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APPM 4600

1.)

1. Assume the error in an integration formula has the asymptotic expansion

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots$$

Generalize the Richardson extrapolation process to obtain an estimate of I with an error of order $\frac{1}{n^2\sqrt{n}}$. Assume that three values I_n, $I_{n/2}$ and $I_{n/4}$ have been computed.

① $I = I_n + \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots$

② $I = I_{n/2} + \frac{C_1}{\frac{n}{2}\sqrt{\frac{n}{2}}} + \frac{C_2}{(\frac{n}{2})^2} + \frac{C_3}{(\frac{n}{2})^2\sqrt{\frac{n}{2}}} + \frac{C_4}{(\frac{n}{2})^3} + \cdots$

$= I_{n/2} + \frac{2\sqrt{2}C_1}{n\sqrt{n}} + \frac{4C_2}{n^2} + \frac{4\sqrt{2}C_3}{n^2\sqrt{n}} + \frac{8C_4}{n^3} + \cdots$

② $- 2\sqrt{2}$① eliminates the $\frac{1}{n\sqrt{n}}$ term. Note: all constants are arbitrary, i.e $\alpha C_i = C_i$

$\therefore\ I - 2\sqrt{2}I = I_n - 2\sqrt{2}I_n + \cancel{\frac{2\sqrt{2}C_1}{n\sqrt{n}}} + \frac{4C_2}{n^2} + \frac{4\sqrt{2}C_3}{n^2\sqrt{n}} + \frac{8C_4}{n^3} - 2\sqrt{2}\left(\cancel{\frac{C_1}{n\sqrt{n}}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3}\right) + \cdots$

$I = \frac{I_{n/2} - 2\sqrt{2}I_n}{1 - 2\sqrt{2}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots$

③ $I = \frac{I_{n/4} - 2\sqrt{2}I_{n/2}}{1 - 2\sqrt{2}} + \frac{C_2}{(\frac{n}{2})^2} + \frac{C_3}{(\frac{n}{2})^2\sqrt{\frac{n}{2}}} + \frac{C_4}{(\frac{n}{2})^3} + \cdots$

$= \frac{I_{n/4} - 2\sqrt{2}I_{n/2}}{1 - 2\sqrt{2}} + \frac{4C_2}{n^2} + \frac{4\sqrt{2}C_3}{n^2\sqrt{n}} + \frac{8C_4}{n^3} + \cdots$

③ $- 4\cdot$② eliminates the $\frac{1}{n^2}$ term.

$I - 4I = \frac{I_{n/4} - 2\sqrt{2}I_{n/2}}{1 - 2\sqrt{2}} - \frac{4(I_{n/2} - 2\sqrt{2}I_n)}{1 - 2\sqrt{2}} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots$

$$\therefore\ \boxed{I = \frac{8\sqrt{2}I_n - (2\sqrt{2}-4)I_{n/2} + I_{n/4}}{6\sqrt{2} - 3} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots}$$

2.)

2. Use the transformation $t = x^{-1}$ and Composite Simpson's rule with 5 nodes to approximate

$$\int_1^\infty \frac{\cos(x)}{x^3}dx.$$

$t = x^{-1}$ $\qquad x = \infty \Rightarrow t = 0$

$dt = -x^{-2}dx$ $\qquad x = 1 \Rightarrow t = 1$

$dt = -x^{-2}\,dx \qquad x = 1 \Rightarrow t = 1$

$$\therefore \; I = \int_1^0 \frac{\cos(\frac{1}{t})}{x^3} \cdot -x^2\,dt = \int_0^1 \frac{\cos(\frac{1}{t})}{(\frac{1}{t})}\,dt = \int_0^1 t\cos(\tfrac{1}{t})\,dt$$

Using Composite Simpson's Rule w/ 5 nodes:

Output:

```
$ python3 Problem2.py
C:\Users\tonys\Documents\APPM4600\Samour_APPM4600\Homework\Homework11\Problem2.p
y:23: IntegrationWarning: The maximum number of subdivisions (50) has been achie
ved.
  If increasing the limit yields no improvement it is advised to analyze
  the integrand in order to determine the difficulties.  If the position of a
  local difficulty can be determined (singularity, discontinuity) one will
  probably gain from splitting up the interval and calling the integrator
  on the subranges.  Perhaps a special-purpose integrator should be used.
  Ieval = quad(f, a, b)
Composite Simpsons Approximation:  0.0016884969113066994 Num Evaluations:  11
Simpsons Error:  [0.01642898 0.00168772]
Built-In Quadrature:  (0.01811747827025022, 7.80629135889484e-07)
```

Notes:

- I was running into a divide by 0 error due to the $\cos(\frac{1}{x})$. I could not figure out a way to rewrite this, so I set my lower bound to 1e-10.
- Simpson's Method cannot be used on an odd number of nodes, so I used $2(5) = 10$ nodes.

4.) 4. Given the linear system

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

(a) Find the value(s) of α for which the system has no solutions.
(b) Find the value(s) for α for which the system has an infinite number of solutions.
(c) Assuming a unique solution exists for a given α, find the solution.

(a) $$\begin{bmatrix} 2 & -6\alpha \\ 3\alpha & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -6\alpha & 3 \\ 3\alpha & -1 & \frac{3}{2} \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{cc|c} 1 & -3\alpha & \frac{3}{2} \\ 3\alpha & -1 & \frac{3}{2} \end{array}\right] \xrightarrow{R_2 - 3\alpha R_1} \left[\begin{array}{cc|c} 1 & -3\alpha & \frac{3}{2} \\ 0 & -1+9\alpha^2 & \frac{3}{2} - \frac{9\alpha}{2} \end{array}\right] \Rightarrow$$

There are no solutions when $-1 + 9\alpha^2 = 0$

$\therefore \; 9\alpha^2 = 1$

$$\boxed{\alpha = -\tfrac{1}{3}}$$

(b)

$$\left[\begin{array}{cc|c} 2 & -6\alpha & 3 \\ 3\alpha & -1 & \frac{3}{2} \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{cc|c} 1 & -3\alpha & \frac{3}{2} \\ 3\alpha & -1 & \frac{3}{2} \end{array}\right] \Rightarrow$$

there is no solution solution when $1 - 3\alpha = 3\alpha - 1 \Rightarrow \boxed{\alpha = \tfrac{1}{3}}$

(c) $Ax = b \Rightarrow x = A^{-1}b$

$$\det(A) = -2 - (-18\alpha^2) = -2 + 18\alpha^2$$

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} -1 & 6\alpha \\ -3\alpha & 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{18\alpha^2-2} & \frac{6\alpha}{18\alpha^2-2} \\ \frac{-3\alpha}{18\alpha^2-2} & \frac{2}{18\alpha^2-2} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = \frac{-3}{18\alpha^2-2} - \frac{9\alpha}{18\alpha^2-2}$$

$$x_2 = \frac{-9\alpha}{18\alpha^2-2} + \frac{3}{18\alpha^2-2} \quad \Rightarrow \quad \boxed{x = \frac{-9\alpha \pm 3}{18\alpha^2-2}}$$