1a) Using iteration given on (0.1)

```
$ python3 Problem1aNew.py
[0.5        0.8660254]
the error message reads: 0
took this many seconds: 0.00016028881072998046
number of iterations is: 33
```

1c) Using Newton's Method for (x0,y0) = (1,1)

```
$ python3 Problem1c.py
[0.5        0.8660254]
Newton: the error message reads: 0
Newton: took this many seconds: 0.0003135967254638671 7
Netwon: number of iterations is: 5
```

3b) Point on ellipsoid for (x0,y0,z0) = (1,1,1)

```
[1.09364232 1.36032838 1.36032838]
Newton: the error message reads: 0
Newton: took this many seconds: 0.0
Netwon: number of iterations is: 4
```

The error between iterations is given below (used to prove quadratic convergence)

```
0.5181138552112181
0.09123718502632691
0.0020187814410449 71
1.036232733013136e-06
5.253550961348586e-09
```

Tony Samow | Homework 5
APPM 4600
6 OCT 2023

1.) $(x_0, y_0) = (1,1)$, $f(x,y) = 3x^2 - y^2 = 0$, $g(x,y) = 3xy^2 - x^3 - 1 = 0$ $\}(0.1)$

(a) Iterate the following system in python:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix} \cdot \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \quad n = 0, 1, 2, \ldots$$

Code provided in PDF submission.

Output: $(x^*, y^*) = (0.5, 0.866)$, in 33 iterations.

(b) Provide some motivation for the (2×2) matrix used in (a)

$f_x(x,y) = 6x$
$f_y(x,y) = -2y$
$g_x(x,y) = 3y^2 - 3x^2$
$g_y(x,y) = 6xy$

Create Jacobian, $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x,y)=(1,1)} = \begin{bmatrix} 6 & -2 \\ 0 & 6 \end{bmatrix}$

$\det(J(1,1)) = 6 \cdot 6 - (-2)(0) = 36$

$\therefore (J(1,1))^{-1} = \frac{1}{36} \cdot \begin{bmatrix} 6 & 2 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix}$, which is the matrix in (a)

So, this (2×2) matrix is the inverse Jacobian solved at $(x_0, y_0) = (1,1)$

(c) Iterate on (0.1) using Newton's Method using $(x_0, y_0) = (1,1)$

Code provided in PDF submission.

Output: $(x^*, y^*) = (0.5, 0.866)$, in 5 iterations

(d) From my results in part (a) and (c), it seems like the solution is $(0.5, 0.866)$

From system, we can create the following substitution:

$3x^2 = y^2$

$\Rightarrow g(x,y) = 3x \cdot (3x^2) - x^3 - 1 = 0 \Rightarrow 9x^3 - x^3 - 1 = 0 \Rightarrow 8x^3 - 1 = 0$

$\Rightarrow (2x)^3 - 1 = 0 \Rightarrow x = \frac{1}{2}$, or x is a complex number, which we will not consider.

If $x = \frac{1}{2}$, $y^2 = 3(\frac{1}{2})^2 \Rightarrow y = \pm \frac{\sqrt{3}}{2}$.

Therefore, there are 2 possible choices, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ or $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Since we are finding the solution closer to $(1,1)$, our answer is $(\frac{1}{2}, \frac{\sqrt{3}}{2}) \approx (0.5, 0.866)$ ✓

2.) Consider the nonlinear system of equations:

$$\begin{cases} x = \frac{1}{\sqrt{2}}\sqrt{1+(x+y)^2} - \frac{2}{3} \\ y = \frac{1}{\sqrt{2}}\sqrt{1+(x-y)^2} - \frac{2}{3} \end{cases}$$

<u>Theorem 10.6:</u>

Let $D = \{(x_1, x_2, \dots, x_n)^T : a_i \le x_i \le b_i\}$ some collection of constants $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Suppose that G is a continuous function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with the property that $G(x) \in D$ whenever $x \in D$. Then G has a fixed point in D. Moreover, supposed that all the component functions of G have continous partial derivative and a constant $K \le 1$ exists with

$$\left| \frac{\partial g_i(x)}{\partial x_j} \right| \le \frac{K}{n}$$

whenever $x \in D$, for each $j = 1, \dots, n$ and each component function g_i. Then the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrary selected $x^{(0)}$ in D and generated by

$$x^{(k)} = G(x^{(k-1)}), \text{ for each } k \ge 1$$

converges to the unique fixed point $p \in D$ and

$$\|x^{(k)} - p\|_\infty \le \frac{K^k}{1-K} \|x^{(1)} - x^{(0)}\|_\infty$$

Based on this theorem, find a region D in the xy-plane for which the fixed point iteration

$$\begin{cases} x_{n+1} = \frac{1}{\sqrt{2}}\sqrt{1+(x_n+y_n)^2} - \frac{2}{3} \\ y_{n+1} = \frac{1}{\sqrt{2}}\sqrt{1-(x_n-y_n)^2} - \frac{2}{3} \end{cases}$$

Let G be defined, then, as the Jacobian of the nonlinear system of equations. That is, let

$$G = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}$$ for the (2×2) nonlinear system.

From Theorem 10.6, it suffices that $\|G(\bar{p})\| < 1$, where $\bar{p}$ is the vector that defines the fixed point in $\mathbb{R}^2$, and $\|\cdot\|$ defines the l_2 norm.

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{2}}\sqrt{1+(x+y)^2}\cdot\frac{2}{3}\right) = \frac{x+y}{\sqrt{2}\cdot\sqrt{1+(x+y)^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{2}}\cdot\sqrt{1+(x+y)^2}-\frac{2}{3}\right) = \frac{x+y}{\sqrt{2}\sqrt{1+(x+y)^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{2}}\sqrt{1+(x-y)^2}-\frac{2}{3}\right) = \frac{x-y}{\sqrt{2}\sqrt{1+(x-y)^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{2}}\sqrt{1+(x-y)^2}-\frac{2}{3}\right) = \frac{-(x-y)}{\sqrt{2}\sqrt{1+(x-y)^2}}$$

$$\therefore G = \begin{bmatrix} \frac{x+y}{\sqrt{2}\cdot\sqrt{1+(x+y)^2}} & \frac{x+y}{\sqrt{2}\cdot\sqrt{1+(x+y)^2}} \\ \frac{x-y}{\sqrt{2}\sqrt{1+(x-y)^2}} & \frac{-(x-y)}{\sqrt{2}\cdot\sqrt{1+(x-y)^2}} \end{bmatrix}$$

$$\|G(\bar{p})\| = \|G(x_0, y_0)\|$$

This fixed point iteration must converge when $\|G(x_0, y_0)\| < 1$

3.) Let $f(x,y)$ be a smooth function such that $F(x,y)=0$ defines a smooth curve in the xy-plane. Want to Find a point on the curve in the neighborhood of a starting guess (x_0, y_0) that is off the curve.

(a) Derive the iteration scheme

$$\begin{cases} x_{n+1} = x_n - dF_x \\ y_{n+1} = y_n - dF_y \end{cases}, \quad \text{where } d = \frac{f}{(F_x^2 + F_y^2)}$$

, and $f, F_x,$ and F_y are evaluated at (x_n, y_n)

Hint: Find (x_{n+1}, y_{n+1}) that lies on gradient line (x_n, y_n) and obeys $f(x,y)=0$

$$\nabla F(x_n, y_n) = \left[\frac{\partial F(x_n)}{\partial x_n}, \frac{\partial F(y_n)}{\partial y_n}\right] = \left[F_x(x_n), F_y(y_n)\right]$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \Delta x_n \\ \Delta y_n \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_n \\ \Delta y_n \end{bmatrix} = -J(x_n, y_n)^{-1} \cdot \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$, applying Newton's Method

Not quite sure where to continue from here

(b) $x^2 + 4y^2 + 4z^2 = 16 \Rightarrow F(x,y,z) = 0 \Rightarrow x^2 + 4y^2 + 4z^2 - 16 = 0$

$x_0 = y_0 = z_0 = 1, \therefore \bar{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Created the given iteration in Python to solve numerically

$F_x(x,y,z) = 2x$

$F_y(x,y,z) = 8y$

$F_z(x,y,z) = 8z$

Output: ier = 0

[1.0936, 1.3603, 1.3603]

# of iterations: 4

In my code, it prints the error of each iteration, If we take the difference in error between iterations, it becomes apparent that the function converges quadratically.