Homework 1

Wednesday, September 6, 2023 11:13 AM

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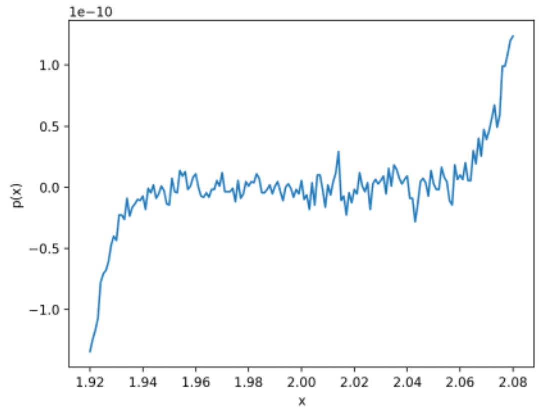
APPM 4600

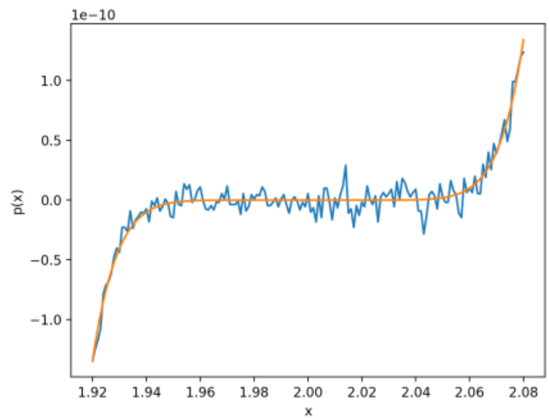
1.) 1. Consider the polynomial
$p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512.$

i. Plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ (i.e. $x = [1.920 : 0.001 : 2.080]$;) evaluating p via its coefficients.

* All code attached separately



ii. Produce the same plot again, now evaluating p via the expression $(x-2)^9$.



iii. What is the difference? What is causing the discrepancy? Which plot is correct?

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# iii.) The first plot of the expanded form is very sharp and has a lot of jumps,
# whereas the second plot (in red) of the binomial is very smooth. This discrepancy
# is due to the machine having to compute the multiple terms, add and subtract them,
# all while keeping up to four significant figures. This extra computation leads to the
# loss of accuracy. We can tell that the (x-2)**9 plot is correct because we know that
# there is only 1 root at x = 2, whereas the expanded plot has multiple roots.
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2.) 2. How would you perform the following calculations to avoid cancellation? Justify your answers.

i. Evaluate $\sqrt{x+1}-1$ for $x \simeq 0$.

To avoid cancellation, I would multiply by the conjugate to eliminate the square root.

$$(\sqrt{x+1}-1)\cdot\frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)} = \frac{x+1-1}{\sqrt{x+1}+1} = \boxed{\frac{x}{\sqrt{x+1}+1}}$$

, which no longer has cancellation issues when $x \approx 0$.

ii. Evaluate $\sin(x)-\sin(y)$ for $x \simeq y$.

In this instance, multiplying by the conjugate results in the following:

$$\frac{\sin^2(x)-\sin^2(y)}{\sin(x)+\sin(y)}$$

, which still results in cancellation when $x \approx y$

Instead, I will try using trig identities.

$$\sin(x)-\sin(y) = 2\cos\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x-y}{2}\right),$$

which is no longer problematic since the $x-y$ is contained within the function.

iii. Evaluate $\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$.

$$\frac{1-\cos(x)}{\sin(x)}\cdot\frac{(1+\cos(x))}{(1+\cos(x))}$$

Multiply by conjugate of numerator

$$\frac{1-\cos^2(x)}{\sin(x)+\sin(x)\cos(x)}$$

3) 3. Find the second degree Taylor polynomial $P_2(x)$ for $f(x) = (1 + x + x^3)\cos(x)$ about $x_0 = 0$.

Taylor Polynomial Formula: $P_n(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(c)}{n!}(x-c)^n$

$f(0) = 1$

$F'(x) = (1+3x^2)\cos(x) - \sin(x)(1+x+x^3)\Big|_{x_0=0} = 1$

$f''(x) = \left[(6x)\cdot\cos(x) - (1+3x^2)\sin(x)\right] - \left[\cos(x)(1+x+x^3) + \sin(x)(1+3x^2)\right]\Big|_{x_0=0} = -1$

$\therefore P_2(x) = 1 + x - \frac{1}{2}x^2$

(a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for the error $|f(0.5) - P_2(0.5)|$ using the error formula and compare it to the actual error.

$P_2(0.5) = 1 + 0.5 - \frac{1}{2}(0.5)^2 = 1.375$

$R_2(x) = \left|\frac{F^{n+1}(c)}{(n+1)!}\cdot(x-c)^{n+1}\right|$ — Lagrange error bound

$F'''(x) = \left[(6\cdot\cos(x) - (6x)\cdot\sin(x)) - (6x\sin(x) + (1+3x^2)\cos(x))\right]$

$- \left[(-\sin(x)\cdot(1+x+x^3) + (1+3x^2)(\cos(x)) + (\cos(x)(1+3x^2) + (6x\sin(x))\right]\Big|_{x=0}$

$= 6 - 0 - 0 - 1 - (-0) - 1 - 1 + 0 = 3$

$F'''(0.5) = -2.878 \qquad \therefore M = 3$

$R_2(0.5) \leq \frac{3}{3!}(0.5-0)^3 = 0.0625$ (Lagrange Error Bound)

$|f(0.5) - P_2(0.5)| = |1.62494 - 1.375| = 0.249938$ (abs. error)

$\frac{|F(0.5) - P_2(0.5)|}{|P_2(0.5)|} = \frac{|1.62494 - 1.375|}{1.375} = 0.181773$ (relative error)

(b) Find a bound for the error $|f(x) - P_2(x)|$ when $P_2(x)$ is used to approximate $f(x)$. This will be a function of x.

$\frac{|F(x) - P_2(x)|}{|P_2(x)|} = \left|\frac{(1+x+x^3)\cdot\cos(x) - (1+x-\frac{x^2}{2})}{(1+x-\frac{x^2}{2})}\right|$

(c) Approximate $\int_0^1 f(x)dx$ using $\int_0^1 P_2(x)dx$.

$$\int_0^1 f(x)\,dx \approx \int_0^1 P_2(x)\,dx$$

$$\int_0^1 P_2(x)\,dx = \int_0^1 \left(1 + x - \frac{x^2}{2}\right)dx$$

$$= \left(x + \frac{x^2}{2} - \frac{x^3}{6}\right)\Big|_0^1 = \left(1 + \frac{1}{2} - \frac{1}{6}\right) - 0$$

$$= \frac{8}{6} \text{ or } 1.\overline{3}$$

$$\boxed{\int_0^1 P_2(x)\,dx = 1.\overline{3}}$$

(d) Estimate the error in the integral.

Since we integrate the polynomial, the error will increase to be ≈ 0.5

4.) 4. Consider the quadratic equation $ax^2 + bx + c = 0$ with $a = 1, b = -56, c = 1$.

(a) Assume you can calculate the square root with 3 correct decimals (e.g. $\sqrt{(2)} \approx 1.414 \pm \frac{1}{2}10^{-3}$) and compute the relative errors for the two roots to the quadratic when computed using the standard formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$r_{1,2} = \frac{-(-56) \pm \sqrt{(-56)^2 - 4\cdot(1)(1)}}{2(1)}$$

$$= \frac{56 \pm \sqrt{3136 - 4}}{2}$$

$$= \frac{56 \pm 55.964}{2}$$

$r_1 = 55.982$, $r_2 = 0.018$

(b) A better approximation for the "bad" root can be found by manipulating $(x - r_1)(x - r_2) = 0$ so that r_1 and r_2 can be related to a, b, c. Find such relations (there are two) and see if either can be used to compute the "bad" root more accurately.

$x^2 - 56x + 1 = 0$ $\qquad r_2 = 0.018$ (Good root)

$(x^2 - 56x + 784) - 783 = 0$

$x^2 - 56x + 1 = 0$

$(x^2 - 56x + 784) - 783 = 0$

$(x-28)^2 - 783 = 0$

$(x-28)^2 = 783$

$r_1, r_2 = \pm\sqrt{783} + 28$

$r_1 = 55.9821$

$r_2 = 0.017863$

$r_2 = 0.018$ (Good root)

$r_2 = \frac{-b + \sqrt{b^2-4ac}}{2a} \cdot \frac{(-b - \sqrt{b^2-4ac})}{(-b-\sqrt{b^2-4ac})}$

$= \frac{2c}{-b-\sqrt{b^2-4ac}}\Big|_{a=1, b=-56, c=1} = 55.9821$

Since there is no difference within the root, completing the square is more computationally accurate

5) 5. **Cancellation of terms**. Consider computing $y = x_1 - x_2$ with $\tilde{x}_1 = x_1 + \Delta x_1$ and $\tilde{x}_2 = x_2 + \Delta x_2$ being approximations to the exact values. If the operation $x_1 - x_2$ is carried out exactly we have $\tilde{y} = y + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$.

Play with different values of x. One really small value (< 1) and one large value $> 10^5$.

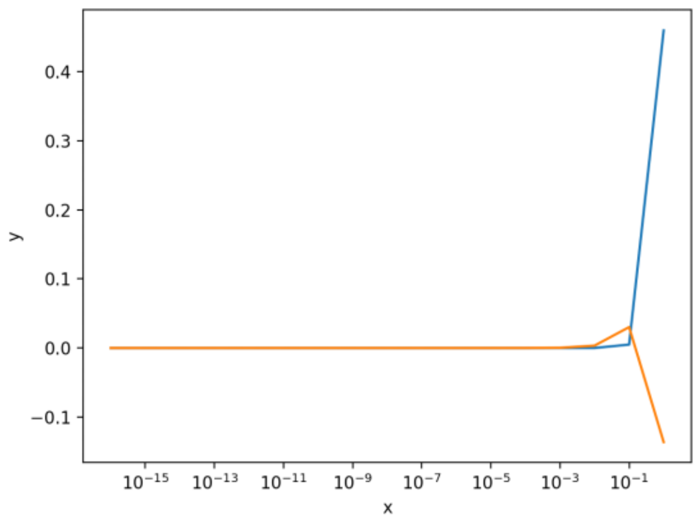
(a) Find upper bounds on the absolute error $|\Delta y|$ and the relative error $|\Delta y|/|y|$, when is the relative error large?

Code Provided.

(b) First manipulate $\cos(x+\delta) - \cos(x)$ into an expression without subtraction. Pick two values of x; say $x = \pi$ and $x = 10^6$. Then for each x, tabulate or plot the difference between your expression and $\cos(x+\delta) - \cos(x)$ for $\delta = 10^{-16}, 10^{-15}, \cdots, 10^{-2}, 10^{-1}, 10^0$ (note that you can use your logx command to uniformly distribute δ on the x-axis).

$\cos(x+\delta) = (\cos(x)\cos(\delta) - \sin(x)\sin(\delta)) - \cos(x)$

$x = \pi, \; x = 10^6$



(c) Taylor expansion yields $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\xi)$, $\xi \in [x, x+\delta]$. Use this expression to create your own algorithm for approximating $\cos(x+\delta) - \cos(x)$. Explain why you chose the algorithm. Then compare the approximation from your algorithm with the techniques in part (b). Use the same values for x and δ.

I would use a for loop to insert different values into the expansion, and then evaluate the error to compare to part b.