Dynamics of Free Flying Space Manipulators

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Contents

1 Notation

m_i	mass of link i
m_b	mass of the base
μ_i	mass fraction of link i
N	number of manipulator links
FFSM	free-flying space manipulator
M	total mass of combined system
CM	centre of mass
$\vec{\boldsymbol{\rho}}_{\boldsymbol{i}}$	physical vector from manipulator CM to CM of link i
$\vec{\boldsymbol{d}}_{\boldsymbol{i}}$	physical vector from inertial origin to CM of link i
$\boldsymbol{r_{CM}}$	location of manipulator CM in inertial frame
$\boldsymbol{r_E}$	location of camera link in inertial frame
$\vec{\boldsymbol{r}}_{\boldsymbol{i}}$	body-fixed vector from CM of link i to joint i+1
$\vec{\boldsymbol{\ell}}_{\boldsymbol{i}}$	body-fixed vector from CM of link i to joint i
$\vec{\boldsymbol{c}}_{\boldsymbol{i}}$	body-fixed vector from CM of link i to barycentre of link i
$\vec{\boldsymbol{r}}^*_{\boldsymbol{i}}, \vec{\boldsymbol{\ell}}^*_{\boldsymbol{i}}, \vec{\boldsymbol{c}}^*_{\boldsymbol{i}}$	body-fixed barycentric vectors of link i, see Fig. 2
$\boldsymbol{\tau}$	joint torque vector
$\vec{\boldsymbol{q}}$	generalized coordinates, including manipulator joint angles
$\vec{\boldsymbol{\omega}}_{\boldsymbol{i}}$	angular velocity of link i
ε, η	Vector and scalar parts of the base attitude quaternion
$\boldsymbol{R_0}$	rotation matrix between the spacecraft base and the inertial frame
${}^{j}\boldsymbol{R_i}$	rotation matrix relating frame i to frame j
${}^{\boldsymbol{0}}\boldsymbol{R_i}$	rotation matrix between base and i^{th} frame
$\tilde{\boldsymbol{R}}$	diag($\boldsymbol{R}, \boldsymbol{R}$) where R is any rotation matrix
$\boldsymbol{A_{i,i-1}}$	twist-propagation matrix from frame $i-1$ to frame i
${}^{\boldsymbol{0}}\boldsymbol{I_i}$	Inertia tensor of link i expressed in spacecraft frame
$\boldsymbol{I_i}$	inertia tensor of link i expressed in frame i (Constant)
$\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}$	dyadic product of vectors $\vec{a}$ and $\vec{b}$
$\mathbf{1}$	identity matrix
$\mathbf{0}$	zero matrix

Vectors with arrows are physical vectors; matrices and dyadics are always written in bold without arrows. When written in bold face without an arrow, vectors are expressed in a given frame and the vectrix notation is used. Often a left prescript will accompany a vector or matrix which has been resolved into a frame. If there is no prescript, vectors without arrows are assumed to be expressed in the inertial frame unless specified otherwise. In vectrix notation, the dot product $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}$ is replaced by the inner product $\boldsymbol{v^T v}$ and the dyadic product $\vec{\boldsymbol{b}}\vec{\boldsymbol{b}}$ by the outer product $\boldsymbol{bb^T}$.

2 Introduction

There is a considerable body of literature surrounding the dynamics of free-flying space manipulators (FFSM). These robotic arms have a base which can translate and rotate in space, in environments where the acceleration due

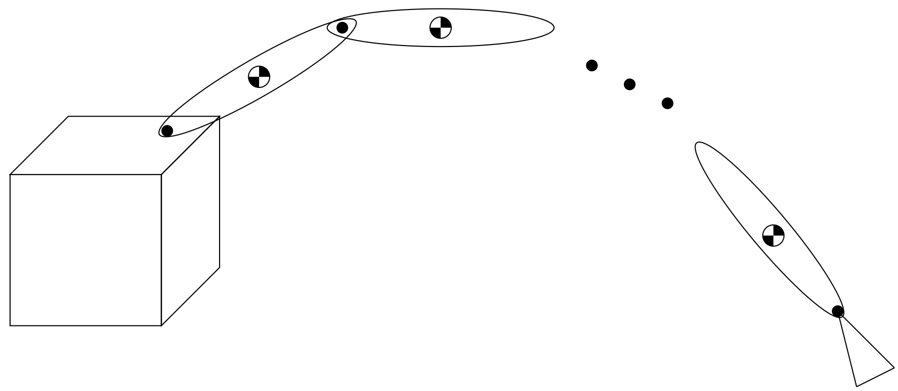

Figure 1: A free-flying space manipulator with camera

to gravity is low or zero. The base is powered by gas jet thrusters and its orientation is controlled by momentum exchange reaction wheels. A major difference between space manipulators and fixed-base terrestrial manipulators is that the state of each link, ie. its position and velocity, is affected by all of the links in the chain, not only by those which come before it. As well, the dynamics of the arm and the base are coupled, ie. motion of the arm causes motion of the base. In this report, the links and base are assumed to be rigid, and flexibility is not considered. All joints are assumed to be revolute. The inertia of the motors is not considered. The robot to be considered is an inspection robot, which must closely observe a satellite in orbit. To achieve this task, a camera is placed at the end of the arm, instead of an end effector.

There are several competing formulations for these dynamics and each one is carefully considered. The Lagrangian and quasi-Lagrangian momentum formulations are considered first through the barycentric vectors initially derived in [10]. The second is the Generalized Jacobian approach of [16]. Last but not least is the Decoupled Natural Orthogonal Complement (DeNOC) matrices formulation, which builds on the recursive Newton-Euler dynamics. Details of this can be found in [12] and [5]. This formulation is used in the MATLAB software package SPART [18]. There are still other formulations, for example the direct-path method of [7], however the DeNOC formulation is the simplest and most computationally efficient [17]. A later report may consider the direct path method, especially as it relates to the effect of gravity gradient torque during orbital change manoeuvres.

3 Barycentric Vector Formulation

Before implementing thrusters and reaction wheels, it is crucial to calculate the kinematics of a *free-floating* space manipulator, ie. one where the base is free to move in response to the commanded motions of the joints in the arm. There is a considerable body of literature around this topic. Wilde et al. [19] use the familiar Denavit-Hartenberg parameters for a treatment which is straightforward and easy to understand. Unfortunately, it does not adequately address the rotational terms in the Lagrangian. A better treatment of rotation and angular momentum is given by Nanos and Papadopoulos [8]. To reduce the amount of calculation, their barycentric vector formulation is used in this section of the report.

3.1 Barycentric Vectors

Figure 2 is a representation of the barycentre and the vectors $\vec{\boldsymbol{\ell}}_{\boldsymbol{i}}^{*}, \vec{\boldsymbol{r}}_{\boldsymbol{i}}^{*}$ and $\vec{\boldsymbol{c}}_{\boldsymbol{i}}^{*}$.

With the notation defined as in Section 1, the inertial position of the camera can be defined as

$$\vec{\boldsymbol{r}}_{\boldsymbol{E}} = \vec{\boldsymbol{r}}_{\boldsymbol{CM}} + \vec{\boldsymbol{\rho}}_{\boldsymbol{N}} + \vec{\boldsymbol{r}}_{\boldsymbol{N}} \tag{1}$$

Furthermore, displacements between links' CMs can be written in terms of the body-fixed vectors $\boldsymbol{\ell_i}$ and $\boldsymbol{r_i}$

$$\vec{\boldsymbol{\rho}}_{\boldsymbol{i}} - \vec{\boldsymbol{\rho}}_{\boldsymbol{i-1}} = \vec{\boldsymbol{r}}_{\boldsymbol{k-1}} - \vec{\boldsymbol{\ell}}_{\boldsymbol{k}} \tag{2}$$

As the $\vec{\boldsymbol{\rho}}_{\boldsymbol{i}}$ vectors are defined in terms of the system CM, by definition

$$\sum_{i=0}^{N} m_i \vec{\boldsymbol{\rho}}_{\boldsymbol{i}} = 0 \tag{3}$$

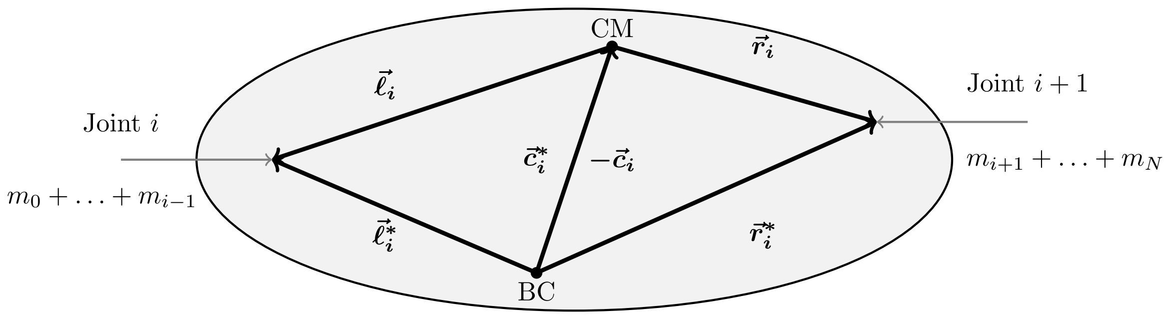

Figure 2: Body i

Solving equations 2 and 3 together results in [11]

$$\vec{\boldsymbol{\rho}}_{\boldsymbol{k}} = \sum_{i=1}^{k} (\vec{\boldsymbol{r}}_{\boldsymbol{i-1}} - \vec{\boldsymbol{\ell}}_{\boldsymbol{i}})\mu_i - \sum_{i=k+1}^{N} (\vec{\boldsymbol{r}}_{\boldsymbol{i-1}} - \vec{\boldsymbol{\ell}}_{\boldsymbol{i}})(1-\mu_k) \qquad i = 0, ..., N \tag{4}$$

where the mass distribution μ_i is given by

$$\mu_i = \begin{cases} 0 & i = 0 \\ \displaystyle\sum_{j=0}^{i-1} \frac{m_j}{M} & i = 1, ..., N \\ 1 & i = N+1 \end{cases} \tag{5}$$

Simplify equation 4 by using the barycentric vectors as defined in Figure 2 and described in [11] and [10]. The barycentre of body i is defined as the centre of mass of the augmented body created by adding a point mass equal to the total mass of the preceding links ($M\mu_i$) to joint i and a similar point mass equal to the total mass of the subsequent links ($M(1-\mu_i)$), to joint $i+1$. With this definition, the barycentre's location with respect to the link's CM is given by

$$\vec{\boldsymbol{c}}_{\boldsymbol{i}} = \vec{\boldsymbol{\ell}}_{\boldsymbol{i}}\mu_i + \vec{\boldsymbol{r}}_{\boldsymbol{i}}(1-\mu_i) \qquad i = 0, ..., N \tag{6}$$

The barycentric vectors are then defined by

$$\vec{\boldsymbol{c}}_{\boldsymbol{i}}^{*} = -\vec{\boldsymbol{c}}_{\boldsymbol{i}} \tag{7}$$

$$\vec{\boldsymbol{r}}_{\boldsymbol{i}}^{*} = \vec{\boldsymbol{r}}_{\boldsymbol{i}} - \vec{\boldsymbol{c}}_{\boldsymbol{i}} \tag{8}$$

$$\vec{\boldsymbol{\ell}}_{\boldsymbol{i}}^{*} = \vec{\boldsymbol{\ell}}_{\boldsymbol{i}} - \vec{\boldsymbol{c}}_{\boldsymbol{i}} \tag{9}$$

With these equations and equation 6 one can rewrite equation 4 in a much more compact format[10]:

$$\vec{\boldsymbol{\rho}}_{\boldsymbol{k}} = \sum_{i=0}^{N} \vec{\boldsymbol{v}}_{\boldsymbol{ik}} \qquad k = 0, ..., N \tag{10}$$

Here, ${}^{i}\boldsymbol{v}_{\boldsymbol{ik}}$ is defined as a body-fixed vector in frame i by selection as one of the barycentric vectors.

$$\vec{\boldsymbol{v}}_{\boldsymbol{ik}} = \begin{cases} \vec{\boldsymbol{r}}_{\boldsymbol{i}}^{*} & i < k \\ \vec{\boldsymbol{c}}_{\boldsymbol{i}}^{*} & i = k \\ \vec{\boldsymbol{\ell}}_{\boldsymbol{i}}^{*} & i > k \end{cases} \tag{11}$$

3.2 Forward Kinematics

With this formulation it is very straightforward to determine the position of the camera in inertial space. [8] defines the body-fixed vector ${}^{i}\boldsymbol{v}_{\boldsymbol{iN,E}} = {}^{i}\boldsymbol{v}_{\boldsymbol{iN}} + \delta_{iN}{}^{i}\boldsymbol{r}_{\boldsymbol{N}}$, where δ is the Kronecker delta. With this notation, equation 1 can be rewritten in the inertial frame as

$$\boldsymbol{r}_{\boldsymbol{E}} = \boldsymbol{r}_{\boldsymbol{CM}} + \boldsymbol{R}_{\boldsymbol{0}} \sum_{i=0}^{N} {}^{\boldsymbol{0}}\boldsymbol{R}_{\boldsymbol{i}}{}^{i}\boldsymbol{v}_{\boldsymbol{iN,E}} \tag{12}$$

where $\boldsymbol{r}_{\boldsymbol{CM}}$ is the location of the combined system centre of mass.

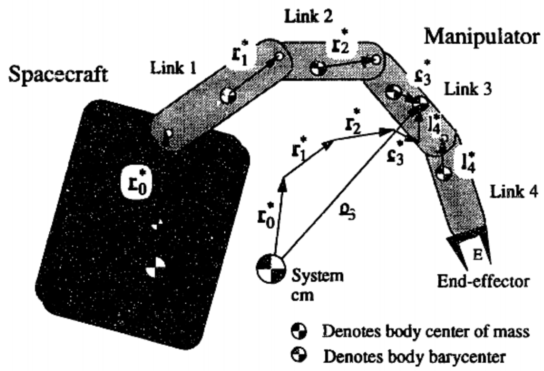

Figure 2.3. Construction of ρ_k for k=3 and N=4.

Figure 3: Construction of $\vec{\rho_k}$ from barycentric vectors [10]

3.3 Attitude Kinematics

The attitude of the base spacecraft with respect to the inertial frame is described by the quaternion $(\boldsymbol{\varepsilon}, \eta)$ and its associated rotation matrix

$$\boldsymbol{R_0} = (1 - 2\boldsymbol{\varepsilon^T}\varepsilon)\mathbf{1} + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon^T} + 2\eta(\boldsymbol{\varepsilon})^{\times} \tag{13}$$

. The full rotation matrix of the camera is given by

$$\boldsymbol{R_i}(\varepsilon, \eta, \vec{\boldsymbol{q}}) = \boldsymbol{R_0}{}^{\mathbf{0}}\boldsymbol{R_i}(\vec{\boldsymbol{q}}) \tag{14}$$

$$= \boldsymbol{R_0}{}^{\mathbf{0}}\boldsymbol{R_1}(q_1){}^{\mathbf{1}}\boldsymbol{R_2}(q_2)...{}^{\boldsymbol{i-1}}\boldsymbol{R_i}(q_i) \tag{15}$$

$$\tag{16}$$

In this analysis the constant Denavit-Hartenberg parameter α_i is used to relate the rotation axes of successive joints.

$${}^{\boldsymbol{i-1}}\boldsymbol{R_i}(q_i) = \begin{bmatrix} \cos(q_i) & -\cos(\alpha_i)\sin(q_i) & \sin(\alpha_i)\sin(q_i) \\ \sin(q_i) & \cos(\alpha_i)\cos(q_i) & -\cos(q_i)\sin(\alpha_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix} \tag{17}$$

The quaternion corresponding to the orientation of the camera can be easily extracted from the final rotation matrix.

3.4 Velocity Kinematics

Differentiating an equation as compact as (12) is straightforward. The vectors fixed in body i rotate with angular velocity $\vec{\boldsymbol{\omega}}_{\boldsymbol{i}}$. This means that

$$\dot{\vec{\boldsymbol{\rho}}}_{\boldsymbol{k}} = \sum_{i=0}^{N} \vec{\boldsymbol{\omega}}_{\boldsymbol{i}} \times \vec{\boldsymbol{v}}_{\boldsymbol{ik}} \tag{18}$$

Differentiate equation (1) with respect to time in the inertial frame and combine with this result to find the linear velocity of the camera. The Jacobian terms ${}^{\mathbf{0}}\boldsymbol{J_{11}}, {}^{\mathbf{0}}\boldsymbol{J_{12}}$ and ${}^{\mathbf{0}}\boldsymbol{J_{22}}$ are defined below.

$$\dot{\boldsymbol{r_E}} = \dot{\boldsymbol{r}}_{\boldsymbol{CM}} + \sum_{i=0}^{N} \boldsymbol{\omega_i^{\times}} \boldsymbol{v_{iN}} + \boldsymbol{\omega_N^{\times}} \boldsymbol{r_N} \tag{19}$$

$$= \dot{\boldsymbol{r}}_{\boldsymbol{CM}} + \boldsymbol{R_0}({}^{\mathbf{0}}\boldsymbol{J_{11}}{}^{\mathbf{0}}\boldsymbol{\omega_0} + {}^{\mathbf{0}}\boldsymbol{J_{12}}\dot{\boldsymbol{q}}) \tag{20}$$

The angular velocity of the ith link is the sum of the relative angular velocities of the previous links,

$$\vec{\boldsymbol{\omega}}_{\boldsymbol{i}} = \vec{\boldsymbol{\omega}}_{\boldsymbol{0}} + \sum_{j=1}^{i} {}_{\boldsymbol{j-1}}\vec{\boldsymbol{\omega}}_{\boldsymbol{j}} \qquad j = 1, ..., N \tag{21}$$

The angular velocity of the camera is that of the last link in the chain, so in the inertial frame this becomes

$$\boldsymbol{\omega_E} = \boldsymbol{\omega_N} = \boldsymbol{R_0}({}^{\mathbf{0}}\boldsymbol{\omega_0} + {}^{\mathbf{0}}\boldsymbol{J_{22}}\dot{\boldsymbol{q}}) \tag{22}$$

3.4.1 Jacobian of the camera frame

The Jacobian terms which simplify the analysis are defined as follows. They only depend on the system configuration $\vec{q}$. First, define a matrix

$$ {}^{0}\boldsymbol{F_k} \equiv [{}^{0}\boldsymbol{R_1}{}^{1}\boldsymbol{w_1} \quad {}^{0}\boldsymbol{R_2}{}^{2}\boldsymbol{w_2} \quad \boldsymbol{\ldots} \quad {}^{0}\boldsymbol{R_k}{}^{k}\boldsymbol{w_k} \quad \boldsymbol{0}], k = 1, ..., N \tag{23} $$

where ${}^{k}\boldsymbol{w_k}$ is "the unit column vector in frame k parallel to the revolute axis through joint k."[8]

Then,

$$ {}^{0}\boldsymbol{J_{11}} \equiv -\sum_{i=1}^{N} [{}^{0}\boldsymbol{R_i}{}^{i}\boldsymbol{v_{iN,E}}]^{\times} \tag{24} $$

$$ {}^{0}\boldsymbol{J_{12}} \equiv -\sum_{i=1}^{N} [{}^{0}\boldsymbol{R_i}{}^{i}\boldsymbol{v_{iN,E}}]^{\times}\,{}^{0}\boldsymbol{F_i} \tag{25} $$

$$ {}^{0}\boldsymbol{J_{22}} \equiv {}^{0}\boldsymbol{F_N} \tag{26} $$

These equations can be combined to form a Jacobian for the camera frame.

$$ \dot{\vec{\boldsymbol{x}}} = \begin{bmatrix} \dot{\boldsymbol{r}}_{\boldsymbol{E}} \\ {}^{0}\boldsymbol{\omega_E} \end{bmatrix} = \boldsymbol{J^+} \begin{bmatrix} {}^{0}\dot{\boldsymbol{r}}_{\boldsymbol{CM}} \\ {}^{0}\boldsymbol{\omega_0} \\ \dot{\boldsymbol{q}} \end{bmatrix} \tag{27} $$

where $\boldsymbol{J^+}(\varepsilon, \eta, \vec{\boldsymbol{q}}) = \tilde{\boldsymbol{R_0}}{}^{0}\boldsymbol{J^+}(\vec{\boldsymbol{q}})$ and

$$ {}^{0}\boldsymbol{J^+}(\vec{\boldsymbol{q}}) = \begin{bmatrix} \boldsymbol{1} & {}^{0}\boldsymbol{J_{11}} & {}^{0}\boldsymbol{J_{12}} \\ \boldsymbol{0} & \boldsymbol{1} & {}^{0}\boldsymbol{J_{22}} \end{bmatrix} \tag{28} $$

3.4.2 Jacobian of the base spacecraft

It is possible to define Jacobians for an arbitrary point on an arbitrary link. This allows a Jacobian to be developed for the CM of the base spacecraft .

To define the velocity of an arbitrary point m in body k, first define $\vec{\boldsymbol{v}}_{\boldsymbol{ik,m}} = \vec{\boldsymbol{v}}_{\boldsymbol{ik}} + \boldsymbol{\delta_{im}}\vec{\boldsymbol{r}}_{\boldsymbol{k,m}}$. Then, the position of the point can be described as

$$ \vec{\boldsymbol{d}}_{\boldsymbol{k,m}} = \vec{\boldsymbol{r}}_{\boldsymbol{CM}} + \vec{\boldsymbol{\rho}}_{\boldsymbol{k}} + \vec{\boldsymbol{r}}_{\boldsymbol{k,m}} \tag{29} $$

$$ = \vec{\boldsymbol{r}}_{\boldsymbol{CM}} + \sum_{i=0}^{N} \vec{\boldsymbol{v}}_{\boldsymbol{ik,m}} \tag{30} $$

Differentiate this with respect to time in the inertial frame analogously to equation 20 to define

$$ \dot{\boldsymbol{d}}_{\boldsymbol{k,m}} = \dot{\boldsymbol{r}}_{\boldsymbol{CM}} + \boldsymbol{R_0}({}^{0}\boldsymbol{J_{11k,m}}{}^{0}\boldsymbol{\omega_0} + {}^{0}\boldsymbol{J_{12k,m}}\dot{\boldsymbol{q}}) \tag{31} $$

$$ \boldsymbol{\omega_k} = \boldsymbol{R_0}({}^{0}\boldsymbol{\omega_0} + {}^{0}\boldsymbol{J_{22k,m}}\dot{\boldsymbol{q}}) \tag{32} $$

where, much as for the camera

$$ {}^{0}\boldsymbol{J_{11k,m}} \equiv -\sum_{i=0}^{N} [{}^{0}\boldsymbol{R_i}{}^{i}\boldsymbol{v_{ik,m}}]^{\times} \tag{33} $$

$$ {}^{0}\boldsymbol{J_{12k,m}} \equiv -\sum_{i=1}^{N} [{}^{0}\boldsymbol{R_i}{}^{i}\vec{\boldsymbol{v}}_{\boldsymbol{ik,m}}]^{\times}\,{}^{0}\boldsymbol{F_i} \tag{34} $$

$$ {}^{0}\boldsymbol{J_{22k,m}} \equiv {}^{0}\boldsymbol{F_k} \tag{35} $$

To find the Jacobian of the base spacecraft, simply set k=0 and m=CM ("S") to find ${}^{0}\boldsymbol{J_{11S}}, {}^{0}\boldsymbol{J_{12S}}, {}^{0}\boldsymbol{J_{22S}}$ and ${}^{0}\boldsymbol{J_S^+}$.

3.5 Quaternion kinematics

As the spacecraft rotates, $\boldsymbol{R_0(\varepsilon}, \eta)$ will change according to the well-known equations [6]

$$\dot{\vec{\varepsilon}} = 1/2(\varepsilon^{\times} + \eta\mathbf{1})\,{}^{0}\omega_0 \tag{36}$$

$$\dot{\eta} = -1/2(\boldsymbol{\varepsilon^{T0}\omega_0}) \tag{37}$$

It is this equation which must be integrated to find the orientation of the base spacecraft.

3.6 Momentum

In a free-floating system, both linear and angular momentum are conserved. This is not true of a free-flying system.

3.6.1 Linear Momentum

The linear momentum of the system centre of mass in the inertial frame is given by

$$\boldsymbol{p} = M\dot{\boldsymbol{r}}_{\boldsymbol{CM}} \tag{38}$$

The rate of change of linear momentum $\boldsymbol{\dot{p} = f_{ext}}$, where $\boldsymbol{f_{ext}}$ is the sum of all external forces acting on the system, will be explored in Section 4.2.

3.6.2 Angular Momentum

The angular momentum of the system in the inertial frame is given by

$$\boldsymbol{h = r_{CM}^{\times}p} + \sum_{i=0}^{N}(\boldsymbol{I_i\omega_i} + m_i\boldsymbol{\rho_i^{\times}\dot{\rho}_i}) \tag{39}$$

It is shown in [11] and [10] that the second term in this equation can be written as

$$\boldsymbol{h_{CM}} = \sum_{j=0}^{N}\sum_{i=0}^{N}\boldsymbol{D_{ij}\omega_j} \tag{40}$$

where the $\boldsymbol{D_{ij}}$ are the inertia dyadics. Therefore, the total angular momentum of the manipulator system is given by

$$\boldsymbol{h = r_{CM}^{\times}p} + \sum_{j=0}^{N}\sum_{i=0}^{N}\boldsymbol{D_{ij}\omega_j} \tag{41}$$

3.7 Inertia Dyadics

These special matrices are formed from the barycentric vectors. They allow for a very compact formulation of energy, momentum and dynamics. When expressed in the spacecraft's base frame,

$$\boldsymbol{{}^{0}D_{ij}} = \begin{cases} -M(\boldsymbol{{}^{0}\ell_j^{*T}{}^{0}r_i^{*})1 - {}^{0}\ell_j^{*}{}^{0}r_i^{*T}} & i < j \\ \boldsymbol{{}^{0}I_i} + \sum_{k=0}^{N} m_k(\boldsymbol{{}^{0}v_{ik}^{T}\,{}^{0}v_{ik})1 - {}^{0}v_{ik}\,{}^{0}v_{ik}^{T}} & i = j \\ -M(\boldsymbol{{}^{0}r_j^{*T}\,{}^{0}\ell_i^{*})1 - {}^{0}r_j^{*}\,{}^{0}\ell_i^{*T}} & i > j \end{cases} \tag{42}$$

The other inertia-type matrices can be derived from this expression.

$$ {}^{0}\boldsymbol{D_j} \equiv \sum_{i=0}^{N} {}^{0}\boldsymbol{D_{ij}} \tag{43} $$

$$ {}^{0}\boldsymbol{D} \equiv \sum_{j=0}^{N} {}^{0}\boldsymbol{D_j} \tag{44} $$

$$ {}^{0}\boldsymbol{D_q} \equiv \sum_{j=1}^{N} {}^{0}\boldsymbol{D_j}{}^{0}\boldsymbol{F_j} \tag{45} $$

$$ {}^{0}\boldsymbol{D_{qq}} \equiv \sum_{j=1}^{N}\sum_{i=1}^{N} {}^{0}\boldsymbol{F_i^T}{}^{0}\boldsymbol{D_{ij}}{}^{0}\boldsymbol{F_j} \tag{46} $$

$$ \tag{47} $$

Using these matrices, following [10], the angular momentum of the system in the inertial frame is

$$ \boldsymbol{h_{CM}} = \boldsymbol{R_0}({}^{0}\boldsymbol{D}{}^{0}\boldsymbol{\omega_0} + {}^{0}\boldsymbol{D_q}\dot{\boldsymbol{q}}) \tag{48} $$

4 Dynamics of a Free-Floating System

In this section, the effects of chaser-target orbital dynamics and gravity from celestial bodies are neglected. The total energy of the system is reduced to the kinetic energy. The Euler-Lagrange equation is used to derive the joint rates, and the momentum formulation is used to derive quasi-Lagrangian equations for the rotational motion.

4.1 Kinetic Energy

The total kinetic energy is given by the equation

$$ T = \frac{1}{2}\sum_{i=0}^{N}(m_i {}^{0}\boldsymbol{v_i^T}{}^{0}\boldsymbol{v_i} + {}^{0}\boldsymbol{\omega_0^T}{}^{0}\boldsymbol{I_i}{}^{0}\boldsymbol{\omega_0}) \tag{49} $$

In the absence of gravity, this is equivalent to the Lagrangian, L. Using the inertia matrices [8], the Lagrangian can be expressed as

$$ L = \frac{1}{2}{}^{0}\boldsymbol{\omega_0^T}{}^{0}\boldsymbol{D}{}^{0}\boldsymbol{\omega_0} + {}^{0}\boldsymbol{\omega_0^T}{}^{0}\boldsymbol{D_q}\dot{\boldsymbol{q}} + \boldsymbol{1/2}\dot{\boldsymbol{q}}^{\boldsymbol{T}}{}^{0}\boldsymbol{D_{qq}}\dot{\boldsymbol{q}} \tag{50} $$

4.2 Equations of Motion for a Free-Floating System

The translational equations of motion of the system CM are given by the rate of change of linear momentum. For a free-floating case, linear momentum is conserved, so this term vanishes.

$$ \dot{\boldsymbol{p}} = M\ddot{\boldsymbol{r}}_{\boldsymbol{CM}} = \boldsymbol{f_{ext}} = 0 \tag{51} $$

The rotational equations of motion are not so easily found. Applying the momentum formulation results in quasi-Lagrangian equations [8] for the free-floating case

$$ \frac{d}{dt}\Big(\frac{\partial L}{\partial {}^{0}\boldsymbol{\omega_0}}\Big) + {}^{0}\boldsymbol{\omega_0}^{\times}\frac{\partial L}{\partial {}^{0}\boldsymbol{\omega_0}} = \boldsymbol{R_0^T n_{CM}} \tag{52} $$

while the joint rates $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are related by an Euler-Lagrange equation

$$ \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\Big) - \frac{\partial L}{\partial \boldsymbol{q}} = \boldsymbol{\tau} \tag{53} $$

When there is no moment of external forces acting on the spacecraft, $\boldsymbol{g_{CM}} = 0$ and it can be shown [8] that

$$ \boldsymbol{H(q)}\ddot{\boldsymbol{q}} + \boldsymbol{c_h} = \boldsymbol{\tau} \tag{54} $$

where $\mathbf{H(q)}$ is the reduced system inertia matrix and the vector $\boldsymbol{c_h}$ comprises the nonlinear terms. In the free-floating case,

$$\boldsymbol{H}(\boldsymbol{q}) = {}^{\mathbf{0}}\boldsymbol{D_{qq}} - {}^{\mathbf{0}}\boldsymbol{D_q}^{\boldsymbol{T}\,\mathbf{0}}\boldsymbol{D}^{-\mathbf{1}\,\mathbf{0}}\boldsymbol{D_q} \tag{55}$$

and

$$\boldsymbol{c_h} = \boldsymbol{c_2^*} - {}^{\mathbf{0}}\boldsymbol{D_q^T}{}^{\mathbf{0}}\boldsymbol{D}^{-1}\boldsymbol{c_1^*} \tag{56}$$

where $\boldsymbol{c_2^*}$ and $\boldsymbol{c_1^*}$ are functions of many partial derivatives of the Lagrangian described in equation 52. For a more detailed treatment and simulation of these equations for the planar and spatial cases described in [8], please see the attached MATLAB code.

4.3 Dynamic Singularities and the Generalized Jacobian

A relationship between joint space and task space velocities is desired. To find it, the system inertia matrix must be partitioned. In [19], [15], and [16] a generalized inertia matrix is developed in the form

$$\boldsymbol{H}^\star = \boldsymbol{H_m} - \boldsymbol{H_{0m}^T}\boldsymbol{H_0^{-1}}\boldsymbol{H_{0m}} \tag{57}$$

where $\boldsymbol{H_m}$ represents the inertia matrix of the manipulator, $\boldsymbol{H_0}$ that of the base spacecraft and $\boldsymbol{H_{0m}}$ the dynamic coupling inertia matrix. This is analogous to the relationship in equation 55, with $\boldsymbol{H_m} \to {}^{\mathbf{0}}\boldsymbol{D_{qq}}$, $\boldsymbol{H_0} \to {}^{\mathbf{0}}\boldsymbol{D}$ and $\boldsymbol{H_{0m}} \to {}^{\mathbf{0}}\boldsymbol{D_q}$. With this partitioning, it is possible to define a generalized Jacobian for a free floating system such that

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \boldsymbol{J}^*\dot{\boldsymbol{q}} \quad \text{where} \quad \boldsymbol{J}^* = \boldsymbol{J_m} - \boldsymbol{J_0}\boldsymbol{H_0^{-1}}\boldsymbol{H_{0m}} \tag{58}$$

While [19] and [16] formulate their generalized Jacobian in terms of the base and manipulator Jacobians $\boldsymbol{J_m}$ and $\boldsymbol{J_0}$, [8] defines instead a ${}^{\mathbf{0}}\boldsymbol{J_{11}}$ and ${}^{\mathbf{0}}\boldsymbol{J_{12}}$, and combines conservation of linear and angular momentum into a matrix equation.

$$\boldsymbol{A}\begin{bmatrix} {}^{\mathbf{0}}\boldsymbol{\omega_0} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R_0^T}\boldsymbol{h_{cm}} \\ \mathbf{0} \end{bmatrix} \quad \text{where} \quad \boldsymbol{A} = \begin{bmatrix} {}^{\mathbf{0}}\boldsymbol{D} & {}^{\mathbf{0}}\boldsymbol{D_q} \\ {}^{0}\boldsymbol{J_{11}} & {}^{0}\boldsymbol{J_{12}} \end{bmatrix} \tag{59}$$

Dynamic singularities occur when this matrix loses full rank, ie. when

$$\det(\boldsymbol{A}) = \det({}^{\mathbf{0}}\boldsymbol{D})\det(-{}^{\mathbf{0}}\boldsymbol{J_{11}}{}^{\mathbf{0}}\boldsymbol{D}^{-\mathbf{1}\,\mathbf{0}}\boldsymbol{D_q} + {}^{\mathbf{0}}\boldsymbol{J_{12}}) = 0 \tag{60}$$

According to [8], ${}^{\mathbf{0}}\boldsymbol{D}$ is always invertible, and as a result A is only singular when $\det(-{}^{\mathbf{0}}\boldsymbol{J_{11}}{}^{\mathbf{0}}\boldsymbol{D}^{-\mathbf{1}\,\mathbf{0}}\boldsymbol{D_q} + {}^{\mathbf{0}}\boldsymbol{J_{12}}) = 0$. This forms a sort of generalized Jacobian, as the vector of joint rates is then given by

$$\dot{\boldsymbol{q}} = -({}^{\mathbf{0}}\boldsymbol{J_{11}}{}^{\mathbf{0}}\boldsymbol{D}^{-\mathbf{1}\,\mathbf{0}}\boldsymbol{D_q} + {}^{\mathbf{0}}\boldsymbol{J_{12}})^{-\mathbf{1}\,\mathbf{0}}\boldsymbol{J_{11}}{}^{\mathbf{0}}\boldsymbol{D}^{-1}\boldsymbol{R_0^T}\boldsymbol{h_{cm}} \tag{61}$$

Here, the similarity ends. The Jacobian formulated in [11] is nothing like that in [19], where the velocity kinematics is derived from the forward kinematics of the Denavit-Hartenberg parameter formulation popularized by [13] and others. Unfortunately, this is just a first glance at the hidden complexity of the barycentric vector formulation, which is very convenient for a free-floating base, especially when momentum is conserved.

5 Base motion

When the base is actuated by thrusters and spinning reaction wheels, it will have a nonzero velocity which is independent of $\boldsymbol{q}$. In this case, momentum will not be conserved. Following [6], the quasi-Lagrangian equations for a rigid body with linear velocity $\vec{\boldsymbol{v}}$ and angular velocity $\boldsymbol{\omega}$ expressed in the spacecraft base frame are as follows:

$$\frac{d}{dt}(\frac{\partial T}{\partial \boldsymbol{v}}) + \boldsymbol{\omega}^\times(\frac{\partial T}{\partial \boldsymbol{v}}) = \boldsymbol{f} \tag{62}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \boldsymbol{\omega}}) + \boldsymbol{\omega}^\times(\frac{\partial T}{\partial \boldsymbol{\omega}}) + \boldsymbol{v}^\times(\frac{\partial T}{\partial \boldsymbol{v}}) = \boldsymbol{g} \tag{63}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ represent the sum of all external forces and moments acting on the spacecraft, expressed in the spacecraft base frame.

5.1 Forces and moments from the thrusters

There are very few external forces, as the weight of the camera is almost negligible, possibly less than 1kg, and the robot will never contact a payload. The only forces acting on the base are the discrete forces from the jet thrusters, each one providing a force $\vec{\boldsymbol{f}}_{\boldsymbol{k}}(t)$ acting at $\vec{\boldsymbol{r}}_{\boldsymbol{k}}$. According to [6]

$$\vec{\boldsymbol{f}}(\boldsymbol{t}) = \sum_k \vec{\boldsymbol{f}}_{\boldsymbol{k}}(\boldsymbol{t}) \tag{64}$$

$$\vec{\boldsymbol{g}}(\boldsymbol{t}) = \sum_k \vec{\boldsymbol{r}}_{\boldsymbol{k}} \times \vec{\boldsymbol{f}}_{\boldsymbol{k}}(\boldsymbol{t}) \tag{65}$$

5.2 Effect of the Momentum Exchange Wheels

The effect of a wheel which is allowed to rotate around an axis $\vec{\boldsymbol{a}}$ inside a rigid body B of mass m_b is to generate a moment with which to stabilize the spacecraft's orientation. The purpose of these wheels is to store angular momentum, and this is transferred to the wheels by the control torque applied by an electric motor. When the wheel is fixed in direction and the magnitude of the momentum is changed, it is known as a reaction wheel. When the magnitude of the momentum is fixed, but the axis of the wheel is free to move, it is known as a control moment gyroscope. This report will focus on the dynamics of the reaction wheel. In this case, the spacecraft is called a gyrostat [9]. For a single wheel whose centre of mass CW is located at $\vec{\boldsymbol{b}}$ with respect to an arbitrary point O with mass m_w, the first and second moments of inertia about O are given by

$$\vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}_{\boldsymbol{b}} + m_w \vec{\boldsymbol{b}} \tag{66}$$

$$\boldsymbol{J} = \boldsymbol{J}_{\boldsymbol{b}} + \boldsymbol{I}_{\boldsymbol{w}} + m_w(b^2\mathbf{1} - \vec{\boldsymbol{b}}\vec{\boldsymbol{b}}) \tag{67}$$

where $\vec{\boldsymbol{c}}_{\boldsymbol{b}}$ and $\boldsymbol{J}_{\boldsymbol{b}}$ are the moments of inertia of B about O and $\boldsymbol{I}_{\boldsymbol{w}}$ is the moment of inertia of the wheel about its centre of mass. The axis of rotation $\vec{\boldsymbol{a}}$ is a unit vector fixed in B, and the wheel itself is free to rotate. Since the wheel is axisymmetric, its moment of inertia can be expressed in the form

$$\boldsymbol{I}_{\boldsymbol{w}} = I_t\mathbf{1} + (I_s - I_t)\vec{\boldsymbol{a}}\vec{\boldsymbol{a}} \tag{68}$$

where I_s is the moment of inertia about the symmetry axis and I_t is the moment of inertia about any other axis. [6]

5.2.1 Momentum

After some derivation, the total momentum of the system is

$$\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_{\boldsymbol{b}} + \vec{\boldsymbol{p}}_{\boldsymbol{w}} = m\vec{\boldsymbol{v}} - \vec{\boldsymbol{c}} \times \vec{\boldsymbol{\omega}} \tag{69}$$

where $\vec{\boldsymbol{v}}$ is the absolute velocity of O and $\vec{\boldsymbol{\omega}}$ the absolute angular velocity of the body B. Here, $m = m_b + m_w$ is the combined mass of the system. The rotor spin rate $\vec{\boldsymbol{\omega}}_{\boldsymbol{s}} = \vec{\boldsymbol{\omega}}_{\boldsymbol{w}} - \vec{\boldsymbol{\omega}} = \omega_s\vec{\boldsymbol{a}}$ is the difference between the absolute angular velocity of the wheel, $\vec{\boldsymbol{\omega}}_{\boldsymbol{w}}$ and that of B. The velocity at a point $\vec{\boldsymbol{r}} \in B$ is $\vec{\boldsymbol{v}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ and the velocity at a point $\vec{\boldsymbol{r}}_{\boldsymbol{w}} \in W$ is $\vec{\boldsymbol{v}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{b}} + \vec{\boldsymbol{\omega}}_{\boldsymbol{w}} \times \vec{\boldsymbol{r}}_{\boldsymbol{w}}$. With this, the total angular momentum about O is that of the base and the wheels, $\vec{\boldsymbol{h}}_{\boldsymbol{b}} + \vec{\boldsymbol{h}}_{\boldsymbol{w}}$, where

$$\vec{\boldsymbol{h}}_{\boldsymbol{b}} = \int_{\boldsymbol{B}} \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{v}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})dm = \vec{\boldsymbol{c}}_{\boldsymbol{b}} \times \vec{\boldsymbol{v}} + \boldsymbol{J}_{\boldsymbol{b}} \cdot \vec{\boldsymbol{\omega}} \tag{70}$$

and

$$\vec{\boldsymbol{h}}_{\boldsymbol{w}} = \int_{\boldsymbol{W}} \vec{\boldsymbol{r}}_{\boldsymbol{w}} \times (\vec{\boldsymbol{v}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{b}} + \vec{\boldsymbol{\omega}}_{\boldsymbol{w}} \times \vec{\boldsymbol{r}}_{\boldsymbol{w}} dm = \boldsymbol{I}_{\boldsymbol{w}} \cdot \vec{\boldsymbol{\omega}}_{\boldsymbol{w}} \tag{71}$$

The sum of these is

$$\vec{\boldsymbol{h}} = \vec{\boldsymbol{c}} \times \vec{\boldsymbol{v}} + \boldsymbol{J} \cdot \vec{\boldsymbol{\omega}} + \vec{\boldsymbol{a}} I_s \omega_s \tag{72}$$

The axial component of $\vec{\boldsymbol{h}}$, h_a is given by

$$h_a \equiv \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{h}}_{\boldsymbol{w}} = I_s \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{\omega}} + I_s \omega_s \tag{73}$$

Let the force and torque acting on W due to B be $\vec{f}_{bw}$ and $\vec{g}_{bw}$. The equal and opposite force and torque act on B due to W, and there is an external force $\vec{\boldsymbol{f}}$ and torque $\vec{\boldsymbol{g}}$ on R due to the thrusters as explained in the previous section. Therefore

$$\dot{\vec{\boldsymbol{p}}}_{\boldsymbol{b}} = \vec{\boldsymbol{f}} - \vec{\boldsymbol{f}}_{\boldsymbol{bw}} \tag{74}$$

$$\dot{\vec{\boldsymbol{p}}}_{\boldsymbol{w}} = \vec{\boldsymbol{f}}_{\boldsymbol{bw}} \tag{75}$$

$$\rightarrow \dot{\vec{\boldsymbol{p}}} = \vec{\boldsymbol{f}} \tag{76}$$

$$\tag{77}$$

The angular momentum equations can be simplified:

$$\dot{\vec{\boldsymbol{h}}}_{\boldsymbol{b}} - \vec{\boldsymbol{v}} \times (\vec{\boldsymbol{c}}_{\boldsymbol{b}} \times \vec{\boldsymbol{\omega}}) = \vec{\boldsymbol{g}} - \vec{\boldsymbol{g}}_{\boldsymbol{bw}} - \vec{\boldsymbol{b}} \times \vec{\boldsymbol{f}}_{\boldsymbol{bw}} \tag{78}$$

$$\dot{\vec{\boldsymbol{h}}}_{\boldsymbol{w}} = \vec{\boldsymbol{g}}_{\boldsymbol{bw}} \tag{79}$$

Add these two equations and substitute 75 as well as $\dot{\vec{\boldsymbol{b}}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{b}}$ to obtain the final form:

$$\dot{\vec{\boldsymbol{h}}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{p}} = \vec{\boldsymbol{g}} \tag{80}$$

The axial torque on the wheel is defined as $g_a \equiv \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{g}}_{\boldsymbol{bw}}$. With a motor, g_a can be used to control ω_s. Indeed, the equation of motion for the wheel is derived in [6] and is quite simple:

$$\dot{h}_a = g_a \tag{81}$$

5.2.2 Energy

The kinetic energy is, after some simplification, given by

$$T = \frac{1}{2} m \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} + \frac{1}{2} \vec{\boldsymbol{\omega}} \cdot \boldsymbol{J} \cdot \vec{\boldsymbol{\omega}} + \frac{1}{2} I_s \omega_s^2 - \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{c}} \times \vec{\boldsymbol{\omega}} + I_s \omega_s \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{\omega}} \tag{82}$$

and its rate of change is given by

$$\dot{T} = \vec{\boldsymbol{f}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{\omega}} + g_a \omega_s \tag{83}$$

In the spacecraft base frame, the kinetic energy in equation 82 can be expressed as

$$T = \frac{1}{2} m \boldsymbol{v}^{\boldsymbol{T}} \boldsymbol{v} + \frac{1}{2} \boldsymbol{\omega}^{\boldsymbol{T}} \boldsymbol{J} \boldsymbol{\omega} + \frac{1}{2} I_s \omega_s^2 - \boldsymbol{v}^{\boldsymbol{T}} \boldsymbol{c}^{\times} \boldsymbol{\omega} + I_s \omega_s \boldsymbol{a}^{\boldsymbol{T}} \boldsymbol{\omega} \tag{84}$$

$$\frac{\partial T}{\partial \boldsymbol{v}} = \boldsymbol{p} \qquad \frac{\partial T}{\partial \boldsymbol{\omega}} = \boldsymbol{h} \qquad \frac{\partial T}{\partial \omega_s} = h_a \tag{85}$$

5.2.3 Equations of Motion

Solving equations 69, 72 and 73 in their vectrix form simultaneously with 84 results in a matrix relationship with a matrix M which is constant, symmetric and positive definite. As well, the first moment of inertia is constant, since $\vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}_{\boldsymbol{b}} + m_w \vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}_{\boldsymbol{b}} = \int_R \vec{\boldsymbol{r}} dm$ is constant. As a result, it is possible to shift the point of reference without loss of generality. Setting the centre of mass of the overall system as the reference point allows the translational and rotational motions to be decoupled without any loss of generality. When expressed in that frame, $\boldsymbol{c} = \boldsymbol{0}, \quad \boldsymbol{p} = m\boldsymbol{v}, \quad \boldsymbol{J} \rightarrow \boldsymbol{I}$ and the momentum equations can then be written in the very compact form

$$\boldsymbol{p} = m\boldsymbol{v} \tag{86}$$

$$\boldsymbol{h} = \boldsymbol{I}\boldsymbol{\omega} + I_s \omega_s \boldsymbol{a} \tag{87}$$

$$h_a = I_s(\boldsymbol{a}^{\boldsymbol{T}} \boldsymbol{\omega} + \omega_s) \tag{88}$$

$$\dot{\boldsymbol{p}} = -\boldsymbol{\omega}^{\times} \boldsymbol{p} + \boldsymbol{f} \tag{89}$$

$$\dot{\boldsymbol{h}} = -\boldsymbol{\omega}^{\times} \boldsymbol{h} + \boldsymbol{g} \tag{90}$$

$$\dot{h}_a = g_a \tag{91}$$

The motion equations 89 and 90 can be written in a quasi-Lagrangian form as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{v}}\right) + \boldsymbol{\omega}^{\times}\left(\frac{\partial T}{\partial \boldsymbol{v}}\right) = \boldsymbol{f} \tag{92}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\omega}}\right) + \boldsymbol{\omega}^{\times}\left(\frac{\partial T}{\partial \boldsymbol{\omega}}\right) = \boldsymbol{g} \tag{93}$$

Define $h_s = I_s\omega_s$ and$\boldsymbol{h_s} = h_s\boldsymbol{a}$, the extra angular momentum in 87 which is caused by the spin of the wheel. Substitute 87 into 90 to find the rotational equation of motion:

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}^{\times}(\boldsymbol{I}\boldsymbol{\omega} + h_s\boldsymbol{a}) + \boldsymbol{g} \tag{94}$$

5.3 Multiple Wheels

On the inspection spacecraft, there are likely to be multiple wheels. Let there be n_w of them. According to [6], the angular momentum of this system about the combined centre of mass is

$$\boldsymbol{h} = \boldsymbol{I}\boldsymbol{\omega} + \sum_{n=1}^{n_w} h_{sn}\boldsymbol{a_n} \tag{95}$$

where each wheel contributes $h_{sn}\boldsymbol{a_n}$ to the total angular momentum. The combined kinetic energy will be the sum of the individual wheels' kinetic energy, so

$$T = \frac{1}{2}(m_b + \sum_{j=1}^{n_w} m_{wj})\boldsymbol{v^T}\boldsymbol{v} + \frac{1}{2}\boldsymbol{\omega^T}\boldsymbol{J}\boldsymbol{\omega} + \frac{1}{2}\sum_{j=1}^{n_w}(I_{sj}\omega_{sj}^2 + I_{sj}\omega_{sj}\boldsymbol{a_j^T}\boldsymbol{\omega}) - \boldsymbol{v^T}\boldsymbol{c}^{\times}\boldsymbol{\omega} \tag{96}$$

If the centre of mass is taken as the reference point for the system, $\boldsymbol{c} = 0$ and the last term in this equation vanishes. Here, define the augmented inertia of each wheel as I_{aj}.

$$\boldsymbol{J} = \boldsymbol{J_b} + \sum_{j=1}^{n_w}(\boldsymbol{I_{wj}} + m_{wj}(b_j^2\boldsymbol{1} - \boldsymbol{b_j}\boldsymbol{b_j^T})) \tag{97}$$

$$\boldsymbol{J} \equiv \boldsymbol{J_b} + \sum_{j=1}^{n_w}\boldsymbol{I_{aj}} \tag{98}$$

$$\tag{99}$$

Therefore, the combined kinetic energy of a base with n_w spinning reaction wheels is

$$T = \frac{1}{2}\sum_{j=1}^{n_w} m_{wj}\boldsymbol{v^T}\boldsymbol{v} + \frac{1}{2}\boldsymbol{\omega^T}\boldsymbol{J_b}\boldsymbol{\omega} + \sum_{j=1}^{n_w}(\frac{1}{2}I_{sj}\omega_{sj}^2 + I_{sj}\omega_{sj}\boldsymbol{a_j^T}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega^T}\boldsymbol{I_{aj}}\boldsymbol{\omega}) \tag{100}$$

where

$$\boldsymbol{I} = \boldsymbol{J_b} + \frac{1}{2}\sum_{j=1}^{n_w}\boldsymbol{\omega^T}\boldsymbol{I_{aj}}\boldsymbol{\omega} \tag{101}$$

6 Combination of Base and Links

The above equations are formulated in the spacecraft's base frame. This is located at the overall centre of mass of the base, which now includes reaction wheels. As a result they may be easily combined with the earlier equations for the links, also derived in the spacecraft's base frame. In this frame, the total energy of the combined system is the sum of the links' energy, calculated in section 4.1 and the wheels', calculated in equation 84. The first moment of inertia about this point vanishes, which simplifies the calculation immensely. To rectify the notation, the ω in section 5 is replaced by ${}^0\omega_0$ as in section 4.1. The mass and inertia of the base are accounted for in the first two terms, when $i = 0$, ie. $J_b = I_0$ which is included in the inertia terms ${}^0D, {}^0D_q$ and ${}^0D_q q$.

6.1 Energy and Momentum

The kinetic energy of the entire system is given by:

$$T = \frac{1}{2}M\boldsymbol{v_0^T v_0} + \boldsymbol{\frac{1}{2}}{}^{\boldsymbol{0}}\boldsymbol{\omega_0^{T\,0} D^0 \omega_0} + {}^{\boldsymbol{0}}\boldsymbol{\omega_0^{T\,0} D_q \dot{q}} + \frac{1}{2}\boldsymbol{\dot{q}^{T\,0} D_{qq} \dot{q}} + \sum_{j=1}^{n_w}[\frac{1}{2}I_{sj}\omega_{sj}^2 + I_{sj}\omega_{sj}\boldsymbol{a_j^{T\,0}\omega_0} + \frac{1}{2}{}^{\boldsymbol{0}}\boldsymbol{\omega_0^T I_{aj}}{}^{\boldsymbol{0}}\boldsymbol{\omega_0}] \tag{102}$$

The total linear momentum of the base centre of mass is given by the sum of the links, the base and the wheels' momentum. This must be expressed in the spacecraft base frame. To avoid confusion with the combined system's centre of mass, the original definition of $\boldsymbol{p_{links}} = \sum_{k=1}^{N} m_k \boldsymbol{\dot{\rho}_k}$ is used.

$$\boldsymbol{p} = \sum_{k=1}^{N} m_k \sum_{i=0}^{N} \boldsymbol{\omega_i^\times v_{ik}} + M\boldsymbol{v_0} \tag{103}$$

The total angular momentum in the spacecraft base frame is given by

$$\boldsymbol{h} = {}^{\boldsymbol{0}}\boldsymbol{D}^{\boldsymbol{0}}\boldsymbol{\omega_0} + {}^{\boldsymbol{0}}\boldsymbol{D_q \dot{q}} + \sum_{j=1}^{n_w}(\boldsymbol{I_{aj}}{}^{\boldsymbol{0}}\boldsymbol{\omega_0} + \boldsymbol{a_j} I_{sj}\omega_{sj}) \tag{104}$$

The change in the linear momentum is given by

$$\boldsymbol{\dot{p}} + {}^{\boldsymbol{0}}\boldsymbol{\omega_0}^\times \boldsymbol{p} = \boldsymbol{f} \tag{105}$$

where the external force is that of the thrusters, given in section 5.1.

The change in the angular momentum is given by

$$\boldsymbol{\dot{h}} + \boldsymbol{v}^\times \boldsymbol{p} + {}^{\boldsymbol{0}}\boldsymbol{\omega_0}^\times \boldsymbol{h} = \boldsymbol{g} \tag{106}$$

6.2 Combined Equations of Motion

6.2.1 Linear motion

$$\frac{\partial T}{\partial \boldsymbol{v}} = M\boldsymbol{v} = \boldsymbol{p} \tag{107}$$

Therefore, substituting into equation 105 and dividing through by the total mass M $= m_b + \sum_i m_i + \sum_j m_{wj} + \sum_k m_{tk}$:

$$\boldsymbol{\dot{v}} + \boldsymbol{\omega}^\times \boldsymbol{v} = \frac{1}{M}\boldsymbol{f} \tag{108}$$

6.2.2 Rotational Motion

$$h = \frac{\partial T}{\partial {}^{\boldsymbol{0}}\boldsymbol{\omega_0}} = {}^{\boldsymbol{0}}\boldsymbol{D}^{\boldsymbol{0}}\boldsymbol{\omega_0} + {}^{\boldsymbol{0}}\boldsymbol{D_q \dot{q}} + \sum_{j}^{n_j}[I_{sj}\omega_{sj}\boldsymbol{a_j} + \boldsymbol{I_{aj}}{}^{\boldsymbol{0}}\boldsymbol{\omega_0}] \tag{109}$$

To take the time derivative of the D matrices, use the chain rule:

$$\frac{d}{dt}\boldsymbol{D(q)} = \frac{d}{d\boldsymbol{q}}\boldsymbol{D(q)\dot{q}} = \boldsymbol{\dot{q}^T}\frac{d\boldsymbol{D(q)}}{d\boldsymbol{q}} \tag{110}$$

This derivative forms a three-dimensional array, since a matrix is being differentiated by a vector. Evaluation of this will be done element-wise, ie.

$$(\frac{d\boldsymbol{D(q)}}{d\boldsymbol{q}})_{ijk} = \frac{\partial(\boldsymbol{D(q)})_{ij}}{\partial q_k} \tag{111}$$

The following analysis will omit the "0" frame of reference for brevity:

$$\begin{aligned} \frac{d}{dt}(\frac{\partial T}{\partial \boldsymbol{\omega}}) &= \boldsymbol{\dot{D}\omega + D\dot{\omega} + \dot{D}_q\dot{q} + D_q\ddot{q}} + \sum_{j}^{n_w}\dot{\omega}_{sj}\boldsymbol{a_j} \\ &= \boldsymbol{\dot{q}^T}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega} + \boldsymbol{D\dot{\omega}} + \boldsymbol{\dot{q}^T}\frac{d\boldsymbol{D_q}}{d\boldsymbol{q}}\boldsymbol{\dot{q}} + \boldsymbol{D_q\ddot{q}} + \sum_{j}^{n_w}\dot{\omega}_{sj}\boldsymbol{a_j} \end{aligned} \tag{112}$$

Substitute this equation into 106 to create an equation of motion.

$$\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}+\boldsymbol{D}\dot{\boldsymbol{\omega}}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_q}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}+\boldsymbol{D_q}\ddot{\boldsymbol{q}}+\sum_j^{n_w}\dot{\omega}_{sj}\boldsymbol{a_j}+\boldsymbol{\omega}^{\times}(\boldsymbol{D}\boldsymbol{\omega}+\boldsymbol{D_q}\dot{\boldsymbol{q}}+\sum_j^{n_w}[I_{sj}\omega_{sj}\boldsymbol{a_j}+\boldsymbol{I_{aj}}\boldsymbol{\omega}])+\boldsymbol{v}^{\times}(M\boldsymbol{v})=\boldsymbol{g_{CM}} \tag{113}$$

After cancelling and simplifying, this becomes

$$\boldsymbol{D}\dot{\boldsymbol{\omega}}+\boldsymbol{D_q}\ddot{\boldsymbol{q}}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_q}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}+\boldsymbol{\omega}^{\times}\boldsymbol{D_q}\dot{\boldsymbol{q}}+\sum_j^{n_j}(\dot{\omega}_{sj}+I_{sj}\omega_{sj})\boldsymbol{\omega}^{\times}\boldsymbol{a_j}=\boldsymbol{g_{CM}} \tag{114}$$

6.2.3 Joint Rates

To find the equation of joint rate motion, use the Euler-Lagrange equation.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}-\frac{\partial L}{\partial \boldsymbol{q}}=\boldsymbol{\tau} \tag{115}$$

where $\boldsymbol{\tau}$ is the vector of joint torques. Expanding these terms results in:

$$\frac{\partial L}{\partial \boldsymbol{q}}=\frac{1}{2}\boldsymbol{\omega}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}+\boldsymbol{\omega}^{\boldsymbol{T}}\frac{d\boldsymbol{D_q}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}+\frac{1}{2}\dot{\boldsymbol{q}}^{T}\frac{d\boldsymbol{D_{qq}}}{d\boldsymbol{q}}\dot{\boldsymbol{q}} \tag{116}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}}=\boldsymbol{D_q}\boldsymbol{\omega}+\boldsymbol{D_{qq}}\dot{\boldsymbol{q}} \tag{117}$$

Taking the time derivative of this equation results in

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}=\boldsymbol{D_q}\dot{\boldsymbol{\omega}}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_q}}{d\boldsymbol{q}}\boldsymbol{\omega}+\boldsymbol{D_{qq}}\ddot{\boldsymbol{q}}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_{qq}}}{d\boldsymbol{q}}\dot{\boldsymbol{q}} \tag{118}$$

Substituting into equation 115:

$$\boldsymbol{D_q}\dot{\boldsymbol{\omega}}+\boldsymbol{D_{qq}}\ddot{\boldsymbol{q}}+\frac{1}{2}\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_{qq}}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}-\frac{1}{2}\boldsymbol{\omega}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}=\boldsymbol{\tau} \tag{119}$$

6.2.4 Wheel Rates

For reaction wheels, this equation is relatively straightforward. For reaction wheel j, with rotor spin rate ω_{sj} as its generalized coordinate,

$$\frac{\partial T}{\partial \omega_{sj}}=I_{sj}\omega_{sj}+I_{sj}\boldsymbol{a_j^T}\boldsymbol{\omega}=h_{aj} \tag{120}$$

where h_a is the axial component of angular momentum as explained in section 5.2. Using equation 81, find

$$I_{sj}\dot{\omega}_{sj}+I_{sj}\boldsymbol{a_j^T}\dot{\boldsymbol{\omega}}=g_{aj} \tag{121}$$

6.2.5 Combined Equation of Motion

These equations are coupled and highly nonlinear. To combine them, solve for $\ddot{\boldsymbol{q}}$ and $\dot{\omega}_{sj}$ in terms of $\dot{\boldsymbol{\omega}}$.

$$\dot{\omega}_{sj}=-\boldsymbol{a_j^T}\dot{\boldsymbol{\omega}}+I_{sj}^{-1}g_{aj} \tag{122}$$

$$\ddot{\boldsymbol{q}}=-\boldsymbol{D_{qq}^{-1}}[\boldsymbol{D_q}\dot{\boldsymbol{\omega}}+\boldsymbol{\frac{1}{2}}\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{\boldsymbol{dD_{qq}}}{\boldsymbol{dq}}\dot{\boldsymbol{q}}-\boldsymbol{\frac{1}{2}}\boldsymbol{\omega}^{\boldsymbol{T}}\frac{\boldsymbol{dD}}{\boldsymbol{dq}}\boldsymbol{\omega}-\boldsymbol{\tau}] \tag{123}$$

Substitute these into equation 114:

$$\begin{aligned}\boldsymbol{D}\dot{\boldsymbol{\omega}}-\boldsymbol{D_q}\boldsymbol{D_{qq}^{-1}}[\boldsymbol{D_q}\dot{\boldsymbol{\omega}}+\frac{1}{2}\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_{qq}}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}-\frac{1}{2}\boldsymbol{\omega}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}-\boldsymbol{\tau}]+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_q}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}\\ +\boldsymbol{\omega}^{\times}\boldsymbol{D_q}\dot{\boldsymbol{q}}+\sum_j^{n_w}(-\boldsymbol{a_j^T}\dot{\boldsymbol{\omega}}+I_{sj}^{-1}g_{aj}+I_{sj}\omega_{sj})\boldsymbol{\omega}^{\times}\boldsymbol{a_j}=\boldsymbol{g_{CM}}\end{aligned} \tag{124}$$

which can be simplified somewhat to yield

$$(\boldsymbol{D}-\boldsymbol{D_q}\boldsymbol{D_{qq}^{-1}}\boldsymbol{D_q}-\sum_j^{n_w}\boldsymbol{a_j^T})\dot{\boldsymbol{\omega}}-\frac{1}{2}\boldsymbol{D_q}\boldsymbol{D_{qq}^{-1}}\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_{qq}}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}-\frac{1}{2}\boldsymbol{D_q}\boldsymbol{D_{qq}^{-1}}\boldsymbol{\omega}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D}}{d\boldsymbol{q}}\boldsymbol{\omega}+\dot{\boldsymbol{q}}^{\boldsymbol{T}}\frac{d\boldsymbol{D_q}}{d\boldsymbol{q}}\dot{\boldsymbol{q}}$$

$$+\boldsymbol{\omega}^{\times}\boldsymbol{D_q}\dot{\boldsymbol{q}}+\sum_j^{n_w}I_{sj}\omega_{sj}\boldsymbol{\omega}^{\times}\boldsymbol{a_j}=\boldsymbol{g_{CM}}-\boldsymbol{D_q}\boldsymbol{D_{qq}^{-1}}\boldsymbol{\tau}-\sum_j^{n_w}I_{sj}^{-1}g_{aj} \quad (125)$$

The complexity of this equation makes it difficult to simulate and it will likely be very computationally expensive. A recursive approach may be more prudent.

7 Decoupled Natural Orthogonal Complement Matrices Formulation

This is an adaptation of the recursive Newton-Euler algorithm, originally derived in [1]. It returns a closed-form solution for the forward and inverse dynamics of a serial chain manipulator on a mobile base.

7.1 Twists and Twist Propagation

The twist, aka the linear and angular velocity, of the origin of a rigid body can be expressed as [12] [14]

$$\boldsymbol{t_O}=\begin{bmatrix}\boldsymbol{v_O}\\ \boldsymbol{\omega_O}\end{bmatrix} \quad (126)$$

The twist at any point P can be expressed as

$$\boldsymbol{t_P}=\begin{bmatrix}\boldsymbol{v_P}\\ \boldsymbol{\omega_P}\end{bmatrix}=\begin{bmatrix}\boldsymbol{v_O}+\boldsymbol{\omega_O^{\times}}\boldsymbol{p}\\ \boldsymbol{\omega_O}\end{bmatrix}=\begin{bmatrix}\boldsymbol{I_3} & -\boldsymbol{p}^{\times}\\ \boldsymbol{0_3} & \boldsymbol{I_3}\end{bmatrix}\begin{bmatrix}\boldsymbol{v_O}\\ \boldsymbol{\omega_O}\end{bmatrix}=\boldsymbol{A_{p,O}}\boldsymbol{t_O} \quad (127)$$

With this, the twist vector of any rigid link of a chain of rigid bodies can be expressed as the propagated twist of the previous link plus the twist generated by the rotation at the joint.

$$^{i}\boldsymbol{t_i}={}^{i}\boldsymbol{A_{i,i-1}}{}^{i-1}\boldsymbol{t_{i-1}}+{}^{i}\boldsymbol{p_i}\dot{\theta}_i \quad (128)$$

Here, the twist-propagation matrix ${}^{i}\boldsymbol{A_{i,i-1}}$ is defined as

$$^{i-1}\boldsymbol{A_{i,i-1}}=\begin{bmatrix}\boldsymbol{1} & -(\boldsymbol{a_{i-1,i}})^{\times}\\ \boldsymbol{0} & \boldsymbol{1}\end{bmatrix} \quad (129)$$

where $\boldsymbol{a_{i-1,i}}$ is the length of link $i-1$. Since ${}^{i-1}\boldsymbol{A_{i,i-1}}$ is defined in frame $i-1$ it must be rotated to bring it into frame i:

$$^{i}\boldsymbol{A_{i,i-1}}={}^{i}\boldsymbol{\tilde{R}_{i-1}}{}^{i-1}\boldsymbol{A_{i,i-1}}=\begin{bmatrix}{}^{i}\boldsymbol{R_{i-1}} & \boldsymbol{0}\\ \boldsymbol{0} & {}^{i}\boldsymbol{R_{i-1}}\end{bmatrix}{}^{i-1}\boldsymbol{A_{i,i-1}} \quad (130)$$

The recursive equation 128 can be rewritten in matrix form, where $\boldsymbol{t_m}=[{}^{1}\boldsymbol{t_1}^{\boldsymbol{T}}\boldsymbol{...}{}^{i}\boldsymbol{t_i}^{\boldsymbol{T}}\boldsymbol{...}{}^{n}\boldsymbol{t_n}^{\boldsymbol{T}}]^{\boldsymbol{T}}$ and $\dot{\boldsymbol{q}}_{\boldsymbol{m}}=[\dot{\theta}_1...\dot{\theta}_i...\dot{\theta}_n]^T$. All vectors and matrices without prescript are represented in their base frame, eg. $\boldsymbol{A_{i,i-1}}={}^{i}\boldsymbol{A_{i,i-1}}$

$$\begin{bmatrix}\boldsymbol{t_1}\\ \boldsymbol{t_2}\\ \boldsymbol{t_3}\\ \vdots\\ \boldsymbol{t_n}\end{bmatrix}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0}\\ \boldsymbol{A_{2,1}} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A_{3,2}} & \boldsymbol{0} & \dots & \boldsymbol{0}\\ \vdots & \vdots & \vdots & \ddots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A_{n,n-1}} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{t_1}\\ \boldsymbol{t_2}\\ \boldsymbol{t_3}\\ \vdots\\ \boldsymbol{t_n}\end{bmatrix}+\begin{bmatrix}\boldsymbol{p_1} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{p_2} & \boldsymbol{0} & \dots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{p_3} & \dots & \boldsymbol{0}\\ \vdots & \vdots & \vdots & \ddots & 0\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{p_n}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \vdots\\ \dot{\theta}_k\\ \vdots\\ \dot{\theta}_n\end{bmatrix} \quad (131)$$

ie. $\boldsymbol{t_m}=\boldsymbol{A_m}\boldsymbol{t_m}+\boldsymbol{N_d}\dot{\boldsymbol{q}}_{\boldsymbol{m}}$.

7.2 Kinematics using NOC matrices

Using Natural Orthogonal Complement matrices [12] this relationship can be expressed in a closed form, $\boldsymbol{t_m} = \boldsymbol{N\dot{q}_m} = \boldsymbol{N_l N_d \dot{q}_m}$. Here,

$$\boldsymbol{N_l} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{A_{2,1}} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{A_{3,1}} & \boldsymbol{A_{3,2}} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \mathbf{0} \\ \boldsymbol{A_{n,1}} & \boldsymbol{A_{n,2}} & \dots & \boldsymbol{A_{n,n-1}} & \mathbf{1} \end{bmatrix} \tag{132}$$

and

$$\boldsymbol{N_d} = \begin{bmatrix} \boldsymbol{p_1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{p_2} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{p_3} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{p_n} \end{bmatrix} \tag{133}$$

To find the twist of a link and its Jacobian take the corresponding slice of the block-partitioned expansion, ie.

$$t_i = \begin{bmatrix} \boldsymbol{v_i} \\ \boldsymbol{\omega_i} \end{bmatrix} = \boldsymbol{J_i(q_m)\dot{q}_m} = \begin{bmatrix} \boldsymbol{A_{i,1}} & \dots & \boldsymbol{A_{i,i-1}} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \boldsymbol{N_d \dot{q}_m} \tag{134}$$

In this formulation the camera/end effector is considered frame $n+1$. Its twist is given by

$$\boldsymbol{t_{ee}} = \begin{bmatrix} \boldsymbol{v_{ee}} \\ \boldsymbol{\omega_{ee}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A_{n,1}} & \dots & \boldsymbol{A_{n+1,n}} \end{bmatrix} \boldsymbol{N_d \dot{q}_m} \tag{135}$$

7.3 Kinematics of Combined System

Let the coordinates of the base COM be $\boldsymbol{x_b} = [\boldsymbol{r_{CM}^T} \quad \boldsymbol{\varepsilon^T} \quad \eta]^T$ and let the twist of the base be $\boldsymbol{t_b} = [\boldsymbol{v_{CM}^T} \quad \boldsymbol{\omega_{CM}^T}]^{\boldsymbol{T}}$. The generalized coordinates are therefore $\boldsymbol{q} = [\boldsymbol{x_b^T} \quad \boldsymbol{q_m^T}]^{\boldsymbol{T}}$ and the velocities are $\boldsymbol{\dot{q}} = [\boldsymbol{t_b^T} \quad \boldsymbol{\dot{q}_m^T}]^{\boldsymbol{T}}$.

The manipulator is anchored with position and orientation relative to the base given by the homogeneous transform matrix ${}^{\boldsymbol{b}}\boldsymbol{T_a}$, so ${}^{\boldsymbol{b}}\boldsymbol{T_i} = {}^{\boldsymbol{b}}\boldsymbol{T_a}{}^{\boldsymbol{a}}\boldsymbol{T_i(q_m)}$. Also, the twist propagation matrix ${}^{\boldsymbol{b}}\boldsymbol{A_{a,b}}$ can be written analogously to equation 127. It must be rotated [14] using ${}^{\boldsymbol{a}}\boldsymbol{A_{a,b}} = {}^{\boldsymbol{a}}\boldsymbol{\tilde{R}_b}{}^{\boldsymbol{b}}\boldsymbol{A_{a,b}}$. Therefore, write

$$\boldsymbol{t_m} = \boldsymbol{A_m t_m} + \boldsymbol{N_{m,d}\dot{q}_m} + \boldsymbol{A_b t_b} \tag{136}$$

where $\boldsymbol{A_m}$ and $\boldsymbol{N_{m,d}}$ are those matrices defined in equation 131, and

$$\boldsymbol{A_b} = \begin{bmatrix} \boldsymbol{A_{1,b}} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \tag{137}$$

Here, $\boldsymbol{A_{1,b}}$ is the twist-propagation from the base to the first link, as found by ${}^{\mathbf{1}}\boldsymbol{A_{1,b}} = {}^{\mathbf{1}}\boldsymbol{A_{1,a}}{}^{\boldsymbol{a}}\boldsymbol{A_{a,b}}$. This can be rewritten as [14]

$$\begin{aligned} \boldsymbol{t_m} &= \boldsymbol{N_{m,l}N_{m,d}\dot{q}_m} + \boldsymbol{N_{b,l}N_{b,d}\dot{q}_b} \\ &= \boldsymbol{N_m\dot{q}_m} + \boldsymbol{N_b\dot{q}_b} \end{aligned} \tag{138}$$

where $\boldsymbol{N_{m,l}}$ and $\boldsymbol{N_{m,d}} = \boldsymbol{N_d}$ were defined previously in equations 132 and 133. Since $\boldsymbol{t_b} = \boldsymbol{\dot{q}_b}$, $\boldsymbol{N_{b,d}} = \mathbf{1}$ and $\boldsymbol{N_{b,l}}$ has the form [5]

$$\boldsymbol{N_{b,l}} = \begin{bmatrix} \boldsymbol{A_{1,b}} \\ \boldsymbol{A_{2,b}} \\ \vdots \\ \boldsymbol{A_{n,b}} \end{bmatrix} \tag{139}$$

Then the matrix equation can be written as

$$\begin{bmatrix} \boldsymbol{t_b} \\ \boldsymbol{t_m} \end{bmatrix} = \begin{bmatrix} \boldsymbol{N_{b,d}} & \mathbf{0} \\ \boldsymbol{N_b} & \boldsymbol{N_m} \end{bmatrix} \begin{bmatrix} \boldsymbol{\dot{q}_b} \\ \boldsymbol{\dot{q}_m} \end{bmatrix} \tag{140}$$

This can be expressed as

$$\boldsymbol{t} = \boldsymbol{N}\dot{\boldsymbol{q}} \tag{141}$$

. where the matrix $\boldsymbol{N}$ can be further decoupled into $\boldsymbol{N} = \boldsymbol{N_l}\boldsymbol{N_d}$,

$$\boldsymbol{N_l} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \boldsymbol{N_{b,l}} & \boldsymbol{N_{m,l}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{N_d} = \begin{bmatrix} \boldsymbol{N_{b,d}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{N_{m,d}} \end{bmatrix} \tag{142}$$

As previously, a link's Jacobian is found from the corresponding slice of the expansion and that of the camera can be found by setting $i = n + 1$. This results in the equation

$$\boldsymbol{t_{ee}} = \boldsymbol{J_{ee,m}}\dot{\boldsymbol{q}}_{\boldsymbol{m}} + \boldsymbol{J_{ee,b}}\dot{\boldsymbol{q}}_{\boldsymbol{b}} = \begin{bmatrix} \boldsymbol{J_{ee,m}} & \boldsymbol{J_{ee,b}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_{\boldsymbol{b}} \\ \dot{\boldsymbol{q}}_{\boldsymbol{m}} \end{bmatrix} \equiv \boldsymbol{J_{ee}}\dot{\boldsymbol{q}} \tag{143}$$

where

$$\boldsymbol{J_{ee,m}} = [\boldsymbol{A_{n+1,1}} \quad \ldots \boldsymbol{A_{n+1,n}}]\boldsymbol{N_{m,d}} \qquad \boldsymbol{J_{ee,b}} = \boldsymbol{A_{n+1,b}}\boldsymbol{N_{b,d}} \tag{144}$$

7.4 Dynamics of Manipulator

The Newton-Euler equations of motion for the i^{th} link are expressed as

$$\boldsymbol{I_i}\dot{\boldsymbol{\omega}}_{\boldsymbol{i}} + m_i\boldsymbol{d_i^\times}\dot{\boldsymbol{v}}_{\boldsymbol{i}} + \boldsymbol{\omega_i^\times}(\boldsymbol{I_i}\boldsymbol{\omega_i}) = \boldsymbol{n_i} \tag{145}$$

$$m_i\dot{\boldsymbol{v}}_{\boldsymbol{i}} - m_i\boldsymbol{d_i^\times}\dot{\boldsymbol{\omega}}_{\boldsymbol{i}} - \boldsymbol{\omega_i^\times}(m_i\boldsymbol{d_i^\times}\boldsymbol{\omega_i}) = \boldsymbol{f_i} \tag{146}$$

Here, $\boldsymbol{I_i}$ is the moment of inertia about the origin of the rigid body, O. This equation can be rewritten [12] as

$$\boldsymbol{M_i}\dot{\boldsymbol{t}}_{\boldsymbol{i}} + \boldsymbol{\Omega_k}\boldsymbol{M_i}\boldsymbol{E_i}\boldsymbol{t_i} = \boldsymbol{w_i} \tag{147}$$

where the mass matrix $\boldsymbol{M}$, the angular velocity matrix $\boldsymbol{\Omega}$ and the coupling matrices $\boldsymbol{E}$ are

$$\boldsymbol{M_i} = \begin{bmatrix} m_i\mathbf{1} & -m_i\boldsymbol{d_i^\times} \\ m_i\boldsymbol{d_i^\times} & \boldsymbol{I_i} \end{bmatrix} \quad \boldsymbol{\Omega_i} = \begin{bmatrix} \boldsymbol{\omega_i^\times} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega_i^\times} \end{bmatrix} \quad \boldsymbol{E_i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{148}$$

The wrench $\boldsymbol{w_i} = [\boldsymbol{f_i^T} \quad \boldsymbol{n_i^T}]^{\boldsymbol{T}}$ comes from three sources: $\boldsymbol{w_i^D}$the wrench due to driving moments and forces, $\boldsymbol{w_i^C}$, the wrench due to constrained moments and forces and $\boldsymbol{w_i^F}$the wrench due to external moments and forces. The constrained wrench is found with Newton's Third Law: [14] $\boldsymbol{w_{i,e}} = -\boldsymbol{w_{i+1}}$ and hence

$$\boldsymbol{w_i} = \boldsymbol{w_i^D} - \boldsymbol{A_{i+1,i}^T}\boldsymbol{w_{i+1}} \tag{149}$$

To find the twist rate, take the time-derivative of equation 131:

$${}^{i}\dot{\boldsymbol{t}}_{\boldsymbol{i}} = {}^{i}\dot{\boldsymbol{A}}_{\boldsymbol{i,i-1}}{}^{\boldsymbol{i-1}}\boldsymbol{t_{i-1}} + {}^{i}\boldsymbol{A_{i,i-1}}{}^{\boldsymbol{i-1}}\dot{\boldsymbol{t}}_{\boldsymbol{i-1}} + {}^{i}\dot{\boldsymbol{p}}_{\boldsymbol{i}}\dot{\theta}_i + {}^{i}\boldsymbol{p_i}\ddot{\theta}_i \tag{150}$$

Since the vector $\boldsymbol{p}$ and matrix $\boldsymbol{A}$ are constant with respect to their local frame, their derivatives are given by $\dot{\boldsymbol{p}}_{\boldsymbol{i}} = \boldsymbol{\Omega_i}\boldsymbol{p_i}$ and

$$\dot{\boldsymbol{A}} = \boldsymbol{\omega}^\times\boldsymbol{A} - \boldsymbol{A}\boldsymbol{\omega}^\times \tag{151}$$

which, when rotated properly, results in

$$\dot{\boldsymbol{A}}_{\boldsymbol{i,i-1}} = {}^{i}\tilde{\boldsymbol{R}}_{\boldsymbol{i-1}}(\boldsymbol{\Omega_i}{}^{\boldsymbol{i-1}}\boldsymbol{A_{i,i-1}} - {}^{\boldsymbol{i-1}}\boldsymbol{A_{i,i-1}}\boldsymbol{\Omega_i}) = {}^{i}\tilde{\boldsymbol{R}}_{\boldsymbol{i-1}} \begin{bmatrix} \mathbf{0} & -(\boldsymbol{\omega_i^\times}\boldsymbol{a_{i-1,i}^\times}) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{152}$$

The acceleration of the anchor point is $\dot{\boldsymbol{t}}_{\boldsymbol{a}} = \dot{\boldsymbol{A}}_{\boldsymbol{a,b}}\boldsymbol{t_b} + \boldsymbol{A_{a,b}}\dot{\boldsymbol{t}}_{\boldsymbol{b}}$, where $\dot{\boldsymbol{A}}_{\boldsymbol{a,b}}$ is defined analogously to $\dot{\boldsymbol{A}}_{\boldsymbol{i,i-1}}$.

The generalized effort vector is $\boldsymbol{\tau_m} = [\tau_1 \quad \ldots \quad \tau_n]^T$ where τ_i represents the torque applied by the motor at the i^{th} revolute joint. It is shown in [12] that motor wrench is only effective along its axis, ie.

$$\tau_i = \boldsymbol{p_i^T}\boldsymbol{w_i} \tag{153}$$

7.5 Compact form of the Dynamics Equations

Combining equation 147 as it relates to all of the links, results in a compact form for the dynamic equation for the entire manipulator:

$$\bar{\boldsymbol{M}}_{\boldsymbol{m}}\dot{\bar{\boldsymbol{t}}}_{\boldsymbol{m}} + \bar{\boldsymbol{\Omega}}_{\boldsymbol{m}}\bar{\boldsymbol{M}}_{\boldsymbol{m}}\bar{\boldsymbol{E}}_{\boldsymbol{m}}\bar{\boldsymbol{t}}_{\boldsymbol{m}} = \bar{\boldsymbol{w}}_{\boldsymbol{m}} \tag{154}$$

where

$$\bar{\boldsymbol{M}}_{\boldsymbol{m}} \equiv diag(\boldsymbol{M_1},\ldots,\boldsymbol{M_n}) \qquad \bar{\boldsymbol{\Omega}}_{\boldsymbol{m}} \equiv diag(\boldsymbol{\Omega_1},\ldots,\boldsymbol{\Omega_n}) \qquad \bar{\boldsymbol{E}}_{\boldsymbol{m}} \equiv diag(\boldsymbol{E_1},\ldots,\boldsymbol{E_n}) \qquad \bar{\boldsymbol{w}}_{\boldsymbol{m}} = \begin{bmatrix} \boldsymbol{w_1} \\ \vdots \\ \boldsymbol{w_n} \end{bmatrix} \tag{155}$$

A similar equation can be derived for the spacecraft itself,

$$\boldsymbol{M_b}\dot{\boldsymbol{t}}_{\boldsymbol{b}} + \boldsymbol{\Omega_b}\boldsymbol{M_b}\boldsymbol{E_b}\boldsymbol{t_b} = \boldsymbol{w_b} \tag{156}$$

where $\boldsymbol{w_b} = \boldsymbol{w_b^D} + \boldsymbol{w_b^E} + \boldsymbol{w_b^C}$ is the total wrench. $\boldsymbol{w_b^D}$ represents the wrench from the base actuators, $\boldsymbol{w_b^E}$ represents the wrench from the environment, and $\boldsymbol{w_b^C}$ the constraint wrench from the manipulator's anchor points. This implies that the control wrench is an external force - internal momentum sources must be treated differently. These forces would be considered intra-modular forces in the approach used by [12]. The effect of the momentum exchange wheels in the DeNOC formulation will be discussed in a later addendum to this report.

Equation 154 can be combined with equation 156 to further compact the dynamics:

$$\bar{\boldsymbol{M}}\dot{\boldsymbol{t}} + \bar{\boldsymbol{\Omega}}\bar{\boldsymbol{M}}\bar{\boldsymbol{E}}\boldsymbol{t} = \bar{\boldsymbol{w}} \tag{157}$$

where

$$\bar{\boldsymbol{M}} \equiv diag(\boldsymbol{M_b}, \bar{\boldsymbol{M}}_{\boldsymbol{m}}) \qquad \bar{\boldsymbol{\Omega}} \equiv diag(\boldsymbol{\Omega_b}, \bar{\boldsymbol{\Omega}}_{\boldsymbol{m}}) \qquad \bar{\boldsymbol{E}} \equiv diag(\boldsymbol{E_b}, \bar{\boldsymbol{E}}_{\boldsymbol{m}}) \qquad \bar{\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{w_b} \\ \bar{\boldsymbol{w}}_{\boldsymbol{m}} \end{bmatrix} \tag{158}$$

Differentiating 127, the twist rate is

$$\dot{\boldsymbol{t}} = \boldsymbol{N}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{N}}\dot{\boldsymbol{q}} \tag{159}$$

where $\dot{\boldsymbol{N}} = \dot{\boldsymbol{N}}_{\boldsymbol{l}}\boldsymbol{N_d} + \boldsymbol{N_l}\dot{\boldsymbol{N}}_{\boldsymbol{d}}$. From the original definition of $\boldsymbol{N_l}$, it is equivalent to $(\boldsymbol{1}-\boldsymbol{A})^{-1}$ and hence its derivative $\dot{\boldsymbol{N}}_{\boldsymbol{l}} = \boldsymbol{N_l}\dot{\boldsymbol{A}}\boldsymbol{N_l}$, where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A_b} & \boldsymbol{A_m} \end{bmatrix} \rightarrow \dot{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \dot{\boldsymbol{A}}_{\boldsymbol{b}} & \dot{\boldsymbol{A}}_{\boldsymbol{m}} \end{bmatrix} \tag{160}$$

where

$$\dot{\boldsymbol{A}}_{\boldsymbol{m}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \dot{\boldsymbol{A}}_{\boldsymbol{2,1}} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \dot{\boldsymbol{A}}_{\boldsymbol{3,2}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dot{\boldsymbol{A}}_{\boldsymbol{n,n-1}} & \boldsymbol{0} \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{A}}_{\boldsymbol{b}} = \begin{bmatrix} \dot{\boldsymbol{A}}_{\boldsymbol{1,b}} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix} \tag{161}$$

Also,

$$\dot{\boldsymbol{N}}_{\boldsymbol{d}} = \begin{bmatrix} \dot{\boldsymbol{N}}_{\boldsymbol{b,d}} & \boldsymbol{0} \\ \boldsymbol{0} & \dot{\boldsymbol{N}}_{\boldsymbol{m,d}} \end{bmatrix} \quad \text{where} \quad \dot{\boldsymbol{N}}_{\boldsymbol{m,d}} = \begin{bmatrix} \dot{\boldsymbol{p}}_{\boldsymbol{1}} & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \dot{\boldsymbol{p}}_{\boldsymbol{n}} \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{N}}_{\boldsymbol{b,d}} = \dot{\boldsymbol{P}}_{\boldsymbol{b}} = \boldsymbol{\Omega_b}\boldsymbol{P_b} \tag{162}$$

The derivatives of the twist-propagation matrices can be found using equation 152. Before substituting equations 141 and 159 into equation 157, it is important to mention the power due to constraint wrenches. It is known [12] that constraint wrenches are orthogonal to the velocity transformation matrix and hence the product vanishes, ie. $\boldsymbol{N^T}\bar{\boldsymbol{w}}^{\boldsymbol{C}} = \boldsymbol{0}$. To exploit this, premultiply equation 157 by $\boldsymbol{N^T}$. This results in

$$\boldsymbol{N^T}\bar{\boldsymbol{M}}(\boldsymbol{N}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{N}}\dot{\boldsymbol{q}}) + \boldsymbol{N^T}\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{M}}\bar{\boldsymbol{E}}(\boldsymbol{N}\dot{\boldsymbol{q}}) = \boldsymbol{N^T}(\bar{\boldsymbol{w}}^{\boldsymbol{D}} + \bar{\boldsymbol{w}}^{\boldsymbol{C}}) \tag{163}$$

which simplifies to

$$\boldsymbol{H}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{\tau} \tag{164}$$

where

$$\boldsymbol{H}(\boldsymbol{q}) = \boldsymbol{N^T}\bar{\boldsymbol{M}}\boldsymbol{N} \qquad \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{N^T}\bar{\boldsymbol{M}}\dot{\boldsymbol{N}} + \boldsymbol{N^T}\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{M}}\bar{\boldsymbol{E}}\boldsymbol{N} \qquad \boldsymbol{\tau} = \boldsymbol{N^T}\bar{\boldsymbol{w}}^{\boldsymbol{D}} \tag{165}$$

7.6 Thrusters and Momentum Exchange Wheels

Forces from the thrusters may be considered external forces, but the internal dynamics of the reaction wheels have not yet been considered.

7.7 Use of SPART

SPART [18] is a software package for mobile-base robotic multibody systems which was developed by researchers at the Naval Postgraduate School. This toolbox uses the DeNOC method [17] to solve for the motion of any robot defined either by the Universal Robot Description Format or from a set of DH parameters. The toolbox can produce

- Forward kinematics
- Twist propagation matrices and vectors
- Operational-space velocities
- Jacobians of any point on any link
- Time derivatives of Jacobians
- Accelerations of links and base
- Inertia matrices
- Mass composite body matrix $\bar{\boldsymbol{M}}$
- Generalized inertia matrix $\boldsymbol{H}$
- Convective inertia matrix $\boldsymbol{C}$
- Forward dynamics
- Inverse dynamics for a flying base
- Inverse dynamics and base acceleration for a floating base
- Plots of kinematic manipulability ellipsoids and reachable workspace
- Quaternion kinematic integration

There are some subtle differences between the method just presented and the method used in SPART. Most importantly, the dynamic equations in SPART use the link centre of mass frame instead of the origin of the rigid body. Also, instead of using the Newton-Euler equations for the acceleration, [17] sets the time derivative of the momentum equal to the wrench:

$$\boldsymbol{M_i}\dot{\boldsymbol{t}}_{\boldsymbol{i}} + \dot{\boldsymbol{M}}_{\boldsymbol{i}}\boldsymbol{t_i} = \boldsymbol{w_i} \quad \text{where} \quad \boldsymbol{M_i} = \begin{bmatrix} m_i\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I_i} \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{M}}_{\boldsymbol{i}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_i^{\times}\boldsymbol{I_i} \end{bmatrix} \tag{166}$$

This leads to a simpler equation for the convective inertia matrix,

$$\boldsymbol{C} = \boldsymbol{N}^{\boldsymbol{T}}(\boldsymbol{M}\dot{\boldsymbol{N}} + \dot{\boldsymbol{M}}\boldsymbol{N}) \tag{167}$$

While the equations used by the toolbox are relatively straightforward, it has been published in several journal articles and papers. [4] [2] [3]

8 The Effect of Gravity

A spacecraft is not entirely unaffected by gravity. This section will study the effect of gravitational forces and torques on the base of the spacecraft. It is assumed that only one nearby celestial body needs to be considered, that this body is spherically symmetric and that the spacecraft is small compared to its distance from the centre of the body. [6] For a spacecraft orbiting the Earth, where now, $\vec{\boldsymbol{r}}_{\boldsymbol{CM}}$ represents the displacement from Earth's centre and $\vec{\boldsymbol{d}}$ the displacement of mass element dm. Here, $\mu = Gm_{\oplus}$ is the standard gravitational parameter of Earth, $\mu \approx 3.986 \times 10^{14} N \cdot m^2/kg$.

$$\vec{\boldsymbol{d}} = \vec{\boldsymbol{r}}_{\boldsymbol{CM}} + \vec{\boldsymbol{r}} \tag{168}$$

$$\vec{\boldsymbol{f}} = -\mu \int_B \frac{\vec{\boldsymbol{d}}}{d^3} dm \tag{169}$$

$$\vec{\boldsymbol{g}}_{\boldsymbol{c}} = -\mu \int_B \frac{\vec{\boldsymbol{r}} \times \vec{\boldsymbol{d}}}{d^3} dm \tag{170}$$

$$\tag{171}$$

It is clear that $\vec{\boldsymbol{g}}_{\boldsymbol{c}} \cdot \vec{\boldsymbol{r}}_{\boldsymbol{CM}} = 0$ when the integral is taken about the centre of mass of the spacecraft. This indicates that there is no gravitational torque on B about the local vertical [6]. Substitute equation 168 into equations 169 and 170 and implement the assumption that $r/r_{CM} << 1$, where these scalars represent the magnitude of their respective vectors. Using the binomial approximation results in

$$\vec{\boldsymbol{f}} = -\frac{\mu m}{r_{CM}^3} \vec{\boldsymbol{r}}_{\boldsymbol{CM}} \tag{172}$$

$$\vec{\boldsymbol{g}}_{\boldsymbol{c}} = -\frac{3\mu}{r_{CM}^5} \times \int_B \vec{\boldsymbol{r}}\vec{\boldsymbol{r}} dm \cdot \vec{\boldsymbol{r}}_{\boldsymbol{CM}} \tag{173}$$

$$\tag{174}$$

In essence the gravitational field has been replaced by the value at the centre of mass and a gravity gradient force field. There is no gradient term in the equation for force - therefore there is no net force on B due to the gravity gradient field. However, the gravity gradient does result in a torque. Define the unit vector $\hat{\boldsymbol{k}} = -\vec{\boldsymbol{r}}_{\boldsymbol{CM}}/r_{CM}$ which indicates a "Down" direction . This allows the gravity gradient torque to be written as

$$\vec{\boldsymbol{g}}_{\boldsymbol{c}} = 3\frac{\mu}{r_{CM}^3} \hat{\boldsymbol{k}} \times \boldsymbol{I} \cdot \hat{\boldsymbol{k}} \tag{175}$$

It is shown in Hughes [6] p. 247 that for a system of N bodies, the gravity gradient torque on link i calculated about its CM is

$$g_{ci} = \frac{3\mu}{r_{CM}^3} \hat{\boldsymbol{k}} \times \boldsymbol{I_i} \cdot \hat{\boldsymbol{k}} \tag{176}$$

This torque is usually negligible. This is because the inspection spacecraft will follow an orbital path for most of its journey. During an orbit, the linear velocity of the spacecraft satisfies the simple dynamic equation [5]

$$m_b \dot{\vec{\boldsymbol{v}}} = \boldsymbol{f_g} + \boldsymbol{f_{dist}} + \boldsymbol{f_c} \tag{177}$$

Without external forces or control forces the spacecraft must only follow gravity, ie.

$$m_b \dot{\vec{\boldsymbol{v}}}_{\boldsymbol{g}} = \boldsymbol{f_g} \tag{178}$$

where $\vec{\boldsymbol{v}}_{\boldsymbol{g}}$ is defined as the nominal speed on orbit. Let it be perturbed, ie. $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_{\boldsymbol{g}} + \delta\vec{\boldsymbol{v}}$ Then the equations can be decoupled, ie.

$$m_b \delta\dot{\vec{\boldsymbol{v}}} = \boldsymbol{f_{dist}} + \boldsymbol{f_c} \tag{179}$$

While on an orbit, the gravity force may be considered constant and one may confidently study only the perturbation from the nominal orbital velocity. This assumption is valid when the translational motion is small compared to the distance from Earth. It will be valid throughout a given orbit. It may be violated during changes in orbit, which will be necessary to best inspect the target spacecraft. During these manoeuvres, the general gravity gradient field must be considered. This will be included in a later report.

References

[1] Jorge Angeles and Sang Koo Lee. “The Formulation of Dynamical Equations of Holonomic Mechanical Systems Using a Natural Orthogonal Complement”. eng. In: *Journal of applied mechanics* 55.1 (1988), pp. 243–244. ISSN: 0021-8936.

[2] Lorenzo Capra and Michele Lavagna. “Design Of A Versatile Space Manipulator Functional Engineering Simulator For IOS”. In: ().

[3] ST Kwok Choon et al. “Astrobatics: A hopping-maneuver experiment for a spacecraft-manipulator system on board the international space Station”. In: *Proceedings of the International Astronautical Congress, IAC*. 2019.

[4] Christelle Cumer et al. “MODELLING AND ATTITUDE CONTROL DESIGN FOR AUTONOMOUS IN-ORBIT ASSEMBLY”. In: *ESA GNC 2021*. Sopot, Poland, June 2021. URL: `https://hal.science/hal-03424038`.

[5] Vincent Dubanchet. “Modeling and Control of a Flexible Space Robot to Capture a Tumbling Debris”. eng. PhD thesis. Ecole Polytechnique du Montreal, 2016. ISBN: 9798209707356.

[6] Peter C. Hughes. *Spacecraft Attitude Dynamics*. eng. 1st ed. Dover Books on Aeronautical Engineering. Newburyport: Dover Publications, 2012. ISBN: 0-486-14013-X.

[7] S. Ali A. Moosavian and Evangelos Papadopoulos. “On the kinematics of multiple manipulator space free-flyers and their computation”. eng. In: *Journal of robotic systems* 15.4 (1998), pp. 207–216. ISSN: 0741-2223.

[8] Kostas Nanos and Evangelos Papadopoulos. “On the use of free-floating space robots in the presence of angular momentum”. In: *Intelligent service robotics* 4.1 (2011), pp. 3–15. ISSN: 1861-2776.

[9] Michael Paluszek. *ADCS : Spacecraft Attitude Determination and Control*. eng. Amsterdam, Netherlands: Elsevier Inc., 2023. ISBN: 9780323985413.

[10] E. Papadopoulos. “On the Dynamics and Control of Space Manipulators”. PhD thesis. Massachusetts Institute of Technology, 1990.

[11] E. Papadopoulos and S. Dubowsky. “Dynamic Singularities in Free-Floating Space Manipulators”. eng. In: *Journal of dynamic systems, measurement, and control* 115.1 (1993), pp. 44–52. ISSN: 0022-0434.

[12] Suril Vijaykumar Shah, Subir Kumar Saha, and Jayanta Kumar Dutt. *Dynamics of tree-type robotic systems*. eng. 1st ed. 2013. International series on intelligent systems, control and automation–science and engineering; v. 62. New York: Springer, 2013. ISBN: 1-283-93815-4.

[13] Bruno Siciliano et al. *Robotics: Modelling, Planning and Control*. eng. 1st ed. Advanced Textbooks in Control and Signal Processing. London: Springer Nature, 2008. ISBN: 9781846286421.

[14] Charles Sirois. “Nonlinear MPC for Free-Flying Space Manipulator”. MA thesis. Politecnico Milano, 2022.

[15] Y. Umetani and K. Yoshida. “Resolved motion rate control of space manipulators with generalized Jacobian matrix”. eng. In: *IEEE transactions on robotics and automation* 5.3 (1989), pp. 303–314. ISSN: 1042-296X.

[16] Y. Umetani and K. Yoshida. “Control of Space Manipulators with Generalized Jacobian Matrix”. In: Springer, 1993. Chap. 7.

[17] Josep Virgili-Llop et al. “Spacecraft Robotics Toolkit: an Open-Source Simulator for Spacecraft Robotic Arm Dynamic Modeling And Control”. In: *6th International Conference on Astrodynamics Tools and Techniques (ICATT)*. Mar. 2016.

[18] Josep Virgili-Llop et al. *SPART: an open-source modeling and control toolkit for mobile-base robotic multibody systems with kinematic tree topologies*. `https://github.com/NPS-SRL/SPART`.

[19] Markus Wilde et al. “Equations of Motion of Free-Floating Spacecraft-Manipulator Systems: An Engineer’s Tutorial”. In: *Frontiers in robotics and AI* 5 (2018). ISSN: 2296-9144.