Homework 4

Tony Samour
APPM 4600
29 SEP 2023

1.) $$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right),$$ Where T_s is const. temp., T_i is initial temp of soil, α is therm. cond.

Assume $T_i = 20°C$, $T_s = -15°C$, $\alpha = 0.138 \cdot 10^{-6}\ m^2/s$

(a) Determine depth ($x = ?$) of water main so that it will only freeze after 60 days @ this temp.

Formulate this as a root finding problem finding $f(x) = 0$.

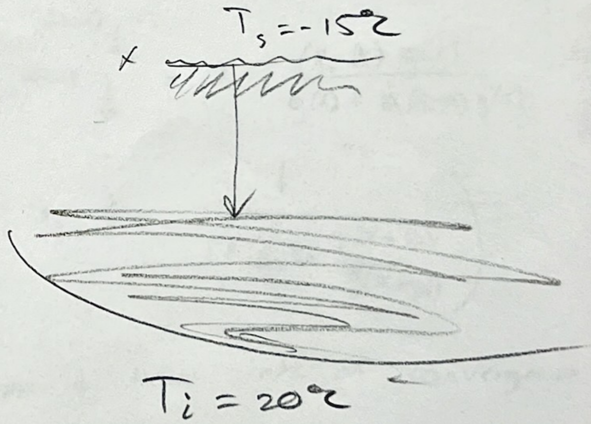
$$60 \text{ days}\left(\frac{24 \text{ hours}}{\text{day}}\right)\left(\frac{60 \text{ min}}{\text{hr}}\right)\left(\frac{60 \text{ sec}}{\text{min}}\right) = 5.184 \cdot 10^{6} \text{ seconds}$$

$$T(x,t) = T_s + (T_i - T_s) \cdot \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Plot $$\underbrace{T(x, 5.184e6)}_{=0°C} = -15°C + (35°C) \cdot \text{erf}\left(\frac{x}{2\sqrt{0.138 \cdot 10^{-6} \cdot 5.184 \cdot 10^{6}}}\right)$$

$$f(x) = -15 + 35 \cdot \text{erf}(0.59115 \cdot x)$$

$$f'(x) = \frac{35 \cdot e^{-x^2/4\alpha \cdot t}}{\sqrt{\pi} \cdot \sqrt{\alpha t}}$$



Plotted for $[0, 5]$

(b) Compute an approximate depth using Bisection w/ $a_0 = 0m$, $b_0 = \bar{x}\ m$

Output included

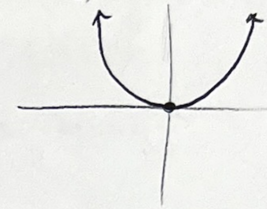
(c) Compute using Newton's method

Output included for both $p_0 = 0.01m$ and $p_0 = 5m$

2.) Let $f(x)$ denote a function with root α of multiplicity m

(a) Write a formal mathematical def. of what this means.

For the example of a parabola w/ root mult. 2:



This means that $f(\alpha) = 0$ and $f'(\alpha) = 0$

Expanding this to a more general case,

The multiplicity m of any root α of some function $f(x)$ implies that $f^{(m)}(\alpha) \neq 0$ for $m \in \mathbb{Z}^+$, but $f^{(k)}(\alpha) = 0 \ \forall k \in \mathbb{Z}^+$ s.t. $k < m$.

(b) Show that Newton's Method applied to $f(x)$ only converges linearly to α

Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $f(x) = (x-\alpha)^m \cdot g(x)$, where $g(\alpha) \neq 0$. If we consider a modified Newton's Method:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = x_n - \frac{m(x_n-\alpha)^m \cdot g(x_n)}{m(x_n-\alpha)^{m-1} g(x_n) + g'(x_n)(x_n-\alpha)^m} \cdot \frac{(x_n-\alpha)^{-(m-1)} \cdot \frac{1}{m}}{(x_n-\alpha)^{-(m-1)} \cdot \frac{1}{m}} = x_n - \frac{(x_n-\alpha)\, g(x_n)}{g(x_n) + \frac{1}{m}(x_n-\alpha)\, g'(x_n)} \cdot \frac{\frac{1}{g(x_n)}}{\frac{1}{g(x_n)}}$$

$$= x_n - \frac{(x_n-\alpha)}{1 + \frac{1}{m}(x_n-\alpha)\frac{g'(x_n)}{g(x_n)}}$$

Set $y_n = x_n - \alpha$

$$\Rightarrow y_{n+1} = y_n\left(1 - \frac{1}{1 + \frac{1}{m} y_n \frac{g'(\alpha+y_n)}{g(\alpha+y_n)}}\right)$$

First order approximation gives $y_{n+1} \approx y_n^2 \cdot \frac{1}{m} \cdot \frac{g'(\alpha+y_n)}{g(\alpha+y_n)}$

$y_{n+1} \approx y_n\left(\frac{m-1}{m} + \frac{1}{m} y_n \frac{g'(x+y_n)}{g(\alpha+y_n)}\right) \approx \frac{m-1}{m} y_n$, which implies a linear rate of convergence

(c) Show that fixed-point iter. applied to $g(x) = x - m \cdot \frac{f(x)}{f'(x)}$ is 2^{nd} order conv.

$$g'(x) = 1 - m\left(\frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}\right)\Bigg|_{m=1} \Rightarrow |g'(x)| = \left|\frac{f(x)f''(x)}{[f'(x)]^2}\right|_{x=\alpha} = 0, \text{ if } f'(x) \neq 0$$

$$|g''(x)| = \left|\frac{[f'(x)]^2 \cdot [f'(x)f''(x) + f''(x) \cdot f(x)] - [f(x) \cdot f''(x) \cdot 2[f'(x)] \cdot f''(x)]}{[f'(x)]^2}\right|_{x=\alpha} = \left|\frac{f''(\alpha)}{f'(\alpha)}\right|$$

$$|g''(\alpha)| = \left|\frac{f''(\alpha)}{f'(\alpha)}\right| < M \text{ if } \sup_{x\in I}|f''(x)| \cdot \frac{1}{\sup_{x\in I}|f'(x)|} < M \text{ and } f'(x) \neq, \ \forall x \in I$$

Which, by fixed-point theory must be second-order convergent.

(d) What does (c) provide for roots α with multiplicity >1?

Recalling that roots of multiplicity >1 must converge linearly, Using Fixed point theory, we know that $g(x)$ converges to root α linearly if

$|g'(\alpha)| < 1$, and that the convergence rate will be $|g'(\alpha)|$.

From (c): $$|g'(x)| = \left| 1 - m\left(\frac{[f'(x)]^2 - f(x)\cdot f''(x)}{[f'(x)]^2} \right) \right|$$

$$|g'(\alpha)| < 1 \Rightarrow \left| 1 - m\left(\frac{[f'(x)]^2 - f(x)\cdot f''(x)}{[f'(x)]^2} \right) \right| < 1$$

$$\therefore \; 0 < m\left(\frac{[f'(x)]^2 - f(x)\cdot f''(x)}{[f'(x)]^2} \right) < 1$$

3.) Beginning w/ the definition of order of convergence of a sequence $\{x_k\}_{k=1}^{\infty}$ that converges to α, derive a relationship between $\log(|x_{n+1} - \alpha|)$ and $\log(|x_n - \alpha|)$. What is the order p?

Definition 2.7: $\{x_n\}_{k=1}^{\infty}$ converges to α of order p, with asymptotic error λ

where $$\lim_{k\to\infty} \frac{|x_{k+1} - \alpha|}{|x_n - \alpha|^p} = \lambda$$

Take the natural log of both sides

$$\lim_{k\to\infty} \log\left| \frac{|x_{k+1} - \alpha|}{|x_n - \alpha|^p} \right| = \log|\lambda|$$

$$\Rightarrow \lim_{k\to\infty} \left| \log|x_{k+1} - \alpha| - p\cdot\log|x_n - \alpha| \right| = \log|\lambda|$$, by log properties

$$\Rightarrow \lim_{k\to\infty} \left|\log|x_{k+1} - \alpha|\right| - p\cdot\lim_{k\to\infty}\left|\log|x_n - \alpha|\right| = \log|\lambda|$$, by limit properties

$$\Rightarrow p\cdot\lim_{k\to\infty} \log|x_n - \alpha| = \lim_{k\to\infty} \log|x_{k+1} - \alpha| - \log|\lambda|$$

$$\Rightarrow \boxed{p = \lim_{k\to\infty} \frac{\log\left|\frac{x_{k+1} - \alpha}{\lambda}\right|}{\log|x_k - \alpha|}}$$

4.) Consider finding root of function $f(x)$ where

$f(x) = e^{3x} - 27x^6 + 27x^4 e^x - 9x^2 e^{2x}$ in the interval $(3,5)$

$$f'(x) = 3e^{3x} - 6 \cdot 27x^5 + 27\left[4x^3e^x + x^4e^x\right] - 9\left[2xe^{2x} + 2x^2e^{2x}\right]$$

(i) Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $p_0 = 4$ (center of interval)

(ii) Modified Newton's Method: $x_{n+1} = x_n - m \cdot \dfrac{f(x_n)}{f'(x_n)}$

Visually, from looking at a plot of $f(x)$ on desmos, it appears that the root has multiplicity of $m = 2$. Again, $p_0 = 4.0$

(iii) Let $g(x) = \dfrac{f(x)}{f'(x)} \Rightarrow g'(x) = \dfrac{f'(x) \cdot f'(x) - f(x) \cdot f''(x)}{[f'(x)]^2}$

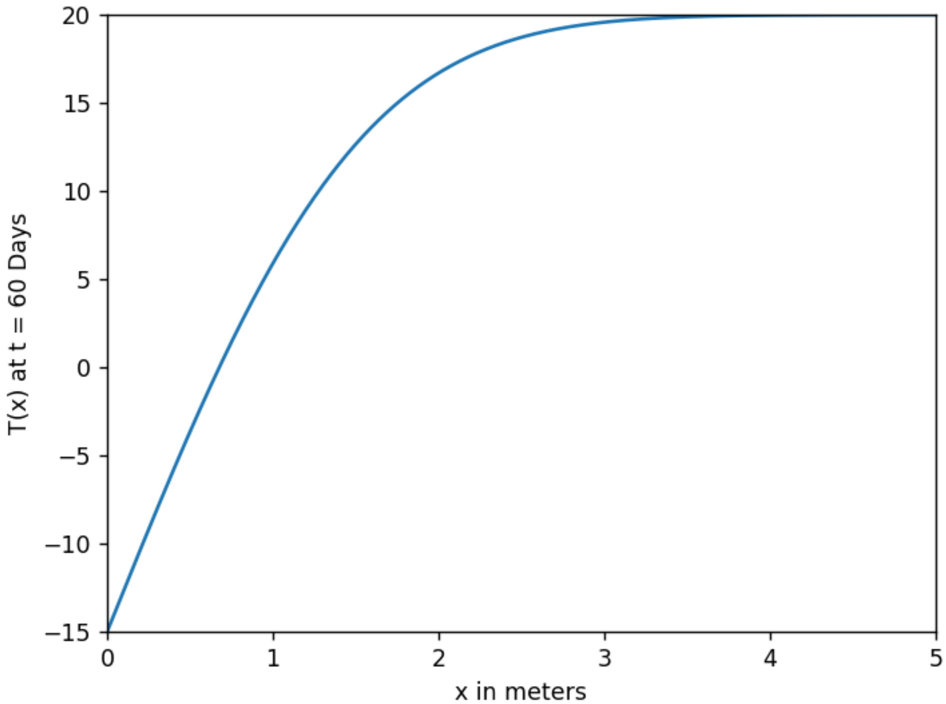
$$f''(x) = 9e^{3x} - 30 \cdot 27 \cdot x^4 + 27\left[12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x\right] - 9\left[2e^{2x} + 4xe^{2x} + 4xe^{2x} + 4x^2e^{2x}\right]$$

Let $p_0 = 4$

$$\therefore x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = \frac{\frac{f(x_n)}{f'(x_n)}}{\frac{f'(x_n)^2 - f(x_n) \cdot f''(x_n)}{f'(x_n)^2}} = x_n - \frac{f(x_n) \cdot f'(x_n)}{f'(x_n)^2 - f(x_n) f''(x_n)}$$

Although (iii) converged the fastest by a long shot, the extra work to find $f''(x)$ was not very efficient in this particular example. For other functions I think that method (iii) would do much better.

1a) Plot from [0,5]:



1b) Approx. Root using Bisection Method

```
$ python3 Problem1b.py
the approximate root is 0.6769618544819167
the error message reads: 0
f(astar) = -3.925748615074535e-13
```

1c) Approx root using Newton's method

For p0 = 0.01:

```
$ python3 Problem1c.py
the approximate root is 6.7696185448193646e-01
the error message reads: 0
Number of iterations: 4
```

For p0 = 5:

```
$ python3 Problem1c.py
C:\Users\tonys\Documents\APPM4600\Samour_APPM4600\Homework\Homework4\Problem1c.p
y:47: RuntimeWarning: divide by zero encountered in scalar divide
  p1 = p0-f(p0)/fp(p0)
C:\Users\tonys\Documents\APPM4600\Samour_APPM4600\Homework\Homework4\Problem1c.p
y:47: RuntimeWarning: invalid value encountered in scalar subtract
  p1 = p0-f(p0)/fp(p0)
the approximate root is              nan
the error message reads: 1
Number of iterations: 99
```

2) No code output for problem 2

3) No code output for problem 3

4)
(i) Newton's method, p0 = 4

```
$ python3 Problem4i.py
the approximate root is 3.7330650107477870e+00
the error message reads: 0
Number of iterations: 27
```

(ii) Modified Newton's Method from problem 2c

```
$ python3 Problem4ii.py
the approximate root is 3.7330785768836017e+00
the error message reads: 0
Number of iterations: 17
```

(iii) Modified Newton's Method using g(x) = f(x)/f'(x)

```
$ python3 Problem4iii.py
the approximate root is 3.7330805671760268e+00
the error message reads: 0
Number of iterations: 6
```