)$$

Equation: taken about the center of mass G:

$$\mathbf{H}_G = \sum_i (\mathbf{r}'_i \times m\mathbf{v}_i) \quad | \; \mathbf{v}_i = \dot{\mathbf{r}}_i = \dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i = \mathbf{v}_G + \mathbf{v}'_i \tag{1}$$

$$= \sum_i \left(\mathbf{r}'_i \times m\left(\mathbf{v}_G + \mathbf{v}'_i\right)\right)$$

$$= \sum_i (\mathbf{r}'_i \times m\mathbf{v}_G) + \sum_i (\mathbf{r}'_i \times m\mathbf{v}'_i) \quad | \; -\mathbf{v}_G \times \sum_i m_i \mathbf{r}'_i = 0$$

$$\mathbf{H}_G = \sum_i (\mathbf{r}'_i \times m\mathbf{v}'_i) \tag{2}$$

Equation (1) is the ***absolute angular momentum***, whereas equation (2) is the ***relative angular momentum***; when taken about G, these quantities are identical.
Time rate of change: wrt fixed point O

$$\frac{\mathrm{d}\mathbf{H}_O}{\mathrm{d}t} = \cancelto{0}{\dot{\mathbf{r}} \times m\mathbf{v}} + \mathbf{r} \times m\dot{\mathbf{v}} \quad | \; \dot{\mathbf{r}} = \mathbf{v}$$
$$= \mathbf{r} \times m\mathbf{a} \quad | \; \mathbf{F} = m\mathbf{a}$$
$$= \mathbf{r} \times \mathbf{F}$$
$$= \mathbf{M}_O$$

- for a system of particles (about fixed point O):

$$\dot{\mathbf{H}}_O = \sum_i \cancelto{0}{\left(\dot{\mathbf{r}}_i \times m_i \mathbf{v}_i\right)} + \sum_i (\mathbf{r}_i \times m_i \mathbf{a}_i)$$
$$= \sum_i (\mathbf{r}_i \times \mathbf{F}_i) + \sum_i M_i \quad | \; M_i = \text{external moments}$$
$$= \mathbf{M}_O$$

Time rate of change: about CoM G:

$$\dot{\mathbf{H}}_G = \sum_i (\mathbf{r}'_i \times m\dot{\mathbf{v}}'_i) = \sum_i (\mathbf{r}'_i \times \mathbf{F}_i) + \sum_i M_i = \mathbf{M}_G,$$

where $\mathbf{M}_G$ is the total moment about G of the applied external forces plus any external moments. Expression valid for any movement of G!
Time rate of change: 3D rigid body, body-fixed frame:

$$\dot{\mathbf{H}} = [\mathbf{I}]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{M}$$
$$M_x = \dot{H}_x - H_y\omega_z + H_z\omega_y$$
$$M_y = \dot{H}_y - H_z\omega_x + H_x\omega_z$$
$$M_z = \dot{H}_z - H_x\omega_y + H_y\omega_x$$

Remark: internal forces and motions **will not** change angular momentum!

Conservation of Angular Momentum

Several ways to express, but the best choices are:
(i) about the center of mass G:

$$I_G\boldsymbol{\alpha} = \mathbf{M}_G = \dot{\mathbf{H}}_G$$

(ii) about fixed point O:

$$I_O\boldsymbol{\alpha} = \mathbf{M}_O$$

Angular Impulse

$$\int_{t_1}^{t_2} \mathbf{M}_O dt = \int_{t_1}^{t_2} \dot{\mathbf{H}}_O dt = \mathbf{H}_O(t_2) - \mathbf{H}_O(t_1) = \Delta\mathbf{H}_O$$

"Useful when dealing with impulsive forces. Possible to calculate the integrated effect of a force on a particle without knowing in detail the actual value of the force as a function of time."

Kinetic Energy T

Equation: $T = T_T + T_R$, translational and rotational parts
Equation: for a system of particles:

$$T = \sum_i T_i = \sum_i \left(\frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i\right) \quad | \text{ relative and } G \text{ parts}$$
$$= \frac{1}{2} m v_G^2 + \sum_i \frac{1}{2} m_i {v'_i}^2$$

Equation: for a 3D rigid body:

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H}_G$$
$$= \underbrace{\frac{1}{2}\mathbf{v}_G \cdot \mathbf{L}_G}_{\text{translation}} + \underbrace{\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_G}_{\text{rotation}}$$

Equation: fxd pt O about which body rotates:

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H}_O$$

Equation: instantaneous center of rot C

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H}_C$$

Equation: if $\boldsymbol{\omega}$ expressed in principal axes

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right)$$

Remark: can provide considerable insight into the behavior of a rotating body.
Remark: no assurance that energy will be conserved; internal motions can dissipate energy.

General Motion (Chasles' Theorem)

Velocity and acceleration of a point P in a rigid body:

$$\mathbf{v}_P = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r}'_P$$
$$\mathbf{a}_P = \mathbf{a}_{O'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_P),$$

where O' is a known point in the rigid body.

Moments of Inertia

Mass distribution, always positive. $(\cdot)'$ relative to point

$$I_{xx} = \int_m \left({y'}^2 + {z'}^2\right) dm$$
$$I_{yy} = \int_m \left({x'}^2 + {z'}^2\right) dm$$
$$I_{zz} = \int_m \left({x'}^2 + {y'}^2\right) dm$$

Products of Inertia

Imbalances in the mass distribution; pos, neg, or 0. $(\cdot)'$ relative to point

$$I_{xy} = I_{yx} = \int_m x'y'dm$$
$$I_{xz} = I_{zx} = \int_m x'z'dm$$
$$I_{yz} = I_{zy} = \int_m y'z'dm$$

Tensor of Inertia

How mass is distributed in a rigid body. Symmetric matrix. Causes $\mathbf{H}$ and $\boldsymbol{\omega}$ to **not** necessarily be parallel!

$$[\mathbf{I}_G] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Principal Axes of Inertia

Frame of reference for which products of inertia are always 0 (diagonal $[\mathbf{I}_G]$).

$$\mathbf{H}_G = I_x\omega_x\hat{\mathbf{i}} + I_y\omega_y\hat{\mathbf{j}} + I_z\omega_z\hat{\mathbf{k}}$$

Parallel Axis Theorem

Get inertia tensor about different axes (about O). $(\cdot)'$ relative to G

$$(I_{xx})_O = I_{xx} + m(y_G^2 + z_G^2)$$
$$(I_{xy})_O = I_{xy} + m x_G y_G$$

*Principal Axes as Eigenvalue Problem

Center of Mass

$$\mathbf{r}_G = \frac{1}{m}\left(\sum_i m_i \mathbf{r}_i\right); m = \sum_i m_i$$

Tangential Velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

General Eqns for 3D Rotational Motion

The governing equations are those of conservation of:

$$\text{linear momentum } \mathbf{L} = M\mathbf{v}_G,\ \dot{\mathbf{L}} = \mathbf{F}$$

$$\text{angular momentum } \mathbf{H} = [\mathbf{I}]\boldsymbol{\omega},\ \dot{\mathbf{H}}_G = \mathbf{M}_G$$

Euler's Equations

Body-fixed frame aligned with principal axes:

$$\begin{aligned}
M_x &= I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \\
M_y &= I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \\
M_z &= I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y
\end{aligned}$$

Coreolis Theorem

Time derivative of a vector in a rotating frame. Consider W a fixed frame, B a rotating frame with angular velocity $\mathbf{\Omega}$, and a vector $\mathbf{r}$:

$$^{W}\dot{\mathbf{r}} = \frac{\mathrm{d}\,^{W}\mathbf{r}}{\mathrm{d}t} = {}^{B}\dot{\mathbf{r}} + \mathbf{\Omega} \times {}^{B}\mathbf{r}$$

*Space and Body Cones

*Conservation of Energy