Homework2

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APPM 4600

1.) (a) Show that $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$.

$(1+x)^n - nx - 1 = o(x)$ as $x \to 0$ if $\lim_{x\to 0} \frac{|(1+x)^n - nx - 1|}{|x|} = 0$

$$\lim_{x\to 0} \frac{|(1+x)^n - nx - 1|}{|x|} = \frac{|(1+0)^n - n(0) - 1|}{0} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \frac{|n(1+x)^{n-1} - n|}{1} = \frac{|n(1)^{n-1} - n|}{1} = 0 \checkmark$$

(b) Show that $x \sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$.

$x \cdot \sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$ if $\exists$ a positive constant M st

$\lim_{x\to 0} \frac{|x \sin(\sqrt{x})|}{|x^{3/2}|} \le M$ $\forall$ values of x in neighborhood of 0

$$\lim_{x\to 0} \frac{|x \cdot \sin(\sqrt{x})|}{|x^{3/2}|} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \frac{|\sin(\sqrt{x}) + 2\sqrt{x}(\cos(\sqrt{x}))|}{\frac{3}{2}\sqrt{x}} = \lim_{x\to 0} \underbrace{\left|\frac{\sin(\sqrt{x})}{\frac{3}{2}\sqrt{x}}\right|}_{0} + \lim_{x\to 0} \frac{|2\cos(\sqrt{x})|}{\frac{3}{2}}$$

$$= \frac{2 \cdot \cos(\sqrt{0})}{\frac{3}{2}} = \frac{4}{3} \neq 0 \checkmark$$

(c) Show that $e^{-t} = o(\frac{1}{t^2})$ as $t \to \infty$. Following the definition of $o(t)$ as shown is part (a)

$$\lim_{t\to\infty} \frac{|e^{-t}|}{\left|\frac{1}{t^2}\right|} = \lim_{t\to\infty} \frac{|t^2|}{|e^t|} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{t\to\infty} \frac{|2t|}{|e^t|} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{t\to\infty} \frac{2}{|e^t|} = 0 \checkmark$$

(d) Show that $\int_0^\varepsilon e^{-x^2} dx = O(\varepsilon)$ as $\varepsilon \to 0$. Following the definition of $O(\varepsilon)$ as shown in (b)

$$\lim_{\varepsilon\to 0} \frac{\left|\int_0^\varepsilon e^{-x^2} dx\right|}{|\varepsilon|} = \lim_{\varepsilon\to 0} \frac{\left|\frac{\sqrt{\pi}}{2} \operatorname{erf}(\varepsilon)\right|}{|\varepsilon|} = \frac{0}{0} \overset{L'H}{=} \lim_{\varepsilon\to 0} \frac{\left|\frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} e^{-x^2}\right|}{|1|}$$

$$= \lim_{\varepsilon\to 0} \left|\frac{1}{e^{x^2}}\right|$$

2.)