Homework2

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APPM 4600

1.) (a) Show that $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$.

$(1+x)^n - nx - 1 = o(x)$ as $x \to 0$ if $\lim_{x\to 0} \frac{|(1+x)^n - nx - 1|}{|x|} = 0$

$$\lim_{x\to 0} \frac{|(1+x)^n - nx - 1|}{|x|} = \frac{|(1+0)^n - n(0) - 1|}{0} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \frac{|n(1+x)^{n-1} - n|}{1} = \frac{|n(1)^{n-1} - n|}{1} = 0 \checkmark$$

(b) Show that $x \sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$.

$x \cdot \sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$ if $\exists$ a positive constant M st

$\lim_{x\to 0} \frac{|x \sin(\sqrt{x})|}{|x^{3/2}|} \le M$ $\forall$ values of x in neighborhood of 0

$$\lim_{x\to 0} \frac{|x \cdot \sin(\sqrt{x})|}{|x^{3/2}|} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \frac{|\sin(\sqrt{x}) + 2\sqrt{x}(\cos(\sqrt{x}))|}{\frac{3}{2}\sqrt{x}} = \lim_{x\to 0} \frac{|\sin(\sqrt{x})|}{\frac{3}{2}\sqrt{x}} + \lim_{x\to 0} \frac{|2\cos(\sqrt{x})|}{\frac{3}{2}}$$

0

$$= \frac{2 \cdot \cos(\sqrt{0})}{\frac{3}{2}} = \frac{4}{3} \ne 0 \checkmark$$

(c) Show that $e^{-t} = o(\frac{1}{t^2})$ as $t \to \infty$. Following the definition of $o(t)$ as shown is part (a)

$$\lim_{t\to\infty} \frac{|e^{-t}|}{|\frac{1}{t^2}|} = \lim_{t\to\infty} \frac{|t^2|}{|e^t|} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{t\to\infty} \frac{|2t|}{|e^t|} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{t\to\infty} \frac{2}{|e^t|} = 0 \checkmark$$

(d) Show that $\int_0^\varepsilon e^{-x^2} dx = O(\varepsilon)$ as $\varepsilon \to 0$. Following the definition of $O(\varepsilon)$ as shown in (b)

$$\lim_{\varepsilon\to 0} \frac{|\int_0^\varepsilon e^{-x^2} dx|}{|\varepsilon|} = \lim_{\varepsilon\to 0} \frac{|\frac{\sqrt{\pi}}{2} \operatorname{erf}(\varepsilon)|}{|\varepsilon|} = \frac{0}{0} \overset{L'H}{=} \lim_{\varepsilon\to 0} \frac{|\frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} e^{-x^2}|}{|1|}$$

$$= \lim_{\varepsilon\to 0} \left| \frac{1}{e^{x^2}} \right|$$

2.)

2. Consider solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The exact solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the inverse of $\mathbf{A}$ is $\begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$. In this problem we will investigate a perturbation in $\mathbf{b}$ of $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$ and the numerical effects of the condition number.

(a) Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx.

Exact Problem:

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Perturbed problem: $\underset{\text{pert. soln.}}{A\hat{x}} = b + \delta b \Rightarrow \hat{x} = A^{-1}b + A^{-1}\delta b$

$$F(\text{pert.}) - f(\text{original}) = \underbrace{A^{-1}b + A^{-1}\delta b}_{\hat{x}} - \underbrace{A^{-1}b}_{x}$$

$$= A^{-1}\delta b$$

$\therefore$ Perturbed Problem: $$\boxed{\tilde{x} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix} \cdot \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}}$$

(b) What is the condition number of $\mathbf{A}$?

Numerator of relative condition # : $\dfrac{\|A^{-1}\delta b\|}{\|x\|}$

$$\Rightarrow K = \frac{\|A^{-1}\delta b\|}{\|x\|} \cdot \frac{\|b\|}{\|\delta b\|}$$

Bound to get a more practical Formula

$$K = \frac{\|A^{-1}b\|}{\|x\|} \cdot \frac{\|b\|}{\|\delta b\|} \overset{\text{by Cauchy-Schwarz}}{\leq} \frac{\|A^{-1}\|\cdot\|\cancel{\delta b}\|}{\|x\|} \cdot \frac{\overset{= Ax}{\|b\|}}{\cancel{\|\delta b\|}}$$

$$= \frac{\|A^{-1}\|\cdot\|Ax\|}{\|x\|} \overset{\text{by C.S.}}{\leq} \|A^{-1}\|\cdot\|A\| = K(A)$$

$\therefore \; K(A) = \|A^{-1}\|\cdot\|A\|$

$\|x\|$

$\therefore \quad \kappa(A) = \|A^{-1}\| \cdot \|A\|$

$\|A\| = \sigma_1$ (largest sing. value of A)

$\|A^{-1}\| = \frac{1}{\sigma_n}$ $\quad \therefore \quad \kappa(A) = \|A^{-1}\| \cdot \|A\|$

Using MATLAB to evaluate the norm

MATLAB SCRIPT:

```
clear all
close all
clc
format long g

% Creat Matrices A and A^-1
A = [1/2 1/2; (1/2)*(1+10^(-10)) (1/2)*(1-10^(-10))]
A_inv = inv(A)

% Find the singular values of both Matrices
S1 = svd(A)
S2 = svd(A_inv)

% Find condition number
kappa = S1(1) * S2(2)
```

MATLAB OUTPUT

```
A =

                 0.5                 0.5
       0.50000000005       0.49999999995


A_inv =

   -9999999170.59636    9999999171.59636
    9999999172.59636   -9999999171.59636


S1 =

                       1
    5.00000615905423e-11


S2 =

        19999998343.1927
       0.99999987267217


kappa =

       0.99999987267217
```

$$\boxed{\therefore \ \kappa(A) = 0.99999987267217}$$

(c) Let Δb_1 and Δb_2 be of magnitude 10^{-5}; not necessarily the same value. What is the relative error in the solution? What is the relationship between the relative error, the condition number, and the perturbation. Is the behavior different if the perturbations are the same? Which is more realistic: same value of perturbation or different value of perturbation?

Let $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 10^{-5} \\ 2 \cdot 10^{-5} \end{bmatrix}$

Relative error $= \dfrac{|x_1 - \tilde{x}_1|}{|x_1|} = 0.999989999999173$

Relative error $= \dfrac{|x_2 - \tilde{x}_2|}{|x_2|} = 1.00000999980083$

3) 3. Let $f(x) = e^x - 1$

(a) What is the relative condition number $\kappa(f(x))$? Are there any values of x for which this is ill-conditioned?

$$K(f(x)) = \frac{|f'(c)|\cdot|x|}{|f(x)|},$$

$$K(e^x-1) = \frac{|e^c|\cdot|x|}{|e^x-1|}, \text{ pick } c \text{ to be } 0 \therefore \boxed{K(e^x-1) = \frac{|x|}{|e^x-1|}}$$

This is ill-conditioned for values of x near 0

(b) Consider computing $f(x)$ via the following algorithm:

```
1: y = math.e^x
2: return y -1
```

Is this algorithm stable? Justify your answer

After attempting different x values and a small perturbation, I found that small changes to x resulted in changes in y of the same order of magnitude. However, it is unstable near 0, so it is unstable.

(c) Let x have the value $9.999999995000000 \times 10^{-10}$, in which case $f(x)$ is equal to 10^{-9} up to 16 decimal places. How many correct digits does the algorithm listed above give you? Is this expected?

Inputing the above x value yielded the following value for f(x): $1.000000082740371 \cdot 10^{-9}$, which implies that the correct number of digits is 8.

(d) Find a polynomial approximation of $f(x)$ that is accurate to 16 digits for $x = 9.999999995000000\times 10^{-10}$. Hint: use Taylor series, and remember that 16 digits of accuracy is a relative error, not an absolute one.

Taylor expansion of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=1} \frac{x^n}{n!}$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\therefore \text{ Absolute Error } = \left|(e^x-1) - \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\right|$$

$$\text{Relative Error } = \frac{\left|(e^x-1) - \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\right|}{|(e^x-1)|}$$

When using the machine to solve for the number of terms needed, I found that I needed two of the Taylor Polynomial terms

(e) Verify that your answer from part (d) is correct.

Using information from part (g), I used expm1 to verify my results

4) All work from question 4 done in Python. Here are screenshots of Python output:

(a)

```
$ python3 Problem4a.py
The Sum is:  -20.686852364346837
```

(b)