Graphs

7

A graph G = (V, E) is a collection of sets V and E where V is the collection of vertices and E is the collection of edges. An edge is a line or arc connecting two vertices and it is denoted by a pair (i, j) where i, j belong to the set of vertices V. A graph can be of two types - Undirected graph or Directed graph.

7.1 Undirected Graph

A graph, which has unordered pair of vertices, is called undirected graph. If there is an edge between vertices u and v then it can be represented as either (u, v) or (v, u).

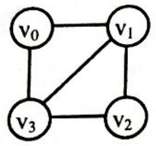

Figure 7.1 Undirected Graph

$V(G) = \{ V_0, V_1, V_2, V_3 \}$
$E(G) = \{ (V_0, V_1), (V_0,V_3), (V_1,V_2), (V_1,V_3), (V_2,V_3) \}$
This graph is undirected; it has 4 vertices and 5 edges.

7.2 Directed Graph

A directed graph or digraph is a graph which has ordered pair of vertices (u,v) where u is the tail and v is the head of the edge. In this type of graph, a direction is associated with each edge i.e. (u,v) and (v,u) represent different edges.

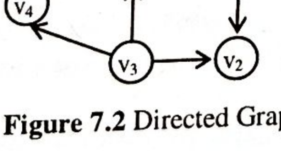

Figure 7.2 Directed Graph

$V(G) = \{ V_0, V_1, V_2, V_3, V_4 \}$
$E(G) = \{ (V_0,V_1), (V_1,V_2), (V_3,V_2),(V_3,V_0),(V_3,V_4),(V_4,V_0) \}$
This graph is directed; it has 5 vertices and 6 edges.

7.3 Graph Terminology

The definitions and terms associated with graph data structures are explained with the help of example graphs shown in figure 7.3.

Weighted graph - A graph is weighted if its edges have been assigned some non negative value as weight. A weighted graph is also known as network. Graph G9 is a weighted graph. The weight on the edge may represent cost, length or distance associated with the edge.

Subgraph - A graph H is said to be a subgraph of another graph G, iff the vertex set of H is subset of vertex set of G and edge set of H is subset of edge set of G.

Adjacency - Adjacency is a relation between two vertices of a graph. A vertex v is adjacent to another vertex u if there is an edge from vertex u to vertex v i.e. edge (u,v) Є E.

In an undirected graph if we have an edge (u,v), it means that there is an edge from u to v and also an edge from v to u. So the adjacency relation is symmetric for undirected graphs, i.e. if (u,v) is an edge then u is adjacent to v and v is adjacent to u. For example in G2, vertex v_0 is adjacent to v_3 and v_3 is adjacent to v_0, vertex v_0 is not adjacent to v_2 since there is no edge between them.

In a digraph if (u,v) is an edge, then v is adjacent *to* u but u is not adjacent to v since there is no edge from v to u. The vertex u is said to be adjacent *from* v. For example in G5, v_1 is adjacent to v_0 but v_0 is not adjacent to v_1. The vertex v_0 is adjacent from v_1.

Incidence - Incidence is a relation between a vertex and an edge of a graph. In an undirected graph the edge (u,v) is incident on vertices u and v. For example in G2 edge(v_0,v_3) is incident on vertices v_0 and v_3.

In a digraph, the edge (u,v) is incident from vertex u and is incident to vertex v. For example in G5, the edge(v_0,v_1) is incident from vertex v_0 and incident to vertex v_1.

Path - A path from vertex u_1 to vertex u_n is a sequence of vertices u_1, u_2, u_3,..............., u_{n-1}, u_n such that u_2 is adjacent to u_1, u_3 is adjacent to u_2, ... , u_n is adjacent to u_{n-1}. In other words we can say that (u_1,u_2), (u_2,u_3) ....(u_{n-1},u_n) are all edges or (u_i , u_{i+1}) Є E for i=1,2,3……n-1. For example in digraph G5, v_3–v_4–v_0–v_1–v_2 is a path, while v_3–v_2–v_1 is not a path since v_1 is not adjacent to v_2. In undirected graph G12, v_5–v_2–v_3 and v_0–v_1–v_4–v_3–v_1 are examples of path.

Length of a path - The length of a path is the total number of edges included in the path. For a path with vertices u_1, u_2, u_3,............, u_{n-1}, u_n , the length of path is n-1. For example the length of path v_3–v_4–v_0–v_1–v_2 in G5 is 4.

Reachable - If there is a path P from vertex u to vertex v, then vertex v is said to be reachable from vertex u via path P. For example in digraph G5, vertex v_2 is reachable from vertex v_4 via path v_4–v_0–v_1–v_2 while vertex v_3 is not reachable from vertex v_4 as there is no path from v_4 to v_3.

Simple path - Simple path is a path in which all the vertices are distinct. For example in graph G12, path v_0–v_1–v_3–v_4–v_6 is a simple path while path v_0–v_1–v_3–v_4–v_6–v_3–v_2 is not a simple path because vertex v_3 is repeated.

Cycle - In a digraph, a path u_1, u_2, ... , u_{n-1},u_n is called a cycle if it has at least two vertices and the first and last vertices are same i.e. $u_1 = u_n$. In graph G7, path v_0–v_2–v_1–v_0 is a cycle and in graph G9 path v_0–v_1–v_0 is a cycle.

In an undirected graph, a path u_1, u_2, ..., u_{n-1}, u_n is called a cycle if it has at least three vertices and the first and last vertices are same i.e. $u_1 = u_n$. In undirected graph if (u,v) is an edge then u–v–u should not be considered a cycle since (u,v) and (v,u) represent the same edge. So for a path to be a cycle in an undirected graph there should be at least three vertices. For example in graph G12, v_1–v_4–v_6–v_3–v_1 is a cycle of length 4, path v_6-v_4-v_3-v_6 is a cycle of length 3.

Simple Cycle - A cycle u_1, u_2, u_3,......,u_{n-1}, u_n is simple if the vertices u_2, u_3, ... ,u_{n-1}, u_n are distinct. For example in graph G12, v_1–v_4–v_3–v_2–v_0–v_1 is a simple cycle but v_1–v_4–v_3–v_6–v_2–v_3–v_1 is not a simple cycle because vertex v_3 is repeated.

Cyclic graph - A graph that has one or more cycles is called a cyclic graph. Graphs G1, G3, G7, G9, G11 and G12 are examples of cyclic graphs.

Acyclic graph - A graph that has no cycle is called an acyclic graph. Graphs G2, G4, G5, G6, G8 and G10 are examples of acyclic graphs.

DAG - A directed acyclic graph is named DAG after its acronym. Graph G5 is an example of a dag.

Degree - In an undirected graph, the degree of a vertex is the number of edges incident on it. In graph G12, degree of vertex v_0 is 2, degree of v_1 is 3, degree of v_2 is 4, degree of v_3 is 4, degree of v_4 is 3, degree of v_5 is 2 and degree of v_6 is 4. In a digraph, each vertex has an indegree and an outdegree. The degree of a vertex in a digraph is the sum of its indegree and outdegree.

1 **Indegree** - The indegree of a vertex v is the number of edges entering the vertex v, or in other words the number of edges incident to vertex v. In graph G8, the indegree of vertices v_0, v_1, v_3 and v_6 are 0, 2, 6 and 1 respectively.

2 **Outdegree** - The outdegree of vertex v is the number of edges leaving the vertex v or in other words the number of edges which are incident from v. In graph G8, outdegrees of vertices v_0, v_1, v_3, v_5, and v_6 are 3, 1, 0, 3, and 2 respectively.

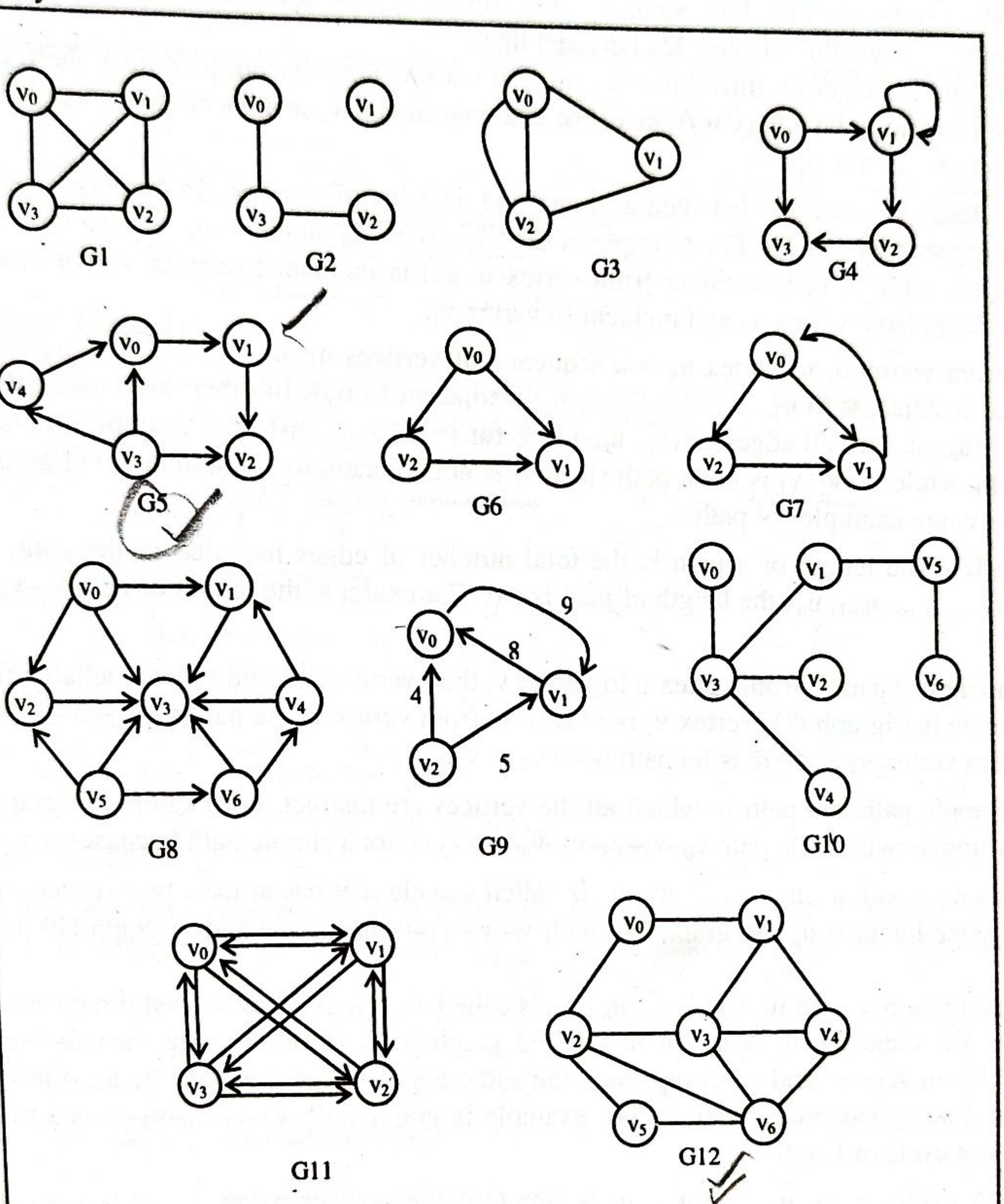

Figure 7.3

Source - A vertex, which has no incoming edges, but has outgoing edges, is called a source. The indegree of a source is zero. In graph G8, vertices v_0 and v_5 are sources.

Sink - A vertex, which has no outgoing edges but has incoming edges, is called a sink. The outdegree of a sink is zero. In graph G8, vertex v_3 is a sink.

Pendant vertex - A vertex in a digraph is said to be pendant if its indegree is equal to 1 and outdegree is equal to 0.

1 → 0

Isolated vertex - If the degree of a vertex is 0, then it is called an isolated vertex. In graph G2, vertex v_1 is an isolated vertex.

Successor and predecessor - In a digraph, if a vertex v is adjacent to vertex u, then v is said to be the successor of u, and u is said to be the predecessor of v. In graph G8, v_0 is predecessor of v_1 while v_1 is successor of v_0.

Maximum edges in a graph - If n is the total number of vertices in a graph, then an undirected graph can have maximum n(n-1)/2 edges and a digraph can have maximum n(n-1) edges. For example an undirected graph with 3 vertices can have maximum 3 edges, and an undirected graph with 4 vertices can have maximum 6 edges. A digraph with 3 vertices can have maximum 6 edges and a digraph with 4 vertices can have maximum 12 edges.

Complete graph - A graph is complete if any vertex in the graph is adjacent to all the vertices of the graph or we can say that there is an edge between any pair of vertices in the graph. A complete graph contains maximum number of edges, so an undirected complete graph with n vertices will have n(n-1)/2 edges and a directed complete graph with n vertices will have n(n-1) edges. Graph G1 is a complete undirected graph and graph G11 is a complete directed graph.

Multiple edges - If there is more than one edge between a pair of vertices then the edges are known as multiple edges or parallel edges. In graph G3, there are multiple edges between vertices v_0 and v_2.

Loop - An edge is called loop or self edge if it starts and ends on the same vertex. Graph G4 has a loop at vertex v_1.

Multigraph - A graph which contains loop or multiple edges is known as multigraph. Graphs G3 and G4 are multigraphs.

Simple graph - A graph which does not have loop or multiple edges is known as simple graph.

Regular graph - A graph is regular if every vertex is adjacent to the same number of vertices. Graph G1 is regular since every vertex is adjacent to 3 vertices.

Planar graph - A graph is called planar if it can be drawn in a plane without any two edges intersecting. Graph G1 is not a planar graph, while graphs G2, G3, G4 are planar graphs.

Null graph - A graph which has only isolated vertices is called null graph.

7.4 Connectivity in Undirected Graph

7.4.1 Connected Graph

An undirected graph is connected if there is a path from any vertex to any other vertex, or any vertex is reachable from any other vertex. A connected graph of n vertices has at least n-1 edges. The following are some examples of connected graphs.

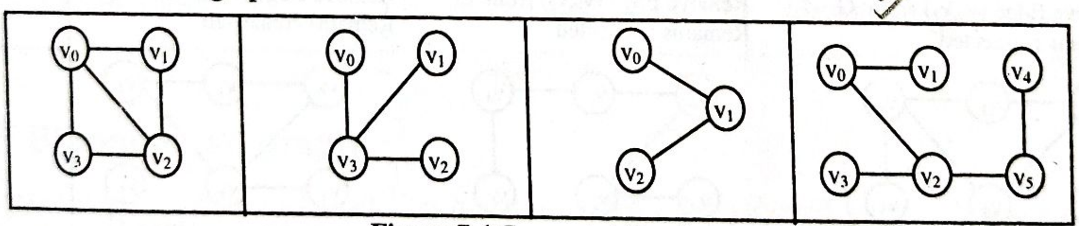

Figure 7.4 Connected Graphs

The three graphs given next are not connected graphs.

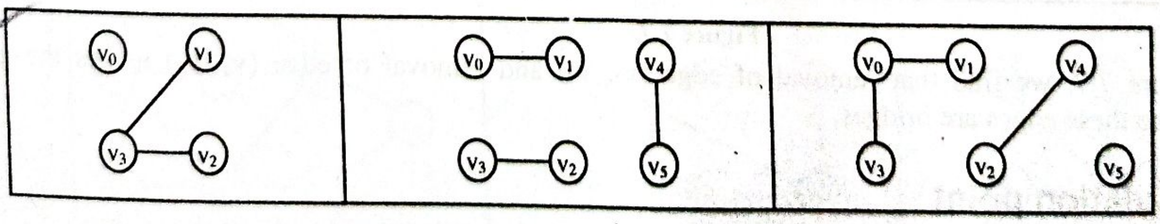

Figure 7.5 Graphs which are not connected