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APPM 4600

1.) 1. Find the least squares solution to the overdetermined linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Theorem: The vector $\bar{x}$ that minimizes $\|Ax - b\|_2$ where $A \in \mathbb{R}^{n \times m}$, $n > m$ is given by the solution of $A^T A \bar{x} = A^T \bar{b}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} u = 1 \\ v = 1/2 \end{matrix}}$$

2.) 2. Find the vector $\boldsymbol{x}$ that minimizes the quantity $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$, when it holds that

$$\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Hint: When we solve a linear system of equations $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, multiplication from the left with a nonsingular matrix will leave the solution unchanged. This is **not** the case when finding the least squares solution to an overdetermined system. Exploit this and multiply the system above with a suitable diagonal matrix, so that the problems becomes a regular least squares problem (for which we can apply the normal equation approach.)

Given that we want to minimize the quantity $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$, it would seem suitable to multiply the system by the diagonal matrix where the entries are the coefficients of E.

i.e $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_2 \\ 5b_3 \\ 3b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_2 \\ 5b_3 \\ 3b_4 \end{bmatrix}$$

Using the Theorem from problem 1, we can use the least squares method to solve

$$\begin{bmatrix} 1 & 12 & 20 & 6 \\ 3 & -2 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 12 & 20 & 6 \\ 3 & -2 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 421 \\ 247 \end{bmatrix}$$

Now solving this linear system for $\bar{x}$ using $\bar{x} = A^{-1}b$,

we arrive at $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.6536 \\ 0.3929 \end{bmatrix} \Rightarrow \boxed{x_1 = 0.6536, \; x_2 = 0.3929}$

MATLAB Code:

```
%% Main Script
A = [1 3;
     6 -1;
     4 0;
     2 7];
C = [1 2 3 4]';

E = [1 0 0 0;
     0 2 0 0;
     0 0 5 0;
     0 0 0 3];

Aadj = E*A
Cadj = E*C

Atrans = Aadj';


M = Atrans*Aadj
Cnew = Atrans*Cadj

M\(Cnew)
```

MATLAB Output:

```
Aadj =

     1     3
    12    -2
    20     0
     6    21


Cadj =

     1
     4
    15
    12


M =

   581   105
   105   454


Cnew =

   421
   247


ans =

    0.6536
    0.3929
```

3.) 3. If a function is identically zero over an interval, all its derivatives must also be identically zero over the same interval. Based on this observation:

(a) Prove that $\{1, x, x^2, \ldots, x^n\}$ are linearly independent.

(b) Show that the function set

$$\{1, \cos(x), \cos(2x), \ldots, \cos(nx), \sin(x), \ldots, \sin(nx)\}$$

is linearly independent (also over any interval).

(a) Prove that $\{1, x, x^2, \ldots, x^n\}$ are linearly independent

Take the Wronskian of the above vector:

$$W = \begin{bmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1 & 2x & 3x^2 & & nx^{n-1} \\ 0 & 0 & 2 & 6x & & n(n-1)x^{n-2} \\ 0 & 0 & 0 & 6 & & \vdots \\ \vdots & & & & & \end{bmatrix}$$

W is a triangular matrix $\Rightarrow \det(W) = \prod_{i=j=0}^{n} w_{ij}$

$\therefore \det(W) = n!(n-1)! \cdots 1 \neq 0$

$$W = \begin{bmatrix} 0 & 0 & 2 & 6x & n(n-1)x^{n-2} \\ 0 & 0 & 0 & 6 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots 0 & n! \end{bmatrix} \qquad \therefore \det(W) = n!(n-1)! \cdots 1 \neq 0$$

Since $\det(W) \neq 0$, the set is linearly independent.

(b) $\{1, \cos(x), \cos(2x) \ldots, \cos(nx), \sin(x), \sin(2x), \ldots \sin(nx)\}$

To show linear independence, we have to show that

$$a_0\cos(0x) + a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + b_0\sin(0x) + b_1\sin(x) + \cdots + b_n\sin(nx) \equiv 0$$

Taking n derivatives of the above functions will result in the multiplication of each term by n^n i.e. $a_1 \cdot 1^n\cos(x) + a_2 \cdot 2^n\cos(2x) + \cdots + a_n n^n\cos(nx) + \cdots + b_n n^n\sin(nx)$ which would result in

$$a_n\cos(nx) + b_n\sin(x) \equiv 0$$

and if we create the Wronskian:

$$\begin{bmatrix} \cos(nx) & \sin(nx) \\ -n\sin(nx) & n\cos(nx) \end{bmatrix}, \quad \det(W) = n\cos^2(nx) - (-n\sin^2(nx)) = n(\cos^2(nx) + \sin^2(nx)) = n \neq 0$$

$\therefore$ these are linearly independent vectors.

Extending this from the $n-1, n-2, \ldots, 1, 0$ satisfies

$$a_0\cos(0x) + a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + b_0\sin(0x) + b_1\sin(x) + \cdots + b_n\sin(nx) \equiv 0,$$

meaning that $\{1, \cos(x), \cos(2x), \ldots, \cos(nx), \sin(x), \sin(2x), \ldots, \sin(nx)\}$ is indeed a linearly independent set of vectors.

4) 4. Prove the three-term recursion formula for orthogonal polynomials:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x)$$

where

$$b_k = \frac{< x\phi_{k-1}, \phi_{k-1} >}{< \phi_{k-1}, \phi_{k-1} >} \quad c_k = \frac{< x\phi_{k-1}, \phi_{k-2} >}{< \phi_{k-2}, \phi_{k-2} >}$$

Hint: Since $\phi_k(x)$ is a polynomial of degree k and of the form $\phi_k = x^k + \{\text{lower order terms}\}$, we can clearly select b_k and c_k so that the right hand side (RHS) of (1) matches $\phi_k(x)$ for powers x^k , x^{k-1} and x^{k-2} . We have no obvious reason to expect that the two sides will match the other lower order terms. Hence, we would expect to need to include a lot more terms in the RHS to get the two sides to become equal:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4}(x) + \cdots + a_0\phi_0(x)\} \quad (1)$$

We now need to show that all these a's are in fact are zero. To show that $a_j = 0$, $j \leq k-3$, we form the scalar product of (1) with $\phi_j(x)$ for $j = 0, \ldots, k-1$. You need to show that everything in (1) apart from $a_j < \phi_j, \phi_j >$ then vanishes, thereby showing that $a_j = 0$, $j \leq k-3$. After that, it remains to determine the values of b_k and c_k . These coefficients follow by again forming suitable scalar products.

From the hint:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4}(x) + a_0\phi_0(x)\}$$

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \sum^{k-3} a_i\phi_i(x)$$

$$\phi_k(x) = (x-b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \sum_{i=0}^{k-3} a_i\phi_i(x)$$

To show $a_j = 0$, $j \le k-3$, form a scalar product of ϕ_k and ϕ_j, for $j = 0, \dots, k-1$

$$\therefore \langle \phi_k, \phi_j \rangle = \langle \phi_{k-1}, x\phi_j \rangle - b_k \langle \phi_{k-1}, \phi_j \rangle - c_k \langle \phi_{k-2}, \phi_j \rangle - \sum_{i=0}^{k-3} a_i \cdot \langle \phi_i, \phi_j \rangle, \quad j = 0, \dots, k-3$$

$x\phi_j(x)$ is a polynomial of degree $j+1$, so we can rewrite as $x\phi_j(x) = \sum_{i=0}^{j+1} d_i \phi_i(x)$

$$\therefore \; 0 = \sum_{i=0}^{j+1} d_i \langle \phi_i, \phi_{k-1} \rangle - 0 - 0 - a_j \langle \phi_j, \phi_j \rangle, \quad j \le k-3$$

$j+1 \le k-2 \Rightarrow$ every term w/in the sum is 0

$$\therefore \quad 0 = a_j \langle \phi_j, \phi_j \rangle, \quad j \le k-3$$

Now we can solve for b_k and c_k.

Take the inner product of ϕ_k and ϕ_{k-1}:

$$\langle \phi_k, \phi_{k-1} \rangle = \langle x\phi_{k-1}, \phi_{k-1} \rangle - b_k \langle \phi_{k-1}, \phi_{k-1} \rangle - c_k \langle \phi_{k-2}, \phi_{k-1} \rangle$$
$$0 = \langle x\phi_{k-1}, \phi_{k-1} \rangle - b_k \langle \phi_{k-1}, \phi_{k-1} \rangle - 0$$

$$\Rightarrow b_k = \frac{\langle x\phi_{k-1}, \phi_{k-1} \rangle}{\langle \phi_{k-1}, \phi_{k-1} \rangle} \quad \checkmark$$

Take the inner product of ϕ_k and ϕ_{k-2}:

$$\langle \phi_k, \phi_{k-2} \rangle = \langle x\phi_{k-1}, \phi_{k-2} \rangle - b_k \langle \phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle$$
$$0 = \langle x\phi_{k-1}, \phi_{k-2} \rangle - 0 - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle$$

$$\Rightarrow c_k = \frac{\langle x\phi_{k-1}, \phi_{k-2} \rangle}{\langle \phi_{k-2}, \phi_{k-2} \rangle} \quad \checkmark$$

5.) 5. One of the many formulas for computing the Chebychev polynomials $T_n(x)$ is

$$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \tag{2}$$

where z is implicitly defined through x via $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Confirm that the formula (2) indeed generates the same polynomials as the standard definition of the Chebychev polynomials.
Hint: One way would be to verify that it produces the correct result for T_0 and T_1 and that it satisfies the 3 term recursion.

$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$, where z is defined implicitly through x via $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$

The standard 3-term recursion for the Chebyshev polynomials is defined as

$T_0 = 1$

$T_0 = 1$

$T_1 = x$

$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x) \Rightarrow T_{n+1}(x) + T_{n-1}(x) = 2x \cdot T_n(x)$

Given formula (2) as defined above,

$$\frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right) + \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right) = \cancel{2} \cdot \overbrace{\frac{1}{\cancel{2}}\left(z + \frac{1}{z}\right)}^{\text{Implicit definition of } x} \cdot \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$$

$$\frac{1}{2}\left[z^n \cdot z^1 + z^{-n} \cdot z^{-1} + z^n \cdot z^{-1} + z^{-n} \cdot z^1\right] = \frac{1}{2}\left[z^1 \cdot z^n + z^{-1} z^n + z^1 z^{-n} + z^{-1} z^{-n}\right]$$

$$\frac{1}{2}\left[z^{n+1} + z^{-(n+1)} + z^{n-1} + z^{-(n-1)}\right] = \frac{1}{2}\left[z^{n+1} + z^{-(n+1)} + z^{n-1} + z^{-(n-1)}\right] \checkmark$$

Therefore, formula (2) as given in the problem statement indeed satisfies the standard 3-term recursion.