Contents

Chapter 10

Surface integrals

This section is devoted to integrals of functions defined over surfaces in $\mathbb{R}^3$. Before we begin, let us be more precise in what we mean by the term *surface*. Until now we have been treating surfaces in $\mathbb{R}^3$ in an intuitive way, either as the graph of a function $f(x, y)$, or as graphs of equations $f(x, y, z) = 0$.

A smooth curve is thought of as a *one-dimensional* object, because points on it can be located by giving one co-ordinate. Therefore, a curve can be defied as the range of a vector valued function of a single real variable. A surface is a *two dimensional* object; points on it can be located by using two co-ordinates, and can be defined as the range of a vector valued function of two real variables. We call such functions a *surface parameterisation*.

10.1 Parameterisation of surfaces

There are usually several ways one can describe a surface in $\mathbb{R}^3$. They can be implicitly defined, for instance,

$$S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$$

or parametrically defined, for example by a parameterisation such as $\mathbf{s}\colon [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

$$\mathbf{s}(\alpha, \beta) = \langle \sin(\alpha)\cos(\beta), \sin(\alpha)\sin(\beta), \cos(\alpha)\rangle,$$

for $\alpha \in [0, \pi]$ and $\beta \in [0, 2\pi]$, see FIGURE 10.1. Indeed, one can generate some very interesting surfaces, such as tubular knots, see FIGURE 10.2.

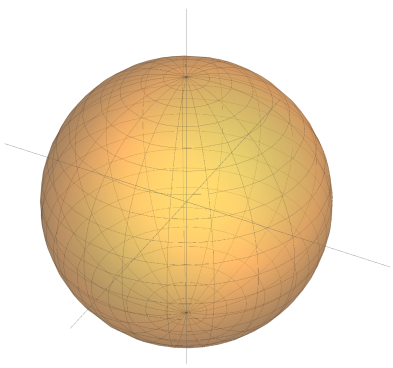


Figure 10.1: Graphical representation of the parameterisation $\mathbf{S}\colon [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ defined by $\mathbf{s}(\alpha, \beta) = \langle \sin(\alpha)\cos(\beta), \sin(\alpha)\sin(\beta), \cos(\alpha)\rangle$, for $\alpha \in [0, \pi]$ and $\beta \in [0, 2\pi]$.

Definition 10.1.1 *Let Ω denote a non-empty connected closed subset of $\mathbb{R}^2$ with non-empty interior. A* ***parameterisation*** *of a surface S in $\mathbb{R}^3$ is a continuous map $\mathbf{r}\colon \Omega \to \mathbb{R}^3$ such that $\mathbf{r}(\Omega) = S$. If a surface exhibits a parameterisation, then we call the surface a* ***parametric surface****.*

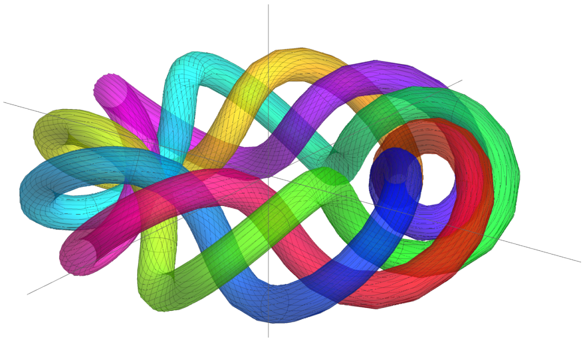


Figure 10.2: Graphical representation of the parameterisation $\mathbf{s}\colon [0,\pi]^2 \to \mathbb{R}^3$ given by $\mathbf{s}(\alpha,\beta) = \mathbf{p}(\alpha) + (\mathbf{n}(\alpha)\cos(\beta) + \mathbf{b}(\alpha)\sin(\beta))/10$, for α and $\beta \in [0,\pi]$. Here $\mathbf{p}\colon [0,\pi] \to \mathbb{R}^3$ is defined by $\mathbf{p}(\alpha) = \langle (1+0.3\cos(11\alpha/3))\cos(\alpha), (1+0.3\cos(11\alpha/3))\sin(\alpha), 0.3\sin(11\alpha/3)\rangle$, and $\mathbf{n}$ and $\mathbf{b}\colon [0,\pi] \to \mathbb{R}^3$ respectively denote the unit normal and unit bi-normal vectors to $\mathbf{p}(t)$, namely $\mathbf{n}(\alpha) = \mathbf{p}''(\alpha)/|\mathbf{p}''(\alpha)|$ and $\mathbf{b}(\alpha) = \mathbf{n} \times \mathbf{p}'(\alpha)/|\mathbf{n} \times \mathbf{p}'(\alpha)|$.

Example 10.1.2

- *A parameterisation of the cone* $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2$ *and* $z \in [0,1]\}$*, is given by*

$$(\alpha,\beta) \mapsto \langle \alpha\cos(\beta), \alpha\sin(\beta), \alpha\rangle,$$

for $\alpha \in [0,1]$ *and* $\beta \in [0,2\pi]$*, see* Figure *10.3.*

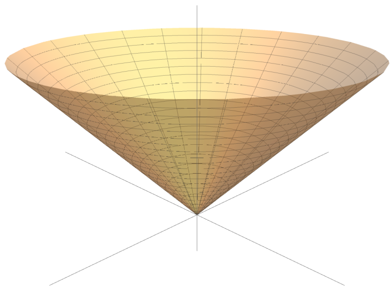


Figure 10.3: Graphical representation of the parameterisation $(\alpha,\beta) \mapsto \langle \alpha\cos(\beta), \alpha\sin(\beta), \alpha\rangle$, for $\alpha \in [0,1]$ and $\beta \in [0,2\pi]$.

- *A parameterisation of the surface* $\{(x,y,z) \in \mathbb{R}^3 \colon \sin(xy) = z\}$*, is given by*

$$(\alpha,\beta) \mapsto \langle \alpha, \beta, \sin(\alpha\beta)\rangle,$$

for α *and* $\beta \in \mathbb{R}$*, see* Figure *10.4.*

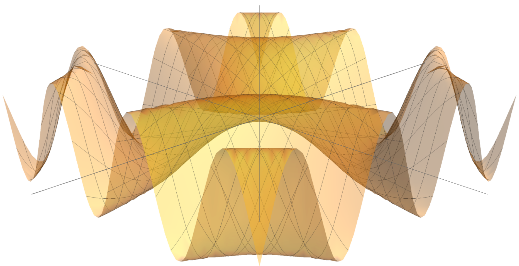


Figure 10.4: Graphical representation of the parameterisation $(\alpha,\beta) \mapsto \langle \alpha, \beta, \sin(\alpha\beta)\rangle$, for α and $\beta \in \mathbb{R}$.

- *A parameterisation of the surface of the ellipsoid* $\{(x, y, z) \in \mathbb{R}^3 : 4x^2 + y^2 + 16z^2 = 16\}$*, is given by*

$$(\alpha, \beta) \mapsto \langle 2\sin(\alpha)\cos(\beta), 4\sin(\alpha)\sin(\beta), \cos(\alpha)\rangle,$$

for $\alpha \in [0, \pi]$ *and* $\beta \in [0, 2\pi]$*, see* FIGURE *10.5.*

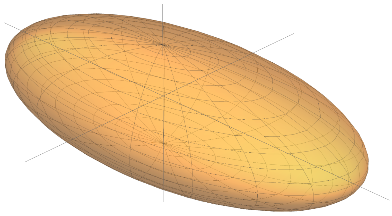


Figure 10.5: Graphical representation of the parameterisation $(\alpha, \beta) \mapsto \langle 2\sin(\alpha)\cos(\beta), 4\sin(\alpha)\sin(\beta), \cos(\alpha)\rangle$, for $\alpha \in [0, \pi]$ and $\beta \in [0, 2\pi]$.

- *A parameterisation of the surface, in cylindrical polars,* $\{(r, \theta, z) \in \mathbb{R}^3 : \theta \in [-\pi, \pi),\ r \in \mathbb{R}_0^+,\ \text{and } z = \theta\}$*, is given by*

$$(\alpha, \beta) \mapsto \langle \alpha\cos(\beta), \alpha\sin(\beta), \beta\rangle,$$

for $\alpha \in \mathbb{R}_0^+$ *and* $\beta \in [-\pi, \pi]$*, see* FIGURE *10.6.*

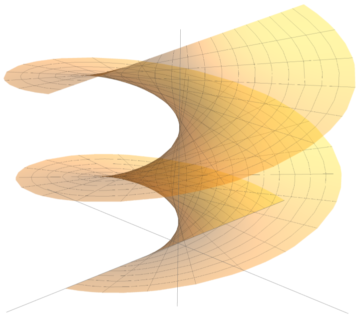


Figure 10.6: Graphical representation of the parameterisation $(\alpha, \beta) \mapsto \langle \alpha\cos(\beta), \alpha\sin(\beta), \beta\rangle$, for $\alpha \in \mathbb{R}_0^+$ and $\beta \in [-\pi, \pi]$.

- *A parameterisation of the surface* $\{(x, y, z) \in \mathbb{R}^3 : x^2 = y^2 z \text{ and } z \geq 0\}$*, is given by*

$$(\alpha, \beta) \mapsto \langle \alpha\beta, \alpha, \beta^2\rangle,$$

for α *and* $\beta \in \mathbb{R}$*, see* FIGURE *10.7. This surface is known as Whitney's umbrella.*

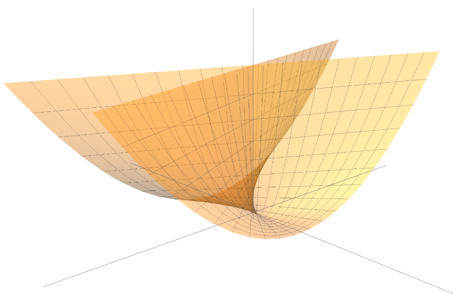


Figure 10.7: Graphical representation of the parameterisation $(\alpha, \beta) \mapsto \langle \alpha\beta, \alpha, \beta^2\rangle$, for α and $\beta \in \mathbb{R}$.

- *A parameterisation of the surface* $\{(x,y,z) \in \mathbb{R}^3 \colon (4-\sqrt{x^2+y^2})^2+z^2=1\}$*, is given by*
$$(\alpha,\beta) \mapsto \langle (4+\cos(\alpha))\cos(\beta), (4+\cos(\alpha))\sin(\beta), \sin(\alpha)\rangle,$$
for α *and* $\beta \in [0,2\pi]$*, see* FIGURE *10.8. This surface is often refereed to as a torus.*

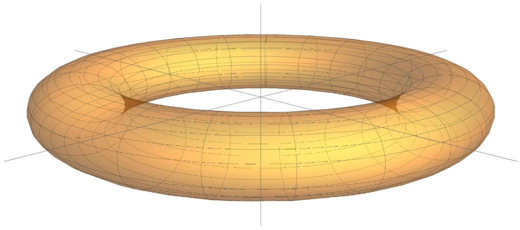


Figure 10.8: Graphical representation of the parameterisation $(\alpha,\beta) \mapsto \langle (4+\cos(\alpha))\cos(\beta), (4+\cos(\alpha))\sin(\beta), \sin(\alpha)\rangle$, for α and $\beta \in [0,2\pi]$.

For a given a surface S there can exist several parameterisations of S, for instance looking at our examples above, a second parameterisation of the torus is
$$(\alpha,\beta) \mapsto \langle (4-\cos(\alpha))\cos(\beta), (4-\cos(\alpha))\sin(\beta), \sin(\alpha)\rangle,$$
for α and $\beta \in [0,2\pi]$.

Let $\mathbf{r}$ denote a parameterisation of a surface S. If $\mathbf{r}$ is injective, then the parametric surface does not self intersect itself. In this case $\mathbf{r}$ maps the boundary of its domain space Ω onto a curve in $\mathbb{R}^3$ called the (**topological**) **boundary of the parametric surface**. For certain parameterisations $\mathbf{r}$ that are not injective, such as the first parameterisation in **Example 10.1.2** and the parameterisation of the sphere given at the start of this section, we can sometimes still define the boundary of the parametric surface. In the first parameterisation of **Example 10.1.2** the boundary of the resulting parametric surface is the curve given by the parameterisation
$$\mathbf{s}(\beta) = \langle \cos(\beta), \sin(\beta), 1\rangle,$$
for $\beta \in [0,2\pi]$. For the parameterisation of the sphere given at the start of this section, the boundary of the resulting parametric surface is empty. Such surfaces, namely bounded surfaces with no *unjoined* edges which may be used to form a boundary, are called **closed surfaces**. Examples of closed surfaces are depicted in FIGURES 10.1, 10.2, 10.5 and 10.8. An example of surfaces which is not closed is depicted in FIGURE 10.3.

Definition 10.1.3 *A subset* S *of* $\mathbb{R}^3$ *is called a* ***closed surface*** *if it is the boundary of a non-empty open connected bounded subset of* $\mathbb{R}^3$.

Exercise 10.1.4 *List the parameterisations in* ***Example 10.1.2*** *which are not injective and and hence the surfaces which self intersect.*

Solution. The first, the third, the fifth and the sixth parameterisations in **Example 10.1.2** are not injective and hence the resulting surfaces self intersect.

10.2 Composite and smooth surfaces

Our goal is to find the value of double integrals over curved surfaces, such as those depicted in the previous section. For this we need our given surface to be smooth or to comprise of smooth components. Therefore, in this section we introduce two concepts: composite surfaces and smooth surfaces.

Composite surfaces

If a finite number of parametric surfaces are joined together along part or all of their boundary, the result is called a **composite surface**. For example, a sphere can be obtained by joining two hemispheres along their boundary circles, and the surface of a cube can be obtained by appropriately joining six square faces along their boundary, see FIGURE 10.9. This latter composite surface is closed since there are no *unjoined* edges to comprise the boundary out of. However, if the top face is removed, then the remaining composite surface forms a cubical box with no top. The top edges of the four side faces form the boundary of this composite surface, see FIGURE 10.9.

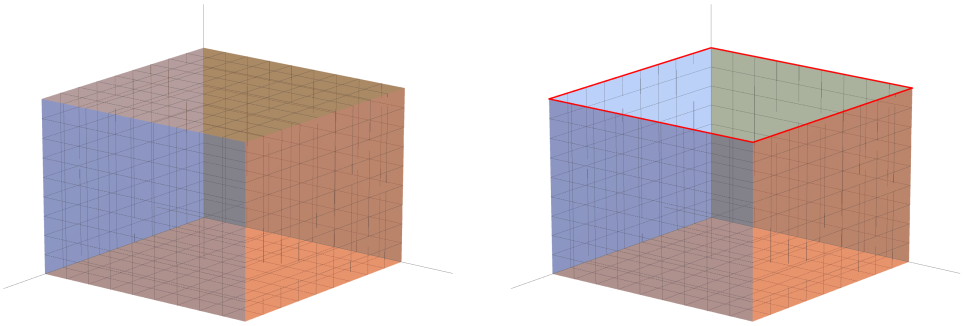


Figure 10.9: LEFT. Surface of the unit cube. RIGHT. Surface of the unit cube with top surface remove. Highlighted in red is the boundary of this composite surface.

Smooth surfaces

Loosely speaking, a surface S is smooth if it has a unique tangent plane at any non-boundary point $p \in S$. A non-zero vector $\mathbf{n}$ normal to such a tangent plane is said to be **normal to the surface at p**. Below we give a formal definition for when parametric and implicitly defined surfaces are smooth.

Definition 10.2.1

- *A parametric surface $S \subseteq \mathbb{R}^3$ is called **smooth**, if there exists a parameterisation $\mathbf{r}$ of S with $\mathbf{r}$ smooth, that is all partial derivatives of all orders of $\mathbf{r}$ are continuous, and $\mathbf{r}_\alpha \times \mathbf{r}_\beta \neq \mathbf{0}$ on the interior of the domain of $\mathbf{r}$, where α and β are the independent variables of $\mathbf{r}$.*
- *Let S be an implicitly defined surface given by $S = \{(x, y, z) \in \Omega \colon F(x, y, z) = 0\}$, for some function F of three variables with domain $\Omega \subseteq \mathbb{R}^3$. We say that S is **smooth** if S is contained in the interior of Ω and F is smooth with $\nabla F \neq \mathbf{0}$ on S.*

The conditions that $\mathbf{r}_\alpha \times \mathbf{r}_\beta \neq \mathbf{0}$ and $\nabla F \neq \mathbf{0}$ in the above definition means that we may always determine a tangent plane at any point on S.

An example of a smooth surface is the cone $K = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 \text{ and } z \in [0, 1]\}$ with the origin removed. Other examples of smooth surfaces include planes, spheres and ellipsoids, as well as open subsets of smooth surfaces.

A surface is said to be **piecewise smooth** if it can be *pieced together* via finitely many smooth surfaces. This is a loose definition, as to define piecewise smooth surfaces formally is in fact quite difficult, and so we shall not do so here. For the most part, all the surfaces we will deal with in the remainder of this course will either be smooth or will be unions of very few smooth pieces which intersect over curves. The later of which are (in most cases) examples of piecewise smooth surfaces.

10.3 Surface integrals of the first kind

Our aim is to develop a method to be able to integrate a function defined on a surface. For this we require knowledge about the area element. This is what we now develop.

Consider a parametric surface S with parameterisation $\mathbf{r} \colon \Omega \to \mathbb{R}^3$. If (α_0, β_0) is a point in the interior of Ω, then $\alpha \mapsto \mathbf{r}(\alpha, \beta_0)$ and $\beta \mapsto \mathbf{r}(\alpha_0, \beta)$ are two curves on S, intersecting at $\mathbf{r}(\alpha_0, \beta_0)$ and having at that point tangent vectors

$$\mathbf{r}_\alpha(\alpha_0, \beta_0) = \frac{\partial \mathbf{r}}{\partial \alpha}(\alpha_0, \beta_0) \qquad \text{and} \qquad \mathbf{r}_\beta(\alpha_0, \beta_0) = \frac{\partial \mathbf{r}}{\partial \beta}(\alpha_0, \beta_0),$$

respectively. Assuming that these two vectors are not parallel, their cross product $\mathbf{n}$ is normal to the surface S at $\mathbf{r}(\alpha_0, \beta_0)$. Further, the area element $\mathrm{d}S$ on S is bounded by the four curves

$$\beta \mapsto \mathbf{r}(\alpha_0, \beta), \quad \beta \mapsto \mathbf{r}(\alpha_0 + \mathrm{d}\alpha, \beta), \quad \alpha \mapsto \mathbf{r}(\alpha, \beta_0) \quad \text{and} \quad \alpha \mapsto \mathbf{r}(\alpha, \beta_0 + \mathrm{d}\beta),$$

and maybe approximated by the parallelogram spanned by the vectors $\mathbf{r}_\alpha \mathrm{d}\alpha$ and $\mathbf{r}_\beta \mathrm{d}\beta$ at (α_0, β_0), and hence the area element is given by

$$\mathrm{d}S = |\mathbf{r}_\alpha \times \mathbf{r}_\beta| \,\mathrm{d}A$$

see FIGURE 10.10. From which we may conclude the following.

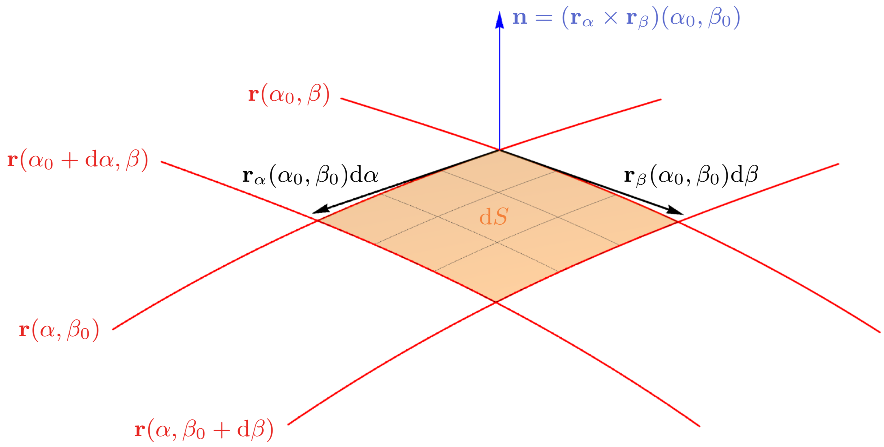


Figure 10.10: An area element $\mathrm{d}S$ on a parametric surface.

Definition/Proposition 10.3.1 *Suppose we have a parametric surface S with a smooth parameterisation $\mathbf{r}\colon \Omega \to \mathbb{R}^3$, for some connected closed and bounded set $\Omega \subset \mathbb{R}^2$ with non-empty interior, and suppose that $\mathbf{r}$ is injective on the interior of Ω.*

- *Given a point $(a, b, c) \in S \setminus \partial S$, a vector equation of the tangent plane to the surface S at (a, b, c) is given by*

$$((\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta)) \cdot \langle x, y, z\rangle = ((\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta)) \cdot \langle a, b, c\rangle,$$

  *where $(\alpha, \beta) \in \Omega$ satisfies $\mathbf{r}(\alpha, \beta) = \langle a, b, c\rangle$. The vector $\mathbf{r}_\alpha \times \mathbf{r}_\beta$ is called the **fundamental vector** of the parameterisation $\mathbf{r}$.*

- *The area of the surface S is equal to*

$$\mathrm{Area}(S) = \iint_\Omega |(\mathbf{r}_\alpha \times \mathbf{r}_\beta)| \,\mathrm{d}A.$$

- *Let $f\colon S \to \mathbb{R}$ denote a smooth scalar field. The **surface integral (of the first kind) of f** is defined by*

$$\iint_S f \,\mathrm{d}S = \iint_\Omega (f \circ \mathbf{r}) \,|\mathbf{r}_\alpha \times \mathbf{r}_\beta| \,\mathrm{d}A.$$

If S is the union of smooth surfaces (such as a composite surface or a piecewise smooth surface), then the surface integral over S, is defined as the sum of the individual surface integrals. Also, note that, the area of a parametric surface S can be computed via the following formula,

$$\mathrm{Area}(S) = \iint_\Omega \sqrt{(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha)(\mathbf{r}_\beta \cdot \mathbf{r}_\beta) - (\mathbf{r}_\alpha \cdot \mathbf{r}_\beta)^2} \,\mathrm{d}A.$$

This result is a direct consequence of Lagrange's identity.

Theorem 10.3.2 (Lagrange's identity) *If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, then*

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2}.$$

Proof. Recall that if $\mathbf{u}$ and $\mathbf{v}$ are two vectors and θ is the angle between them, then $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \,|\mathbf{v}| \,|\sin(\theta)|$ and $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| \,|\mathbf{v}| \,|\cos(\theta)|$ and so

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 \,|\mathbf{v}|^2 \,\sin^2(\theta) = |\mathbf{u}|^2 \,|\mathbf{v}|^2 \,(1 - \cos^2(\theta)) = (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2.$$

■

10.4 Surface integrals of the second kind

A smooth surface S in $\mathbb{R}^3$ is said to be **orientable** if there exists a unit vector field $\mathbf{n}(p)$ defined for all non-boundary points $p \in S$ that varies continuously as p ranges over all non-boundary points of S and that is normal to S at p. Any such vector field $\mathbf{n}$ determines an **orientation** on S. This means that the surface must have two sides since $\mathbf{n}(p)$ can only have one value at each point $p \in S$. The side out of which $\mathbf{n}$ points is called the **positive side**; the other side is called the **negative side**. An **oriented surface** is a surface together with a particular choice of orientating unit normal vector field $\mathbf{n}$.

An oriented surface S **induces an orientation** on any of its boundary curves C. Namely, if we stand on the positive side of the surface S and walk around C in the direction of its orientation, then S will be on our left hand side, see FIGURE 10.11.

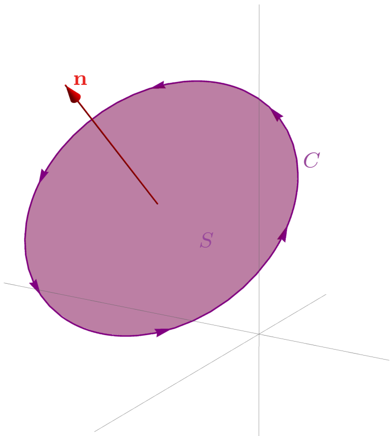


Figure 10.11: A surface S together with a unit normal vector field $\mathbf{n}$ with the induced orientation on the boundary curve C highlighted.

A piecewise smooth surface is said to be **orientable** if, whenever two smooth components surfaces join along a common boundary curve C, then they induce opposite orientations along C. This forces the normal unit filed $\mathbf{n}$ to be on the same side of adjacent components. For instance, the surface of the unit cube is a piecewise smooth closed surface consisting of six smooth surfaces (the square faces) joined along edges, see FIGURE 10.9. If all the faces are oriented so that their normal $\mathbf{n}$ points out of the cube (or if they all point into the cube), then the surface is oriented. Examples of orientable surfaces include planes, spheres and ellipsoids, as well as, open subsets of orientable surfaces.

Definition/Proposition 10.4.1 *Let Ω denote a connected, closed and bounded subset of $\mathbb{R}^2$ with non-empty interior, let $S \subseteq \mathbb{R}^3$ denote an orientable parametric surface with a smooth parameterisation $\mathbf{r} \colon \Omega \to S$ that is injective on its interior, let $U \subseteq \mathbb{R}^3$ be an open set containing S, and let $\mathbf{F} \colon U \to \mathbb{R}^3$ denote a smooth vector field. A unit normal vector field to S is given by*

$$\mathbf{n} = \frac{\mathbf{r}_\alpha \times \mathbf{r}_\beta}{|\mathbf{r}_\alpha \times \mathbf{r}_\beta|},$$

*where α and β are the independent variables of $\mathbf{r}$. The **surface integral (of the second kind)** of $\mathbf{F}$ over S, is defined by*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint_\Omega (\mathbf{F} \circ \mathbf{r}) \cdot \frac{\mathbf{r}_\alpha \times \mathbf{r}_\beta}{|\mathbf{r}_\alpha \times \mathbf{r}_\beta|} |(\mathbf{r}_\alpha \times \mathbf{r}_\beta)| \, \mathrm{d}A = \iint_\Omega (\mathbf{F} \circ \mathbf{r}) \cdot (\mathbf{r}_\alpha \times \mathbf{r}_\beta) \, \mathrm{d}A.$$

If the surface S is piecewise smooth, then the integral of $\mathbf{F}$ over S is defined as the sum of the integrals of $\mathbf{F}$ over the respective smooth pieces.

Exercise 10.4.2 *Let $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1 \, and \, z \geq 0\}$ and let $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ for $(x, y, z) \in \mathbb{R}^3$. Show that*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = 2\pi,$$

where $\mathbf{n}$ *is the unit normal vector arising from the parameterisation* $\mathbf{s}\colon [0,\pi/2]\times[0,2\pi]\to\mathbb{R}^3$ *given for* $\alpha\in[0,\pi/2]$ *and* $\beta\in[0,2\pi]$*, by* $\mathbf{s}(\alpha,\beta)=\langle\sin(\alpha)\cos(\beta),\sin(\alpha)\sin(\beta),\cos(\alpha)\rangle$*.*

Solution. Since S is the upper hemisphere with radius one, it is smooth, and a direct calculation shows that $\mathbf{s}$ is injective on the interior of its domain. Further,

$$\begin{aligned}\mathbf{s}_\alpha(\alpha,\beta)&=\langle\cos(\alpha)\cos(\beta),\cos(\alpha)\sin(\beta),-\sin(\alpha)\rangle,\\ \mathbf{s}_\beta(\alpha,\beta)&=\langle-\sin(\alpha)\sin(\beta),\sin(\alpha)\cos(\beta),0\rangle,\ \text{and}\\ (\mathbf{s}_\alpha\times\mathbf{s}_\beta)(\alpha,\beta)&=\langle\sin^2(\alpha)\cos(\beta),\sin^2(\alpha)\sin(\beta),\cos(\alpha)\sin(\alpha)\rangle,\end{aligned}$$

for $\alpha\in[0,\pi/2]$ and $\beta\in[0,2\pi]$. Thus, as $(\mathbf{s}_\alpha\times\mathbf{s}_\beta)$ is continuous, S is orientable and as $\sim$ is smooth, we have that S is a smooth surface. Observing that

$$\mathbf{F}(\mathbf{s}(\alpha,\beta))\cdot((\mathbf{s}_\alpha\times\mathbf{s}_\beta)(\alpha,\beta))=\sin(\alpha),$$

for $\alpha\in[0,\pi/2]$ and $\beta\in[0,2\pi]$, we may conclude

$$\iint_S\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S=\int_{\beta=0}^{\beta=2\pi}\int_{\alpha=0}^{\alpha=\pi/2}\sin(\alpha)\,\mathrm{d}\alpha\mathrm{d}\beta=\int_{\beta=0}^{\beta=2\pi}1\,\mathrm{d}\beta=2\pi.$$

Note, as our integrand is bounded and continuous on $[0,\pi/2]\times[0,2\pi]$, and as $[0,\pi/2]\times[0,2\pi]$ is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and independent of the order in which the repeated integral is computed.

Exercise 10.4.3 *Let* $S=\{(x,y,z)\in\mathbb{R}^3\colon x^2+y^2+z^2=1\,and\,z\geq 0\}$ *and let* $\mathbf{F}\colon\mathbb{R}^3\to\mathbb{R}^3$ *be defined* $\mathbf{F}(x,y,z)=\langle x,y,z\rangle$ *for* $(x,y,z)\in\mathbb{R}^3$*. Show that*

$$\iint_S\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S=-2\pi,$$

where $\mathbf{n}$ *is the unit normal vector arising from the parameterisation* $\mathbf{s}\colon[0,\pi/2]\times[0,2\pi]\to\mathbb{R}^3$ *given, for* $\alpha\in[0,\pi/2]$ *and* $\beta\in[0,2\pi]$*, by* $\mathbf{s}(\alpha,\beta)=\langle\sin(\alpha)\sin(\beta),\sin(\alpha)\cos(\beta),\cos(\alpha)\rangle$*.*

Solution. Since S is the upper hemisphere with radius one, it is smooth, and a direct calculation shows that $\mathbf{s}$ is injective on the interior of its domain. Further,

$$\begin{aligned}\mathbf{s}_\alpha(\alpha,\beta)&=\langle\cos(\alpha)\sin(\beta),\cos(\alpha)\cos(\beta),-\sin(\alpha)\rangle,\\ \mathbf{s}_\beta(\alpha,\beta)&=\langle\sin(\alpha)\cos(\beta),-\sin(\alpha)\sin(\beta),0\rangle,\ \text{and}\\ (\mathbf{s}_\alpha\times\mathbf{s}_\beta)(\alpha,\beta)&=\langle-\sin^2(\alpha)\sin(\beta),-\sin^2(\alpha)\cos(\beta),-\cos(\alpha)\sin(\alpha)\rangle,\end{aligned}$$

for $\alpha\in[0,\pi/2]$ and $\beta\in[0,2\pi]$. Thus, as $(\mathbf{s}_\alpha\times\mathbf{s}_\beta)$ is continuous, S is orientable and as $\mathbf{s}$ is smooth, we have that S is a smooth surface. Observing that

$$\mathbf{F}(\mathbf{s}(\alpha,\beta))\cdot((\mathbf{s}_\alpha\times\mathbf{s}_\beta)(\alpha,\beta))=-\sin(\alpha),$$

for $\alpha\in[0,\pi/2]$ and $\beta\in[0,2\pi]$, we may conclude

$$\iint_S\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S=\int_{\beta=0}^{\beta=2\pi}\int_{\alpha=0}^{\alpha=\pi/2}-\sin(\alpha)\,\mathrm{d}\alpha\mathrm{d}\beta=\int_{\beta=0}^{\beta=2\pi}-1\,\mathrm{d}\beta=-2\pi.$$

Note, as our integrand is bounded and continuous on $[0,\pi/2]\times[0,2\pi]$, and as $[0,\pi/2]\times[0,2\pi]$ is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and independent of the order in which the repeated integral is computed.

The discrepancy in the values of the two integrals computed above is related to the issue of *clockwise vs counter clockwise* for line integrals. However, here the problem is more difficult. Recall that we defined the unit normal field to a parametric surface, with parameterisation $\mathbf{r}$, by

$$\mathbf{n}=\frac{\mathbf{r}_\alpha\times\mathbf{r}_\beta}{|\mathbf{r}_\alpha\times\mathbf{r}_\beta|}.$$

This can sometimes cause some complications as we shall see in the following examples.

Example 10.4.4 *Consider the surface given by the parameterisation* $\mathbf{r}\colon [-1,1]\times[0,2\pi]\to\mathbb{R}^3$ *given by*

$$\mathbf{r}(\alpha,\beta)=\langle(4+\alpha\cos(\beta/2))\cos(\beta),(4+\alpha\cos(\beta/2))\sin(\beta),\alpha\sin(\beta/2)\rangle,$$

for $\alpha\in[-1,1]$ *and* $\beta\in[0,2\pi]$*, see* Figure *10.12. Such a surface is called a Möbius strip. Observe that* $\mathbf{r}$ *is injective on the interior of its domain and that* $\mathbf{r}(0,0)=\mathbf{r}(0,2\pi)=\langle 4,0,0\rangle$*. Moreover, for* $\alpha\in[-1,1]$ *and* $\beta\in[0,2\pi]$*,*

$$\begin{aligned}\mathbf{r}_\alpha(\alpha,\beta)&=\langle\cos(\beta/2)\cos(\beta),\cos(\beta/2)\sin(\beta),\sin(\beta/2)\rangle \text{ and}\\ \mathbf{r}_\beta(\alpha,\beta)&=\langle-4\sin(\beta)-\alpha\sin(\beta/2)\cos(\beta)/2-\alpha\cos(\beta/2)\sin(\beta),\\ &\qquad 4\cos(\beta)-\alpha\sin(\beta/2)\sin(\beta)/2-\alpha\cos(\beta/2)\cos(\beta),\alpha\cos(\beta/2)/2\rangle.\end{aligned}$$

With this at hand and with some effort one can also show that

$$\begin{aligned}(\mathbf{r}_\alpha\times\mathbf{r}_\beta)(\alpha,\beta)=\langle&-(\alpha\cos(\beta/2)+4)\cos(\beta)\sin(\beta/2)+\alpha\sin(\beta)/2,\\ &-(\alpha\cos(\beta/2)+4)\sin(\beta)\sin(\beta/2)-\alpha\cos(\beta)/2,\\ &\alpha\cos^2(\beta/2)+4\cos(\beta/2)\rangle,\end{aligned}$$

from which we may conclude that

$$(\mathbf{r}_\alpha\times\mathbf{r}_\beta)(0,0)=\langle 0,0,4\rangle\quad\text{and}\quad(\mathbf{r}_\alpha\times\mathbf{r}_\beta)(0,2\pi)=\langle 0,0,-4\rangle.$$

This poses a problem because the unit normal vector field (induced by $\mathbf{r}$) is not well defined at every point. Surfaces for which this occurs are called **non-orientable**. Asides from the Möbius strip (and slight variation) there are not too many examples of non-orientable surfaces. Most others are self intersecting, such examples include the Klein bottle and the Roman surface, see Figure 10.13.

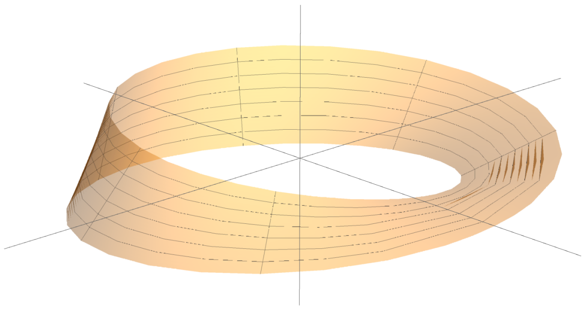


Figure 10.12: Graphical representation of a Möbius strip with parameterisation $(\alpha,\beta)\mapsto\langle(4+\alpha\cos(\beta/2))\cos(\beta),(4+\alpha\cos(\beta/2))\sin(\beta),\alpha\sin(\beta/2)\rangle$, for $\alpha\in[0,2]$ and $\beta\in[0,2\pi]$.

Exercise 10.4.5 *Consider the cylinder* $S=\{(x,y,z)\in\mathbb{R}^3\colon x^2+y^2=1 \text{ and } z\in[-1,1]\}$ *and the vector field* $\mathbf{F}\colon\mathbb{R}^3\to\mathbb{R}^3$ *given by* $\mathbf{F}(x,y,z)=\langle xz,yz,z\rangle$ *for* $(x,y,z)\in\mathbb{R}^3$*. Compute the value of the integral*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S,$$

where $\mathbf{n}$ *is the unit normal vector field pointing away from the* z*-axis.*

Solution. A smooth parameterisation of the given surface is $\mathbf{r}\colon[0,2\pi]\times[-1,1]\to\mathbb{R}^3$ given by

$$\mathbf{r}(\alpha,\beta)=\langle\cos(\alpha),\sin(\alpha),\beta\rangle,$$

for $\alpha\in[0,2\pi]$ and $\beta\in[-1,1]$. A direct calculation shows that $\mathbf{r}$ is injective on the interior of its domain and that

$$\mathbf{r}_\alpha(\alpha,\beta)=\langle-\sin(\alpha),\cos(\alpha),0\rangle\quad\text{and}\quad\mathbf{r}_\beta(\alpha,\beta)=\langle 0,0,1\rangle,$$

whence

$$(\mathbf{r}_\alpha\times\mathbf{r}_\beta)(\alpha,\beta)=\langle\cos(\alpha),\sin(\alpha),0\rangle,$$

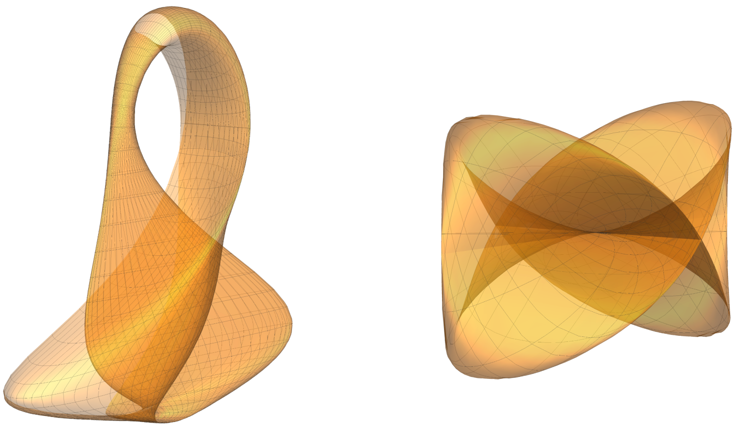


Figure 10.13: LEFT. Illustration of a Klein bottle. RIGHT. Illustration of a Roman surface.

for $\alpha \in [0, 2\pi]$ and $\beta \in [-1, 1]$. This vector indeed points away from the z-axis. Observing that the component functions of $\mathbf{r}_\alpha \times \mathbf{r}_\beta$ are of trigonometric functions, the normal vector field $\mathbf{r}_\alpha \times \mathbf{r}_\beta$ is continuous and so, under the parameterisation $\mathbf{r}$, our surface S is orientable, and as $\diagdown$ is smooth we have that the cylinder S is smooth. Further, since the components of the given vector field are polynomials, our vector field is smooth, we may conclude that

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S &= \int_{\beta=-1}^{\beta=1} \int_{\alpha=0}^{\alpha=2\pi} \langle \beta\cos(\alpha), \beta\sin(\alpha), \beta \rangle \cdot \langle \cos(\alpha), \sin(\alpha), 0 \rangle \, \mathrm{d}\alpha \mathrm{d}\beta \\ &= \int_{\beta=-1}^{\beta=1} \int_{\alpha=0}^{\alpha=2\pi} \beta\cos^2(\alpha) + \beta\sin^2(\alpha) \, \mathrm{d}\alpha \mathrm{d}\beta = \int_{\beta=-1}^{\beta=1} \int_{\alpha=0}^{\alpha=2\pi} \beta \, \mathrm{d}\alpha \mathrm{d}\beta = 0.\end{aligned}$$

Note, as our integrand is bounded and continuous on $[0, \pi/2] \times [0, 2\pi]$, and as $[0, \pi/2] \times [0, 2\pi]$ is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and independent of the order in which the repeated integral is computed.

Exercise 10.4.6 *Let S denote the surface of the pyramid that is formed by the co-ordinate planes and the plane $\{(x, y, z) \in \mathbb{R}^3 \colon x + y + z = 1\}$. Compute the value of the integral*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

where $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field $\mathbf{F}(x, y, z) = \langle x + y, y + z, z + x \rangle$, for $(x, y, z) \in \mathbb{R}^3$, and where $\mathbf{n}$ is the unit normal vector field on S that points out of the pyramid.

Solution. We compute the integrals

$$\iint_{S_i} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

for $i \in \{1, 2, 3, 4\}$, where $S_1, \ldots, S_4$ denote the planar regions

$$\begin{aligned} S_1 &= \{(x, y, z) \in \mathbb{R}^3 \colon x \in [0, 1],\ y \in [0, 1 - x] \text{ and } z = 0\}, \\ S_2 &= \{(x, y, z) \in \mathbb{R}^3 \colon x \in [0, 1],\ y = 0 \text{ and } z \in [0, 1 - x]\}, \\ S_3 &= \{(x, y, z) \in \mathbb{R}^3 \colon x = 0,\ y \in [0, 1] \text{ and } z \in [0, 1 - y]\}, \text{ and} \\ S_4 &= \{(x, y, z) \in \mathbb{R}^3 \colon x \in [0, 1],\ y \in [0, 1 - x] \text{ and } z = 1 - x - y\}. \end{aligned}$$

To this end let $\Omega = \{(x, y) \in \mathbb{R}^2 \colon x \in [0, 1] \text{ and } y \in [0, 1 - x]\}$ and observing that the component functions of $\mathbf{F}$ are linear functions, we have that $\mathbf{F}$ is smooth. Further, since the surfaces S_1, S_2, S_3, S_4 are planar regions, they are subsets of smooth surfaces and so smooth.

We start by parameterising S_1. The natural linear injective parameterisation is given by $\mathbf{r}(\alpha, \beta) = \langle \alpha, \beta, 0 \rangle$, for $(\alpha, \beta) \in \Omega$. The normal vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta)$, at $\mathbf{r}(\alpha, \beta)$, is $\langle 0, 0, 1 \rangle$, which points towards the inside of the pyramid. We therefore use the linear injective parameterisation $\mathbf{r}\colon \Omega \to \mathbb{R}^3$ given by $\mathbf{r}(\alpha,\beta) = \langle \beta, \alpha, 0\rangle$ for $(\alpha,\beta) \in \Omega$, which yields the normal vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha,\beta) = \langle 0, 0, -1\rangle$ at $\mathbf{r}(\alpha,\beta)$ and thus

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} \langle \beta+\alpha, \alpha, \beta\rangle \cdot \langle 0,0,-1\rangle \,\mathrm{d}\beta\mathrm{d}\alpha \\ &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} -\beta\,\mathrm{d}\beta\mathrm{d}\alpha = \int_{\alpha=0}^{\alpha=1} \frac{-(1-\alpha)^2}{2}\,\mathrm{d}\alpha = \frac{(1-\alpha)^3}{6}\bigg|_{\alpha=0}^{\alpha=1} = \frac{-1}{6}.\end{aligned}$$

Similarly, a linear injective parameterisation of S_2 is $\mathbf{r}\colon \Omega \to \mathbb{R}^3$ given by $\mathbf{r}(\alpha,\beta) = \langle \alpha, 0, \beta\rangle$, for $(\alpha,\beta)\in\Omega$, which yields the normal vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha,\beta) = \langle 0,-1,0\rangle$ at $\mathbf{r}(\alpha,\beta)$, and thus,

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} \langle \alpha, \beta, \alpha+\beta\rangle \cdot \langle 0,-1,0\rangle \,\mathrm{d}\beta\mathrm{d}\alpha \\ &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} -\beta\,\mathrm{d}\beta\mathrm{d}\alpha = \int_{\alpha=0}^{\alpha=1} \frac{-(1-\alpha)^2}{2}\,\mathrm{d}\alpha = \frac{(1-\alpha)^3}{6}\bigg|_{\alpha=0}^{\alpha=1} = \frac{-1}{6}.\end{aligned}$$

We parameterisation S_3 by $\mathbf{r}\colon \Omega \to \mathbb{R}^3$ defined by $\mathbf{r}(\alpha,\beta) = \langle 0, \beta, \alpha\rangle$, for $(\alpha,\beta)\in\Omega$, which is linear and injective and yields the normal vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha,\beta) = \langle -1,0,0\rangle$ at $\mathbf{r}(\alpha,\beta)$, and thus,

$$\begin{aligned}\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} \langle \beta, \alpha+\beta, \alpha\rangle \cdot \langle -1,0,0\rangle \,\mathrm{d}\beta\mathrm{d}\alpha \\ &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} -\beta\,\mathrm{d}\beta\mathrm{d}\alpha = \int_{\alpha=0}^{\alpha=1} \frac{-(1-\alpha)^2}{2}\,\mathrm{d}\alpha = \frac{(1-\alpha)^3}{6}\bigg|_{\alpha=0}^{\alpha=1} = \frac{-1}{6}.\end{aligned}$$

Finally, we parameterisation S_4 by $\mathbf{r}\colon \Omega \to \mathbb{R}^3$ defined by $\mathbf{r}(\alpha,\beta) = \langle \alpha, \beta, 1-\alpha-\beta\rangle$, for $(\alpha,\beta)\in\Omega$, which is linear and injective and yields the normal vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha,\beta) = \langle 1,1,1\rangle$ at $\mathbf{r}(\alpha,\beta)$, and thus,

$$\begin{aligned}\iint_{S_4} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} \langle \alpha+\beta, 1-\alpha, 1-\beta\rangle \cdot \langle 1,1,1\rangle \,\mathrm{d}\beta\mathrm{d}\alpha \\ &= \int_{\alpha=0}^{\alpha=1}\int_{\beta=0}^{\beta=1-\alpha} 2\,\mathrm{d}\beta\mathrm{d}\alpha = \int_{\alpha=0}^{\alpha=1} 2(1-\alpha)\mathrm{d}\alpha = 1.\end{aligned}$$

Note, in each of the above calculations, since the normal vector field $\mathbf{r}_\alpha \times \mathbf{r}_\beta$ is continuous, we have that each of the faces of the pyramid is orientable. Further, as in each of the above integrals, our integrand is bounded and continuous on Ω, and as Ω is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and independent of the order in which the repeated integral is computed.

Combing the above, and using the linearity of the surface integral, we have that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \frac{-1}{6} + \frac{-1}{6} + \frac{-1}{6} + 1 = \frac{1}{2}.$$

Chapter 11

Theorems of Gauss and Stokes

Gauss' Divergence Theorem is one of three important generalisations of the fundamental theorem of calculus. The other two being Green's Theorem and Stokes' Theorem. In Section 11.1 we state and prove Gauss' Divergence Theorem and in Section 11.2 we discuss Stokes' Theorem.

11.1 Gauss' Divergence Theorem

Gauss' Divergence Theorem is a result that relates the net outward flux of a vector field through a closed surface S to the behaviour of the vector field on the solid enclosed by S.

Theorem 11.1.1 (Gauss' Divergence Theorem) *Let V denote a non-empty closed solid in $\mathbb{R}^3$ whose surface S is a smooth orientable and closed, and let $\mathbf{n}$ denote the unit normal vector field to S pointing outward from V. Let $\Omega \subseteq \mathbb{R}^3$ denote an open set containing V. If $\mathbf{F}\colon \Omega \to \mathbb{R}^3$ is a smooth vector field, then*

$$\underbrace{\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S}_{\substack{\text{Net outward flux}\\ \text{of } \mathbf{F} \text{ across } S}} = \underbrace{\iiint_V \operatorname{div}(\mathbf{F})\,\mathrm{d}V.}_{\substack{\text{Divergence}\\ \text{integral}}}$$

We note that Gauss' divergence theorem also extends to the situation when the surface S is piecewise smooth, for instance, to the surface of a cube or the surface of a pyramid.

Proof (Sketch). We focus on the special case when $\mathbf{F} = \langle 0, 0, R\rangle$, for some scalar field $R\colon \Omega \to \mathbb{R}$, in which case the conclusion of Gauss' Divergence Theorem becomes

$$\iint_S (R\,\mathbf{k})\cdot\mathbf{n}\,\mathrm{d}S = \iiint_V \frac{\partial R}{\partial z}\,\mathrm{d}V. \tag{11.1}$$

To obtain the general case, when $\mathbf{F} = \langle P, Q, R\rangle$, for given scalar fields P, Q and $R\colon \Omega \to \mathbb{R}$, one sums three such identities, one for each component P, Q and R of $\mathbf{F}$. If the region V happens to be **vertically simple**, namely of the form $V = \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in U \text{ and } z_1(x,y) \leq z \leq z_2(x,y)\}$, for some simply connected closed domain $U \subset \mathbb{R}^2$ and continuous functions z_1 and $z_2\colon \mathbb{R}^2 \to \mathbb{R}$ with $z_1(x,y) \leq z_2(x,y)$ for $(x,y) \in U$, then we may proceed as follows. By the Fundamental Theorem of Calculus and Fubini-Tonelli's Theorem, the right-hand side of (11.1) can be simplified to

$$\iiint_V \frac{\partial R}{\partial z}\,\mathrm{d}V = \iint_U \left(\int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial}{\partial z} R(x,y,z)\,\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y = \iint_U \left(R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right)\mathrm{d}x\mathrm{d}y.$$

On the other hand, the left-hand side of (11.1) is a surface integral which has three distinct components: $T = \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in U \text{ and } z = z_2(x,y)\}$, $B = \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in U \text{ and } z = z_1(x,y)\}$ and $W = \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in \partial U \text{ and } z \in [z_1(x,y), z_2(x,y)]\}$. The surface T is given by the graph of the function z_2, and as such, it follows, from **Definition/Proposition 10.4.1** and by an application of Fubini-Tonelli's Theorem, that

$$\mathbf{n}\,\mathrm{d}S = \left\langle -\frac{\partial z_2}{\partial x}, -\frac{\partial z_2}{\partial y}, 1\right\rangle \mathrm{d}x\mathrm{d}y.$$

In particular, $\langle 0, 0, R\rangle \cdot \mathbf{n}\,\mathrm{d}S = R\,\mathrm{d}x\mathrm{d}y$, and so

$$\iint_T (R\,\mathbf{k}) \cdot \mathbf{n}\,\mathrm{d}S = \iint_U R(x, y, z_2(x, y))\,\mathrm{d}x\mathrm{d}y.$$

In a similar way, on B we have that

$$\mathbf{n}\,\mathrm{d}S = \left\langle \frac{\partial z_1}{\partial x}, \frac{\partial z_1}{\partial y}, -1 \right\rangle \mathrm{d}x\mathrm{d}y,$$

and therefore

$$\iint_B R \cdot (\mathbf{k} \cdot \mathbf{n})\,\mathrm{d}S = \iint_U -R(x, y, z_1(x, y))\,\mathrm{d}x\mathrm{d}y.$$

Finally, since the sides of V are vertical, that is parallel to the z-axis, the vector field $\mathbf{F} = \langle 0, 0, R\rangle$ is perpendicular to the normal vector to W, and so the surface integral of $\mathbf{F}$ over W vanishes. This verifies (11.1) for vertically simple region V.

If V is not vertically simple, then we may decompose it into finitely many vertically simple pieces. Applying the result for each of them, and then summing the various contributions, completes the proof. ■

11.2 Stokes' Theorem

If we regard a region R in the x-y plane as a surface in $\mathbb{R}^3$ with normal vector field $\mathbf{n} = \mathbf{k}$ and boundary curve C, and if we let $\mathbf{F}$ denote a vector field defined on R, then Green's Theorem states that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S$$

(assuming the conditions of Green's Theorem are met). Stokes' Theorem generalises this result.

Theorem 11.2.1 (Stokes' Theorem) *Let $S \subset \mathbb{R}^3$ denote a smooth orientable surface whose boundary is a simple piecewise smooth, closed curve C, and let Ω denote an open subset of $\mathbb{R}^3$ containing S. For a smooth vector field $\mathbf{F}\colon \Omega \to \mathbb{R}^3$ we have that*

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \iint_S \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S,$$

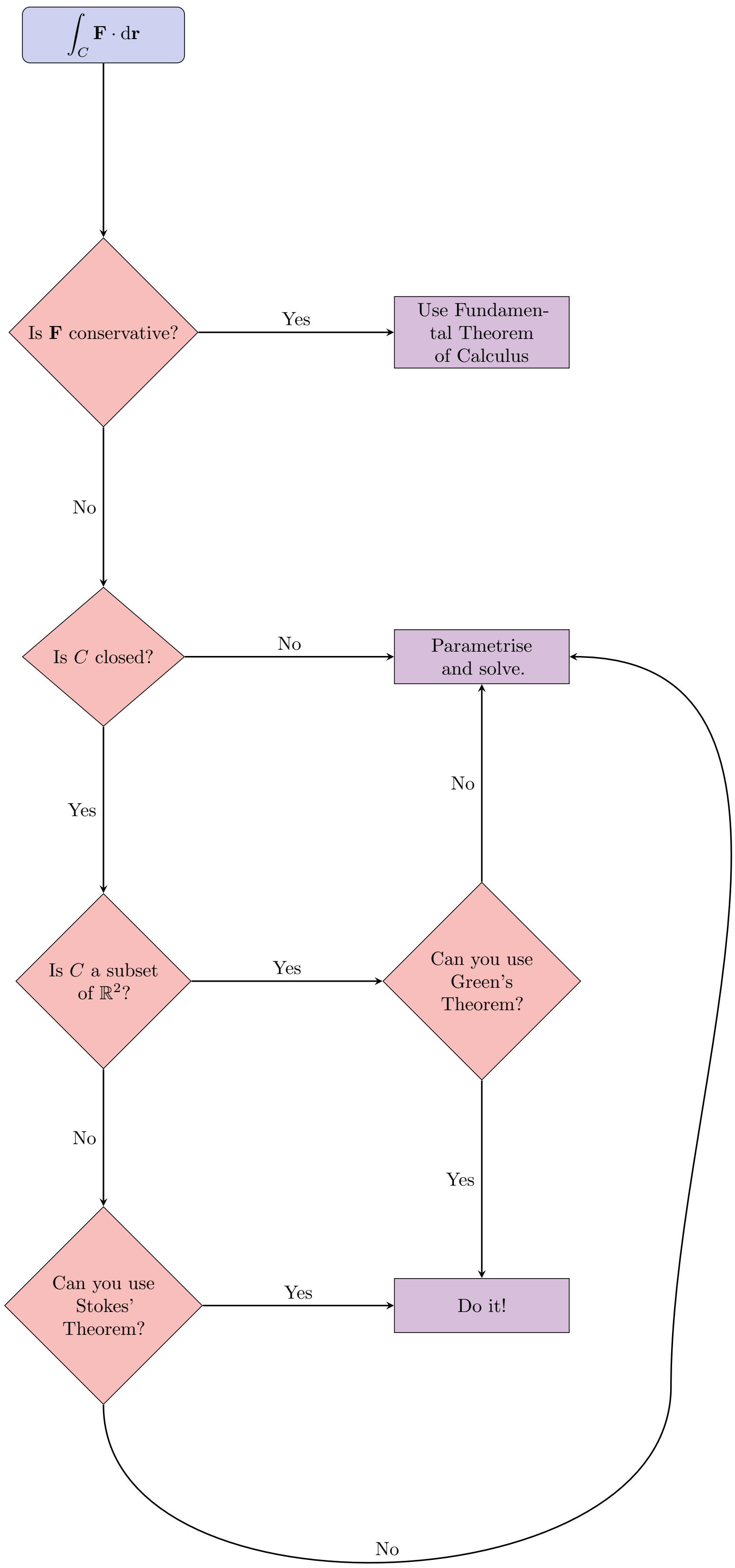
where $\mathbf{n}$ denotes a continuous unit normal vector field to the surface S, and $\mathbf{s}$ denotes a piecewise smooth parameterisation of C that traverses C once in a positive direction with respect to $\mathbf{n}$.

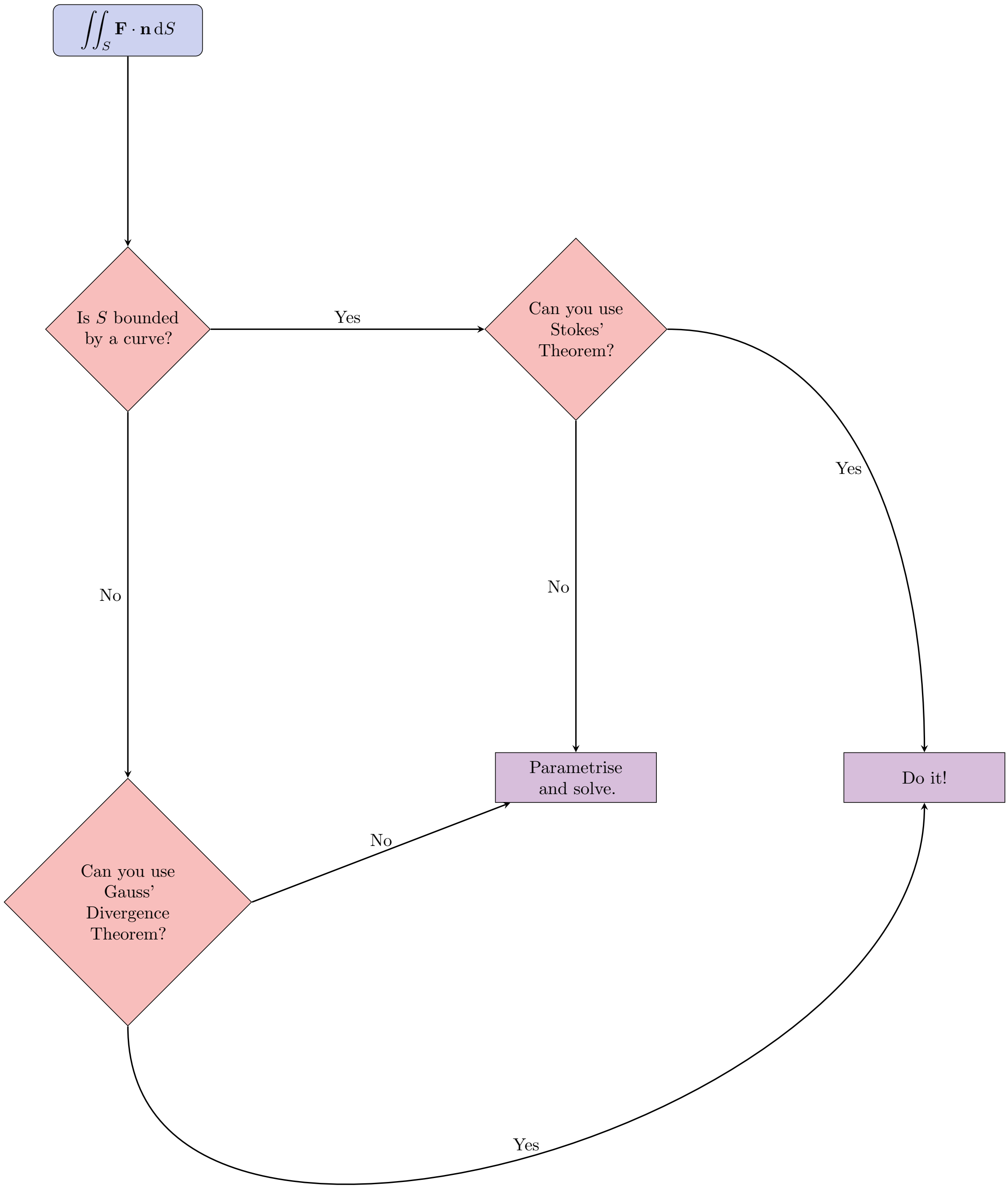
A rigorous proof of Stokes' Theorem lies beyond the scope of the present course. Having said this let us indicate a few reasons as to why one could expect such a statement to hold.

- We have just seen that Stokes' Theorem holds if the curve C and the surface S both lie on the x-y-plane – in fact this is just Green's Theorem. A similar reasoning takes care of the cases when C and S both lie in the y-z-plane, or both lie in the x-z-plane.

- As a matter of fact, Stokes' Theorem holds if the curve C and the surface S both lie in any given plane (not necessarily a co-ordinate plane). A proof of this statement uses the fact that work done (line integrals) and the circulation density all make sense independently of the specific coordinate system which we choose to work in.

- Let S be any given surface in three-dimensional space, and denote its boundary curve by C. Loosely speaking, the surface S can be decomposed into tiny, almost flat pieces. The sum of the work done by a given vector field $\mathbf{F}$ around each little piece equals the total work done along C. Similarly, the sum of the circulation density of $\mathbf{F}$ through each little piece equals the total circulation density through S. Therefore Stokes' Theorem for flat pieces loosely implies the general form of Stokes' Theorem.


$\int_C \mathbf{F} \cdot d\mathbf{r}$
Is **F** conservative?
Yes
Use Fundamental Theorem of Calculus
No
Is C closed?
No
Parametrise and solve.
Yes
Is C a subset of $\mathbb{R}^2$?
Yes
Can you use Green's Theorem?
No
Yes
No
Can you use Stokes' Theorem?
Yes
Do it!
No


$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$
Is S bounded by a curve?
Yes
Can you use Stokes' Theorem?
Yes
No
No
Parametrise and solve.
Do it!
No
Can you use Gauss' Divergence Theorem?
Yes

Chapter E

Formative assessments

E.1 Formative assessment – Week 15

(1) Use Green's Theorem to compute the areas of regions bounded by a curve C determined the parameterisation $\mathbf{r}$ when

(a) $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle 2\cos(t), 3\sin(t)\rangle$ for $t \in [0, 2\pi]$,

(b) $\mathbf{r}\colon [0, \pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle 2\cos(t)\sin(t), 3\sin(t)\rangle$ for $t \in [0, \pi]$

(c) $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle \cos(t) - 2\sin(t), \cos(t) + \sin(t)\rangle$ for $t \in [0, 2\pi]$,

(d) $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle -\cos(2t) + 2\cos(t), -\sin(2t) + 2\sin(t)\rangle$ for $t \in [0, 2\pi]$.

(2) For each of the surfaces S list below, find a parameterisation $\mathbf{r}$ of two variables α and β, and compute the vector valued functions

$$\mathbf{r}_\alpha, \quad \mathbf{r}_\beta \quad \text{and} \quad \mathbf{r}_\alpha \times \mathbf{r}_\beta.$$

Further, find an equation for the tangent plane at the given point P.

(a) $S = \{(x, y, z) \in \mathbb{R}^3\colon x^2 - y^2 + z^2 = 0 \text{ and } y \in [0, 1]\}$ and $P = (1/2, 1/2, 0)$

(b) $S = \{(x, y, z) \in \mathbb{R}^3\colon x^2 + y^2 - 2x + z^2 = 0\}$ and $P = (1, 1, 0)$

(c) $S = \{(x, y, z) \in \mathbb{R}^3\colon x^2 + y^2 - z^2 = 2y + 2z \text{ and } -1 \leq z \leq 0\}$ and $P = (1/2, 1, -1/2)$

(d) $S = \{(x, y, z) \in \mathbb{R}^3\colon (4 - \sqrt{x^2 + y^2})^2 + z^2 = 1\}$ and $P = (4, 0, 1)$

(3) Compute the area of each of the surfaces described in Question 2.

(4) For each of the surfaces described in Question 2 compute the value of the surface integral,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S,$$

where $\mathbf{r}$ is the parameterisation you have given in your answer to Question 2, $\mathbf{n} = (\mathbf{r}_\alpha \times \mathbf{r}_\beta)/|\mathbf{r}_\alpha \times \mathbf{r}_\beta|$, and where $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$, for $(x, y, z) \in \mathbb{R}^3$.

(5) Consider the surface M determined by $\mathbf{r}\colon [0, 2\pi] \times [-1, 1] \to \mathbb{R}^3$ defined by

$$\mathbf{r}(\alpha, \beta) = \cos(2\alpha)(2 - \beta\cos(\alpha))\mathbf{i} + \beta\sin(\alpha)\mathbf{j} + \sin(2\alpha)(\beta\cos(\alpha) + 2)\mathbf{k},$$

for $\alpha \in [0, 2\pi]$ and $\beta \in [-1, 1]$. Determine if the normal vector field $\mathbf{n}$ to M determined by $\mathbf{r}$ is continuous as a vector valued function of two variables α and β. If you are feeling adventurous try sketching M.

(6) (a) Let C be the unit circle in the x-y plane centred at the origin, oriented counter clockwise, let $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x,y) = x^2\mathbf{i} + xy\mathbf{j}$ and set

$$I = \oint_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s,$$

where $\mathbf{n}$ is unit normal vector to C pointing away from the origin. Describe geometrically and/or sketch the vector field $\mathbf{F}$. Which portions of C contribute positively to the line integral I and which portions of C contribute negatively to the line integral I?

(b) Evaluating the line integral I given in Part (a), without using Green's Theorem, and justify your answer using Part (a).

(c) Evaluating the line integral I given in Part (a) using Green's Theorem.

(7) (a) Let S denote the part of the upper hemisphere determined by the equations $x^2 + y^2 + z^2 = 1$ and $z \geq 0$, and that lies within the quadric surface determined by the identity $z^2 = x^2 + y^2$. Evaluate the following surface integral.

$$\iint_S y^2\,\mathrm{d}S.$$

(b) Let S denote the part of the half-cylinder determined by the equation $x^2 + z^2 = 1$ with $z \geq 0$, that lies between the planes given by $y = 0$ and $y = 2$. Evaluate the following surface integral.

$$\iint_S x + y + z\,\mathrm{d}S.$$

(8) Evaluate the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

for the given vector fields $\mathbf{F}$ and oriented surfaces S.

(a) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ for $(x,y,z) \in \mathbb{R}^3$, let S be the part of the paraboloid determined by $z = 4 - x^2 - y^2$, that lies above the square $[0,1]^2$, and equip S with an upward orientation, namely so that the unit normal vector $\mathbf{n}$ to S satisfies $\mathbf{n}\cdot\mathbf{k} \geq 0$.

(b) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\mathbf{F}(x,y,z) = -x\mathbf{i} - y\mathbf{j} + z^3\mathbf{k}$, let S be the part of the cone determined by $z^2 = x^2 + y^2$, that lies between the planes $z = 1$ and $z = 3$, and equip S with a downward orientation, namely so that the unit normal vector $\mathbf{n}$ to S satisfies $\mathbf{n}\cdot\mathbf{k} \leq 0$.

(c) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$, let S be the upper hemisphere determined by $x^2 + y^2 + z^2 = 4$, and equip S with a downward orientation, namely so that the unit normal vector $\mathbf{n}$ to S satisfies $\mathbf{n}\cdot\mathbf{k} \leq 0$.

(9) Let a and $b \in \mathbb{R}$ be such that $a < b$, and let $f\colon \mathbb{R} \to \mathbb{R}$ be a positive smooth function. Denote by S the surface in $\mathbb{R}^3$ obtained by rotating the curve $C = \{(x,y,z) \in \mathbb{R}^3\colon z = 0,\ y = f(x) \text{ and } x \in [a,b]\}$ about the x-axis through 2π radians. By finding a suitable parameterisation of S and evaluating an appropriate surface integral, prove that the surface area of S is given by

$$\int_a^b 2\pi f(s)\sqrt{1 + (f'(s))^2}\,\mathrm{d}s.$$

E.2 Formative assessment – Week 16

(1) In each of the cases below, the given curve C is oriented counter clockwise when viewed from high above the x-y plane.

(a) Let C be the curve which arises from the intersection of the plane $\{(x, y, z) \in \mathbb{R}^3 \colon y = z\}$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 2y\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane. Use Stokes' Theorem to find the value of the following integral.

$$\int_C (x + 2y)\,\mathrm{d}x + (2z + 2x)\,\mathrm{d}y + (z + y)\,\mathrm{d}z$$

(b) Let C be the curve which arises from the intersection of the plane $\{(x, y, z) \in \mathbb{R}^3 \colon x - z = 1\}$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 1\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane.. Use Stokes' Theorem to find the value of the following integral.

$$\int_C (y - z)\,\mathrm{d}x + (z - x)\,\mathrm{d}y + (x - y)\,\mathrm{d}z$$

(c) Let C be the curve which arises from the intersection of the plane $\{(x, y, z) \in \mathbb{R}^3 \colon y = z\}$ and the sphere $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane.. Use Stokes' Theorem to find the value of the following integral.

$$\int_C (x + 2y)\,\mathrm{d}x + (2z + 2x)\,\mathrm{d}y + (z + y)\,\mathrm{d}z$$

(d) Let C be the curve which arises from the intersection of the half space $\{(x, y, z) \in \mathbb{R}^3 \colon z \geq 0\}$, the sphere $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = x/2\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane. Use Stokes' Theorem to find the value of the following integral.

$$\int_C x\,\mathrm{d}x + (x - 2yz)\,\mathrm{d}y + (x^2 + z)\,\mathrm{d}z$$

Before applying Stoke's Theorem, be sure to check that the conditions of the theorem are satisfied.

(2) Use Gauss' Divergence Theorem to compute the value of the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S$$

where $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ and S are as follows and $\mathbf{n}$ is the outward pointing unit normal vector field to S.

(a) $S = \partial\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 1\}$ and $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$, for $(x, y, z) \in \mathbb{R}^3$

(b) $S = \partial\{(x, y, z) \in \mathbb{R}^3 \colon 0 \leq x, y, z \leq 1\}$ and $\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2\rangle$, for $(x, y, z) \in \mathbb{R}^3$

(c) $S = \{(x, y, z) \in R^3 \colon x^2 + y^2 + z^2 = 1\}$ and $\mathbf{F}(x, y, z) = \langle xz - z^2, -yz, z + x + y\rangle$, for $(x, y, z) \in \mathbb{R}^3$

You may assume that the conditions of Gauss' Divergence Theorem are met, but remember if such a question appears on your final exam, unless otherwise stated, you must check that the conditions are met before applying the result.

(3) Prove or disprove the following statement. Let $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field and let $\mathbb{S}_\epsilon \subset \mathbb{R}^3$ denote a circle of radius $\epsilon > 0$ centred at a point $P_0 \in \mathbb{R}^3$, enclosing a disk $\mathbb{D}_\epsilon$ with unit normal vector field $\mathbf{n}$. If $\mathbb{S}_\epsilon$ is positively orientated with respect to $\mathbf{n}$, see FIGURE F.14, then

$$\lim_{\epsilon \to 0} \frac{1}{\pi\epsilon^2} \oint_{C_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \mathbf{n} \cdot \operatorname{curl}(\mathbf{F})(P_0),$$

where $\mathbf{r}$ is a parameterisation of $\mathbb{S}_\epsilon$ that is smooth and traverses $\mathbb{S}_\epsilon$ once positively with respect to $\mathbf{n}$.

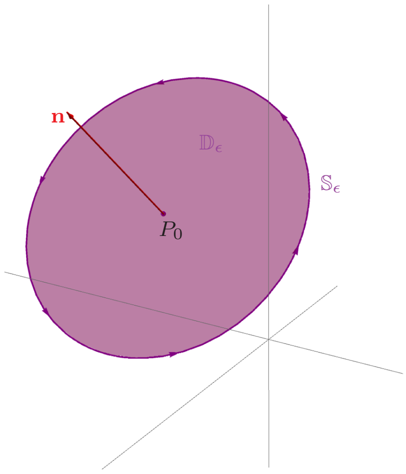


FIGURE E.1: Graph of a disc $\mathbb{D}_\epsilon \subseteq \mathbb{R}^3$ centred at a point $P_0 \in \mathbb{R}^3$, together with the boundary curve $\mathbb{S}_\epsilon$ positively orientated with respect to a given unit normal vector field $\mathbf{n}$.

(4) Let $P_0 \in \mathbb{R}^3$ be a fixed point, and for each $\epsilon > 0$ let $\mathbb{S}_\epsilon \subseteq \mathbb{R}^3$ denote the sphere centred at P_0 with radius ϵ. Prove that

$$\lim_{\epsilon \to 0} \frac{3}{4\pi\epsilon^3} \iint_{\mathbb{S}_\epsilon} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \mathrm{div}(\mathbf{F})(P_0),$$

where $\mathbf{n}$ is the outward pointing unit normal vector field to $\mathbb{S}_\epsilon$ and $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field.

(5) Let $\mathbf{F}$ denote a vector field whose component functions have continuous first-order partial derivatives throughout a non-empty open subset of $\mathbb{R}^3$ containing a connected non-empty region D whose boundary is a smooth connected orientable closed surface S. Prove, if $|\mathbf{F}(x, y, z)| \leq 1$, for all $(x, y, z) \in D$, then

$$\iiint_D \nabla \cdot \mathbf{F} \, \mathrm{d}V$$

is bounded above by the surface area of S.

(6) Let $\phi \colon \mathbb{R}^3 \to \mathbb{R}$ denote a smooth scalar field and let $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ denote a smooth vector field. Proving any necessary vector identities, show that

$$\iiint_D \phi \, \mathrm{div}(\mathbf{F}) \, \mathrm{d}V + \iiint_D \nabla(\phi) \cdot \mathbf{F} \, \mathrm{d}V = \iint_S \phi \, \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S.$$

Here, D is a connected non-empty region in $\mathbb{R}^3$ whose boundary S is a connected smooth closed orientable surface, and $\mathbf{n}$ is the outward unit normal vector field of S.

Chapter F

Formative assessment solutions

F.1 Formative assessment – Week 15 – Solutions

(1) Use Green's Theorem to compute the areas of regions bounded by a curve C determined the parameterisation $\mathbf{r}$ when

(a) $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle 2\cos(t), 3\sin(t)\rangle$ for $t \in [0, 2\pi]$,
(b) $\mathbf{r}\colon [0, \pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle 2\cos(t)\sin(t), 3\sin(t)\rangle$ for $t \in [0, \pi]$
(c) $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle \cos(t) - 2\sin(t), \cos(t) + \sin(t)\rangle$ for $t \in [0, 2\pi]$,
(d) $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ is defined by $\mathbf{r}(t) = \langle -\cos(2t) + 2\cos(t), -\sin(2t) + 2\sin(t)\rangle$ for $t \in [0, 2\pi]$.

Solution.

(a) Let us plot the given curve C determined by the path $\mathbf{r}$ and shade the region of interest, see Figure F.1. The parameterisation $\mathbf{r}$ is smooth since its component functions are linear combinations of trigonometric functions. The curve C is traversed once in a counter clockwise direction by $\mathbf{r}$ and is simple and closed, see Figure F.1. In fact, the curve C is an ellipse and so letting D denote the planar region enclosed by C, we have that it is regular. Defining $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = \langle -y/2, x/2\rangle$, our corollary to Green's Theorem implies

$$\begin{aligned}\text{Area of } D &= \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^{2\pi} \langle -3\sin(t), 2\cos(t)\rangle \cdot \langle -2\sin(t), 3\cos(t)\rangle\,\mathrm{d}t = 3\int_0^{2\pi} 1\,\mathrm{d}t = 6\pi.\end{aligned}$$

(b) Let us plot the given curve C determined by the path $\mathbf{r}$ and shade the region of interest, see Figure F.2. The parameterisation $\mathbf{r}$ is smooth since its component functions are linear combinations of products of trigonometric functions. The curve C is traversed once in a counter clockwise direction by $\mathbf{r}$ and is simple and closed, see Figure F.2. Letting D denote the planar region enclosed by C, from this and Figure F.2, we see that it is regular. Defining $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = \langle -y/2, x/2\rangle$, our corollary to Green's Theorem implies

$$\begin{aligned}\text{Area of } D &= \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_0^{\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^{\pi} \langle -3\sin(t), 2\cos(t)\sin(t)\rangle \cdot \langle 2\cos^2(t) - 2\sin^2(t), 3\cos(t)\rangle\,\mathrm{d}t \\ &= 3\int_0^{\pi} -\sin(t)\cos^2(t) + (1 - \cos^2(t))\sin(t) + \cos^2(t)\sin(t)\,\mathrm{d}t \\ &= 3\int_0^{\pi} -2\sin(t)\cos^2(t) + \sin(t) + \cos^2(t)\sin(t)\,\mathrm{d}t = 2\cos^3(t) - 3\cos(t) - \cos^3(t)\Big|_0^{\pi} = 4.\end{aligned}$$

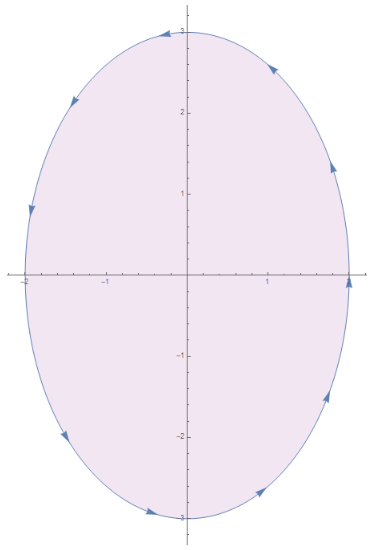


FIGURE F.1: Graph of the curve C determined by the path $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ defined by $\mathbf{r}(t) = \langle 2\cos(t), 3\sin(t)\rangle$ for $t \in [0, 2\pi]$.

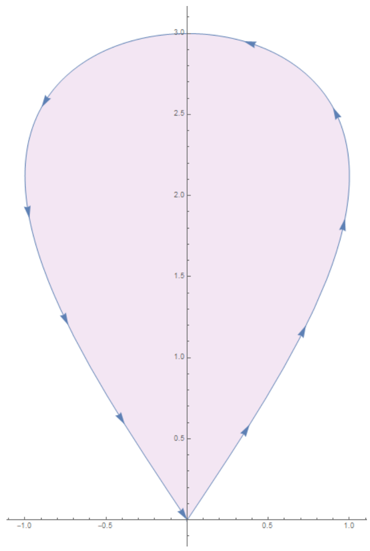


FIGURE F.2: Graph of curve C determined by the path $\mathbf{r}\colon [0, \pi] \to \mathbb{R}^2$ defined by $\mathbf{r}(t) = \langle 2\cos(t)\sin(t), 3\sin(t)\rangle$ for $t \in [0, \pi]$.

(c) Let us plot the given curve C determined by the path $\mathbf{r}$ and shade the region of interest, see FIGURE F.3. The parameterisation $\mathbf{r}$ is smooth since its component functions are linear combinations of trigonometric functions. The curve C is traversed once in a counter clockwise direction by $\mathbf{r}$ and is simple and closed, see FIGURE F.3. Letting D denote the planar region enclosed by C, from this and FIGURE F.3, we see that it is regular. Defining $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = \langle -y/2, x/2\rangle$, our corollary to Green's Theorem implies

$$\begin{aligned}
\text{Area of } D &= \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \langle -\cos(t) - \sin(t), \cos(t) - 2\sin(t)\rangle \cdot \langle -\sin(t) - 2\cos(t), -\sin(t) + \cos(t)\rangle\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} (-\cos(t) - \sin(t))(-\sin(t) - 2\cos(t)) + (\cos(t) - 2\sin(t))(-\sin(t) + \cos(t))\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} 3\,\mathrm{d}t = 3\pi
\end{aligned}$$

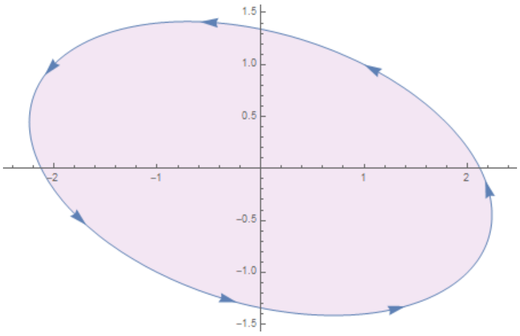


Figure F.3: Graph of the curve C determined by the path $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ defined by $\mathbf{r}(t) = \langle \cos(t) - 2\sin(t), \cos(t) + \sin(t)\rangle$ for $t \in [0, 2\pi]$.

(d) Let us plot the given curve C determined by the path $\mathbf{r}$ and shade the region of interest, see Figure F.4. The parameterisation $\mathbf{r}$ is smooth since its component functions are linear combinations and compositions of trigonometric and linear functions. The curve C is traversed once in a counter clockwise direction by $\mathbf{r}$ and is simple and closed, see Figure F.4. Letting D denote the planar region enclosed by C, from this and Figure F.3, we see that it is regular. Defining $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = \langle -y/2, x/2\rangle$, our corollary to Green's Theorem implies

$$
\begin{aligned}
\text{Area of } D &= \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \langle \sin(2t) - 2\sin(t), -\cos(2t) + 2\cos(t)\rangle \cdot \langle 2\sin(2t) - 2\sin(t), -2\cos(2t) + 2\cos(t)\rangle\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} (\sin(2t) - 2\sin(t))(2\sin(2t) - 2\sin(t)) + (-\cos(2t) + 2\cos(t))(-2\cos(2t) + 2\cos(t))\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} 2\sin^2(2t) - 6\sin(2t)\sin(t) + 4\sin^2(t) + 2\cos^2(2t) - 6\cos(t)\cos(2t) + 4\cos^2(t)\,\mathrm{d}t \\
&= 3\int_0^{2\pi} 1 - \cos(t)\,\mathrm{d}t = 3(t - \sin(t))\Big|_0^{2\pi} = 6\pi.
\end{aligned}
$$

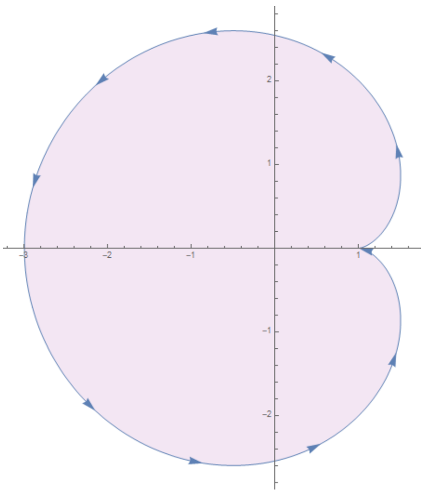


Figure F.4: Graph of the curve C determined by the path $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^2$ defined by $\mathbf{r}(t) = \langle -\cos(2t) + 2\cos(t), -\sin(2t) + 2\sin(t)\rangle$ for $t \in [0, 2\pi]$.

(2) For each of the surfaces S list below, find a parameterisation $\mathbf{r}$ of two variables α and β, and compute the vector valued functions

$$\mathbf{r}_\alpha, \quad \mathbf{r}_\beta \quad \text{and} \quad \mathbf{r}_\alpha \times \mathbf{r}_\beta.$$

Further, find an equation for the tangent plane at the given point P.

(a) $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 - y^2 + z^2 = 0 \text{ and } y \in [0, 1]\}$ and $P = (1/2, 1/2, 0)$
(b) $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 - 2x + z^2 = 0\}$ and $P = (1, 1, 0)$
(c) $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 - z^2 = 2y + 2z \text{ and } -1 \leq z \leq 0\}$ and $P = (1/2, 1, -1/2)$
(d) $S = \{(x, y, z) \in \mathbb{R}^3 \colon (4 - \sqrt{x^2 + y^2})^2 + z^2 = 1\}$ and $P = (4, 0, 1)$

Solution.

(a) If $(x, y, z) \in S$, then $x^2 + z^2 = y^2$. Thus, setting $y = \alpha$, we have that $x^2 + z^2 = \alpha^2$. The set of points satisfying this latter equation lie on a circle of radius α, and so, we can set $x = \alpha\cos(\beta)$ and $z = \alpha\sin(\beta)$. Hence, a smooth parameterisation of the given surface is $\mathbf{r}\colon [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$ defined by $\mathbf{r}(\alpha, \beta) = \langle \alpha\cos(\beta), \alpha, \alpha\sin(\beta)\rangle$, for $\alpha \in [0, 1]$ and $\beta \in [0, 2\pi]$. With this parameterisation we have that, $\mathbf{r}_\alpha(\alpha, \beta) = \langle \cos(\beta), 1, \sin(\beta)\rangle$ and $\mathbf{r}_\beta(\alpha, \beta) = \langle -\alpha\sin(\beta), 0, \alpha\cos(\beta)\rangle$, whence,

$$(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\beta) & 1 & \sin(\beta) \\ -\alpha\sin(\beta) & 0 & \alpha\cos(\beta) \end{pmatrix} = \alpha\cos(\beta)\mathbf{i} - \alpha\mathbf{j} + \alpha\sin(\beta)\mathbf{k}.$$

Observe that $P \in S$ since $1/2^2 - 1/2^2 + 0^2 = 0$. If α and β are such that $\mathbf{r}(\alpha, \beta) = \langle 1/2, 1/2, 0\rangle$, then one choice for α and β is $\alpha = 1/2$ and $\beta = 0$. Since the vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(1/2, 0) = \langle 1/2, -1/2, 0\rangle$ is normal to the surface S at the point $P = (1/2, 1/2, 0)$ we have that the tangent plane is determined by the equation

$$((1 - x)\mathbf{i} + (1 - y)\mathbf{i} + (0 - z)\mathbf{k}) \cdot ((\mathbf{r}_\alpha \times \mathbf{r}_\beta)(1, 0)) = 0,$$

or equivalently $((1 - x)\mathbf{i} + (1 - y)\mathbf{i} + (0 - z)\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j}) = 0$. Therefore, the equation of the tangent plane is $y - x = 0$, see FIGURE F.5.

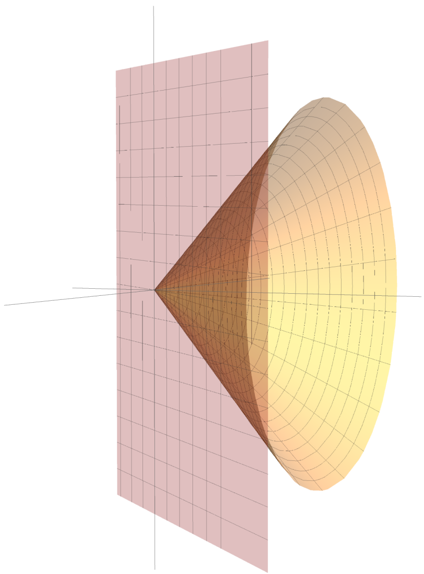


FIGURE F.5: Graph of the surface $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 - y^2 + z^2 = 0 \text{ and } y \in [0, 1]\}$ and the tangent plane at the point $P = (1, 1, 0)$.

(b) If $(x, y, z) \in S$, then, by completing the square, $(x-1)^2+y^2+z^2 = 1$. This is the sphere with radius 1 and centre at $(1, 0, 0)$. Thus, a smooth parameterisation of the given surface is $\mathbf{r}\colon [-\pi/2, \pi/2] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

$$\mathbf{r}(\alpha, \beta) = \langle \cos(\alpha)\cos(\beta) + 1, \cos(\alpha)\sin(\beta), \sin(\alpha)\rangle,$$

for $\alpha \in [-\pi/2, \pi/2]$ and $\beta \in [0, 2\pi]$. With this parameterisation we have that,

$$\mathbf{r}_\alpha(\alpha, \beta) = \langle -\sin(\alpha)\cos(\beta), -\sin(\alpha)\sin(\beta), \cos(\alpha)\rangle$$

and

$$\mathbf{r}_\beta(\alpha, \beta) = \langle -\cos(\alpha)\sin(\beta), \cos(\alpha)\cos(\beta), 0\rangle,$$

whence,

$$\begin{aligned}(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta) & \cos(\alpha) \\ -\cos(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta) & 0 \end{pmatrix} \\ &= -\cos^2(\alpha)\cos(\beta)\mathbf{i} - \cos^2(\alpha)\sin(\beta)\mathbf{j} - \sin(\alpha)\cos(\alpha)\mathbf{k}.\end{aligned}$$

Observe that $P \in S$ since $1^2 + 1^2 - 2(1) + 0^2 = 0$. If α and β are such that $\mathbf{r}(\alpha, \beta) = \langle 1, 1, 0\rangle$, then one choice for the values of α and β is $\alpha = 0$ and $\beta = \pi/2$. Since $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(0, \pi/2) = \langle 0, -1, 0\rangle$ is normal to the surface S at the point $P = (1, 1, 0)$ we have that the tangent plane is determined by the equation

$$((1 - x)\mathbf{i} + (1 - y)\mathbf{j} + (0 - z)\mathbf{k}) \cdot ((\mathbf{r}_\alpha \times \mathbf{r}_\beta)(0, \pi/2)) = 0,$$

or equivalently

$$((1 - x)\mathbf{i} + (1 - y)\mathbf{j} + (0 - z)\mathbf{k}) \cdot (-\mathbf{j}) = 0.$$

Therefore, the equation of the tangent plane is $y = 1$, see Figure F.6.

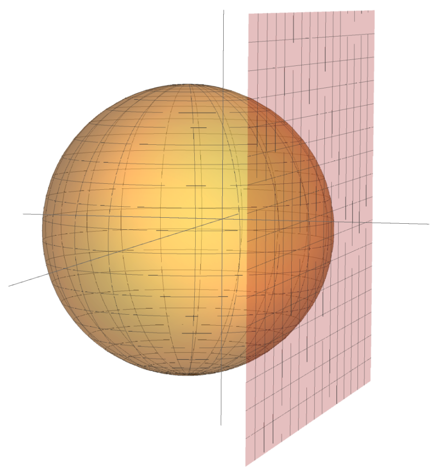


Figure F.6: Graph of the surface $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 - y^2 + z^2 = 0 \text{ and } y \in [0, 1]\}$ and the tangent plane at the point $P = (1, 1, 0)$.

(c) If $(x, y, z) \in S$, then, by completing the square, $x^2 + (y - 1)^2 = (z + 1)^2$ where $-1 \le z \le 0$. This is a cone in the negative half space with apex at $(0, 1, -1)$ and at height $\beta - 1$, for $\beta \in [0, 1]$, the cross section, parallel to the x-y plane, of the cone is a circle of radius β. Thus, a smooth parameterisation of the given surface is $\mathbf{r}\colon [0, 2\pi] \times [0, 1] \to \mathbb{R}^3$ defined by

$$\mathbf{r}(\alpha, \beta) = \langle \beta\cos(\alpha), \beta\sin(\alpha) + 1, \beta - 1\rangle,$$

for $\alpha \in [0, 2\pi]$ and $\beta \in [0, 1]$. With this parameterisation we have that,

$$\mathbf{r}_\alpha(\alpha, \beta) = \langle -\beta\sin(\alpha), \beta\cos(\alpha), 0\rangle \quad \text{and} \quad \mathbf{r}_\beta(\alpha, \beta) = \langle \cos(\alpha), \sin(\alpha), 1\rangle,$$

whence,

$$(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\beta \sin(\alpha) & \beta \cos(\alpha) & 0 \\ \cos(\alpha) & \sin(\alpha) & 1 \end{pmatrix} = \beta \cos(\alpha)\mathbf{i} + \beta \sin(\alpha)\mathbf{j} - \beta\mathbf{k}.$$

Observe that $P \in S$ since $1/2^2 + 1^2 - (-1/2)^2 = 2(1) - 2(-1/2)$. If α and β are such that $\mathbf{r}(\alpha, \beta) = \langle 1/2, 1, -1/2\rangle$, then one choice for the values of α and β is $\alpha = 0$ and $\beta = 1/2$. Since the vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(0, 1) = \langle 1/2, 0, -1/2\rangle$ is normal to the surface S at the point $P = (1, 1, 0)$ we have that the tangent plane is determined by the equation

$$((1/2 - x)\mathbf{i} + (1 - y)\mathbf{i} + (-1/2 - z)\mathbf{k}) \cdot ((\mathbf{r}_\alpha \times \mathbf{r}_\beta)(0, 1)) = 0,$$

or equivalently

$$((1/2 - x)\mathbf{i} + (1 - y)\mathbf{i} + (-1/2 - z)\mathbf{k}) \cdot (\mathbf{i} - \mathbf{k}) = 0.$$

Therefore, the equation of the tangent plane is $z = x - 1$, see FIGURE F.7.

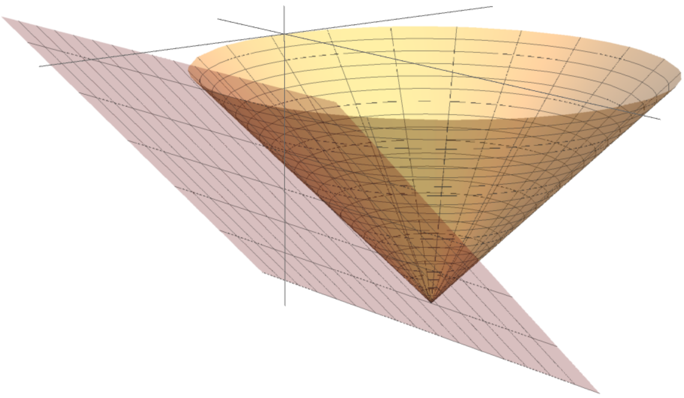


FIGURE F.7: Graph of the surface $S = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2-z^2 = 2y+2x \text{ and } -1 \leq z \leq 1\}$ and the tangent plane at the point $P = (1, 1, 0)$.

(d) If $(x, y, z) \in S$, setting $p = 4-\sqrt{x^2 + y^2}$, we have that $p^2+z^2 = 1$. Using the standard parameterisation for a circle, we may choose $p = \cos(\alpha)$ and $z = \sin(\alpha)$, for $\alpha \in [0, 2\pi]$. In which case $4 - \sqrt{x^2 + y^2} = \cos(\alpha)$, or equivalently $x^2 + y^2 = (4 - \cos(\alpha))^2$. Once again, using the standard parameterisation for a circle, we may choose $x = (4 - \cos(\alpha)) \cos(\beta)$ and $y = (4 - \cos(\alpha)) \sin(\beta)$, for $\beta \in [0, 2\pi]$. Thus, a smooth parameterisation of the given surface is $\mathbf{r}\colon [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

$$\mathbf{r}(\alpha, \beta) = \langle (4 + \cos(\alpha)) \cos(\beta), (4 + \cos(\alpha)) \sin(\beta), \sin(\alpha)\rangle,$$

for $\alpha \in [0, 2\pi]$ and $\beta \in [0, 2\pi]$. With this parameterisation we have that,

$$\mathbf{r}_\alpha(\alpha, \beta) = \langle -\cos(\beta) \sin(\alpha), -\sin(\alpha) \sin(\beta), \cos(\alpha)\rangle$$

and

$$\mathbf{r}_\beta(\alpha, \beta) = \langle -(\cos(\alpha) + 4) \sin(\beta), (\cos(\alpha) + 4) \cos(\beta), 0\rangle,$$

whence,

$$\begin{aligned}(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos(\beta) \sin(\alpha) & -\sin(\alpha) \sin(\beta) & \cos(\alpha) \\ -(\cos(\alpha) + 4) \sin(\beta) & (\cos(\alpha) + 4) \cos(\beta) & 0 \end{pmatrix} \\ &= -(\cos(\alpha) + 4)(\cos(\alpha) \cos(\beta)\mathbf{i} + \cos(\alpha) \sin(\beta)\mathbf{j} + \sin(\alpha)\mathbf{k}).\end{aligned}$$

Observe that $P \in S$, since $(4 - \sqrt{4^2 - 0^2})^2 + 1^2 = 1$. If α and β are such that $\mathbf{r}(\alpha, \beta) = \langle 4, 0, 1\rangle$, then one choice for the values of α and β is $\alpha = \pi/2$ and $\beta = 0$. Since the vector $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\pi/2, 0) = \langle 0, 0, -4\rangle$

is normal to the surface S at the point $P = (4, 0, 1)$ we have that the tangent plane is determined by the equation

$$((4-x)\mathbf{i} + (0-y)\mathbf{i} + (1-z)\mathbf{k}) \cdot ((\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\pi/2, 0)) = 0,$$

or equivalently $((4-x)\mathbf{i} + (0-y)\mathbf{i} + (1-z)\mathbf{k}) \cdot (-4\mathbf{k}) = 0$. Therefore, the equation of the tangent plane is $z = 1$, see FIGURE F.8.

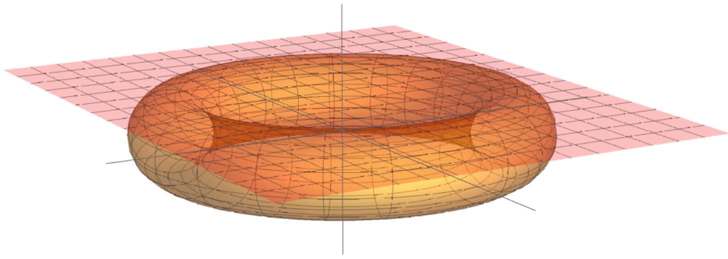


FIGURE F.8: Graph of the surface $S = \{(x, y, z) \in \mathbb{R}^3 : (4 - \sqrt{x^2 + y^2})^2 + z^2 = 1\}$ and the tangent plane at the point $P = (4, 0, 1)$.

(3) Compute the area of each of the surfaces described in Question 2.

Solution.

In the previous question, we have seen that each of the given surfaces exhibit a smooth parameterisation, each of which has a connected, closed and bounded domain with non-empty interior. Further, from their definitions, one can see that they are all injective on the interior of their domains. Thus we may apply Definition/Proposition 10.3.1 to find the required surface areas.

(a) Since $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) = \alpha \cos(\beta)\mathbf{i} - \alpha\mathbf{j} + \alpha \sin(\beta)\mathbf{k}$, we have that

$$|\mathbf{r}_\alpha \times \mathbf{r}_\beta| = \alpha\sqrt{2}.$$

Therefore, the area of the given surface is equal to

$$\iint_D |\mathbf{r}_\alpha \times \mathbf{r}_\beta| \, \mathrm{d}\alpha \mathrm{d}\beta = \int_0^{2\pi} \int_0^1 \alpha\sqrt{2} \, \mathrm{d}\alpha \mathrm{d}\beta = \int_0^{2\pi} \frac{\alpha^2}{\sqrt{2}}\bigg|_0^1 \mathrm{d}\beta = \int_0^{2\pi} \frac{1}{\sqrt{2}} \, \mathrm{d}\beta = \pi\sqrt{2}.$$

As our integrand is bounded and continuous on the domain D, and as D is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and is independent of the order in which the repeated integral is computed.

(b) Since $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) = -\cos^2(\alpha)\cos(\beta)\mathbf{i} - \cos^2(\alpha)\sin(\beta)\mathbf{j} - \cos(\alpha)\sin(\alpha)\mathbf{k}$, it follows that $|\mathbf{r}_\alpha \times \mathbf{r}_\beta| = |\cos(\alpha)|$. Therefore, the area of the given surface is equal to

$$\begin{aligned}\iint_D |\mathbf{r}_\alpha \times \mathbf{r}_\beta| \, \mathrm{d}\alpha \mathrm{d}\beta &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} |\cos(\alpha)| \, \mathrm{d}\alpha \mathrm{d}\beta \\ &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos(\alpha) \, \mathrm{d}\alpha \mathrm{d}\beta = \int_0^{2\pi} \sin(\alpha)\bigg|_{-\pi/2}^{\pi/2} \mathrm{d}\beta = \int_0^{2\pi} 2 \, \mathrm{d}\beta = 4\pi.\end{aligned}$$

As our integrand is bounded and continuous on the domain D, and as D is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and is independent of the order in which the repeated integral is computed.

(c) Since $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) = \beta\cos(\alpha)\mathbf{i} + \beta\sin(\alpha)\mathbf{j} - \beta\mathbf{k}$, we have that $|\mathbf{r}_\alpha \times \mathbf{r}_\beta| = \sqrt{2}\beta$. Therefore, the area of the given surface is equal to

$$\iint_D |\mathbf{r}_\alpha \times \mathbf{r}_\beta| \,\mathrm{d}\alpha\mathrm{d}\beta = \int_0^1 \int_0^{2\pi} \sqrt{2}\beta \,\mathrm{d}\alpha\mathrm{d}\beta = \sqrt{2}\pi.$$

As our integrand is bounded and continuous on the domain D, and as D is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and is independent of the order in which the repeated integral is computed.

(d) Since $(\mathbf{r}_\alpha \times \mathbf{r}_\beta)(\alpha, \beta) = -(\cos(\alpha)+4)(\cos(\alpha)\cos(\beta)\mathbf{i} + \cos(\alpha)\sin(\beta)\mathbf{j} + \sin(\alpha)\mathbf{k})$, we have that $|\mathbf{r}_\alpha \times \mathbf{r}_\beta| = 4 + \cos(\alpha)$. Therefore, the area of the given surface is equal to

$$\iint_D |\mathbf{r}_\alpha \times \mathbf{r}_\beta| \,\mathrm{d}\alpha\mathrm{d}\beta = \int_0^{2\pi} \int_0^{2\pi} 4 + \cos(\alpha) \,\mathrm{d}\alpha\mathrm{d}\beta = \int_0^{2\pi} 4\alpha + \sin(\alpha)\Big|_0^{2\pi} \mathrm{d}\beta = \int_0^{2\pi} 8\pi \,\mathrm{d}\beta = 16\pi^2.$$

As our integrand is bounded and continuous on the domain D, and as D is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a reported integral and is independent of the order in which the repeated integral is computed.

(5) Consider the surface M determined by $\mathbf{r}\colon [0, 2\pi] \times [-1, 1] \to \mathbb{R}^3$ defined by

$$\mathbf{r}(\alpha, \beta) = \cos(2\alpha)(2 - \beta\cos(\alpha))\mathbf{i} + \beta\sin(\alpha)\mathbf{j} + \sin(2\alpha)(\beta\cos(\alpha) + 2)\mathbf{k},$$

for $\alpha \in [0, 2\pi]$ and $\beta \in [-1, 1]$. Determine if the normal vector field $\mathbf{n}$ to M determined by $\mathbf{r}$ is continuous as a vector valued function of two variables α and β. If you are feeling adventurous try sketching M.

Solution.

The given parameterisation satisfies $\mathbf{r}(\alpha + \pi, \beta) = \mathbf{r}(\alpha, -\beta)$ for $\alpha \in [0, \pi]$ and $\beta \in [0, 1]$. Therefore, the surface M is traced out twice as α and β vary over their range. In particular, $\mathbf{r}(0, \pi) = \mathbf{r}(0, 0)$. Noting that

$$\begin{aligned}\mathbf{r}_\alpha(\alpha, \beta) = {} & (\beta\sin(\alpha)\cos(2\alpha) - 2\sin(2\alpha)(2 - \beta\cos(\alpha))\mathbf{i} \\ & + \beta\cos(\alpha)\mathbf{j} + (2\cos(2\alpha)(\beta\cos(\alpha) + 2) - \beta\sin(\alpha)\sin(2\alpha))\mathbf{k}\end{aligned}$$

and

$$\mathbf{r}_\beta(\alpha, \beta) = -\cos(\alpha)\cos(2\alpha)\mathbf{i} + \sin(\alpha)\mathbf{j} + \sin(2\alpha)\cos(\alpha)\mathbf{k},$$

we have that $\mathbf{r}_\alpha(\pi, 0) = 4\mathbf{k} = \mathbf{r}_\alpha(0, 0)$ and $\mathbf{r}_\beta(\pi, 0) = \mathbf{i} = -\mathbf{r}_\beta(0, 0)$. Thus, the normal vector field $\mathbf{n}$ to M determined by $\mathbf{r}$ at $\mathbf{r}(\pi, 0) = \mathbf{r}(0, 0)$ is not well defined and so cannot be continuous. This can also be seen from the sketch the surface M given in FIGURE F.9.

(7) (a) Let S denote the part of the upper hemisphere determined by the equations $x^2 + y^2 + z^2 = 1$ and $z \geq 0$, and that lies within the quadric surface determined by the identity $z^2 = x^2 + y^2$. Evaluate the following surface integral.

$$\iint_S y^2 \,\mathrm{d}S.$$

(b) Let S denote the part of the half-cylinder determined by the equation $x^2 + z^2 = 1$ with $z \geq 0$, that lies between the planes given by $y = 0$ and $y = 2$. Evaluate the following surface integral.

$$\iint_S x + y + z \,\mathrm{d}S.$$

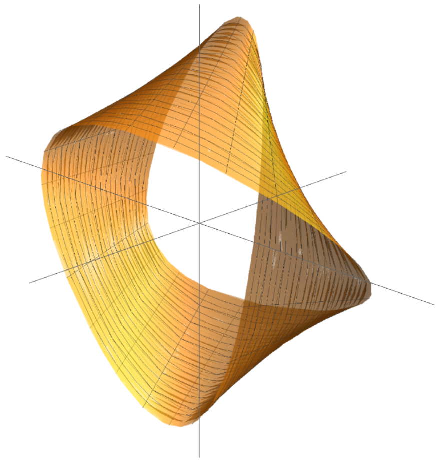


FIGURE F.9: Graph of the surface determined by $\mathbf{r}\colon [0,2\pi]\times[-1,1]\to\mathbb{R}^3$ where $\mathbf{r}(\alpha,\beta) = \cos(2x)(2-y\cos(x))\mathbf{i} + y\sin(x)\mathbf{j} + \sin(2x)(y\cos(x)+2)\mathbf{k}$ for $\alpha\in[0,2\pi]$ and $\beta\in[-1,1]$. This surface is more commonly known as a Möbius strip.

Solution.

(a) Since the outward pointing unit normal vector to the sphere has norm one, and since in terms of spherical co-ordinates, if $(1,\phi,\theta)\in S$, then $\theta\in[0,2\pi]$ and $\phi\in[0,\pi/4]$, if $\Omega=[0,2\pi]\times[0,\pi/4]$, then

$$\begin{aligned}\iint_S y^2\,\mathrm{d}S = \iint_\Omega y^2\cdot 1\,\mathrm{d}A &= \int_0^{2\pi}\int_0^{\pi/4}(\sin(\phi)\sin(\theta))^2\sin(\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \left(\int_0^{\pi/4}\sin^3(\phi)\,\mathrm{d}\phi\right)\left(\int_0^{2\pi}\sin^2(\theta)\,\mathrm{d}\theta\right).\end{aligned}$$

Note, as our integrand is bounded and continuous on the domain Ω, and as Ω is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and independent of the order in which the repeated integral is computed. Further,

$$\int_0^{\pi/4}\sin^3(\phi)\,\mathrm{d}\phi = \int_0^{\pi/4}(1-\cos^2(\phi))\sin(\phi)\,\mathrm{d}\phi = -\cos(\phi)-\frac{\cos^3(\phi)}{3}\bigg|_0^{\pi/4} = \frac{4}{3}-\frac{7}{6\sqrt{2}},$$

and

$$\int_0^{2\pi}\sin^2(\theta)\,\mathrm{d}\theta = \int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,\mathrm{d}\theta = \pi.$$

Thus, in conclusion

$$\iint_S y^2\,\mathrm{d}S = \frac{4\pi}{3}-\frac{7\pi}{6\sqrt{2}}.$$

(b) The central axis of the given cylinder is the y-axis, and its radius is of length one. Therefore, as we have $z\geq 0$ and $0\leq y\leq 2$ a parameterisation is given by $\mathbf{r}\colon[0,\pi]\times[0,2]$ defined by

$$\mathbf{r}(\alpha,\beta) = \langle\cos(\alpha),\beta,\sin(\alpha)\rangle,$$

for $\alpha\in[0,\pi]$ and $\beta\in[0,2]$. This parameterisation is smooth and has a connected, closed and bounded domain with non-empty interior. Further, from its definition, one can see that it is injective on the interior of its domains. Thus we may apply Definition/Proposition 10.3.1 to evaluate the given surface integral. To this end observe that

$$\mathbf{r}_\alpha(\alpha,\beta) = \langle-\sin(\alpha),0,\cos(\alpha)\rangle \quad\text{and}\quad \mathbf{r}_\beta(\alpha,\beta) = \langle 0,1,0\rangle,$$

whence

$$\mathbf{r}_\alpha \times \mathbf{r}_\beta(\alpha,\beta) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(\alpha) & 0 & \cos(\alpha) \\ 0 & 1 & 0 \end{pmatrix} = -\cos(\alpha)\mathbf{i} - \sin(\alpha)\mathbf{k}.$$

Thus, $|\mathbf{r}_\alpha \times \mathbf{r}_\beta| = 1$ and so, setting $\Omega = [0,\pi]\times[0,2]$,

$$\begin{aligned}\iint_S (x+y+z)\,\mathrm{d}S = \iint_\Omega (x+y+z)\,\mathrm{d}A &= \int_0^2\int_0^\pi \cos(\alpha)+\beta+\sin(\alpha)\,\mathrm{d}\alpha\mathrm{d}\beta \\ &= \int_0^2 \pi\beta + 2\mathrm{d}y = \frac{\pi\beta^2}{2} + 2\beta\bigg|_0^2 = 2\pi+4.\end{aligned}$$

Note, as our integrand is bounded and continuous on the domain Ω, and as Ω is closed and bounded, here we have applied Fubini-Tonelli's Theorem, namely the value of the double integral is equal to a repeated integral and independent of the order in which the repeated integral is computed.

(9) Let a and $b \in \mathbb{R}$ be such that $a < b$, and let $f\colon \mathbb{R}\to\mathbb{R}$ be a positive smooth function. Denote by S the surface in $\mathbb{R}^3$ obtained by rotating the curve $C = \{(x,y,z)\in\mathbb{R}^3 \colon z = 0,\ y = f(x) \text{ and } x\in[a,b]\}$ about the x-axis through 2π radians. By finding a suitable parameterisation of S and evaluating an appropriate surface integral, prove that the surface area of S is given by

$$\int_a^b 2\pi f(s)\sqrt{1+(f'(s))^2}\,\mathrm{d}s.$$

Solution.

Let (x,y,z) be a point on the surface S. Consider the cross section at $x = x^*$, for some $x^*\in[a,b]$; this is a circle with radius $r = f(x^*)$. A set of parametric equations for this circle is given by

$$x(\theta) = x^*, \quad y(\theta) = r\cos(\theta) = f(x^*)\cos(\theta), \quad \text{and} \quad z(\theta) = r\sin(\theta) = f(x^*)\sin(\theta),$$

for $\theta\in[-\pi,\pi]$. Thus, a set of parametric equations for the surface S is given by

$$x(s,\theta) = s, \quad y(s,\theta) = f(s)\cos(\theta), \quad \text{and} \quad z(s,\theta) = f(s)\sin(\theta),$$

for $(s,\theta)\in[a,b]\times[-\pi,\pi]$, yielding that $\mathbf{r}\colon[a,b]\times[-\pi,\pi]$ given by $\mathbf{r}(s,\theta) = s\mathbf{i} + f(s)\cos(\theta)\mathbf{j} + f(s)\sin(\theta)\mathbf{k}$ for $(s,\theta)\in[a,b]\times[-\pi,\pi]$, is a parameterisation of S. With this at hand, the surface area of S is given by

$$\iint_\Omega |\mathbf{r}_s\times\mathbf{r}_\theta|\,\mathrm{d}A,$$

where $\Omega = [a,b]\times[-\pi,\pi]$. Since

$$\mathbf{r}_s(s,\theta) = \mathbf{i} + f'(s)\cos(\theta)\mathbf{j} + f'(s)\sin(\theta)\mathbf{k} \quad \text{and} \quad \mathbf{r}_\theta(s,\theta) = -f(s)\sin(\theta)\mathbf{j} + f(s)\cos(\theta)\mathbf{k}$$

we have that

$$(\mathbf{r}_s\times\mathbf{r}_\theta)(s,\theta) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(s)\cos(\theta)\mathbf{j} & f'(s)\sin(\theta) \\ 0 & -f(s)\sin(\theta) & f(s)\cos(\theta) \end{vmatrix} = f(s)f'(s)\mathbf{i} - f(s)\cos(\theta)\mathbf{j} - f(s)\sin(\theta)\mathbf{k}.$$

and so $|(\mathbf{r}_s\times\mathbf{r}_\theta)(s,\theta)|^2 = ((f(s)f'(s))^2 + (f(s))^2\cos^2(\theta) + (f(s))^2\sin^2(\theta)) = (f(s))^2(1+(f'(s))^2)$. Thus, $|\mathbf{r}_s\times\mathbf{r}_\theta|$ is a bounded continuous function, and since $[a,b]\times[-\pi,\pi]$ is a bounded region of $\mathbb{R}^2$, by Fubini-Tonelli's theorem, we have

$$\iint_\Omega |\mathbf{r}_s\times\mathbf{r}_\theta|\,\mathrm{d}A = \int_{\theta=-\pi}^{\theta=\pi}\int_{s=a}^{s=b} |(\mathbf{r}_s\times\mathbf{r}_\theta)(s,\theta)|\,\mathrm{d}s\,\mathrm{d}\theta = \int_a^b 2\pi f(s)\sqrt{1+(f'(s))^2}\,\mathrm{d}s,$$

as required.

F.2 Formative assessment – Week 16 – Solutions

(1) In each of the cases below, the given curve C is oriented counter clockwise when viewed from high above the x-y plane.

(a) Let C be the curve which arises from the intersection of the plane $\{(x, y, z) \in \mathbb{R}^3 \colon y = z\}$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 2y\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane. Use Stokes' Theorem to find the value of the following integral.

$$\int_C (x + 2y)\,\mathrm{d}x + (2z + 2x)\,\mathrm{d}y + (z + y)\,\mathrm{d}z$$

(b) Let C be the curve which arises from the intersection of the plane $\{(x, y, z) \in \mathbb{R}^3 \colon x - z = 1\}$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 1\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane.. Use Stokes' Theorem to find the value of the following integral.

$$\int_C (y - z)\,\mathrm{d}x + (z - x)\,\mathrm{d}y + (x - y)\,\mathrm{d}z$$

(c) Let C be the curve which arises from the intersection of the plane $\{(x, y, z) \in \mathbb{R}^3 \colon y = z\}$ and the sphere $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane.. Use Stokes' Theorem to find the value of the following integral.

$$\int_C (x + 2y)\,\mathrm{d}x + (2z + 2x)\,\mathrm{d}y + (z + y)\,\mathrm{d}z$$

(d) Let C be the curve which arises from the intersection of the half space $\{(x, y, z) \in \mathbb{R}^3 \colon z \geq 0\}$, the sphere $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = x/2\}$, and equipped with a counter clockwise orientation, when viewed from high above the x-y plane. Use Stokes' Theorem to find the value of the following integral.

$$\int_C x\,\mathrm{d}x + (x - 2yz)\,\mathrm{d}y + (x^2 + z)\,\mathrm{d}z$$

Before applying Stoke's Theorem, be sure to check that the conditions of the theorem are satisfied.

Solution.

(a) Let E denote the elliptic region enclosed by the curve C in the plane $P = \{(x, y, z) \in \mathbb{R}^3 \colon y = z\}$, see FIGURE F.10. Observe that E is a smooth orientable surface as it is a subset of a plane and since E is bounded by a smooth curve (specifically an ellipse). Letting $D = \{(x, y) \in \mathbb{R}^2 \colon (x, y, y) \in E\}$ denote the projection of E onto the x-y plane, a smooth parameterisation $\mathbf{s} \colon D \to \mathbb{R}^3$ of E is given by $\mathbf{s}(\alpha, \beta) = \alpha\mathbf{i} + \beta\mathbf{j} + \beta\mathbf{k}$, for $(\alpha, \beta) \in D$. Since $E \subseteq \mathbb{R}^3$ is a non-trivial elliptic region in the plane determined by the equation $y = z$, it follows that D is a non-trivial elliptic region of $\mathbb{R}^2$, and as such is a connected closed and bounded subset of $\mathbb{R}^2$ with non-empty interior. Moreover, by construction, $\mathbf{s}$ is injective, and setting $\Gamma = \partial D$, we have have that Γ is an is a smooth simple closed curve, and $C = \mathbf{s}(\Gamma)$. A unit normal vector $\mathbf{n}$ to to the plane P, and hence E, is given by $\mathbf{n} = (\nabla(z - y))/(|\nabla(z - y)|) = (-\mathbf{j} + \mathbf{k})/(\sqrt{2})$. The positive orientation induced on C by this normal vector field is counter clockwise, when viewed from high above the x-y plane, and coincides with orientations it inherits from $\mathbf{s}$. Letting $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{F} = (x + 2y)\mathbf{i} + (2z + 2x)\mathbf{j} + (z + y)\mathbf{k}$, for $(x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{F}$ is a smooth vector field since its co-ordinate functions are polynomials. Thus, the conditions of Stokes' Theorem are satisfied.

By Stokes' Theorem, since

$$(\mathrm{curl}(\mathbf{F}))(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y & 2z + 2x & z + y \end{vmatrix} = -\mathbf{i},$$

for $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned}\int_C (x + 2y)\,\mathrm{d}x + (2z + 2x)\,\mathrm{d}y + (z + y)\,\mathrm{d}z &= \iint_E \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S \\ &= \iint_D (-\mathbf{i}) \cdot ((-\mathbf{j} + \mathbf{k})/\sqrt{2})\,|\mathbf{s}_\alpha \times \mathbf{s}_\beta|\,\mathrm{d}A = \iint_D 0\,\mathrm{d}A = 0.\end{aligned}$$

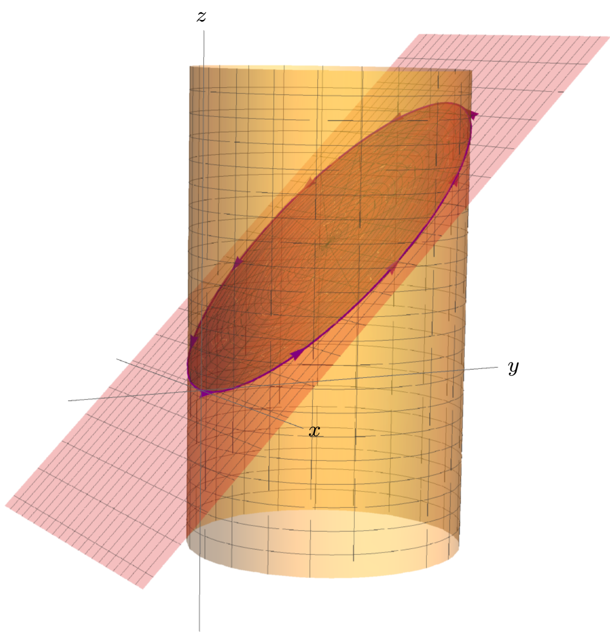


FIGURE F.10: Graph of the elliptic region lying in the plane $P = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$ bounded by the curve C, highlighted in purple, which lies at the intersection of P and the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 2y\}$.

(b) Let E be the elliptic region enclosed by the curve C in the plane $P = \{(x, y, z) \in \mathbb{R}^3 : x - z = 1\}$, see FIGURE F.11. Observe that E is a smooth orientable surface as it is a subset of a plane and since E is bounded by a smooth curve (specifically an ellipse). Letting $D = \{(x, y) \in \mathbb{R}^2 : (x, y, x - 1) \in E\}$ denote the projection of E on to the x-y plane, a smooth parameterisation $\mathbf{s} : D \to \mathbb{R}^3$ of E is given by $\mathbf{s}(\alpha, \beta) = \alpha\mathbf{i} + \beta\mathbf{j} + (\alpha - 1)\mathbf{k}$, for $(\alpha, \beta) \in D$. Since $E \subseteq \mathbb{R}^3$ is a non-trivial elliptic region in the plane determined by $z = x - 1$, it follows that D is a non-trivial elliptical region of $\mathbb{R}^2$, and as such is a connected closed and bounded subset of $\mathbb{R}^2$ with non-empty interior. Moreover, by construction, $\mathbf{s}$ is injective, and setting $\Gamma = \partial D$, we have have that Γ is a smooth simple closed curve, and $C = \mathbf{s}(\Gamma)$. A unit normal vector $\mathbf{n}$ to the plane P, and hence to E, is given by

$$\mathbf{n} = \frac{\nabla(z - x + 1)}{|\nabla(z - x + 1)|} = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}.$$

The positive orientation induced on C by this normal vector field is counter clockwise, when viewed from high above the x-y plane, and coincides with orientations it inherits from $\mathbf{s}$. Letting $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}$, for $(x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{F}$ is a smooth vector field since its co-ordinate functions are polynomials. Thus, the conditions of Stokes' Theorem are satisfied.

By Stokes' Theorem, since

$$(\operatorname{curl}(\mathbf{F}))(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k},$$

for $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned} \int_C (y - z)\,\mathrm{d}x + (z - x)\,\mathrm{d}y + (x - y)\,\mathrm{d}z &= \iint_E \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S \\ &= \iint_D (-2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \cdot ((-\mathbf{j} + \mathbf{k})/\sqrt{2})\,|\mathbf{s}_\alpha \times \mathbf{s}_\beta|\,\mathrm{d}A \\ &= \iint_D 0\,\mathrm{d}A = 0. \end{aligned}$$

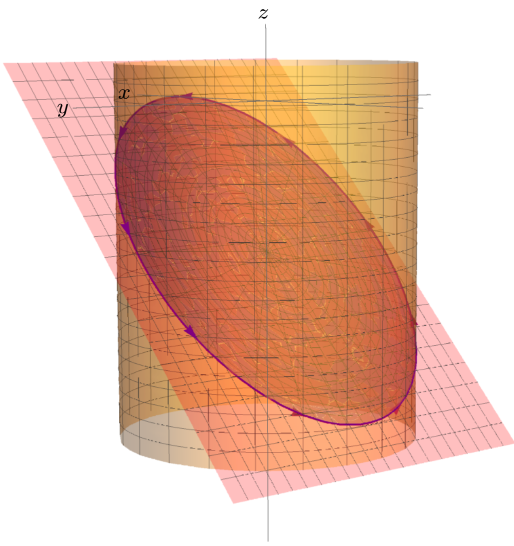


FIGURE F.11: Graph of the elliptic region lying in the plane $P = \{(x, y, z) \in \mathbb{R}^3 : x - z = 1\}$ bounded by the curve C, highlighted in purple, which lies at the intersection of P and the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

(c) Let E denote the circular region enclosed by the curve C in the plane

$$P = \{(x, y, z) \in \mathbb{R}^3 : y = z\},$$

see FIGURE F.12. Observe that E is a smooth orientable surface as it is a subset of a plane and since E is bounded by a smooth curve (specifically a circle). Letting

$$D = \{(x, y) \in \mathbb{R}^2 : (x, y, y) \in E\}$$

denote the projection of E onto the x-y plane, a smooth parameterisation $\mathbf{s} : D \to \mathbb{R}^3$ of E is given by $\mathbf{s}(\alpha, \beta) = \alpha\mathbf{i} + \beta\mathbf{j} + \beta\mathbf{k}$, for $(\alpha, \beta) \in D$. Since $E \subseteq \mathbb{R}^3$ is a non-trivial disk in the plane determined by the equation $y = z$, it follows that D is a connected closed and bounded subset of $\mathbb{R}^2$ with non-empty interior. In fact D is an ellipse with major axis of length 2 and minor axis of length $\sqrt{2}$, meaning that the area of D is equal to $\sqrt{2}\,\pi$. Moreover, by construction, $\mathbf{s}$ is injective, and setting $\Gamma = \partial D$, we have have that Γ is an ellipse, which is a smooth simple closed curve, and $C = \mathbf{s}(\Gamma)$. A unit normal vector $\mathbf{n}$ to E at $\mathbf{s}(\alpha, \beta)$ is determined by $(\mathbf{s}_\alpha \times \mathbf{s}_\beta)(\alpha, \beta) = -\mathbf{j} + \mathbf{k}$, for $(\alpha, \beta) \in D$, namely $\mathbf{n} = \mathbf{s}_\alpha \times \mathbf{s}_\beta / |\mathbf{s}_\alpha \times \mathbf{s}_\beta|$. The positive orientation induced on C by this normal vector field is counter clockwise, when viewed from high above the x-y plane, and coincides with orientations it inherits from $\mathbf{s}$. Letting $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{F} = (x + 2y)\mathbf{i} + (2z + 2x)\mathbf{j} + (z + y)\mathbf{k}$, for $(x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{F}$ is a smooth vector field since its co-ordinate functions are polynomials. Thus, the conditions of Stokes' Theorem are satisfied.

By Stokes' Theorem, since

$$(\mathrm{curl}(\mathbf{F}))(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y & 2z - 2x & z + y \end{vmatrix} = -\mathbf{i} - 2\mathbf{k},$$

for $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned} \int_C (y - z)\,\mathrm{d}x + (z - x)\,\mathrm{d}y + (x - y)\,\mathrm{d}z &= \iint_E \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S \\ &= \iint_D (-\mathbf{i} - 2\mathbf{k}) \cdot (-\mathbf{j} + \mathbf{k})\,\mathrm{d}A \\ &= \iint_D -2\,\mathrm{d}A = -2\,\mathrm{Area}(D) = -2\sqrt{2}\,\pi. \end{aligned}$$

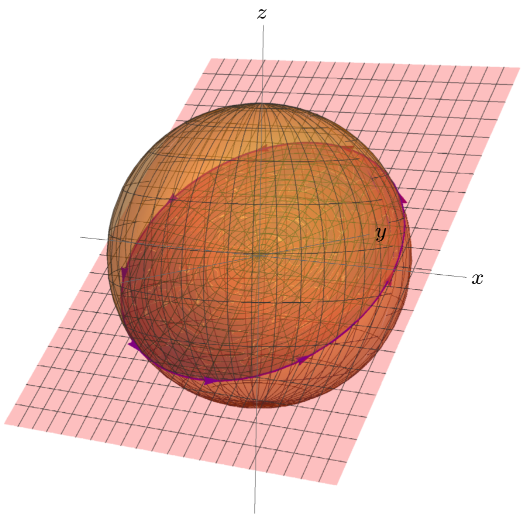


FIGURE F.12: Graph of the circular region lying in the plane $P = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$ bounded by the curve C, highlighted in purple, which lies at the intersection of P and the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

(d) Let E be the region enclosed by the curve C lying on upper hemisphere

$$H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\},$$

see FIGURE F.13. Observe that E is a smooth orientable surface as it is a subset of a sphere with a smooth boundary curve C. Indeed, a parameterisation of C is $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ given by

$$\mathbf{r}(t) = ((\cos(t) + 1)/4)\mathbf{i} + (\sin(t)/4)\mathbf{j} + \sqrt{1 - (\cos(t) + 1)/4}\mathbf{k},$$

for $t \in [0, 2\pi]$; and a smooth parameterisation of E is $\mathbf{s} : B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq x/2\} \to \mathbb{R}^3$ given by

$$\mathbf{s}(\alpha, \beta) = \alpha\mathbf{i} + \beta\mathbf{j} + \sqrt{1 - \alpha^2 - \beta^2}\mathbf{k},$$

for $(\alpha, \beta) \in B$. Since $B \subset \mathbb{R}^2$ is a non-trivial disk, it is a connected, closed and bounded subset of $\mathbb{R}^2$ with non-empty interior. Moreover, by construction, $\mathbf{s}$ is injective, and, setting $\Gamma = \{(x, y) : x^2 + y^2 = x/2\}$, namely the circle centred at $1/2$ with radius $1/4$ in the x-y plane, we have that Γ is a smooth simple closed curve, and $C = \mathbf{r}([0, 2\pi]) = \mathbf{s}(\Gamma)$. A normal vector field $\mathbf{n}$ to to the surface H at $\mathbf{s}(\alpha, \beta)$, for some $(\alpha, \beta) \in B$, is given by

$$(\mathbf{s}_\alpha \times \mathbf{s}_\beta)(\alpha, \beta) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\alpha/\sqrt{1 - \alpha^2 - \beta^2} \\ 0 & 1 & -\beta/\sqrt{1 - \alpha^2 - \beta^2} \end{vmatrix} = \frac{\alpha}{\sqrt{1 - \alpha^2 - \beta^2}}\mathbf{i} + \frac{\beta}{\sqrt{1 - \alpha^2 - \beta^2}}\mathbf{j} + \mathbf{k}.$$

The positive orientation induced on C by this normal vector field is counter clockwise, when viewed from high above the x-y plane, and coincides with orientations it inherits from $\mathbf{r}$. Letting $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{F} = x\mathbf{i} + (x - 2yz)\mathbf{j} + (x^2 + z)\mathbf{k}$, for $(x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{F}$ is a smooth vector field since its co-ordinate functions are polynomials. Thus, the conditions of Stokes' Theorem are satisfied.

By Stokes' Theorem, since

$$(\text{curl}(\mathbf{F}))(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x - 2yz & x^2 + z \end{vmatrix} = 2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k},$$

for $(x, y, z) \in \mathbb{R}^3$, and thus $(\text{curl}(\mathbf{F}))(\mathbf{s}(\alpha, \beta)) = 2\beta\mathbf{k} - 2\alpha\mathbf{j} + \mathbf{k}$, for $(\alpha, \beta) \in B$, we have that

$$\int_C (x + 2y)\,\mathrm{d}x + (2z + 2x)\,\mathrm{d}y + (z + y)\,\mathrm{d}z = \iint_E \text{curl}(\mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S$$

$$= \iint_B (2\beta\mathbf{i} - 2\alpha\mathbf{j} + \mathbf{k}) \cdot \left(\frac{\alpha}{\sqrt{1 - \alpha^2 - \beta^2}}\mathbf{i} + \frac{\beta}{\sqrt{1 - \alpha^2 - \beta^2}}\mathbf{j} + \mathbf{k}\right)\mathrm{d}A = \iint_B 1\,\mathrm{d}A = \text{Area}(B) = \pi/16.$$

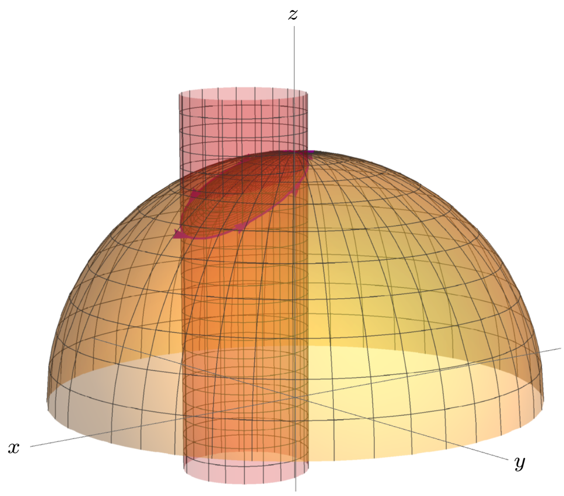


FIGURE F.13: Graph of the region lying in the upper hemisphere $H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$ bounded by the curve C, highlighted in purple, which lies at the intersection of H and the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = x/2\}$.

(3) Prove or disprove the following statement. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field and let $\mathbb{S}_\epsilon \subset \mathbb{R}^3$ denote a circle of radius $\epsilon > 0$ centred at a point $P_0 \in \mathbb{R}^3$, enclosing a disk $\mathbb{D}_\epsilon$ with unit normal vector field $\mathbf{n}$. If $\mathbb{S}_\epsilon$ is positively orientated with respect to $\mathbf{n}$, see FIGURE F.14, then

$$\lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \oint_{C_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \mathbf{n} \cdot \operatorname{curl}(\mathbf{F})(P_0),$$

where $\mathbf{r}$ is a parameterisation of $\mathbb{S}_\epsilon$ that is smooth and traverses $\mathbb{S}_\epsilon$ once positively with respect to $\mathbf{n}$.

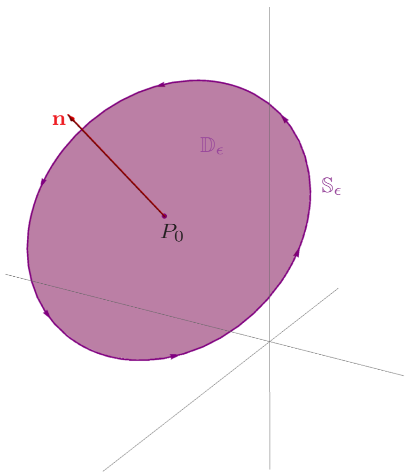


FIGURE F.14: Graph of a disc $\mathbb{D}_\epsilon \subseteq \mathbb{R}^3$ centred at a point $P_0 \in \mathbb{R}^3$, together with the boundary curve $\mathbb{S}_\epsilon$ positively orientated with respect to a given unit normal vector field $\mathbf{n}$.

Solution.

Since $\mathbb{S}_\epsilon$ is a circle, $\mathbf{F}$ is smooth and $\mathbf{r}$ is smooth and traverse $\mathbb{S}_\epsilon$ once positively with respect to $\mathbf{n}$, we may apply Stokes' Theorem to obtain, for a fixed $\epsilon > 0$, that

$$\begin{aligned}
\frac{1}{\pi \epsilon^2} \oint_{C_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S \\
&= \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} \operatorname{curl}(\mathbf{F})(P_0) \cdot \mathbf{n} \, \mathrm{d}S + \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} (\operatorname{curl}(\mathbf{F}) - \operatorname{curl}(\mathbf{F})(P_0)) \cdot \mathbf{n} \, \mathrm{d}S \\
&= \operatorname{curl}(\mathbf{F})(P_0) \cdot \mathbf{n} + \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} (\operatorname{curl}(\mathbf{F}) - \operatorname{curl}(\mathbf{F})(P_0)) \cdot \mathbf{n} \, \mathrm{d}S
\end{aligned}$$

Thus, it is sufficient to show that

$$\lim_{\epsilon \to 0} \left| \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} (\mathrm{curl}(\mathbf{F}) - \mathrm{curl}(\mathbf{F})(P_0)) \cdot \mathbf{n} \, \mathrm{d}S \right| = 0.$$

To this end, since $\mathbf{F}$ is smooth, we have that $\mathrm{curl}(\mathbf{F})$ is continuous and so

$$\begin{aligned}
&\lim_{\epsilon \to 0} \left| \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} (\mathrm{curl}(\mathbf{F}) - \mathrm{curl}(\mathbf{F})(P_0)) \cdot \mathbf{n} \, \mathrm{d}S \right| \\
&\leq \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} |(\mathrm{curl}(\mathbf{F}) - \mathrm{curl}(\mathbf{F})(P_0)) \cdot \mathbf{n}| \, \mathrm{d}S \\
&\leq \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \iint_{\mathbb{S}_\epsilon} |\mathrm{curl}(\mathbf{F}) - \mathrm{curl}(\mathbf{F})(P_0)| \, \mathrm{d}S \\
&\leq \lim_{\epsilon \to 0} \sup\{|\mathrm{curl}(\mathbf{F})(Q) - \mathrm{curl}(\mathbf{F})(P_0)| \colon Q \in \mathbb{S}_\epsilon\} = 0,
\end{aligned}$$

as required.

(5) Let $\mathbf{F}$ denote a vector field whose component functions have continuous first-order partial derivatives throughout a non-empty open subset of $\mathbb{R}^3$ containing a connected non-empty region D whose boundary is a smooth connected orientable closed surface S. Prove, if $|\mathbf{F}(x, y, z)| \leq 1$, for all $(x, y, z) \in D$, then

$$\iiint_D \nabla \cdot \mathbf{F} \, \mathrm{d}V$$

is bounded above by the surface area of S.

Solution.

The hypotheses of the question gives us that the conditions of Gauss' divergence theorem are met. Letting $\mathbf{n}$ denote the outward unit normal vector field to the surface S, and since $|\mathbf{F}| \leq 1$, we have $|\mathbf{F} \cdot \mathbf{n}| \leq |\mathbf{F}||\mathbf{n}| \leq 1$. Therefore,

$$\iiint_D \nabla \cdot \mathbf{F} \, \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S \leq \iint_S |\mathbf{F} \cdot \mathbf{n}| \, \mathrm{d}S \leq \iint_S 1 \, \mathrm{d}S.$$

Recalling that

$$\iint_S 1 \, \mathrm{d}S$$

is equal to the surface area of S, we have that the given triple integral is bounded above by the surface area of S, as required.