Results Math 72 (2017), 1823–1837

1422-6383/17/041823-15
published online September 13, 2017
DOI 10.1007/s00025-017-0746-9

Results in Mathematics

Coarse Ricci Curvature as a Function on $M \times M$

Antonio G. Ache and Micah W. Warren

Abstract. We use the framework used by Bakry and Emery in their work on logarithmic Sobolev inequalities to define a notion of coarse Ricci curvature on smooth metric measure spaces alternative to the notion proposed by Y. Ollivier. This function can be used to recover the Ricci tensor on smooth Riemannian manifolds by the formula

$$\mathrm{Ric}(\gamma'(0), \gamma'(0)) = \frac{1}{2}\frac{d^2}{ds^2}\mathrm{Ric}_{\triangle_g}(x, \gamma(s))$$

for any curve $\gamma(s)$.

Mathematics Subject Classification. 53C23 (60D05, 90C56, 51K10).

Keywords. Ricci curvature, metric measure space, discrete metric space, metric geometry, clustering.

Contents

The Antonio G. Ache was partially supported by a postdoctoral fellowship of the National Science Foundation, Award No. DMS-1204742. The Micah W. Warren is partially supported by NSF Grant DMS-1438359.

Birkhäuser

1. Introduction

Given Riemannian manifold (M, g), the metric tensor g can be recovered by differentiating the metric function d_g^2 as follows. Given a curve $\gamma(s)$ in M, with $\gamma'(0) \in T_xM$ (the tangent space to M at x)

$$g(\gamma'(0), \gamma'(0)) = \frac{1}{2}\frac{d^2}{ds^2} d_g^2(x, \gamma(s)).$$

In this note, we construct and discuss a similar potential for the Ricci tensor: A function

$$\mathrm{cRic} : M \times M \diagdown \mathfrak{C} \to \mathbb{R}$$

(here $\mathfrak{C}$ is a "cut locus" to be defined) such that

$$\mathrm{Ric}_x(\gamma'(0), \gamma'(0)) = \frac{1}{2}\frac{d^2}{ds^2}\mathrm{cRic}(x, \gamma(s)).$$

Here Ric_x is the classical Ricci curvature on M at x. We would like our coarse Ricci curvature function $\mathrm{cRic}(x, y)$ to be have a strong agreement with the Ricci curvature on Riemannian manifolds, so in particular we will be interested in functions which have the following *coarse Ricci curvature lower bound* property: for any $K \in \mathbb{R}$ we have

$$\mathrm{cRic}(x, y) \geq K d_g^2(x, y) \text{ for all } x, y \in M$$

if and only if

$$\mathrm{Ric} \geq Kg \text{ on } M,$$

where Ric is the classical Ricci curvature

There is an abundance of functions that satisfy this property. We propose one in particular, grounded in the Bakry–Emery Γ_2 calculus. In particular, we appeal to the Bochner formula

$$\Gamma_2(f, f) = \mathrm{Ric}(\nabla f, \nabla f) + \|\nabla^2 f\|^2.$$

In [6], Belkin and Niyogi show that the graph Laplacian of a point cloud of data samples taken from a submanifold in Euclidean space converges to the Laplace–Beltrami operator on the underlying manifold. Following this reasoning, in [2] we define a robust family of coarse Ricci curvature operators on metric measure spaces which depend on a scale parameter t. Motivated by problems in manifold learning, in [2] we show that on smooth submanifolds of Euclidean space, these Ricci curvatures recover the intrinsic Ricci curvature in the limit. Explicit convergence rates and applications to manifold learning are explored in [3].

The original motivation for this project was to determine if the Ricci curvature could be recovered by using approximations of Laplace operators. For this reason we desire a coarse Ricci function that can be constructed from the Laplace operator and the distance function, without appealing to any tensor calculus. While there are other, perhaps more naive functions which satisfy the above properties, we use a particular one: see (2.10).

1.1. Background and Motivation

The motivation for the paper stems from both the theory of Ricci lower bounds on metric measure spaces and the theory of manifold learning.

1.1.1. Ricci Curvature Lower Bounds on Metric Measure Spaces. One of our main sources of intuition for understanding Ricci curvature is the problem of reinterpreting lower bounds on the Ricci curvature in such a way that it becomes stable under Gromov–Hausdorff limits and thus defining "weak" notion of lower bounds on Ricci curvature. This theory has undergone significant development over the last 10–15 years. A major achievement is the work of Lott–Villani [16] and Sturm [25,26], which defines Ricci curvature lower bounds on metric measure spaces. These definitions work quite beautifully provided the metric space is also a length space, but fail to be useful on discrete spaces. The underlying calculus for this theory lies in optimal transport: given a metric measure space (X, d, μ) one can consider the space $\mathcal{B}(X)$ of all Borel probability measures with the 2-Wasserstein distance, which we will denote by W_2, and is given by

$$W_2(\mu_1, \mu_2) = \sqrt{\inf_{\gamma \in \Pi} \int_{X \times X} d^2(x, y) d\gamma(x, y)}, \tag{1.1}$$

where Π is the set of all probability measures in $\mathcal{B}(X \times X)$ whose marginals are the measures μ_1 and μ_2, i.e. if $P_i : X \times X \to X$ for $i = 1, 2$, are the projections onto the first and second factors respectively, then $\mu_i = (P_i)_* \gamma$ (the push-forward of γ by P_i). One of the crucial ideas in the work of Lott–Villani and Sturm is to use tools from optimal transport to define a notion of convexity with respect to 2-Wasserstein geodesics (i.e., geodesics with respect to the Wasserstein distance W_2) such that lower bounds on the Ricci curvature are equivalent to the geodesic convexity of well-chosen functionals. Further, they show that this convexity property is stable under measured Gromov–Hausdorff limits. In particular, fixing a background measure ν they define the entropy of μ with respect to ν as

$$E(\mu|\nu) = \int \frac{d\mu}{d\nu} \log\left(\frac{d\mu}{d\nu}\right) d\nu, \tag{1.2}$$

and define a space to have non-negative Ricci curvature if the entropy (1.2) is convex along Wasserstein geodesics. This convexity property along Wasserstein geodesics, or *displacement convexity,* used by Lott–Villani was in turn inspired by the work of Robert McCann in [18]. Before the work of Lott–Villani and Sturm, it was well known that lower bounds on sectional curvature are stable under Gromov–Hausdorff limits. For more on the theory of Alexandrov spaces, see for example [4].

The work of Lott–Villani and Sturm provides a precise definition of lower bounds on Ricci Curvature on length spaces that a priori may be very rough. More recently, Aaron Naber in [20] used stochastic analysis on manifolds to characterize two-sided bounds on non-smooth spaces. Naber's approach also relies heavily on the existence of Lipschitz geodesics. On the other hand, it is easy to show (for example, consider the space with two positively measured points unit distance apart) that the Wasserstein space W_2 for discrete spaces does not admit any Lipschitz geodesics. To overcome the lack of Wasserstein geodesics on discrete space, Bonciocat and Sturm [7] introduced the notion of an"h-rough geodesic," which behaves like a geodesic up to an error h. Then using optimal transport methods, they are able to define a lower bound on Ricci curvature, which depends on the scale h.

The notion of *coarse Ricci curvature* was developed to characterize Ricci curvature, including lower bounds, on a more general class of spaces. The first definition of coarse Ricci curvature was proposed by Ollivier [21] as a function on pairs of points in a metric measure space. Heuristically, two points should have positive coarse Ricci curvature if geodesics balls near the points are "closer" to each other than the points themselves. One possible precise definition of "close" is given by (1.1). Comparing the distance between two points to the distance of normalized unit balls allows one to extract useful information about the geometry of the space. It is also natural to consider the heat kernel for some small positive time around each given point. In fact, the intuition becomes blatant in the face of [29, Cor. 1.4 (x)], which states that the distance (1.1) decays at an exponential rate given by the Ricci lower bound, as mass spreads out from two points via Brownian motion.

Even earlier, it was shown by Jordan, Kinderlehrer and Otto [13] that the gradient flow of the entropy function (1.2) on W_2-space agrees with the heat flow on L^2. On nice metric measure spaces, one expects this flow to converge to an invariant measure, the same one from which the entropy was originally defined. Thus in principle, the behavior of the entropy functional (1.2) and the generator of heat flow are fundamentally related. This deep relationship is explored in generality in the recent paper [1]. It is natural then to attempt to define Ricci curvature in terms of a Markov process. In fact, Ollivier's idea was to compare the distance between two points to the distance between the point masses after one step in the Markov process. This is also essentially the idea in Lin–Lu–Yau [15], where a lower bound on Ricci curvature of graphs is defined. In contrast, a Γ_2 approach was used by Lin–Yau to define Ricci curvature lower bounds on graphs in [17].

Recently, Erbas and Maas [10] and Mielke [19] have provided a very natural way to define Ricci curvature for arbitrary Markov chains. This involves creating a new Wasserstein space, called the discrete transportation metric, in which one can implement the idea of Lott–Sturm–Villani, i.e., relating lower bounds on Ricci curvature to geodesic convexity of certain entropy functional.

Gigli and Maas [11] show that in the limit these transportation metrics converge to the Wasserstein space of the manifold in question as the mesh size goes to zero, at least on the torus. Thus this notion of Erbas and Maas recovers the Ricci curvature in the limit.

Any discussion of lower bounds on Ricci curvature would not be complete without a discussion of isoperimetric inequalities, most notably the log-Sobolev inequality. These inequalities, which hold for positive Ricci lower bounds, were generalized by Lott–Villani. In the coarse Ricci setting, log-Sobolev inequalities have been proved by Ollivier, Lin–Yau and Erbar–Maas for their respective definitions of Ricci curvature lower bounds. We will mention the log-Sobolev inequality that holds in general when a Bakry–Emery condition is present. This Bakry–Emery type condition in principle, should be related to coarse Ricci curvature, however, it is not clear to us in the non-Riemannian case what the relationship should be.

For further introduction to concepts of coarse Ricci curvature see the survey of Ollivier [22].

1.1.2. The Manifold Learning Problem. Roughly speaking, the manifold learning problem deals with inferring or predicting geometric information from a manifold if one is only given a point cloud on the manifold, i.e., a sample of points drawn from the manifold at random according to a certain distribution, without any further information. From a pure mathematical perspective, a point cloud can be a metric measure space that approximates a manifold in a measured Gromov–Hausdorff sense despite being a discrete set. Belkin and Niyogi showed in [6] that given a uniformly distributed point cloud on Σ there is a 1-parameter family of operators L_t, which converge to the Laplace–Beltrami operator Δ_g on the submanifold. More generally, the results in [8] and [27] show that it is possible to recover a whole family of operators that include the Fokker–Planck operator and the weighted Laplacian $\Delta_\rho f = \Delta f - \langle \nabla \rho, \nabla f \rangle$ associated to the smooth metric measure space $(M, g, e^{-\rho} d\text{vol})$, where ρ is a smooth function.

In [2] we consider the problem of learning the Ricci curvature of an embedded submanifold Σ of $\mathbb{R}^N$ at a point from purely measure and distance considerations. In [3] we will show that one expects these notions to converge almost surely with an appropriate choice of scale.

2. Carré Du Champ and Bochner Formula

Let P_t be a 1-parameter family of operators of the form

$$P_t f(x) = \int_M f(y) p_t(x, dy),$$

where f is a bounded measurable function defined on M and $p_t(x, dy)$ is a non-negative kernel. We assume that P_t satisfies the semi-group property, i.e.

$$P_{t+s} = P_t \circ P_s. \tag{2.1}$$

$$P_0 = \mathrm{Id}. \tag{2.2}$$

In $\mathbb{R}^n$, an example of P_t is the *Brownian motion*, defined by the density

$$p_t(x, dy) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x-y|^2}{2t}} dy, \quad t \geq 0.$$

If now P_t is a diffusion semi-group defined on (M, g), we let L be the infinitesimal generator of P_t, which is densely defined in L^2 by

$$Lf = \lim_{t \to 0} t^{-1}(P_t f - f). \tag{2.3}$$

We consider a bilinear form which has been introduced in potential theory by Roth [24] and by Kunita in probability theory [14] and measures the failure of L from satisfying the Leibnitz rule. This bilinear form, the *Carré du Champ* , is defined as

$$\Gamma(L, u, v) = \frac{1}{2} \left(L(uv) - L(u)v - uL(v) \right).$$

When L is the rough Laplacian with respect to the metric g, then

$$\Gamma(\Delta_g, u, v) = \langle \nabla u, \nabla v \rangle.$$

We will also consider the *iterated Carré du Champ* introduced by Bakry and Emery [5] denoted by Γ_2 and defined by

$$\Gamma_2(L, u, v) = \frac{1}{2} \left(L(\Gamma(L, u, v)) - \Gamma(L, Lu, v) - \Gamma(L, u, Lv) \right).$$

Note that if we restrict our attention to the case $L = \Delta_g$ the Bochner formula yields

$$\Gamma_2(\Delta_g, u, v) = \frac{1}{2} \Delta \langle \nabla u, \nabla v \rangle_g - \frac{1}{2} \langle \nabla \Delta_g u, \nabla v \rangle_g - \frac{1}{2} \langle \nabla u, \nabla \Delta_g v \rangle_g \tag{2.4}$$

$$= \mathrm{Ric}(\nabla u, \nabla v) + \langle \mathrm{Hess}_u, \mathrm{Hess}_v \rangle_g. \tag{2.5}$$

The fundamental observation of Bakry and Emery is that the properties of Ricci curvature lower bounds can be observed and exploited by using the bilinear form Γ_2. With this in mind, they define a curvature-dimension condition for an operator L on a space X as follows. If there exist measurable functions $k : X \to \mathbb{R}$ and $N : X \to [1, \infty]$ such that for every f on a set of functions dense in $L^2(X, d\nu)$ the inequality

$$\Gamma_2(L, f, f) \geq \frac{1}{N}(Lf)^2 + k\Gamma(L, f, f) \tag{2.6}$$

holds, then the space X together with the operator L satisfies the $CD(k, N)$ *condition*, where k stands for curvature and N for dimension. In particular, when considering a smooth metric measure space $(M^n, g, e^{-\rho} d\text{vol})$ one has the natural diffusion operator

$$\Delta_\rho u = \Delta u - \langle \nabla \rho, \nabla u \rangle, \tag{2.7}$$

corresponding to the variation of the Dirichlet energy with respect to the measure $e^{-\rho} d\text{vol}$. By studying the properties of Δ_ρ, Bakry and Emery arrive at the following dimension and weight dependent definition of the Ricci tensor:

$$\text{Ric}_N = \begin{cases} \text{Ric} + \text{Hess}_\rho & \text{if } N = \infty, \\ \text{Ric} + \text{Hess}_\rho - \frac{1}{N-n}(d\rho \otimes d\rho) & \text{if } n < N < \infty, \\ \text{Ric} + \text{Hess}_\rho - \infty(d\rho \otimes d\rho) & \text{if } N = n, \\ -\infty & \text{if } N < n, \end{cases} \tag{2.8}$$

and moreover, they showed the equivalence between the $CD(k, N)$ condition (2.6) and the bound $\text{Ric}_N \geq k$.

2.1. Iterated Carré du Champ and Coarse Ricci Curvature

In this section we provide a definition of coarse Ricci curvature on general metric spaces with a given operator. In most cases of interest, this operator will be invariant with respect to a measure on the space. The definition will provide a coarse Ricci function given any operator.

Consider the function

$$f_{x,y}(z) = \frac{1}{2}\left(d^2(x, y) - d^2(y, z) + d^2(z, x)\right) \tag{2.9}$$

One can check that, in the Euclidean case, this is simply the linear function with gradient $y - x$. One can also check that for x very near y on a Riemannian manifold, this is (up to high order) the corresponding function in normal coordinates at x determined by the normal coordinates of y. This leads us to the following definition of coarse Ricci curvature.

First, we need the following definition of cut locus, which will work for an arbitrary metric space with an operator L.

Definition 2.1. *For a distance function on an arbitrary metric space with an operator L, we define the cut locus as*

$$\mathfrak{C} = \{(x, y) \in X \times X : \Gamma_2(L, f_{x,y}, f_{x,y}) \text{ is not defined at either of } x \text{ or } y\}$$

Remark 2.2. *This agrees with the definition of cut locus on Riemannian manifolds.*

Definition 2.3. *Given an operator L we define the coarse Ricci curvature for L as follows. If $(x, y) \notin \mathfrak{C}$ we define*

$$\text{Ric}_L(x, y) = \Gamma_2(L, f_{x,y}, f_{x,y})(x). \tag{2.10}$$

In order to check that this is consistent with the classical notions, note that this defines a coarse Ricci curvature on a Riemannian manifold as

$$\text{Ric}_{\triangle_g}(x, y) = \Gamma_2(\Delta_g, f_{x,y}, f_{x,y})(x).$$

We record that, in particular, Ric_{Δ_g} has the following property of coarse Ricci curvature on smooth Riemannian manifolds.

Theorem 2.4. Suppose that M is a smooth Riemannian manifold. Let $\gamma(s)$ be a smooth curve with $\gamma'(0) \in T_x M$. Then

$$\mathrm{Ric}(\gamma'(0), \gamma'(0)) = \frac{1}{2}\frac{d^2}{ds^2}\mathrm{Ric}_{\triangle_g}(x, \gamma(s)). \tag{2.11}$$

Remark 2.5. The function $\mathrm{Ric}_L(x, y)$ produced by (2.10) need not be symmetric. Also note that (2.9) does not require that the distance function be symmetric.

A classical result of Synge in [28] provides an expansion for the square of the geodesic distance at a point in normal coordinates. This allows us to prove the following proposition. The proof of Proposition 2.4 will follow.

Proposition 2.6. Given points $x, y \in M$ let Y represent the normal coordinate of y in the tangent space at x, i.e. $y = \exp_x(Y)$. Then,

$$\mathrm{Ric}_{\Delta_g}(x, y) = \mathrm{Ric}(Y, Y) + \tilde{G}(Y, x)$$

where $\tilde{G}(Y, x)$ vanishes to fourth order at $Y = 0$.

Proof. In general, for a function h, we can compute in normal coordinates at $(x = 0)$

$$\Gamma_2(h, h)(0) = \sum_{i,j}\left(h_{ij}^2 + \mathrm{Ric}_{ij}h_i h_j\right)$$

In normal coordinates at x, for $f_{x,y}$ given by (2.9) the coordinate derivatives can be computed as

$$\begin{aligned}(f_{x,y})_i &= -\left(\frac{1}{2}d^2(y, z)\right)_i + z_i\\ (f_{x,y})_{ij} &= -\left(\frac{1}{2}d^2(y, z)\right)_{ij} + \delta_{ij}\end{aligned}$$

so we have at the origin $z = 0$

$$\begin{aligned}-\left(\frac{1}{2}d^2(y, z)\right)_i &= Y^i\\ \Gamma_2(f_{x,y}, f_{x,y})(0) &= \sum_{i,j}\left(\left[\delta_{ij} - \left(\frac{1}{2}d^2(y, z)\right)_{ij}\right]^2 + \mathrm{Ric}_{ij}Y^i Y^j\right)\end{aligned}$$

in other words,

$$\mathrm{Ric}_{\Delta_g}(x, y) = \mathrm{Ric}(Y, Y) + \sum_{i,j}\left[\delta_{ij} - \left(\frac{1}{2}d^2(y, z)\right)_{ij}\right]^2$$

where Y is the coordinate of y in normal coordinates at x.

Following for example, [23, 158–161] or originally [28] , we see that

$$\left(\frac{1}{2}d^2(y,z)\right)_{ij} = \delta_{ij} + O(Y^2)$$

and the conclusion follows. □

We now prove Theorem 2.4.

Proof. Compute using normal coordinates with $Y = \gamma(s)$:

$$\mathrm{Ric}_{\triangle_g}(x,\gamma(s)) = \mathrm{Ric}(\gamma(s),\gamma(s)) + \sum_{i,j}\left[\delta_{ij} - \left(\frac{1}{2}d^2(\gamma(s),z)\right)_{ij}\right]^2.$$

Differentiate this

$$\frac{d}{ds}\mathrm{Ric}_{\triangle_g}(x,\gamma(s)) = 2\mathrm{Ric}(\gamma'(s),\gamma(s)) + \frac{d}{ds}\left(\tilde{G}(\gamma(s),x)\right),$$

and again

$$\frac{d^2}{ds^2}\mathrm{Ric}_{\triangle_g}(x,\gamma(s)) = 2\mathrm{Ric}(\gamma'(s),\gamma'(s)) + 2\mathrm{Ric}(\gamma(s),\gamma''(s)) \tag{2.12}$$

$$+ \frac{d^2}{ds^2}\left(\tilde{G}(\gamma(s),x)\right). \tag{2.13}$$

Then plugging in $\gamma(0) = 0$ we get

$$\frac{d^2}{ds^2}\mathrm{Ric}_{\triangle_g}(x,\gamma(s)) = 2\mathrm{Ric}(\gamma'(s),\gamma'(s)).$$

□

The following is natural, considering the analogy between coarse Ricci curvature and the distance function.

Theorem 2.7. Suppose that (M, g) is a Riemannian manifold. Then

$$\mathrm{Ric} \geq K \tag{2.14}$$

if and only if

$$\mathrm{Ric}_{\Delta_g}(x,y) \geq K d^2(x,y).$$

Proof. First we show that the first condition implies the second. It follows from the Bochner formula that (2.14) implies the Bakry–Emery condition

$$\Gamma_2(f,f) \geq K\Gamma(f,f)$$

in particular

$$\begin{aligned}\mathrm{Ric}_{\Delta_g}(x,y) &= \Gamma_2(\Delta_g, f_{x,y}, f_{x,y})(x)\\ &\geq K\Gamma(\Delta_g, f_{x,y}, f_{x,y})(x)\\ &= \left|\nabla f_{x,y}\right|^2(x).\end{aligned}$$

In normal coordinates at x it is easy to compute the gradient

$$\nabla f_{x,y}(x) = -Y$$

so

$$\|\nabla f_{x,y}(x)\|^2 = \|Y\|^2 = d^2(x,y)$$

and the conclusion follows.

The reverse implication follows from (2.11). $\square$

Theorem 2.8. Let

$$\Delta_\rho v = \Delta_g v - \langle \nabla\rho, \nabla v\rangle_g$$

be the weighted Laplacian and let

$$\mathrm{Ric}_\infty = \mathrm{Ric} + \nabla_g^2 \rho.$$

Then

$$\mathrm{Ric}_\infty(\gamma'(0), \gamma'(0)) = \frac{1}{2}\frac{d^2}{ds^2}\mathrm{Ric}_{\Delta_\rho}(x, \gamma(s)).$$

and

$$\mathrm{Ric}_\infty \geq K \tag{2.15}$$

if and only if

$$\mathrm{Ric}_{\Delta_\rho}(x,y) \geq K d^2(x,y).$$

Proof. This follows by gently modifying the above proofs, considering that the Bochner formula becomes

$$\Gamma_2(\Delta_\rho, f, f) = \left\|\nabla_g^2 f\right\|^2 + \mathrm{Ric}_\infty(\nabla f, \nabla f).$$

$\square$

3. Final Remarks

At this point we can make comparisons to other definitions of coarse Ricci curvature. Lin and Yau [17], following Chung and Yau [9], consider curvature dimension lower bounds of the form (2.6), in the particular case where L is the graph Laplacian using the standard distance function on graphs. This class of metric spaces is quite restricted - in their setting they are able to show that every locally finite graph satisfies a $CD(2, -1)$ condition. The approach by Lin, Lu and Yau in [15] follows the ideas of Ollivier [21], using the 1-Wasserstein distance (denoted by W_1) instead of the 2-Wasserstein distance. With this metric, they compare mass distributions after short diffusion times, and take the limit as the diffusion time approaches zero. They are able to show a Bonnet–Myers type theorem which holds for graphs with positive Ricci curvature. While the Bonnet–Myers result holds in the classical Riemannian setting, it fails for general notions of Ricci curvature, for example, when the Ricci curvature is derived from the standard Ornstein–Uhlenbeck process. Ollivier's definition [21, Definition 3] is very general, and also uses the W_1-Wasserstein metric. Ollivier uses ε-geodesics to obtain local-to-global results. Note that the notion of ε-geodesics is stronger than h-rough geodesics seen in [7].

3.1. Other Definitions on Riemannian Manifolds

One may also consider the following

$$\mathrm{cRic}(x,y) = \int_0^1 \mathrm{Ric}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))dt \tag{3.1}$$

where $\gamma : [0,1]$ is the unique constant speed geodesic starting at x and ending at y. One can check that this satisfies the coarse Ricci curvature lower bound property. Further it satisfies the much nicer property that if g evolves by Ricci flow, i.e.

$$\frac{d}{dt} g_{ij} = -2\mathrm{Ric}_{ij}$$

then

$$\begin{aligned}\frac{d}{dt} d^2(x,y) &= \frac{d}{dt}\int_0^1 g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))dt \\ &= -\int_0^1 2\mathrm{Ric}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))dt = -2\mathrm{cRic}(x,y)\end{aligned}$$

so this definition behaves nicely with respect to the Ricci flow. This definition assumes existence and full knowledge of the Ricci tensor, so it may be of little use for many purposes.

3.2. Lower Bounds, Log Sobolev Inequalities, and Other General Remarks

From the theory of Gross in [12] and Bakry–Emery, one can show that whenever a heat semi-group together with the invariant measure satisfies standard properties (self adjointness, ergodicity, Leibnitz rule) a Bakry–Emery condition of the form

$$\Gamma_2(f,f) \geq K\Gamma(f,f) \tag{3.2}$$

implies a log-Sobolev inequality of the form

$$\int f \log f d\mu \leq \frac{1}{2K}\int \frac{\Gamma(f,f)}{f} d\mu$$

for all $f > 0$ with $\int f d\mu = 1$.

The condition (3.2) is quite general, as it holds pointwise, for every function. In comparison

$$\mathrm{Ric}_L(x,y) \geq K d^2(x,y) \tag{3.3}$$

will follow from (3.2) by evaluating only on a subset of functions at certain points, provided that the distance function satisfies a condition of the form

$$\Gamma(L, f_{x,y}, f_{x,y})(x) \geq d^2(x,y). \tag{3.4}$$

We are allowing for rather general operators L without imposing structure. It is not clear to us when the condition (3.4) holds or whether the condition (3.3) implies a condition of the form (3.2).

Without additional conditions, there is no local-to-global type result, such as Proposition 19 in [21]: Consider, for example the sets

$$\Omega = \left\{(x, y) : \varepsilon^2 \leq x^2 + \varepsilon^2 y^2 \leq 2\varepsilon^2\right\}$$
$$\tilde{\Omega} = \left\{(x, y) : x^2 + \varepsilon^2 y^2 \leq 2\varepsilon^2\right\}.$$

By putting a Euclidean metric and the annular region Ω and putting a metric with very large negative curvature on the inside region $\tilde{\Omega}\backslash\Omega$ we have a manifold in which large negative curvature contributes to the distance function of points whose minimizing geodesic crosses the middle region. The Laplace operator and distance operators will relate to each other accordingly. However, by restricting the distance function and Laplace operators to the annular Ω we obtain a distance function that is not path generated. Despite the fact that the Ricci curvature derived from Δ should be nonnegative on small balls inside of Ω, the condition (3.3) will not hold for pairs of points lying on opposite sides of the narrow band. In this example, the requirement that the space is a length space is blatantly violated.

It should also be noted that without a condition such as (3.4) serving as a nexus between the distance function and the operator L, other expected properties can go awry. For example, let M^2 be a submanifold of $\mathbb{R}^3$, diffeomorphic to a sphere, that contains large flat pieces, say

$$P_1 = \left\{(x, y, z) : z = 1, |x| + |y| < \frac{1}{2}\right\}$$
$$P_2 = \left\{(x, y, z) : z = -1, |x| + |y| < \frac{1}{2}\right\}.$$

(Think of this as a sphere, flattened at the top and bottom.) By uniformization, we can choose a round metric on M, so that M is isometric to $\mathbb{S}^2$. The spectral properties of Δ_g, the Laplace–Betrami operator on $\mathbb{S}^2$, are well understood. However, we can choose the extrinsic $\mathbb{R}^3$ distance function

$$d_E(p, q) = \|p - q\|_{\mathbb{R}^3}$$

as a distance function on M^2. By considering

$$p = (0, 0, -1)$$
$$q = (0, 0, 1)$$

we can compute

$$\begin{aligned} f_{p,q}(s) &= \frac{1}{2}\left(|p - q|^2 - |q - s|^2 + |p - s|^2\right) \\ &= 2 + 2(0, 0, 1) \cdot s. \end{aligned}$$

Notice that $f_{p,q}(s) \equiv 0$ on the region P_1, so in particular, both $D^2 f_{p,q}$ and $\nabla f_{p,q}$ vanish at p. It follows that

$$\Gamma_2(\Delta_g, f_{p,q}, f_{p,q})(p) = \left\| D^2 f_{p,q} \right\|^2 (p) + Ric_{\mathbb{S}^2}(\nabla f_{p,q}, \nabla f_{p,q})(x) = 0$$

where we are taking $\mathrm{Ric}_{\mathbb{S}^2}$ to be the Ricci curvature associated to the round metric (and accordingly, the round Δ_g) on M^2. It follows that (3.3) only holds on (M, d_E, Δ_g) with $K = 0$. On the other hand, (3.3) only holds with $K = 1$ for the round distance function on (M, d_g, Δ_g). For this reason we do not expect to find a relation between the lower bound K and the spectral gap similar to that seen in [21, Proposition 30].

References

[1] Ambrosio, L., Gigli, N., Savaré, G.: Bakry–Émery curvature-dimension condition and Riemannian Ricci curvature bound. Ann. Probab. **43**(1), 339–404 (2015)

[2] Ache, A., Warren, M.: Approximating coarse Ricci curvature with applications to submanifolds of Euclidean space. arXiv:1410.3351

[3] Ache, A., Warren, M.: Coarse Ricci curvature and the manifold learning problem. arXiv:1410.3351

[4] Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry, Graduate Studies in Mathematics, vol. 33. American Mathematical Society, Providence (2001)

[5] Bakry, D., Émery, M.: Diffusions hypercontractives, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, pp. 177–206. Springer, Berlin (1985)

[6] Belkin, M., Niyogi, P.: Towards a theoretical foundation for Laplacian-based manifold methods. J. Comput. Syst. Sci. **74**(8), 1289–1308 (2008)

[7] Bonciocat, A.-I., Sturm, K.-T.: Mass transportation and rough curvature bounds for discrete spaces. J. Funct. Anal. **256**(9), 2944–2966 (2009)

[8] Coifman, R.R., Lafon, S.: Diffusion maps. Appl. Comput. Harmon. Anal. **21**(1), 5–30 (2006)

[9] Chung, F.R.K., Yau, S.-T.: Logarithmic Harnack inequalities. Math. Res. Lett. **3**(6), 793–812 (1996)

[10] Erbar, M., Maas, J.: Ricci curvature of finite Markov chains via convexity of the entropy. Arch. Ration. Mech. Anal. **206**(3), 997–1038 (2012)

[11] Gigli, N., Maas, J.: Gromov–Hausdorff convergence of discrete transportation metrics. SIAM J. Math. Anal. **45**(2), 879–899 (2013)

[12] Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**(4), 1061–1083 (1975)

[13] Jordan, R., Kinderlehrer, D., Otto, F.: The variational formulation of the Fokker–Planck equation. SIAM J. Math. Anal. **29**(1), 1–17 (1998)

[14] Kunita, H.: Absolute continuity of Markov processes and generators. Nagoya Math. J. **36**, 1–26 (1969)

[15] Lin, Y., Lu, L., Yau, S.-T.: Ricci curvature of graphs. Tohoku Math. J. **63**(4), 605–627 (2011)

[16] Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. Math. **169**(3), 903–991 (2009)

[17] Lin, Y., Yau, S.-T.: Ricci curvature and eigenvalue estimate on locally finite graphs. Math. Res. Lett. **17**(2), 343–356 (2010)

[18] McCann, R.J.: A convexity principle for interacting gases. Adv. Math. **128**(1), 153–179 (1997)

[19] Mielke, A.: Geodesic convexity of the relative entropy in reversible Markov chains. Calc. Var. Part. Differ. Equ. **48**(1–2), 1–31 (2013)

[20] Naber, A.: Characterization of bounded Ricci curvature on smooth and nonsmooth settings. arXiv:1306.6512 (2013)

[21] Ollivier, Y.: Ricci curvature of Markov chains on metric spaces. J. Funct. Anal. **256**(3), 810–864 (2009)

[22] Ollivier, Y.: A survey of Ricci curvature for metric spaces and Markov chains. In: Probabilistic Approach to Geometry. Adv. Stud. Pure Math., vol. 57, pp. 343–381. Math. Soc. Japan, Tokyo (2010)

[23] Petersen, P.: Riemannian Geometry, Graduate Texts in Mathematics, vol. 171, 2nd edn. Springer, New York (2006)

[24] Roth, J.-P.: Opérateurs carré du champ et formule de Lévy-Kinchine sur les espaces localement compacts. C. R. Acad. Sci. Paris Sér. A **278**, 1103–1106 (1974)

[25] Sturm, K.-T.: On the geometry of metric measure spaces. I. Acta Math. **196**(1), 65–131 (2006)

[26] Sturm, K.-T.: On the geometry of metric measure spaces II. Acta Math. **196**(1), 133–177 (2006)

[27] Singer, A., Wu, H.-T.: Vector diffusion maps and the connection Laplacian. Commun. Pure Appl. Math. **65**(8), 1067–1144 (2012)

[28] Synge, J.L.: A characteristic function in Riemannian space and its application to the solution of geodesic triangles. Proc. Lond. Math. Soc. **S2–32**(1), 241 (1931)

[29] von Renesse, M.-K., Sturm, K.-T.: Transport inequalities, gradient estimates, entropy, and Ricci curvature. Commun. Pure Appl. Math **58**(7), 923–940 (2005)

Antonio G. Ache
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame
IN 46556
USA
e-mail: aache@nd.edu

Micah W. Warren
Department of Mathematics
University of Oregon
Eugene
OR 97403
USA
e-mail: micahw@oregon.edu

Received: January 24, 2017.
Accepted: August 24, 2017.