C1 Differentiation and Integration 1

1a/ $f(x) = \dfrac{(3-4\sqrt{x})^2}{\sqrt{x}}$

$= \dfrac{(3-4x^{1/2})(3-4x^{1/2})}{x^{1/2}}$

$= \dfrac{9-24x^{1/2}+16x}{x^{1/2}}$

$= 9x^{-1/2} - 24 + 16x^{1/2}$ $\quad A=16 \quad B=-24$

b/ $f'(x) = -\frac{9}{2}x^{-3/2} + 8x^{-1/2}$

c/ $f'(9) = -\frac{9}{2}(9)^{-3/2} + 8(9)^{-1/2}$

$= -\frac{9}{2}\left(\frac{1}{27}\right) + 8\left(\frac{1}{3}\right)$

$= \dfrac{-1}{6} + \dfrac{8}{3}$

$= \frac{15}{6} = \underline{\underline{\frac{5}{2}}}$

2/ $\int 12x^5 - 8x^3 + 3 \; dx$

$\dfrac{12x^6}{6} - \dfrac{8x^4}{4} + 3x + c$

$2x^6 - 2x^4 + 3x + c$

3a/ $y = 2x^3 + 3x^{-2}$

$\dfrac{dy}{dx} = 6x^2 - 6x^{-3}$

b/ $\int y \, dx = \dfrac{2x^4}{4} + \dfrac{3x^{-1}}{-1} + c$

$= \frac{1}{2}x^4 - 3x^{-1} + c$

4/ $f'(x) = 3x^2 - 3x^{1/2} - 7$ $(4, 22)$

$f(x) = \frac{3x^3}{3} - \frac{3x^{3/2}}{3/2} - 7x + c$

$= x^3 - 2x^{3/2} - 7x + c$

$22 = (4)^3 - 2(4)^{3/2} - 7(4) + c$

$22 = 64 - 16 - 28 + c$

$c = 2$

$\therefore f(x) = x^3 - 2x^{3/2} - 7x + 2$

5/ $\frac{2x^2 - x^{3/2}}{x^{1/2}}$

$2x^{3/2} - x$ $p = 3/2$ $q = 1$

b/ $y = 5x^4 - 3 + 2x^{3/2} - x$

$\frac{dy}{dx} = 20x^3 + 3x^{1/2} - 1$

6/ $\int 3x^2 + 4x^5 - 7 \, dx$

~~6x~~

$\frac{3x^3}{3} + \frac{4x^6}{6} - 7x + c$

$x^3 + \frac{2}{3}x^6 - 7x + c$

7/ $\int 2 + 5x^2 \, dx$

$2x + \frac{5x^3}{3} + c$

8/ $f(x) = 3x + x^3$

$f'(x) = 3 + 3x^2$

b/ $3 + 3x^2 = 15$

$3x^2 = 12$

$x^2 = 4$

$x = 2$

9/ $y = 4x^3 - 1 + 2x^{1/2}$

$\frac{dy}{dx} = 12x^2 + x^{-1/2}$

10/ $\frac{dy}{dx} = \frac{(x^2+3)^2}{x^2}$

$= \frac{(x^2+3)(x^2+3)}{x^2}$

$= \frac{x^4+6x^2+9}{x^2}$

$= x^2 + 6 + 9x^{-2}$

b/ $y = \frac{x^3}{3} + 6x + \frac{9x^{-1}}{-1} + c$

$y = \frac{1}{3}x^3 + 6x - 9x^{-1} + c$

$(3,20)$ $20 = \frac{1}{3}(3)^3 + 6(3) - \frac{9}{3} + c$

$20 = 9 + 18 - 3 + c$

$c = -4$

$y = \frac{1}{3}x^3 + 6x - 9x^{-1} - 4$

11 a/ $\frac{2x^{1/2} + 3}{x}$

$\underline{2x^{-1/2} + 3x^{-1}}$

b) $y = 5x - 7 + 2x^{-1/2} + 3x^{-1}$

$\frac{dy}{dx} = 5 - x^{-3/2} - 3x^{-2}$

12) $y = 3x^2 + 4x^{1/2}$

a) $\frac{dy}{dx} = 6x + 2x^{-1/2}$

b) $\frac{d^2y}{dx^2} = 6 - x^{-3/2}$

c) $\int y\,dx = \frac{3x^3}{3} + \frac{4x^{3/2}}{3/2} + c$

$= x^3 + \frac{8}{3}x^{3/2} + c$

13a) $(4 + 3x^{1/2})(4 + 3x^{1/2})$

$16 + 12x^{1/2} + 12x^{1/2} + 9x$

$16 + 24x^{1/2} + 9x$

b) $16x + \frac{24x^{3/2}}{3/2} + \frac{9x^2}{2} + c$

$16x + 16x^{3/2} + 9/2\, x^2 + c$

14a) $y = x^4 + 6x^{1/2}$

$\frac{dy}{dx} = 4x^3 + 3x^{-1/2}$

b) $\frac{(x+4)(x+4)}{x} = \frac{x^2 + 8x + 16}{x} = x + 8 + 16x^{-1}$

$\frac{dy}{dx} = 1 - 16x^{-2}$

ii) $\int 1 + 3x^{1/2} - x^{-2}\,dx$

$x + \frac{3x^{3/2}}{3/2} - \frac{x^{-1}}{-1} + c$

$x + 2x^{3/2} + x^{-1} + c$

15. $y = 5x^3 + 7x + 3$

a) $\frac{dy}{dx} = 15x^2 + 7$

b) $\frac{d^2y}{dx^2} = 30x$

ii) $\int 1 + 3x^{1/2} - x^{-2} \, dx$

$x + \frac{3x^{3/2}}{3/2} - \frac{x^{-1}}{-1} + c$

$x + 2x^{3/2} + x^{-1} + c$

16a) $\frac{(3 - x^{1/2})(3 - x^{1/2})}{x^{1/2}}$

$\frac{9 - 6x^{1/2} + x}{x^{1/2}}$

$9x^{-1/2} - 6 + x^{1/2}$

b) $\frac{dy}{dx} = 9x^{-1/2} - 6 + x^{1/2}$ $(1, 2/3)$

$y = \frac{9x^{1/2}}{1/2} - 6x + \frac{x^{3/2}}{3/2} + c$ $y = 18x^{\frac{1}{2}} - 6x + \frac{2}{3}x^{\frac{3}{2}} + c$

$2/3 = \frac{9(1)^{1/2}}{1/2} - 6(1) + \frac{(1)^{3/2}}{3/2} + c$

$\frac{2}{3} = 18 - 6 + \frac{2}{3} + c$

$c = -12$

$y = 18x^{\frac{1}{2}} - 6x + \frac{2}{3}x^{\frac{3}{2}} - 12$

17 $y = 2x^2 - 6x^{-3}$

$\frac{dy}{dx} = 4x + 18x^{-4}$

$\int y \, dx = \frac{2x^3}{3} - \frac{6x^{-2}}{-2} + c = \underline{\frac{2}{3}x^3 + 3x^{-2} + c}$

18/ $f'(x) = 3 + 5x^{3/2} + 2x^{-1/2}$ $(1, 6)$

$f(x) = 3x + \frac{5x^{5/2}}{5/2} + \frac{2x^{1/2}}{1/2} + c$

$f(x) = 3x + 2x^{5/2} + 4x^{1/2} + c$

$6 = 3(1) + 2(1)^{5/2} + 4(1)^{1/2} + c$

$c = -3$

$f(x) = 3x + 2x^{5/2} + 4x^{1/2} - 3$

19/ $y = 6x - 4x^{-2}$

$\frac{dy}{dx} = 6 + 8x^{-3}$

$\int y\,dx = \frac{6x^2}{2} - \frac{4x^{-1}}{-1} + c$

$= 3x^2 + 4x^{-1} + c$

C1 Differentiation and Integration 2

1) $y = kx^3 - x^2 + x - 5$

$\frac{dy}{dx} = 3kx^2 - 2x + 1$

b) $2y - 7x + 1 = 0$

$2y = 7x - 1$

$y = \frac{7}{2}x - \frac{1}{2}$

$m = 7/2 \quad \therefore \frac{dy}{dx} = 7/2$

$3kx^2 - 2x + 1 = \frac{7}{2}$

$3k\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) + 1 = 7/2$

$\frac{3k}{4} + 1 + 1 = 7/2$

$\frac{3k}{4} = \frac{3}{2}$

$\underline{k = 2}$

c) $y = 2x^3 - x^2 + x - 5$

$y = 2\left(\frac{-1}{2}\right)^3 - \left(-\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right) - 5$

$y = -\frac{2}{8} - \frac{1}{4} - \frac{1}{2} - 5$

$\underline{y = -6}$

2) $y = 9 - 4x - 8x^{-1}$

$\frac{dy}{dx} = -4 + 8x^{-2}$

when $x = 2$

$\frac{dy}{dx} = -4 + \frac{8}{2^2}$

$= -2$

$y = -2x + c$

$-3 = -2(2) + c$

$-3 = -4 + c$

$1 = c$

$\underline{\underline{y = -2x + 1}}$

$y = 9 - 4(2) - \frac{8}{2}$

$= 9 - 8 - 4$

$= -3$

b) $(2,-3)$ $\quad m = \frac{1}{2}$

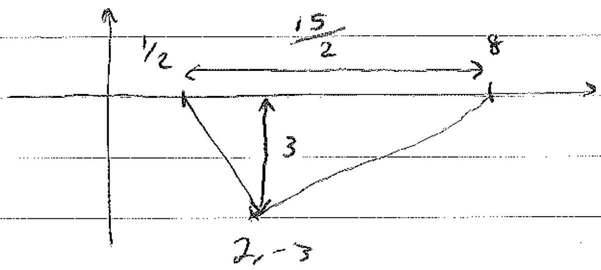
$$y = \tfrac{1}{2}x + c$$
$$-3 = \tfrac{1}{2}(2) + c$$
$$-3 = 1 + c$$
$$c = -4$$
$$\underline{y = \tfrac{1}{2}x - 4}$$

c) line meets x axis when $y=0$

A: $0 = 1 - 2x$ $\qquad$ B: $0 = \tfrac{1}{2}x - 4$

$2x = 1$ $\qquad$ $4 = \tfrac{1}{2}x$

$x = \tfrac{1}{2}$ $\qquad$ $x = 8$

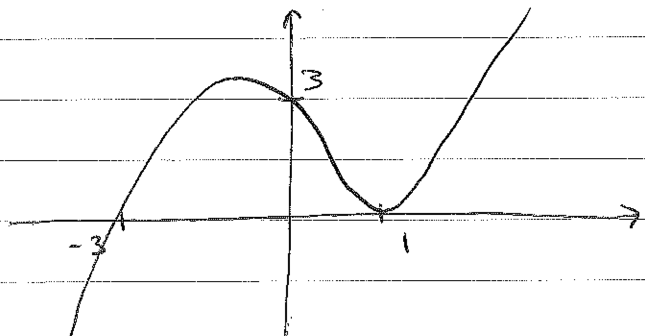


$$\text{Area} = \tfrac{1}{2}\cdot \text{base}\cdot\text{height}$$
$$= \tfrac{1}{2}\cdot \tfrac{15}{2}\cdot 3$$
$$= \underline{\tfrac{45}{4}}\ \text{units}^2$$

3a) $y = (x+3)(x-1)^2$

when $y=0$ $\quad x=-3 \quad x=1$

when $x=0$ $\quad y=3$



b/ $y = (x+3)(x-1)(x-1)$

$y = (x+3)(x^2-2x+1)$

$y = x^3 - 2x^2 + x + 3x^2 - 6x + 3$

$= x^3 + x^2 - 5x + 3$

$\underline{k = 3}$

c/ $\frac{dy}{dx} = 3x^2 + 2x - 5$

$3 = 3x^2 + 2x - 5$

$0 = 3x^2 + 2x - 8$

$0 = (3x-4)(x+2)$

$\underline{x = 4/3} \quad \underline{x = -2}$

4 $f'(x) = 4x - 6x^{1/2} + 8x^{-2}$ $(4, 1)$

$f(x) = \frac{4x^2}{2} - \frac{6x^{3/2}}{3/2} + \frac{8x^{-1}}{-1} + c$

$f(x) = 2x^2 - 4x^{3/2} - 8x^{-1} + c$

$1 = 2(4)^2 - 4(4)^{3/2} - 8(4)^{-1} + c$

$1 = 32 - 32 - 2 + c$

$c = 3$

$f(x) = 2x^2 - 4x^{3/2} - 8x^{-1} + 3$

b/ $f'(x) = 4x - 6\sqrt{x} + \frac{8}{x^2}$

$f'(4) = 4(4) - 6\sqrt{4} + \frac{8}{4^2}$

$= 16 - 12 + 1/2$

$= 9/2$

gradient of normal $= -2/9$

$y = -2/9\, x + c$

$1 = -2/9(4) + c$

$1 = -8/9 + c$

$c = \frac{17}{9}$

$\underline{y = -2/9\, x + 17/9}$

5 $f'(x) = 6x^2 - 10x - 12$ $(5, 65)$

$f(x) = \frac{6x^3}{3} - \frac{10x^2}{2} - 12x + c$

$f(x) = 2x^3 - 5x^2 - 12x + c$

$65 = 2(5)^3 - 5(5)^2 - 12(5) + c$

$65 = 250 - 125 - 60 + c$

$65 = 65 + c$

$c = 0$

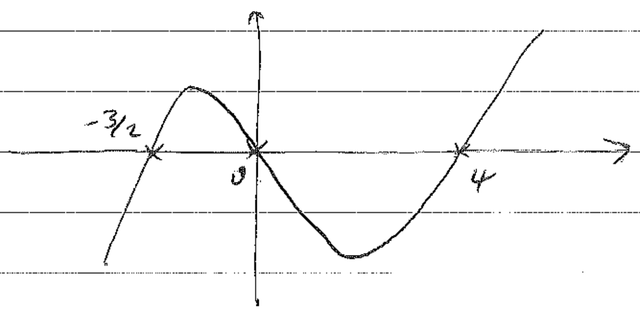
$f(x) = 2x^3 - 5x^2 - 12x$

b/ $f(x) = x(2x^2 - 5x - 12)$

$= x(2x+3)(x-4)$

c/ when $y=0$ $x=0$ $x=-3/2$ $x=4$

when $x=0$ $y=0$



6 $y = x^2(x-6) + 4/x$

when $x=1$

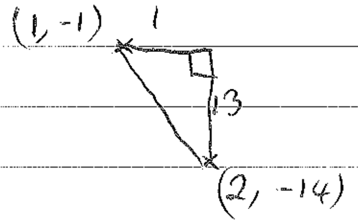
$y = (1)^2(1-6) + \frac{4}{1}$

$= -1$

when $x=2$

$y = (2)^2(2-6) + 4/2$

$= -14$



$PQ = \sqrt{1^2 + 13^2}$

$PQ = \sqrt{170}$

b/ $y = x^2(x-6) + 4x^{-1}$

$\cancel{\frac{dy}{dx}} = x^3 - 6x^2 + 4x^{-1}$

$\frac{dy}{dx} = 3x^2 - 12x - 4x^{-2}$

when $x = 1$

$\frac{dy}{dx} = 3(1)^2 - 12(1) - 4(1)^{-2}$

$= 3 - 12 - 4$

$= \underline{-13}$

when $x = 2$

$\frac{dy}{dx} = 3(2)^2 - 12(2) - 4(2)^{-2}$

$= 12 - 24 - 1$

$= \underline{-13}$

c/ gradient of normal $= \frac{1}{13}$ $(1, -1)$

$y = \frac{1}{13}x + c$

$-1 = \frac{1}{13}(1) + c$

$c = \frac{-14}{13}$

$\therefore y = \frac{1}{13}x + \frac{-14}{13}$

$13y = x + -14$

$\underline{x - 13y - 14 = 0}$

7 $f'(x) = 3x^2 - 6 - 8x^{-2}$ $(2, 1)$

$f(x) = \frac{3x^3}{3} - 6x - \frac{8x^{-1}}{-1} + c$

$f(x) = x^3 - 6x + 8x^{-1} + c$

$1 = (2)^3 - 6(2) + 8(2)^{-1} + c$

$1 = 8 - 12 + 4 + c$

$c = 1$

$f(x) = x^3 - 6x + 8x^{-1} + 1$

b/ $f'(2) = 3(2)^2 - 6 - 8(2)^{-2}$

$= 12 - 6 - 2$

$= 4$

$y = 4x + c$

$1 = 4(2) + c$

$c = -7$

$\underline{y = 4x - 7}$

8 a) $y = 4x + 3x^{3/2} - 2x^2$

$\frac{dy}{dx} = 4 + \frac{9}{2}x^{1/2} - 4x$

b) $8 = 4(4) + 3(4)^{3/2} - 2(4)^2$

$8 = 16 + 24 - 32$

$8 = 40 - 32$

$\underline{8 = 8}$ Shown

c) at $x = 4$ $\quad \frac{dy}{dx} = 4 + \frac{9}{2}(4)^{1/2} - 4(4)$

$= 4 + 9 - 16$

$= -3$

gradient of normal $= \frac{1}{3}$

$y = \frac{1}{3}x + c \qquad (4, 8)$

$8 = \frac{1}{3}(4) + c$

$8 = \frac{4}{3} + c$

$c = \frac{20}{3}$

$y = \frac{1}{3}x + \frac{20}{3}$

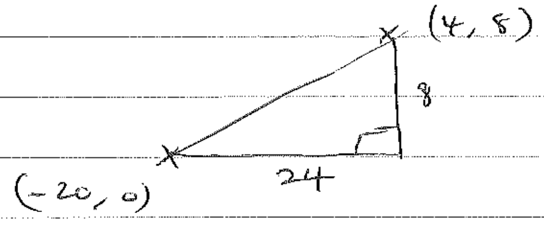
$\underline{3y = x + 20}$

d) cuts x axis when $y = 0$

$0 = x + 20$

$x = -20$

$(-20, 0) \qquad (4, 8)$



$PQ = \sqrt{24^2 + 8^2}$

$= \sqrt{576 + 64}$

$= \sqrt{640}$

$= \sqrt{64}\sqrt{10}$

$= \underline{8\sqrt{10}}$

9 $f'(x) = 2x + 3x^{-2}$ $\quad (3, \frac{15}{2})$

$f(x) = \frac{2x^2}{2} + \frac{3x^{-1}}{-1} + c$

$f(x) = x^2 - 3x^{-1} + c$

$\frac{15}{2} = (3)^2 - 3(3)^{-1} + c$

$\frac{15}{2} = 9 - 1 + c$

$\frac{15}{2} = 8 + c$

$c = -\frac{1}{2}$

$f(x) = x^2 - 3x^{-1} - \frac{1}{2}$

b/ $f(-2) = (-2)^2 - 3(-2)^{-1} - \frac{1}{2}$

$= 4 + \frac{3}{2} - \frac{1}{2}$

$\underline{= 5}$

c/ $f'(-2) = 2(-2) + 3(-2)^{-2}$

$= -4 + \frac{3}{4}$

$= \cancel{-15}\ \frac{-13}{4}$

$y = \frac{-13}{4}x + c$ $\quad (-2, 5)$

$5 = \frac{-13}{4}(-2) + c$

$5 = \frac{26}{4} + c$

$c = -\frac{3}{2}$

$y = \frac{-13}{4}x - \frac{3}{2}$

$4y = -13x - 6$

$\underline{13x + 4y + 6 = 0}$

10a/ $y = 4x^2 + 5x^{-1} - 1$

$\frac{dy}{dx} = 8x - 5x^{-2}$

when $x = 1$

$\frac{dy}{dx} = 8(1) - 5(1)^{-2}$

$\underline{= 3}$

b/ $y = 3x + c$ when $x = 1$

$y = 4(1)^2 + \frac{5-1}{1}$

$= 8$

$8 = 3 + c$

$c = 5$

$\underline{y = 3x + 5}$

c/ meets x axis when $y = 0$

$0 = 3x + 5$

$-5 = 3x$

$x = -5/3$

11/ $\frac{dy}{dx} = (3x-1)(3x-1)$

$= 9x^2 - 6x + 1$

a/ when $x = 1$ $\frac{dy}{dx} = 9 - 6 + 1 = \underline{4}$

gradient of normal $= -1/4$

$y = -1/4 x + c$ $(1, 4)$

$4 = -1/4(1) + c$

$17/4 = c$

$y = -1/4 x + 17/4$

b/ $y = \frac{9x^3}{3} - \frac{6x^2}{2} + x + c$

$y = 3x^3 - 3x^2 + x + c$

$4 = 3(1)^3 - 3(1)^2 + 1 + c$

$c = 3$

$y = 3x^3 - 3x^2 + x + 3$

c/ $\frac{dy}{dx} = (3x-1)^2$ $m = -2$

$(3x-1)^2 = -2$

no solutions as you cannot sqroot a negative no.

12a/ P: $x = -2$ $y = (x-1)(x+2)(x-2)$

Q $x = 2$

b/ $y = (x-1)(x^2-4)$

$= x^3 - x^2 - 4x + 4$

$\frac{dy}{dx} = 3x^2 - 2x - 4$

c/ $(-1, 6)$

when $x = -1$ $\frac{dy}{dx} = 3(-1)^2 - 2(-1) - 4$

$= 3 + 2 - 4$

$= 1$

$M = 1$

$y = x + c$

$6 = -1 + c$

$c = 7$

$y = x + 7$

d/ $\frac{dy}{dx} = 1$

$3x^2 - 2x - 4 = 1$

$3x^2 - 2x - 5 = 0$

$(3x - 5)(x + 1) = 0$

$x = \frac{5}{3}$ $x = -1$

when $x = 5/3$ $y = (5/3 - 1)\left(\left(\frac{5}{3}\right)^2 - 4\right)$

$= (2/3)\left(\frac{25}{9} - \frac{36}{9}\right)$

$= (2/3)\left(\frac{-11}{9}\right)$

$= -\frac{22}{27}$

$\left(\frac{5}{3}, \frac{-22}{27}\right)$

13a/ $y = \frac{1}{3}x^3 - 4x^2 + 8x + 3$

$0 = 1/3(3)^3 - 4(3)^2 + 8(3) + 3$

$0 = 9 - 36 + 24 + 3$

$0 = 0$

b) $y = \frac{1}{3}x^3 - 4x^2 + 8x + 3$

$\frac{dy}{dx} = x^2 - 8x + 8$

when $x = 3$

$\frac{dy}{dx} = (3)^2 - 8(3) + 8$

$= 9 - 24 + 8$

$= -7$

$\therefore m = -7$

$y = -7x + c$

$0 = -7(3) + c$

$c = 21$

$\underline{y = -7x + 21}$

c) $\frac{dy}{dx} = -7$

$x^2 - 8x + 8 = -7$

$x^2 - 8x + 15 = 0$

$(x-3)(x-5) = 0$

$x = 3 \quad x = 5$

when $x = 5 \quad y = \frac{1}{3}(5)^3 - 4(5)^2 + 8(5) + 3$

$= \frac{125}{3} - 100 + 40 + 3$

$= \frac{125}{3} - \frac{300}{3} + \frac{120}{3} + \frac{9}{3}$

$= \frac{-46}{3}$

14 $y = x^3 - 2x^2 - x + 9$

$7 = (2)^3 - 2(2)^2 - (2) + 9$

$7 = 8 - 8 - 2 + 9$

$\underline{7 = 7}$

b) $\frac{dy}{dx} = 3x^2 - 4x - 1$

when $x = 2 \quad \frac{dy}{dx} = 3(2)^2 - 4(2) - 1 \quad = 3$

$m = 3 \qquad (2, 7)$

$y = 3x + c$

$7 = 3(2) + c$

$c = 1$

$\underline{\underline{y = 3x + 1}}$

c/ $\frac{dy}{dx} = -\frac{1}{3}$

$3x^2 - 4x - 1 = -\frac{1}{3}$

$3x^2 - 4x - \frac{2}{3} = 0$

$9x^2 - 12x - 2 = 0$

$a = 9 \quad b = -12 \quad c = -2$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(9)(-2)}}{2(9)}$

$= \frac{12 \pm \sqrt{144 + 72}}{18}$

$= \frac{12 \pm \sqrt{216}}{18} \qquad \sqrt{216} = \sqrt{36} \cdot \sqrt{6}$

$= \frac{12 \pm 6\sqrt{6}}{18}$

$= \frac{2 \pm \sqrt{6}}{3}$

$= \frac{1}{3}(2 \pm \sqrt{6})$

x is positive $\therefore x = \frac{1}{3}(2 + \sqrt{6})$