CHAPTER 1: SETS, RELATIONS and LANGUAGES

1.1 SETS

1.1.1 (a) **True**. In fact every set is a subset of itself

(b) **False**. $\emptyset$ has no members

(c) **True**. $\{\emptyset\}$ has one member, which is $\emptyset$.

(d) **True**. In fact $\emptyset$ is a subset of any set

(e) **True**. $\{a, b, c, \{a, b\}\}$ has four elements, one of which is the set $\{a, b\}$

1.2 RELATIONS AND FUNCTIONS

1.2.1 (a) $\{(1,1,1),(1,1,2),(1,1,3),(1,2,1),(1,2,2),(1,2,3)\}$

(b) $\emptyset$

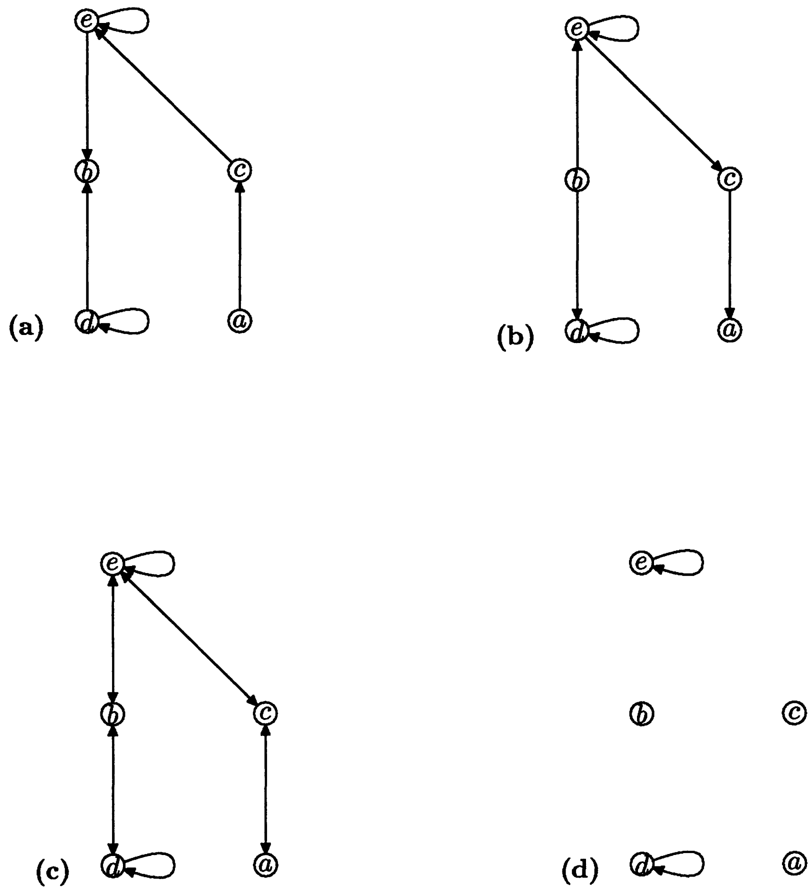
(c) $= \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \times \{1,2\}$
$= \{(\emptyset,1),(\{1\},1),(\{2\},1),(\{1,2\},1),(\emptyset,2),(\{1\},2),(\{2\},2),(\{1,2\},2)\}$

1.2.2

- $R \cdot R = \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c)\}$
- $R^{-1} = \{(b,a),(a,a),(d,c),(a,a),(a,b)\}$
- None of $R, R \cdot R, R^{-1}$ is a function.

1.3 SPECIAL TYPES OF BINARY RELATIONS

1.3.1 See below



1.3.2 (a) R is not reflexive, not symmetric and not transitive. S is symmetric, but not reflexive or transitive

(b) $R \cup S$ is reflexive, but is neither symmetric not transitive

1.3.4 (a) R is not reflexive

(b) R is symmetric

(c) R is anti-symmetric

(d) R is transitive

1.3.5 $R = \{(a, b) \mid f(a) = f(b)\}$. R is an equivalence relation because:

- R is reflexive:
- R is transitive:
- R is symmetric:

1.3.6 (a) R is a partial order because

  – R is reflexive because for any $a \in \mathbf{Z}^+$ (the set of positive integers), $a \, mod \, a = 0$ and hence $(a, a) \in R$

– R is transitive because if $(a,b),(b,c) \in R$ then $(a,c) \in R$
– R is anti-symmetric because if $(a,b) \in R$ and $a \neq b$ then $a > b$ and hence $(b,a) \notin R$

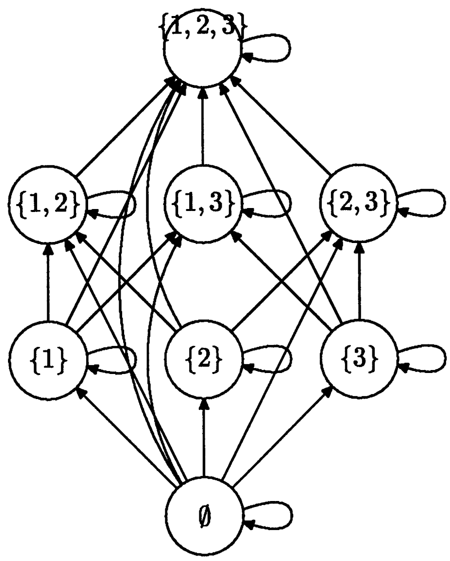
R is not a total order since for many pairs (a,b), a is not divisible by b and b is not dividable by a, for example when $a=2, b=3$. Hence neither (a,b) nor (b,a) are in R.

(b) R is not a partial order (and so not a total order) since it is not anti-symmetric: $((1,2),(2,1)) \in R$ and $((2,1),(1,2)) \in R$ but $(1,2) \neq (2,1)$.

(c) R is not a partial order (and hence not a total order) since it is not transitive: $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$

1.3.7
- $R_1 \cap R_2$ is reflexive: for any $a \in A$, $(a,a) \in R_i$ (since R_i is reflexive) and hence $(a,a) \in R_1 \cap R_2$.
- $R_1 \cap R_2$ is transitive:
- $R_1 \cap R_2$ is anti-symmetric:

1.3.8 (a)



(b)

1.3.9 when for each node, there is exactly one edge going out.

1.3.10 Let $f : A \to A$ where A is finite. Since $\{f^i(a) \mid i \in \mathbf{N}\} \subseteq A$, there exists $i < j \in \mathbf{N}$ such that $f^i(a) = f^j(a)$ ....

1.4 FINITE AND INFINITE SETS

- Exercises in lecture notes (p 32)

- *Every subset of a finite set is finite*: Let $B \subseteq A$. If A is finite, then there exists a bijection $f : \{1, 2, \ldots n\} \to A$ for some $n \in \mathbf{N}$, or equivalently we can enumerate A as $A = \{a_1, a_2, \ldots, a_n\}$ where $a_i = f(i)$. If for each $a_i \notin B$, we delete a_i from this enumeration, then we obtain an enumeration for B. Hence B is finite.
- *Every subset of a countably infinite set is finite or countably infinite:* similar to the above proof.
- ***N***$\times$ ***N*** *is countably infinite:* (proof in the text book, page 21) We can enumerate $\mathbf{N} \times \mathbf{N}$ by "dovetailing" as follows.

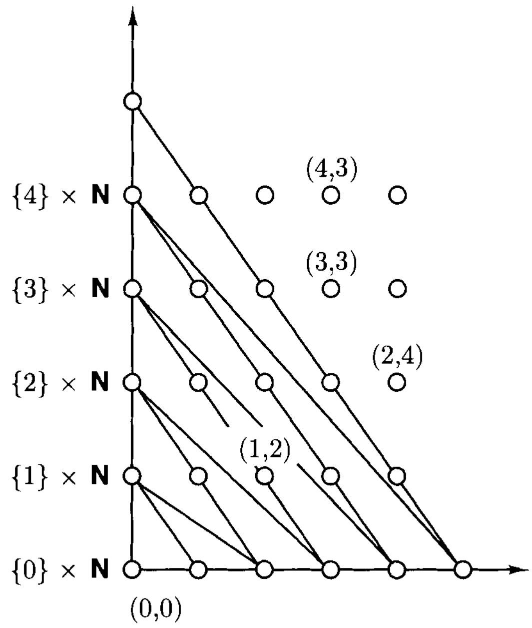


Figure 1-8

Alternatively, we can show that $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ where $f(i, j) = \frac{1}{2}((i + j)^2 + 3i + j)$ is a bijection.
- **Theorem** in p 33: *The union of countably infinite collection of countably infinite sets is again countable infinite.*
Let $\mathcal{A} = \{A_1, A_2, \ldots\}$ be a countably infinite set where each A_i is countably infinite.
There is a bijection from A_i to $\{i\} \times \mathbf{N}$
Hence there is a bijection from $A_1 \cup A_2 \cup \ldots$ to $\{1\} \times \mathbf{N} \cup \{2\} \times \mathbf{N} \cup \ldots$
There there is a bijection from $\bigcup \mathcal{A}$ to $\{1, 2, 3, \ldots\} \times \mathbf{N} = N \times N$

Hence $\bigcup \mathcal{A}$ and $\mathbf{N}^2$ are equinumerous. Since $\mathbf{N}^2$ is countable, $\bigcup \mathcal{A}$ is also countable.

1.4.1 (b) Let $X = \{Y \in 2^N \mid Y \text{ is finite}\}$

For $n \in \mathbf{N}$, let $\mathbf{N}_n = \{Y \in X \mid sum(Y) = n\}$. It is easy to see that $\mathbf{N}_n$ is finite.

Consider $Y \in X$. Since Y is finite, there exists $n \in \mathbf{N}$ such that $sum(Y) = n$. Hence $Y \in \mathbf{N}_n$. Hence $Y \in \mathbf{N}_1 \cup \mathbf{N}_2 \cup \cdots = \bigcup\{\mathbf{N}_n \mid n \in \mathbf{N}\}$. Hence $X \subseteq \bigcup\{\mathbf{N}_n \mid n \in \mathbf{N}\}$. Note that $\bigcup\{\mathbf{N}_n \mid n \in \mathbf{N}\}$ is an union of countably infinite collection of finite sets, and hence is countably infinite.

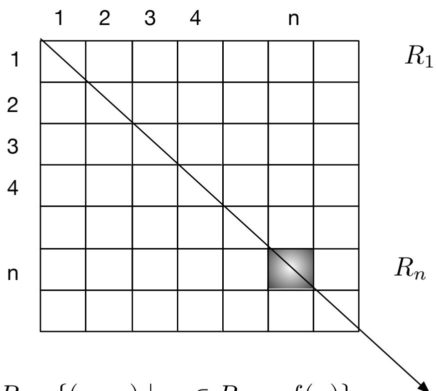
1.4.2 (a) $f(n) = 2n + 1$

(b) $f(x) = 2x - 1$ if $x > 0$; $-2x$ otherwise.

1.5 THREE PROOF TECHNIQUES

(a) Exercise in p 36 of lecture note:

$f : \{1, 2, \ldots, n, \ldots\} \to 2^{\mathbf{N}}$
$2^{\mathbf{N}} = R_1 \cup R_2 \cup \cdots \cup R_n \cup \ldots$



$R_1 = \{m \mid m \in f(1)\}$

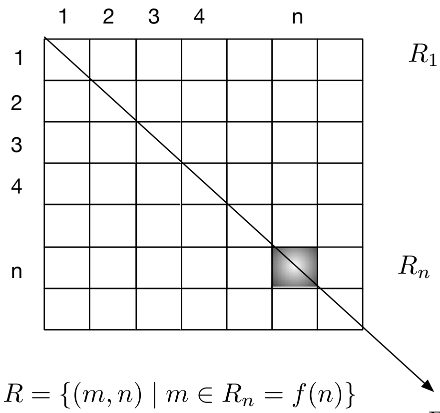
$R_n = \{m \mid m \in f(n)\}$

$R = \{(m, n) \mid m \in R_n = f(n)\}$

$D \neq R_n$ for any $n \in \mathbf{N}$
and hence f is not an onto $\mathbf{N}$ function

(b) **Theorem**: The set $2^{\mathbf{N}}$ is uncountable.

$f : 2^{\mathbf{N}} \to \{1, 2, \ldots, n, \ldots\}$
$2^{\mathbf{N}} = R_1 \cup R_2 \cup \cdots \cup R_n \cup \ldots$



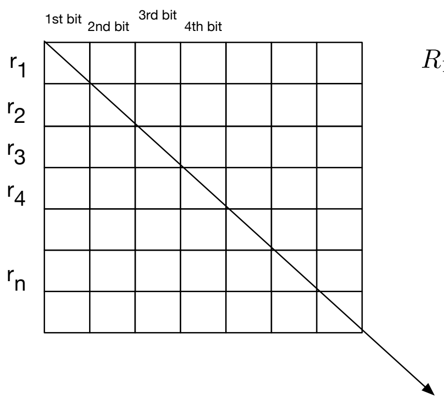
$R_1 = \{m \mid m \in f(1)\}$

$R_n = \{m \mid m \in f(n)\}$

$R = \{(m, n) \mid m \in R_n = f(n)\}$

$D \neq R_n$ for any $n \in \mathbf{N}$
and hence f is not an onto $\mathbf{N}$ function

1.5.11 • The set of real numbers in the interval $[0, 1]$ is uncountable.
Suppose $[0, 1] = \{r_1, r_2, r_3, \ldots, r_n, \ldots\}$



R_1 is binary representation of r_1

for any r_n, $D \neq R_n$, and hence $D \notin [0, 1]$

• Exercise: The set of floats in the interval $[0, 1]$ is countable.

1.7 ALPHABET AND LANGUAGES

1.7.4 a $\{e\}^* = \{e\}$
By definition, $\{e\}^* = \{\omega_1 \ldots \omega_n \mid \omega_i \in \{e\}\}$. $\omega_i \in \{e\}$ means that $\omega_i = e$ and thus $\{e\}^* = \{e^n \mid n \geq 0\} = \{e\}$

b It is clear that for any $L_1 \subseteq L_2$, $L_1^* \subseteq L_2^*$

* $L \subseteq L^*$ and hence $L^* \subseteq (L^*)^*$

- $(L^*)^* \subseteq L^*$?

c
- $\{a,b\}^* \subseteq \{a\}^*(\{b\}\{a\}^*)^*$ because any language over $\Sigma = \{a,b\}$ is a subset of Σ^*
- $\{a\}^*(\{b\}\{a\}^*)^* \subseteq (\{b\}\{a\}^*)^*$??

1.7.5 (a) $abbaab \in L$ but $ababab \notin L$

(b) $e \in L$ but $a \notin L$

(c) $L = \{a,b\}^*$ (i.e. any string is in L)

(d) $aa \in L$ but $a \notin L$

1.7.6 $L^+ = L^*$ iff $e \notin L$

1.8 FINITE REPRESENTATION OF LANGUAGES

Page 49 example: $\alpha = (a \cup b)^* a$

$L(\alpha) = L((a \cup b)^*)L(a)$

$= L((a \cup b)^*)\{a\}$

$= L((a \cup b))^*\{a\}$

$= (L(a) \cup L(b))^*\{a\}$

$= (\{a\} \cup \{b\})^*\{a\}$

$= (\{a,b\})^*\{a\} = \{\omega \mid \omega \in \{a,b\}^* \text{ and } \omega \text{ ends with an } a\}$

1.8.1 The language $\{a,b\}^*b$ consisting of only symbols a, b and end with b

1.8.2 (a) $(a \cup b)^*$. The language is Σ^*

(b) Also $(a \cup b)^*$

(c) Also $(a \cup b)^*$

(d) $\Sigma^* a \Sigma^*$

1.8.3 (a) $b^* \cup b^*ab^* \cup b^*ab^*ab^* \cup b^*ab^*ab^*ab^*$

(b) $b^*(ab^*ab^*ab^*)b^*$

1.8.5 (a) True; (b) True; (c) False; (d) False