Homework3

Monday, September 18, 2023 1:30 PM

Tony Seamour
APPM 4600

1.) 1. Consider the equation $2x - 1 = \sin x$.

(a) Find a closed interval $[a, b]$ on which the equation has a root r, and use the Intermediate Value Theorem to prove that r exists.

$2x - 1 = 0$ for $x = 1$, $\sin(x) = 0$ for $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Let $y = 2x - 1 - \sin(x)$ for $x = (-\frac{\pi}{2})$ $\therefore$ $y = 2(-\frac{\pi}{2}) - 1 - \sin(-\frac{\pi}{2}) = -\pi < 0$

for $x = (\frac{\pi}{2})$ $\therefore$ $y = 2(\frac{\pi}{2}) - 1 - \sin(\frac{\pi}{2}) = \pi - 2 > 0$

$\therefore$ let $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$. Since $y = 2x - 1 - \sin(x) < 0$ for $x = -\frac{\pi}{2}$ and $y = 2x - 1 - \sin(x) > 0$ for $x = \frac{\pi}{2}$, by the Intermediate Value Theorem, there must exist some root r on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(b) Prove that r from (a) is the only root of the equation (on all of $\mathbb{R}$).

$2x - 1 - \sin(x) = 0$

$\frac{\partial}{\partial x}\left(2x - 1 - \sin(x)\right) = 2 - \cos(x) = 0$

Since $\cos(x)$ is bounded by $|1|$, there are no values of $x \in \mathbb{R}$ where another root exists.

(c) Use the bisection code from class (or your own) to approximate r to eight correct decimal places. Include the calling script, the resulting final approximation, and the total number of iterations used.

Code:

```
# import libraries
import numpy as np

def driver():

# use routines
    f = lambda x: 2*x - 1 - np.sin(x)
    a = -np.pi/2
    b = np.pi/2

    tol = 1e-8

    [astar,ier] = bisection(f,a,b,tol)
    print('the approximate root is',astar)
    print('the error message reads:',ier)
    print('f(astar) =', f(astar))




# define routines
def bisection(f,a,b,tol):

#    Inputs:
#     f,a,b       - function and endpoints of initial interval
#      tol  - bisection stops when interval length < tol

#    Returns:
#      astar - approximation of root
#      ier   - error message
#            - ier = 1 => Failed
```

```
#    Returns:
#      astar - approximation of root
#      ier   - error message
#            - ier = 1 => Failed
#            - ier = 0 == success

#    first verify there is a root we can find in the interval

    fa = f(a)
    fb = f(b);
    if (fa*fb>0):
       ier = 1
       astar = a
       return [astar, ier]

#   verify end points are not a root
    if (fa == 0):
      astar = a
      ier =0
      return [astar, ier]

    if (fb ==0):
      astar = b
      ier = 0
      return [astar, ier]

    count = 0
    d = 0.5*(a+b)
    while (abs(d-a)> tol):
      fd = f(d)
      if (fd ==0):
        astar = d
        ier = 0
        return [astar, ier]
      if (fa*fd<0):
         b = d
      else:
        a = d
        fa = fd
      d = 0.5*(a+b)
      count = count +1
    print('abs(d-a) = ', abs(d-a))
    print('Number of iterations: ', count)
    astar = d
    ier = 0
    return [astar, ier]

driver()
```

Output:

```
tonys@Tonys-Surface MINGW64 /c/users/tonys/Documents/APPM4600/testrep/Homework/Homework3 (main)
$ python3 Problem1c.py
abs(d-a) =  5.851672257861651e-09
Number of iterations:  28
the approximate root is 0.8878622154822129
the error message reads: 0
f(astar) = 5.354353072029028e-09
```

2.) 2. The function $f(x) = (x-5)^9$ has a root (with multiplicity 9) at $x = 5$ and is monotonically increasing (decreasing) for $x > 5$ ($x < 5$) and should thus be a suitable candidate for your function above. Use `a=4.82` and `b=5.2` and `tol = 1e-4` and use `bisection` with:

* Using code included in part (a) *

(a) $f(x) = (x-5)^9$.

```
$ python3 Problem2a.py
abs(d-a) =  9.277343750024869e-05
Number of iterations:  11
the approximate root is 5.000073242187501
the error message reads: 0
f(astar) = 6.065292655789404e-38
```

(b) The expanded expanded version of $(x-5)^9$, that is, $f(x) = x^9 - 45x^8 + \ldots - 1953125$.

```
$ python3 Problem2b.py
the approximate root is 5.12875
the error message reads: 0
f(astar) = 0.0
```

(c) Explain what is happening.

As we found in homework 1, the expanded version takes in more values of x, increasing the error with each iteration.

4.) 4. **Definition 1** *Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n. If there exists postive constants λ and α such that*

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=1}^{\infty}$ converges to p with an order α and asympotic error constant λ. If $\alpha = 1$ and $\lambda < 1$ then the sequence converges linearly. *If $\alpha = 2$, the sequence is* quadratically *convergent.*

Which of the following iterations will converge to the indicated fixed point x_* (provided x_0 is sufficiently close to x_*)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

(a) (10 points) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $x_* = 2$

Fixed point where $f(x) = x$

$\therefore$ given $f(x) = -16 + 6x + \frac{12}{x}$, $f(x^*=2) = -16 + 6(2) + \frac{12}{(2)} = -16 + 12 + 6 = 2$

$f(x^*) = x^* = 2 \Rightarrow x^* = 2$ is a fixed point ✓

To find order of convergence, Taylor expand $f(x)$ about point p

$$f(x) = f(p) + f'(p)\cdot(x-p) + \frac{f''(\eta_k)}{2}\cdot(x-p)^2$$

for some η_k between x & p

Plug in x_k

$$f(x_k) = f(p) + f'(p)\cdot(x_k - p) + \frac{f''(\eta_k)}{2}(x_k - p)^2$$

$$x_{k+1} = p + f'(p)(x_k - p) + \frac{f''(\eta_k)}{2}(x_k - p)^2$$

$$\underbrace{x_{k+1} - p}_{\text{numerator}} = f'(p)\underbrace{(x_k - p)}_{\text{denom.}} + \frac{f''(\eta_k)}{2}\cdot(x-p)^2$$

need to correct since we are going to toss quadratic term

$$\frac{x_{k+1} - p}{x_k - p} = f'(\beta_k) \text{ for some } \beta_k \text{ between } x_k \text{ and } p$$

$$\lim_{k\to\infty}\left|\frac{x_{k+1} - p}{x_k - p}\right| = \lim_{k\to\infty}|f'(\beta_k)| = |f'(p)|$$

$\therefore$ 1st order as long as $|F'(p)| < 1$ and the convergence rate is $|F'(p)|$

Using this, we find

$$f'(x) = 6 - \frac{1}{x^2}\Big|_{x^* = 2} = 6 - \frac{1}{4} = 5\tfrac{3}{4} > 1 \therefore$$

this sequence is not First order convergent

(b) (10 points) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $x_* = 3^{1/3}$

$$f(x) = \frac{2}{3}x + \frac{1}{x^2}\Big|_{x^* = 3^{\frac{1}{3}}} = \frac{2}{3}(3^{1/3}) + (3^{1/3})^{-2}$$

$$= \frac{2}{3}(3^{1/3}) + (3^{-2/3})$$

$$= \frac{2}{3}(3^{1/3}) + (3^{-}$$