Let X be a continuous random variable with values uniformly distributed over the interval [0, 20].

(a) Find the mean and variance of X.

Lets first find the c.d.f., p.d.f., expected value and variance for a general Uniform random variable Y on interval (a, b).

We know that $f_Y(y)$ is of the form $f_Y(y) = \begin{cases} c & a \leq y \leq b \\ 0 & otherwise \end{cases}$ for some constant value c. We find the value of c by integrating the p.d.f and setting to 1:

$1 = \int_{-\infty}^{\infty} f_Y(y)dy = \int_a^b c\, dy = cy]_a^b = c(b-a) \;\Rightarrow\; c = \frac{1}{(b-a)}$

So $f_Y(y) = \begin{cases} \frac{1}{(b-a)} & a \leq y \leq b \\ 0 & otherwise \end{cases}$

Now $F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy = \int_a^y \frac{1}{(b-a)}dy = \frac{1}{(b-a)}y]_a^y = \frac{y-a}{(b-a)}$ for $a \leq y \leq b$

So $F_Y(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{(b-a)} & a \leq y \leq b \\ 1 & y > b \end{cases}$

To find the expectation we use the formula $E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_a^b \frac{1}{(b-a)} y dy = \frac{1}{(b-a)} \frac{y^2}{2}]_a^b = \frac{1}{(b-a)} \frac{b^2-a^2}{2} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{(a+b)}{2}$.

To find the Varience we first to calculate $E(Y^2)$. $E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_a^b \frac{1}{(b-a)} y^2 dy = \frac{1}{(b-a)} \frac{y^3}{3}]_a^b = \frac{1}{(b-a)} \frac{b^3-a^3}{3} = \frac{(b^2+ab+a^2)(b-a)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$.

Now $Var(Y) = E(Y^2) - E^2(Y) = \frac{b^2+ab+a^2}{3} - (\frac{(a+b)}{2})^2 = \frac{4(b^2+ab+a^2)}{12} - \frac{3(a^2+2ab+b^2)}{12} = \frac{(a^2-2ab+b^2)}{12} = \frac{(a-b)^2}{12}$

We can now conclude the following about X:

$f_X(x) = \begin{cases} \frac{1}{20} & 0 \leq x \leq 20 \\ 0 & otherwise \end{cases}$

$P(X \leq x) = F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{20} & 0 \leq x \leq 20 \\ 1 & x > 20 \end{cases}$

E(X) = (20+0)/2 = 10

Var(X) = (20-0)^2/12 = 33.33