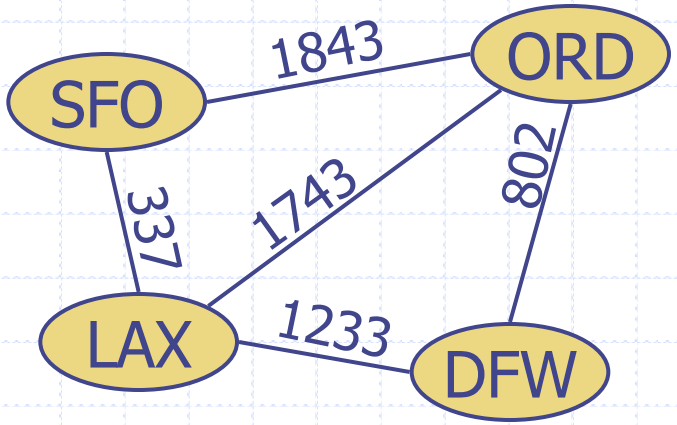
Lesson 11
Graphs:
Combinatorics of Pure Intelligence

Wholeness of the Lesson

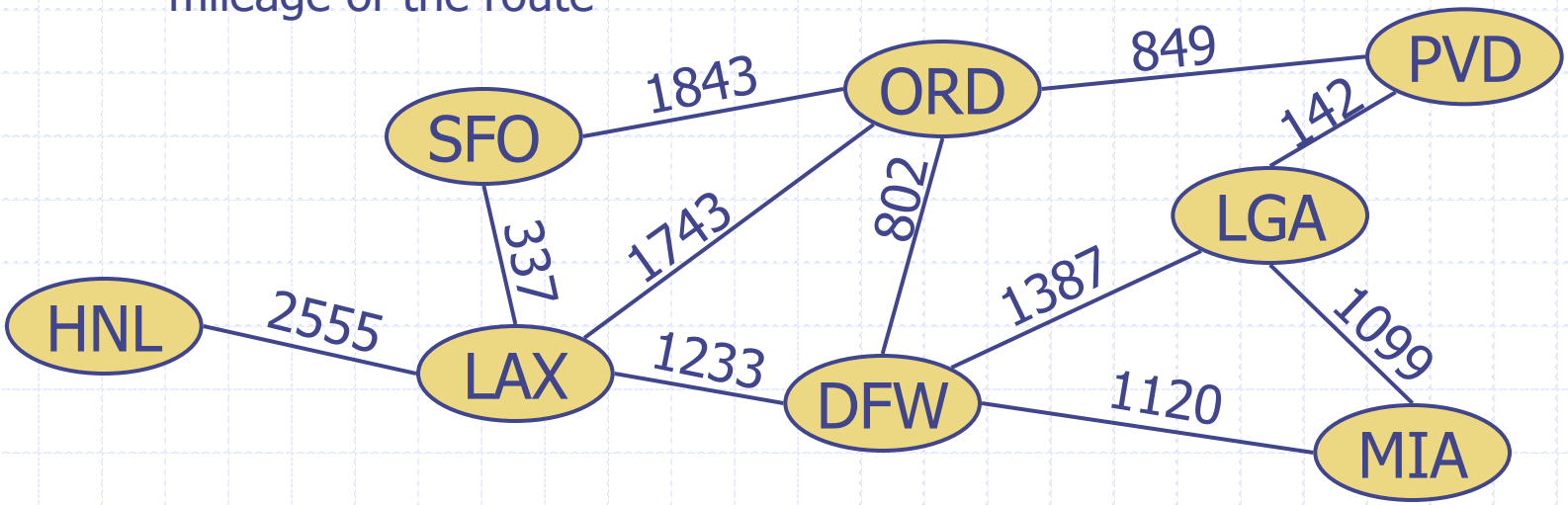
Graphs are data structures that do more than simply store and organize data; they are used to model interactions in the world. This makes it possible to make use of the extensive mathematical knowledge from the theory of graphs to solve problems abstractly, at the level of the model, resulting in a solution to real-world problems.

Science of Consciousness: Our own deeper levels of intelligence exhibit more of the characteristics of Nature's intelligence than our own surface level of thinking. Bringing awareness to these deeper levels, as the mind dives inward, engages Nature's intelligence, Nature's know-how, and this value is brought into daily activity. The benefit is greater ability to solve real-world problems, meet challenges, and find the right path for success.



Graph

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges can be implemented so that they store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route

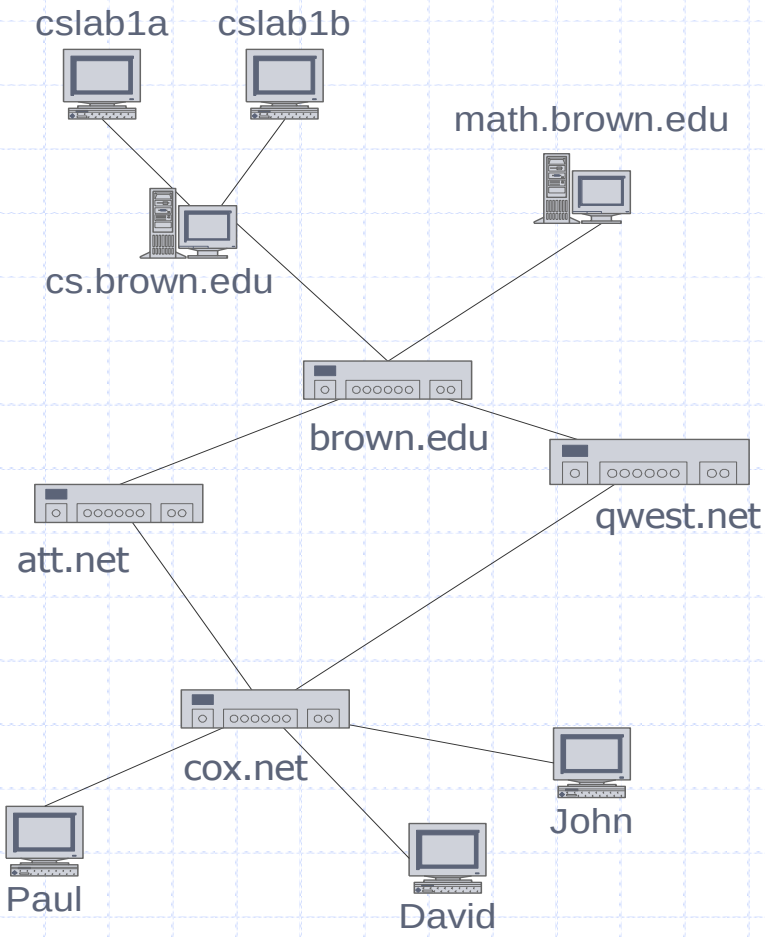


Edge Types

- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- Directed graph
  - all the edges are directed
  - e.g., flight network
- Undirected graph
  - all the edges are undirected
  - e.g., route network

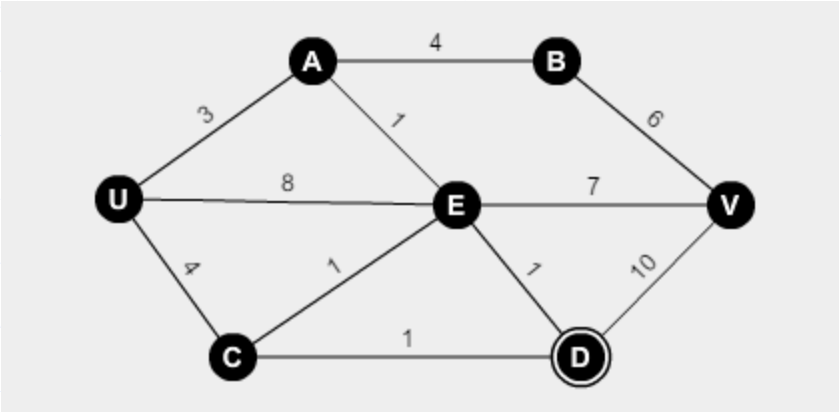
Applications

- Electronic circuits
  - Printed circuit board
    (nodes = junctions, edges are the traces)
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram
- Physics / Chemistry
  - Atomic structure simulations (e.g. shortest path algs)
  - Model of molecule -- atoms/bonds



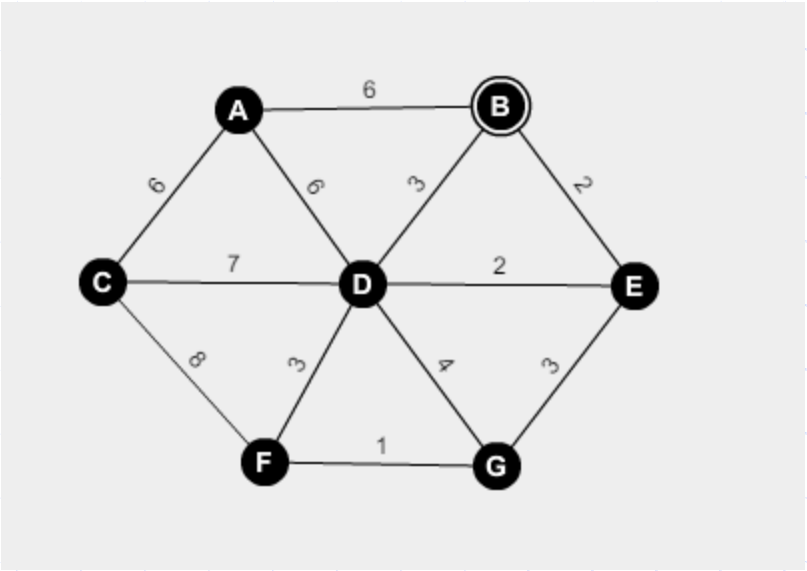
Examples

SHORTEST PATH. The diagram below schematically represents a railway network between cities; each numeric label represents the distance between respective cities. What is the shortest path from city U to city V? Devise an algorithm for solving such a problem in general.



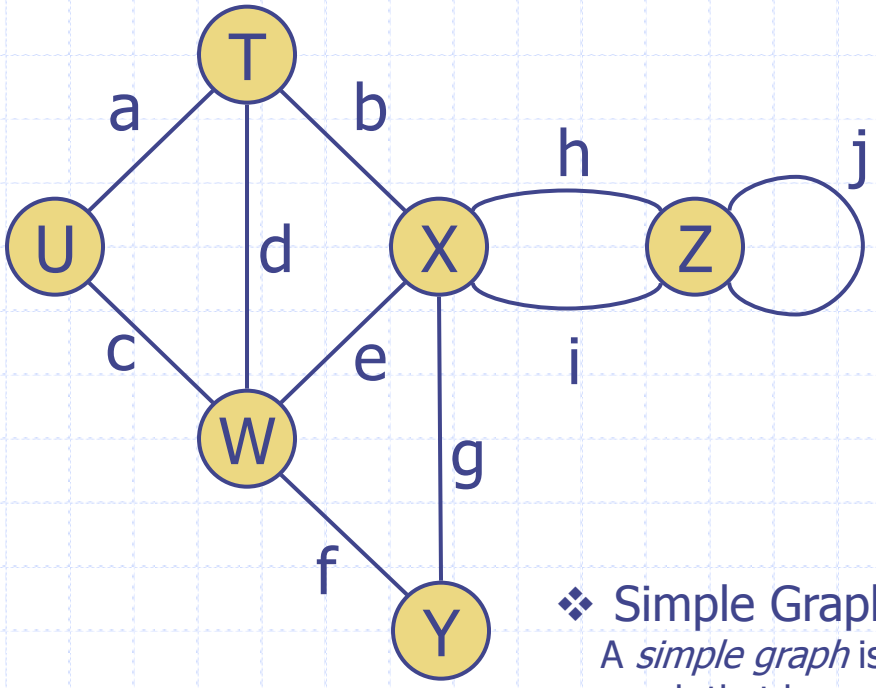
Examples (continued)

CONNECTOR. The diagram below schematically represents potential railway paths between cities; a numeric label represents the cost to lay the track between the respective cities. What is the least costly way to build the railway network in this case, given that it must be possible to reach any city from any other city by rail? Devise an algorithm for solving such a problem in general.



Terminology

- |V| (also ν or n) is the number of vertices of G; |E| (also ε or m) is the number of edges. (ν = "nu", ε = "epsilon")
- End vertices (or endpoints) of an edge
  - U and T are the endpoints of a
- Edges incident to a vertex
  - a, d, and b are incident to T
- Adjacent vertices
  - U and T are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop

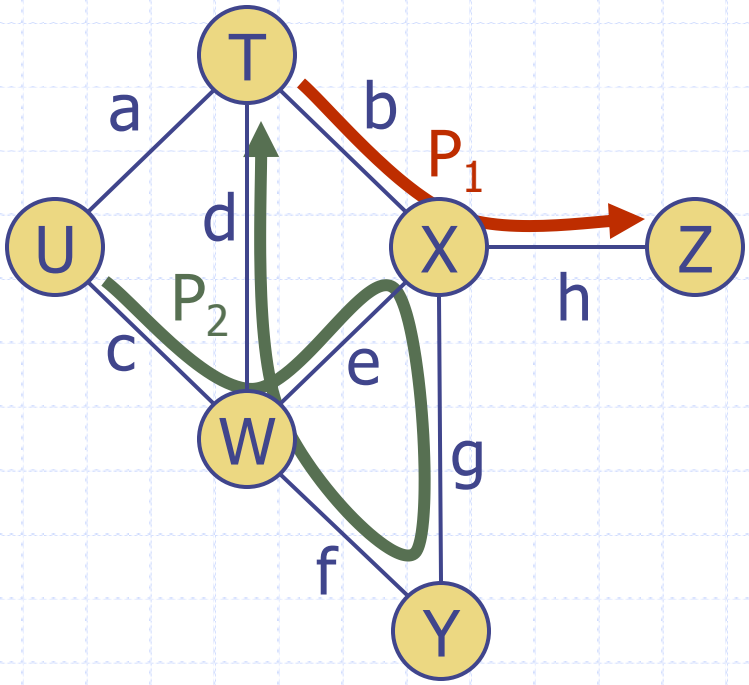


❖ Simple Graph

A *simple graph* is a graph that has no self-loops or parallel edges

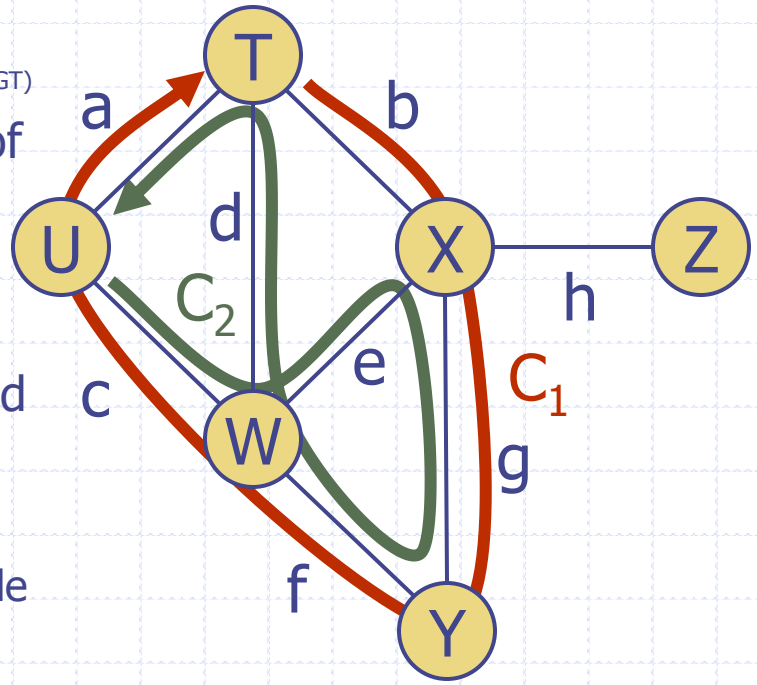
Terminology (cont.)

- Path
  - sequence of alternating vertices and edges – in a simple graph, can omit edges (called a "walk" in graph theory)
  - begins and ends with a vertex
  - *length* of a path is number of edges
- Simple path
  - path such that all its vertices and edges are distinct (called a "path" in GT)
  - Fact: If a graph has a path p from u to v, there is a subpath of p that is a simple path from u to v
- Examples
  - P_1=(T, X, Z) is a simple path
  - P_2=(U, W, X, Y, W, T) is a path that is not simple



Terminology (cont.)

- Cycle
  - path in which all edges are distinct and whose first and last vertex are the same (called "closed trail" in GT)
  - *length* of a cycle is the number of edges in the cycle
- Simple cycle
  - path such that all vertices and edges are distinct except first and last vertex (called a "cycle" in GT)
- Examples
  - C_1=(T, X, Y, W, U, T) is a simple cycle
  - C_2=(U, W, X, Y, W, T, U) is a cycle that is not simple



Details About Cycles

- Example: Not every path with identical first and last vertex is a cycle (edges must be distinct)

consider X - Y - Z - Y - X

- Facts
  - In a simple graph, the shortest possible cycle has length 3
  - If an edge belongs to a cycle in G, it belongs to a simple cycle
  - If a simple graph contains a cycle with odd length, it contains a *simple* cycle with odd length.

Optional: Simple Paths

Theorem: Suppose G is a simple graph, u, v are distinct vertices, and there is a path p in G that starts at u and ends at v. Then there is a simple subpath of p in G that starts at u and ends at v.

Proof: It is enough to show that whenever there is a path p in G from u to v that is not simple, there is another shorter subpath of p from u to v in G.

Suppose p: $u = v_1, v_2, \ldots, v_k = v$ is a path from u to v in G that is not simple.

Case 1: *Some vertex occurs twice in p*. Suppose $i < j$ and $v_i = v_j$ in p. Then notice that $u = v_1, v_2, \ldots, v_i, v_{j+1}, \ldots, v_k = v$ is another shorter path from u to v.

Case 2: *Some edge occurs twice in p*. In this case, some vertex must also occur twice, and we return to the argument in Case 1.

Optional: Simple Cycles

Theorem: Suppose G is a simple graph and e is an edge in G. Suppose G contains a cycle C, and e is an edge in C. Then some simple cycle in G also has e as an edge; in fact, such a simple cycle can be found that is a subpath of C.

Proof: Suppose G is a simple graph having edge e, and that G has a cycle C of length n that contains e. We show e belongs to a simple cycle in G that is a subpath of C, by induction on n.

Base Case: n = 3. In that case, C must already be simple.

Induction Step: Assume that whenever a cycle C' of length < n in G contains e, e belongs to a simple cycle in G that is a subpath of C'. Let C be a cycle of length n in G, and write C as

$$v=v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n = v$$

and let e be the edge (v_i, v_{i+1}). Assume C is not simple, so there are $j < k$ where $v_j = v_k$

There are 3 cases:

(continued)

Case 1 v_j precedes v_i and v_{i+1} precedes v_k

$$v=v_1, v_2, \ldots, v_j, \ldots, v_i, v_{i+1}, \ldots, v_k, \ldots, v_n = v$$

Then since the section from v_j to v_k is a cycle C′ containing e, there is a simple cycle C″, which is a subpath of C′, which contains e. Now C″ must also be a subpath of C

Case 2 Both v_j and v_k precede v_i

$$v=v_1, v_2, \ldots, v_j, \ldots, v_k, \ldots, v_i, v_{i+1}, \ldots, v_n = v$$

Then if $1 < j$, we can delete the section from v_j to v_k (include v_j not v_k) and the resulting path is a shorter cycle containing e, so there is a simple cycle in G containing e, which again must be a subpath of C. If $j = 1$, then if we delete v_j up to v_k (include v_j not v_k), it follows that $v_1 = v_k = v$, so the path from v_k to v_n is a shorter cycle containing e, so again there is a simple cycle, also a subpath of C.

Case 3 Both v_j and v_k occur after v_{i+1}

$$v=v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_j, \ldots, v_k, \ldots, v_n = v$$

If $k < n$, delete section from v_j up to (not including) v_k; again we have a shorter cycle containing e. If $k = n$, if we delete from v_j up to (not including v_k), since $v_j = v$, we again have a shorter cycle containing e. Either way, there is a simple cycle in G containing e which is also a subpath of C.

Optional: Odd Cycles

Theorem: Suppose G is a simple graph that has an odd cycle C (a cycle with odd length). Then G has an odd *simple* cycle that is a subpath of C.

Proof: We show that whenever C is an odd cycle of length n in G, either C is simple or some subpath of C is an odd simple cycle; we proceed by induction on n.

Base Case: n = 3. In that case, C must already be simple.

Induction Step: Assume that whenever C' is an odd cycle of length < n in G, C' contains an odd simple subcycle. Let C be an odd cycle of length n. Write C as

$$v=v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n = v.$$

Assuming C is not simple, let $i < j$ be such that $v_i = v_j$. Then both

$v_i, v_{i+1}, \ldots, v_j$ and $v_1, v_2, \ldots, v_i\ v_{j+1}, \ldots, v_n$ are subcycles of C with no common edges. Therefore, one of these cycles must be odd. By the induction hypothesis, the odd cycle must have an odd simple subcycle.

Properties

Convention: Focus on simple undirected graphs at first

Property 1

$$\Sigma_{\mathbf{v}} \deg(\mathbf{v}) = 2\varepsilon$$

Proof: Let $E_v = \{e \mid e \text{ incident to } v\}$.

Then

$$\Sigma_{\mathbf{v}} \deg(\mathbf{v}) = \Sigma_{\mathbf{v}} |E_v|$$

Notice every edge (v,w) belongs to just two of these sets: E_v and E_w. So every edge is counted exactly twice.

Property 2

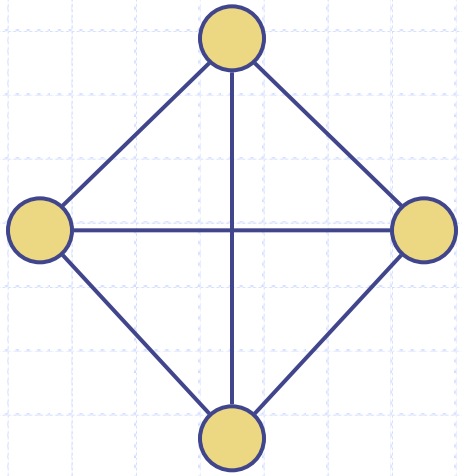
$$\varepsilon \leq \nu(\nu - 1)/2$$

Proof: max number of edges is

$$C(\nu, 2) = C_{\nu,2} = \binom{\nu}{2}$$

Notation

ν	number of vertices
ε	number of edges
$\deg(\mathbf{v})$	degree of vertex $\mathbf{v}$

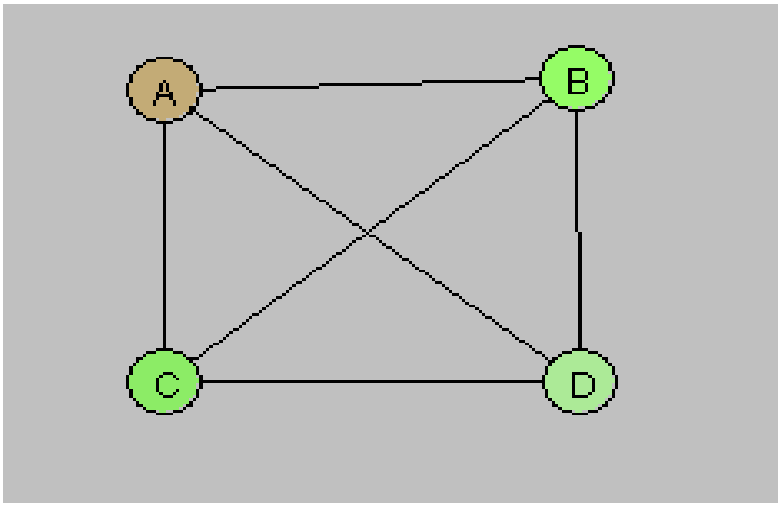


Example

- $\nu = 4$
- $\varepsilon = 6$
- $\deg(\mathbf{v}) = 3$

Complete Graphs and Bipartite Graphs

- A graph G is **complete** if for every pair of vertices u, v, there is an edge $e \in E$ that is incident to u, v. In other words, for every u, v, $(u, v) \in E$.
- This is the complete graph on 4 vertices, denoted K_4.

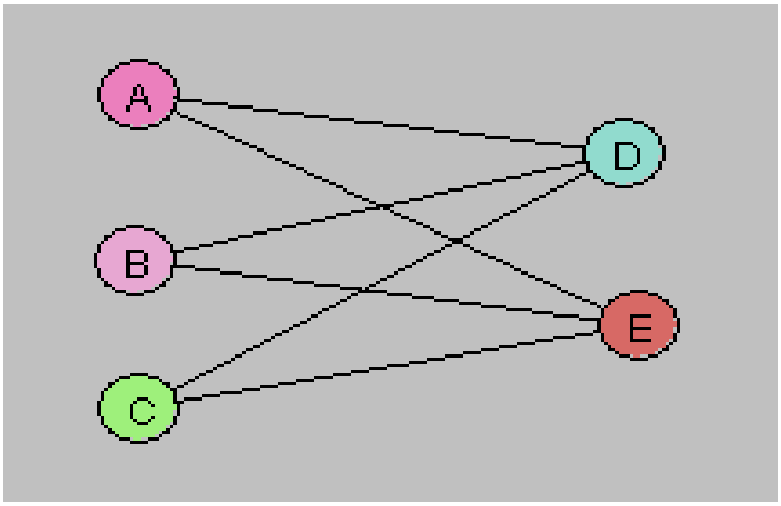


- In general, the complete graph on n vertices is denoted K_n. The one-point graph is K_1.
- **Exercise**. For a complete graph G,

$$\epsilon = \frac{\nu(\nu - 1)}{2}.$$

  Therefore, K_n has exactly $n(n-1)/2$ edges.

- A graph G is **bipartite** if there exist sets X, Y that are disjoint and subsets of V with the property that every $e \in E$ is incident with a vertex in X and a vertex in Y. For such graphs, (X, Y) is called a **bipartition** of G. A **complete bipartite graph** G with bipartition (X, Y) is a bipartite graph with the property that for every $u \in X$ and every $v \in Y$, $(u, v) \in E$.
- This an example of a complete bipartite graph.



- A simple cycle is *odd* (*even*) if the length of the cycle is odd (*even*).
- **Exercise**. Show that a graph is bipartite if and only if it contains no odd (simple) cycle.

Subgraphs

- A graph $H = (V_H, E_H)$ is a **subgraph** of a graph $G = (V_G, E_G)$ if both of the following are true:
  a. $V_H \subseteq V_G$ and $E_H \subseteq E_G$, and
  b. for every edge (u, v) belonging to E_H, both u and v belong to V_H.

  H is called a **spanning subgraph** if $V_H = V_G$.

- Suppose $G = (V, E)$ is a graph and $W \subseteq V$, where W is nonempty. Then the subgraph of G that is **induced by** W, denoted $G[W]$, is the subgraph of G whose vertices are W and whose edges are just those edges in E both of whose ends lie in W.

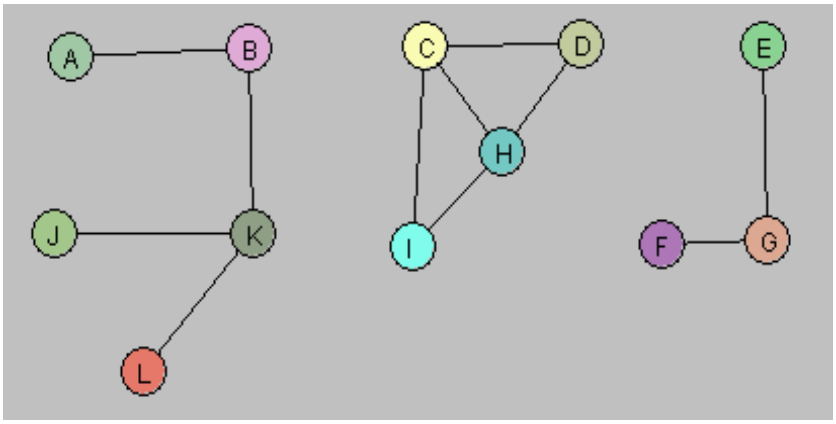
  Equivalently, the subgraph induced by W is the subgraph whose vertex set is W and which has the maximum possible number of edges.

Connected Graphs And Connected Components

- A graph is **connected** if for any two vertices u, v in G, there is a path from u to v.
- **Observation** If a graph $G = (V, E)$ is not connected, then V can be partitioned into sets $V_1, V_2, \ldots, V_k$ (all disjoint) so that each of the subgraphs $G[V_1], G[V_2], \ldots, G[V_k]$ is connected; they are called the **connected components** of G.
- **In-Class Exercise**. Show that if $G = (V, E)$ is a graph, then G is connected whenever

$$\epsilon > \binom{\nu - 1}{2}.$$

- The following graph has 3 connected components.



- The **complement** of a graph $G = (V, E)$, denoted G^c, is the graph whose vertices are V and whose edges are precisely those (u, v) that do not belong to E, where $u, v \in V$.
- **In-Class Exercise**. Show that if G is disconnected, then G^c is connected.

Trees

- A graph is **acyclic** if it contains no simple cycle.
- An acyclic connected graph is called a **tree**.
- A **rooted tree** is a tree having a distinguished vertex r.

  The usual levels we have in trees as data structures can be defined for rooted trees. Level 0 contains only the root. Level 1 contains all vertices adjacent to the root. Level 2 contains all vertices adjacent to a Level 1 vertex other than the Level 0 vertex. In general, Level $i+1$ contains vertices adjacent to the Level i vertices that do not appear at previous levels.

  Facts About Levels Suppose T is a rooted tree with levels defined as above.

  A. No two vertices at the same level are adjacent
  B. If v belongs to Level i and w belongs to Level j and $|i-j| > 1$ then v and w are not adjacent.
  C. Every vertex but the root has exactly one parent.
  D. If v is in Level i and w is in Level j, $i < j$, and w is a descendant of v, then a path from v to w includes one vertex from every level between Levels i and j. (A *path* in a rooted tree is a path in the usual sense except that if (v_i, v_{i+1}) is an edge in the path, then v_{i+1} is a *child* of v_i in the tree.)

- A subgraph T of G is a **spanning tree** if T is a spanning subgraph and also a tree.
- **Theorem**. In a tree, any two vertices are connected by a unique simple path. [In-class Exercise]
- **Theorem**. If G is a tree, $\epsilon = \nu - 1$.
- **Exercise**. Show that every tree having 2 or more vertices has at least two vertices of degree one.
- **Exercise**. Suppose G is a graph with $\nu - 1$ edges. Show that the following are equivalent:

  a. G is connected
  b. G is acyclic
  c. G is a tree

Hamiltonian Graphs And Vertex Covers

- A **Hamiltonian cycle** in a graph G is a simple cycle that contains every vertex of G. A graph is a **Hamiltonian graph** if it contains a Hamiltonian cycle.
- **Examples.** The Herschel graph is not a Hamiltonian graph.
- If $G = (V, E)$ is a graph, a **vertex cover for** G is a set $C \subseteq V$ such that for every $e \in E$, at least one end of e lies in C.
- **Fact**. The known algorithms for determining whether a graph is Hamiltonian, and for computing the smallest size of a vertex cover, run in exponential time.

Verifying Hard Problems Belong to NP

- Recall from the beginning of the course that we classified the computable functions as being either feasible or infeasible. The feasible ones were those whose running time was bounded by a polynomial in n, and the infeasible ones admitted no such bound.
- In Complexity Theory, *decision problems* rather than *algorithms* are classified in this way. Recall that a decision problem is formulated so that its output is always "true" or "false" (recall the Knapsack problem).
- A decision problem is polynomial-time solvable, and is said to belong to the class $\mathcal{P}$, if there is an alogirthm for solving all instances of the problem of size n in $O(n^k)$ time (for some fixed k).
- Decision problems can be classified in another way (and this classification is useful in cases where no polynomial time solutions are known). A decision problem for which a correct solution can be verified in polynomial time is said to belong to the class $\mathcal{NP}$. Knapsack, VertexCover, HamiltonianCycle can be shown to belong to $\mathcal{NP}$ even though there is no known polynomial-time solution to any of them.

HamiltonianCycle is $\mathcal{NP}$

- We verify that HamiltonianCycle belongs to $\mathcal{NP}$.
- Recall that the input for this problem is a graph $G = (V, E)$, where $|G| = n$. The decision problem to solve is, Does G contain a Hamiltonian cycle?
- To verify that the problem is $\mathcal{NP}$, we assume we have a solution C, which is a subgraph of G. We determine the running time required to verify that C is indeed a Hamiltonian cycle. Here is what must be verified:
  - Check: As a graph, C is a simple cycle.
  - Check: C contains all vertices of G.
  - Check: All edges of C are also edges of G.
- It is not hard to see that these verifications can be performed in $O(n^3)$ steps. Therefore, HamiltonianCycle belongs to $\mathcal{NP}$.