Contents

Chapter 7

Vector fields

This chapter is concerned with vector valued functions of a several variable. Such functions are frequently called *vector fields*. Applications of vector fields often involve integrals taken, not along axes or over regions in $\mathbb{R}^2$ or $\mathbb{R}^3$, but rather over curves and surfaces. We will introduce such line and surface integrals in what follows.

7.1 Vector and scalar fields

Let $n \in \mathbb{N}$. A vector valued function with domain $\Omega \subseteq \mathbb{R}^n$ is called a **vector field**. That is, a vector field $\mathbf{F}\colon \Omega \to \mathbb{R}^n$ associates a vector $\mathbf{F}(x)$ to each point $x \in \Omega$. The components of $\mathbf{F}$ are real-valued functions $F_i\colon \Omega \to \mathbb{R}$, for $i \in \{1, 2, \ldots, n\}$, and $\mathbf{F}(x)$ can be expressed in terms of the standard basis $\{\mathbf{e}_i\colon i \in \{1, \ldots, n\}\}$ for $\mathbb{R}^n$ as

$$\mathbf{F}(x) = F_1(x)\mathbf{e}_1 + F_2(x)\mathbf{e}_2 + \cdots + F_n(x)\mathbf{e}_n.$$

We will frequently make use of position vectors in the argument of vector fields; namely denoting the position vector of a point $x \in \mathbb{R}^n$ by $\mathbf{x}$, and write $\mathbf{F}(\mathbf{x})$ in replace of $\mathbf{F}(x)$. A scalar-valued function is called a **scalar field**. Thus, the components of a vector field are scalar fields.

Many of the results we prove about vector fields require that the fields be *smooth* in some sense. We call a vector field **smooth** whenever its component scalar fields have continuous partial derivatives of all orders. However, for most results continuity of the first order partial derivatives would be sufficient, such vector fields are called C^1; in the case when we require continuity of the first and the second order partial derivatives, then we call such vector fields C^2.

Vector fields arise in many situation in the natural sciences. Examples include the following.

- The gravitational field $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ due to some object is the force of attraction that the object exerts on a unit mass located at position $(x, y) \in \mathbb{R}^2$, see FIGURE 7.1.
- The electrostatic force field $\mathbf{E}\colon \mathbb{R}^2 \to \mathbb{R}^2$ due to an electrically charged object is the electrical force that the object exerts on a unit charge at position $(x, y) \in \mathbb{R}^2$.
- The velocity vector field $\mathbf{v}\colon \mathbb{R}^3 \to \mathbb{R}^3$ in moving a fluid (or solid) is the velocity of motion of the particle at position $(x, y, z) \in \mathbb{R}^3$, see FIGURE 7.1 for a two-dimensional illustration. If the motion is not steady state, then the velocity field will also depend on time.
- The gradient vector $\nabla f\colon \mathbb{R}^3 \to \mathbb{R}^3$ of a scalar field $f\colon \mathbb{R}^3 \to \mathbb{R}$ whose partial derivatives all exist, gives the direction and magnitude of the greatest rate of change of f at $(x, y, z) \in \mathbb{R}^3$. In particular, a temperature gradient ∇T is a vector field giving the direction and magnitude of the greatest rate of change of temperature T at a point $(x, y, z) \in \mathbb{R}^3$ in a heat-conducting medium. Pressure gradients provide similar information about the variation of pressure in a fluid.
- The unit radial and unit transverse vectors $\mathbf{r}$ and θ are examples of vector fields in the x-y plane. Both are defined at all points of the plane except at the origin.

The graphical representations of vector fields such as those shown in FIGURE 7.1 suggest a pattern of motion through space or in the plane. Whether or not the field is a velocity field, we can interpret it as such and ask what trajectory will be followed by a corresponding particle, initially at some point, whose velocity is given by

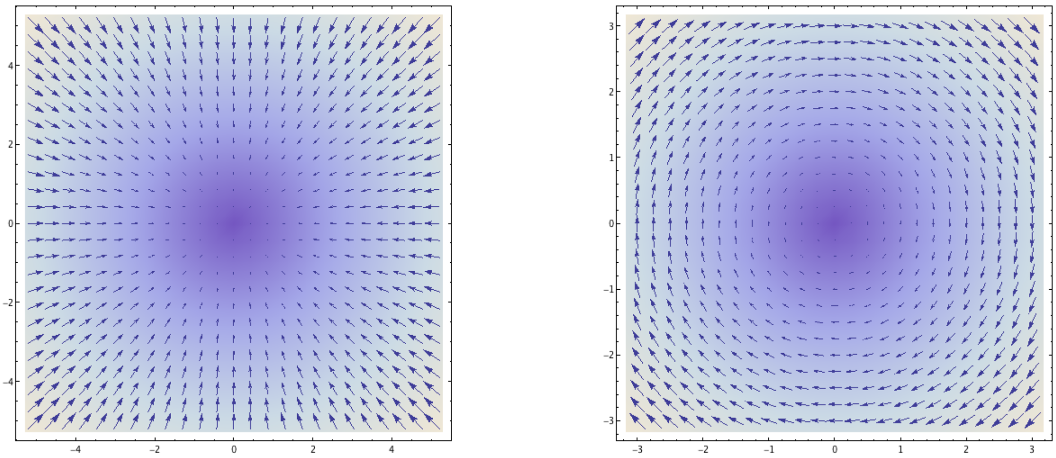

Figure 7.1: LEFT: The gravitational field $\mathbf{F}(x, y) = -x\mathbf{i} - y\mathbf{j}$ of a point mass located at the origin. RIGHT: The velocity field $\mathbf{v}(x, y) = y\mathbf{i} - x\mathbf{j}$ of a rigid body rotating about the origin.

the field. The trajectory will be a curve to which the field is tangent at every point. Such curves are called **field lines**, **integral curves** or **trajectories** for the given vector field, see FIGURE 7.2. In the specific case where the vector field gives the the velocity in a fluid flow, the field lines are also called **streamlines** or **flow lines** of the flow. For a force field, the field lines are called **lines of force**.

The field lines of a vector field $\mathbf{F}$ do not depend on the magnitude of $\mathbf{F}$ at any point, but only on the direction of the field. If the field line through some point has parametric equation $\mathbf{r}(t)$, then its tangent vector $d\mathbf{r}/dt$ at t must be parallel to $\mathbf{F}(\mathbf{r}(t))$ for all t. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \lambda(t)\mathbf{F}(\mathbf{r}(t)),$$

for some scalar field λ. For some vector fields this differential equation can be integrated to find the field lines explicitly.

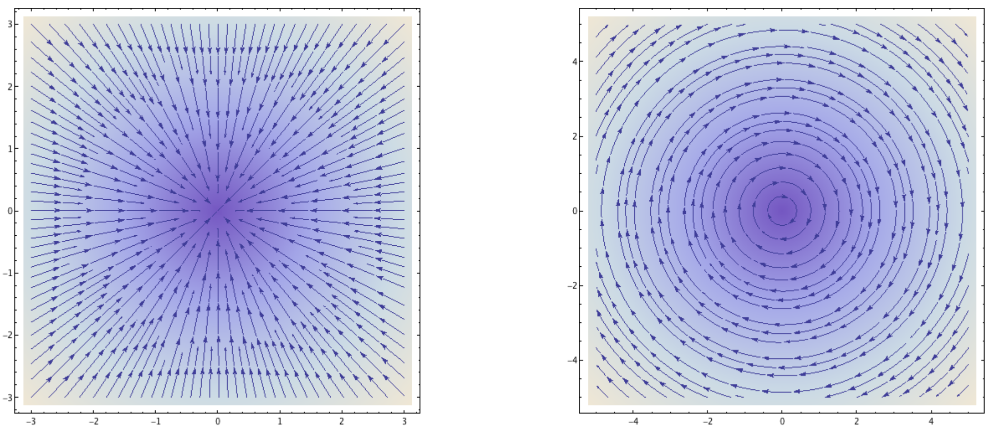

Figure 7.2: LEFT: Force field lines of the gravitational field $\mathbf{F}(x, y) = -x\mathbf{i} - y\mathbf{j}$ of a point mass located at the origin. RIGHT: Vector field lines of the velocity field $\mathbf{v}(x, y) = y\mathbf{i} - x\mathbf{j}$ of a rigid body rotating about the origin.

Exercise 7.1.1 *Find the field lines of* $\mathbf{v}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *a velocity vector field, given by* $\mathbf{v}(x, y) = -y\mathbf{i} + x\mathbf{j}$*, of a rigid body rotating about the origin.*

Solution. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ denote a parameterisation of the field line of $\mathbf{v}$ through the point $(x, y) = (x(t), y(t))$. By definition the field lines satisfy the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda(t)G_1(x(t), y(t)), \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}y(t) = \lambda(t)G_2(x(t), y(t)),$$

where G_1 and $G_2\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $G_1(x,y) = -y$ and $G_2(x,y) = x$; or equivalently

$$\int x \, \mathrm{d}x = \int -y \, \mathrm{d}y,$$

provided that $(x,y) \neq (0,0)$. Solving these integrals gives $x^2 + y^2 = c$. Thus, the field lines are circles centred about the origin in the x-y plane. Additionally, if we regard $\mathbf{v}$ as a vector field in $\mathbb{R}^3$, then we find that the field lines are horizontal circles centred on the z-axis, namely $x^2 + y^2 = c_1$ and $z = c_2$.

Exercise 7.1.2 *Let $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and let m denote a positive real number. Find the field lines of $\mathbf{F}\colon \mathbb{R}^3 \setminus \{P_0\} \to \mathbb{R}^3$, a gravitational force field given by*

$$\mathbf{F}(x,y,z) = -m\frac{(x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{3/2}},$$

due to a point mass located at P_0 having position vector $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$.

Solution. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ denote a parameterisation of the field line of $\mathbf{F}$ through the point $(x,y,z) = (x(t), y(t), z(t))$. The vector in the numerator of $\mathbf{F}$ gives the direction of $\mathbf{F}$. Therefore, the field lines satisfy the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda(t)G_1(x(t),y(t),z(t)), \qquad \frac{\mathrm{d}}{\mathrm{d}t}y(t) = \lambda(t)G_2(x(t),y(t),z(t)), \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}z(t) = \lambda(t)G_3(x(t),y(t),z(t)),$$

where G_1, G_2 and $G_3\colon \mathbb{R}^3 \setminus \{P_0\} \to \mathbb{R}$ are given by $G_1(x,y,z) = x - x_0$, $G_2(x,y,z) = y - y_0$ and $G_3(x,y,z) = z - z_0$; or equivalently,

$$\int \frac{1}{x - x_0} \, \mathrm{d}x = \int \frac{1}{y - y_0} \, \mathrm{d}y = \int \frac{1}{z - z_0} \, \mathrm{d}z.$$

Solving these integrals gives

$$\ln|x - x_0| + \ln(c_1) = \ln|y - y_0| + \ln(c_2) = \ln|z - z_0| + \ln(c_3),$$

where $\ln(c_1)$, $\ln(c_2)$ and $\ln(c_3)$ are constance of integration. Taking exponentials yields,

$$c_1(x - x_0) = c_2(y - y_0) = c_3(z - z_0).$$

This represents three families of planes all passing though $P_0 = (x_0, y_0, z_0)$. The field lines are the intersection of the planes from each of the families, so they are straight lines through the point P_0.

7.2 Differential operators

One of our aims is to develop a two and three dimensional analogue of the fundamental theorem of calculus. For this two differential linear operators that will be important are what are called the *divergence* and the *circulation density*, and are related to the gradient operator ∇. Geometrically the divergence of a vector field measures how much the vector field streams *out* from a given point and the circulation density measures how much the field *swirls* around a certain point.

Definition 7.2.1 *Let $n \geq 2$ denote a natural number, $\Omega \subseteq \mathbb{R}^n$ and let $\mathbf{u}\colon \Omega \to \mathbb{R}^n$,*

$$\mathbf{u}(x_1, x_2, \ldots, x_n) = \langle u_1(x_1, x_2, \ldots, x_n), u_2(x_1, x_2, \ldots, x_n), \ldots, u_n(x_1, x_2, \ldots, x_n) \rangle$$

be a C^1-vector field. The ***divergence*** *of $\mathbf{u}$ is the scalar field*

$$\operatorname{div}(\mathbf{u}) \coloneqq \nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial u_n}{\partial x_n}.$$

When $n = 3$, the ***circulation density****, or* ***curl*** *for short, of $\mathbf{u}$ is the vector field*

$$\operatorname{curl}(\mathbf{u}) \coloneqq \nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\mathbf{k}.$$

In this course, we define the operator curl only for 3-dimensional C^1-vector fields, whereas the operator div is defined for C^1-vector fields in all dimensions. Note however, if we are given a 2-dimensional vector field $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$, then we may define $\operatorname{curl}(\mathbf{F})$ by $\operatorname{curl}(\mathbf{F} + 0\mathbf{k})$, in which case,

$$\operatorname{curl}(\mathbf{F}) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Exercise 7.2.2 *Find the divergence and curl of the following vector fields.*

- $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ *given by* $\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 - z^2)\mathbf{j} + yz\mathbf{k}$
- $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $\mathbf{F}(x, y) = xe^y\mathbf{i} - ye^x\mathbf{j}$

Solution.

- The divergence of the vector field $\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 - z^2)\mathbf{j} + yz\mathbf{k}$ is given by

$$\operatorname{div}(\mathbf{F})(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) = y + 2y + y = 4y.$$

  The curl of the vector field $\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 - z^2)\mathbf{j} + yz\mathbf{k}$ is given by

$$\begin{aligned}
\operatorname{curl}(\mathbf{F})(x, y, z) &= \nabla \times \mathbf{F}(x, y, z) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 - z^2 & yz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - z^2)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(y^2 - z^2) - \frac{\partial}{\partial y}(xy)\right)\mathbf{k} \\
&= 3z\mathbf{i} - x\mathbf{k}.
\end{aligned}$$

- The divergence of the vector field $\mathbf{F}(x, y) = xe^y\mathbf{i} - ye^x\mathbf{j}$ is given by

$$\operatorname{div}(\mathbf{F})(x, y) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}(xe^y) - \frac{\partial}{\partial y}(ye^x) = e^y - e^x.$$

  The curl of the vector field $\mathbf{F}(x, y) = xe^y\mathbf{i} - ye^x\mathbf{j}$ is given by

$$\operatorname{curl}(\mathbf{F})(x, y) = \left(\frac{\partial}{\partial x}(-ye^x) - \frac{\partial}{\partial y}(xe^y)\right)\mathbf{k} = -(ye^x + xe^y)\mathbf{k}.$$

In our next proposition, we give several properties of ∇, div and curl and demonstrate connections between the operators.

Proposition 7.2.3 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are* C^2*-vector fields, and if* w *is a scalar field whose first and second order partial derivatives are continuous, then the following equalities hold.*

- $\nabla \cdot (\mathbf{u} + \mathbf{v}) = \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{v}$ *or equivalently* $\operatorname{div}(\mathbf{u} + \mathbf{v}) = \operatorname{div}(\mathbf{u}) + \operatorname{div}(\mathbf{v})$
- $\nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times \mathbf{u} + \nabla \times \mathbf{v}$ *or equivalently* $\operatorname{curl}(\mathbf{u} + \mathbf{v}) = \operatorname{curl}(\mathbf{u}) + \operatorname{curl}(\mathbf{v})$
- $\nabla \cdot (w\mathbf{u}) = \nabla(w) \cdot \mathbf{u} + w\nabla \cdot \mathbf{u}$
- $\nabla \times (w\mathbf{u}) = \nabla w \times \mathbf{u} + w\nabla \times \mathbf{u}$
- $\operatorname{curl}(\nabla(w)) = \nabla \times \nabla(w) = \mathbf{0}$
- $\operatorname{div}(\operatorname{curl}(\mathbf{u})) = \nabla \cdot (\nabla \times \mathbf{u}) = 0$
- $\operatorname{div}(\nabla(w)) = \Delta(w)$, *where* Δ *denotes the Laplacian*

Proof. We will prove the third and the fifth statements for 3-dimensional vector fields defined on all of $\mathbb{R}^3$, and leave the remaining statements to the reader.

- To show that $\nabla \cdot (w\mathbf{u}) = \nabla(w) \cdot \mathbf{u} + w\nabla \cdot \mathbf{u}$, let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and observe, by the product rule, for all $(x, y, z) \in \mathbb{R}^3$, that

$$\begin{aligned}
\nabla \cdot (w\mathbf{u})(x,y,z) &= \frac{\partial}{\partial x}(w(x,y,z)u_1(x,y,z)) + \frac{\partial}{\partial y}(w(x,y,z)u_2(x,y,z)) + \frac{\partial}{\partial z}(w(x,y,z)u_3(x,y,z)) \\
&= \left(\frac{\partial}{\partial x}w(x,y,z)\right)u_1(x,y,z) + w(x,y,z)\left(\frac{\partial}{\partial x}u_1(x,y,z)\right) \\
&\quad + \left(\frac{\partial}{\partial y}w(x,y,z)\right)u_2(x,y,z) + w(x,y,z)\left(\frac{\partial}{\partial y}u_2(x,y,z)\right) \\
&\quad + \left(\frac{\partial}{\partial z}w(x,y,z)\right)u_3(x,y,z) + w(x,y,z)\left(\frac{\partial}{\partial z}u_3(x,y,z)\right) \\
&= (\nabla(w) \cdot \mathbf{u})(x,y,z) + w(x,y,z)(\nabla \cdot \mathbf{u})(x,y,z).
\end{aligned}$$

- To show that $\operatorname{curl}(\nabla(w)) = \nabla \times \nabla(w) = \mathbf{0}$, we appeal to the definition of curl and ∇, and observe

$$\begin{aligned}
\operatorname{curl}(\nabla(w)) &= \operatorname{curl}\left(\frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} + \frac{\partial w}{\partial z}\mathbf{k}\right) \\
&= \left(\frac{\partial^2 w}{\partial y \partial z} - \frac{\partial^2 w}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 w}{\partial z \partial x} - \frac{\partial^2 w}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}.
\end{aligned}$$

  The last equality is an application of Clairaut's Theorem.

■

Chapter 8

Line integrals

In mathematics, a *line integral* is an integral where the function to be integrated is evaluated along a curve. The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by a scalar field determined by the curvature of the curve. This weighting distinguishes line integrals from those we have thus far encountered.

In qualitative terms, a line integral can be thought of as a measure of the total effect of a given vector field along a given curve; and the line integral over a scalar field $f\colon \mathbb{R}^2 \to \mathbb{R}$ can be interpreted as the area under the field carved out by a particular curve. This can be visualised as the surface created by $z = f(x, y)$ and a curve C in the x-y plane. The line integral of f would be the net area of the *curtain* created when the points of the surface that are directly over C are carved out, see Figure 8.1.

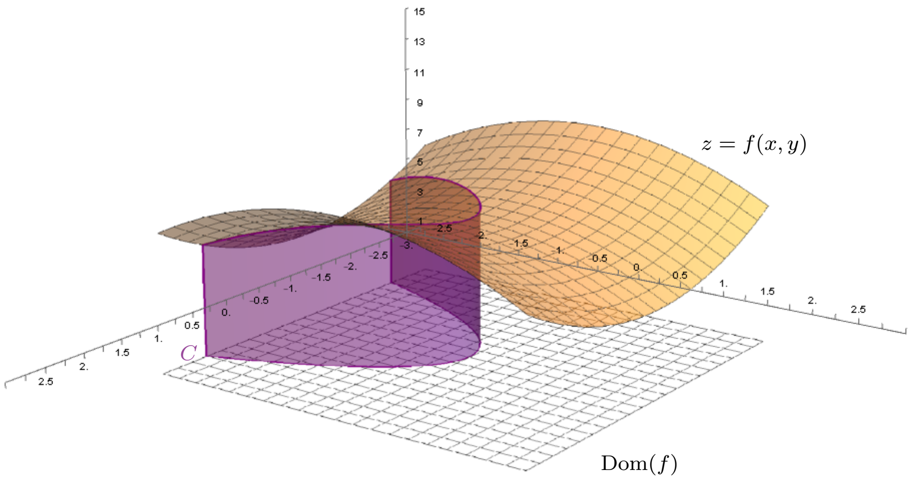

Figure 8.1: Graph of scalar field $f(x, y) = 8 + 3(x^2/4 - y^2/4)$ with domain $[-2, 2]^2$ and a curve C given by $y = -x^2$ in the x-y plane. The purple shaded region represents the line integral of the scale field f over the curve C.

8.1 Line integrals

Definition 8.1.1 *Let $a, b \in \mathbb{R}$ be such that $a < b$. A* ***path*** *is a continuous vector valued function $\mathbf{r}$ with domain $[a, b]$. The path is called* C^1 *if the derivative $\mathbf{r}'$ of $\mathbf{r}$ exists and is continuous on $[a, b]$, and if derivatives of all order of $\mathbf{r}$ are continuous then we say that the path is* ***smooth****. It is called* ***piecewise*** C^1*, respectively* ***piecewise smooth****, if one can decompose $[a, b]$ into finitely many subintervals on which $\mathbf{r}$ is C^1, respectively smooth. Further, if $\mathbf{r}(a) = \mathbf{r}(b)$, then we call $\mathbf{r}$ a* ***closed path****.*

A ***curve*** *C in $\mathbb{R}^n$, for some $n \in \mathbb{N}$ with $n \geq 2$, is a collection of ordered tuples $(f_1(t), f_2(t), \ldots, f_n(t))$, where f_i is a continuous function on an interval I, for all $i \in \{1, 2, \ldots, n\}$.*

Given a curve C in $\mathbb{R}^n$, for some $n \in \mathbb{N}$ with $n \geq 2$, if $\mathbf{r}$ is a path whose image generates C, then we call $\mathbf{r}$ a ***parameterisation*** *of C; note, a parameterisation of a curve is not necessarily unique, that is, there can exist many different parameterisations of a given curve. Further, if we image that we are standing on C, then a parameterisation tells us how to traverse (walk across) C.*

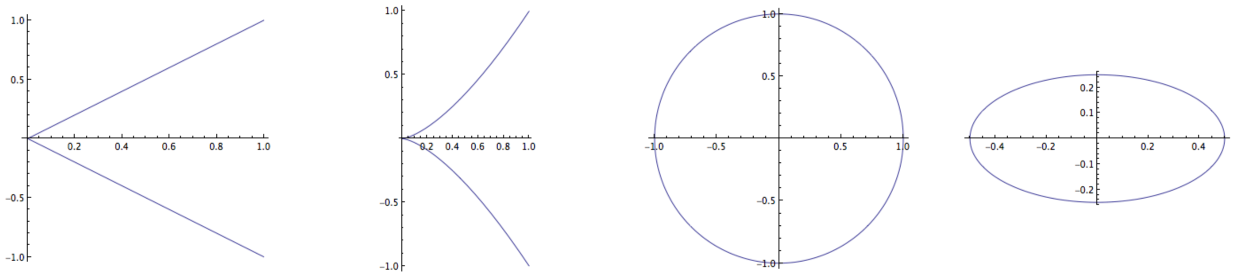

Figure 8.2: Graph of the paths $t \mapsto \langle |t|, t\rangle$, $t \mapsto \langle t^2, t^3\rangle$, $t \mapsto \langle \cos(t), \sin(t)\rangle$ and $t \mapsto \langle \cos(t)/2, \sin(t)/4\rangle$, respectively.

Example 8.1.2

- *The map* $\mathbf{r}\colon [-1,1] \to \mathbb{R}^2$ *given by* $\mathbf{r}(t) = \langle |t|, t\rangle$ *is a piecewise* C^1 *path. However, it is not* C^1*. See* Figure 8.2 *for a graphical representation.*
- *The map* $\mathbf{r}\colon [-1,1] \to \mathbb{R}^2$ *given by* $\mathbf{r}(t) = \langle t^2, t^3\rangle$ *is a* C^1 *path, see* Figure 8.2 *for a graphical representation.*
- *The standard parameterisation of the circle* $\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ *as a path is given by* $\mathbf{r}\colon [-\pi, \pi] \to \mathbb{R}^2$ *where* $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$*. This path is continuous and* C^1*, see* Figure 8.2 *for a graphical representation.*
- *Let* $a, b \in \mathbb{R}$ *be positive. The standard parameterisation of the ellipse* $\{(x,y) \in \mathbb{R}^2 \colon (ax)^2 + (by)^2 = 1\}$ *as a path is given by* $\mathbf{r}\colon [-\pi, \pi] \to \mathbb{R}^2$ *where* $\mathbf{r}(t) = \langle \cos(t)/a, \sin(t)/b\rangle$*. This path is continuous and* C^1*, see* Figure 8.2 *for a graphical representation.*
- *The curve* L *resulting from the intersection between the sphere* $S = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ *and the plane* $P = \{(x,y,z) \in \mathbb{R}^3 \colon x = z\}$ *may be parameterised as follows. Since* $L = S \cap P$*, if* $(x,y,z) \in L$*, it follows that* $2x^2 + y^2 = 1$ *and we can use the standard ellipse parametrisation to obtain the parametrisation* $\mathbf{r}\colon [-\pi, \pi] \to \mathbb{R}^3$ *given by*

  $$\mathbf{r}(t) = \langle \cos(t)/\sqrt{2}, \sin(t), \cos(t)/\sqrt{2}\rangle.$$

  See Figure 8.3 *for a graphical representation.*

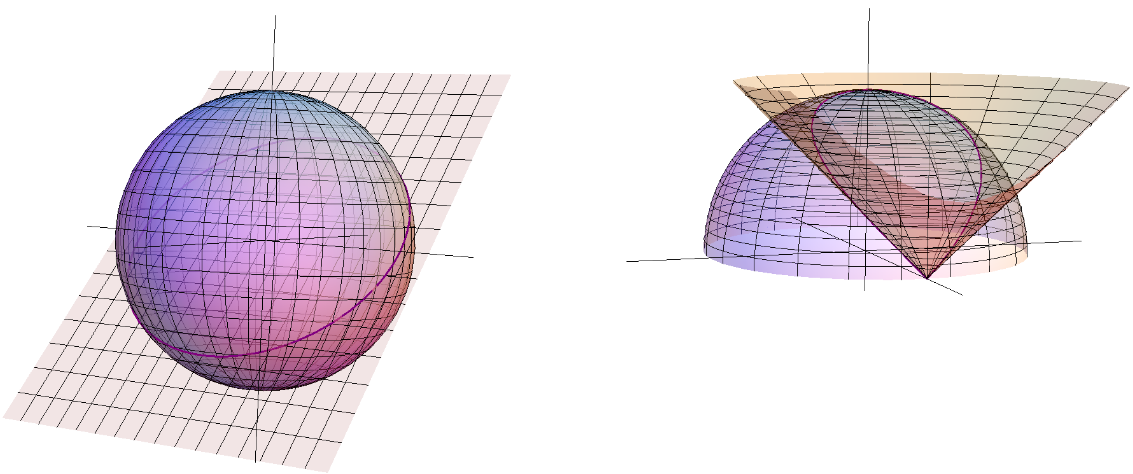

Figure 8.3: Left: Graphical representation of the sphere $S = \{(x,y,z) \in \mathbb{R}^3 \colon x^2+y^2+z^2 = 1\}$, the plane $P = \{(x,y,z) \in \mathbb{R}^3 \colon x = z\}$ and the curve $L = S \cap P$. Right: Graphical representation of the sphere $S = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$, the cone $K = \{(x,y,z) \in \mathbb{R}^3 \colon (x-1)^2 + y^2 = z^2\}$ and the curve $L = S \cap K$ in the positive upper half space.

Exercise 8.1.3 *Find a parameterisation of the curve which lies at the intersection of the sphere* $S = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ *and the cone* $K = \{(x,y,z) \in \mathbb{R}^3 \colon (x-1)^2 + y^2 = z^2\}$ *in the positive upper half space, see* Figure 8.3.

Solution. Let L be the set of points which lie at the intersection of S and K. If $(x, y, z) \in L$, then $x^2 + y^2 + z^2 = 1$ and $(x-1)^2 + y^2 = z^2$, and so $y^2 = -x(x-1)$ implying that $z^2 = -(x-1)$. This yields that $y^2 = xz^2$, and so setting $z = \cos(\theta)$, for $\theta \in [-\pi/2, \pi/2]$, we have that $x = \sin^2(\theta)$ and $y = \sin(\theta)\cos(\theta)$. Hence, a parameterisation of the curve L is $\mathbf{r}\colon [-\pi/2, \pi/2] \to \mathbb{R}^3$ given by

$$\mathbf{r}(t) = \langle \sin^2(\theta), \sin(\theta)\cos(\theta), \cos(\theta) \rangle.$$

To motivate the definition of a line integral, let us consider the following. Consider a force field $\mathbf{F}$ acting on a particle and transporting it along a curve C – how should one define the work done by $\mathbf{F}$ along C?

To answer this question, recall, if the force field is a constant vector $\mathbf{F}$ and the curve is a straight segment $\overrightarrow{AB}$, by definition, the work done is the dot product $\mathbf{F} \cdot \overrightarrow{AB}$; indeed work done is equal to the force in the direction of motion times the distance moved. Now assume that the curve is approximated by piecewise linear segments $\mathbf{V}_1 - \mathbf{V}_0$, $\mathbf{V}_2 - \mathbf{V}_1$, ..., $\mathbf{V}_n - \mathbf{V}_{n-1}$ so that, on the segments $\mathbf{V}_{i+1} - \mathbf{V}_i$, the force field is constantly equal to $\mathbf{F}_i$, see Figure 8.4. With this structure in mind, it follows that the total work done is the sum of the work done on each segment, namely

$$\sum_{i=0}^{n-1} \mathbf{F}_i \cdot (\mathbf{V}_{i+1} - \mathbf{V}_i).$$

To relate this quantity to an integral, let us parameterise the piecewise linear segments $\mathbf{V}_1 - \mathbf{V}_0$, $\mathbf{V}_2 - \mathbf{V}_1$, ..., $\mathbf{V}_n - \mathbf{V}_{n-1}$. To this end, let $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$, essentially a decomposition of the unit interval into n pieces. We define a parameterisation of our curve as follows.

$$\mathbf{r}(t) = \frac{t - t_i}{t_{i+1} - t_i}\mathbf{V}_{i+1} + \frac{t_{i+1} - t}{t_{i+1} - t_i}\mathbf{V}_i,$$

if $t_i \leq t \leq t_{i+1}$; so that $\mathbf{r}|_{[t_i, t_{i+1}]}$ is a parameterisation of the segment $\mathbf{V}_{i+1} - \mathbf{V}_i$. Observe that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(\xi_i) = \frac{1}{t_{i+1} - t_i}(\mathbf{V}_{i+1} - \mathbf{V}_i)$$

for $\xi_i \in [t_i, t_{i+1}]$ and $i \in \{0, 1, \ldots, n-1\}$. Thus, the total work done can be computed as follows.

$$\sum_{i=0}^{n-1} \mathbf{F}_i \cdot (\mathbf{V}_{i+1} - \mathbf{V}_i) = \sum_{i=0}^{n-1} \mathbf{F}_i \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(\xi_i)(t_{i+1} - t_i).$$

(Note the sum is independent of the choice of $\xi_i \in [t_i, t_{i+1}]$ for $i \in \{0, 1, \ldots, n-1\}$.) This is a standard approximation for an integral, suggesting the following definition for the line integral.

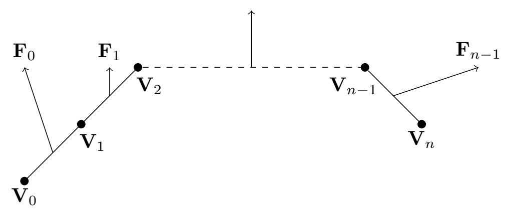

Figure 8.4: Piecewise linear curve with a constant force acting on each linear segment.

Definition 8.1.4 *Let $n \in \mathbb{N}$, Ω denote a connected non-empty open subset of $\mathbb{R}^n$, and a and $b \in \mathbb{R}$ with $a < b$.*

- *If $\mathbf{r}\colon [a, b] \to \Omega$ is a C^1 parametrisation of a curve C and $\mathbf{F}\colon \Omega \to \mathbb{R}^n$ is a continuous vector field, then the **line integral of $\mathbf{F}$ along C with respect to $\mathbf{r}$** is given by*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} := \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t.$$

- *If $\mathbf{r}\colon [a, b] \to \Omega$ is a C^1 parametrisation of a curve C and if $f\colon \Omega \to \mathbb{R}$ is a continuous scalar field, then the **line integral of f along C with respect to $\mathbf{r}$** is given by*

$$\int_C f(\mathbf{r}(t))\, \mathrm{d}t := \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, \mathrm{d}t$$

If $\mathbf{r}$ is piecewise C^1, then the corresponding line integral is the sum of the line integrals on the C^1 pieces of $\mathbf{r}$.

If, in the above definition, the curve C is closed, that is $\mathbf{r}(a) = \mathbf{r}(b)$, then it is common to represent the line integral as

$$\oint_C f(\mathbf{r}(t))\,\mathrm{d}t \quad \text{or} \quad \oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}.$$

If in the above definition $\mathbf{F} = \langle F_1, \cdots, F_n\rangle$ and $\mathbf{r}(t) = \langle x_1(t), \cdots, x_n(t)\rangle$ is a continuous piecewise C^1 parameterisation of C, that traverses C once, then the line integral is also sometimes written as

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C F_1\,\mathrm{d}x_1 + \cdots + F_n\,\mathrm{d}x_n.$$

In particular, in this case, if $n = 2$ or if $n = 3$, then it is common to write

$$\int_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y \quad \text{or} \quad \int_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z,$$

respectively.

Exercise 8.1.5 *Consider the curve C given by the path $\mathbf{r}(t) = (t+1)\mathbf{i} + t^2\mathbf{j}$ defined on the interval $[0,1]$ and the vector field $\mathbf{F}(x,y) = (x-1)^3\mathbf{i} - \sqrt{|y|}\mathbf{j}$ defined on $\mathbb{R}^2$. Compute the value of the line integral*

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}.$$

Solution. Observe that $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$ and that $\mathbf{F}(\mathbf{r}(t)) = t^3\mathbf{i} - t\mathbf{j}$. Thus, we have that

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^1 (t^3\mathbf{i} - t\mathbf{j})\cdot(\mathbf{i} + 2t\mathbf{j})\,\mathrm{d}t = \int_0^1 t^3 - 2t^2\,\mathrm{d}t = \frac{1}{4} - \frac{2}{3}.$$

Two natural questions to ask here are the following. Suppose we have two C^1 paths connecting a point v to a point w and a continuous vector field $\mathbf{F}$. If C_1 denotes the curve obtained from the first path and C_2 denotes the curve obtained from the second path, then is line integral of $\mathbf{F}$ over C_1 equal to the line integral of $\mathbf{F}$ over C_2? Also, what if we have two paths with the same range. Do the line integrals with respect to the different parameterisations differ? Let us try to attempt to answer these questions.

Consider two curves which connect the points $(0,0)$ and $(1,1)$. The first, denoted by C_1, has parametrisation $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, for $t \in [0,1]$, and the second, denoted by C_2, has parametrisation $\mathbf{s}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, for $t \in [0,1]$. Also, consider the two vector fields $\mathbf{F}(x,y) = xy\mathbf{i} + y\mathbf{j}$ and $\mathbf{G}(x,y) = y\mathbf{i} + x\mathbf{j}$ defined on $\mathbb{R}^2$. In this case $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$ and $\mathbf{s}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$ and so we have the following.

- $\displaystyle\int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^1 (t^2\mathbf{i} + t\mathbf{j})\cdot(\mathbf{i} + \mathbf{j})\,\mathrm{d}t = \int_0^1 t^2 + t\,\mathrm{d}t = \frac{1}{3} + \frac{1}{2}$
- $\displaystyle\int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{s} = \int_0^1 (t^5\mathbf{i} + t^3\mathbf{j})\cdot(2t\mathbf{i} + 3t^2\mathbf{j})\,\mathrm{d}t = \int_0^1 (2t^6 + 3t^5)\,\mathrm{d}t = \frac{2}{7} + \frac{3}{6}$
- $\displaystyle\int_{C_1} \mathbf{G}\cdot \mathrm{d}\mathbf{r} = \int_0^1 (t\mathbf{i} + t\mathbf{j})\cdot(\mathbf{i} + \mathbf{j})\,\mathrm{d}t = \int_0^1 2t\,\mathrm{d}t = 1$
- $\displaystyle\int_{C_2} \mathbf{G}\cdot \mathrm{d}\mathbf{s} = \int_0^1 (t^3\mathbf{i} + t^2\mathbf{j})\cdot(2t\mathbf{i} + 3t^2\mathbf{j})\,\mathrm{d}t = \int_0^1 (2t^4 + 3t^4)\,\mathrm{d}t = \int_0^1 5t^4\,\mathrm{d}t = 1$

The conclusion we may make from this, is that the line integral only sometimes depends on the chosen path. We shall see that there are certain vector fields, like $\mathbf{G}$, for which the line integral only depends on the end points of the curve traced out by the path. Such vector fields are called *conservative* and are related to physical concept known as *conservation of energy*, as we shall soon see.

Many paths can be thought of as the *same*. Indeed, if $\mathbf{r}\colon [0,1] \to \mathbb{R}^n$ is a parameterisation of a curve C, then $\mathbf{s}(t) = \mathbf{r}(t^2)$ is also a parameterisation of C. In such a case, we want to think of paths like these as being equivalent.

Definition 8.1.6 *Let $n \in \mathbb{N}$ with $n \geq 2$, and let a, b, c and $d \in \mathbb{R}$ be such that $a < b$ and $c < d$. Suppose $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$ and $\mathbf{s}\colon [c,d] \to \mathbb{R}^n$. We say that $\mathbf{r}$ and $\mathbf{s}$ are* ***equivalent*** *if there exists a continuous piecewise C^1 surjective function $u\colon [c,d] \to [a,b]$ whose derivative is non-zero when it exists such that $\mathbf{s}(t) = \mathbf{r}(u(t))$. Note, this means that $\mathbf{r}$ and $\mathbf{s}$ have the same image. If $u'(t) > 0$ for all $t \in [c,d]$ for which $u'(t)$ exists, then we say that the two equivalent parameterisations $\mathbf{r}$ and $\mathbf{s}$ have the* ***same orientation****; if $u'(t) < 0$ for all $t \in [c,d]$ for which $u'(t)$ exists, then we say that the two equivalent parameterisations $\mathbf{r}$ and $\mathbf{s}$ have* ***opposite orientation****.*

Proposition 8.1.7 *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\mathbf{r}$ and $\mathbf{s}$ denote equivalent C^1 parameterisations of a given curve(s) C and let $\mathbf{F}$ denote a continuous vector field defined on a connected open subset of $\mathbb{R}^n$ containing C.*

- *If $\mathbf{r}$ and $\mathbf{s}$ have the same orientation, then* $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}$.
- *If $\mathbf{r}$ and $\mathbf{s}$ have opposite orientations, then* $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}$.

Proof. Without loss of generality we can assume that the paths are C^1 (otherwise we decompose them into a union of C^1 paths, prove the result on each piece and use the additive properties of integrals). By our hypothesis, there exists a function u whose domain is equal to that of $\mathbf{s}$ and whose range is equal to the domain of $\mathbf{r}$ and such that $\mathbf{s}(t) = \mathbf{r}(u(t))$. In which case, $\mathbf{s}'(t) = \mathbf{r}'(u(t))u'(t)$ and so the conclusion follows by a direct application of the substitution rules for integrals. ∎

Example 8.1.8 *Consider $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, for $t \in [1/4, 1]$, and $\mathbf{s}(t) = t^2\mathbf{i} + t^2\mathbf{j}$, for $t \in [1/2, 1]$. In this case letting $u(t) = t^2$, we have that $u'(t) = 2t > 0$ and $\mathbf{r}(u(t)) = \mathbf{s}(t)$. Thus, $\mathbf{r}$ and $\mathbf{s}$ are equivalent and have the same orientation.*

Note that it is quite hard to give a general *independence of parametrisation* argument. Just knowing the image is not enough. For example, if we think about a line integral on the unit circle with the parametrisation

$$\mathbf{r_n}(t) = \cos(2\pi nt)\mathbf{i} + \sin(2\pi nt)\mathbf{j},$$

for $t \in [0,1]$ and some fixed $n \in \mathbb{N}$, then the result will depend on the given n. In fact, for a piecewise continuous vector field $\mathbf{F}$ defined on an open connected neighbourhood of $\mathbb{S}^1$,

$$\int_{\mathbb{S}^1} \mathbf{F} \cdot \mathrm{d}\mathbf{r_n} = n \int_{\mathbb{S}^1} \mathbf{F} \cdot \mathrm{d}\mathbf{r_1}.$$

Indeed, if C_1 and C_2 are two curves in $\mathbb{R}^n$, for some $n \geq 2$ a natural number, with respective C^1 parameterisations $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$ and $\mathbf{s}\colon [c,d] \to \mathbb{R}^n$, such that $\mathbf{r}(b) = \mathbf{s}(c)$ and where a, b, c and $d \in \mathbb{R}$ with $a < b$ and $c < d$, then for a continuous vector field $\mathbf{F}$ defined on an open connected region of $\mathbb{R}^n$ containing C_1 and C_2,

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{w},$$

where $C = C_1 \cup C_2$ and $\mathbf{w}\colon [a, b + (d-c)] \to \mathbb{R}^n$ is given by

$$\mathbf{w}(t) = \begin{cases} \mathbf{r}(t) & \text{if } t \in [a,b], \\ \mathbf{s}(t - b + c) & \text{if } t \in [b, b + (d-c)]. \end{cases}$$

This result follows from the definition of a path and that of a line integral.

Analogous results to those above also hold for line integrals over scalar fields.

8.2 Conservative vector fields

Let us begin this section by recalling the Fundamental Theorem of Calculus.

Theorem 8.2.1 (Fundamental theorem of calculus) *Let a and $b \in \mathbb{R}$ be such that $a < b$. If g is a differentiable function with continuous derivative on the interval $[a,b]$, then*

$$\int_a^b g'(t)\,\mathrm{d}t = g(b) - g(a).$$

In other words the integral of the derivative of g only depends on the values g takes when evaluated at the end points of the interval, namely the boundary of the interval.

Theorem 8.2.2 (Fundamental theorem of calculus for line integrals) *Let a and $b \in \mathbb{R}$ with $a < b$, let $n \geq 2$ denote a natural number, let Ω denote a connected non-empty open subset of $\mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}$ be a scalar field of the variables $x_1, x_2, \ldots, x_n$, such that all of its partial derivatives are continuous. If $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is a continuous piecewise C^1 parametrisation of a curve C contained in Ω, then*

$$\int_C \nabla(f) \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. Without loss of generality we can assume that the given path is C^1 (otherwise we may decompose it into a union of C^1 paths, prove the theorem on each piece and use the additive properties of integrals). Consider the function $g : [a, b] \to \mathbb{R}$ given by $g(t) = f(\mathbf{r}(t))$. By the chain rule we have

$$g'(t) = \frac{\mathrm{d}}{\mathrm{d}t} g(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{r}(t)) \frac{\mathrm{d}}{\mathrm{d}t} r_i(t) = \nabla(f)(\mathbf{r}(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t),$$

where r_i is the i-th component function of $\mathbf{r}$, that is $\mathbf{r}(t) = \langle r_1(t), r_2(t), \ldots, r_n(t)\rangle$. Thus,

$$\int_C \nabla(f) \cdot \mathrm{d}\mathbf{r} = \int_a^b \nabla(f)(\mathbf{r}(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t)\,\mathrm{d}t = \int_a^b g'(t)\,\mathrm{d}t = g(b) - g(a) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

∎

Corollary 8.2.3 *Let a and $b \in \mathbb{R}$ be such that $a < b$, let $n \in \mathbb{N}$ with $n \geq 2$, let Ω denote a connected non-empty open subset of $\mathbb{R}^n$, and let $\mathbf{r} : [a, b] \to \mathbb{R}^n$ denote a continuous piecewise C^1 parametrisation of a curve C contained in Ω. If $\mathbf{F} = \nabla(f)$ where f is a C^1 scalar field defined on Ω, then the line integral*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

(if it exists) does not depend on C but just on the value that f takes at the end points of C.

We conclude this section by defining two important concepts.

Definition 8.2.4

- *If $\mathbf{F} = \nabla(f)$ for a given scalar field f with continuous first order partial derivatives, then f is called a **potential** of $\mathbf{F}$.*
- *Let a, b, c and $d \in \mathbb{R}$ with $a < b$ and $c < d$, let $n \in \mathbb{N}$ with $n \geq 2$, and let Ω denotes a connected non-empty open subset of $\mathbb{R}^n$. A continuous vector field $\mathbf{F} : \Omega \to \mathbb{R}^n$ is called **conservative** if for any two curves C_1 and C_2 in Ω with respective continuous piecewise C^1 parameterisations $\mathbf{r} : [a, b] \to \Omega$ and $\mathbf{s} : [c, d] \to \Omega$ where $\mathbf{r}(a) = \mathbf{s}(c)$ and $\mathbf{r}(b) = \mathbf{s}(d)$,*

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

8.3 Conservation of energy

Suppose that $\mathbf{F}$ is a force field in the plane $\mathbb{R}^2$ (or 3-space $\mathbb{R}^3$) which has f as a potential, that is $\mathbf{F} = \nabla(f)$. Recall that the work done by the force $\mathbf{F}$ in moving a body with mass m along a curve C parametrised by a continuous piecewise C^1 path $\mathbf{r} : [a, b] \to \mathbb{R}^n$, for some a and $b \in \mathbb{R}$ with $a < b$, is given by

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

By Newton's laws of motion, if $\mathbf{v}(t)$ is the velocity and $\mathbf{a}(t)$ is the acceleration of the body at $\mathbf{r}(t)$, then $\mathbf{r}'(t) = \mathbf{v}(t)$ and $\mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t)$, and so $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{a}(t) = m\mathbf{v}'(t) = m\mathbf{r}''(t)$. Noting that

$$(\mathbf{v}(t) \cdot \mathbf{v}(t))' = 2\mathbf{v}'(t) \cdot \mathbf{v}(t),$$

if we denote by $k(t) = m(\mathbf{v}(t)\cdot\mathbf{v}(t))/2$ the kinetic energy of the body at $\mathbf{r}(t)$, then

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_a^b m\mathbf{v}'(t)\cdot\mathbf{v}(t)\,\mathrm{d}t = \frac{1}{2}m\int_a^b ((\mathbf{v}(t)\cdot\mathbf{v}(t))'\,\mathrm{d}t = k(b) - k(a).$$

Thus, the work done is equal to the difference in kinetic energy of the body at $\mathbf{r}(a)$ and $\mathbf{r}(b)$. On the other hand, by the fundamental theorem of calculus for line integrals, the work done by the force $\mathbf{F}$ in moving a body from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ only depends on the values of f at the end points of C, and equals $f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. In other words, $k(b) - k(a) = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$, or equivalently $k(b) - f(\mathbf{r}(b)) = k(a) - f(\mathbf{r}(a))$. In particular, the quantity $k - f\circ\mathbf{r}$ is constant. The value $-f(\mathbf{r}(t))$ is often called the **potential energy** of the body at $\mathbf{r}(t)$. With this in mind, if $\mathbf{F}$ is a force field which admits a potential f on a region Ω, then the total energy is independent of one's position in Ω.

8.4 Conservative vector fields are gradients

Here we discuss how to decipher when a vector field is a gradient vector field and how to compute its potential. By **Theorem 8.2.2**, it is necessary that a line integral of a gradient vector field $\mathbf{F}$ does not depend on the chosen path between two given points. In this section we will investigate if the converse holds.

Recall, a set $\Omega \subseteq \mathbb{R}^n$, for some $n \in \mathbb{N}$, is called **path connected** if for any x and $y \in \Omega$, there exists a path $\mathbf{r}\colon [0,1] \to \Omega$ with $\mathbf{r}(0) = \mathbf{x}$ and $\mathbf{r}(1) = \mathbf{y}$. (Note: If Ω is open, then it is path connected if and only if it is connected.)

Theorem 8.4.1 *Let $n \in \mathbb{N}$ with $n \geq 2$ and suppose that $\mathbf{F}$ is a continuous conservative vector field defined on a path connected non-empty open set $\Omega \subset \mathbb{R}^n$. Let $a \in \Omega$ be fixed and for $x = (x_1, x_2, \dots, x_n)$ set*

$$f(x) = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r},$$

where $\mathbf{r}$ is a continuous piecewise C^1 parametrisation of a curve C in Ω connecting a and x with a the initial point of C and x the terminal point of C. The scalar field f has continuous first order partial derivatives and $\nabla(f) = \mathbf{F}$.

Proof. Suppose that $\mathbf{F} = \langle F_1, \cdots, F_n\rangle$. It suffices to show, for all $i \in \{1, \dots, n\}$, that

$$\frac{\partial f}{\partial x_i} = F_i.$$

By definition and conservatively,

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_i) - f(\mathbf{x})}{h} = \lim_{h\to 0}\frac{1}{h}\int_{C_h}\mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

where $\mathbf{r}$ is a continuous piecewise C^1 parametrisation of a curve C_h connecting $\mathbf{x}$ and $\mathbf{x} + h\mathbf{e}_i$, and where $\mathbf{e}_i$ denotes the i-th standard coordinate vector of $\mathbb{R}^n$. One possible choice for $\mathbf{r}$, and hence C_h, is $\mathbf{r}(t) = \mathbf{x} + t\mathbf{e}_i$ on the interval $[0, h]$. Here, we need to take care as we require $\{\mathbf{x} \pm t\mathbf{e}_i : t \in [0, h]\}$ to be contained in Ω, but this is possible since Ω is open and we are only interested in the case when h is arbitrarily small. With the above at hand, it follows that $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{e}_i\,\mathrm{d}t = F_i(\mathbf{x} + t\mathbf{e}_i)\,\mathrm{d}t$. Note, $t \mapsto F_i(\mathbf{x} + t\mathbf{e}_i)$ is continuous and so there exists a function $g_i\colon [-h, h] \to \mathbb{R}$ so that $g_i'(t) = F_i(\mathbf{x} + t\mathbf{e}_i)$, i.e.

$$g_i(t) = \int_0^t F_i(\mathbf{x} + s\mathbf{e}_i)\,\mathrm{d}s.$$

Thus, we obtain that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h\to 0}\frac{1}{h}\int_0^h F_i(\mathbf{x} + t\mathbf{e}_i)\,\mathrm{d}t = \lim_{h\to 0}\frac{g_i(h) - g_i(0)}{h} = g_i'(0) = F_i(\mathbf{x}).$$

■

This leads us to the following result, which gives sufficient and necessary conditions for when a vector field is conservative.

Theorem 8.4.2 *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\mathbf{F}$ denote a vector field that is continuous on a non-empty path connected open subset $\Omega \subseteq \mathbb{R}^n$. The following are equivalent.*

- *There exists a scalar field f such that $\mathbf{F} = \nabla(f)$ on Ω.*
- *$\mathbf{F}$ is conservative on Ω.*
- *The line integral of $\mathbf{F}$ along all closed piecewise C^1 paths in Ω is equal to zero.*

Proof. The fundamental theorem of calculus for line integrals shows that the first statement implies the second statement and **Theorem 8.4.1** shows that the second statement implies the the first statement. The proof that the second statement implies the third statement follows from the definition of conservative, the equivalence of the first and second statements and the fundamental theorem of calculus for line integrals.

To complete the proof, consider two curves C_1 and C_2 contained in Ω with respective continuous C^1 parameterisations $\mathbf{r}$ and $\mathbf{s}\colon [0,1] \to \Omega$ so that $\mathbf{r}(0) = \mathbf{s}(0)$ and $\mathbf{r}(1) = \mathbf{s}(1)$. Also, consider a third path $\mathbf{w}$ defined as follows.

$$\mathbf{w}(t) = \begin{cases} \mathbf{r}(2t) & \text{if } t \in [0, 1/2] \\ \mathbf{s}(2-2t) & \text{if } t \in [1/2, 1] \end{cases}$$

This is a closed piecewise C^1 path. By our hypothesis,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{w} = 0.$$

However,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{w} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s} \quad \text{and so} \quad \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

∎

We end the section with a few properties of conservative vector fields.

Proposition 8.4.3 *If $\mathbf{F}$ and $\mathbf{G}$ are conservative vector fields with a common non-empty path connected open domain and α and $\beta \in \mathbb{R}$, then $\alpha\mathbf{F} + \beta\mathbf{G}$ is a conservative vector field.*

Proof. This follows from the additive properties of line integrals. ∎

Note, if $\mathbf{G}$ is a conservative vector field and if C is a curve contained in the domain of $\mathbf{G}$ determined by a closed C^1 path denoted by $\mathbf{r}$, then

$$\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = 0$$

and so for any other continuous vector field $\mathbf{F}$ with the same domain as $\mathbf{G}$, we have that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\mathbf{F} - \mathbf{G}) \cdot \mathrm{d}\mathbf{r}.$$

In particular, if f is a scalar field with a non-empty path connected open domain $\Omega \subset \mathbb{R}^n$, for some $n \in \mathbb{N}$ with $n \geq 2$, such that all of its partial derivatives are continuous, if C is a curve in Ω determined by a closed C^1 path denoted by $\mathbf{r}$, and if $\mathbf{F}$ is a continuous vector field also with domain Ω, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\mathbf{F} - \nabla(f)) \cdot \mathrm{d}\mathbf{r}.$$

Chapter 9

Green's Theorem

We have seen that the fundamental theorem of calculus is an extremely powerful and useful result. It tells us that if we know the value of the function $g\colon \mathbb{R} \to \mathbb{R}$ at the boundary points of the interval $[a, b]$, for some a and $b \in \mathbb{R}$ with $a \leq b$, then we know the value of the integral of the derivative of g over the interval $[a, b]$. A natural question to ask is, does there exist a higher dimensional analog? We have already seen that it generalises to line integrals, but can we relate the integral of some type of derivative of a two-variable function f over a two-dimensional (bounded) region R to the values which f takes on the boundary of R? The answer is yes and is given by Green's Theorem. But why is it interesting to have such a relation? Let's say that we did not have the use of a modern computer and we wanted to accurately measure

- the area of a cross-section of a tumour,
- the area of a leaf or
- the area of a wing of a wasp (or some other insect),

all of which have an intricate boundary. How could we do this? Historically, one would use a *planimeter*, a measuring device pioneered by Johann Martin Hermann in 1814, see FIGURE 9.1. How does such a device work? One arm is fixed, or weighted down, and with the other arm you trace around the object of interest in a counter clockwise direction. You see a series of dials moving in some wacky way, and in the end, once you have traced around the object, it tells you its area. So how does this device work and what is the mathematics behind the device? It relies purely on Green's Theorem, which is the main focus of this section.

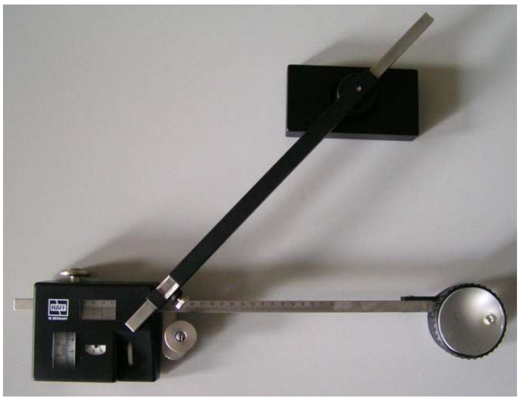

Figure 9.1: Image of a planimeter. Source: `www.britannica.com/science/planimeter` (2019)

9.1 Necessary conditions

Recall that, if $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a scalar field, whose first and second order partial derivatives are continuous, then $\text{curl}(\nabla f) = \mathbf{0}$, and so, if $\mathbf{F} = \nabla(f)$, then $\text{curl}(\mathbf{F}) = \mathbf{0}$. In particular, since a vector field is conservative if and only if it is a gradient vector field, this means that if $\mathbf{F}$ is a three dimensional C^1 conservative vector field, then $\text{curl}(\mathbf{F}) = \mathbf{0}$. Therefore, a natural question to ask is the following.

Question 9.1.1 *Does the converse to this statement hold true? Namely, is it true that if the circulation density of a three dimensional vector field is equal to the zero vector field, then it is conservative? And since we have only defined the circulation density for three dimensional vector fields, does there exist higher dimensional analogues of these statements?*

The following proposition gives us a partial solution to these questions.

Proposition 9.1.2 *Let $n \in \mathbb{N}$ with $n > 1$, let Ω denote a path connected non-empty open subset of $\mathbb{R}^n$ and let $\mathbf{F}\colon \Omega \to \mathbb{R}^n$ denote a conservative vector field whose component functions $F_i\colon \Omega \in \mathbb{R}^n$, for $i \in \{1, 2, \ldots, n\}$, are C^1. If $\mathbf{F}(x_1, \ldots, x_n) = \langle F_1(x_1, \ldots, x_n), \cdots, F_n(x_1, \ldots, x_n)\rangle$, then*

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

for all i and $j \in \{1, \cdots, n\}$.

Proof. Since $\mathbf{F}$ is conservative, we have that $\mathbf{F} = \nabla(f)$ for some scalar field $f\colon \Omega \to \mathbb{R}$ whose first and second order partial derivatives are continuous. In particular,

$$F_i = \frac{\partial f}{\partial x_i},$$

for all $i \in \{1, 2, \ldots, n\}$, and so, by Clairaut's theorem,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial F_j}{\partial x_i},$$

for all i and $j \in \{1, 2, \ldots, n\}$. ∎

Question 9.1.3 *What about the converse?*

To answer this question let us consider the following exercise.

Exercise 9.1.4 *Let $\mathbf{F}\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ be defined by*

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle,$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Show that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$

where F_1 and F_2 are, respectively, the first and second component functions of $\mathbf{F}$. Further, determine if the vector field $\mathbf{F}$ conservative.

Solution. By the quotient rule we have that

$$\frac{\partial F_1}{\partial y} = -\frac{x^2 - y^2}{x^4 + 2x^2y^2 + y^4} = \frac{\partial F_2}{\partial x}$$

and so we see that $\mathbf{F}$ satisfies the conclusion of **Proposition 9.1.2**. However, letting $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ be the parameterisation of the unit circle given by $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$, for $t \in [0, 2\pi]$, we have that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \frac{-\sin(t)}{\sin^2(t) + \cos^2(t)}(-\sin(t)) + \frac{\cos(t)}{\sin^2(t) + \cos^2(t)} \cos(t)\, \mathrm{d}t = \int_0^{2\pi} 1\, \mathrm{d}t = 2\pi.$$

Thus, $\mathbf{F}$ is not conservative.

This exercise shows that the converse of **Proposition 9.1.2** does not always hold. The issue here is that the domain $\mathbf{F}$, in our exercise, is $\mathbb{R}^2 \setminus \{(0,0)\}$ which is not a nice set topologically. Spaces such as these are called not simply connected. Therefore, in order to obtain a converse of the above result we need to at least add a restriction to the shape of the domain of the vector field. This is the main point of discussion in our next section.

9.2 Sufficient conditions for specific types of domains

To present the main result of this section, **Theorem 9.2.2** we require the following definition.

Definition 9.2.1 *Let $n \geq 2$ denote a natural number. A subset Ω of $\mathbb{R}^n$ is called **star-shaped** if there exists a $\mathbf{x_0} \in \Omega$ such that the image of the path $t \mapsto t\mathbf{x} + (1-t)\mathbf{x_0}$, for $t \in [0,1]$, lies entirely in Ω, for all $\mathbf{x} \in \Omega$.*

Star-shaped subsets are examples of what one refers to, in topology, as a simply connected domain, and are a generalisation of convex sets.

Theorem 9.2.2 *Let $n \geq 2$ be a natural number. If $\Omega \subset \mathbb{R}^n$ is a star-shaped set and if $\mathbf{F} = \langle F_1, \dots, F_n \rangle \colon \Omega \to \mathbb{R}^n$ is a C^1 vector field in the variables $x_1, x_2, \dots, x_n$, such that*

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

*for all i and $j \in \{1, 2, \dots, n\}$, then $\mathbf{F}$ is conservative. Moreover, since Ω is star-shaped, letting $\mathbf{x_0}$ be fixed, and as in **Definition 9.2.1**, a potential of $\mathbf{F}$ is $f \colon \Omega \to \mathbb{R}$ given by*

$$f(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x} + (1-t)\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})\, \mathrm{d}t.$$

Proof. Without loss of generality, let us assume that $\mathbf{x_0}$ in the definition of the star-shaped region Ω is the zero vector. Let $f \colon \Omega \to \mathbb{R}$ be defined by

$$f(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \cdot \mathbf{x}\, \mathrm{d}t$$

and observe that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \int_0^1 \frac{\partial}{\partial x_i}(\mathbf{F}(t\mathbf{x}) \cdot \mathbf{x})\, \mathrm{d}t.$$

An application of the product and chain rule yields

$$\begin{aligned}\frac{\partial}{\partial x_i}(\mathbf{F}(t\mathbf{x}) \cdot \mathbf{x}) &= \frac{\partial}{\partial x_i} \sum_{j=1}^{n} F_j(t\mathbf{x}) x_j \\ &= \left(\sum_{j=1}^{n} \left(t \frac{\partial F_j}{\partial x_i}(t\mathbf{x}) \right) x_j \right) + \left(\sum_{j=1}^{n} F_j(t\mathbf{x}) \frac{\partial x_j}{\partial x_i} \right) = \left(\sum_{j=1}^{n} \left(t \frac{\partial F_j}{\partial x_i}(t\mathbf{x}) \right) x_j \right) + F_i(t\mathbf{x}).\end{aligned}$$

Recall that, by our hypothesis, we have that

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

and so

$$\frac{\partial}{\partial x_i}(\mathbf{F}(t\mathbf{x}) \cdot \mathbf{x}) = t \left(\sum_{j=1}^{n} x_j \frac{\partial F_i}{\partial x_j}(t\mathbf{x}) \right) + F_i(t\mathbf{x}).$$

The above in tandem with the fact that

$$\frac{\partial}{\partial t} F_i(t\mathbf{x}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} F_i(t\mathbf{x}) \frac{\partial}{\partial t} t x_j = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} F_i(t\mathbf{x})$$

yields

$$\frac{\partial}{\partial x_i}(\mathbf{F}(t\mathbf{x}) \cdot \mathbf{x}) = t \frac{\partial}{\partial t} F_i(t\mathbf{x}) + F_i(t\mathbf{x}) = \frac{\partial}{\partial t}(t F_i(t\mathbf{x}))$$

Thus, we may conclude that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \int_0^1 \frac{\partial}{\partial x_i}(\mathbf{F}(t\mathbf{x}) \cdot \mathbf{x})\, \mathrm{d}t = \int_0^1 \frac{\partial}{\partial t}(t F_i(t\mathbf{x}))\, \mathrm{d}t = t F_i(t\mathbf{x}) \Big|_{t=0}^{t=1} = F_i(\mathbf{x}).$$

In other words the co-ordinate functions of $\nabla(f)$ are, respectively, equal to those of $\mathbf{F}$, which implies that $\nabla(f) = \mathbf{F}$; as required. ∎

Exercise 9.2.3 *Let* $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ *denote the vector field given by*

$$\mathbf{F}(x,y,z) = \langle yz\cos(xyz), z+2y+xz\cos(xyz), xy\cos(xyz)+y+z^3\rangle,$$

for $(x,y,z) \in \mathbb{R}^3$. *Decipher whether* $\mathbf{F}$ *is conservative or not. If* $\mathbf{F}$ *is conservative, then determine a potential for* $\mathbf{F}$.

Solution 1 – An application of Theorem 9.2.2. Observe that

$$\begin{aligned}
\frac{\partial F_2}{\partial x}(x,y,z) &= \frac{\partial F_1}{\partial y}(x,y,z) = z\cos(xyz) - yz^2x\sin(xyz),\\
\frac{\partial F_1}{\partial z}(x,y,z) &= \frac{\partial F_3}{\partial x}(x,y,z) = y\cos(xyz) - xy^2z\sin(xyz), \text{ and}\\
\frac{\partial F_2}{\partial z}(x,y,z) &= \frac{\partial F_3}{\partial y}(x,y,z) = 1 + x\cos(xyz) - x^2yz\sin(xyz).
\end{aligned}$$

Thus, since $\mathbb{R}^3$ is star-shaped with $\mathbf{x_0} = \langle 0,0,0\rangle$, **Theorem 9.2.2** yields that $\mathbf{F}$ is conservative. Moreover, a potential for $\mathbf{F}$ is given by

$$\begin{aligned}
g(x,y,z) &= \int_0^1 \mathbf{F}(tx,ty,tz)\cdot\langle x,y,z\rangle\,\mathrm{d}t\\
&= \int_0^1 \langle t^2yz\cos(t^3xyz), tz+t2y+t^2xz\cos(t^3xyz), t^2xy\cos(t^3xyz)+ty+t^3z^3\rangle\cdot\langle x,y,z\rangle\,\mathrm{d}t\\
&= \int_0^1 3t^2xyz\cos(t^3xyz)+2tyz+2ty^2+t^3z^4\,\mathrm{d}t\\
&= \sin(t^3xyz)+t^2yz+t^2y^2+\frac{t^4z^4}{4}\bigg|_{t=0}^{t=1}\\
&= \sin(xyz)+yz+y^2+\frac{z^4}{4}.
\end{aligned}$$

Solution 2 – A direct inductive solution using step by step value integration. Let F_1, F_2 and $F_3\colon \mathbb{R}^3 \to \mathbb{R}$ denote the coordinate functions of $\mathbf{F}$, namely be such that

$$\mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$$

for all $(x,y,z) \in \mathbb{R}^3$. Our aim is to show the existence of an $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. To this end suppose that such an f exists. Integrating F_1 with respect to x, regarding y and z as constants, yields

$$f(x,y,z) = \int F_1(x,y,z)\,\mathrm{d}x = \sin(xyz) + g(y,z),$$

for some function g dependent purely on y and z, and so,

$$\frac{\partial f}{\partial x} = F_1 \quad\text{and}\quad \frac{\partial f}{\partial y}(x,y,z) = xz\cos(xyz) + \frac{\partial g}{\partial y}(y,z).$$

Now the condition $\partial f/\partial y = F_2$ implies that $(\partial g/\partial y)(y,z) = z+2y$. Integrating both sides with respect to y, regarding z as constant, yields

$$g(y,z) = \int (z+2y)\,\mathrm{d}y, = zy + y^2 + h(z),$$

for some function h dependent purely on z, and so

$$\frac{\partial f}{\partial z}(x,y,z) = xy\cos(xyz) + y + \frac{\partial h}{\partial z}(z).$$

Now the condition $\partial f/\partial z = F_3$, implies that $(\partial h/\partial z)(z) = z^3$ and so,

$$f(x,y,z) = \sin(xyz) + zy + y^2 + \frac{z^4}{4} + c,$$

has the property that $\nabla f = \mathbf{F}$. Namely, $\mathbf{F}$ is a gradient vector with potential f, and thus, $\mathbf{F}$ is conservative.

9.3 Green's Theorem

We begin this section by recalling that a curve in $\mathbb{R}^2$ parameterised by $\mathbf{r}\colon [a,b] \to \mathbb{R}^2$, for some a and $b \in \mathbb{R}$ with $a < b$, is called a **simple closed curve** if $\mathbf{r}$ is continuous and injective on $[a,b)$ and $\mathbf{r}(a) = \mathbf{r}(b)$. In other words, $\mathbf{r}$ is continuous and the image of $\mathbf{r}$ is a loop which does not self intersect.

Green's Theorem can be thought of as a two dimensional version of the fundamental theorem of calculus. It expresses the double integral of a certain kind of derivative of a two dimensional vector field $\mathbf{F}$, namely the $\mathbf{k}$-component of curl($\mathbf{F}$), over a region R in the x-y plane as a line integral of the tangential components of $\mathbf{F}$ around the curve C which is the oriented boundary of R, in symbols

$$\iint_R \operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} \, \mathrm{d}A = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

For this formula to hold it is important C is the oriented boundary of R considered as a surface with orientation provided by the normal vector $\mathbf{N}$. Since we are living in the x-y plane, we have that $\mathbf{N} = \mathbf{k}$ – note here $\mathbf{N}$ is the unit normal vector to the surface. Thus, C is oriented with R on the left as we move around C in the direction of the orientation. We will call such a curve **positively oriented** with respect to R; in the case that R has no holes, we sometime use the terms positively oriented and oriented counter clockwise interchangeably. If the region R has holes, then in order for the boundary of R to be positively oriented, the boundary of the holes must be oriented clockwise, see Figure 9.2. In either case, the unit tangent vector $\mathbf{T}$ and the unit normal vector $\mathbf{n}$ on C satisfy $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ – note here $\mathbf{n}$ is the unit normal vector to the curve, which is distinct to the unit normal vector to the surface $\mathbf{N} = \mathbf{k}$.

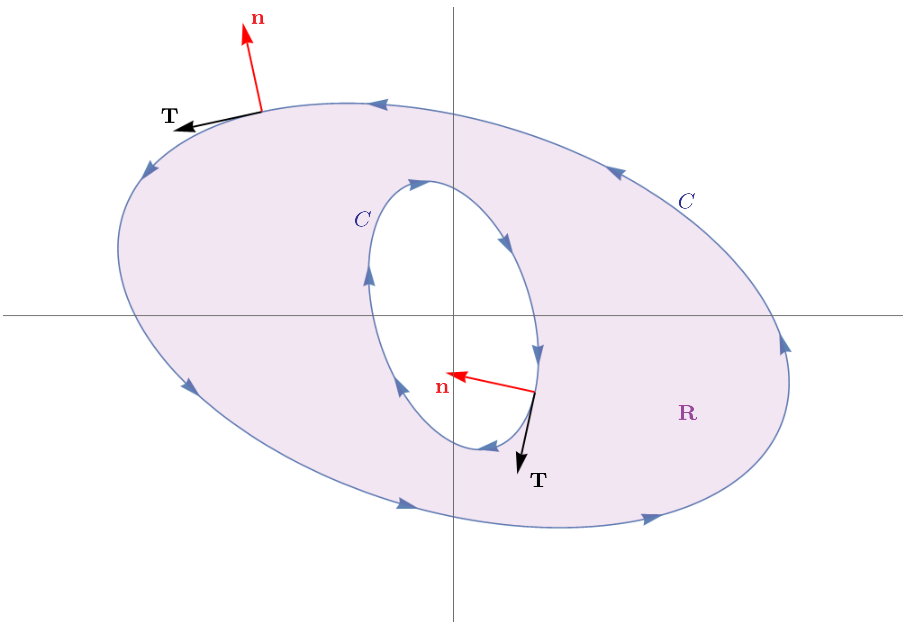

Figure 9.2: A planar domain R with positively oriented boundary C.

Before we can formally state Green's Theorem we need one more concept, and that is what is known as a *regular domain*. We say that a domain $D \subset \mathbb{R}^2$ is **y-simple** if it is bounded by two vertical lines $x = a$ and $x = b$, and two continuous functions $y = c(x)$ and $y = d(x)$ between these two lines. Similarly, we say that a domain $D \subset \mathbb{R}^2$ is **x-simple** if it is bounded by two horizontal lines $y = c$ and $y = d$, and two continuous functions $x = a(y)$ and $x = b(y)$ between these two lines. Domains which are unions of finitely many non-overlapping (except possibly at their boundary) sub-domains that are either x-simple, y-simple or both x-simple and y-simple are called **regular**. Indeed, any polygon in the plane is an example of a regular domain.

Theorem 9.3.1 (Green's Theorem) *Let R be a regular closed region in the x-y plane whose boundary C consists of one or more piecewise C^1 simple closed curves that are positively oriented with respect to R. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a C^1 vector field on R, then*

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A.$$

where $\mathbf{r}$ is a piecewise C^1 parameterisation of C which traverses C once in alignment with the orientation induced by R.

Note, if C consists of two or more piecewise C^1 simple closed curves, C_1, C_2, …, C_n say, then we set

$$\oint_C F_1 \,\mathrm{d}x + F_2 \,\mathrm{d}y = \sum_{i=1}^{n} \oint_{C_i} F_1 \,\mathrm{d}x + F_2 \,\mathrm{d}y.$$

Proof (Sketch) of Green's Theorem. If C_1 is a curve in the plane parallel to the x-axis at height β_1 with start point (α_1, β_1) and end point (α_2, β_1), as in Figure 9.3, a parameterisation of C_1 is given by $\mathbf{r_1}(t) = t\mathbf{i} + \beta_1\mathbf{j}$, for $t \in [\alpha_1, \alpha_2]$, and so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r_1} = \int_{\alpha_1}^{\alpha_2} F_1(t, \beta_1) \,\mathrm{d}t.$$

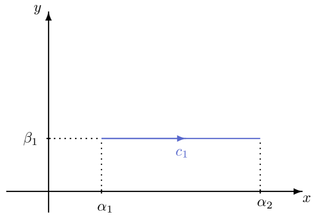

Figure 9.3: Plot of C_1.

Similarly, if C_2 is a curve in the plane parallel to the y-axis at horizontal distance α_2 with start point (α_2, β_1) and end point (α_2, β_2), as in Figure 9.4, a parameterisation of C_2 is given by $\mathbf{r_2}(t) = \alpha_2\mathbf{i} + t\mathbf{j}$, for $t \in [\beta_1, \beta_2]$, and so

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r_2} = \int_{\beta_1}^{\beta_2} F_2(\alpha_2, t) \,\mathrm{d}t,$$

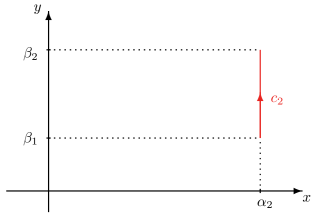

Figure 9.4: Plot of C_2.

With this in mind we now compute the line integral over the curve C depicted in Figure 9.5. (Here $\mathbf{r}$ denotes a parameterisation of C and $\mathbf{r}_\ell$ denotes the parameterisation of the curve C_ℓ, for $\ell \in \{1, 2, 3, 4\}$, given above.)

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r_1} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r_2} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r_3} + \int_{C_4} \mathbf{F} \cdot \mathrm{d}\mathbf{r_4} \\
&= \int_{\alpha_1}^{\alpha_2} F_1(t, \beta_1) \,\mathrm{d}t + \int_{\beta_1}^{\beta_2} F_2(\alpha_2, t) \,\mathrm{d}t \\
&\quad + \int_{\alpha_2}^{\alpha_1} F_1(t, \beta_2) \,\mathrm{d}t + \int_{\beta_2}^{\beta_1} F_2(\alpha_1, t) \,\mathrm{d}t \\
&= \int_{\beta_1}^{\beta_2} F_2(\alpha_2, t) - F_2(\alpha_1, t) \,\mathrm{d}t - \int_{\alpha_1}^{\alpha_2} F_1(t, \beta_2) - F_1(t, \beta_1) \,\mathrm{d}t.
\end{aligned}$$

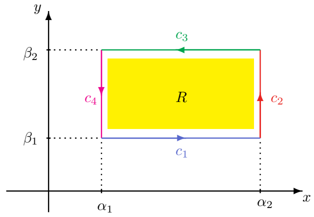

Figure 9.5: Plot of a simple, closed curve $C = C_1 + C_2 + C_3 + C_4$ encompassing a rectangular region R.

Applying the fundamental theorem of calculus, we have that

$$F_2(\alpha_2, t) - F_2(\alpha_1, t) = \int_{\alpha_1}^{\alpha_2} \frac{\partial}{\partial x} F_2(x, t) \,\mathrm{d}x \quad \text{and} \quad F_1(t, \beta_2) - F_1(t, \beta_1) = \int_{\beta_1}^{\beta_2} \frac{\partial}{\partial y} F_1(t, y) \,\mathrm{d}y.$$

Combining the above with Fubini-Tonelli's Theorem, it follows that

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{\beta_1}^{\beta_2} \int_{\alpha_1}^{\alpha_2} \frac{\partial}{\partial x} F_2(x, y) \,\mathrm{d}x\mathrm{d}y - \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \frac{\partial}{\partial y} F_1(x, y) \,\mathrm{d}y\mathrm{d}x \\
&= \iint_R \frac{\partial}{\partial x} F_2(x, y) - \frac{\partial}{\partial y} F_1(x, y) \,\mathrm{d}A.
\end{aligned}$$

This is precisely Green's Theorem for the case when C is the curve encompassing a rectangular region. ∎

Let us return to the planimeter and answer the question, how does it apply Green's Theorem to compute the area of an object by tracing around its boundary? This is given by the following corollary.

Corollary 9.3.2 *If C is a simple closed piecewise C^1 positively oriented planar curve and if Ω, the region bounded by C, is regular, then the area of Ω is*

$$\mathrm{Area}(\Omega) = \oint_C x \,\mathrm{d}y = -\oint_C y \,\mathrm{d}x = \frac{1}{2} \oint_C -y \,\mathrm{d}x + x \,\mathrm{d}y.$$

Proof. This is an application of Green's Theorem, where we consider the vector fields

$$\mathbf{F}(x, y) = x\mathbf{j}, \qquad \mathbf{G}(x, y) = -y\mathbf{i}, \quad \text{and} \quad \mathbf{H}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{2},$$

respectively. ∎

Chapter C

Formative assessments

C.1 Formative assessment – Week 13

(1) Using vector operations, find the area of the triangle $\triangle ABC$ where $A = (1,0,1)$, $B = (2,1,2)$ and $C = (1,1,1)$.

(2) Show that the vectors $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ are perpendicular and find a third vector that is perpendicular to both $\mathbf{v}$ and $\mathbf{u}$.

(3) Consider a point particle moving in the x-y plane via the rule

$$\mathbf{r}(t) = a\cos(bt)\mathbf{i} + b\sin(bt)\mathbf{j},$$

where a and b are some fixed constants. Compute the velocity and the acceleration of the particle.

(4) Compute the divergence and circulation density of the following vector fields.

(a) $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathbf{F}(x,y,z) = x\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$

(b) $\mathbf{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathbf{G}(x,y,z) = x^2\mathbf{i} - xy\mathbf{j} + z\mathbf{k}$

(c) $\mathbf{H}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathbf{H}(x,y,z) = x^2yz\mathbf{i} + \sin(xy)\mathbf{j} + zx\mathbf{k}$

(5) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Compute the second order partial derivatives $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

(6) Find a parametrisation for the following curves.

(a) The curve arising from the intersection of the cylinder $\{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2 = 2x\}$ and the hemisphere $\{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+z^2 = 1 \text{ and } z \geq 0\}$.

(b) The curve arising from the intersection of the cone $\{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2 = z^2 \text{ and } z \geq 0\}$ and the sphere $\{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+z^2 = 1\}$.

(c) The curve arising from the intersection of the cylinder $\{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2 = x\}$ and the plane $\{(x,y,z) \in \mathbb{R}^3\colon y = z\}$.

(d) The curve arising from the intersection of the sphere $\{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+z^2 = 1\}$ and the plane $\{(x,y,z) \in \mathbb{R}^3\colon y = z\}$.

(e) The curve arising from the intersection of the cone $\{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2 = z^2 \text{ and } z \geq 0\}$ and the plane $\{(x,y,z) \in \mathbb{R}^3\colon 3z = -x+4\}$.

(7) Compute the value of the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathbf{F}$ and $\mathbf{r}$ are as follows.

(a) $\mathbf{F}(x, y) = \langle x^2 y, y^2 x \rangle$ and $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [-\pi, \pi]$

(b) $\mathbf{F}(x, y) = \langle \cos(x), \sin(x) \rangle$ and $\mathbf{r}(\mathbf{t}) = \langle t, t \rangle$ for $t \in [0, 1]$

(c) $\mathbf{F}(x, y) = \langle xy - z, y^2 xz, xyz \rangle$ and $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ for $t \in [0, 2\pi]$

(8) Let $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the force field given by

$$\mathbf{F}(x, y, z) = (x - yz)\mathbf{i} + (y - xz)\mathbf{j} + (x(1 - y) + z^2)\mathbf{k},$$

for $(x, y, z) \in \mathbb{R}^3$. Compute the work done by $\mathbf{F}$ in moving a particle once around the triangle with vertices $A = (0, 0, 0)$, $B = (1, 1, 1)$ and $C = (1, 1, 0)$, with orientation given by starting in A, traversing to B, then to C and returning to A.

(9) Calculate the work done by the force field $\mathbf{F}(x, y, z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ along the curve C which lies at the intersection of the sphere $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$, the cylinder $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 2x\}$ and the half-space $\{(x, y, z) \in \mathbb{R}^3 \colon z \geq 0\}$ and which is traversed in a direction that appears clockwise when viewed from high above the x-y plane.

(10) Let $a, b \in \mathbb{R}$ with $a < b$ and let $n \geq 2$ denote a natural number. If the position vector $\mathbf{r} \colon [a, b] \to \mathbb{R}^n$ of a given curve is smooth and satisfies $\mathbf{r}(t) \neq \mathbf{0}$, for all $t \in [a, b]$, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

If in addition, $\mathbf{r}$ has the property that $\mathbf{r}(t)$ is always perpendicular to it's velocity vector $\mathbf{r}'(t)$, show that the image of $\mathbf{r}$ lies on a sphere centred at the origin.

(11) Suppose that the position of a particle at time t is given, for $t \in [0, 2\pi]$, by

$$x_1(t) = 3\sin(t) \quad \text{and} \quad y_1(t) = 2\cos(t)$$

and the position of a second particle is given, for $t \in [0, 2\pi]$, by

$$x_2(t) = -3 + \cos(t) \quad \text{and} \quad y_2(t) = 1 + \sin(t).$$

(a) Graph the paths of both particles. How many points of intersection are there?

(b) Are any of these points of intersection collision points? In other words, are the particles ever at the same place at the same time? If so, find the collision points.

(c) Describe what happens if the path of the second particle is given, for $t \in [0, 2\pi]$, by

$$x_2(t) = 3 + \cos(t) \quad \text{and} \quad y_2(t) = 1 + \sin(t).$$

(12) Let a and $b \in \mathbb{R}^+$ be fixed, and let $\mathbf{v} \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the velocity vector field given by $\mathbf{v}(x, y) = y\,\mathbf{i} - x\,\mathbf{j}$, for $(x, y) \in \mathbb{R}^2$. Prove that the streamline of $\mathbf{v}$ passing through (a, b) is a circle centred at the origin.

C.2 Formative assessment – Week 14

(1) Use Theorem 9.2.2 to find a potential for the following vector fields.

(a) $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x, y) = \langle x^2 \cos(x) - y\cos(xy) + 2x\sin(x), -4y^3 - x\cos(xy)\rangle$

(b) $\mathbf{L} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{L}(x, y) = \langle xy^2 \cos(xy) + y\sin(xy), x^2 y\cos(xy) + x\sin(xy)\rangle$

(c) $\mathbf{M}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{M}(x, y) = \langle x\cos(x) + ye^{xy} + \sin(x), xe^{xy} + 2y\sin(y^2)\rangle$

(d) $\mathbf{P} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{P}(x, y) = \langle 3x^2 y - y^3 - y\sin(xy), x^3 - 3xy^2 - x\sin(xy)\rangle$

(2) Use Theorem 9.2.2 to find a potential for the following vector fields.

(a) $\mathbf{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{G}(x, y, z) = \langle -3x^2 + yz, -4y^3 + xz, xy\rangle$

(b) $\mathbf{H}\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{H}(x, y, z) = \langle z\cos(x)\cos(y), -z\sin(x)\sin(y), \cos(y)\sin(x)\rangle$

(c) $\mathbf{K}\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{K}(x, y, z) = \langle x^2 yz\cos(xyz) + 2x\sin(xyz), x^3 z\cos(xyz), x^3 y\cos(xyz)\rangle$

(3) Use Green's Theorem to find the area enclosed by the curves determined by following equations.

(a) $x^2 + 2xy + 5y^2 = 1$

(b) $x^{2/3} + y^{2/3} = 1$

(c) $x(t) = 3\cos(t) - \cos(3t)$ and $y(t) = 3\sin(t) - \sin(3t)$, where $t \in [0, 2\pi]$

(4) Use Green's Theorem to compute the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F}$, C and $\mathbf{r}$ are as follows.

(a) $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined, for $(x, y) \in \mathbb{R}^2$, by

$$\mathbf{F}(x, y) = \langle x^2 \cos(x) - y\cos(xy) + 2x\sin(x), -4y^3 - x\cos(xy)\rangle$$

and C the curve parameterised by $r\colon [0, 2\pi] \to \mathbb{R}^2$ given by $\mathbf{r}(t) = \langle\cos(t), \sin(t)\rangle$, for $t \in [0, 2\pi]$

(b) $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined, for $(x, y) \in \mathbb{R}^2$, by

$$\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle = \langle x^2 y\cos(x) + 2xy\sin(x), x^2 \sin(x)\rangle$$

and C the curve parameterised by $r\colon [0, 2\pi] \to \mathbb{R}^2$ given by $\mathbf{r}(t) = \langle 3\cos(t), 2\sin(t)\rangle$, for $t \in [0, 2\pi]$

(c) $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined, for $(x, y) \in \mathbb{R}^2$, by

$$\mathbf{F}(x, y) = \langle\cos(x)\cos(y), -\sin(x)\sin(y) + x\rangle,$$

C the border of the unit square and $\mathbf{r}$ a piecewise linear parameterisation of C which traverses C once counter clockwise

(d) $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined, for $(x, y) \in \mathbb{R}^2$, by

$$\mathbf{F}(x, y) = \langle (x^2 - y)y\cos(xy) + 2x\sin(xy), (x^2 - y)x\cos(xy) + xy - \sin(xy)\rangle,$$

C the border of the unit square and $\mathbf{r}$ a piecewise linear parameterisation of C which traverses C once counter clockwise

(5) Let C denote a piecewise smooth simply closed curve in the plane, let $\mathbf{r}$ denote a parameterisation of C with positive orientation. Also, assume that the planar region D enclosed by C is regular. Further, assume that C does not pass through the origin and let $\mathbf{F}\colon \mathbb{R}^2 \setminus \{(0, 1)\} \to \mathbb{R}^2$ be defined by

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(a) Prove, if $(0,0) \notin D$, then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$

(b) Prove, if $(0,0) \in D$, then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi.$$

(6) Using Green's Theorem, determine if the following vector fields are conservative or not conservative.

(a) $\mathbf{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x,y) = (x^2 - yx)\mathbf{i} + (y^2 - xy)\mathbf{j}$

(b) $\mathbf{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x,y) = (2xe^{xy} + x^2ye^{xy})\mathbf{i} + (x^3e^{xy} + 2y)\mathbf{j}$

(7) Let D denote a closed regular region in the plane whose boundary C consists of one or more piecewise smooth simple closed curves, see for instance see Figure D.11. Let $\mathbf{n}$ denote the unit outward (from D) normal field on C. If $\mathbf{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector field on D, then show that

$$\iint_D \mathrm{div}(\mathbf{F})\,\mathrm{d}A = \oint_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s.$$

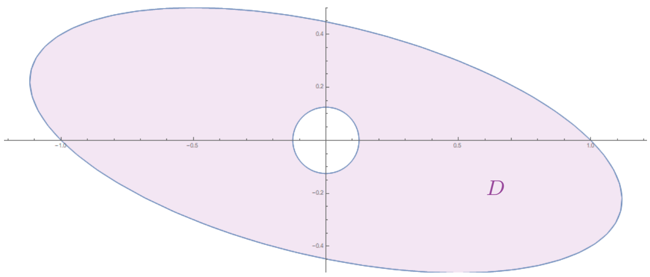

Figure C.1: An illustration of a domain D in the plane whose boundary consists of two piecewise smooth simple closed curves.

(8) Let D denote planar region with boundary curve C, let $\mathbf{n}$ denote the unit outward (from D) normal field on C and let Ω denote an open subset of $\mathbb{R}^2$ containing D. Assuming that D and C satisfy the conditions of Green's Theorem, and letting f and $g \colon \Omega \to \mathbb{R}$ denote two C^2 functions, use Green's Theorem to prove

(a) **Green's first identity**, namely that

$$\iint_D f \cdot \Delta(g)\,\mathrm{d}A = \oint_C (f \cdot \nabla(g)) \cdot \mathbf{n}\,\mathrm{d}s - \iint_D \nabla(f) \cdot \nabla(g)\,\mathrm{d}A,$$

(b) **Green's second identity**, namely that

$$\oint_C (f \cdot \nabla(g) - g \cdot \nabla(f)) \cdot \mathbf{n}\,\mathrm{d}s = \iint_D f \cdot \Delta(g) - g\Delta(f)\,\mathrm{d}A.$$

Recall that Δ denotes the Laplacian, namely $\Delta = \nabla \cdot \nabla$.

(9) The *streamlines* of a smooth planar fluid flow are the smooth curves traced by the fluid's individual particles. The vectors of the flow's velocity field are the tangent vectors of the streamlines. Suppose that the smooth planar flow takes place over a non-empty open simply connected region $R \subseteq \mathbb{R}^2$, the flow's velocity field $\mathbf{F}$ is smooth throughout R, and the divergence of $\mathbf{F}$ is non-zero throughout R. Under these assumptions, prove that none of the streamlines of the smooth planar flow in R is closed.

(10) Assume the hypotheses of Question 8. A C^2 function $g \colon D \to \mathbb{R}$ is called **harmonic** on D if it satisfies Laplace's equation, namely $\Delta(g)(x,y) = 0$ for all $(x,y) \in D$. Use Green's first identity to prove the following two statements.

(a) If the directional derivative $D_{\mathbf{n}}(g)$ exists and if g is harmonic on D, then

$$\oint_C D_{\mathbf{n}}(g)\,\mathrm{d}s = 0.$$

(b) If g is harmonic on D, and if $g(x,y) = 0$ on C, then

$$\iint_D |\nabla(g)|^2\,\mathbf{d}A = 0.$$

(11) Consider the following statement.

The vector field $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\mathbf{F}(x,y) = x\mathbf{i} - xy\mathbf{j}$, is conservative.

State if the following justification for the above statement for the above statement is complete and correct; and provide a brief explanation for your answer.

Let a and b be positive real numbers, and consider the ellipse $C_{a,b} \subseteq \mathbb{R}^2$ with parameterisation $\mathbf{r_{a,b}}\colon [0,2\pi] \to \mathbb{R}^2$ given by $\mathbf{r_{a,b}}(t) = a\sin(t)\mathbf{i} + b\cos(t)\mathbf{j}$ for $t \in [0,2\pi]$. By definition and integration by substitution,

$$\begin{aligned}
\oint_{C_{a,b}} \mathbf{F}\cdot \mathrm{d}\mathbf{r_{a,b}} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r_{a,b}}(t)) \cdot \mathbf{r_{a,b}}'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \langle a\sin(t), -ab\sin(t)\cos(t)\rangle \cdot \langle a\cos(t), -b\sin(t)\rangle\,\mathrm{d}t \\
&= \int_0^{2\pi} a^2\sin(t)\cos(t) + ab^2\sin^2(t)\cos(t)\,\mathrm{d}t \\
&= \frac{a^2\sin^2(t)}{2} + \frac{ab^2\sin^3(t)}{3}\Bigg|_{t=0}^{t=2\pi} = 0.
\end{aligned}$$

Therefore, since the line integral of $\mathbf{F}$ over any ellipse is equal to zero, we have that $\mathbf{F}$ is conservative.

(12) Assuming that all the necessary derivatives exist and are continuous, prove the following statement. If $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies Laplace's equation $\Delta(f) = 0$, then

$$\oint_C \frac{\partial f}{\partial y}\,\mathrm{d}x - \frac{\partial f}{\partial x}\,\mathrm{d}y = 0$$

for all smooth simple closed curves $C \subseteq \mathbb{R}^2$ that are positively orientated.

Chapter D

Formative assessment solutions

D.1 Formative assessment – Week 13 – Solutions

(1) Using vector operations, find the area of the triangle $\triangle ABC$ where $A = (1, 0, 1)$, $B = (2, 1, 2)$ and $C = (1, 1, 1)$.

Solution.

The area of the triangle is given by half the area of the parallelogram determined by the vectors $\overrightarrow{\mathbf{AB}} = \langle 2-1, 1-0, 2-1\rangle = \langle 1, 1, 1\rangle$ and $\overrightarrow{\mathbf{AC}} = \langle 1-1, 1-0, 1-1\rangle = \langle 0, 1, 0\rangle$. Thus,

$$\text{Area of} \triangle\, ABC = \frac{1}{2}\,|\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}| = \frac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}\right| = \frac{1}{2}\,|{-\mathbf{i}} + \mathbf{k}| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

(3) Consider a point particle moving in the x-y plane via the rule

$$\mathbf{r}(t) = a\cos(bt)\mathbf{i} + b\sin(bt)\mathbf{j},$$

where a and b are some fixed constants. Compute the velocity and the acceleration of the particle.

Solution.

The vector $\mathbf{r}(t)$ represents the position of the particle at time t, and so the velocity and the acceleration of the particle at time t are respectively given by the derivative $\mathbf{r}'$ of $\mathbf{r}$ with respect to t and the second derivative $\mathbf{r}''$ of $\mathbf{r}$ with respect to t. Thus, the velocity of the particle at time t is given by $\mathbf{r}'(t) = -ab\sin(bt)\mathbf{i} + b^2\cos(bt)\mathbf{j}$ and the acceleration of the particle at time t is given by $\mathbf{r}''(t) = -ab^2\cos(bt)\mathbf{i} - b^3\sin(bt)\mathbf{j}$.

(5) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Compute the second order partial derivatives $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Solution.

For ease of notation, let $F(x, y) = f_x(x, y)$ and $G(x, y) = f_y(x, y)$, and note, by the quotient rule, provided that $(x, y) \neq (0, 0)$,

$$F(x, y) = \frac{x^4y + 4x^2y^3 - y^5}{x^4 + 2x^2y^2 + y^4} \quad \text{and} \quad G(x, y) = \frac{x^5 - 4x^3y^2 - xy^4}{x^4 + 2x^2y^2 + y^4}.$$

By definition, $f(h,0) = 0 = f(0,h)$, for all $h \in \mathbb{R}$, and so

$$F(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0 \quad \text{and} \quad G(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0.$$

Combining the above, we conclude that

$$F(x,y) = \begin{cases} \dfrac{x^4y + 4x^2y^3 - y^5}{x^4 + 2x^2y^2 + y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G(x,y) = \begin{cases} \dfrac{x^5 - 4x^3y^2 - xy^4}{x^4 + 2x^2y^2 + y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $F(0,h) = -h$ and $G(h,0) = h$, and so by definition, we have the following.

$$\frac{\partial^2}{\partial x \partial y} f(0,0) = \frac{\partial}{\partial x} G(0,0) = \lim_{h\to\infty} \frac{G(h,0) - G(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

$$\frac{\partial^2}{\partial y \partial x} f(0,0) = \frac{\partial}{\partial y} F(0,0) = \lim_{h\to\infty} \frac{F(0,h) - F(0,0)}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} -1 = -1$$

(6) Find a parametrisation for the following curves.

(a) The curve arising from the intersection of the cylinder $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 = 2x\}$ and the hemisphere $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$.

(b) The curve arising from the intersection of the cone $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 \text{ and } z \geq 0\}$ and the sphere $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$.

(c) The curve arising from the intersection of the cylinder $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 = x\}$ and the plane $\{(x,y,z) \in \mathbb{R}^3 \colon y = z\}$.

(d) The curve arising from the intersection of the sphere $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ and the plane $\{(x,y,z) \in \mathbb{R}^3 \colon y = z\}$.

(e) The curve arising from the intersection of the cone $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 \text{ and } z \geq 0\}$ and the plane $\{(x,y,z) \in \mathbb{R}^3 \colon 3z = -x + 4\}$.

Solution.

(a) Since $x^2 + y^2 = 2x$ and $-x^2 - y^2 + 1 = z^2$ we have that $z^2 = 1 - 2x$. Further, by completing the square we have that the equality $x^2 + y^2 = 2x$ is equivalent to $(x-1)^2 + y^2 = 1$. Thus, the x-y co-ordinates of the curve in question lie on a circle with radius 1 and centre at $(1,0)$. Combining the above, a parametrisation $\mathbf{r}\colon [2\pi/3, 4\pi/3] \to \mathbb{R}^3$ is given by

$$\mathbf{r}(t) = \langle \cos(t) + 1, \sin(t), \sqrt{1 - 2(\cos(t) + 1)} \rangle,$$

for $[2\pi/3, 4\pi/3]$. The restriction on the domain arises as we require $0 \leq \cos(t) + 1 \leq 1/2$, also we take the positive square root in the z co-ordinate, since, by our hypothesis, $z \geq 0$. See Figure D.1 for a graphical representation.

(b) Since $x^2 + y^2 = z^2$ and $x^2 + y^2 = 1 - z^2$, we have that $z^2 = 1 - z^2$ and so, as $z \geq 0$, we have that $z = 1/\sqrt{2}$. Thus, the equality $x^2 + y^2 = z^2$ becomes $x^2 + y^2 = 1/2$. In other words, the x-y co-ordinates of the curve in question lie on a circle with radius $1/\sqrt{2}$ and center at $(0,0)$. Combining the above, a parametrisation $\mathbf{r}\colon [-\pi, \pi] \to \mathbb{R}^3$ is given by

$$\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{2}} \cos(t), \frac{1}{\sqrt{2}} \sin(t), \frac{1}{\sqrt{2}} \right\rangle,$$

for $[-\pi, \pi]$. See Figure D.2 for a graphical representation.

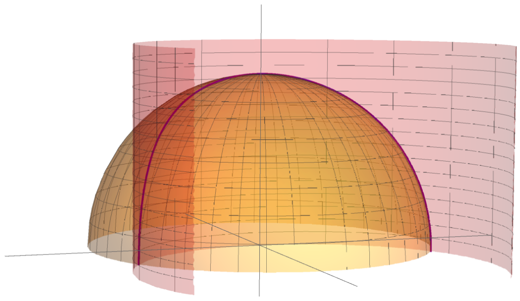

FIGURE D.1: Graph of the cylinder $K = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 2x\}$, the hemisphere $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$ and the curve $C = K \cap S$.

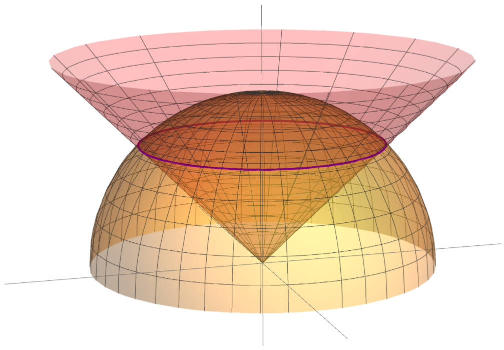

FIGURE D.2: Graph of the cone $K = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 \text{ and } z \geq 0\}$, the sphere $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ and the curve $C = K \cap S$.

(c) By completing the square the equality $x^2 + y^2 = x$ becomes $(x - 1/2)^2 + y^2 = 1/4$. In other words, the x-y co-ordinates of the curve in question lie on a circle with radius $1/2$ and center at $(1/2, 0)$. Further, as $y = z$, the z co-ordinate of our parameterisation will be the same as the y co-ordinate. Combining the above, a parametrisation $\mathbf{r} \colon [-\pi, \pi] \to \mathbb{R}^3$ is given by

$$\mathbf{r}(t) = \left\langle \frac{1}{2}(\cos(t) + 1), \frac{1}{2}\sin(t), \frac{1}{2}\sin(t) \right\rangle,$$

for $[-\pi, \pi]$. See FIGURE D.3 for a graphical representation.

(d) Since $x^2 + y^2 + z^2 = 1$ and $y = z$, we have that $x^2 + 2y^2 = 1$. In other words, the x-y co-ordinates of the curve in question lie on an ellipse with major axis along the x-axis of length 2, minor axis along the y-axis of length $\sqrt{2}$ and center at $(0, 0)$. Further, as $y = z$, the z co-ordinate of our parameterisation will be the same as the y co-ordinate. Combining the above, we may conclude that a parametrisation $\mathbf{r} \colon [-\pi, \pi] \to \mathbb{R}^3$ would be given by

$$\mathbf{r}(t) = \left\langle \cos(t), \frac{1}{\sqrt{2}}\sin(t), \frac{1}{\sqrt{2}}\sin(t) \right\rangle,$$

for $[-\pi, \pi]$. See FIGURE D.4 for a graphical representation.

(e) Since $x^2 + y^2 = z^2$ and $3z = -x + 4$. we have that

$$x^2 + y^2 = \frac{(-x + 4)^2}{9}.$$

When expanding and completing the square, this equality is equivalent to

$$\frac{4}{9}\left(x + \frac{1}{2}\right)^2 + \frac{1}{2}y^2 = 1.$$

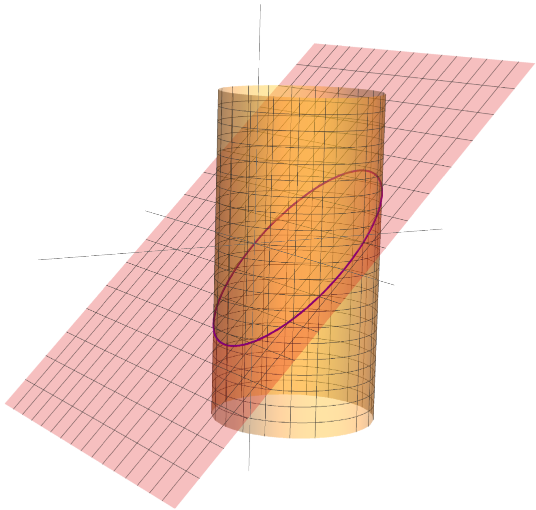

FIGURE D.3: Graph of the cylinder $K = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = x\}$, the plane $P = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$ and the curve $C = K \cap P$.

In other words, the x-y co-ordinates of the curve in question lie on an ellipse with major axis along the x-axis of length 3, minor axis along the y-axis of length $2\sqrt{2}$ and centre at $(-1/2, 0)$. Combining the above, a parametrisation $\mathbf{r} : [-\pi, \pi] \to \mathbb{R}^3$ is given by

$$\mathbf{r}(t) = \left\langle \frac{3}{2}\cos(t) - \frac{1}{2}, \sqrt{2}\sin(t), -\frac{1}{2}\cos(t) + \frac{3}{2} \right\rangle,$$

for $[-\pi, \pi]$. Here, to obtain the z co-ordinate we have used the fact that $z = (-x+4)/3$. See FIGURE D.5 for a graphical representation.

(7) Compute the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F}$ and $\mathbf{r}$ are as follows.

(a) $\mathbf{F}(x, y) = \langle x^2 y, y^2 x \rangle$ and $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [-\pi, \pi]$
(b) $\mathbf{F}(x, y) = \langle \cos(x), \sin(x) \rangle$ and $\mathbf{r}(\mathbf{t}) = \langle t, t \rangle$ for $t \in [0, 1]$
(c) $\mathbf{F}(x, y) = \langle xy - z, y^2 xz, xyz \rangle$ and $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ for $t \in [0, 2\pi]$

Solution.

(a) The given parameterisation traverses it image exactly once and that the component functions of $\mathbf{F}(x, y) = \langle x^2 y, y^2 x \rangle$ and $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$ are continuous. Thus.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-\pi}^{\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_{-\pi}^{\pi} \langle \cos^2(t)\sin(t), \sin^2(t)\cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle\, dt \\ &= \int_{-\pi}^{\pi} -\cos^2(t)\sin^2(t) + \sin^2(t)\cos^2(t)\, dt = \int_{-\pi}^{\pi} 0\, dt = 0.\end{aligned}$$

(b) The given parameterisation traverses it image exactly once and that the component functions of $\mathbf{F}(x, y) = \langle \cos(x), \sin(x) \rangle$ and $\mathbf{r}'(t) = \langle 1, 1 \rangle$ are continuous. Thus,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt &= \int_0^1 \langle \cos(t), \sin(t) \rangle \cdot \langle 1, 1 \rangle\, dt \\ &= \int_0^1 \cos(t) + \sin(t)\, dt = \sin(t) - \cos(t)|_0^1 = \sin(1) - \cos(1) + 1.\end{aligned}$$

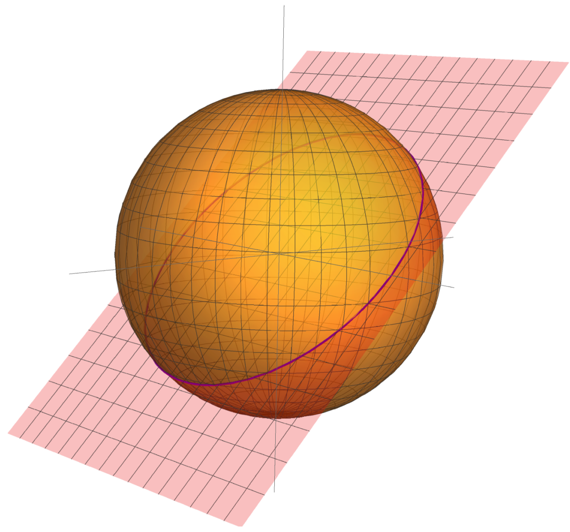

Figure D.4: Graph of the sphere $K = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, the plane $P = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$ and the curve $C = K \cap P$.

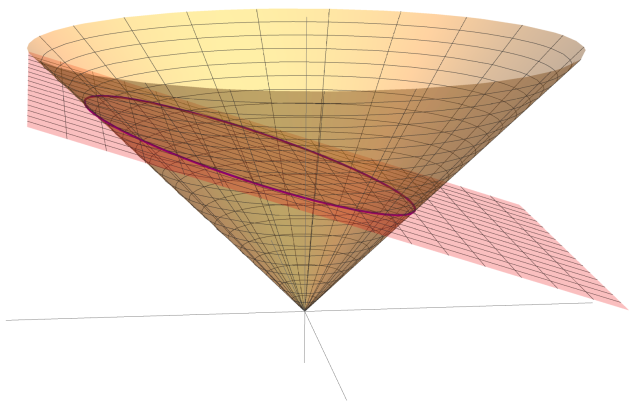

Figure D.5: Graph of the cone $K = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 \text{ and } z \geq 0\}$, the plane $P = \{(x, y, z) \in \mathbb{R}^3 : 3z = -x + 4\}$ and the curve $C = K \cap P$.

(c) The given parameterisation traverses it image exactly once and that the component functions of $\mathbf{F}(x, y) = \langle xy - z, y^2xz, xyz\rangle$ and $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$ are continuous. Thus,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^{2pi} \langle 0, t^8, t^6\rangle \cdot \langle 1, 2t, 3t^2\rangle\,\mathrm{d}t$$
$$= \int_0^{2\pi} 2t^9 + 3t^8\,\mathrm{d}t = \frac{1}{5}t^{10} + \frac{1}{3}t^9\Big|_0^{2\pi} = \frac{1024\pi^{10}}{5} + \frac{512\pi^9}{3}.$$

(9) Calculate the work done by the force field $\mathbf{F}(x, y, z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ along the curve C which lies at the intersection of the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 2x\}$ and the half-space $\{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ and which is traversed in a direction that appears clockwise when viewed from high above the x-y plane.

Solution.

Here we want to compute the value of the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathbf{r}$ is a path which traverses C in a direction that appears clockwise when viewed from high above the x-y plane. To this end we first find a parameterisation of the C and second, using this, compute the value of the line integral given above.

By completing the square, we have that the equality $x^2 + y^2 = 2x$ is equivalent to $(x-1)^2 + y^2 = 1$. In other words, the x-y co-ordinates of the curve in question lie on a circle with radius 1 and center at $(1, 0)$. Further, since $z^2 = 1 - x^2 - y^2$ and $x^2 + y^2 = 2x$, we have that $z^2 = 1 - 2x$, and thus, as $z > 0$, it follows that $z = \sqrt{1-2x}$, in particular $x < 1/2$. Combining the above, a smooth parametrisation that traverses C exactly once is given by

$$\mathbf{r}(t) = \left\langle \cos(t) + 1, \sin(t), \sqrt{1 - 2(\cos(t) + 1)} \right\rangle$$

with domain $[2\pi/3, 4\pi/3]$. However, note that this parameterisation gives a counter clockwise motion when the curve C is viewed from high above the x-y plane. Therefore, to compute the work done, we wish to solve

$$W = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_{2\pi/3}^{4\pi/3} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t.$$

A direct calculation shows that

$$\mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \frac{\sin(t)}{\sqrt{-1 - 2\cos(t)}}\mathbf{k}.$$

The $\mathbf{k}$ component looks rather tricky to integrate, and so let us try to modify the force field without changing the value of the line integral by subtracting a gradient vector. Indeed, recall, if f is a scalar field such that $\nabla(f)$ exists, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\mathbf{F} - \nabla(f)) \cdot \mathrm{d}\mathbf{r} + (f(\mathbf{r}(4\pi/3)) - f(\mathbf{r}(2\pi/3))).$$

Setting $f(x, y, z) = x^2 z$, we have $(\mathbf{F} - \nabla(f))(x, y, z) = (y^2 - 2xz)\mathbf{i} + z^2\mathbf{j}$ and

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\mathbf{F} - \nabla(f)) \cdot \mathrm{d}\mathbf{r} + (f(\mathbf{r}(4\pi/3)) - f(\mathbf{r}(2\pi/3))) = \int_C (\mathbf{F} - \nabla(f)) \cdot \mathrm{d}\mathbf{r}.$$

In the last equality we have used that the $\mathbf{k}$ component of $\mathbf{r}(2\pi/3)$ and $\mathbf{r}(4\pi/3)$ is equal to zero and so $f(\mathbf{r}(4\pi/3)) = f(\mathbf{r}(2\pi/3)) = 0$. Combining the above yields,

$$\begin{aligned}
&\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\mathbf{F} - \nabla(f)) \cdot \mathrm{d}\mathbf{r} = \int_{2\pi/3}^{4\pi/3} (\mathbf{F} - \nabla(f)) \cdot \mathbf{r}'(t)\, \mathrm{d}t \\
&= \int_{2\pi/3}^{4\pi/3} \left(\left(\sin^2(t) - 2(\cos(x) + 1)\sqrt{-1 - 2\cos(x)} \right) \mathbf{i} + (-1 - 2\cos(t))\, \mathbf{j} \right) \\
&\qquad \cdot \left(-\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \frac{\sin(t)}{\sqrt{-1 - 2\cos(t)}}\mathbf{k} \right) \mathrm{d}t \\
&= \int_{2\pi/3}^{4\pi/3} -\left(\sin^2(t) - 2(\cos(x) + 1)\sqrt{-2\cos(t) - 1} \right) \sin(t) - (2\cos(t) + 1)\cos(t)\, \mathrm{d}t \\
&= \int_{2\pi/3}^{4\pi/3} -\left(1 - \cos^2(t) - 2(\cos(t) + 1)\sqrt{-2\cos(t) - 1} \right) \sin(t) - 1 - \cos(2t) - \cos(t)\, \mathrm{d}t \\
&= \cos(t) - \frac{\cos^3(t)}{3} - \frac{2}{5}(-2\cos(t) - 1)^{5/2} + \frac{1}{3}(-2\cos(t) - 1)^{3/2} - t - \frac{1}{2}\sin(2t) - \sin(t) \Big|_{2\pi/3}^{4\pi/3} = -\frac{2\pi}{3} + \sqrt{3},
\end{aligned}$$

and so $W = 2\pi/3 - \sqrt{3}$.

(11) Suppose that the position of a particle at time t is given, for $t \in [0, 2\pi]$, by

$$x_1(t) = 3\sin(t) \quad \text{and} \quad y_1(t) = 2\cos(t)$$

and the position of a second particle is given, for $t \in [0, 2\pi]$, by

$$x_2(t) = -3 + \cos(t) \quad \text{and} \quad y_2(t) = 1 + \sin(t).$$

(a) Graph the paths of both particles. How many points of intersection are there?
(b) Are any of these points of intersection collision points? In other words, are the particles ever at the same place at the same time? If so, find the collision points.
(c) Describe what happens if the path of the second particle is given, for $t \in [0, 2\pi]$, by

$$x_2(t) = 3 + \cos(t) \quad \text{and} \quad y_2(t) = 1 + \sin(t).$$

Solution.

(a) From the figure given below (Figure D.6), we see that there are two intersection points, one at $(-3, 0)$ and the other near to $(-2.1, 1.4)$.
(b) A collision point corresponds to t such that $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$, that is when $3\sin(t) = -3 + \cos(t)$ and $2\cos(t) = 1 + \sin(t)$. From the first equality we obtain that $\cos(t) = 3 + 3\sin(t)$, and substituting this into the second equation we obtain $5 + 5\sin(t) = 0$, or equivalently, $\sin(t) = -1$, which corresponds to $t = 3\pi/2$. Thus, the intersection point $(-3, 0)$ is a collision point.

(c) The image of the new second path is a circle centred at $(3, 1)$ with radius 1. Thus, as before there are two points of intersection, one at $(3, 0)$ and the other near to $(2.1, 1.4)$. However, in this case, there are no collision points, since the equations $3\sin(t) = 3 + \cos(t)$ and $2\cos(t) = 1 + \sin(t)$ imply that $\sin(t) = 7/5$ and $\cos(t) = 6/5$, for which no solution exists.

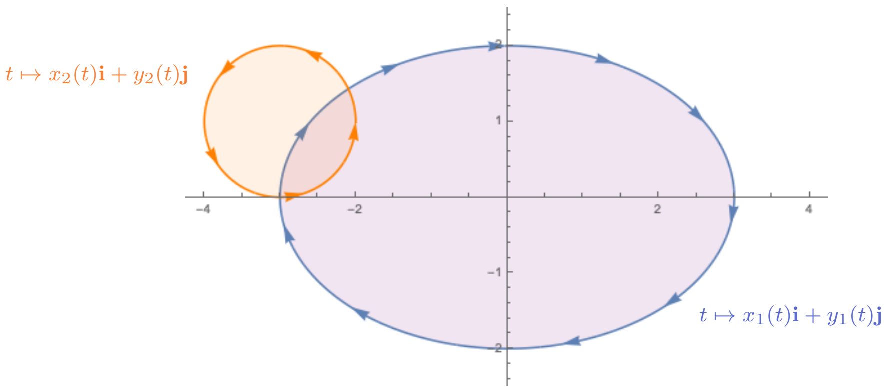

Figure D.6: Graphical representation of the parameterisations $t \mapsto x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $t \mapsto x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$.

D.2 Formative assessment – Week 14 – Solutions

(1) Use Theorem 9.2.2 to find a potential for the following vector fields.

(a) $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x, y) = \langle x^2 \cos(x) - y\cos(xy) + 2x\sin(x), -4y^3 - x\cos(xy)\rangle$

(b) $\mathbf{L} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{L}(x, y) = \langle xy^2 \cos(xy) + y\sin(xy), x^2 y\cos(xy) + x\sin(xy)\rangle$

(c) $\mathbf{M} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{M}(x, y) = \langle x\cos(x) + ye^{xy} + \sin(x), xe^{xy} + 2y\sin(y^2)\rangle$

(d) $\mathbf{P} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{P}(x, y) = \langle 3x^2 y - y^3 - y\sin(xy), x^3 - 3xy^2 - x\sin(xy)\rangle$

Solution.

Theorem 9.2.2 states the following. If $n \geq 2$ is a natural number, $\Omega \subseteq \mathbb{R}^n$ is a star shaped set with respect to $x_0 \in \Omega$ and $\mathbf{F} = \langle F_1, F_2, \ldots, F_n\rangle : \Omega \to \mathbb{R}^n$ is a C^1 vector field with the property that

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_i}{\partial xj}$$

for all i and $j \in \{1, 2, \ldots, n\}$, then $\mathbf{F}$ is conservative. Further, a potential $f : \Omega \to \mathbb{R}$ of $\mathbf{F}$ is given by

$$f(x) = \int_0^1 \mathbf{F}((1-t)\mathbf{x}_0 + t\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)\,\mathrm{d}t,$$

for $x \in \Omega$ and where $\mathbf{x}$ and $\mathbf{x}_0$ denote the vectors with initial point the origin and terminal point x and x_0 respectively. In the following, since $\Omega = \mathbb{R}^2$, which is star shaped with respect to the origin, we set $x_0 = (0, 0)$.

(a) Let F_1 and $F_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F_1(x, y) = x^2 \cos(x) - y\cos(xy) + 2x\sin(x) \quad \text{and} \quad F_2(x, y) = -4y^3 - x\cos(xy),$$

for $(x, y) \in \mathbb{R}^2$. Since F_1 and F_2 are sums of products and compositions of polynomials and trigonometric functions, they are smooth. Further, since

$$\frac{\partial F_1}{\partial y}(x, y) = xy\sin(xy) - \cos(xy) = \frac{\partial F_2}{\partial x}(x, y),$$

we may apply the result stated above, yielding that a potential $f : \mathbb{R}^2 \to \mathbb{R}$ for $\mathbf{F}$ is given by

$$\begin{aligned} f(x, y) &= \int_0^1 \mathbf{F}(tx, ty) \cdot \langle x, y\rangle\,\mathrm{d}t \\ &= \int_0^1 \langle t^2x^2\cos(tx) - ty\cos(t^2xy) + 2tx\sin(tx), -4t^3y^3 - tx\cos(t^2xy)\rangle \cdot \langle x, y\rangle\,\mathrm{d}t \\ &= \int_0^1 t^2x^3\cos(tx) + 2tx^2\sin(tx) - 4t^3y^4 - 2txy\cos(t^2xy)\,\mathrm{d}t \\ &= t^2x^2\sin(tx) - t^4y^4 - \sin(t^2xy)\big|_0^1 = x^2\sin(x) - y^4 - \sin(xy). \end{aligned}$$

(b) Let L_1 and $L_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$L_1(x, y) = xy^2\cos(xy) + y\sin(xy) \quad \text{and} \quad L_2(x, y) = x^2y\cos(xy) + x\sin(xy),$$

for $(x, y) \in \mathbb{R}^2$. Since L_1 and L_2 are sums of products and compositions of polynomials and trigonometric functions, they are smooth. Further, since

$$\frac{\partial L_1}{\partial y}(x, y) = 3xy\cos(xy) - x^2y^2\sin(xy) + \sin(xy) = \frac{\partial L_2}{\partial x}(x, y),$$

a potential $l : \mathbb{R}^2 \to \mathbb{R}$ for $\mathbf{L}$ is given, for $(x, y) \in \mathbb{R}^2$, by

$$\begin{aligned} l(x, y) &= \int_0^1 \mathbf{L}(tx, ty) \cdot \langle x, y\rangle\,\mathrm{d}t \\ &= \int_0^1 \langle t^3xy^2\cos(t^2xy) + ty\sin(t^2xy), t^3x^2y\cos(t^2xy) + tx\sin(t^2xy)\rangle \cdot \langle x, y\rangle\,\mathrm{d}t \\ &= \int_0^1 2t^3x^2y^2\cos(t^2xy) + 2txy\sin(t^2xy)\,\mathrm{d}t \\ &= t^2xy\sin(t^2xy)\big|_0^1 = xy\sin(xy). \end{aligned}$$

(c) Let M_1 and $M_2\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$M_1(x, y) = x\cos(x) + ye^{xy} + \sin(x) \quad \text{and} \quad M_2(x, y) = xe^{xy} + 2y\sin(y^2),$$

for $(x, y) \in \mathbb{R}^2$. Since M_1 and M_2 are sums of products and compositions of polynomials, and trigonometric and exponential functions, they are smooth. Further, since

$$\frac{\partial M_1}{\partial y}(x, y) = e^{xy} + xye^{xy} = \frac{\partial M_2}{\partial x}(x, y),$$

a potential $m\colon \mathbb{R}^2 \to \mathbb{R}$ for $\mathbf{M}$ is given, for $(x, y) \in \mathbb{R}^2$, by

$$\begin{aligned}
m(x, y) &= \int_0^1 \mathbf{M}(tx, ty) \cdot \langle x, y\rangle \, \mathrm{d}t \\
&= \int_0^1 \langle tx\cos(tx) + tye^{t^2xy} + \sin(tx), txe^{t^2xy} + 2ty\sin(t^2y^2)\rangle \cdot \langle x, y\rangle \, \mathrm{d}t \\
&= \int_0^1 tx^2\cos(tx) + x\sin(tx) + 2txye^{t^2xy} + 2ty^2\sin(t^2y^2) \, \mathrm{d}t \\
&= tx\sin(tx) + e^{t^2xy} - \cos(t^2y^2)\big|_0^1 \\
&= x\sin(x) + e^{xy} - \cos(y^2).
\end{aligned}$$

(d) Let P_1 and $P_2\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$P_1(x, y) = 3x^2y - y^3 - y\sin(xy) \quad \text{and} \quad P_2(x, y) = x^3 - 3xy^2 - x\sin(xy),$$

for $(x, y) \in \mathbb{R}^2$. Since P_1 and P_2 are sums of products and compositions of polynomials and trigonometric functions, they are both smooth. Further, since

$$\frac{\partial P_1}{\partial y}(x, y) = 3x^2 - 3y^2 - \sin(xy) - xy\cos(xy) = \frac{\partial P_2}{\partial x}(x, y),$$

a potential $p\colon \mathbb{R}^2 \to \mathbb{R}$ for $\mathbf{P}$ is given, for $(x, y) \in \mathbb{R}^2$, by

$$\begin{aligned}
p(x, y) &= \int_0^1 \mathbf{P}(tx, ty) \cdot \langle x, y\rangle \, \mathrm{d}t \\
&= \int_0^1 \langle 3t^3x^2y - t^3y^3 - ty\sin(t^2xy), t^3x^3 - 3t^3xy^2 - tx\sin(t^2xy)\rangle \cdot \langle x, y\rangle \, \mathrm{d}t \\
&= \int_0^1 4t^3x^3y - 2txy\sin(t^2xy) - 4t^3xy^3 \, \mathrm{d}t \\
&= t^4x^3y + \cos(t^2xy) - t^4xy^3\big|_0^1 = x^3y + \cos(xy) - xy^3 - 1.
\end{aligned}$$

(3) Use Green's Theorem to find the area enclosed by the curves determined by following equations.

(a) $x^2 + 2xy + 5y^2 = 1$

(b) $x^{2/3} + y^{2/3} = 1$

(c) $x(t) = 3\cos(t) - \cos(3t)$ and $y(t) = 3\sin(t) - \sin(3t)$, where $t \in [0, 2\pi]$

Solution.

(a) Let us first plot the given curve and shade the region of interest, see Figure D.7. From this plot we see that the curve C is simple and that the planar region D that it encompasses is regular.

Letting $\mathbf{r}$ denote a smooth parameterisation of the ellipse C so that C is traversed once in a counter clockwise direction, letting D denote the region enclosed by C, and letting $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{F}(x, y) = \langle 0, x\rangle$ for $(x, y) \in \mathbb{R}^2$, Green's Theorem yields

$$\text{Area of } D = \iint_D 1 \, \mathrm{d}x\mathrm{d}y = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, \mathrm{d}x\mathrm{d}y = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

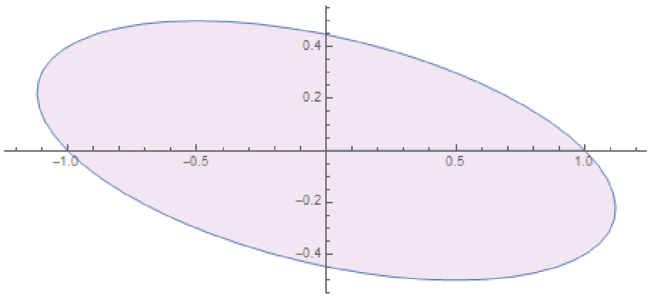

FIGURE D.7: Graph of the curve C determined by the equation $x^2 + 2xy + 5y^2 = 1$.

To this end, note that the equation $x^2 + 2xy + 5y^2 = 1$ is equivalent to $(x + y)^2 + 4y^2 = 1$. Since $\cos^2(t) + \sin^2(t) = 1$, setting $\cos(t) = x + y$ and $\sin(t) = 2y$, we obtain that a parameterisation $\mathbf{r}\colon [0, 2\pi]$ of C is given by $\mathbf{r}(t) = \langle \cos(t) - \sin(t)/2, \sin(t)/2\rangle$ for $t \in [0, 2\pi]$. This parameterisation traverses C once in a counter clockwise direction and is smooth. Therefore,

$$
\begin{aligned}
\text{Area of } D = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \left\langle 0, \cos(t) - \frac{\sin(t)}{2} \right\rangle \cdot \left\langle -\sin(t) - \frac{\cos(t)}{2}, \frac{\cos(t)}{2} \right\rangle \mathrm{d}t \\
&= \int_0^{2\pi} \frac{\cos^2(t)}{2} - \frac{\sin(t)\cos(t)}{4}\,\mathrm{d}t \\
&= \int_0^{2\pi} \frac{1 + \cos(2t)}{4} - \frac{\sin(2t)}{8}\,\mathrm{d}t \\
&= \frac{t}{4} + \frac{\sin(2t)}{8} + \frac{\cos(2t)}{16}\bigg|_0^{2\pi} = \frac{\pi}{2}.
\end{aligned}
$$

(b) Let us first plot the given curve and shade the region of interest, see FIGURE D.8. From this plot we see that the curve C is simple and that the planar region D that it encompasses is regular.

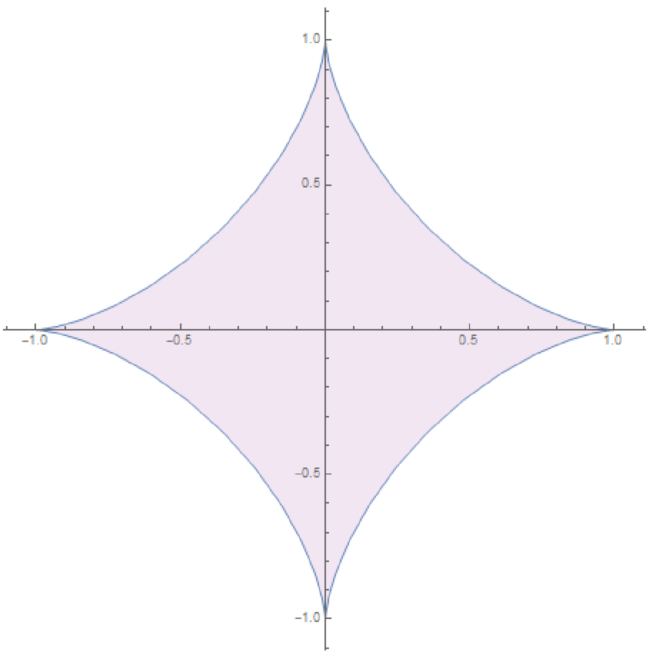

FIGURE D.8: Graph of the curve C determined by the equation $x^{2/3} + y^{2/3} = 1$.

Letting $\mathbf{r}$ denote a piecewise smooth parameterisation of the star shaped curve C so that C is traversed once in a counter clockwise direction, letting D denote the region enclosed by C, and letting $\mathbf{F}\colon \mathbb{R}^1 \to \mathbb{R}$ be defined by $\mathbf{F}(x, y) = \langle 0, x\rangle$ for $(x, y) \in \mathbb{R}^2$, Green's Theorem yields

$$\text{Aread of } D = \iint_D 1 \, \mathrm{d}x\mathrm{d}y = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, \mathrm{d}x\mathrm{d}y = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Since $\cos^2(t) + \sin^2(t) = 1$, setting $x = \cos^3(t)$ and $y = \sin^3(t)$, we obtain that a parameterisation $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}$ of C is given by $\diagdown(t) = \langle\cos^3(t), \sin^3(t)\rangle$ for $t \in [0, 2\pi]$. This parameterisation traverses C once in a counter clockwise direction and is smooth. Therefore,

$$\begin{aligned}
\text{Area of } D &= \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t \\
&= \int_0^{2\pi} \langle 0, \cos^3(t)\rangle \cdot \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)\rangle \, \mathrm{d}t \\
&= 3\int_0^{2\pi} \cos^4(t)\sin^2(t) \, \mathrm{d}t \\
&= 3\int_0^{2\pi} \cos^2(t)(\cos(t)\sin(t))^2 \, \mathrm{d}t \\
&= 3\int_0^{2\pi} \frac{1 + \cos(2t)}{2} \frac{\sin^2(2t)}{4} \, \mathrm{d}t \\
&= 3\int_0^{2\pi} \frac{1}{16} - \frac{\cos(4t)}{16} + \frac{\sin^2(2t)\cos(2t)}{8} \, \mathrm{d}t \\
&= \frac{3t}{16} - \frac{3\sin(4t)}{64} + \frac{3\sin^3(2t)}{48}\bigg|_0^{2\pi} = \frac{3\pi}{8}
\end{aligned}$$

(c) Let us first plot the given curve and shade the region of interest, see FIGURE D.9 below. From this plot we see the curve C is simple and that the planar region D that it encompasses is regular.

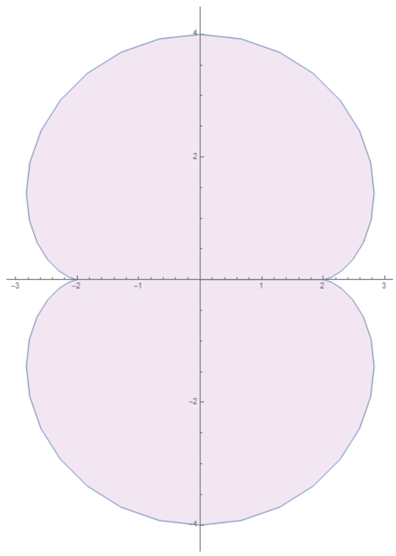

FIGURE D.9: Graph of the curve C determined by the parametric equations $t \mapsto \langle 3\cos(t) - \cos(3t), 3\sin(t) - \sin(3t)\rangle$.

Letting $\mathbf{r}$ denote a piecewise smooth parameterisation of the star shaped curve C so that C is traversed once in a counter clockwise direction, letting D denote the region enclosed by C, and letting $\mathbf{F}\colon \mathbb{R}^1 \to \mathbb{R}$ be defined by $\mathbf{F}(x, y) = \langle 0, x\rangle$ for $(x, y) \in \mathbb{R}^2$, Green's Theorem yields

$$\text{Area of } D = \iint_D 1 \, \mathrm{d}x\mathrm{d}y = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, \mathrm{d}x\mathrm{d}y = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Let $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ defined by $\mathbf{r}(t) = \langle 3\cos(t) - \cos(3t), 3\sin(t) - \sin(3t)\rangle$, for $t \in [0, 2\pi]$, denote the given parameterisation of C. This parameterisation traverses C once in a counter clockwise direction and is smooth. Therefore,

$$\begin{aligned}
\text{Area of } D &= \iint_D 1 \,\mathrm{d}x\mathrm{d}y = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \,\mathrm{d}x\mathrm{d}y \\
&= \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \,\mathrm{d}t \\
&= \int_0^{2\pi} \langle 0, 3\cos(t) - \cos(3t)\rangle \cdot \langle -3\sin(t) + 3\sin(3t), 3\cos(t) - 3\cos(3t)\rangle \,\mathrm{d}t \\
&= \int_0^{2\pi} (3\cos(t) - \cos(3t))(3\cos(t) - 3\cos(3t)) \,\mathrm{d}t \\
&= \int_0^{2\pi} 9\cos^2(t) - 12\cos(t)\cos(3t) + 3\cos^2(3t) \,\mathrm{d}t \\
&= \int_0^{2\pi} \frac{9(1 + \cos(2t))}{2} - 6\cos(4t) - 6\cos(2t) + \frac{3(1 + \cos(6t))}{2} \,\mathrm{d}t \\
&= \frac{9(2t + \sin(2t))}{4} - \frac{6\sin(4t)}{4} - 3\sin(2t) + \frac{3(6t + \sin(6t))}{12}\Big|_0^{2\pi} = 12\pi.
\end{aligned}$$

(5) Let C denote a piecewise smooth simply closed curve in the plane, let $\mathbf{r}$ denote a parameterisation of C with positive orientation. Also, assume that the planar region D enclosed by C is regular. Further, assume that C does not pass through the origin and let $\mathbf{F}\colon \mathbb{R}^2 \setminus \{(0, 1)\} \to \mathbb{R}^2$ be defined by

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(a) Prove, if $(0, 0) \notin D$, then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$

(b) Prove, if $(0, 0) \in D$, then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi.$$

Solution.

(a) Observe, by our hypothesis the conditions of Green's Theorem are satisfied. Let F_1 and $F_2\colon \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ be such that $\mathbf{F} = \langle F_1, F_2\rangle$. If $(0, 0) \notin D$, then we have that

$$\frac{\partial F_1}{\partial y}(x, y) = \frac{\partial}{\partial y} \frac{-y}{x^2 + y^2} = \frac{-(x^2 + y^2) + 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial F_2}{\partial x}(x, y)$$

and thus,

$$\frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y) = 0,$$

for all $(x, y) \in D$. Therefore, if $(0, 0) \notin D$, then by Green's Theorem,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y} \,\mathrm{d}x\mathrm{d}y = 0.$$

(b) Observe, by our hypothesis the conditions of Green's Theorem are satisfied. Suppose that $(0, 0) \in D$. Since it is assumed that the origin does not lie on the curve C, it must be an interior point of D. The interior of D is open, and so there exists an $\epsilon > 0$ such that the circle $S_\epsilon = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \epsilon^2\}$

belongs to the interior of D. Let $\mathbf{r}_\epsilon \colon [0, 2\pi] \to \mathbb{R}^2$ be defined by $\mathbf{r}_\epsilon(t) = \epsilon\cos(t)\mathbf{i} - \epsilon\sin(t)\mathbf{j}$, for $t \in [0, 2\pi]$, be a parameterisation of S_ϵ. This parameterisation traverses S_ϵ once in a clockwise direction and is smooth. Additionally, since S_ϵ is a circle centred at the origin with radius ϵ, it a simple closed curve. With this at hand, we have

$$\begin{aligned}\oint_{S_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r}_\epsilon &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}_\epsilon(t))\mathbf{r}_\epsilon'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} \left\langle \frac{\sin(t)}{\epsilon}, \frac{\cos(t)}{\epsilon} \right\rangle \cdot \langle -\epsilon\sin(t), -\epsilon\cos(t)\rangle\,\mathrm{d}t \\ &= \int_0^{2\pi} -1\,dt = -2\pi.\end{aligned}$$

Together, C and S_ϵ form a positively oriented boundary of a region R that excludes the origin, see for instance FIGURE D.10. Thus, by an identical argument used in Part (a), we have that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \oint_{S_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r}_\epsilon = 0,$$

and hence,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\oint_{S_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r}_\epsilon = 2\pi.$$

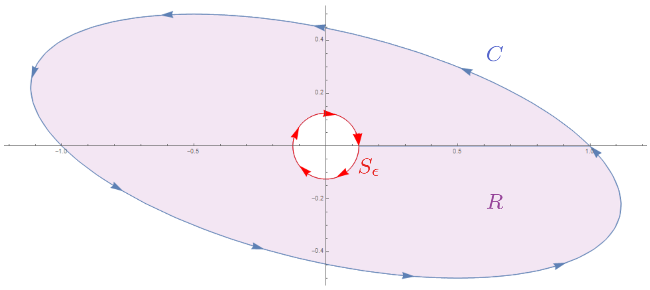

FIGURE D.10: An illustration of a curve C and S_ϵ with the orientation of their parameterisations highlighted.

(7) Let D denote a closed regular region in the plane whose boundary C consists of one or more piecewise smooth simple closed curves, see for instance see FIGURE D.11. Let $\mathbf{n}$ denote the unit outward (from D) normal field on C. If $\mathbf{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector field on D, then show that

$$\iint_D \operatorname{div}(\mathbf{F})\,\mathrm{d}A = \oint_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s.$$

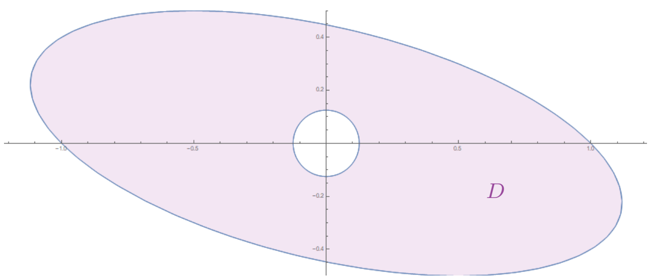

FIGURE D.11: An illustration of a domain D in the plane whose boundary consists of two piecewise smooth simple closed curves.

Solution.

Observe that $\mathbf{n} = \mathbf{T} \times \mathbf{k}$, where $\mathbf{T}$ is the unit tangent vector field of C. If T_1 and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are such that $\mathbf{T} = T_1\mathbf{i} + T_2\mathbf{j}$, then $\mathbf{n} = T_2\mathbf{i} - T_1\mathbf{j}$. Let $\mathbf{G}$ be the vector field with components $G_1 = -F_2$ and $G_2 = F_1$, in which case $\mathbf{G} \cdot \mathbf{T} = \mathbf{F} \cdot \mathbf{n}$. Since by our hypothesis, the conditions of Green's Theorem are satisfied, letting $\mathbf{r} \colon [0, 1] \to C$ denote a piecewise smooth parameterisation of C, we have

$$\begin{aligned}\iint_D \operatorname{div}(\mathbf{F})\,\mathrm{d}A &= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)\mathrm{d}A = \iint_D \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\mathrm{d}A \\ &= \oint_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \cdot \mathrm{d}t \\ &= \int_0^1 \mathbf{G}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\,\mathrm{d}t \\ &= \int_0^1 \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{T}(\mathbf{r}(t))|\mathbf{r}'(t)|\,\mathrm{d}t = \oint_C \mathbf{G} \cdot \mathbf{T}\,\mathrm{d}s = \oint_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s.\end{aligned}$$

This result is what is often referred to as the divergence theorem in the plane.

(9) The *streamlines* of a smooth planar fluid flow are the smooth curves traced by the fluid's individual particles. The vectors of the flow's velocity field are the tangent vectors of the streamlines. Suppose that the smooth planar flow takes place over a non-empty open simply connected region $R \subseteq \mathbb{R}^2$, the flow's velocity field $\mathbf{F}$ is smooth throughout R, and the divergence of $\mathbf{F}$ is non-zero throughout R. Under these assumptions, prove that none of the streamlines of the smooth planar flow in R is closed.

Solution.

Assume that a particle has a closed smooth trajectory in R, and let us denote the curve traced out by this trajectory by C. By construction the region D enclosed by C is simply connected and C is a smooth, simple closed curve.

Since the velocity vectors at a point on C are tangent to C, by the definition of a streamline, letting $\mathbf{n}$ denote the unit outward normal vector field of D, we have $\mathbf{F} \cdot \mathbf{n} = 0$, and so

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}t = 0.$$

Noting that the conditions of Green's theorem are met, we may apply the result to obtain,

$$0 = \oint_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}t = \iint_D \operatorname{div}(\mathbf{F})\,\mathrm{d}A = \iint_D \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,\mathrm{d}A,$$

where M and N are the component functions of $\mathbf{F}$, namely where $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, for all $(x, y) \in R$. On the other hand, since $\mathbf{F}$ is smooth throughout R,

$$\operatorname{div}(\mathbf{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

is continuous and, by our hypotheses, non-zero throughout R. Thus, the divergence of $\mathbf{F}$ is strictly positive or strictly negative in the region R, meaning

$$\iint_D \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,\mathrm{d}A$$

is strictly positive or strictly negative, yielding a contradiction to our assumption on the existence of C.

(11) Consider the following statement.

The vector field $\mathbf{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\mathbf{F}(x, y) = x\mathbf{i} - xy\mathbf{j}$, is conservative.

State if the following justification for the above statement for the above statement is complete and correct; and provide a brief explanation for your answer.

Let a and b be positive real numbers, and consider the ellipse $C_{a,b} \subseteq \mathbb{R}^2$ with parameterisation $\mathbf{r_{a,b}} \colon [0, 2\pi] \to \mathbb{R}^2$ given by $\mathbf{r_{a,b}}(t) = a\sin(t)\mathbf{i} + b\cos(t)\mathbf{j}$ for $t \in [0, 2\pi]$. By definition and integration by substitution,

$$\begin{aligned}
\oint_{C_{a,b}} \mathbf{F} \cdot \mathrm{d}\mathbf{r_{a,b}} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r_{a,b}}(t)) \cdot \mathbf{r_{a,b}}'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \langle a\sin(t), -ab\sin(t)\cos(t)\rangle \cdot \langle a\cos(t), -b\sin(t)\rangle\,\mathrm{d}t \\
&= \int_0^{2\pi} a^2\sin(t)\cos(t) + ab^2\sin^2(t)\cos(t)\,\mathrm{d}t \\
&= \frac{a^2\sin^2(t)}{2} + \frac{ab^2\sin^3(t)}{3}\bigg|_{t=0}^{t=2\pi} = 0.
\end{aligned}$$

Therefore, since the line integral of $\mathbf{F}$ over any ellipse is equal to zero, we have that $\mathbf{F}$ is conservative.

Solution.

This justification is incomplete and incorrect as it only considers curves which are ellipses centred at the origin. To show that a vector field is conservative we must consider that all piecewise C^1 closed curves. Alternatively, since we know that if a vector field is conservative, then it is a gradient vector field, this means that if $\mathbf{F}$ were conservative, then $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$. Since $\operatorname{curl}(\mathbf{F}) \neq \mathbf{0}$, by a contrapositive argument we know that $\mathbf{F}$ is not conservative.