Dynamics and Control of Inverted Pendulum Systems

1 Standard Inverted Pendulum on a Cart

1.1 System Description

- Cart mass: M
- Pendulum mass: m
- Pendulum length to CoM: l
- Gravity: g
- Horizontal force input: F
- x: horizontal position of the cart
- θ: pendulum angle from vertical (clockwise positive)

1.2 Nonlinear Equations of Motion

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F \tag{1}$$

$$ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mgl\sin\theta = 0 \tag{2}$$

1.3 Nonlinear State-Space Model

Define state vector and input:

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad u = F$$

Express the system as:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = f(\mathbf{x}, u)$$

To solve for $\ddot{x}$ and $\ddot{\theta}$, rewrite (1), (2) in matrix form:

$$\underbrace{\begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}}_{D(\theta)} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + ml\dot{\theta}^2 \sin\theta \\ mgl\sin\theta \end{bmatrix}$$

Then:

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = D(\theta)^{-1} \begin{bmatrix} u + ml\dot{\theta}^2 \sin\theta \\ mgl\sin\theta \end{bmatrix}$$

Hence the nonlinear state-space model is:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x}(\mathbf{x}, u) \\ \dot{\theta} \\ \ddot{\theta}(\mathbf{x}, u) \end{bmatrix}$$

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2 Wheeled Inverted Pendulum (Self-Balancing Robot)

2.1 System Description

- Wheel mass: M, radius R, inertia I_w
- Pendulum mass: m, pendulum CoM length: l, inertia I_p
- Gravity: g
- Torque input: τ (applied to wheel)
- x: horizontal position of wheel
- θ: pendulum angle from vertical (clockwise positive)

2.2 Nonlinear Equations of Motion

$$\left(M + m + \frac{I_w}{R^2}\right)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = \frac{\tau}{R} \tag{3}$$

$$ml\ddot{x}\cos\theta + \left(I_p + ml^2\right)\ddot{\theta} - mgl\sin\theta = 0 \tag{4}$$

2.3 Nonlinear State-Space Model

Define

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad u = \tau$$

Rewrite (3), (4) as:

$$D(\theta) = \begin{bmatrix} M + m + \frac{I_w}{R^2} & ml\cos\theta \\ ml\cos\theta & I_p + ml^2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{u}{R} + ml\dot{\theta}^2 \sin\theta \\ mgl\sin\theta \end{bmatrix}$$

Then

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = D(\theta)^{-1}\mathbf{b}$$

The nonlinear state-space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x}(\mathbf{x}, u) \\ \dot{\theta} \\ \ddot{\theta}(\mathbf{x}, u) \end{bmatrix}$$

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3 Linearized Models and State-Space Form

3.1 Linearized Dynamics for Cart Pendulum

Linearize around $\theta = 0$, $\dot{\theta} = 0$:

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1$$

Linearized equations:

$$\begin{bmatrix} M + m & ml \\ ml & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u \\ mgl\theta \end{bmatrix}$$

Explicitly:

$$(M + m)\ddot{x} + ml\ddot{\theta} = u$$
$$ml\ddot{x} + ml^2\ddot{\theta} - mgl\theta = 0$$

3.2 Linearized State-Space Form for Cart Pendulum

Define state:

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad u = F$$

The linear state-space model is:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mgl}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix}$$

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3.3 Linearized Dynamics for Wheeled Inverted Pendulum

Linearize around $\theta = 0$, $\dot{\theta} = 0$:

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1$$

$$D = \begin{bmatrix} \alpha & ml \\ ml & \beta \end{bmatrix}$$

with

$$\alpha = M + m + \frac{I_w}{R^2}, \quad \beta = I_p + ml^2$$

Linearized equations:

$$\begin{bmatrix} \alpha & ml \\ ml & \beta \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{u}{R} \\ mgl\theta \end{bmatrix}$$

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3.4 Linearized State-Space Form for Wheeled Pendulum

State vector:

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad u = \tau$$

State-space model:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m^2 g l^2}{\Delta R} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m g l \alpha}{\Delta} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{\beta}{\Delta R} \\ 0 \\ -\frac{ml}{\Delta} \end{bmatrix}$$

where the determinant

$$\Delta = \alpha\beta - (ml)^2$$

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4 Controller Design

4.1 PID Control

For both systems, a PID controller can be used to regulate the pendulum angle:

$$u(t) = K_p e(t) + K_i \int e(t)dt + K_d \frac{de(t)}{dt}, \quad e(t) = \theta_{\text{ref}} - \theta(t)$$

Tuning methods (Ziegler-Nichols, manual) apply.

4.2 LQR Control

Given linearized models:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu, \quad u = -K\mathbf{x}$$

The gain matrix K minimizes

$$J = \int_0^\infty \left(\mathbf{x}^T Q \mathbf{x} + u^T R u\right) dt$$

where Q and R are weighting matrices chosen by the designer.
K is computed by solving the algebraic Riccati equation.
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