Quantum Computing Cheat Sheet

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bra-ket notation: Quantum states are represented by column vectors, called kets

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

To get a bra from a given ket, we take the conjugate transpose of the ket. (this is known as dagger)

$$\langle\psi| = |\psi\rangle^\dagger = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^\dagger = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$$

Inner Product: Multiplying a bra and a ket is an inner product that yields the projection, or amplitude, of the states onto each other. The inner product yields a scalar output.

$$\langle\psi|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^\dagger \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
$$\langle\phi|\psi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}^\dagger \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \gamma^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

- A state whose inner product with itself equals 1 is normalized.
- States with inner product equaling 0 are orthogonal.

Outer Product: Multiplying a ket and a bra is an outer product, which is a square matrix.

Global phase factor: A complex number z can be written as $z = re^{i\theta}$ in the case where $r = 1$ and θ is a real number, then z is known as a phase factor and it can be simply ignored, as in it will not affect the state.

Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Quantum gates & Matrices:

- Quantum gates are unitary matrices, which satisfy: $UU^\dagger = U^\dagger U = I$
- Unitary matrices are always reversible with: $U^{-1} = U^\dagger$

The dagger operation on a matrix, is the conjugate transpose: $A^\dagger = (\overline{A})^T$
Transpose of a Matrix A:

$$A_{3\times 3} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

An example of dagger operation:

$$A_{2\times 2} = \begin{bmatrix} 1+i & 3i \\ 0 & 3-i \end{bmatrix} \qquad A^\dagger = \begin{bmatrix} 1-i & 0 \\ -3i & 3+i \end{bmatrix}$$

- **Trace of a matrix** is the sum of all diagonal elements.

Common gates:

Identity and Hadamard gates:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H|0\rangle = |+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$
$$H|1\rangle = |-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$

Pauli gates:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Gate identities:

Pauli identities:

1. $XY = iZ$
2. $YZ = iX$
3. $ZX = iY$
4. $XZ = -iY$
5. $ZY = -iZ$
6. $YX = -iZ$

Hadamard identities:

1. $HZH = X$
2. $HYN = -Y$
3. $HXN = Z$

S identities:

1. $SXS^\dagger = Y$
2. $SYS^\dagger = -X$
3. $SZS^\dagger = Z$

Qubit basis states:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$H|0\rangle = |+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$
$$H|1\rangle = |-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$

Density Matrix: A general way of expressing a quantum state. When state vectors and wave functions can only describe pure states, a density matrix can describe mixed states as well.

A density matrix is defined as:

$$\rho = \sum_j P_j |\psi_j\rangle\langle\psi_j|$$

where P_j is the probability of obtaining the state $|\psi_j\rangle$.

Every density matrix can also be written as a linear combination of Pauli gates, as follows:

$$\rho = \frac{1}{2}I + c_1 X + c_2 Y + c_3 Z$$

where c_1, c_2, c_3 give us the Bloch vector in cartesian coordinates:

$$\vec{r} = (x, y, z) = (2c_1, 2c_2, 2c_3)$$

State Purity: a scalar quantity to measure how pure a state is:

$$purity = \gamma = tr(\rho^2)$$

such that γ can be between $\frac{1}{d}$ and 1, where d is the dimension of the Hilbert space.

$$\gamma = 1 \rightarrow \text{pure state}$$
$$\gamma \neq 1 \rightarrow \text{mixed state}$$
$$\gamma = \frac{1}{d} \rightarrow \text{completely mixed state}$$

Note: Pure states lie on the surface of a Bloch sphere, whereas mixed states lie within it. so if the length of a Bloch vector ($\sqrt{x^2 + y^2 + z^2}$) equals 1, then the state is pure.

Fidelity: a scalar value to measure the closeness of two quantum states, for example the initial state compared with the final state after going through quantum gates, defined as:

$$F(|\psi_i\rangle, |\psi_f\rangle) = |\langle\psi_i|\psi_f\rangle|^2$$

Fidelity holds the following properties:

- **Symmetery:** $F(|\psi_i\rangle, |\psi_f\rangle) = F(|\psi_f\rangle, |\psi_i\rangle)$
- **Bounded values:** $0 \leq F(|\psi_i\rangle, |\psi_f\rangle) \leq 1$ and $F(|\psi\rangle, |\psi\rangle) = 1$
