Contents

Chapter 4

Lagrange multipliers

Here we will discuss some of the ways partial derivatives contribute to the understanding and solutions of problems in pure and applied mathematics. Many such problems can be put in the context of determining maximum or minimum values of functions of several variables.

4.1 Stationary points

Let us begin by extending the definition of a local/global minimum/maximum of a function of one variable to a function of several variables.

Definition 4.1.1 *Let $n \in \mathbb{N}$ with $n \geq 2$, let $D \subseteq \mathbb{R}^n$ have non-empty interior, and let $f \colon D \to \mathbb{R}$. A point $a \in D$ is said to be a:*

- ***local maximum*** *if there exists an open ball $U \ni a$ such that $f(x) \leq f(a)$ for all $x \in U \cap D$;*
- ***local minimum*** *if there exists an open ball $U \ni a$ such that $f(x) \geq f(a)$ for all $x \in U \cap D$:*
- ***global maximum*** *if $f(x) \leq f(a)$ for all $x \in D$;*
- ***global minimum*** *if $f(x) \geq f(a)$ for all $x \in D$;*
- ***local*** *or* ***global extreme*** *if it is a local or global maximum or minimum;*
- ***stationary point*** *if a is an interior point of D and $\nabla f(a) = \mathbf{0}$;*
- ***singular point*** *if a is an interior point of D and ∇f does not exist at a;*
- ***critical point*** *if a is an interior point of D and is either a stationary point or a singular point;*
- ***saddle point*** *if a is a stationary point which is neither a local maximum nor a local minimum.*

If f is a differentiable function of several variables, and if $\nabla f(x) = \mathbf{0}$, for some $x \in \mathrm{Dom}(f)$, then the function f is not changing in any direction at x, since $\nabla f(x)$ gives the direction of the maximum rate of change. Thus, we expect this quantity to play a key rôle in finding local and global extreme of f, see Figure 4.1 for examples.

Theorem 4.1.2 *Let $n \in \mathbb{N}$ with $n \geq 2$, let $D \subseteq \mathbb{R}^n$ have non-empty interior and let $f \colon D \to \mathbb{R}$. If f has a local or global extreme at the point $a \in D$, then a must be either*

- *a stationary point of f,*
- *a singular point of f, or*
- *a boundary point of D.*

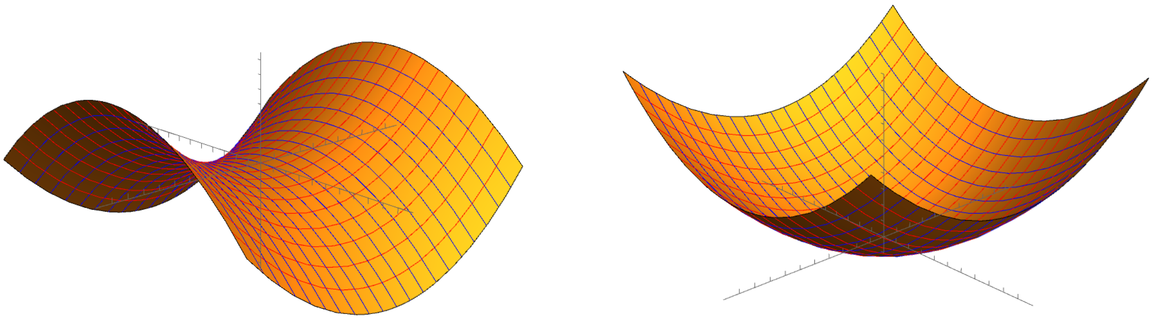

Figure 4.1: Left: Example of a graph of a function with a saddle point. Right: Example of a graph of a function with a global minimum.

Proof. If a is not a singular point nor a boundary point, then $\nabla f(a)$ exists since f is differentiable. By definition, if a is not a stationary point, then $\nabla f(a) \neq \mathbf{0}$ and so f has a positive directional derivative at a in the direction of $\nabla f(a)$ and negative directional derivative in the direction of $-\nabla f(a)$, these directional derivatives exist as f is differentiable. However, this means that f increases and decreases near to a. Thus, a cannot be a maximum or minimum. ∎

We recall that if f is a continuous function of a single variable whose domain is closed and bounded, then f attains its maximum and minimum. The same is true for a function of several variables. Namely, if f is a continuous function on a closed bounded set, then f is bounded and attains its bounds. Combining this fact, with the above theorem, to find the local or global extreme of a function of several variables we only need to consider the stationary points, the singular points and the boundary points.

Exercise 4.1.3 *Find the global extremum and saddle points (if they exist) of the following functions.*

- $f\colon \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 42\} \to \mathbb{R}$ *where* $f(x,y) = x^2 + y^2$
- $f\colon \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\} \to \mathbb{R}$ *where* $f(x,y) = x^2 - y^2$
- $f\colon [-1,1]^2 \to \mathbb{R}$ *where* $f(x,y) = x^3$
- $f\colon \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 100\} \to \mathbb{R}$ *where* $f(x,y) = y^2 + x^2y + x^4$
- $f\colon \mathbb{R}^3 \to \mathbb{R}$ *where* $f(x,y,z) = x^2 + y^2 - z^2$

Solution.

- Let $D = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 42\}$ and note the boundary ∂D of D is $\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 42\}$ and the interior of D is the set of point $\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 42\}$. Observe that $\nabla f(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$, and that it exists at every point in the interior of D. Therefore, f has no singular points and has a stationary point at $(0,0)$. Further,
  - $f(0,0) = 0$,
  - $f(x,y) \in (0,42)$, for all $(x,y) \in D \setminus \partial D$, and
  - $f(x,y) = 42$ for all $(x,y) \in \partial D$.

  Therefore, every point on the boundary is a global maximum, $(0,0)$ is a global minimum and there exist no saddle points.

- Let $D = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$ and note the boundary ∂D of D is $\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ and the interior of D is the set $\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$. Observe that $\nabla f(x,y) = 2x\mathbf{i} - 2y\mathbf{j}$, and that it exists at every point in the interior of D. Therefore, f has no singular points and has a stationary point at $(0,0)$. Further,
  - $f(0,0) = 0$,
  - $f(x,0) > 0$ and $f(0,y) < 0$ for all $(x,0)$ and $(0,y) \in D$ with $x \neq 0$ and $y \neq 0$,

- $f(x, y) \in (-1, 1)$, for all $(x, y) \in D \setminus \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$, and
- $f(1, 0) = f(-1, 0) = 1$ and $f(0, 1) = f(0, -1) = -1$.

Thus, f has global maxima at $(\pm 1, 0)$, global minima at $(0, \pm 1)$ and a saddle point at $(0, 0)$.

- Let $D = [-1, 1]^2$ and note the boundary ∂D of D is $\{-1, 1\} \times [-1, 1] \cup [-1, 1] \times \{-1, 1\}$ and the interior of D is $(-1, 1) \times (-1, 1)$. Observe that $\nabla f(x, y) = 3x^2\mathbf{i}$ and that it exists at every point in the interior of D. Therefore, f has no singular points and has stationary points at $(0, y)$, for all $y \in [-1, 1]$. Further,

  - $f(0, y) = 0$, for all $y \in [-1, 1]$,
  - $f(x, y) > 0$, for all $(x, y) \in D$ with $x > 0$, and $f(x, y) < 0$, for all $(x, y) \in D$ with $x < 0$,
  - $f(x, y) \in (-1, 1)$, for all $(x, y) \in D \setminus (\{\pm 1\} \times [-1, 1])$, and
  - $f(\pm 1, y) = \pm 1$, for all $y \in [-1, 1]$.

  Therefore, f has global maxima at $(1, y)$, for $y \in [-1, 1]$, and has global minima at $(-1, y)$, for $y \in [-1, 1]$, and saddle points at $(0, y)$, for $y \in (-1, 1)$.

- Let $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 100\}$ and note the boundary ∂D of D is $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 100\}$ and the interior of D is the set $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 100\}$. Observe that $\nabla f(x, y) = (2xy + 4x^3)\mathbf{i} + (2y + x^2)\mathbf{j}$ and that it exists at every point in the interior of D. Therefore, f has no singular points and has a stationary point at $(0, 0)$. Further, for all $(x, y) \in D \setminus \{(0, 0)\}$, since $f(x, y) = y^2 + x^2y + x^4 = (y + x^2/2)^2 + 3x^4/4$, we have that $f(x, y)$ is positive. Therefore, since $f(0, 0) = 0$, we have that $(0, 0)$ is a global minimum. Next we need to check the boundary points. For this we parameter D using polar coordinates, by $(r, t) \mapsto (r\cos(2\pi t), r\sin(2\pi t))$, where $r \in [0, 10]$ and $t \in [0, 1)$ and observe that for a fixed $t \in [0, 1]$ the function

  $$r \mapsto f(r\cos(2\pi t), r\sin(2\pi t)) = r^2\sin^2(2\pi t) + r^3\cos^2(2\pi t)\sin(2\pi t) + r^4\cos^4(2\pi t)$$

  is maximised when $r = 10$. Therefore, the global maximum of f is obtained on the boundary of D. Thus, to complete the solution to this problem we would need to study the function $t \mapsto f(10\sin(2\pi t), 10\cos(2\pi t))$ and use known methods for finding extrema of functions of a single variable. (Please note, on an exam, unless otherwise stated, you would be expected to complete this step.)

- Observe that $\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$ and that it exists at every point in the domain of f. Thus, f has no singular points and has a stationary point at $(0, 0, 0)$. Since the domain of f is $\mathbb{R}^3$, it has no boundary points. Further, as $f(x, 0, 0) > 0$ for all non-zero $x \in \mathbb{R}$, $f(0, 0, z) < 0$ for all non-zero $z \in \mathbb{R}$, and

  $$\lim_{x \to \infty} f(x, 0, 0) = \infty \quad \text{and} \quad \lim_{z \to \infty} f(0, 0, z) = -\infty,$$

  we have that f has saddle point at $(0, 0, 0)$ and no global extreme.

We have seen examples of classifying stationary points by hand. However, this requires some thought. Thus, it is natural to ask, is there a test to determine the nature of stationary points, similar to second derivative test for functions of a single variable. To answer this question we introduce the Hessian.

Definition 4.1.4 *Let $D \subseteq \mathbb{R}^2$ have non-empty interior. If $f \colon D \to \mathbb{R}$ is such that all its second order partial derivatives exist at the point $(a, b) \in D$, then the* ***Hessian matrix of f at (a, b)*** *is*

$$H(a, b) = \begin{pmatrix} f_{11}(a, b) & f_{12}(a, b) \\ f_{21}(a, b) & f_{22}(a, b) \end{pmatrix}.$$

Let $D \subseteq \mathbb{R}^3$ have non-empty interior. If $f \colon D \to \mathbb{R}$ is such that all its second order partial derivatives exist at the point $(a, b, c) \in D$, then the ***Hessian of f at (a, b, c)*** *is*

$$H(a, b, c) = \begin{pmatrix} f_{11}(a, b, c) & f_{12}(a, b, c) & f_{13}(a, b, c) \\ f_{21}(a, b, c) & f_{22}(a, b, c) & f_{23}(a, b, c) \\ f_{31}(a, b, c) & f_{32}(a, b, c) & f_{33}(a, b, c) \end{pmatrix}.$$

Note that the Hessian can be defined for functions of any number of variables in an analogous way.

A recap from linear algebra

If $A = (a_{i,j})_{i,j=1}^n$, then the **leading minors** of A are the values

$$\det(A_1) = |a_{11}|, \quad \det(A_2) = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}, \quad \dots, \quad \det(A_n) = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix}.$$

Recall that if A is an $n \times n$ matrix, letting

$$\mathbf{a_1} = A\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{a_2} = A\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \mathbf{a_n} = A\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

then A maps the unit n-dimensional cube to the n-dimensional parallelotope P defined by the vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\dots$,$\mathbf{a_n}$, namely the region $P = \{c_1\mathbf{a_1} + \cdots + c_n\mathbf{a_n} \colon 0 \le c_i \le 1 \text{ for all } i \in \{1, 2, \dots n\}\}$. The determinant gives the signed n-dimensional volume of P, that is, $\det(A) = \pm\operatorname{vol}(P)$, and hence describes n-dimensional volume scaling factor of the linear transformation A. The sign shows whether the transformation preserves or reverses orientation. In particular, if the determinant is zero, then P has zero n-dimensional volume and thus the dimension of the image of A is strictly less than n. This means that A produces a linear transformation which is not injective, and so is not invertible.

A symmetric $n \times n$ matrix A is said to be **positive definite** if the scalar $\mathbf{z}^t A\mathbf{z}$ is strictly positive for every non-zero column vector $\mathbf{z}$. Similarly, a symmetric $n \times n$ matrix A is said to be **negative definite** if the scalar $\mathbf{z}^t A\mathbf{z}$ is strictly negative for every non-zero column vector $\mathbf{z}$. Indeed, if A is a positive or negative definite matrix, then $\det(A) \neq 0$. Further, A is positive definite if and only if $\det(A_r) > 0$ for all $r \in \{1, 2, \dots, n\}$; A is negative definite if and only if $(-1)^r \det(A_r) > 0$ for all $r \in \{1, 2, \dots, n\}$. Another equivalent definition of A being positive definite, is to say that all of the eigenvalues of A are positive; and thus an equivalent definition of A being negative definite, is to say that all of the eigenvalues of A are negative.

Theorem 4.1.5 (The leading minor test) *Let $D \subseteq \mathbb{R}^2$ have non-empty interior. Suppose $f \colon D \to \mathbb{R}$ is such that all its second order partial derivatives are continuous at a point $(a, b) \in D$, ant that (a, b) is a stationary point of f. If $H = H(a, b)$ denotes the Hessian of f at (a, b) and if $\det(H) \neq 0$, then (a, b) is*

- *a local maximum if $\det(H_1) < 0$ and $\det(H) > 0$;*
- *a local minimum if $\det(H_1) > 0$ and $\det(H) > 0$;*
- *a saddle point if neither of the above hold.*

Let $D \subseteq \mathbb{R}^3$ have non-empty interior. Suppose $f \colon D \to \mathbb{R}$ is such that all its second order partial derivatives are continuous at a point $(a, b, c) \in D$, and that (a, b, c) is a stationary point of f. If $H = H(a, b, c)$ denote the Hessian of f at (a, b, c) and if $\det(H) \neq 0$, then (a, b, c) is

- *a local maximum if $\det(H_1) < 0$, $\det(H_2) > 0$ and $\det(H_3) < 0$;*
- *a local minimum if $\det(H_1) > 0$, $\det(H_2) > 0$ and $\det(H_3) > 0$;*
- *a saddle point if neither of the above hold.*

In each of the above cases, if $\det(H) = 0$, then p can be either a local extremum or a saddle point

Exercise 4.1.6 *Find and classify the stationary points, if they exist, of the following functions.*

- *$f \colon \mathbb{R}^3 \to \mathbb{R}$ where $f(x, y, z) = xy + x^2z - x^2 - y - z^2$*
- *$f \colon \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = y^2 + x^2y + x^4$*
- *$f \colon \mathbb{R}^3 \to \mathbb{R}$ where $f(x, y, z) = x^4 + x^2y + y^2 + z^2 + xz + 1$*
- *$f \colon \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = y^2 + (x + 1)^2y + (x + 1)^4$*

Solution.

- Observe that $\nabla f(x,y,z) = (y+2xz-2x)\mathbf{i} + (x-1)\mathbf{j} + (x^2-2z)\mathbf{k}$. Solving $\nabla f(x,y,z) = \mathbf{0}$, we obtain that f has a single stationary point at $(1,1,1/2)$. Further,

$$\begin{aligned} &f_{xx}(x,y,z) = 2z-2, && f_{xy}(x,y,z) = f_{yx}(x,y,z) = 1,\\ &f_{xz}(x,y,z) = f_{zx}(x,y,z) = 2x, && f_{yy}(x,y,z) = f_{yz}(x,y,z) = f_{zy}(x,y,z) = 0\\ &f_{zz}(x,y,z) = -2, \end{aligned}$$

  and so the Hessian $H = H(1,1,1/2)$ at $(1,1,1/2)$ is given by

$$H = \begin{pmatrix} -1 & 1 & 2\\ 1 & 0 & 0\\ 2 & 0 & -2 \end{pmatrix}.$$

  Thus, the leading minors are,

$$|-1| = -1 < 0, \qquad \begin{vmatrix} -1 & 1\\ 1 & 0 \end{vmatrix} = -1 < 0 \qquad \text{and} \qquad \begin{vmatrix} -1 & 1 & 2\\ 1 & 0 & 0\\ 2 & 0 & -2 \end{vmatrix} = 2.$$

  By Theorem 4.1.5, the stationary point $(1,1,1/2)$ is a saddle point.

- Observe that $\nabla f(x,y) = (2xy+4x^3)\mathbf{i} + (2y+x^2)\mathbf{j}$. Solving $\nabla f(x,y) = \mathbf{0}$, we obtain that f has a single stationary points at $(0,0)$. Further,

$$f_{xx}(x,y) = 2y+12x^2, \qquad f_{xy}(x,y) = 2x, \qquad f_{yx}(x,y) = 2x, \qquad f_{yy}(x,y) = 2,$$

  and so the Hessian $H = H(0,0)$ at $(0,0)$ is given by

$$H = \begin{pmatrix} 0 & 0\\ 0 & 2 \end{pmatrix}.$$

  Thus, the leading minors are,

$$|0| = 0 \qquad \text{and} \qquad \begin{vmatrix} 0 & 0\\ 0 & 2 \end{vmatrix} = 0,$$

  and so the leading minor test is not applicable here. However, since

$$f(x,y) - f(0,0) = y^2 + x^2y + x^4 = (y + x^2/2)^2 + 3x^4/4 > 0,$$

  for all $(x,y) \neq (0,0)$, we have that $(0,0)$ is a local minimum.

- Observe that $\nabla f(x,y,z) = (4x^3+2xy+z)\mathbf{i} + (x^2+2y)\mathbf{j} + (2z+x)\mathbf{k}$. The stationary points occur when $\nabla f(x,y,z) = \mathbf{0}$, that is, when

$$0 = 4x^3 + 2xy + z, \qquad 0 = x^2 + 2y \qquad \text{and} \qquad 0 = 2z + x.$$

  Using the latter two equalities, to eliminate y and z from first equality, we have $4x^3 - x^3 - x/2 = 0$ or equivalently $x(6x^2-1) = 0$, yielding $x = 0$, $x = 1/\sqrt{6}$ and $x = -1/\sqrt{6}$. Hence, we have three stationary points: $(0,0,0)$, $(1/\sqrt{6}, -1/12, -1/(2\sqrt{6}))$ and $(-1/\sqrt{6}, -1/12, 1/(2\sqrt{6}))$. Further,

$$\begin{aligned} &f_{xx}(x,y,z) = 12x^2 + 2y, && f_{xy}(x,y,z) = f_{yx}(x,y,z) = 2x,\\ &f_{xz}(x,y,z) = f_{zx}(x,y,z) = 1, && f_{yy}(x,y,z) = 2,\\ &f_{yz}(x,y,z) = f_{zy}(x,y,z) = 0, && f_{zz}(x,y,z) = 2, \end{aligned}$$

  and so the Hessian $H = H(x,y,z)$ is given by

$$H = \begin{pmatrix} 12x^2 + 2y & 2x & 1\\ 2x & 2 & 0\\ 1 & 0 & 2 \end{pmatrix}.$$

At the point $(1/\sqrt{6}, -1/12, -1/(2\sqrt{6}))$,

$$H = \begin{pmatrix} 11/6 & \sqrt{6}/3 & 1 \\ \sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

which has leading minors

$$|11/6| = 11/6 > 0, \qquad \begin{vmatrix} 11/6 & \sqrt{6}/3 \\ \sqrt{6}/3 & 2 \end{vmatrix} = 3 > 0 \quad \text{and} \quad \begin{vmatrix} 11/6 & \sqrt{6}/3 & 1 \\ \sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4 > 0.$$

By the leading minor test, we have that the point $(1/\sqrt{6}, -1/12, -1/(2\sqrt{6}))$ is a local minimum of f. At the point $(-1/\sqrt{6}, -1/12, 1/(2\sqrt{6}))$,

$$H = \begin{pmatrix} 11/6 & -\sqrt{6}/3 & 1 \\ -\sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

which has leading minors

$$|11/6| = 11/6 > 0, \qquad \begin{vmatrix} 11/6 & \sqrt{6}/3 \\ \sqrt{6}/3 & 2 \end{vmatrix} = 3 > 0 \quad \text{and} \quad \begin{vmatrix} 11/6 & -\sqrt{6}/3 & 1 \\ -\sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4 > 0.$$

By the leading minor test, we have that the point $(-\sqrt{6}/6, -1/12, \sqrt{6}/12)$ is a local minimum of f. At the point $(0, 0, 0)$,

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Since the first leading minor is 0, by the leading minor test, the point $(0, 0, 0)$ is a saddle point.

- Observe that $\nabla f(x, y) = (2(x+1)y + 4(x+1)^3)\mathbf{i} + (2y + (x+1)^2)\mathbf{j}$, and so the gradient vector exists at every point in the domain of f. The stationary points of f occur when $\nabla f(x, y, z) = \mathbf{0}$, namely when $x = -1$ and $y = 0$. In this case, the Hessian at $(-1, 0)$ has determinant zero, and so the leading minor test is not applicable here. Let

$$D = f(-1 + h, 0 + k) - f(-1, 0) = k^2 + h^2k + h^4.$$

Completing the square, we have that $D = (k + h^2/2)^2 + 3h^4/4$. So for any arbitrarily small values of h and k, that are not both zero, $D > 0$ and thus f has a local minimum at $(-1, 0)$.

Let us now take a look at how **Theorem 4.1.5** generalises to higher dimensions.

Theorem 4.1.7 (The leading minor test) *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $D \subseteq \mathbb{R}^n$ have non-empty interior. Suppose that $f\colon D \to \mathbb{R}$ is such that all of its second order partial derivatives are continuous at a given $p \in D$, and that p is a stationary point of f. If $H = H(p)$ denotes the Hessian of f at p, then p is*

- *a local minimum if H is positive definite;*
- *a local maximum if H is negative definite;*
- *a saddle point if neither of the above holds and* $\det(H) \neq 0$.

Proof. Here we will prove the first statement and leave the latter two statements to the reader. Let $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$ be fixed and such that $p + th \in D$ for all $t \in [-1, 1]$, let $g\colon [-1, 1] \to \mathbb{R}$ be defined by $g(t) := f(p + th)$ for $t \in [-1, 1]$, and denote the column vector $\langle h_1, h_2, \ldots, h_n\rangle$ by $\mathbf{h}$. By the chain rule we have

$$g'(t) = \sum_{i=1}^{n} f_i(p + th)h_i \quad \text{and} \quad g''(t) = \sum_{I=1}^{n}\sum_{j=1}^{n} f_{ij}(p + th)h_i h_j = \mathbf{h}^t H(p + th)\mathbf{h}.$$

By Taylor's remainder theorem, setting $a = 0$ and $x = 1$, there exists $\theta \in [-1, 1]$ such that

$$g(1) = g(0) + g'(0) + \frac{g''(\theta)}{2}.$$

Combining the above with the fact that $\nabla(f)(p) = \mathbf{0}$, it follows that

$$f(p + h) - f(p) = \frac{1}{2}\mathbf{h}^t H(p + \theta h)\mathbf{h}. \tag{4.1}$$

Since $H(p)$ is positive definite, and since the components of the Hessian of f are continuous, for $h \in \mathbb{R}$ with $|\mathbf{h}|$ sufficiently small, $\mathbf{h}^t H(p + \theta h)\mathbf{h}$ is positive. This in tandem with (4.1) yields $f(p + th) - f(p) > 0$ for $h \in \mathbb{R}^n$ with $|\mathbf{h}|$ sufficiently small; namely, that p is a local minimum. ∎

4.2 Constrained extreme and Lagrange multipliers

Given a function f of several variables, one is often interested in the extreme along a curve on the graph of f. For example, if $f(x, y) = x + y - x^3 + xy^3$, what are the extreme on $y = (7\cos(2\pi x/3))/4$, see FIGURE 4.2. To solve such a problem we use what is know as a Lagrange multiplier.

Definition 4.2.1 *Let n and $k \in \mathbb{N}$ with $n \geq 2$, and let $f, g_1, g_2, \ldots, g_k$ denote functions of n variables with the same domain. An extreme value of f subject to the conditions $g_i = 0$ for all $i \in \{1, \ldots, k\}$, is called a **constrained extreme value** and the conditions $g_i = 0$ are called **constraints**.*

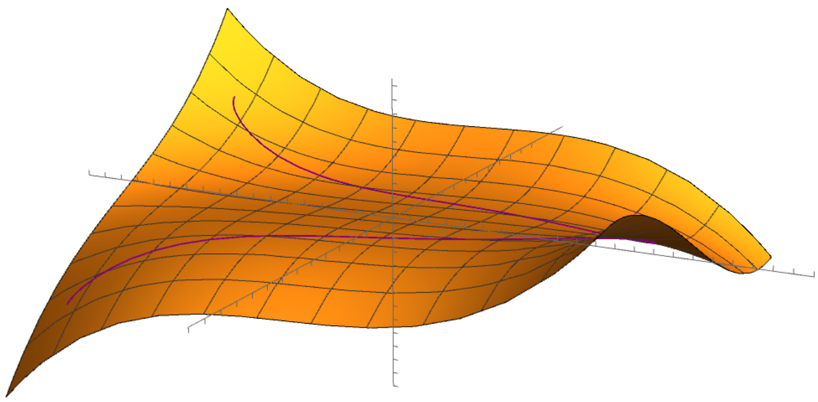

Figure 4.2: Example of curve (drawn in purple) on the a graph of a function.

Definition 4.2.2 *Let n and $k \in \mathbb{N}$ with $n \geq 2$, and let $f, g_1, g_2, \ldots, g_k$ denote functions of n variables with the same domain. The **Lagrangian** of f subject to the constraints $g_i = 0$ for all $i \in \{1, \ldots, k\}$, is the function L of $n + k$ variables defined by*

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k) := f(x_1, \ldots, x_n) + \sum_{i=1}^{k} \lambda_i g_i(x_1, \ldots, x_n).$$

*where $\lambda_i \in \mathbb{R}$ is known as the **i-th Lagrange multiplier**, for $i \in \{1, \ldots, k\}$, and $(x_1, x_2, \ldots, x_n) \in \mathrm{Dom}(f)$.*

The following results shows how we can use Lagrange multiplier to solve constrained extreme value problems.

Theorem 4.2.3 *Assume the setting of Definition 4.2.2 and let*

$$C = \{x \in \mathrm{Dom}(f) \colon g_i(x) = 0 \textit{ for all } i \in \{1, \ldots, k\}\}.$$

Let $p = (p_1, \ldots, p_n) \in C$ be such that f restricted to C has a local extreme at p, and that p is an interior point of $\mathrm{Dom}(f)$. Suppose that both f and g_i have continuous first order partial derivatives throughout an open ball $U \subseteq \mathrm{Dom}(f)$ containing p and, for each $x \in U$, the vectors $\nabla g_1(x)$, $\nabla g_2(x)$, $\ldots$, $\nabla g_n(x)$ are linearly independent. Under these hypotheses, there exist a $\lambda_1, \ldots, \lambda_k$ such that $(p_1, \ldots, p_n, \lambda_1, \ldots, \lambda_k)$ is a stationary point of the Lagrangian L of f subject to the constraints $g_i = 0$.

To gain a better understanding of the statement, let us write out the statement in the simplest case, namely when $n = 2$ and $k = 1$. Let f and $g\colon \mathbb{R}^2 \to \mathbb{R}$ and let $p = (p_1, p_2)$ be a point on the curve C determined by the equation $g(x, y) = 0$, at which f, when restricted to C, has a local extreme. Suppose that both f and g have continuous first order partial derivatives on a neighbourhood of p and that $\nabla g(p_1, p_2) \neq \mathbf{0}$. Then there exists a λ_0 such that (p_1, p_2, λ_0) is a stationary point of the Lagrangian $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. We now present a proof of this statement.

Proof. Since $\nabla g(p) \neq \mathbf{0}$, the curve C has a tangent plane at p with normal vector $\nabla g(p)$. If ∇f is not parallel to ∇g at p, then $\nabla f(p)$ a has non-zero projection $\mathbf{v}$ onto a vector perpendicular to $\nabla g(p)$. Therefore, the function f has a positive directional derivative at p in the direction of $\mathbf{v}$ and a negative directional derivative in the direction $-\mathbf{v}$, or vice versa. Thus, the function f increases and decreases along C as we move away from p, and so p is not an extreme. Since, by our hypothesis, we have that p is a local extreme, it follows that $\nabla f(p)$ and $\nabla g(p)$ are parallel. In other words, there exists a non-zero scalar λ such that $\nabla f(p) = -\lambda \nabla g(p)$. This implies that

$$\frac{\partial L}{\partial x}(p_1, p_2, \lambda) = 0 \qquad \text{and} \qquad \frac{\partial L}{\partial y}(p_1, p_2, \lambda) = 0.$$

The third equation which must be satisfied in order for (p_1, p_2, λ) to be a stationary point of L is that

$$\frac{\partial L}{\partial \lambda}(p_1, p_2, \lambda) = 0.$$

However, by definition we have have that

$$\frac{\partial L}{\partial \lambda}(p_1, p_2, \lambda) = g(p_1, p_2)$$

and since $p \in C$, the result follows. ■

Exercise 4.2.4 *Let a, b and c denote three fixed positive real numbers. Find the rectangular box with the largest volume that fits inside the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

given that it sides are parallel to the (Cartesian) coordinate axes.

Solution. Observe that the rectangular box will have the greatest volume if each of its corners lie on the ellipsoid. Without loss of generality, let one corner of the box be at the point (x, y, z) which belongs to the positive octant of $\mathbb{R}^3$. Using the fact the given ellipsoid is symmetric about the x-axis, the y-axis and z-axis, we have that the rectangular box has corners $(\pm x, \pm y, \pm z)$ and thus its volume is given by $V(x, y, z) = 8xyz$. Our aim here is to maximise V given that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{4.2}$$

(Note that since the constraint surface is bounded a maximum and a minimum does exist.) The Lagrangian of V is subject to (4.2) is given by,

$$L(x, y, z, \lambda) = 8xyz + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$$

and has stationary points when $\nabla L(x, y, z) = \mathbf{0}$, that is when

$$0 = \frac{\partial L}{\partial x}(x, y, z, \lambda) = 8yz + \frac{2\lambda x}{a^2}, \qquad 0 = \frac{\partial L}{\partial y}(x, y, z, \lambda) = 8zx + \frac{2\lambda y}{b^2},$$
$$0 = \frac{\partial L}{\partial z}(x, y, z, \lambda) = 8xy + \frac{2\lambda z}{c^2}, \qquad 0 = \frac{\partial L}{\partial \lambda}(x, y, z, \lambda) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right).$$

As we want to maximise V we can assume that $xyz \neq 0$ so that $x, y, z \neq 0$. Hence,

$$\lambda = \frac{-4a^2 yz}{x} = \frac{-4b^2 zx}{y} = \frac{-4c^2 xy}{z},$$

and so $y^2 a^2 = x^2 b^2$ and $z^2 b^2 = y^2 c^2$. In this case $x^2/a^2 = y^2/b^2 = z^2/c^2$, and thus

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3\frac{x^2}{a^2}$$

or equivalently $x = a/\sqrt{3}$, which implies $y = b/\sqrt{3}$ and $z = c/\sqrt{3}$. Therefore, L has a single stationary point at

$$p = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}}, \frac{-4abc}{\sqrt{3}} \right).$$

Further, we observe that

$$\nabla \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j} + \frac{2z}{c^2}\mathbf{k},$$

which, when evaluated at $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$, is not equal to the zero vector. By **Theorem 4.2.3**, the required maximum is achieved at $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$ and

$$V \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}} \right) = 8 \frac{a}{\sqrt{3}} \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}.$$

The value obtained above is indeed a global maximum for f when restricted to the given ellipsoid, since this domain is closed and bounded and has no boundary. Therefore, f obtains its maximum and minimum when restricted to the given ellipsoid, yielding the required result.

Chapter 5

Taylor series

A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The concept of a Taylor series was formulated by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715. If the Taylor series, of a function of a single variable, is centred at zero, then that series is also called a Maclaurin series, named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series in the 18th century.

A function of a single variable can be approximated by using a finite number of terms of its Taylor series. Taylor's theorem gives quantitative estimates on the error introduced by the use of such an approximation. The polynomial formed by taking some initial terms of the Taylor series is called a Taylor polynomial. Indeed, the Taylor series of a function is the limit of that function's Taylor polynomials as the degree increases, provided that the limit exists. A function may not be equal to its Taylor series, even if it's Taylor series converges at every point. A function that is equal to its Taylor series in an open interval is known as an analytic function.

If the Taylor series of a function f of a single variable converges to f, then it is unique. This means that, in this case, however one builds a power series expansion of f, it will be the Taylor series of f. Therefore, in calculating the Taylor series one should use known power series expansions to calculate Taylor series whenever possible and only calculate the coefficients directly as a last resort, namely as a final course of action.

The question we want to address here is, can we also build a power series representation of a function of several variables? The answer is yes, and we do this by utilising what we know from the one dimensional case and the theory we have thus far built.

Let $n \in \mathbb{N}$ with $n \geq 2$, and suppose that f is a function of n variables that has continuous partial derivatives of all orders on an open connected domain $I \subseteq \mathbb{R}^n$. Let $a = (a_1, a_2, \ldots, a_n) \in I$ and let $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$ be such that $a + th = (a_1 + th_1, a_2 + th_2, \ldots, a_n + th_n) \in I$ for all $t \in [-1, 1]$. Consider the function $t \mapsto F(t) := f(a + th)$, where $t \in [-1, 1]$. By the Chain rule, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(t) = F'(t) &= h_1 f_1(a+th) + h_2 f_2(a+th) + \cdots + h_n f_n(a+th) \\ &= (\mathbf{h} \cdot \nabla) f(a+th),\end{aligned}$$

where $\mathbf{h} = \langle h_1, h_2, \ldots, h_n \rangle$ is the vector with initial point at the origin and terminal point at the point h. Similarly,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}F(t) = F''(t) &= h_1 h_1 f_{11}(a+th) + h_1 h_2 f_{12}(a+th) + \cdots + h_n h_n f_{nn}(a+th) \\ &= (\mathbf{h} \cdot \nabla) \circ (\mathbf{h} \cdot \nabla) f(a+th) = (\mathbf{h} \cdot \nabla)^2 f(a+th).\end{aligned}$$

and in fact one can show, for $j \in \mathbb{N}$, that

$$F^{(j)}(t) = (\mathbf{h} \cdot \nabla)^j f(a+th).$$

Thus, applying Taylor's theorem to the function F, expanding about 0 and evaluating at $t = 1$ we obtain

$$\begin{aligned}f(a+h) &= f(a) + (\mathbf{h} \cdot \nabla) f(a) + \frac{(\mathbf{h} \cdot \nabla)^2 f(a)}{2!} + \cdots + \frac{(\mathbf{h} \cdot \nabla)^m f(a)}{m!} + \frac{(\mathbf{h} \cdot \nabla)^{m+1} f(a+\theta h)}{m+1!} \\ &= \sum_{j=0}^{m} \frac{(\mathbf{h} \cdot \nabla)^j f(a)}{j!} + \frac{(\mathbf{h} \cdot \nabla)^{m+1} f(a+\theta h)}{m+1!},\end{aligned}$$

for some $\theta \in (-1, 1)$ and m a non-negative integer. This leads us to the following definition.

Definition 5.0.1 (Taylor's series - Functions of several variables) *Suppose that all partial derivatives of all order of a function f of two variables exist and are continuous on an open connected domain $I \subseteq \mathbb{R}^2$. Let (a,b) denote a point in I and let $h = (h_1, h_2) \in \mathbb{R}^2$ be such that $(a,b) + t(h_1,h_2) = (a+th_1, b+h_2) \in I$ for all $t \in [-1,1]$. The* ***Taylor series*** *generated by f centred at (a,b) and evaluated at $(a+h_1, b+h_2)$ is the power series*

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{(\mathbf{h}\cdot\nabla)^j f(a,b)}{j!} = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) \\ + \frac{1}{2!}\left(h_1^2 f_{xx}(a,b) + 2h_1h_2 f_{xy}(a,b) + h_2^2 f_{yy}(a,b)\right) \\ + \frac{1}{3!}\left(h_1^3 f_{xxx}(a,b) + 3h_1^2h_2 f_{xxy}(a,b) + 3h_1h_2^2 f_{xyy}(a,b) + h_2^3 f_{yyy}(a,b)\right) + \dots,\end{aligned}$$

where $\mathbf{h} = \langle h_1, h_2 \rangle$ is the vector with initial point at the origin and terminal point at h.

Similarly, suppose that all partial derivatives of all orders of a function f of three variables exist and are continuous on an open connected domain $I \subseteq \mathbb{R}^3$. Let (a,b,c) denote a point of I and let $h = (h_1,h_2,h_3) \in \mathbb{R}^3$ be such that $(a,b,c) + t(h_1,h_2,h_3) = (a+th_1, b+h_2, c+h_3) \in I$ for all $t \in [-1,1]$. The ***Taylor series*** *generated by f centred at (a,b,c) and evaluated at $(a+h_1, b+h_2, c+h_3)$ is the power series*

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{(\mathbf{h}\cdot\nabla)^j f(a,b,c)}{j!} = f(a,b,c) + h_1 f_x(a,b,c) + h_2 f_y(a,b,c) + h_3 f_z(a,b,c) \\ + \frac{1}{2!}\Big[h_1^2 f_{xx}(a,b,c) + h_2^2 f_{yy}(a,b,c) + h_3^2 f_{zz}(a,b,c) \\ + 2h_1h_2 f_{xy}(a,b,c) + 2h_1h_3 f_{xz}(a,b,c) + 2h_2h_3 f_{yz}(a,b,c)\Big] + \dots,\end{aligned}$$

where $\mathbf{h} = \langle h_1, h_2, h_3 \rangle$ is the vector with initial point at the origin and terminal point at h.

In the analogous way one can also define a Taylor series of a function of n variables, with $n > 3$ a natural number.

If the Taylor series of a function f converges to f, namely if the Taylor series of f and f itself coincide, then it is unique. This means that, in this case, however one builds a power series expansion of f, it will be the Taylor series generated by f. Therefore, one typically uses known series to calculate Taylor series where ever possible and only calculate the coefficients directly as a last resort, namely as a final course of action.

Exercise 5.0.2 *Find the Taylor series centred at $(1,0)$, up to and including order three terms, generated by the function $f\colon \{(x,y) \in \mathbb{R}^2 \colon |xy^2 - 2y| < 1\} \to \mathbb{R}$ given by*

$$f(x,y) = \frac{e^x}{1 - xy^2 + 2y}.$$

Solution. Here, we want an expansion centred at $(1,0)$, and so, we want an expansion expressed in terms of $(x-1)$ and $(y-0) = y$. To this end, observe that

$$\frac{e^x}{1-xy^2+2y} = \frac{e^{(x-1)+1}}{1-((x-1)y^2+y^2-2y)} = \frac{e^1 e^{x-1}}{1-((x-1)y^2+y^2-2y)}.$$

Using the Taylor series of the exponential function and the known value of the sum of a geometric series, the Taylor series, up to order three terms, generated by f centred at $(1,0)$ and evaluated at a point $(x,y) \in \mathrm{Dom}(f)$ is

$$\begin{aligned}&e^1\left(\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}\right)\left(\sum_{n=0}^{\infty}((x-1)y^2+y^2-2y)^n\right) \\ &= e^1\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \cdots\right)\left(1 + ((x-1)y^2+y^2-2y) + \cdots\right) \\ &= e^1\left(1 - 2y + (x-1) + 5y^2 - 2(x-1)y + \frac{(x-1)^2}{2} - 12y^3 + 5(x-1)y^2 - y(x-1)^2 \dots\right).\end{aligned}$$

Exercise 5.0.3 *Find the Taylor series centred at* $(2,-1)$*, up to and including order three terms, of the function* $f\colon\{(x,y)\in\mathbb{R}^2\colon |xy-y|<1\}\to\mathbb{R}$ *given by*

$$f(x,y)=\frac{e^x}{1-xy+y}.$$

Solution. Here, we want an expansion centred at $(2,-1)$ and so we want an expansion expressed in terms of $(x-2)$ and $(y-(-1))=(y+1)$. Using the fact that $x=(x-2)+2$ and $y=(y+1)-1$,

$$\frac{e^x}{1-xy+y}=\frac{e^{(x-2)+2}}{1-((x-2)+2)((y+1)-1)+((y+1)-1)}=\frac{e^2e^{x-2}}{2-((x-2)(y+1)-(x-2)+(y+1))}$$

Using our knowledge on the value of the sum of a geometric series, and the fact that $(x,y)\in\mathrm{Dom}(f)$, we have that

$$w=(x-2)(y+1)-(x-2)+(y+1)\in(-1,1),$$

and

$$\frac{1}{2-((x-2)(y+1)-(x-2)+(y+1))}=\frac{1}{2-w}=\frac{1}{2}\left(\frac{1}{1-(w/2)}\right)=\frac{1}{2}\left(1+\frac{w}{2}+\frac{w^2}{4}+\frac{w^3}{8}+\dots\right).$$

Using the fact that the Taylor series about zero of e^u is equal to

$$1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots$$

and converges to e^u, setting $u=x-2$ we have that

$$\begin{aligned}f(x,y)&=\frac{e^2(e^{x-2})}{2-((x-2)(y+1)-(x-2)+(y+1))}\\&=\frac{e^2}{2}\left(1+u+\frac{u^2}{2}+\frac{u^3}{6}+\cdots\right)\left(1+\frac{w}{2}+\frac{w^2}{4}+\frac{w^3}{8}+\dots\right)\qquad(5.1)\\&=\frac{e^2}{2}\left(1+u+\frac{w}{2}+\frac{u^2}{2}+\frac{uw}{2}+\frac{w^2}{4}+\frac{u^3}{6}+\frac{u^2w}{4}+\frac{uw^2}{4}+\frac{w^3}{8}+\dots\right).\end{aligned}$$

Since $u=x-2$ and $w=(x-2)(y+1)-(x-2)+(y+1)$, we have the following equalities.

$$\begin{aligned}uw=&(x-2)^2(y+1)-(x-2)^2+(x-2)(y+1),\\u^2w=&(x-2)^3(y+1)-(x-2)^3+(x-2)^2(y+1)\\w^2=&(x-2)^2(y+1)^2+(x-2)^2+(y+1)^2\\&-2(x-2)^2(y+1)+2(x-2)(y+1)^2-2(x-2)(y+1)\\uw^2=&(x-2)^3(y+1)^2+(x-2)^3+(x-2)(y+1)^2\\&-2(x-2)^3(y+1)+2(x-2)^2(y+1)^2-2(x-2)^2(y+1)\\w^3=&-(x-2)^3+3(x-2)^2(y+1)+3(x-2)^3(y+1)-3(x-2)(y+1)^2-6(x-2)^2(y+1)^2\\&-3(x-2)^3(y+1)^2+(y+1)^3+3(x-2)(y+1)^3+3(x-2)^2(y+1)^3+(x-2)^3(y+1)^3\end{aligned}$$

Substituting all this into (5.1) and rearranging (eventually) leads to the following Taylor series generated by f centred at $(2,-1)$ and evaluated at a point $(x,y)\in\mathrm{Dom}(f)$.

$$\begin{aligned}f(x,y)=\frac{e^2}{4}\Bigg(&2+(x-2)+(y+1)\\&+\frac{(x-2)^2}{2}+(x-2)(y+1)+\frac{(x-2)(y+1)}{2}\\&+\frac{(x-2)^3}{12}+\frac{(x-2)^2(y+1)}{4}+\frac{3(x-2)(y+1)^2}{4}+\frac{(y+1)^3}{4}+\dots\Bigg)\end{aligned}$$

Exercise 5.0.4 *Find the Taylor series centred at* $(0,0)$*, up to and including order three terms, of the function* $f\colon\{(x,y)\in\mathbb{R}^2\colon -\pi/2<x+y<\pi/2\}\to\mathbb{R}$ *given by*

$$f(x,y)=\frac{1}{1-\sin(x+y)}.$$

Solution. Here we want an expansion centred at $(0,0)$, and so we want an expansion expressed in term of $(x-0)=x$ and $(y-0)=y$. For $(x,y)\in\mathrm{Dom}(f)$, we have that $\sin(x+y)\in(-1,1)$ and so, using our knowledge on the known value of the sum of a geometric series, we have that

$$\frac{1}{1-\sin(x+y)} = 1+\sin(x+y)+\sin^2(x+y)+\sin^3(x+y)+\dots$$

Using the fact that the Taylor series about zero of $\sin(u)$ is equal to

$$u-\frac{u^3}{3!}+\frac{u^5}{5!}-\frac{u^7}{7!}+O(u^9)$$

and converges to $\sin(u)$, and setting $u=x+y$, we have that the Taylor series, up to and including order three terms, generated by f centred at $(0,0)$ and evaluated at a point $(x,y)\in\mathrm{Dom}(f)$, is given by

$$\begin{aligned}
\frac{1}{1-\sin(x+y)} &= 1+\left((x+y)-\frac{(x+y)^3}{3!}+\frac{(x+y)^5}{5!}-\frac{(x+y)^7}{7!}+\dots)\right)\\
&\quad+\left((x+y)-\frac{(x+y)^3}{3!}+\frac{(x+y)^5}{5!}-\frac{(x+y)^7}{7!}+\dots\right)^2\\
&\quad+\left((x+y)-\frac{(x+y)^3}{3!}+\frac{(x+y)^5}{5!}-\frac{(x+y)^7}{7!}+\dots\right)^3+\dots\\
&= 1+(x+y)-\frac{(x+y)^3}{3!}+(x+y)^2+(x+y)^3+\dots\\
&= 1+x+y+x^2+2xy+y^2+\frac{5x^3}{6}+\frac{5x^2y}{2}+\frac{5xy^2}{2}+\frac{5y^3}{6}+\dots
\end{aligned}$$

Chapter 6

Multiple integrals

6.1 Multiple integrals

Here and the remaining sections of the chapter, we extend the concept of the definite integral of functions of a single variable to functions of several variables. Defined as limits of Riemann sums, like the one-dimensional definite integral, such multiple integrals can be evaluated using successive single definite integrals. They are used to represent and calculate quantities specified in terms of densities in regions of the plane or spaces of higher dimensions. In the simplest instance, the volume of a three dimensional region is given by a double integral of its height over a two dimensional planar region that is its base. Applications of multiple integrals include finding surface area, computing gravitational potential, as well as moments and centre of mass.

Let a, b, c and $d \in \mathbb{R}$ with $a \leq b$ and $c \leq d$. Let f denote a real-valued function of two variables defined on the rectangular domain $D = [a,b] \times [c,d]$. Let $P_1 = \{x_0, x_1, \ldots, x_n\}$ denote a partition of $[a,b]$ and let $P_2 = \{y_0, y_1, \ldots, y_m\}$ denote a partition of $[c,d]$. Using P_1 and P_2 we form a partition P of D, by letting P consist of the nm rectangles $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, for $(i,j) \in \{(k,l) \in \mathbb{N}^2 \colon 1 \leq k \leq n \text{ and } 1 \leq l \leq m\}$. For each $(i,j) \in \{(k,l) \in \mathbb{N}^2 \colon 1 \leq k \leq n \text{ and } 1 \leq l \leq m\}$, choose $(x_{i,j}^*, y_{i,j}^*) \in R_{i,j}$ and let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$.

The rectangle $R_{i,j}$ has area $\Delta A_{i,j} = \Delta x_i \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1})$ and diameter

$$\operatorname{diam}(R_{i,j}) = \sqrt{\Delta x_i^2 + \Delta y_j^2}.$$

We define the **norm** of the partition P by

$$|P| = \max\{\operatorname{diam}(R_{i,j}) \colon (i,j) \in \mathbb{N}^2 \text{ with } 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

With this at hand we define the **Riemann sum** of f associated with P and the choice of $(x_{i,j}^*, y_{i,j}^*)$ by

$$R(f,P) = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{i,j}^*, y_{i,j}^*) \Delta A_{i,j}.$$

If $f(x_{i,j}^*, y_{i,j}^*) \geq 0$, then the term $f(x_{i,j}^*, y_{i,j}^*) \Delta A_{i,j}$ is the volume of the rectangular box whose base is given by the set of points $\mathcal{R}_{i,j} = \{(x,y,z) \in \mathbb{R}^3 \colon z = 0 \text{ and } (x,y) \in R_{i,j}\}$, and whose height is the value of f at $(x_{i,j}, y_{i,j})$, see Figure 6.1. Therefore, for positive functions f, the Riemann sum $R(f,P)$ approximates the volume of the solid $\{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in D \text{ and } 0 \leq z \leq f(x,y)\}$, or loosely speaking, the solid lying above $\mathcal{D} = \{(x,y,z) \in \mathbb{R}^3 \colon z = 0 \text{ and } (x,y) \in D\}$ and under the graph of f. Thus, as in the case of a function of a single variable, we define the double definite integral over a rectangular region as follows: we say that a function f of two variables x and y is **integrable** over a rectangular region D and has **double integral**

$$I = \iint_D f \, \mathrm{d}A = \iint_D f(x,y) \, \mathrm{d}A$$

if D is contained in the domain of f, and if for every positive real number ϵ, there exists a real number δ such that $|R(f,P) - I| < \epsilon$ for every partition P of D satisfying $|P| < \delta$ and for all choices of points $(x_{i,j}^*, y_{i,j}^*)$ in the sub-rectangles of P.

It is often necessary to define a double integral of a function f of two variables over a domains D that is bounded but not rectangular. To deal with such cases let a, b, c and $d \in \mathbb{R}$ be such that $D \subseteq [a,b] \times [c,d]$ and let P be a

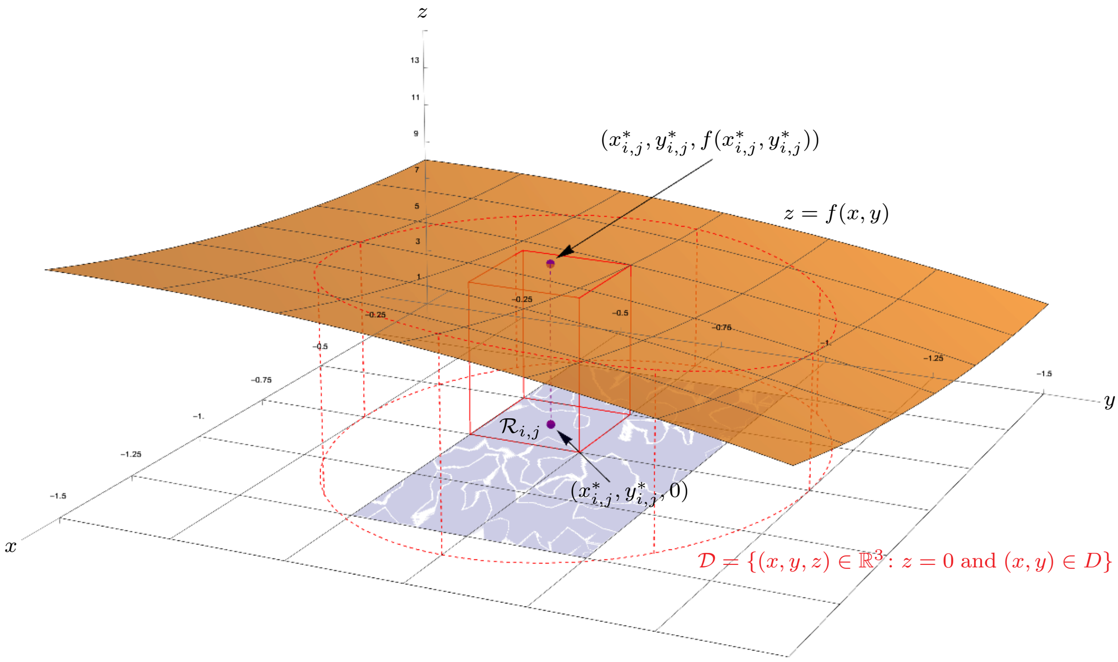

Figure 6.1: In orange is a visualisation of the graph of a function f of two variables, the set of points $\mathcal{D}$ in the x-y-plane contained inside the red dotted circle gives a visualisation in $\mathbb{R}^3$ of the non-rectangular domain we would like to integrate over, and shaded in blue is a visualisation in $\mathbb{R}^3$, of partition elements completely contained within the domain D.

partition of $[a, b] \times [c, d]$. In the Riemann sum of f associated with P we only include the rectangles that lie entirely inside D, see FIGURE 6.1. Note as the partition becomes finer and finer, namely as $|P| \to 0$, this union approximates the region D.

One important fact, which we will repeatedly use is the following. As for functions of a single variable, functions of two variables that are continuous and bounded on a bounded domain D are integrable over D.

Now that we have seen how to extend definite integration to functions of two variables, the extension to functions of three (or more) variables is straight forward. Indeed, for a function f of three variables x, y and z defined on a cuboid D with sides parallel to the co-ordinate planes, the **triple integral** of f over D

$$\iiint_D f \, \mathrm{d}V = \iiint_D f(x, y, z) \, \mathrm{d}V,$$

can be defined as a suitable limit of Riemann sums corresponding to partitions of D consisting of cuboids with sides parallel to the sides of D. Triple integrals over bounded domains are defined in a similar manner to how double integrals over bounded domains are defined, namely, to consider only those partition elements which are completely contained in the given domain. Note, as with double integrals, functions of three variables that are continuous and bounded on D are integrable.

The triple integral of a positive function f of three variables over a bounded domain $D \subset \mathbb{R}^3$ can be interpreted as the **hyper-volume**, that is the 4-dimensional volume, of a region in 4-space having its *base* be given by $\mathcal{D} = \{(x, y, z, w) \in \mathbb{R}^4 \colon (x, y, z) \in \mathbb{R}^3 \text{ and } w = 0\}$, and having its *roof* be given by the graph of f. Another interpretation of the triple integral is where we view the integrand, that is f, as a density (mass per unit volume) of a substance occupying the domain D in three space, in which case the mass of the D is given by

$$\iiint_D f \, \mathrm{d}V.$$

We have now seen how we may define double and triple integrals over bounded domains, let us investigate how we can extend this definition to unbounded domains. To this end let us first consider the two dimensional case. Let f denote a real valued function of two variables defined on an unbounded domain $D \subseteq \mathbb{R}^2$, and for $R \in \mathbb{R}^+$ set $D_R = \overline{\mathbb{B}}^2((0, 0), R) \cap D$. We define the double integral of f over D as

$$\lim_{R \to \infty} \iint_{D_R} f \, \mathrm{d}A,$$

provided that the involved limit and integrals exist. Similarly, letting f denote a real valued function of three variables defined on an unbounded domain $D \subseteq \mathbb{R}^3$, and for $R \in \mathbb{R}$ setting $D_R = \overline{\mathbb{B}}^3((0,0,0), R) \cap D$, we define the triple integral of f over D as

$$\lim_{R \to \infty} \iiint_{D_R} f \, \mathrm{d}V,$$

provided that the involved limit and integrals exist. We note that this is just one way to define double and triple integrals over unbounded domains, and that there exists several other equivalent and non-equivalent definitions.

We remark that some double and triple integrals can be evaluated by inspection, using symmetry and known areas and volumes. Before looking at some specific examples let us collect some facts and properties of double and triple integrals.

Proposition 6.1.1

- *If D is a bounded subset of $\mathbb{R}^2$, then*

$$\iint_D 1 \, \mathrm{d}A = \textit{Area of } D.$$

- *If S is a bounded subset of $\mathbb{R}^3$, then*

$$\iiint_S 1 \, \mathrm{d}V = \textit{Volume of } S.$$

- *If D denotes a subset of $\mathbb{R}^2$, a and $b \in \mathbb{R}$, and f and g are functions of two variables x and y with domains containing D, then*

$$\iint_D af(x,y) + bg(x,y) \, \mathrm{d}A = a \iint_D f(x,y) \, \mathrm{d}A + b \iint_D g(x,y) \, \mathrm{d}A,$$

  provided that the involved integrals exist.

- *If D denotes a bounded subset of $\mathbb{R}^3$, a and $b \in \mathbb{R}$, and f and g are functions of three variables x, y and z with domains containing D, then*

$$\iiint_D af(x,y,z) + bg(x,y,z) \, \mathrm{d}V = a \iiint_D f(x,y,z) \, \mathrm{d}V + b \iiint_D g(x,y,z) \, \mathrm{d}V,$$

  provided that the involved integrals exist.

- *Let $D = D_1 \cup D_2$ be a subset of $\mathbb{R}^2$, where D_1 and D_2 are non-overlapping regions in $\mathbb{R}^2$. If f is a function of two variables x and y whose domain contains D, then*

$$\iint_D f(x,y) \, \mathrm{d}A = \iint_{D_1} f(x,y) \, \mathrm{d}A + \iint_{D_2} f(x,y) \, \mathrm{d}A,$$

  provided that the involved integrals exist.

- *Let $S = S_1 \cup S_2$ be a subset of $\mathbb{R}^3$, where S_1 and S_2 are non-overlapping regions in $\mathbb{R}^3$. If f is a function of three variables x, y and z whose domain contains S, then*

$$\iiint_S f(x,y,z) \, \mathrm{d}V = \iiint_{S_1} f(x,y,z) \, \mathrm{d}V + \iiint_{S_2} f(x,y,z) \, \mathrm{d}V,$$

  provided that the involved integrals exist.

- *Let $D = \{(x,y) \in \mathbb{R}^2 \colon a \leq x \leq b \text{ and } g(x) \leq y \leq h(x)\}$, where g and $h \colon [a,b] \to \mathbb{R}$ are continuous, $g(x) \leq h(x)$ for all $x \in [a,b]$, and a and $b \in \mathbb{R}$ with $a < b$. If $f \colon D \to \mathbb{R}$ is a continuous function of two variables x and y, then*

$$\iint_D f(x,y) \, \mathrm{d}A = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y) \, \mathrm{d}y \right) \mathrm{d}x.$$

  An analogous result also holds true for triple integral.

- *Let $D = \{(x, y) \in \mathbb{R}^2 \colon a \leq y \leq b$ and $g(y) \leq x \leq h(y)\}$, where g and $h \colon [a, b] \to \mathbb{R}$ are continuous, $g(y) \leq h(y)$ for all $y \in [a, b]$, and a and $b \in \mathbb{R}$ with $a < b$. If $f \colon D \to \mathbb{R}$ is a continuous function of two variables x and y, then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \left(\int_{g(y)}^{h(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

  An analogous result also holds true for triple integral.

Our next result is a special case of the famous Fubini-Tonelli Theorem and generalises the last two properties of the above proposition. This result allows us to write double and triple integrals as repeated integrals, and allows us to swap the order of integration within a repeated integral.

Theorem 6.1.2 (Fubini-Tonelli's Theorem) *Let D denote a region of $\mathbb{R}^2$ and let f denote a real-valued function of two variables x and y, with domain D. If one of*

$$\iint_D |f(x,y)|\,\mathrm{d}A, \qquad \iint_D (|f(x,y)|\,\mathrm{d}x)\,\mathrm{d}y, \qquad \iint_D (|f(x,y)|\,\mathrm{d}y)\,\mathrm{d}x$$

exist, then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_D (f(x,y)\,\mathrm{d}x)\,\mathrm{d}y = \iint_D (f(x,y)\,\mathrm{d}y)\,\mathrm{d}x.$$

In other words, the double integral of f over D is equal to a repeated integral where we first integrate with respect to x and then with respect to y, which in turn is equal to the repeated integral where we first integrate with respect to y and then with respect to x.

Let S denote a bounded region of $\mathbb{R}^3$ and let f denote a real-valued function of three variables x, y and z, with domain S. If one of

$$\begin{aligned}
\iiint_S |f(x,y,z)|\,\mathrm{d}V, \quad &\iiint_S (|f(x,y,z)|\,\mathrm{d}x)\,\mathrm{d}y)\,\mathrm{d}z, \quad \iiint_S (|f(x,y,z)|\,\mathrm{d}y)\,\mathrm{d}x)\,\mathrm{d}z,\\
&\iiint_S (|f(x,y,z)|\,\mathrm{d}x)\,\mathrm{d}z)\,\mathrm{d}y, \quad \iiint_S (|f(x,y,z)|\,\mathrm{d}z)\,\mathrm{d}y)\,\mathrm{d}x,\\
&\iiint_S (|f(x,y,z)|\,\mathrm{d}z)\,\mathrm{d}x)\,\mathrm{d}y, \quad \iiint_S (|f(x,y,z)|\,\mathrm{d}y)\,\mathrm{d}z)\,\mathrm{d}x
\end{aligned}$$

exist, then

$$\begin{aligned}
&\iiint_S f(x,y,z)\,\mathrm{d}V\\
&= \iiint_S (f(x,y,z)\,\mathrm{d}x)\,\mathrm{d}y)\,\mathrm{d}z = \iiint_S (f(x,y,z)\,\mathrm{d}y)\,\mathrm{d}x)\,\mathrm{d}z = \iiint_S (f(x,y,z)\,\mathrm{d}x)\,\mathrm{d}z)\,\mathrm{d}y\\
&= \iiint_S (f(x,y,z)\,\mathrm{d}z)\,\mathrm{d}y)\,\mathrm{d}x = \iiint_S (f(x,y,z)\,\mathrm{d}z)\,\mathrm{d}x)\,\mathrm{d}y = \iiint_S (f(x,y,z)\,\mathrm{d}y)\,\mathrm{d}z)\,\mathrm{d}x.
\end{aligned}$$

Exercise 6.1.3 *Find the value of the following integrals over the given regions R.*

- $\displaystyle\iint_R x^2y^2\,\mathrm{d}A$ *where* $R = \left\{(x,y) \in \mathbb{R}^2 \colon 1 \leq y \leq 2 \text{ and } 0 \leq x \leq 3\right\}$

- $\displaystyle\iint_R x^2y^2\,\mathrm{d}A$ *where* $R = \left\{(x,y) \in \mathbb{R}^2 \colon 1 \leq y \leq 2 \text{ and } 0 \leq x \leq y\right\}$

- $\displaystyle\iiint_R 1\,\mathrm{d}V$ *where R is the tetrahedron bounded by four planes respectively determined by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$*

- $\displaystyle\iiint_R z\,\mathrm{d}V$ *where R is the tetrahedron bounded by four planes planes respectively determined by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$*

Solution.

- Here we apply Fubini-Tonelli's Theorem. To this end observe that the region R is a bounded subset of the positive quadrant of $\mathbb{R}^2$ see FIGURE 6.2. Further, the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2y^2$ is continuous positive and bounded on the region R. Therefore, we know that the double integral of $|f|$ over R exists, and so the conditions of Fubini-Tonelli's Theorem are satisfied. Hence,

$$\iint_R x^2y^2\,\mathrm{d}A = \int_{y=1}^{y=2}\left(\int_{x=0}^{x=3} x^2y^2\,\mathrm{d}x\right)\mathrm{d}y = \int_{y=1}^{y=2}\left[y^2\frac{x^3}{3}\right]_{x=0}^{x=3}\mathrm{d}y = \int_{y=1}^{y=2} 9y^2\,\mathrm{d}y = 21.$$

- Here we apply Fubini-Tonelli's Theorem. To this end observe that the region R is a bounded subset of the positive quadrant of $\mathbb{R}^2$ see FIGURE 6.2. Further, the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2y^2$ is continuous positive and bounded on the region R. Therefore, we know that the double integral of $|f|$ over R exists, and so the conditions of Fubini-Tonelli's Theorem are satisfied. We can solve the integral in two ways, first integrating with respect to x and then with respect to y, and second, integrating with respect to y and then with respect to x. In the former cases we have

$$\iint_R x^2y^2\,\mathrm{d}A = \int_{y=1}^{y=2}\int_{x=0}^{x=y} x^2y^2\,\mathrm{d}x\mathrm{d}y = \int_{y=1}^{y=2}\left[\frac{x^3y^2}{3}\right]_{x=0}^{x=y}\mathrm{d}y = \int_{y=1}^{y=2}\frac{y^5}{3}\,\mathrm{d}y = \left[\frac{y^6}{18}\right]_{y=1}^{y=2} = \frac{7}{2}.$$

In the latter cases, we divide the region R into two regions $R_1 = \{(x,y)\colon 0 \le x \le 1 \,\text{and}\, 1 \le y \le 2\}$ and $R_2 = \{(x,y)\colon 1 \le x \le 2 \,\text{and}\, x \le y \le 2\}$. In which case we have that

$$\begin{aligned}
\iint_R x^2y^2\,\mathrm{d}A &= \iint_{R_1} x^2y^2\,\mathrm{d}A + \iint_{R_1} x^2y^2\,\mathrm{d}A \\
&= \int_{x=0}^{x=1}\left(\int_{y=1}^{y=2} x^2y^2\,\mathrm{d}y\right)\mathrm{d}x + \int_{x=1}^{x=2}\left(\int_{y=x}^{y=2} x^2y^2\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{x=0}^{x=1}\left[\frac{x^2y^3}{3}\right]_{y=1}^{y=2}\mathrm{d}x + \int_{x=1}^{x=2}\left[\frac{x^2y^3}{3}\right]_{y=x}^{y=2}\mathrm{d}x \\
&= \int_{x=0}^{x=1}\frac{7x^2}{3}\,\mathrm{d}x + \int_{x=1}^{x=2}\frac{8x^2}{3} - \frac{x^5}{3}\,\mathrm{d}x = \left[\frac{7x^3}{9}\right]_{x=0}^{x=1} + \left[\frac{8x^3}{9} - \frac{x^6}{18}\right]_{x=1}^{x=2} = \frac{7}{2}.
\end{aligned}$$

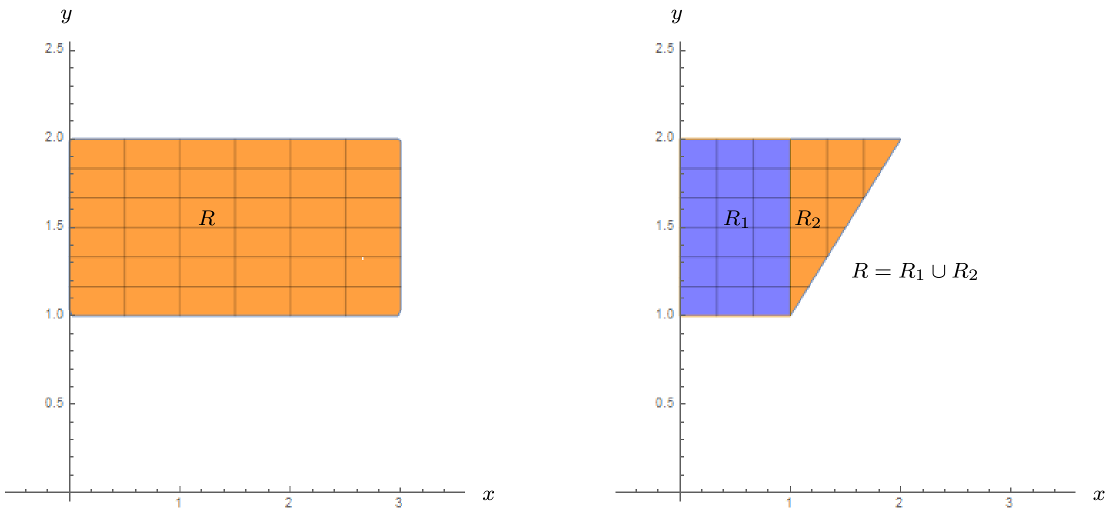

Figure 6.2: LEFT: Plot of the region $\{(x,y)\colon 1 \le y \le 2 \,\text{and}\, 0 \le x \le 3\}$. LEFT: Plot of the region $\{(x,y)\colon 1 \le y \le 2 \,\text{and}\, 0 \le x \le 3\}$.

- Here we apply Fubini-Tonelli's Theorem. To this end observe that the region R is a bounded subset of the first octant of $\mathbb{R}^3$ see FIGURE 6.3. Further, the constant function with value 1 is continuous positive and bounded on the region R. Therefore, we know that the triple integral of $|1|$ over R exists, and so the conditions of Fubini-Tonelli's Theorem are satisfied. With this at hand, we can now solve the integral in several ways.

Here, let us solve it by first integrating with respect to x, then with with respect to z and finally with respect to y. Hence,

$$\begin{aligned}
\iiint_R 1\,\mathrm{d}V &= \int_{y=0}^{y=1}\left(\int_{z=0}^{z=1-y}\left(\int_{x=0}^{x=1-y-z} 1\,\mathrm{d}x\right)\mathrm{d}z\right)\mathrm{d}y\\
&= \int_{y=0}^{y=1}\left(\int_{z=0}^{z=1-y}[x]_{x=0}^{x=1-y-z}\,\mathrm{d}z\right)\,\mathrm{d}y\\
&= \int_{y=0}^{y=1}\left(\int_{z=0}^{z=1-y}(1-y-z)\,\mathrm{d}z\right)\,\mathrm{d}y\\
&= \int_{y=0}^{y=1}\left[z-yz-\frac{z^2}{2}\right]_{z=0}^{z=1-y}\,\mathrm{d}y\\
&= \int_{y=0}^{y=1}(1-y)-y(1-y)-\frac{(1-y)^2}{2}\,\mathrm{d}y\\
&= \left[y-\frac{y^2}{2}-\frac{y^2}{2}+\frac{y^3}{3}+\frac{(1-y)^3}{6}\right]_{y=0}^{y=1} = \frac{1}{6}.
\end{aligned}$$

- Here we apply Fubini-Tonelli's Theorem. To this end observe that the region R is a bounded subset of the first octant of $\mathbb{R}^3$ see FIGURE 6.3. Further, the function $f\colon R\to\mathbb{R}$ given by $f(x,y,z)=z$ is continuous positive and bounded on the region R. Therefore, we know that the triple integral of $|f|$ over R exists, and so the conditions of Fubini-Tonelli's Theorem are satisfied. With this at hand, we can now solve the integral in several ways.

  Here, let us solve it by first integrating with respect to x, then with with respect to y and finally with respect to z. Hence,

$$\begin{aligned}
\iiint_R z\,\mathrm{d}V &= \int_{z=0}^{z=1}\left(\int_{y=0}^{y=1-z}\left(\int_{x=0}^{x=1-y-z} z\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z\\
&= \int_{z=0}^{z=1}\left(\int_{y=0}^{y=1-z} z(1-y-z)\mathrm{d}y\right)\mathrm{d}z\\
&= \int_{z=0}^{z=1} z\left((1-z)-\frac{(1-z)^2}{2}-(1-z)z\right)\,\mathrm{d}z\\
&= \int_{z=0}^{z=1}\frac{z-2z^2+z^3}{2}\,\mathrm{d}z = \left[\frac{z^2}{4}-\frac{z^3}{3}+\frac{z^4}{8}\right]_{z=0}^{z=1} = \frac{1}{24}.
\end{aligned}$$

In certain cases one can also write a multiple integral as the product of single integrals. This is the case when we can write the integrand as a product of functions of a single variable and when the limits of integration are independent of the variables of integration. For example, when f is a function of two variables x and y, which can be written as a product of two functions of a single variable. Namely, if there exists functions g and h of a single variable such that $f(x,y)=g(x)h(y)$. With this trick at hand, one can often use multiple integrals to find the value of single integrals.

Exercise 6.1.4 *Show that*

$$I=\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Solution. Let $f(x,y)=e^{-(x^2+y^2)}$ and, for $R\in\mathbb{R}^+$ set $E_R=\{(x,y)\in\mathbb{R}^2\colon x \text{ and } y\in[0,R]\}$. Let

$$I=\int_0^\infty e^{-x^2}\,\mathrm{d}x,$$

and observe that, since f is continuous and bounded, and since for a fixed R the set E_R is bounded, by Fubini-Tonelli's Theorem, for $R\in\mathbb{R}^+$

$$\iint_{E_R} f(x,y)\,\mathrm{d}A = \int_0^R\int_0^R e^{-x^2}e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x = \left(\int_0^R e^{-x^2}\,\mathrm{d}x\right)\left(\int_0^R e^{-y^2}\,\mathrm{d}y\right) = \left(\int_0^\infty e^{-x^2}\,\mathrm{d}x\right)^2.$$

Thus, if we can find the value of the double integral

$$\iint_{E_R} f(x,y)\,\mathrm{d}A,$$

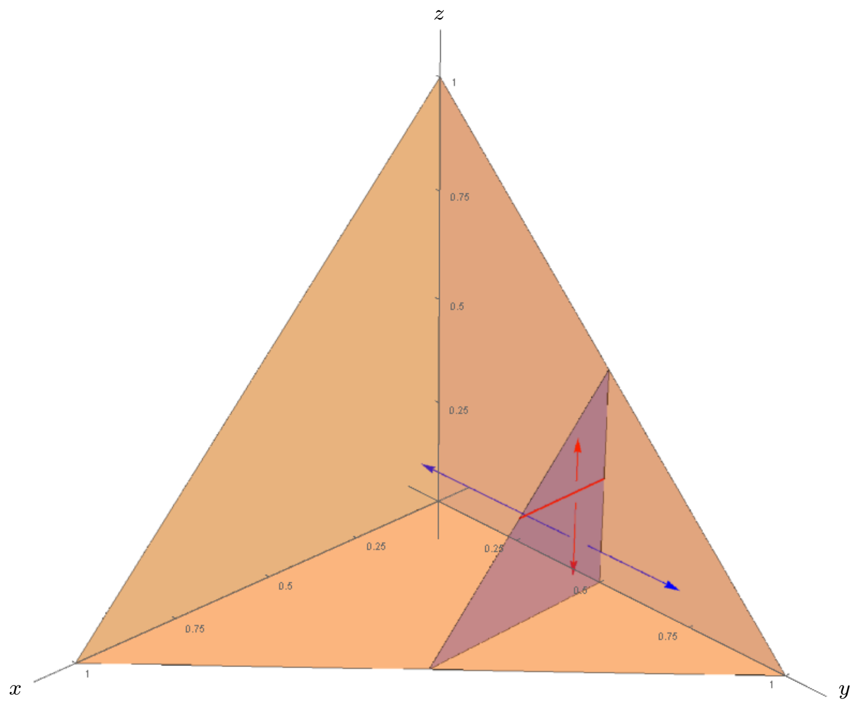

Figure 6.3: The length of the red line is given by the value $\int_{x=0}^{x=1-y-z} 1\,\mathrm{d}x$ where $y = 0.5$ and $z = 0.25$; the area of the blue shaded region is given by the value $\int_{z=0}^{z=1-y} \left(\int_{x=0}^{x=1-y-z} 1\,\mathrm{d}x\right) \mathrm{d}z$ where $y = 0.5$; and the volume of the orange shaded region is given by $\int_{y=0}^{y=1} \left(\int_{z=0}^{z=1-y} \left(\int_{x=0}^{x=1-y-z} 1\,\mathrm{d}x\right) \mathrm{d}z\right) \mathrm{d}y$.

and evaluate the limit as R tends to infinity, then we would be able to solve the problem. To do this efficiently, we appeal to the results of the following sections. Thus, we will return to this exercise later on.

The above results, examples and exercises show how we can evaluate integrals over regions bounded by straight lines. However, in reality, when we are required to compute the value of a multiple integral, our region of integration may not be as such, for instance it could be a sphere or an ellipsoid. Indeed, when solving integrals of functions of a single variable, sometime it helps to apply a change of variable. Also, as we have seen, when solving PDEs, changing our co-ordinate system helps to simplify the problem and thus find a solution. In the following sections we will examine how we can apply such methods to multiple integrals in order to be able to evaluate a multiple integral over a region bounded by curves that are not straight lines.

6.2 Co-ordinate systems

Recall, a point in two dimensions can be thought of as lying at the corner of a rectangle (Cartesian) or on a circle (polar).

- **Cartesian co-ordinates in 2-dimensions**: A point P in $\mathbb{R}^2$ may be specified by two Cartesian co-ordinates, (x, y), where x and y belong to $\mathbb{R}$.
- **Polar co-ordinates in 2-dimensions**: A point P in $\mathbb{R}^2$ may be specified by two polar coordinates, (r, θ), where r is the radial distance from the origin to P, and θ is the directed angle measured counter-clockwise from the positive x-axis to the radial line connecting the origin to P. To ensure a unique representation, we usually take r to be a non-negative real number and θ in the interval $(-\pi, \pi]$.

Further, recall that, 2-dimensional Cartesian and polar co-ordinates are related as follows.

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$
$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

A point in 3-dimensions can be thought of as lying at a corner of a (Cartesian) rectangle, on the surface of a cylinder with axis running along the z-axis (cylindrical polar) or on a sphere with centre at the origin (spherical polar).

- **Cartesian co-ordinates in 3-dimensions**: A point P in $\mathbb{R}^3$ may be specified by three Cartesian co-ordinates, (x, y, z), where x, y and z belong to $\mathbb{R}$.
- **Cylindrical polar co-ordinates**: A point P in $\mathbb{R}^3$, equal to (x, y, z) in Cartesian co-ordinates, may be specified by three cylindrical co-ordinates, (r, θ, z), where r and θ are the two dimensional polar co-ordinates specifying x and y, and $z \in \mathbb{R}$ is the usual Cartesian co-ordinate specifying the height of P above the x-y plane. To ensure a unique representation, we usually take r to be a non-negative real number and θ in the interval $(-\pi, \pi]$.
- **Spherical polar co-ordinates**: A point P in $\mathbb{R}^3$, equal to (x, y, z) in Cartesian co-ordinates, may be specified by three spherical polar co-ordinates, (ρ, ϕ, θ), where ρ is the radial distance from P to O – the origin, ϕ is the angle the line OP makes with the positive z-axis, and θ is the angle in the x-y plane that the line from the origin to the point $(x, y, 0)$ makes with the positive x-axis. To ensure a unique representation, we usually take ρ to be a non-negative real number, ϕ in the interval $[0, \pi]$, and θ in the interval $(-\pi, \pi]$.

Three dimensional Cartesian and cylindrical polar co-ordinates are related as follows.

$$\begin{aligned} x &= r\cos(\theta) & y &= r\sin(\theta) & z &= z \\ r^2 &= x^2 + y^2 & \tan(\theta) &= \frac{y}{x} & z &= z \end{aligned}$$

Applying trigonometric identities shows 3-dimensional Cartesian and spherical polar co-ordinates are related as follows.

$$\begin{aligned} x &= \rho\sin(\phi)\cos(\theta) & y &= \rho\sin(\phi)\sin(\theta) & z &= \rho\cos(\phi) \\ \rho^2 &= x^2 + y^2 + z^2 & \tan(\theta) &= \frac{y}{x} & \tan(\phi) &= \frac{\sqrt{x^2+y^2}}{z} \end{aligned}$$

Further, one can show that cylindrical polar and spherical polar co-ordinates are related as follows.

$$\begin{aligned} r &= \rho\sin(\phi) & \theta &= \theta & z &= \rho\cos(\phi) \\ \rho^2 &= r^2 + z^2 & \theta &= \theta & \tan(\phi) &= \frac{r}{z} \end{aligned}$$

Warning: some authors may use r for what we have called ρ, or use θ for ϕ and ϕ for θ when specifying spherical polar co-ordinates. Therefore, care should be taken to determine which convention is being used.

6.3 The Jacobian and change of variable for multiple integrals

For many double integrals, either the domain of integration, the integrand, or both maybe more easily expressed in terms of polar co-ordinates, rather than in Cartesian co-ordinates. For instance consider the problem of finding the volume of the solid region lying above the x-y plane and beneath the parabola $z = 1 - x^2 - y^2$. Since the parabola intersects the x-y plane in the circle $x^2 + y^2 = 1$, applying Fubini-Tonelli's theorem, gives that the volume, in Cartesian co-ordinates, can be obtained from the integral

$$V = \iint_D (1 - x^2 - y^2)\,\mathrm{d}A = \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (1 - x^2 - y^2)\,\mathrm{d}y\mathrm{d}x,$$

where $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$. Evaluating this integral would require some effort. However, we can represent the same volume in terms of polar co-ordinates as

$$V = \iint_D (1 - r^2)\,\mathrm{d}A.$$

In order to evaluate this integral we need to understand the area element $\mathrm{d}A$ in polar co-ordinates. Indeed, in the Cartesian formula for V, the area element $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ represents the area of the *infinitesimal* region bounded by the co-ordinate lines at x, $x + \mathrm{d}x$, y and $y + \mathrm{d}y$, see FIGURE 6.4. In polar form the area element $\mathrm{d}A$ should represent the area of the *infinitesimal* region bounded by the co-ordinate circles with radius r and $r + \mathrm{d}r$ and co-ordinate rays from the origin at angles θ and $\theta + \mathrm{d}\theta$, see FIGURE 6.4. Notice, the area element $\mathrm{d}A$ in polar co-ordinates is approximately the area of a rectangle with dimensions $\mathrm{d}r$ and $r\mathrm{d}\theta$. The error in this approximation becomes negligible compared with the size of $\mathrm{d}A$ as $\mathrm{d}r$ and $\mathrm{d}\theta$ approaches zero. Therefore, in transforming a double integral from Cartesian to polar co-ordinates, the area element transforms according to the formula

$$\mathrm{d}x\,\mathrm{d}y = \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

In order to compute the value of V considered above, we also need to understand the region we are integrating over. In the above case we have that D is the area encompassed by the unit disc centred at the origin. Thus, $D = \{(r, \theta) \in \mathbb{R}^2 \colon 0 \ \leq r \leq 1 \text{ and } -\pi \leq \theta \leq \pi\}$ and so, this together with the fact that the conditions of Fubini-Tonelli's theorem are satisfied, since D is bounded and our integrand is positive, continuous and bounded, yields

$$V = \iint_D (1 - r^2)\,\mathrm{d}A = \int_{\theta=-\pi}^{\theta=\pi} \int_{r=0}^{r=1} (1 - r^2) r\,\mathrm{d}r\mathrm{d}\theta = \int_{\theta=-\pi}^{\theta=\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=1} \mathrm{d}\theta = \int_{-\pi}^{\pi} \frac{1}{4}\,\mathrm{d}\theta = \frac{\pi}{2}.$$

With the above at hand we can also complete what we started in **Exercise 6.1.4**. Indeed, here we have that $f(x, y) = e^{-(x^2+y^2)}$, $E_R = \{(x, y) \in \mathbb{R}^2 \colon x \text{ and } y \in [0, R]\}$, for $R \in \mathbb{R}^+$, and

$$I = \int_0^\infty e^{-x^2}\,\mathrm{d}x.$$

Further, via an application of Fubini-Tonelli's Theorem, we showed that

$$\begin{aligned}
\lim_{R\to\infty} \iint_{E_R} f(x, y)\,\mathrm{d}A &= \lim_{R\to\infty} \int_0^R \int_0^R e^{-x^2} e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \lim_{R\to\infty} \left(\int_0^R e^{-x^2}\,\mathrm{d}x \right) \left(\int_0^R e^{-y^2}\,\mathrm{d}y \right) \\
&= \lim_{R\to\infty} \left(\int_0^R e^{-x^2}\,\mathrm{d}x \right)^2 \\
&= \left(\lim_{R\to\infty} \int_0^R e^{-x^2}\,\mathrm{d}x \right)^2 = I^2.
\end{aligned}$$

To find the value of I, for $R \in \mathbb{R}^+$, we set $D_R = \{(x, y) \in (\mathbb{R}_0^+)^2 \colon x^2 + y^2 \leq R^2\}$. Using the above, together with the fact that the conditions of Fubini-Tonelli's theorem are satisfied, since D_R is bounded and f is positive, continuous and bounded on D_R, we observe that

$$\begin{aligned}
\iint_{D_R} f(x, y)\,\mathrm{d}A &= \iint_{D_R} e^{-r^2} r\,\mathrm{d}r\mathrm{d}\theta = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=R} e^{-r^2} r\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=0}^{\theta=\pi/2} \left[-\frac{e^{-r^2}}{2} \right]_{r=0}^{r=R} \mathrm{d}\theta = \int_{\theta=0}^{\theta=\pi/2} \frac{1 - e^{-R^2}}{2}\,\mathrm{d}\theta = \frac{\pi(1 - e^{-R^2})}{4}.
\end{aligned}$$

Combining the above together with the fact that $D_R \subset E_R \subset D_{\sqrt{2}R}$ and that $\mathrm{e}^{(-x^2-y^2)/2}$ is positive for $(x, y) \in \mathbb{R}$, for a given $R \in \mathbb{R}^+$, yields

$$\left(\int_0^R \mathrm{e}^{-x^2/2}\,\mathrm{d}x \right)^2 = \iint_{E_R} \mathrm{e}^{(-x^2-y^2)/2}\,\mathrm{d}A \begin{cases} \geq \displaystyle\iint_{D_R} \mathrm{e}^{(-x^2-y^2)/2}\,\mathrm{d}A, \\ \leq \displaystyle\iint_{D_{\sqrt{2}\,R}} \mathrm{e}^{(-x^2-y^2)/2}\,\mathrm{d}A. \end{cases}$$

Taking the limit as R tends to infinity, and observing that the square root function is continuous on $\mathbb{R}^+$, we obtain that $\sqrt{2\pi}$ is both an upper bound and a lower bound for

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x,$$

and thus, it follows that $I = \sqrt{\pi}/2$. We emphasis here that this integral plays a very important rôle in probability theory and statistics.

What we have developed thus far in this section is a special case of a what is known as a Jacobian which is a key ingredient required to develop integration by substitution for multiple integrals.

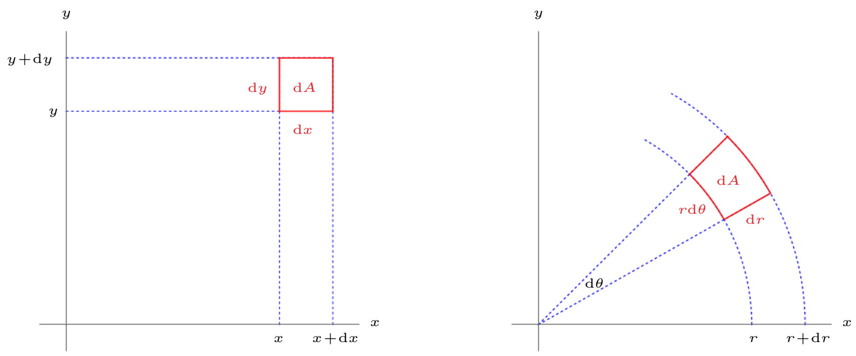

Figure 6.4: Left: Illustration of the area element $\mathrm{d}A$ in Cartesian co-ordinates. Left: Illustration of the area element $\mathrm{d}A$ in polar co-ordinates.

Definition 6.3.1 *Suppose that x and y are functions of two variables u and v, whose partial derivatives exist, then the* ***Jacobian*** *of $x = x(u, v)$ and $y = y(u, v)$ with respect to u and v is*

$$\frac{\partial(x, y)}{\partial(u, v)} := \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

Similarly, suppose that x, y and z are functions of three variables u, v and w, whose partial derivatives exist, then the ***Jacobian*** *of $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ with respect to u, v and w is*

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} := \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

Exercise 6.3.2 *Show that the Jacobian J associated to changing from Cartesian to polar co-ordinates is given by $J(r, \theta) = r$.*

Solution. Recall that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and therefore,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r(\cos^2(\theta) + \sin^2(\theta)) = r.$$

Theorem 6.3.3 (Integration by substitution) *Under the assumption that the hypotheses of Fubini-Tonelli's theorem are satisfied, we have the following.*

- *If $x = x(u, v)$ and $y = y(u, v)$ are functions of two variables u and v, whose partial derivatives exist and are continuous, and provided the Jacobian of x and y with respect to u and v does not change sign, then*

$$\mathrm{d}x\mathrm{d}y = \mathrm{d}A = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\mathrm{d}v.$$

  Moreover, if D is a bounded planar region, $f\colon D \to \mathbb{R}$, and the Jacobian of x and y with respect to u and v does not change sign in D, then

$$\iint_D f\,\mathrm{d}A = \iint_D f(x, y)\,\mathrm{d}x\mathrm{d}y = \iint_D f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\mathrm{d}v,$$

- *If $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ are functions of three variables u, v and w, whose partial derivatives exist and are continuous, and provided the Jacobian of x, y and z with respect to u, v and w does not change sign, then*

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = \mathrm{d}V = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

  Moreover, if S is a bounded region of 3-space, $f\colon S \to \mathbb{R}$, and the Jacobian of x, y and z with respect to u, v and w does not change sign in S, then

$$\iiint_S f\,\mathrm{d}V = \iiint_S f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

With this at hand, and under the assumption that the hypotheses of Fubini-Tonelli's theorem are satisfied, we have the following.

- If $D \subseteq \mathbb{R}^2$ is bounded and $f\colon D \to \mathbb{R}$, then

$$\iint_D f \,\mathrm{d}A = \iint_D f(x, y)\,\mathrm{d}x\mathrm{d}y = \iint_D f(x(r, \theta), y(r, \theta))\, r\,\mathrm{d}r\mathrm{d}\theta.$$

- If $S \subseteq \mathbb{R}^3$ is bounded and $f\colon S \to \mathbb{R}$, then

$$\iiint_S f\,\mathrm{d}V = \iiint_S f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_S f(x(r, \theta, z), y(r, \theta, z), z(r, \theta, z))\, r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

- If $S \subseteq \mathbb{R}^3$ is bounded and $f\colon S \to \mathbb{R}$, then

$$\iiint_S f\,\mathrm{d}V = \iiint_S f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_S f(x(r, \phi, \theta), y(r, \phi, \theta), z(r, \phi, \theta))\, \rho^2 \sin(\phi)\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta.$$

Exercise 6.3.4

- *Find the value of* $\iint_D \sqrt{x^2 + y^2}\,\mathrm{d}A$ *where* $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 2x\}$.

- *Let C denote the solid cone in $\mathbb{R}^3$ determined by $x^2 + y^2 \leq (1 - z)^2$ and $0 \leq z \leq 1$. If the density of C at a point (x, y, z) is z, find the mass of C.*

- *Using a triple integral, find the volume of a cone whose height is h and whose base has radius a.*

Solution.

- We observe that by completing the square we have that the set D is the set of points within the circle centred at $(1, 0)$ with radius 1, namely

$$D = \{(x, y) \in \mathbb{R}^2 \colon (x - 1)^2 + y^2 \leq 1\}.$$

Further, the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \sqrt{x^2 + y^2}$ is continuous positive and bounded on the region D. Therefore, we know that the double integral of $|f|$ over D exists, and so the conditions of Fubini-Tonelli's Theorem are satisfied. Additionally we observe that working in polar co-ordinates will be easier than working in Cartesian coordinates, as the region we are integrating over is a solid disc, namely $D = \{(r, \theta) \in \mathbb{R}^2 \colon -\pi/2 \leq \theta \leq \pi/2 \,\text{and}\, 0 \leq r \leq 2\cos(\theta)\}$ and in polar co-ordinates our function f becomes $(r, \theta) \mapsto r$. With this at hand we have that

$$\begin{aligned}
\iint_D \sqrt{x^2 + y^2}\,\mathrm{d}A = \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=2\cos(\theta)} r^2\,\mathrm{d}r\mathrm{d}\theta &= \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{8\cos^3(\theta)}{3}\,\mathrm{d}\theta \\
&= \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{8\cos(\theta)(1 - \sin^2(\theta))}{3}\,\mathrm{d}\theta \\
&= \int_{u=-1}^{u=1} \frac{8(1 - u^2)}{3}\,\mathrm{d}u \\
&= \left[\frac{8(3u - u^3)}{9}\right]_{u=-1}^{u=1} = \frac{32}{9},
\end{aligned}$$

where $u = \sin(\theta)$. Notice, integrating with respect to θ first would be significantly more difficult.

- If the density of a solid $D \subset \mathbb{R}^3$ is given by a function $g\colon D \to \mathbb{R}$, then the mass of D is equal to,

$$\iiint_D g\,\mathrm{d}V.$$

Notice, in cylindrical polars, $C = \{(r,\theta,z) \in \mathbb{R}^3 : 0 \le r \le 1-z, 0 \le z \le 1 \,\text{and}\, -\pi \le \theta \le \pi\}$ and that on C the function $(r,\theta,z) \mapsto z$ is continuous positive and bounded, and so the conditions of Fubini-Tonelli's Theorem are meet and therefore

$$\begin{aligned}
\text{Mass of C} &= \int_{z=0}^{z=1} \int_{\theta=-\pi}^{\theta=\pi} \int_{r=0}^{r=1-z} zr \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= \int_{z=0}^{z=1} \int_{\theta=-\pi}^{\theta=\pi} \left[\frac{zr^2}{2}\right]_{r=0}^{r=1-z} \mathrm{d}\theta\mathrm{d}z \\
&= \int_{z=0}^{z=1} \int_{\theta=-\pi}^{\theta=\pi} \frac{z(1-z)^2}{2} \,\mathrm{d}\theta\mathrm{d}z \\
&= \int_{z=0}^{z=1} \pi(z^3 - 2z^2 + z)\,\mathrm{d}z \\
&= \pi\left[\frac{z^4}{4} - \frac{2z^3}{3} + \frac{z^2}{2}\right]_{z=0}^{z=1} = \frac{\pi}{12}.
\end{aligned}$$

- Here it is useful to view the cone with the apex at the origin and central axes inline with the z-axes. In spherical polar coordinates, such a cone of height h and radius a is given by

$$C = \{(\rho,\phi,\theta) : -\pi \le \theta \le \pi, 0 \le \phi \le \arctan(a/h) \,\text{and}\, 0 \le \rho \le h\sec(\phi)\}.$$

Since the cone C is bounded, and since a constant function is continuous and bounded, the conditions of Fubini-Tonelli's Theorem are meet and therefore

$$\begin{aligned}
\text{Volume of } C = \iiint_C 1\,\mathrm{d}V &= \int_{\phi=0}^{\phi=\arctan(a/h)} \int_{\theta=-\pi}^{\theta=\pi} \int_{\rho=0}^{\rho=h\sec(\phi)} \rho^2 \sin(\phi)\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \\
&= \int_{\phi=0}^{\phi=\arctan(a/h)} \int_{\theta=-\pi}^{\theta=\pi} \frac{h^3\sec^3(\phi)\sin(\phi)}{3}\,\mathrm{d}\theta\mathrm{d}\phi \\
&= \int_{\phi=0}^{\phi=\arctan(a/h)} \frac{2\pi h^3 \sec^3(\phi)\sin(\phi)}{3}\,\mathrm{d}\phi \\
&= \left[\frac{2\pi h^3}{3}\frac{\sec^2(\phi)}{2}\right]_{\phi=0}^{\phi=\arctan a/h} \\
&= \left[\frac{\pi h^3(\tan^2(\phi)+1)}{3}\right]_{\phi=0}^{\phi=\arctan a/h} \\
&= \frac{\pi h^3(a^2/h^2+1)}{3} - \frac{\pi h^3}{3} = \frac{\pi h a^2}{3}.
\end{aligned}$$

Chapter A

Formative assessments

A.1 Formative assessment – Week 7

(1) Find the Taylor series, if it exists, for the given functions about the indicated point.

(a) $f\colon \{(x,y) \in \mathbb{R}^2 : x \geq 0\} \to \mathbb{R}$ given by $f(x,y) = 1/(2+xy^2)$ at $(0,0)$

(b) $f\colon \{(x,y) \in \mathbb{R}^2 : -1 \leq x(1+y) \leq 1\} \to \mathbb{R}$ given by $f(x,y) = \arctan(x+xy)$ at $(0,-1)$

(c) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = e^{x^2+y^2}$ at $(0,0)$

(2) State all the coefficients of the Taylor series of $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \int_0^{x-y^2} e^{-t^2}\,\mathrm{d}t$, about the point $(0,0)$, up to and including those of degree three terms only.

(3) State all the coefficients of the Taylor series, about the indicated points, up to and including those of degree three terms only, of the following functions.

(a) $f\colon \{(x,y) \in \mathbb{R}^2 : x < 2y < 2+x\} \to \mathbb{R}$ given by $f(x,y) = 1/(2+x-2y)$ at $(2,1)$

(b) $f\colon \{(x,y) \in \mathbb{R}^2 : x^2+y^4 < 1\} \to \mathbb{R}$ given by $f(x,y) = (1+x)/(1+x^2+y^4)$ at $(0,0)$

(4) Find all the stationary points of $f\colon \{(x,y) \in \mathbb{R}^2 : x,y > 0\} \to \mathbb{R}$ given by

$$f(x,y) = (3/(2x) - x/2 - y)^2 + x^2 + y^2.$$

(5) Find all stationary points of the following functions. Also, investigate their nature, that is, if they are a local extreme or a saddle point.

(a) $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x^4 + x^2y + y^2 + z^2 + xz$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^4 + 2y^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1$

(c) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^4 + y^4 + 6x^2y^2 + 8x^3$

(6) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = e^{(y+x\sin(y))}$ has infinitely many stationary points but that none are local maximum nor minimum.

(7) Find the dimensions (height, length and width) of a rectangular box, without a lid, which would contain the maximum amount of water, if the surface area of its walls and base add up to $108\,\mathrm{m}^2$.

(8) Find the extreme values of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

(9) Which points on the sphere $x^2 + y^2 + z^2 = 4$ are closest to and farthest from the point $(3,1,-1)$.

(10) Find the minimum value of $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x + 2y + 3z$ on the curve C formed by the intersection of the plane determine by $x - y + z = 1$ and the cylinder determined by $x^2 + y^2 = 1$.

(11) Let $n \in \mathbb{N}$ be fixed and recall that $\mathbb{R}^+$ denotes the set of all positive real numbers and that $(\mathbb{R}^+)^n$ denotes the set of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_i \in \mathbb{R}^+$ for all $i \in \{1, 2, \ldots, n\}$. Let $(a_1, a_2, \ldots, a_n) \in (\mathbb{R}^+)^n$ be fixed throughout this question.

(a) Find all $x = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^+)^n$ which maximise the sum $\sum_{i=1}^n a_i x_i$ subject to the constraint $\sum_{i=1}^n x_i^2 = 1$.

(b) For the points $x = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^+)^n$ located in Part (i), find the value of the sum $\sum_{i=1}^n a_i x_i$ in terms of $a_1, a_2, \ldots, a_n$.

(12) (a) Let $r \in \mathbb{R}$ be positive and let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^2 y^2 z^2$. Find the maximum value of f on a sphere in $\mathbb{R}^3$ centred at the origin and of radius r.

(b) Using Part (a), prove that

$$(abc)^{1/3} \leq \frac{a + b + c}{3},$$

for all a, b and $c \in \mathbb{R}^+$.

(13) Consider a terrestrial space probe in the shape of a solid ellipsoid, given by $4x^2 + y^2 + 4z^2 \leq 16$. As it enters the atmosphere of the alien planet *Cybertron*, it begins to heat. After one Earth hour, the temperature at the point (x, y, z) on the probe is given by $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point(s) on the probe's surface. Please remember to show all of your working. In your solution, you may use that $24\sqrt{3} \approx 41.569$.

(14) Suppose that you are in charge of erecting a radio telescope on the newly discovered planet *Vogsphere*. To minimise the interference with Vogsphere's natural magnetic field, the telescope must be placed where the magnetic field is weakest. The planet is spherical, with radius 6 units and, based on a Cartesian coordinate system whose origin is at the centre of Vogsphere, the strength of the magnetic field at a point $(x, y, z) \in \mathbb{R}^3$ is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Determine the best location(s) for the radio telescope on Vogsphere. Please remember to show all of your working.

A.2 Formative assessment – Week 8

(1) Evaluate the double integral

$$\iint_R (x-y)\,\mathrm{d}A$$

as a repeated integral where R is the triangle with vertices at $(3,10)$, $(3,1)$, and $(-2,1)$.

(2) By evaluating an appropriate integral, find the area enclosed by the graphs of $x=y^2$ and $x=2y-y^2$.

(3) Sketch the region $R=\{(x,y,z)\in\mathbb{R}^3\colon 0\le z\le 1-|x|-|y|\}$ and evaluate the integral

$$\iiint_R (xy+z^2)\,\mathrm{d}V.$$

(4) (a) Find the Cartesian and cylindrical polar co-ordinates of the point P which has spherical co-ordinates $(3,\pi/4,\pi/3)$.

(b) Find the spherical polar co-ordinates of the point Q which has Cartesian co-ordinates $(1,1,1)$.

(c) The surface, S, of a sphere of radius R centred the origin can be expressed in spherical polars co-ordinates as the set

$$\{(\rho,\phi,\theta)\colon \rho=R,\ 0\le\phi\le\pi \text{ and } -\pi\le\theta<\pi\}.$$

Find an expressions for S in terms of Cartesian and cylindrical polar co-ordinates.

(d) The surface C of a cylinder of radius R with axis lying along the z-axis can be expressed in terms of cylindrical polar co-ordinates as the set

$$\{(r,\theta,z)\colon r=R,\ -\pi\le\theta<\pi \text{ and } -\infty<z<\infty\}.$$

Find expressions for C in terms of Cartesian and spherical polar co-ordinates.

(e) Let K be the surface of an infinite cone with circular cross section, vertex at the origin and axis lying along the positive z-axis. If the angle between the z-axis and the surface of the cone is α, find expressions for K in terms of Cartesian, spherical and cylindrical polar co-ordinates.

(5) Verify the following statements.

(a) The element of area $\mathrm{d}A=\mathrm{d}x\mathrm{d}y$ in Cartesian co-ordinates is equal to $r\mathrm{d}r\mathrm{d}\theta$ in polar co-ordinates.

(b) The element of volume $\mathrm{d}V=\mathrm{d}x\mathrm{d}y\mathrm{d}z$ in Cartesian co-ordinates is equal to $r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$ in cylindrical polar co-ordinates.

(c) The element of volume $\mathrm{d}V=\mathrm{d}x\mathrm{d}y\mathrm{d}z$ is equal to $\rho^2\sin(\phi)\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$ in spherical polar co-ordinates.

(6) Evaluate each of the following double integrals over the given regions.

(a) $\displaystyle\iint_R (y+2x)\,\mathrm{d}A$ where R is the rectangle with vertices at $(-1,-1)$, $(2,-1)$, $(2,4)$ and $(-1,4)$

(b) $\displaystyle\iint_S (x^2+y^2)\,\mathrm{d}A$ where, in polar co-ordinates, $S=\{(r,\theta)\colon 1<r<2 \text{ and } \pi/3<\theta<\pi/2\}$

(7) (a) Evaluate the following integral.

$$I=\int_{y=0}^{y=\pi}\int_{x=0}^{x=2} xy^2+x^2\sin(y)\,\mathrm{d}x\mathrm{d}y$$

(b) Let $D = \{(x,y) \in \mathbb{R}^2 \colon 0 \le x, 0 \le y \,\text{and}\, x^2 + y^2 \le 1\}$. Using Cartesian co-ordinates, evaluate

$$I = \iint_D xy \,\mathrm{d}A.$$

(c) Let D denote the region that lies inside the circle $x^2 + y^2 = 2x$ and above the x-axis. By integrating with respect to polar co-ordinates, find the value of

$$I = \iint_D y\sqrt{x^2 + y^2}\,\mathrm{d}A.$$

(d) Let T be the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ and suppose that T has density xy at the point (x, y, z). Calculate the mass of T.

(8) Evaluate each of the following integrals as a repeated integral using an appropriate co-ordinate systems.

(a) $\iiint_C xy + z^2 \,\mathrm{d}V$ where C is the cube with corners $(\pm 2, \pm 2, \pm 2)$

(b) $\iiint_R ze^{-(x^2+y^2+z^2)}\,\mathrm{d}V$, where $R = \{(x,y,z) \in \mathbb{R}^3 \colon -\infty < x, y < \infty \,\text{and}\, 0 \le z < 1\}$

(c) $\iiint_S x^2 \,\mathrm{d}V$ where, in spherical polar co-ordinates, $S = \{(\rho, \phi, \theta) \in \mathbb{R}_0^+ \times [0, \pi) \times [-\pi, \pi) \colon \rho < 1\}$

(9) Using the substitutions $x = (u + v)/2$ and $y = (u - v)/2$, evaluate the integral

$$\iint_D e^{x+y}\,\mathrm{d}A,$$

where $D = \{(x, y) \in \mathbb{R}^2 \colon |x| + |y| \le 1\}$.

(10) By making the change of variables $x = (u + v)/2$ and $y = (u - v)/2$, or otherwise, calculate

$$\iint_T \cos\left(\frac{x-y}{x+y}\right)\,\mathrm{dA},$$

where T is the triangular region with corners $(0, 0)$, $(1, 0)$ and $(0, 1)$.

(11) A function $g \colon \mathbb{R} \to \mathbb{R}$ is said to be a **density** of a probability distribution if and only if **(i)** $g(x) \ge 0$ for all $x \in \mathbb{R}$, and **(ii)**

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1.$$

Let $k = (\ln(3)/(18\pi))^{1/2}$ and let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = k\, 3^{-x^2/18}$, for $x \in \mathbb{R}$. Determine if f is a density of some probability distribution.

(12) Let $\phi \colon \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\phi(s) = \int_0^{\infty} t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

for $s \in \mathbb{R}^+$. Find the value of ϕ at $s = 1/2$.

Chapter B

Formative assessment solutions

B.1 Formative assessment – Week 7 – Solutions

(1) Find the Taylor series, if it exists, for the given functions about the indicated point.

(a) $f\colon \{(x,y)\in\mathbb{R}^2\colon x\geq 0\}\to\mathbb{R}$ given by $f(x,y)=1/(2+xy^2)$ at $(0,0)$

(b) $f\colon \{(x,y)\in\mathbb{R}^2\colon -1\leq x(1+y)\leq 1\}\to\mathbb{R}$ given by $f(x,y)=\arctan(x+xy)$ at $(0,-1)$

(c) $f\colon \mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=e^{x^2+y^2}$ at $(0,0)$

Solution.

(a) Here we are after an expansion in term of x and y. Using our knowledge on the known value of the sum of a geometric series, for $t\in(-1,1)$, we have that

$$\frac{1}{1+t}=1-t+t^2-t^3+\cdots+(-1)^n t^n+\ldots.$$

Since

$$f(x,y)=\frac{1}{1+xy^2}=\frac{1}{2}\frac{1}{1+(xy^2)/2}$$

it follows that the Taylor series of f at $(0,0)$ is given, for all $(x,y)\in\mathrm{Dom}(f)$, by

$$1-((xy^2)/2)+((xy^2)/2)^2-((xy^2)/2)^3+\cdots+(-1)^n((xy^2)/2)^n+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^n y^{2n}}{2^{n+1}}.$$

(b) Here we are after an expression in terms of x and $y-(-1)$. To this end, recall that the Taylor series of $\arctan(t)$ at zero is given by

$$\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{2n+1}$$

and that the Taylor series of $\arctan(t)$ at zero converges to $\arctan(t)$, for all $t\in[-1,1]$. Therefore, we have, for all $(x,y)\in\mathrm{Dom}(f)$, that

$$\begin{aligned}\arctan(x+xy)&=\sum_{n=0}^{\infty}(-1)^n\frac{(x+xy)^{2n+1}}{2n+1}\\&=\sum_{n=0}^{\infty}(-1)^n\frac{(x(1+y))^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}(y-(-1))^{2n+1}}{2n+1}.\end{aligned}$$

(c) Here we are after an expansion in terms of x and y. To this end recall that the Taylor series of e^t at zero is given by

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}$$

and that the Taylor series of e^t at zero converges to e^t, for all $t \in \mathbb{R}$. Therefore, we have the following chain of equalities.

$$f(x,y) = \sum_{n=0}^{\infty} \frac{(x^2+y^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{2j} y^{2n-2j} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^{2j} y^{2n-2j}}{j!(n-j)!}$$

(3) State all the coefficients of the Taylor series, about the indicated points, up to and including those of degree three terms only, of the following functions.

(a) $f\colon \{(x,y) \in \mathbb{R}^2 \colon x < 2y < 2+x\} \to \mathbb{R}$ given by $f(x,y) = 1/(2+x-2y)$ at $(2,1)$

(b) $f\colon \{(x,y) \in \mathbb{R}^2 \colon x^2+y^4 < 1\} \to \mathbb{R}$ given by $f(x,y) = (1+x)/(1+x^2+y^4)$ at $(0,0)$

Solution.

(a) Here we are after an expansion in terms of $x-2$ and $y-1$. To this end, letting $u = x-2$ and $v = y-1$, observe that

$$f(x,y) = \frac{1}{2+(2+u)-2(v+1)} = \frac{1}{2+u-2v} = \frac{1}{2(1+((u-2v)/2))}.$$

Thus, by our knowledge on the value of the sum of a geometric series, we have the following chain of equalities.

$$\begin{aligned} f(x,y) =& \frac{1}{2(1+((u-2v)/2))} \\ =& \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{u-2v}{2}\right)^n \\ =& \frac{1}{2} - \frac{u}{4} + \frac{v}{2} + \frac{u^2}{8} - \frac{uv}{2} + \frac{v^2}{2} - \frac{u^3}{16} + \frac{3u^2v}{8} - \frac{3uv^2}{4} + \frac{v^3}{2} + \dots \\ =& \frac{1}{2} - \frac{x-2}{4} + \frac{y-1}{2} + \frac{(x-2)^2}{8} - \frac{(x-2)(y-1)}{2} + \frac{(y-1)^2}{2} - \\ & \frac{(x-2)^3}{16} + \frac{3(x-2)^2(y-1)}{8} - \frac{3(x-2)(y-1)^2}{4} + \frac{(y-1)^3}{2} + \dots \end{aligned}$$

(b) Here we are after an expansion in term of x and y. By our knowledge on the value of the sum of a geometric series, for $t \in (-1,1)$, we have that

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots + (-1)^n t^n + \dots = \sum_{n=0}^{\infty} (-t)^n.$$

Thus, for all $(x,y) \in \mathrm{Dom}(f)$, we have the following chain of equalities.

$$f(x,y) = \frac{1+x}{1+x^2+y^4} = \frac{1+x}{1-(-x^2-y^4)} = (1+x)(1-(x^2+y^4)+\dots) = 1+x-x^2-\dots$$

(5) Find all stationary points of the following functions. Also, investigate their nature, that is, if they are a local extreme or a saddle point.

(a) $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x^4 + x^2y + y^2 + z^2 + xz$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^4 + 2y^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1$

(c) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^4 + y^4 + 6x^2y^2 + 8x^3$

Solution.

(a) Recall, (x,y,z) is a stationary point of f if and only if $\nabla f(x,y,z) = \mathbf{0}$, and note that

$$\nabla f(x,y,z) = (4x^3 + 2xy + z)\mathbf{i} + (x^2 + 2y)\mathbf{j} + (2z + x)\mathbf{k}.$$

Hence, $\nabla f(x,y,z) = \mathbf{0}$ if and only if

$$0 = 4x^3 + 2xy + z, \qquad 0 = x^2 + 2y \qquad \text{and} \qquad 0 = 2z + x.$$

The latter two equations give us that $y = -x^2/2$ and $z = -x/2$, substituting these values into the first equation yields $x(6x^2 - 1) = 0$, giving $x = 0$, $x = \sqrt{6}/6$ and $x = -\sqrt{6}/6$. Hence we have three stationary points: $(0,0,0)$, $(\sqrt{6}/6, -1/12, -\sqrt{6}/12)$ and $(-\sqrt{6}/6, -1/12, \sqrt{6}/12)$.

At $(0,0,0)$ we have that $f(0,0,0) = 0$. Further, $f(0,y,z) = y^2 + z^2$ which is strictly positive for all $y, z \in \mathbb{R}$, and $f(2x, -x^2, -x) = x^2(13x^2 - 1)$ which is strictly negative for all x sufficiently small. Hence close to $(0,0,0)$ f both increases and decreases, and so $(0,0,0)$ is a saddle point.

To determine the nature of the other stationary points we calculate the Hessian $H = H(x,y,z)$ of f at (x,y,z). Indeed, since

$$\begin{aligned} f_{xx}(x,y,z) &= 12x^2 + 2y, & f_{xy}(x,y,z) &= f_{yx}(x,y,z) = 2x, \\ f_{xz}(x,y,z) &= f_{zx}(x,y,z) = 1, & f_{yy}(x,y,z) &= 2, \\ f_{yz}(x,y,z) &= f_{zy}(x,y,z) = 0, & f_{zz}(x,y,z) &= 2, \end{aligned}$$

we have that

$$H = \begin{pmatrix} 12x^2 + 2y & 2x & 1 \\ 2x & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

At $(\sqrt{6}/6, -1/12, -\sqrt{6}/12)$,

$$H = \begin{pmatrix} 11/6 & \sqrt{6}/3 & 1 \\ \sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Noting that

$$|11/6| = 11/6 > 0, \qquad \begin{vmatrix} 11/6 & \sqrt{6}/3 \\ \sqrt{6}/3 & 2 \end{vmatrix} = 3 > 0, \quad \text{and} \quad \begin{vmatrix} 11/6 & \sqrt{6}/3 & 1 \\ \sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4 > 0,$$

by the leading minor test, $(\sqrt{6}/6, -1/12, -\sqrt{6}/12)$ is a local minimum. At $(-\sqrt{6}/6, -1/12, \sqrt{6}/12)$,

$$H = \begin{pmatrix} 11/6 & -\sqrt{6}/3 & 1 \\ -\sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Noting that

$$|11/6| = 11/6, \qquad \begin{vmatrix} 11/6 & -\sqrt{6}/3 \\ -\sqrt{6}/3 & 2 \end{vmatrix} = 3 > 0, \quad \text{and} \quad \begin{vmatrix} 11/6 & -\sqrt{6}/3 & 1 \\ -\sqrt{6}/3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4 > 0,$$

by the Leading Minor Test, $(-\sqrt{6}/6, -1/12, \sqrt{6}/12)$ is a local minimum.

(b) Recall, (x,y) is a stationary point of f if and only if $\nabla f(x,y) = \mathbf{0}$, and note that

$$\nabla f(x,y) = (4x^3 + 12x^2y + 12xy^2 + 4y^3)\mathbf{i} + (8y^3 + 4x^3 + 12x^2y + 12xy^2)\mathbf{j}.$$

Hence, $\nabla f(x,y) = \mathbf{0}$ if and only if

$$0 = x^3 + 3x^2y + 3xy^2 + y^3 \qquad \text{and} \qquad 0 = 2y^3 + x^3 + 3x^2y + 3xy^2.$$

Subtracting the fist equation from the second, implies that $y^3 = 0$ and so $y = 0$, in which $x = 0$, so that the only stationary point of f is at $(0,0)$.

To determine the nature of the other stationary points we calculate the Hessian $H = H(0,0)$ of f at $(0,0)$. Indeed, since

$$f_{xx}(x,y) = 12x^2 + 24xy + 12y^2,$$

$$f_{xy}(x,y) = f_{yx}(x,y) = 12x^2 + 24xy + 12y^2,$$
$$f_{yy}(x,y) = 24y^2 + 12x^2 + 24xy,$$

we have that

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the leading minor test is indeterminate. However,

$$f(x,y) - f(0,0) = (x^4 + 2y^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1) - 1 = (x+y)^4 + y^4 > 0$$

for all $(x,y) \neq (0,0)$. Hence $(0,0)$ is a global minimum.

(c) Recall, (x,y) is a stationary point of f if and only if $\nabla f(x,y) = \mathbf{0}$, and note that

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = (4x^3 + 12xy^2 + 24x^2)\mathbf{i} + (4y^3 + 12x^2y)\mathbf{j}.$$

Further, $f_y(x,y) = 0$ if and only if $y(y^2 + 3x^2) = 0$, which in turn is if and only if $y = 0$ or both $x = 0$ and $y = 0$. Further, $f_x(x,y) = 0$ if and only if $x^3 + 3xy^2 + 6x^2 = 0$. Thus, if $y = 0$, we obtain that $f_x(x,y) = 0$ if and only if $x^2(x+6) = 0$. Therefore, we have two stationary points : $(0,0)$ and $(-6,0)$.

To determine the nature of the other stationary points we calculate the Hessian $H = H(0,0)$ of f at $(0,0)$. Indeed, since

$$f_{xx}(x,y) = 12x^2 + 12y^2 + 48x,$$
$$f_{xy}(x,y) = f_{yx}(x,y) = 24xy,$$
$$f_{yy}(x,y) = 12y^2 + 12x^2,$$

we have that

$$H = \begin{pmatrix} 12x^2 + 12y^2 + 48x & 24xy \\ 24xy & 12y^2 + 12x^2 \end{pmatrix}.$$

Thus, at $(0,0)$, the Hessian is the zero matrix, and so the leading minor test is indeterminate. However, $f(0,0) = 0$, and if x is close to 0 and positive, then $f(x,0) = x^4 + 8x^3 > 0$. On the other hand, if x is close to zero (or at least such that $|x| < 8$) and negative, then $f(x,0) = x^4 + 8x^3 < 0$. Hence close to $(0,0)$ the function f both increases and decreases, and so $(0,0)$ is a saddle point. At $(-6,0)$,

$$H = \begin{pmatrix} 144 & 0 \\ 0 & 432 \end{pmatrix}.$$

Thus, the leading minors, are 144 and 144×432 which are both positive, and so, by the leading minor test, we have $(-6,0)$ is a local minimum.

(7) Find the dimensions (height, length and width) of a rectangular box, without a lid, which would contain the maximum amount of water, if the surface area of its walls and base add up to $108\,\mathrm{m}^2$.

Solution.

If the box has dimensions x, y and z, then we wish to maximise $V \colon \{(x,y,z) \in \mathbb{R}^3 \colon x,y,z \geq 0\} \to \mathbb{R}$ given by $V(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = xy + 2xz + 2yz - 108$. In this case the Lagrangian of $V(x,y,z)$ subject to the constraint $g(x,y,z) = 0$ is

$$L(x,y,z,\lambda) = xyz + \lambda(xy + 2xz + 2yz - 108).$$

Provided that $\nabla g(x,y,z) \neq \mathbf{0}$, we have (x,y,z) is an extreme of V if it is stationary point of L. Thus, we wish to solve the system of equations,

$$0 = yz + \lambda(y + 2z),$$
$$0 = xz + \lambda(x + 2z),$$
$$0 = xy + \lambda(2x + 2y),$$
$$0 = xy + 2xz + 2yz - 108.$$

Subtracting the fist of these equations from the second, we obtain that $(y-x)z+\lambda(y-x)=0$, or equivalently $(y-x)(\lambda+z)=0$. Hence, either $\lambda=-z$ or $y=x$. If $\lambda=-z$, then the first of the above displayed equations implies $-2z^2=0$, that is $z=0$. However, this would imply that $V(x,y,z)=0$, but this would be a local minimum, given that the range of V is contained in the non-negative real numbers. If $y=x$, then the third of the above displayed equations becomes $x^2+4\lambda x=0$, so either $x=0$, which again gives $V(x,y,z)=0$, or $x=-4\lambda$. Thus, as we have ruled out the case $x=0$, it follows that $x=-4\lambda$ and $x=y$. With this at hand, the first of the above displayed equations gives $(2z-x)x/4=0$, but as we know $x\neq 0$, it must necessarily be the case that $x=2z$. Since, $g(x,x,x/2)=3x^2-108$ which is equal to zero when $x=\pm 6$, and $\nabla g(6,6,3)=12\mathbf{i}+12\mathbf{j}+24\mathbf{k}\neq\mathbf{0}$. We now claim that the box with maximum volume subject to the constraint has dimensions $x=y=6\,\mathrm{m}^2$ and $z=3\,\mathrm{m}^2$.

Let us now prove our claim. Since we have that $xy+2xz+2yz\leq 108$, and since x,y and z are positive, it follows that $x+y\leq 54/z$ and $xy\leq 108$. Now suppose, by way of contradiction, that $(6,6,3)$ is not the maximum of V subject to the given constraint. This means there exists $(x,y,z)\in\mathbb{R}^3$ such that $xy+2xy+2yz\leq 108$ and $xyz>108$. This latter inequality, in tandem with the fact that $xy\leq 108$, gives that $z\geq 1$. Thus, we have that $x+y\leq 54$ and $xy+2x+2y\leq 108$. Combining the latter two inequalities yields that $xy+108\leq 108$, or equivalently $xy=0$, since we have that x,y and z are positive; contradicting our hypothesis that $xyz\geq 108$.

(9) Which points on the sphere $x^2+y^2+z^2=4$ are closest to and farthest from the point $(3,1,-1)$.

Solution.

We want to maximise and minimise the distance from (x,y,z) to $(3,1,-1)$, that is, we want to find the extreme of

$$d(x,y,z)=\sqrt{(x-3)^2+(y-1)^2+(z+1)^2},$$

subject to the constraint $g(x,y,z)=x^2+y^2+z^2-4$. Since the range of d is a subset of the non-negative real numbers, the extreme of d correspond to the extreme of $d^2=d\circ d$. In this case the Lagrangian of $d^2(x,y,z)$ subject to the constraint $g(x,y,z)=0$ is

$$L(x,y,z,\lambda)=(x-3)^2+(y-1)^2+(z+1)^2+\lambda(x^2+y^2+z^2-4),$$

and so $\nabla L(x,y,z,\lambda)=\langle(2(x-3)+2x\lambda),(2(y-1)+2y\lambda),(2(z+1)+2z\lambda),(x^2+y^2+z^2-4)\rangle$. Thus, $\nabla L(x,y,z,\lambda)=\mathbf{0}$ when

$$\begin{aligned}0&=(x-3)+x\lambda,\\0&=(y-1)+y\lambda,\\0&=(z+1)+z\lambda,\\0&=x^2+y^2+z^2-4.\end{aligned}$$

Note, $\lambda\neq -1$, for if this were the case, then the first displayed equation would yield a contradiction. With this at hand, the first three equations imply $x=3/(1+\lambda)$, $y=1/1+\lambda$ and $z=-1/(1+\lambda)$. Substituting these values into the fourth of the above displayed equations we obtain that

$$\frac{3^2}{(1+\lambda)^2}+\frac{1^2}{(1+\lambda)^2}+\frac{(-1)^2}{(1+\lambda)^2}=4.$$

Thus, $\lambda=-1\pm\sqrt{11}/2$. Substituting these two values back into the expressions for x, y, and z gives two possible extreme p_1 and $p_2$of d^2, where

$$p_1=\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right)\quad\text{and}\quad p_2{=}\left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right).$$

Noting that $\nabla g(x,y,z)=2(x-3)\mathbf{i}+2(y-z)\mathbf{j}+2(z+1)\mathbf{k}$, we have that $\nabla g(p_1)\neq\mathbf{0}$ and $\nabla g(p_2)\neq\mathbf{0}$. Evaluating d^2 at p_1 and p_2, we obtain that p_1 is the closest point and p_2 is the farthest point. Note here, we have also used the fact that the given sphere is closed and bounded, and thus when restricting d^2 to the given sphere, it obtains its maximum and its minimum.

(11) Let $n\in\mathbb{N}$ be fixed and recall that $\mathbb{R}^+$ denotes the set of all positive real numbers and that $(\mathbb{R}^+)^n$ denotes the set of n-tuples $(x_1,x_2,\ldots,x_n)$ with $x_i\in\mathbb{R}^+$ for all $i\in\{1,2,\ldots,n\}$. Let $(a_1,a_2,\ldots,a_n)\in(\mathbb{R}^+)^n$ be fixed throughout this question.

(a) Find all $x = (x_1, x_2, \dots, x_n) \in (\mathbb{R}^+)^n$ which maximise the sum $\sum_{i=1}^n a_i x_i$ subject to the constraint $\sum_{i=1}^n x_i^2 = 1$.

(b) For the points $x = (x_1, x_2, \dots, x_n) \in (\mathbb{R}^+)^n$ located in Part (i), find the value of the sum $\sum_{i=1}^n a_i x_i$ in terms of $a_1, a_2, \dots, a_n$.

Solution.

Here, we want to maximise $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n a_i x_i$ subject to $\sum_{i=1}^n x_i^2 = 1$. Let $L\colon \mathbb{R}^4 \to \mathbb{R}$ be given by $L(x_1, x_2, \dots, x_n, \lambda) = f(x_1, x_2, \dots, x_n) + \lambda(\sum_{i=1}^n x_i^2 - 1)$; in which case, we have that $\nabla L(x_1, x_2, \dots, x_n, \lambda) = \langle a_1 + 2\lambda x_1, a_2 + 2\lambda x_2, \dots, a_n + 2\lambda x_n, \sum_{i=1}^n x_i^2 - 1\rangle$, yielding that $(x_1, x_2, \dots, x_n, \lambda)$ is a stationary point of L if and only if

$$a_1 = -2\lambda x_1, \quad a_2 = -2\lambda x_2, \quad \dots, \quad a_n = -2\lambda x_n, \quad \text{and} \quad \sum_{i=1}^n x_i^2 = 1.$$

Since $a_i \in \mathbb{R}^+$, we have that $\lambda \neq 0$ and so $x_i = -a_i/(2\lambda)$ for all $i \in \{1, 2, \dots, n\}$, implying that

$$\sum_{i=1}^n \frac{a_i^2}{4\lambda^2} = 1, \quad \text{or equivalently} \quad \lambda = \pm\frac{1}{2}\left(\sum_{i=1}^n a_i^2\right)^{1/2}.$$

Since $\nabla\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) = \langle 2x_1, 2x_2, \dots, 2x_n\rangle = \mathbf{0}$ if and only if $x_1 = x_2 = \cdots = x_n = 0$, we have that f is maximised when $x_i = a_i/(2|\lambda|)$ for all $i \in \{1, 2, \dots, n\}$, and takes the value

$$f(a_1/(2|\lambda|), a_2/(2|\lambda|), \dots, a_n/(2|\lambda|)) = \sum_{i=1}^n a_i \frac{a_i}{2|\lambda|} = \frac{1}{2|\lambda|}\sum_{i=1}^n a_i^2 = \left(\sum_{i=1}^n a_i^2\right)^{1/2}.$$

(13) Consider a terrestrial space probe in the shape of a solid ellipsoid, given by $4x^2 + y^2 + 4z^2 \leq 16$. As it enters the atmosphere of the alien planet *Cybertron*, it begins to heat. After one Earth hour, the temperature at the point (x, y, z) on the probe is given by $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point(s) on the probe's surface. Please remember to show all of your working. In your solution, you may use that $24\sqrt{3} \approx 41.569$.

Solution.

Here we want to maximise $T\colon \mathbb{R}^3 \to \mathbb{R}$ given by $T(x, y, z) = 8x^2 + 4yz - 16z + 600$ subject to the constraint $4x^2 + y^2 + 4z^2 = 16$. Let $L\colon \mathbb{R}^4 \to \mathbb{R}$ be defined by $L(x, y, z, \lambda) = T(x, y, z) + \lambda(4x^2 + y^2 + 4z^2 - 16)$; in which case $\nabla L(x, y, z, \lambda) = \langle 16x + 8\lambda x, 4z + 2\lambda y, 4y - 16 + 8\lambda z, 4x^2 + y^2 + 4z^2 - 16\rangle$. If $L(x, y, z, \lambda) = \mathbf{0}$, then

$$x(2 + \lambda) = 0, \quad 2z = -\lambda y, \quad y - 4 = -2\lambda z, \quad \text{and} \quad 4x^2 + y^2 + 4z^2 = 16,$$

which implies $x = 0$ or $\lambda = -2$

Case 1: If $\lambda = -2$, then $2z = 2y$, and so $z = y$, which implies $y - 4 = -2(-2)y$, yielding $z = y = -4/3$; and thus $x = \pm 4/3$.

Case 2: If $x = 0$, then $y^2 + 4z^2 = 16$; meaning that $y \neq 0$, for suppose that $y = 0$, then $z = 0$ yielding a contradiction as $4x^2 + y^2 + 4z^2 = 16$. Hence, $\lambda = -2z/y$, implying $y - 4 = -2(-2z/y)z$, or equivalently, $y^2 - 4y = 4z^2$. Thus, $y^2 - 4y = 16 - y^2$, and so, $y = 4$ or $y = -2$. If $y = 4$, then $z^2 = (4^2 - 4(4))/4 = 0$, and hence, $z = 0$; if $y = -2$, then $z^2 = ((-2)^2 - 4(-2))/4 = 3$, yielding $z = \pm\sqrt{3}$.

Therefore, the stationary points of L are $P_1 = (-4/3, -4/3, -4/3)$, $P_2 = (4/3, -4/3, -4/3)$, $P_3 = (0, 4, 0)$, $P_4 = (0, -2, \sqrt{3})$ and $P_4 = (0, -2, -\sqrt{3})$. Since $\nabla(4x^2 + y^2 + 4z^2) = \mathbf{0}$ if and only if $x = y = z = 0$, and since $T(P_1) = 1928/3$, $T(P_2) = 1928/3$, $T(P_3) = 600$, $T(P_4) = 600 - 24\sqrt{3}$ and $T(P_5) = 600 + 24\sqrt{3}$, the maximum temperature is at $(\pm 4/3, -4/3, -4/3)$.

B.2 Formative assessment – Week 8 – Solutions

(1) Evaluate the double integral

$$\iint_R (x-y)\,\mathrm{d}A$$

as a repeated integral where R is the triangle with vertices at $(3,10)$, $(3,1)$, and $(-2,1)$.

Solution.

We observe that R is a bounded region and that on R the function $(x,y)\mapsto |x-y|$ is continuous and bounded on R, thus the integral

$$\iint_R |x-y|\,\mathrm{d}A$$

exist and so the conditions of Fubini-Tonelli's Theorem are satisfied. The line through $(-2,1)$ and $(3,10)$ has equation

$$(y-10)/(x-3) = (10-1)/(3-(-2)) \quad \text{or equivalently} \quad y = \frac{9}{5}x + \frac{23}{5}.$$

Since we have an expression for y in terms of x, we integrate with respect to y first keeping x fixed between -2 and 3. For a fixed x, observe that y varies from 1 to $9x/5+23/5$, so

$$\begin{aligned}
\iint_R (x-y)\,\mathrm{d}A &= \int_{x=-2}^{x=3}\int_{y=1}^{y=9x/5+23/5} x-y\,\mathrm{d}y\mathrm{d}x \\
&= \int_{x=-2}^{x=3}\left[xy-\frac{y^2}{2}\right]_{y=1}^{y=9x/5+23/5}\mathrm{d}x \\
&= \int_{x=-2}^{x=3}\left(x\left(\frac{9x}{5}+\frac{23}{5}\right)-\frac{1}{2}\left(\frac{9x}{5}+\frac{23}{5}\right)^2 - x + \frac{1}{2}\right)\mathrm{d}x \\
&= \int_{x=-2}^{x=3}\left(x\left(\frac{9}{5}x+\frac{18}{5}\right)-\frac{1}{2}\left(\frac{9}{5}x+\frac{23}{5}\right)^2 + \frac{1}{2}\right)\mathrm{d}x \\
&= \left[\frac{9}{15}x^3+\frac{18}{10}x^2-\frac{5}{54}\left(\frac{9}{5}x+\frac{23}{5}\right)^3+\frac{x}{2}\right]_{-2}^{3} = -60.
\end{aligned}$$

(3) Sketch the region $R=\{(x,y,z)\in\mathbb{R}^3\colon 0\le z\le 1-|x|-|y|\}$ and evaluate the integral

$$\iiint_R (xy+z^2)\,\mathrm{d}V.$$

Solution.

The region is a pyramid with square base, see FIGURE B.1. This is a bounded region in $\mathbb{R}^3$. Indeed, the region is bounded by linear edges and planar faces and has vertices at $(\pm1,0,0)$, $(0,\pm1,0)$ and $(0,0,1)$. Moreover, the function $(x,y,z)\mapsto |xy+z^2|$ is bounded and continuous on R. Thus, the conditions of Fubini-Tonelli's Theorem are meet.

Let $f\colon\mathbb{R}^3\to\mathbb{R}$ be given by $f(x,y,z)=xy+z^2$, set

$$I = \iiint_R f\,\mathrm{d}V$$

and define

$$\begin{aligned}
R_1 &= \{(x,y,z)\in\mathbb{R}^3\colon x\ge 0 \text{ and } y\ge 0\}, & R_2 &= \{(x,y,z)\in\mathbb{R}^3\colon x\le 0 \text{ and } y\ge 0\},\\
R_3 &= \{(x,y,z)\in\mathbb{R}^3\colon x\le 0 \text{ and } y\le 0\}, & R_4 &= \{(x,y,z)\in\mathbb{R}^3\colon x\ge 0 \text{ and } y\le 0\}.
\end{aligned}$$

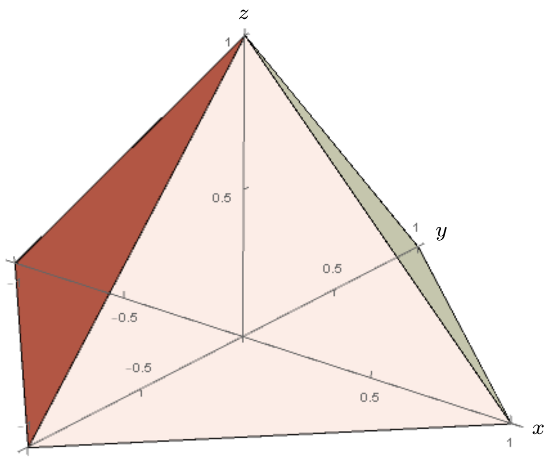

FIGURE B.1: Illustration of the region $R = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 1 - |x| - |y|\}$.

Notice $R = R_1 \cup R_2 \cup R_3 \cup R_4$ and that the interiors of R_1, R_2, R_3 and R_4 are pair-wise disjoint. Therefore, we have that

$$I = \iint_{R_1} f(x, y, z)\,\mathrm{d}V + \iint_{R_2} f(x, y, z)\,\mathrm{d}V + \iint_{R_3} f(x, y, z)\,\mathrm{d}V + \iint_{R_4} f(x, y, z)\,\mathrm{d}V.$$

Further, since $f(-x, -y, z) = f(x, y, z)$ and $f(x, -y, z) = f(-x, y, z)$,

$$\iiint_{R_1} f\,\mathrm{d}V = \iiint_{R_3} f\,\mathrm{d}V \quad \text{and} \quad \iiint_{R_2} f\,\mathrm{d}V = \iiint_{R_4} f\,\mathrm{d}V.$$

Hence,

$$I = 2\iiint_{R_1} f\,\mathrm{d}V + 2\iiint_{R_2} f\,\mathrm{d}V.$$

Let us consider the the first of these integrals. Since x and y are both non-negative,

$$1 - |x| - |y| = 1 - x - y.$$

Here, we integrate first with respect to z, fixing x and y, in which case $0 \leq z \leq 1 - x - y$. Next we integrate with respect to y fixing x, in which case $0 \leq y \leq 1 - x$. Finally we integrate with respect to x which varies between 0 and 1.

$$\begin{aligned}
\iiint_{R_1} f\,\mathrm{d}V &= \iiint_{R_1} xy + z^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} xy + z^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \left[xyz + \frac{z^3}{3}\right]_{z=0}^{z=1-x-y} \mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} xy(1-x-y) + \frac{(1-x-y)^3}{3}\,\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x(1-x)y - xy^2 + \frac{(1-x-y)^3}{3}\,\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{y=1} \left[\frac{x(1-x)y^2}{2} - \frac{xy^3}{3} - \frac{(1-x-y)^4}{12}\right]_{y=0}^{y=1-x} \mathrm{d}x \\
&= \int_{x=0}^{y=1} \frac{x(1-x)(1-x)^2}{2} - \frac{x(1-x)^3}{3} + \frac{(1-x)^4}{12}\,\mathrm{d}x \\
&= \int_{x=0}^{x=1} \frac{x(1-x)^3}{6} + \frac{(1-x)^4}{12}\,\mathrm{d}x
\end{aligned}$$

$$= \frac{1}{12}\int_{x=0}^{x=1} 1 - 2x + 2x^3 - x^4 \,\mathrm{d}x$$
$$= \left[\frac{1}{12}\left(x - x^2 + \frac{x^4}{2} - \frac{x^5}{5}\right)\right]_0^1 = \frac{3}{120}.$$

Let us consider the second integral. Since x is non-positive and y is non-negative, $1 - |x| - |y| = 1 + x - y$. Here, we integrate first with respect to z, fixing x and y, in which case $0 \leq z \leq 1 + x - y$. Next we integrate with respect to y fixing x, in which case $0 \leq y \leq 1 + x$. Finally we integrate with respect to x which varies between -1 and 0.

$$\iiint_{R_2} f \,\mathrm{d}V = \int_{x=-1}^{x=0}\int_{y=0}^{y=1+x}\int_{z=0}^{z=1+x-y} xy + z^2 \,\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$= \int_{x=-1}^{x=0}\int_{y=0}^{y=1+x}\left[xyz + \frac{z^3}{3}\right]_{z=0}^{z=1+x-y} \mathrm{d}y\mathrm{d}x$$
$$= \int_{x=-1}^{x=0}\int_{y=0}^{y=1+x} xy(1+x-y) + \frac{(1+x-y)^3}{3}\,\mathrm{d}y\mathrm{d}x$$
$$= \int_{x=-1}^{x=0}\left[\frac{x(1+x)y^2}{2} - \frac{xy^3}{3} - \frac{(1+x-y)^4}{12}\right]_{x=0}^{y=1+x} \mathrm{d}x$$
$$= \int_{x=-1}^{x=0} \frac{x(1+x)^3}{6} + \frac{(1+x)^4}{12}\,\mathrm{d}x$$
$$= \frac{1}{12}\int_{x=-1}^{x=0} 1 + 6x + 12x^2 + 10x^3 + 3x^4 \,\mathrm{d}x$$
$$= \left[\frac{1}{12}\left(x + 3x^2 + 4x^3 + \frac{5x^4}{2} + \frac{3x^5}{5}\right)\right]_{x=-1}^{x=0} = \frac{1}{120}$$

Hence,

$$I = \frac{6}{120} + \frac{2}{120} = \frac{1}{15}.$$

(5) Verify the following statements.

(a) The element of area $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ in Cartesian co-ordinates is equal to $r\mathrm{d}r\mathrm{d}\theta$ in polar co-ordinates.

(b) The element of volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ in Cartesian co-ordinates is equal to $r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$ in cylindrical polar co-ordinates.

(c) The element of volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is equal to $\rho^2\sin(\phi)\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$ in spherical polar co-ordinates.

Solution.

(a) In two dimensional polar co-ordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Thus, $\partial(x,y)/\partial(r,\theta)$ equals

$$\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Hence,

$$\mathrm{d}A = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}r\mathrm{d}\theta = r\,\mathrm{d}r\mathrm{d}\theta = r\,\mathrm{d}r\mathrm{d}\theta.$$

Note, in polar co-ordinates we assume that $r \geq 0$ and so $r = |r|$.

(b) In cylindrical polars, $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $z = z$. Thus, $\partial(x,y,z)/\partial(r,\theta,z)$ equals

$$\begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Hence,

$$\mathrm{d}V = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = r\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

Note, in cylindrical polars we assume that $r \geq 0$ and so $r = |r|$.

(c) In spherical polars, $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$ and $z = \rho\cos(\phi)$. Thus, $\partial(x,y,z)/\partial(\rho,\phi,\theta)$

$$\begin{aligned}
&\begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{vmatrix} \\
&= \rho^2\cos^2(\phi)\cos^2(\theta)\sin(\phi) + \rho^2\sin^2(\phi)\sin^2(\theta)\sin(\phi) \\
&\quad + \rho^2\sin^2(\theta)\cos^2(\phi)\sin(\phi) + \rho^2\sin^2(\phi)\cos^2(\theta)\sin(\phi) \\
&= \rho^2\sin(\phi)(\cos^2(\phi)\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta) + \sin^2(\phi)\cos^2(\theta) + \cos^2(\phi)\sin^2(\theta)) \\
&= \rho^2\sin(\phi)(\cos^2(\phi)(\cos^2(\theta) + +\sin^2(\theta)) + \sin^2(\phi)(\sin^2(\theta) + \cos^2(\theta))) = \rho^2\sin(\phi).
\end{aligned}$$

(7) (a) Evaluate the following integral.

$$I = \int_{y=0}^{y=\pi}\int_{x=0}^{x=2} xy^2 + x^2\sin(y)\,\mathrm{d}x\mathrm{d}y$$

(b) Let $D = \{(x,y) \in \mathbb{R}^2 \colon 0 \leq x, 0 \leq y \,\text{and}\, x^2+y^2 \leq 1\}$. Using Cartesian co-ordinates, evaluate

$$I = \iint_D xy\,\mathrm{d}A.$$

(c) Let D denote the region that lies inside the circle $x^2+y^2 = 2x$ and above the x-axis. By integrating with respect to polar co-ordinates, find the value of

$$I = \iint_D y\sqrt{x^2+y^2}\,\mathrm{d}A.$$

(d) Let T be the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$ and suppose that T has density xy at the point (x,y,z). Calculate the mass of T.

Solution.

(a)

$$\begin{aligned}
I = \int_{y=0}^{y=\pi}\int_{x=0}^{x=2} xy^2 + x^2\sin(y)\,\mathrm{d}x\mathrm{d}y &= \int_{y=0}^{y=\pi}\left[\frac{x^2y^2}{2} + \frac{x^3\sin(y)}{3}\right]_{x=0}^{x=2}\mathrm{d}y \\
&= \int_{y=0}^{y=\pi} 2y^2 + \frac{8\sin(y)}{3}\,\mathrm{d}y \\
&= \left[\frac{2y^3}{3} - \frac{8\cos(y)}{3}\right]_{y=0}^{y=\pi} = \frac{2(\pi^3+16)}{3}.
\end{aligned}$$

(b) Observe that the region D is bounded and that the function xy is non-negative continuous and bounded on R. Therefore, the conditions of Fubini-Tonelli's Theorem are meet. Further, the region D can be written as the set $\{(x,y) \in \mathbb{R}^2 \colon 0 \leq x \leq \sqrt{1-y^2} \,\text{and}\, 0 \leq y \leq 1\}$. Integrating with respect to x, and then with respect y, yields

$$I = \int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}} xy\,\mathrm{d}x\mathrm{d}y = \int_{y=0}^{y=1}\left[\frac{yx^2}{2}\right]_{x=0}^{x=\sqrt{1-y^2}}\mathrm{d}y = \int_0^1 \frac{y(1-y^2)}{2}\,\mathrm{d}y = \left[\frac{y^2}{4} - \frac{y^4}{8}\right]_0^1 = \frac{1}{8}.$$

(c) The set D in Cartesian co-ordinates is given by $D = \{(x,y) \in \mathbb{R}^2 \colon (x-1)^2 + y^2 < 1 \,\text{and}\, x,y > 0\}$, which is a bounded. Further, the function $(x,y) \mapsto y\sqrt{x^2+y^2}$ is bounded, positive and continuous on D. Therefore, the conditions of Fubini-Tonelli's Theorem hold and moreover we may apply a change of variables. To this end, observe that in polar co-ordinates we have that

$$D = \{(r,\theta) \in \mathbb{R}_0^+ \times [-\pi,\pi] \colon 0 \leq \theta \leq \pi/2 \,\text{and}\, 0 \leq r \leq 2\cos(\theta)\}$$

and hence,

$$\begin{aligned}
I = \iint_D y\sqrt{x^2+y^2}\,\mathrm{d}A &= \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2\cos(\theta)} r\sin(\theta)\sqrt{r^2}r\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2\cos(\theta)} r^3\sin(\theta)\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=0}^{\theta=\pi/2}\left[\frac{r^4}{4}\right]_{r=0}^{r=2\cos(\theta)}\sin(\theta)\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{\theta=\pi/2} 4\cos^4(\theta)\sin(\theta)\,\mathrm{d}\theta = \left[\frac{-4\cos^5(\theta)}{5}\right]_0^{\pi/2} = 4/5.
\end{aligned}$$

(d) Let m denote the mass of T. Since the tetrahedron T is bounded and since the function $(x,y,z)\mapsto xy$ is bounded, non-negative and continuous on T, the conditions of Fubini-Tonelli's Theorem are meet, and so

$$\begin{aligned}
m = \iiint_T xy\,\mathrm{d}V &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} xy\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}[xyz]_{z=0}^{1-x-y}\,\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x(1-x)y - xy^2\,\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{x=1}\left[\frac{x(1-x)y^2}{2} - \frac{xy^3}{3}\right]_{y=0}^{y=1-x}\mathrm{d}x \\
&= \int_{x=0}^{x=1}\frac{x(1-x)^3}{6}\,\mathrm{d}x = \frac{1}{6}\left[\frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}\right]_{x=0}^{x=1} = \frac{1}{120}.
\end{aligned}$$

(9) Using the substitutions $x = (u+v)/2$ and $y = (u-v)/2$, evaluate the integral

$$\iint_D e^{x+y}\,\mathrm{d}A,$$

where $D = \{(x,y)\in\mathbb{R}^2\colon |x|+|y|\le 1\}$.

Solution.

The domain D we are integrating over is given by $\{(x,y)\in\mathbb{R}^2\colon |x|+|y|\le 1\}$. This is a bounded region in the plane. Further, the integrand is positive continuous and bounded on this domain. Therefore, the conditions of Fubini-Tonelli's Theorem are meet and we may apply a change of variables. As suggested, let us use $x = (u+v)/2$ and $y = (u-v)/2$.

The Jacobian $\partial(x,y)/\partial(u,v)$ is given by

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{-1}{2}.$$

The region D we are asked to integrate over is the interior of the square with corners $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$, bounded by the lines $x+y=1$, $x-y$=1, $-x+y=1$ and $-y-x=1$. Substituting $x=(u+v)/2$ and $y=(u-v)/2$ into these equations we obtain the lines $u=1$, $v=1$, $-v=1$ and $-u=1$. Thus, in terms of u and v our region of integration is $\{(u,v)\in\mathbb{R}^2\colon |u|\le 1 \text{ and } |v|\le 1\}$ which is the square with corners at $(\pm1,\pm1)$. Hence,

$$\begin{aligned}
\iint_D e^{x+y}\,\mathrm{d}x\mathrm{d}y = \iint_D e^u\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v &= \int_{v=-1}^{v=1}\int_{u=-1}^{u=1}\frac{e^u}{2}\,\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{2}\int_{v=-1}^{v=1}[e^u]_{u=-1}^{u=1}\,\mathrm{d}v \\
&= \frac{1}{2}\int_{v=-1}^{v=1} e^1 - e^{-1}\,\mathrm{d}v = \frac{1}{2}\left[(e^1-e^{-1})v\right]_{-1}^{1} = 2\sinh(1).
\end{aligned}$$

(11) A function $g\colon \mathbb{R} \to \mathbb{R}$ is said to be a **density** of a probability distribution if and only if **(i)** $g(x) \geq 0$ for all $x \in \mathbb{R}$, and **(ii)**

$$\int_{-\infty}^{\infty} g(x)\, \mathrm{d}x = 1.$$

Let $k = (\ln(3)/(18\pi))^{1/2}$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = k\, 3^{-x^2/18}$, for $x \in \mathbb{R}$. Determine if f is a density of some probability distribution.

Solution.

By observing that $k^x = \mathrm{e}^{\ln(k)x}$ for all $k \in \mathbb{R}^+$ and $x \in \mathbb{R}$, this question is a special case of the more general question: verify that $f\colon \mathbb{R} \to \mathbb{R}$, given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}$$

for $x \in \mathbb{R}$, is a density.

(i) Since the exponential of any real number is positive and since $\sigma\sqrt{2\pi}$ is positive, f is positive.

(ii) If we can show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\, \mathrm{d}x = \sqrt{2\pi}, \tag{B.1}$$

then an application of integration by substitution will yield the required result. By definition

$$\iint_{\mathbb{R}^2} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A = \lim_{R\to\infty} \iint_{D_R} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A,$$

where $D_R = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq R^2\}$. Further, since the integrand $\mathrm{e}^{(-x^2-y^2)/2}$ is continuous and bounded on D_R, and since D_R is a bounded set, for all $R \in \mathbb{R}^+$, by Fubini-Tonelli's theorem

$$\iint_{D_R} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A = \iint_{D_R} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}x\, \mathrm{d}y.$$

Switching to polar coordinates, observing that $x^2 + y^2 = r^2$ and $\mathrm{d}x\, \mathrm{d}y = r\, \mathrm{d}r\, \mathrm{d}\theta$,

$$\iint_{D_R} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A = \int_{\theta=-\pi}^{\theta=\pi} \int_{r=0}^{r=R} \mathrm{e}^{-r^2/2} r\, \mathrm{d}r\, \mathrm{d}\theta = 2\pi \int_{r=0}^{r=R} \mathrm{e}^{-r^2/2} r\, \mathrm{d}r = 2\pi(1 - \mathrm{e}^{-R}),$$

and hence

$$\iint_{\mathbb{R}^2} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A = \lim_{R\to\infty} \iint_{D_R} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A = \lim_{R\to\infty} 2\pi(1 - \mathrm{e}^{-R}) = 2\pi.$$

Further, by a second application of Fubini-Tonelli's theorem and since our integrand is positive, for $R \in \mathbb{R}^+$,

$$\left(\int_{-R}^{R} \mathrm{e}^{-x^2/2}\, \mathrm{d}x\right)^2 = \int_{-R}^{R} \mathrm{e}^{-x^2/2}\, \mathrm{d}x \int_{-R}^{R} \mathrm{e}^{-y^2/2}\, \mathrm{d}y = \iint_{[-R,R]^2} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A \begin{cases} \geq \displaystyle\iint_{D_R} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A, \\ \leq \displaystyle\iint_{D_{\sqrt{2}\,R}} \mathrm{e}^{(-x^2-y^2)/2}\, \mathrm{d}A. \end{cases}$$

Taking the limit as R tends to infinity, and observing that the square root function is continuous on $\mathbb{R}^+$, we obtain that $\sqrt{2\pi}$ is both an upper bound and a lower bound for (B.1), and thus the result follows.