

## Homework 1

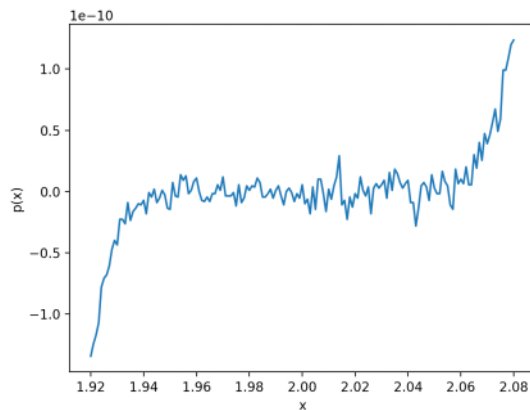
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APPM 4600

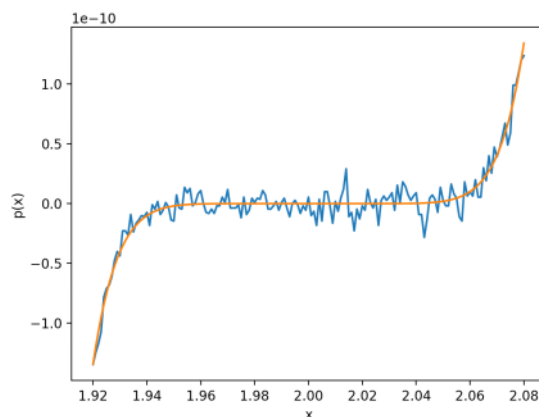
1.) 1. Consider the polynomial  
$$p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512.$$

- i. Plot  $p(x)$  for  $x = 1.920, 1.921, 1.922, \dots, 2.080$  (i.e.  $x = [1.920 : 0.001 : 2.080]$ ;) evaluating  $p$  via its coefficients.

\* All code attached separately



- ii. Produce the same plot again, now evaluating  $p$  via the expression  $(x-2)^9$ .



- iii. What is the difference? What is causing the discrepancy? Which plot is correct?

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# iii.) The first plot of the expanded form is very sharp and has a lot of jumps,  
# whereas the second plot (in red) of the binomial is very smooth. This discrepancy  
# is due to the machine having to compute the multiple terms, add and subtract them,  
# all while keeping up to four significant figures. This extra computation leads to the  
# loss of accuracy. We can tell that the (x-2)**9 plot is correct because we know that  
# there is only 1 root at x = 2, whereas the expanded plot has multiple roots.
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2.) 2. How would you perform the following calculations to avoid cancellation? Justify your answers.

i. Evaluate  $\sqrt{x+1} - 1$  for  $x \approx 0$ .

To avoid cancellation, I would multiply by the conjugate to eliminate the square root.

$$\frac{(\sqrt{x+1} - 1) \cdot (\sqrt{x+1} + 1)}{(\sqrt{x+1} + 1)} = \frac{x+1-1}{\sqrt{x+1} + 1} = \frac{x}{\sqrt{x+1} + 1}, \text{ which no longer has cancellation issues when } x \approx 0.$$

ii. Evaluate  $\sin(x) - \sin(y)$  for  $x \approx y$ .

In this instance, multiplying by the conjugate results in the following:

$$\frac{\sin^2(x) - \sin^2(y)}{\sin(x) + \sin(y)}, \text{ which still results in cancellation when } x \approx y$$

Instead, I will try using trig identities.

$$\sin(x) - \sin(y) = 2 \cos\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{x-y}{2}\right), \text{ which is no longer problematic since the } x-y \text{ is contained within the function.}$$

iii. Evaluate  $\frac{1-\cos(x)}{\sin(x)}$  for  $x \approx 0$ .

$$\frac{1 - \cos(x)}{\sin(x)} \cdot \frac{(1 + \cos(x))}{(1 + \cos(x))} \quad \text{Multiply by conjugate of numerator}$$

$$\frac{1 - \cos^2(x)}{\sin(x) + \sin(x)\cos(x)}$$

3.) Find the second degree Taylor polynomial  $P_2(x)$  for  $f(x) = (1+x+x^3)\cos(x)$  about  $x_0 = 0$ .

Taylor Polynomial Formula:

$$P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

$$f(0) = 1$$

$$f'(x) = (1+3x^2)\cos(x) - \sin(x)(1+x+x^3) \Big|_{x_0=0} = 1$$

$$f''(x) = [(6x) \cdot \cos(x) - (1+3x^2)\sin(x)] - [\cos(x)(1+x+x^3) + \sin(x)(1+3x^2)] \Big|_{x_0=0} = -1$$

$$\therefore P_2(x) = 1 + x - \frac{1}{2} x^2$$

(a) Use  $P_2(0.5)$  to approximate  $f(0.5)$ . Find an upper bound for the error  $|f(0.5) - P_2(0.5)|$  using the error formula and compare it to the actual error.

$$P_2(0.5) = 1 + 0.5 - \frac{1}{2}(0.5)^2 = 1.375$$

$$R_2(x) = \left| \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-c)^{n+1} \right| - \text{Lagrange error bound}$$

$$\begin{aligned} f'''(x) &= [(6 \cdot \cos(x) - (6x) \cdot \sin(x)) - (6x \sin(x) + (1+3x^2)\cos(x))] \\ &\quad - [(-\sin(x) \cdot (1+x+x^3) + (1+3x^2)(\cos(x)) + (\cos(x)(1+3x^2) + (6x \sin(x)))] \Big|_{x=0} \\ &= 6 - 0 - 0 - 1 - (-0) - 1 - 1 + 0 = 3 \end{aligned}$$

$$f'''(0.5) = -2.878 \quad \therefore M = 3$$

$$R_2(0.5) \leq \frac{3}{3!} (0.5-0)^3 = 0.0625 \quad (\text{Lagrange Error Bound})$$

$$|f(0.5) - P_2(0.5)| = |1.62494 - 1.375| = 0.249938 \quad (\text{abs. error})$$

$$\frac{|f(0.5) - P_2(0.5)|}{|P_2(0.5)|} = \frac{|1.62494 - 1.375|}{1.375} = 0.181773 \quad (\text{relative error})$$

(b) Find a bound for the error  $|f(x) - P_2(x)|$  when  $P_2(x)$  is used to approximate  $f(x)$ . This will be a function of  $x$ .

$$\frac{|f(x) - P_2(x)|}{|P_2(x)|} = \left| \frac{(1+x+x^3) \cdot \cos(x) - (1+x - \frac{x^2}{2})}{(1+x - \frac{x^2}{2})} \right|$$

(c) Approximate  $\int_0^1 f(x) dx$  using  $\int_0^1 P_2(x) dx$ .

$$\int_0^1 f(x) dx \approx \int_0^1 P_2(x) dx$$

$$\begin{aligned} \int_0^1 P_2(x) dx &= \int_0^1 \left(1 + x - \frac{x^2}{2}\right) dx \\ &= \left(x + \frac{x^2}{2} - \frac{x^3}{6}\right) \Big|_0^1 = \left(1 + \frac{1}{2} - \frac{1}{6}\right) - 0 \\ &= \frac{8}{6} \text{ or } 1.\bar{3} \end{aligned}$$

$$\boxed{\int_0^1 P_2(x) dx = 1.\bar{3}}$$

(d) Estimate the error in the integral.

Since we integrate the polynomial, the error will increase to be  $\approx 0.5$

4.) 4. Consider the quadratic equation  $ax^2 + bx + c = 0$  with  $a = 1, b = -56, c = 1$ .

(a) Assume you can calculate the square root with 3 correct decimals (e.g.  $\sqrt{2} \approx 1.414 \pm \frac{1}{2}10^{-3}$ ) and compute the relative errors for the two roots to the quadratic when computed using the standard formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_{1,2} = \frac{-(-56) \pm \sqrt{(-56)^2 - 4 \cdot (1)(1)}}{2(1)}$$

$$= \frac{56 \pm \sqrt{3136 - 4}}{2}$$

$$= \frac{56 \pm 55.964}{2}$$

$$r_1 = 55.982, r_2 = 0.018$$

(b) A better approximation for the "bad" root can be found by manipulating  $(x - r_1)(x - r_2) = 0$  so that  $r_1$  and  $r_2$  can be related to  $a, b, c$ . Find such relations (there are two) and see if either can be used to compute the "bad" root more accurately.

$$x^2 - 56x + 1 = 0$$

$$r_{1,2} = \frac{56 \pm \sqrt{3136 - 4}}{2}$$

$$r_2 = 0.018 \quad (\text{Good root})$$

$$x^2 - 56x + 1 = 0$$

$$(x^2 - 56x + 784) - 783 = 0$$

$$(x - 28)^2 - 783 = 0$$

$$(x - 28)^2 = 783$$

$$r_{1,2} = \pm \sqrt{783} + 28$$

$$r_1 = 55.9821$$

$$r_2 = 0.017863$$

$$r_2 = 0.018 \quad (\text{Good root})$$

$$r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{(-b - \sqrt{b^2 - 4ac})}{(-b - \sqrt{b^2 - 4ac})}$$

$$= \frac{2c}{-b - \sqrt{b^2 - 4ac}} \Big|_{a=1, b=-56, c=1} = 55.9821$$

Since there is no difference within the root, completing the square is more computationally accurate

- 5.) 5. **Cancellation of terms.** Consider computing  $y = x_1 - x_2$  with  $\tilde{x}_1 = x_1 + \Delta x_1$  and  $\tilde{x}_2 = x_2 + \Delta x_2$  being approximations to the exact values. If the operation  $x_1 - x_2$  is carried out exactly we have  $\tilde{y} = y + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$ .

Play with different values of  $x$ . One really small value ( $< 1$ ) and one large value  $> 10^5$ .

- (a) Find upper bounds on the absolute error  $|\Delta y|$  and the relative error  $|\Delta y|/|y|$ , when is the relative error large?