SOBOLEV TRACE INEQUALITIES OF ORDER FOUR

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Abstract
We establish sharp trace Sobolev inequalities of order four on Euclidean d-balls for $d \geq 4$. When $d = 4$, our inequality generalizes the classical second-order Lebedev–Milin inequality on Euclidean 2-balls. Our method relies on the use of scattering theory on hyperbolic d-balls. As an application, we characterize the extremal metric of the main term in the log-determinant formula corresponding to the conformal Laplacian coupled with the boundary Robin operator on Euclidean 4-balls, which surprisingly is not the flat metric on the ball.

1. Introduction

In this article we derive some sharp Sobolev trace inequalities on Euclidean d-balls of order four with the $(d-1)$-sphere as boundary. Sobolev inequalities for critical exponents play an important role in problems in conformal geometry. One example is the sharp Sobolev inequality of order two on the standard sphere (S^n, g_{S^n}) for $n > 2$ which takes the form

$$\Big(\frac{1}{\operatorname{vol}(S^n)} \oint_{S^n} |f|^q \, d\sigma\Big)^{2/q} \leq \frac{q-2}{n \cdot \operatorname{vol}(S^n)} \oint_{S^n} |\tilde{\nabla} f|^2 \, d\sigma + \frac{1}{\operatorname{vol}(S^n)} \oint_{S^n} |f|^2 \, d\sigma, \tag{1.1}$$

where $d\sigma$ is the Lebesgue measure induced by g_{S^n}, $\operatorname{vol}(S^n)$ is the volume of S^n with respect to g_{S^n}, $\tilde{\nabla}$ is the gradient operator on (S^n, g_{S^n}), and $q = \frac{2n}{n-2}$. Inequality (1.1) has been crucial in the study of the Yamabe problem on closed manifolds. We remark that, when $n = 2$, (1.1) takes the form of the Moser–Onofri inequality

$$\log\Big(\frac{1}{4\pi} \oint_{S^2} e^f \, d\sigma\Big) \leq \frac{1}{4\pi} \oint_{S^2} f \, d\sigma + \frac{1}{16\pi} \oint_{S^2} |\tilde{\nabla} f|^2 \, d\sigma, \tag{1.2}$$

which is fundamental in the problem of prescribing Gaussian curvature on the sphere S^2 (see [23], [24]; see also [25]–[27]). On manifolds with boundary, a class of

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inequalities analogous to (1.1) and (1.2) is the class of *Sobolev trace inequalities*. A classical example of these inequalities is the following. Consider the unit ball B^d on Euclidean space $\mathbb{R}^d$ with the sphere S^{d-1} as boundary, and let f be a function in $C^\infty(S^{d-1})$. If v is a smooth extension of f to B^d, then we have the inequality

$$\begin{aligned} &\frac{d-2}{2}\operatorname{vol}(S^{d-1})^{\frac{1}{d-1}}\Big(\oint_{S^{d-1}} |f|^{\frac{2(d-1)}{(d-2)}}\,d\sigma\Big)^{\frac{d-2}{d-1}} \\ &\quad\le \int_{B^d} |\nabla v|^2\,dx + \frac{d-2}{2}\oint_{S^{d-1}} f^2\,d\sigma \quad \text{when } d>2. \end{aligned} \tag{1.3}$$

When $d=2$, inequality (1.3) becomes the classical Lebedev–Milin [20] inequality

$$\log\Big(\frac{1}{\pi}\oint_{S^1} e^f\,d\sigma\Big) \le \frac{1}{4\pi}\int_D |\nabla v|^2\,dx + \frac{1}{\pi}\oint_{S^1} f\,d\sigma, \tag{1.4}$$

where now ∇v is the gradient of v with respect to the Euclidean metric on the disk $D=\{x\in\mathbb{R}^2 : |x|\le 1\}$. Inequality (1.3) has been derived by Beckner [1] and also by Escobar [11], who has also applied the inequality to study the Yamabe problem on manifolds with boundary (see [12]); while the Lebedev–Milin inequality (1.4) has been applied to a wide variety of problems in classical analysis, including the Bieberbach conjecture (see [10, p. 1]) and by Osgood, Phillips, and Sarnak (see [25]–[27]) in the study of the compactness of isospectral planar domains. In Section 1.1 below, we further discuss the connection of the inequalities to the isospectral problem.

In this article, we consider extensions of some of the above results to order four, and, as an application, we identify the extremal metric of the main term in the Polyakov–Álvarez-type functional determinant of the conformal Laplacian operator for Euclidean 4-balls. The results are surprising in the sense that the extremal functions and metrics identified in our theorems are not the obvious expected ones—for example, the extremal metric on Euclidean 4-balls in Theorem C is not the flat metric on the ball.

The notion of order mentioned in the title of the article refers to the order of the operator involved in the derivation of the inequalities. For the inequalities cited above, the operators which describe the Euler–Lagrange equation of the extremal functions involved in the proof of the inequality are either the Laplace operator or the conformal Laplace operator, both of order two. In recent years, the role played by these second-order operators has been vastly extended to operators of order four (e.g., the bi-Laplace operator or the Paneitz operator of order four, which we briefly discuss in Section 2). For example, the Sobolev inequality which prescribes the embedding of $W_o^{2,2}$ to $L^{\frac{2n}{n-4}}$ on $\mathbb{R}^n$ for $n>4$ has a fourth-order analogue on S^n; namely,

$$C_n\Big(\frac{1}{\operatorname{vol}(S^n)}\oint_{S^n} |f|^{\frac{2n}{n-4}}\,d\sigma\Big)^{\frac{n-4}{n}} \le \frac{1}{\operatorname{vol}(S^n)}\oint_{S^n} fP_4f\,d\sigma, \tag{1.5}$$

where $C_n = \frac{n(n-2)(n+2)(n-4)}{2^4}$ and P_4 is the fourth-order Paneitz operator on S^n defined as (see [3, Theorem 1.1])

$$P_4 = \Big(-\tilde{\Delta} + \Big(\frac{n}{2}\Big)\Big(\frac{n-2}{2}\Big)\Big)\Big(-\tilde{\Delta} + \Big(\frac{n+2}{2}\Big)\Big(\frac{n-4}{2}\Big)\Big),$$

where $\tilde{\Delta}$ is used to denote the Laplacian with respect to the metric g_{S^n} (see [1, Theorem 6]). When $n = 4$, the inequality becomes an inequality of Moser–Onofri type, just as in (1.2), and takes the form

$$\log\Big(\frac{1}{\mathrm{vol}(S^4)} \oint_{S^4} e^{4(w-\bar{w})}\, d\sigma\Big) \le \frac{1}{3\mathrm{vol}(S^4)} \oint_{S^4} \big((\tilde{\Delta} w)^2 + 2|\tilde{\nabla} w|^2\big)\, d\sigma, \tag{1.6}$$

which is a special case of Beckner's log-Sobolev inequality [1] (see also Chang [6, Lecture 5]). As mentioned above, the main results in this article are extensions of inequalities (1.3) when $d > 2$ and (1.4) above to sharp Sobolev trace inequalities of order four. We adopt the following notational convention.

CONVENTION 1.1
We sometimes use g_0 to denote the Euclidean metric on $\mathbb{R}^d$. The Euclidean metric is also commonly denoted as $|dx|^2$.

We start by stating our result for dimension $d > 4$.

THEOREM A
Given $f \in C^\infty(S^{d-1})$ with $d > 4$, suppose v is a smooth extension of f to the unit ball B^d which also satisfies the Neumann boundary condition

$$\frac{\partial}{\partial n} v\big|_{\partial B^d} = -\frac{(d-4)}{4} f. \tag{1.7}$$

Then we have the inequality

$$\begin{aligned} &c_d \big(\mathrm{vol}(S^{d-1})\big)^{\frac{3}{d-1}} \Big(\oint_{S^{d-1}} |f|^{\frac{2(d-1)}{d-4}}\, d\sigma\Big)^{\frac{d-4}{d-1}} \\ &\quad \le \int_{B^d} (\Delta_{g_0} v)^2\, dx + 2\oint_{S^{d-1}} |\tilde{\nabla} f|^2\, d\sigma + b_d \oint_{S^{d-1}} |f|^2\, d\sigma, \end{aligned} \tag{1.8}$$

where $c_d = \frac{d(d-2)(d-4)}{4}$, $b_d = \frac{d(d-4)}{2}$, and $\tilde{\nabla} f$ is the gradient of f with respect to the round metric $g_{S^{d-1}}$. Moreover, equality holds if and only if v is a biharmonic extension of a function of the form $f_{z_0}(\xi) = c|1 - \langle z_0, \xi\rangle|^{\frac{4-d}{4}}$, where c is a constant, $\xi \in S^{d-1}$, z_0 is some point in B^d, and v satisfies the boundary condition (1.7). When $f \equiv 1$, equality in (1.8) is attained by the function $v(x) = 1 + \frac{d-4}{4}(1 - |x|^2)$.

The corresponding theorem for dimension 4 is the following inequality, which is a generalization of the classical Lebedev–Milin inequality (1.4).

THEOREM B
Given $\varphi \in C^\infty(S^3)$, suppose that w is a smooth extension of φ to the unit ball B^4. If w satisfies the Neumann boundary condition

$$\frac{\partial}{\partial n} w\big|_{\partial B^4} = 0, \tag{1.9}$$

then we have the inequality

$$\log\Big(\frac{1}{2\pi^2}\oint_{S^3} e^{3(\varphi-\bar{\varphi})}\, d\sigma\Big) \leq \frac{3}{16\pi^2}\int_{B^4} (\Delta_{g_0} w)^2\, dx + \frac{3}{8\pi^2}\oint_{S^3} |\tilde{\nabla}\varphi|^2\, d\sigma. \tag{1.10}$$

Moreover, equality holds if and only if w is a biharmonic extension of a function of the form $\varphi_{z_0}(\xi) = -\log|1 - \langle z_0, \xi\rangle| + c$, where c is a constant, $\xi \in S^3$, z_0 is some point in B^4, and w satisfies the boundary condition (1.9).

We remark that the method that we used to discover and establish the above sharp inequalities is nontraditional in the sense that we first derived such inequalities for some metric g^* on B^d which is in the conformal class of the Euclidean metric $g_0 = |dx|^2$, and then we derived the desired inequalities in Theorems A and B through conformal covariant properties between the fourth-order "Paneitz operator" with respect to g^* and the bi-Laplace operator with respect to the flat metric $g_0 = |dx|^2$. The choice of the metric g^* in turn comes from the consideration of the hyperbolic metric g_+ on B^d, and the connection between g^* and g_+ is established through scattering theory. In Section 2 below we explain this connection and why the choice of g^* is a "natural" one for the Sobolev inequalities in Theorems A and B above. We remark that, when we apply the same procedure to the *second-order* conformal Laplacian on B^d, the metric g^* that we obtain agrees with g_0. In this sense, our choice of the metric g^* is natural. We also believe that our method may lead to the formulation of other sharp inequalities.

As an application of Theorems A and B, we derive the extremal metric of a Polyakov–Álvarez–type log-determinant functional on Euclidean 4-balls. This result is stated below as Theorem C.

1.1. Application to Polyakov–Álvarez–type formulas

We conclude by providing an application to the study of the extremals of functional determinants for conformally covariant operators on manifolds of dimension 4 with boundary.

First we recall the classical Polyakov–Álvarez formula for surfaces with boundary: if $(M, \partial M, g)$ is a surface with boundary and we let g_w denote the metric

$$g_w = e^{2w} g,$$

where w is a smooth function on M, then there is a formula relating the functional determinants associated to the Laplace–Beltrami operators Δ_{g_w} and Δ_g and this formula is written entirely in terms of integrals of local invariants of (M, g) and of ∂M. More precisely, we have

$$\begin{aligned} F[w] &:= \log\Big(\frac{\det(\Delta_{g_w})}{\det(\Delta_g)}\Big) \\ &= -\frac{1}{6\pi}\Big(\frac{1}{2}\int_M |\nabla w|_g^2 \, dV_g + \int_M w K_g \, dV_g + \oint_{\partial M} w k_g \, ds_g\Big) \\ &\quad - \frac{1}{4\pi}\Big(\oint_{\partial M} k_{g_w} (ds)_{g_w} - \oint_{\partial M} k_g (ds)_g\Big). \end{aligned} \tag{1.11}$$

Here K_g represents the Gauss curvature of g, k_g is the geodesic curvature of ∂M with respect to g, and $(ds)_g$ is the line element on ∂M with respect to g. In the celebrated work of Osgood, Phillips, and Sarnak (see [25]–[27]) it is shown that there is a relationship between the Lebedev–Milin inequality (1.4) and the study of extremals of the ratio $w \to F[w]$. More precisely, if in (1.11) we isolate the functional determinant coefficient

$$F_1[w] = -\frac{1}{6\pi}\Big(\frac{1}{2}\int_M |\nabla w|_g^2 \, dV_g + \int_M w K_g \, dV_g + \oint_{\partial M} w k_g \, ds_g\Big), \tag{1.12}$$

then $F_1[w]$ is conformally invariant and it is proved in [26] that inequality (1.4) can be applied to show that $F_1[w]$ has a definite sign when the background metric is the Euclidean metric.

A fundamental point in the derivation of (1.11) is that the Laplace–Beltrami operator Δ_g has a conformal invariance property in dimension 2; however, this conformal property fails in higher dimensions. A result extending (1.11) for functional determinants of conformally covariant operators on compact manifolds with boundary in dimension 4 was obtained by Chang and Qing [8] based on an earlier preliminary version of the formula by Branson and Gilkey [4]. Chang and Qing [8] considered the *conformal Laplacian operator* L_g given explicitly in dimension 4 by

$$L_g = -\Delta_g + \frac{1}{6} R_g,$$

and they coupled L_g with a boundary operator, namely, the *Robin operator* defined as

$$\mathfrak{B}_g = \frac{\partial}{\partial n} + \frac{1}{3} H,$$

where n is the outward normal to the boundary ∂M and H is the mean curvature of ∂M with respect to g. The pair $(L_g, \mathfrak{B}_g)$ and its conformal properties are fundamental in the study of the Yamabe problem with boundary (see, e.g., [12]). Let $(M^4, \partial M, g)$ be a compact manifold with boundary, and let $g_w = e^{2w} g$ where w is a smooth function defined on $\overline{M}$. For the pair, $(L_g, \mathfrak{B}_g)$, it is proved in [8] that there is a Polyakov–Álvarez formula of the form

$$\log\Big(\frac{\det(L_{g_w}, \mathfrak{B}_{g_w})}{\det(L_g, \mathfrak{B}_g)}\Big) = \sum_{i=1}^{11} \gamma_i \mathbf{I}_i[g, w], \tag{1.13}$$

where the terms $\mathbf{I}_i[g, w]$ for $i = 1, \dots, 11$ are computed in terms of integrals of local invariants of M and ∂M with respect to the metrics g and g_w. Moreover, the second term $\mathbf{I}_2[g, w]$ is conformally invariant and plays the same role as $F_1[w]$ in (1.11) (where $F_1[w]$ is given by (1.12)). We show an explicit expression for $\mathbf{I}_2[g, w]$ in Section 6 (see (6.8)). This is why we refer to $\mathbf{I}_2[g, w]$ as the *main term in the functional determinant expansion* (1.13). In [8, Corollary 3.4], under two additional geometric assumptions, one can show that $\mathbf{I}_2[g, w]$ has a definite sign in the model case (B^4, S^3, g_0).

As an application of the sharp inequality in Theorem B, we show here that the metric g^* is a minimizer of the functional $\mathbf{I}_2$ over the class of functions with prescribed boundary value and Neumann boundary condition. More precisely, let C_φ and $\tilde{C}_\varphi$ be the following classes of functions:

$$C_\varphi = \Big\{ w \in C^\infty(B^4) : w|_{S^3} = \varphi, \frac{\partial}{\partial n} w\big|_{S^3} = 0 \Big\} \tag{1.14}$$

and

$$\tilde{C}_\varphi = \Big\{ \chi \in C^\infty(B^4) : \chi|_{S^3} = \varphi, \frac{\partial}{\partial n} \chi\big|_{S^3} = -1 \Big\}. \tag{1.15}$$

Observe that any function χ in $\tilde{C}_\varphi$ can be written as $\chi = w + \rho$, where $w \in C_\varphi$. We prove that the main term in the expansion (1.13) is nonnegative for the metric g^*, and we also describe in detail the equality case.

THEOREM C

Given $\varphi \in C^\infty(S^3)$ with $\int_{S^3} e^{3\varphi}\, d\sigma = |S^3|$, then for all $w \in C_\varphi$, we have the following inequalities.

(1) $\mathbf{I}_2[g^*, w] \geq 0$ *with equality if and only if $\varphi(\xi) = -\log|1 - \langle z_0, \xi\rangle| + c$, where $z_0 \in B^4$ is fixed, $\xi \in S^3$, c is a constant, and w is a biharmonic extension of φ in C_φ. It follows that g^* is a global minimizer of $\mathbf{I}_2$ for functions in the class C_φ.*

(2) $\mathbf{I}_2[g_0, \chi] \geq 0$, *where* $\chi \in \tilde{C}_\varphi$. *Equality is attained for* $\varphi = -\log|1 - \langle z_0, \xi\rangle| + c$, *where* $z_0 \in B^4$ *is fixed,* $\xi \in S^3$, c *is a constant, and* χ *is a biharmonic extension of* φ *in* $\tilde{C}_\varphi$.

Remark 1.2
In [8] and [9] an inequality similar to the one stated in Theorem B was proved under very restrictive conditions (see, e.g., [9, Lemma 3.4]).

This article is organized as follows. In Section 2 below, we describe the background, mainly the “adapted metrics” which arise from the scattering theory of conformal geometry, which we use further below in the article. In Section 3, we derive the precise formulas for the adapted metrics in the model case of Euclidean balls in all dimensions. In Section 4, we quote a result of Beckner [1, Theorem 4.2] and apply the result together with the inequalities in Section 3 to prove Theorem A. In Section 5, we prove Theorem B. Finally in Section 6 we incorporate our results to establish Theorem C.

The method we have developed in this article to derive the sharp inequalities in question is new. We believe that this method could be applied in the future to discover and to derive inequalities of fractional orders and inequalities of higher orders for functions defined on some more general classes of manifolds.

2. Adapted metrics

As we mentioned in the Introduction, one of the key steps used in the proof of the inequalities in this article is the consideration of a special metric called the *adapted metric*, which we denote by g^* and which lies in the conformal class of the flat metric $g_0 = |dx|^2$ on the Euclidean unit ball B^d and has totally geodesic boundary. The consideration of the metric g^*, introduced in the work of Case and Chang [5], is natural from a conformal geometry point of view and via the connection of the metric as a preferred conformal compactification (as we explain below) of the hyperbolic metric on B^d.

We now recall the definition of the g^* metric in a general setting and derive its specific form in the special setting of B^d with the sphere as boundary and when the corresponding boundary operator is of order three.

2.1. *Graham, Jenne, Mason, and Sparling (GJMS) operators, scattering theory*

We first recall the definition of the GJMS operators via scattering theory (see [18]). A triple (X^{n+1}, M^n, g_+) is a *Poincaré–Einstein manifold* if

(1) X^{n+1} is (diffeomorphic to) the interior of a compact manifold with boundary $\partial X = M^n$,

(2) (X^{n+1}, g_+) is complete with $\mathrm{Ric}(g_+) = -ng_+$, and

(3) there exists a nonnegative $r \in C^\infty(X)$ such that $r^{-1}(0) = M^n$, $dr \neq 0$ along M, and the metric $g := r^2 g_+$ extends to a smooth metric on X^{n+1}.

A function r satisfying these properties is called a *defining function*, since r is only determined up to multiplication by a positive smooth function on X; it is clear that only the conformal class $[h] := [g|_{TM}]$ on M is well defined for a Poincaré–Einstein manifold.

Given a Poincaré–Einstein manifold (X^{n+1}, M^n, g_+) and a representative h of the conformal boundary, there exists a unique defining function r, called the *geodesic defining function*, such that, locally near M, the metric g_+ takes the form $g_+ = r^{-2}(dr^2 + h_r)$ for h_r a one-parameter family of metrics on M with $h_0 = h$ and having an asymptotic expansion involving only even powers of r, at least up to order n (see [14], [17], [21]).

It is well known (see [18], [22] for more general statements) that, given $f \in C^\infty(M)$ and $s \in C$ such that $\mathrm{Re}(s) > \frac{n}{2}$, $s \notin \frac{n}{2} + \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $s(n-s)$ is not in the pure-point spectrum $\sigma_{\mathrm{pp}}(-\Delta_{g_+})$ of $-\Delta_{g_+}$, the Poisson equation

$$-\Delta_{g_+} u - s(n-s)u = 0 \quad \text{in } X \tag{2.1}$$

has a unique solution of the form

$$u = Fr^{n-s} + Hr^s \quad \text{for } F, H \in C^\infty(X) \text{ and } F|_M = f. \tag{2.2}$$

Indeed, F has an asymptotic expansion

$$F = f_{(0)} + f_{(2)}r^2 + f_{(4)}r^4 + \cdots \tag{2.3}$$

for $f_{(0)} = f$, and all the functions $f_{(2\ell)}$ are determined by f. The *Poisson operator* $\mathcal{P}(s)$ is the operator which maps f to the solution $u = \mathcal{P}(s)f$, and this operator is analytic for $s(n-s) \notin \sigma_{\mathrm{pp}}(-\Delta_{g_+})$. The *scattering operator* is defined by $S(s)f = H|_M$. This defines a meromorphic family of pseudodifferential operators in $\mathrm{Re}(s) > \frac{n}{2}$. Given $\gamma \in (0, \frac{n}{2})$, Graham and Zworski [18] defined the *fractional GJMS operator* $P_{2\gamma}$ as the operator

$$P_{2\gamma} f := d_\gamma S\Big(\frac{n}{2} + \gamma\Big) f \quad \text{for } d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{2.4}$$

For $\gamma \in \mathbb{N}_0$, this definition recovers the GJMS operators (see [16]). For $\gamma \in (0, \frac{n}{2})$, Graham and Zworski showed that $P_{2\gamma}$ is a formally self-adjoint pseudodifferential operator with principal symbol equal to the principal symbol of $(-\Delta)^\gamma$, and, moreover, if $\hat{h} = e^{2\tau}h$ is another choice of the conformal representative of the conformal boundary, then

$$\hat{P}_{2\gamma} f = e^{-\frac{n+2\gamma}{2}\tau} P_{2\gamma}(e^{\frac{n-2\gamma}{2}\tau} f), \tag{2.5}$$

for all $f \in C^\infty(M)$. Together these properties justify the terminology *fractional GJMS operator*.

We adopt the convention that, for $\gamma \in (0, \frac{n}{2})$, the *fractional Q-curvature $Q_{2\gamma}$* is the scalar

$$Q_{2\gamma} := \frac{2}{n-2\gamma} P_{2\gamma}(1). \tag{2.6}$$

In particular, we emphasize that this definition produces a well-defined invariant in the critical case $2\gamma = n$, with the corresponding Q-curvature a generalization of Branson's Q-curvature (in the case when $2\gamma = n = 4$; see [2], [18]).

In this article we only deal with the special case when $\gamma = \frac{3}{2}$ and the corresponding GJMS operator P_3 is of order three. We also treat P_3 as the "boundary operator" of the corresponding GJMS operator. In the case $2\gamma = 4$, the GJMS operator P_4 is also called the *Paneitz operator* and is an elliptic operator of order four defined on the manifold X^d (see [28]) given by

$$(P_4)_g = (-\Delta_g)^2 + \delta_g\big((4A_g - (d-2)J_g g)(\nabla\cdot,\cdot)\big) + \frac{d-4}{2}(Q_4)_g, \tag{2.7}$$

where δ denotes divergence, ∇ denotes gradient on functions, A_g is the Schouten tensor $A = \frac{1}{d-2}(\mathrm{Ric} - J_g g)$, $J_g = \frac{1}{2(d-1)} R_g$, R_g is the scalar curvature of the metric g, and Q_4 is the Q-curvature

$$(Q_4)_g = -\Delta_g J_g + \frac{d}{4} J_g^2 - 2|A_g|_g^2. \tag{2.8}$$

We remark that, in general, properties and formulas for P_{2k} and Q_{2k} have been recursively derived (see [15], [19]). The conformal invariance property of (P_4, Q_4), which is a special case of (2.5), is best expressed as

$$(P_4)_{\hat{g}} U = (\psi)^{\frac{d+4}{d-4}} (P_4)_g(\psi U), \tag{2.9}$$

for all smooth functions U defined on manifolds of dimension $d \geq 5$, where $\hat{g} \in [g]$ denotes the metric $\hat{g} = (\psi)^{\frac{4}{d-4}} g$, and

$$(P_4)_{\hat{g}} U = e^{-4\tau} (P_4)_g(U), \tag{2.10}$$

for all smooth functions U defined on manifolds of dimension $d = 4$, where $\hat{g} \in [g]$ denotes the metric $\hat{g} = e^{2\tau} g$.

2.2. Adapted metrics when $\gamma = \frac{3}{2}$

Case and Chang [5] introduced a special metric called the *adapted metric* to compactify a given conformally compact Poincaré–Einstein manifold $(X^{n+1}, \partial X, g_+)$. The metric was introduced for any parameter $s = \frac{2}{n} + \gamma$ for $\gamma \in (0, \frac{n}{2})$ and for $s = n$ when n is an odd integer. For such an s, we consider the Poisson equation (2.1) with Dirichlet data $f \equiv 1$, and we denote the solution of the equation by v_s. By a result of Lee (see [21, Theorem A]), if one assumes further that the Yamabe constant of the boundary metric h is positive (this happens when the scalar curvature of some metric in $[h]$ is positive), then there is no point spectrum of $-\Delta_{g_+}$ and hence $v_s > 0$ on X. Thus, we can take $y_s := (v_s)^{\frac{1}{n-s}}$ as a defining function, and $g_s = y_s^2 g_+$ is defined as the *adapted metric*. In the limiting case, when $s = n$ and n is an odd integer, that is, when $\gamma = \frac{n}{2}$, the adapted metric is as in the work of Fefferman and Graham [13] (see also [29]) and is defined as

$$\tau = -\frac{d}{ds}\Big|_{s=n} v_s. \tag{2.11}$$

Then τ satisfies

$$-\Delta_{g_+}\tau = n, \tag{2.12}$$

and the adapted metric is defined as $g^* = e^{2\tau} g_+$.

The adapted metric for all s satisfies many good curvature properties stated in terms of a *smooth metric with measures*. Here we just cite the properties when $s = \frac{n}{2} + \frac{3}{2}$ satisfied by the metric $g^* := g_{\frac{n}{2}+\frac{3}{2}}$, which we use in the rest of this article.

PROPOSITION 2.1 (Properties of g^*; see [5, Lemmas 6.2, 7.6])

(a) *g^* is a metric with totally geodesic boundary.*

(b) $Q_4(g^*) \equiv 0$.

(c) *We define the energy of U with respect to the Paneitz operator $(P_4)_g$ as the integral quantity obtained by dropping the boundary terms of the integral $\int_{X^d} \{(P_4)_{g^*}U\} U \, dV_{g^*}$ when integrating by parts; that is,*

$$\begin{aligned} E_4(g^*)[U] = \int_{X^d} &(\Delta_{g^*}U)^2 \\ &- \big(4A_{g^*} - (d-2)J_{g^*}g^*\big)\langle \nabla_{g^*}U, \nabla_{g^*}U\rangle \, dV_{g^*}. \end{aligned} \tag{2.13}$$

For all smooth functions f defined on ∂X^d, we have the identity

$$\frac{1}{2}E_4(g^*)[U_f] = \int_{\partial X} P_3 f f \, d\sigma_{g^*} - \frac{d-4}{2}\int_{\partial X} Q_3(g^*) f^2 \, d\sigma_{g^*}, \tag{2.14}$$

where U_f denotes the (unique) solution of the equation

$$(P_4)_{g^*} U_f = 0 \quad \text{on } X^d, \tag{2.15}$$

$$U_f = f \quad \text{on } \partial X, \tag{2.16}$$

$$\frac{\partial U_f}{\partial n_{g^*}} = 0 \quad \text{on } \partial X. \tag{2.17}$$

2.3. *Explicit formula for the metric g^* on (B^d, S^{d-1}, g_H)*

In this section we derive explicit formulas for the metric g^* in the model case of (B^d, S^{d-1}, g_H), which happens to be computable.

PROPOSITION 2.2

In the model case (B^d, S^{d-1}, g_H), that is, for $g_H = \rho^{-2}|dx|^2$, $\rho = \frac{1-|x|^2}{2}$, we have the following.

(a) *When $d \geq 5$, $g^* = (\psi)^{\frac{4}{d-4}}|dx|^2$, where*

$$\psi = 1 + \frac{d-4}{2}\rho.$$

(b) *When $d = 4$, $g^* = e^{2\rho}|dx|^2 = e^{(1-|x|^2)}|dx|^2$.*

Remark 2.3

The metric g^* in (b) is called the *Fefferman–Graham metric* and has been described in several previous papers including [13], [29], and [5]. The Fefferman–Graham metric is constructed via scattering theory and satisfies several important properties, including the following:

- S^{d-1} is totally geodesic under g^*;
- the Q-curvature of g^* is zero;
- $R_{g^*} > 0$.

Proof

(a) Denote $d = n + 1$, and denote $s = \frac{n}{2} + \frac{3}{2}$. By our definition, $g^* = v_s^{\frac{2}{n-s}} g_H$, where g_H is the hyperbolic metric on B^d and v_s is the unique solution of the Poisson equation with Dirichlet data $f \equiv 1$. Then

$$-\Delta_{g_H} v_s - s(n-s)v_s = 0 \quad \text{in } B. \tag{2.18}$$

We now recall $g_H = \rho^{-2}g_0$, where $\rho = \frac{1-|x|^2}{2}$ and $g_0 = |dx|^2$ is the flat metric on B^d. Using this notation, we find that $g^* = \psi^{\frac{4}{d-4}} g_0$ where $\psi = v_s \rho^{-\frac{d-4}{2}} = v_s \rho^{s-n}$. One of the main observations of Chang and González [7] is that, in the compactified metric $g_0 = \rho^2 g_H$, (2.18) is equivalent to the equation

$$-\delta_{g_0}(\rho^a \nabla\psi) + \mathcal{E}(\rho)\psi = 0, \tag{2.19}$$

where $a = 1 - 2\gamma = -2$ in our case. The error term $\mathcal{E}(\rho)$ is given by

$$\mathcal{E}(\rho) = -\rho^{a/2}(\Delta_{g_0}\rho^{a/2}) + \Big(\gamma^2 - \frac{1}{4}\Big)\rho^{a-2} + \frac{n-1}{4n} R_{g_0}\rho^a, \tag{2.20}$$

which simplifies to

$$\mathcal{E}(\rho) = -\Delta_{g_0}(\rho^{-1})\rho^{-1} + 2\rho^{-4}.$$

Note that

$$\frac{\partial}{\partial r}\rho = -r,$$
$$\frac{\partial^2}{\partial r^2}\rho = -1.$$

Observe also that

$$\Delta_{g_0}\rho^{-1} = \frac{\partial^2}{\partial r^2}\rho^{-1} + (d-1)r^{-1}\frac{\partial}{\partial r}\rho^{-1}$$

and

$$\frac{\partial}{\partial r}\rho^{-1} = r\rho^{-2},$$
$$\frac{\partial^2}{\partial r^2}\rho^{-1} = \rho^{-2} + 2r\rho^{-3},$$

so that $\Delta_{g_0}\rho^{-1} = \frac{2}{\rho^3} + \frac{d-4}{\rho^2}$ and

$$\mathcal{E}(\rho) = -\rho^{-1}\Delta_{g_0}(\rho^{-1}) + 2\rho^{-4} = -\frac{d-4}{\rho^3}.$$

Observe that, in this case, it is reasonable to expect that ψ will be a radial function. Assuming so, we can compute the term

$$\begin{aligned}
-\delta_{g_0}(\rho^{-2}\nabla\psi) &= -2r\rho^{-3}\psi' + \rho^{-2}\Delta_{g_0}\psi \\
&= -2r\rho^{-3}\psi' - \rho^{-2}\Big(\psi'' + \frac{d-1}{r}\psi'\Big) \\
&= \frac{4}{r}\rho^{-3}\Big(\rho - \frac{1}{2}\Big)\psi' - \rho^{-2}\Big(\psi'' + \frac{d-1}{r}\psi'\Big) \\
&= -\rho^{-2}\psi'' - \frac{d-5}{r}\rho^{-2}\psi' - \frac{2}{r}\rho^{-3}\psi',
\end{aligned}$$

and (2.19) is equivalent to

$$-\rho^{-2}\psi'' - \frac{d-5}{r}\rho^{-2}\psi' - \frac{2}{r}\rho^{-3}\psi' - (d-4)\rho^{-3}\psi = 0. \tag{2.21}$$

We can now verify that $\psi = 1 + \frac{d-4}{2}\rho$ is a solution of the ODE (2.21) with the desired boundary condition $\psi|_{\partial B} \equiv 1$. This finishes the proof of (a).

We now prove (b). When $d = 4$, one can either follow a procedure similar to that for (a) to compute explicitly the solution of the PDE $-\Delta_{g_H}\tau = d - 1 = 3$ on B^4 or, after denoting $r = |x|$, find that a radial solution of $-\Delta_{g_H}\tau = d - 1 = 3$ satisfies the equation

$$\frac{\partial^2}{\partial r^2}\tau(r) + \frac{6r}{r^2-4}\frac{\partial}{\partial r}\tau(r) - \frac{2}{r}\frac{\partial}{\partial r}\tau(r) = -\frac{3}{r^2},$$

and a solution to this equation satisfying the boundary condition $e^{2\tau}g_H|_{S^3} = g_{S^3}$ is given by

$$\tau(x) = \log\Big(\frac{1-|x|^2}{2}\Big) + \frac{1-|x|^2}{2}.$$

Our adapted metric will then be $g^* = e^{2\tau}\rho^{-2}g_0 = e^{(1-|x|^2)}g_0$. We remark alternatively that we can find the metric g^* in dimension $d = 4$ by a "dimension continuity" argument of Branson [3], by computing the limit

$$e^{2\tau}\rho^{-2} = \lim_{d\to 4}(\psi_d)^{\frac{4}{d-4}} = e^{2\rho}. \qquad \square$$

Remark 2.4
As pointed out in the Introduction, if in (2.15) we replace the Paneitz operator P_4 by the conformal Laplacian and carry out the construction in Proposition 2.2, we obtain $g^* = g_0$. More precisely, if we consider the system

$$L_{g^*}U_f = 0 \quad \text{in } B^d, \tag{2.22}$$

$$U_f = f \quad \text{on } S^{d-1}, \tag{2.23}$$

$$\frac{\partial}{\partial n_{g^*}}U_f = 0 \quad \text{on } S^{d-1}, \tag{2.24}$$

where $L_{g^*} = -\Delta_{g^*} + \frac{(d-2)R}{4(d-1)}$, then the following hold.

(a) When $d \geq 3$, $g* = (\psi)^{2/(d-2)}g_0$. Then $\psi \equiv 1$ by solving the system (2.22)–(2.24) at $s = n/2 + 1/2$ for the Dirichlet data $f \equiv 1$.

(b) When $d = 2$, by solving (2.12) for $n = 1$ on $\mathbb{H}^2$, $g* = e^{2\tau}g_H = g_0$.

3. Energy identity for the model case in dimension $d > 4$

We now write the energy identity in Proposition 2.1 for the metric g^* in the model case (B^d, S^{d-1}, g_H) for $d > 4$ by using the results of Section 2.3. Recall that in this case the adapted metric is given by $g^* = \psi^{\frac{4}{d-4}} g_0$, where $\psi = 1 + \frac{d-4}{2}\rho$ and $\rho(x) = \frac{1-|x|^2}{2}$. Observe that from the definition of the energy $E(g)$ in Proposition 2.1 as the functional obtained by dropping the boundary terms when integrating by parts in $\int_{X^d} \{(P_4)_g u\} u \, dV_g$ we have

$$E_4(g)[u] = \int_{X^d} u(P_4)_g u \, dV_g + \mathcal{J}_1(g)[u] - \mathcal{J}_2(g)[u] - \mathcal{J}_3(g)[u], \tag{3.1}$$

where $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ are defined for general u by

$$\mathcal{J}_1(g)[u] = \oint_{M^{d-1}} \Delta_g u \frac{\partial}{\partial n_g} u \, d\sigma_g, \tag{3.2}$$

$$\mathcal{J}_2(g)[u] = \oint_{M^{d-1}} u \frac{\partial}{\partial n_g} (\Delta_g u) \, d\sigma_g, \tag{3.3}$$

$$\mathcal{J}_3(g)[u] = \oint_{M^{d-1}} u h(n_g, \nabla u) \, d\sigma_g, \tag{3.4}$$

with $h = 4(A_g - (d-2)Jg)$. In view of (3.1), let us write the energy identity (2.14) in Proposition 2.1 as

$$\int_{B^4} U_f (P_4)_{g^*} U_f \, dV_{g^*} + \mathcal{J}_1(g^*)[U_f] - \mathcal{J}_2(g^*)[U_f] + \mathcal{J}_3(g^*)[U_f] \tag{3.5}$$

$$= 2\oint_{S^{d-1}} P_3 f \, d\sigma - (d-4) \oint_{S^{d-1}} Q_3 f^2 \, d\sigma, \tag{3.6}$$

where U_f satisfies (2.15)–(2.17). For the following three lemmas, we consider functions u (which includes the case when $u = U_f$) that satisfy the following two conditions:

$$u|_{\partial B^4} = f, \tag{3.7}$$

$$\frac{\partial}{\partial n_{g^*}} u = 0. \tag{3.8}$$

Note that on S^{d-1} we have $n_{g_0} = n_{g^*}$. Thus, the Neumann boundary condition (3.8) can also be written in terms of g_0. We start by noting the following lemma, which holds for all dimensions $d \geq 4$.

LEMMA 3.1
Let $d \geq 4$. For the metric g^* and u satisfying (3.7) and (3.8) we have $\mathcal{J}_3(g^*)[u] = 0$ for any function $f \in C^\infty(S^{d-1})$.

Proof
Recall that $\mathcal{I}_3(g^*)[u] = \oint_{S^{d-1}} f h(n_{g^*}, \nabla u)\, d\sigma$, where $h = 4A_{g^*} - (d-2)J_{g^*} g^*$, and by virtue of (2.17) we have $g^*(n_{g^*}, \nabla u) = 0$, so that

$$h(n_{g^*}, \nabla u) = \frac{4}{d-2} \operatorname{Ric}_{g^*}(n_{g^*}, \nabla u).$$

On the other hand, if for $d \geq 4$ we write $g^* = e^{2\tau} g_0$, where of course

$$\tau(x) = \begin{cases} \rho(x) & \text{for } d = 4, \\ \frac{2}{d-4} \log(1 + \frac{d-4}{2}\rho(x)) & \text{for } d \geq 5, \end{cases}$$

then we have by the transformation law for the Ricci tensor the identity

$$\operatorname{Ric}_{g^*} = (d-2)\Big(-\nabla^2_{g_0}\tau - \frac{1}{(d-2)}(\Delta_{g_0}\tau) g_0 + d\tau \otimes d\tau - |\nabla\tau|^2 g_0\Big),$$

and since τ is radial, we have

$$\operatorname{Ric}_{g^*} = \theta\, dn \otimes dn + \omega g_0$$

for radial functions θ and ω, and then clearly $\operatorname{Ric}_{g^*}(n_{g^*}, \nabla u)|_{\partial B^4} = 0$. This implies $\mathcal{I}_3(g^*)[u] = 0$ as needed. □

We now relate the boundary terms $\mathcal{I}_1(g^*)[u]$ and $\mathcal{I}_2(g^*)[u]$ to boundary integrals computed in terms of the Euclidean metric g_0. To be more precise, recall that the operator $(P_4)_g$ satisfies the following conformal covariance property: if we write $g^* = \psi^{\frac{4}{d-4}} g_0$, then

$$(P_4)_{g^*}(u) = \psi^{-\frac{d+4}{d-4}} P_{g_0}(\psi u),$$

and therefore,

$$\begin{aligned}
&\int_{B^d} u(P_4)_{g^*}(u)\, dV_{g^*} \\
&\quad = \int_{B^4} u\psi^{-\frac{d+4}{d-4}} (P_4)_{g_0}(\psi u)\psi^{\frac{2d}{d-4}}\, dV_{g_0} \\
&\quad = \int_{B^d} \psi u (P_4)_{g_0}(\psi u)\, dV_{g_0} \\
&\quad = \oint_{S^{d-1}} f \frac{\partial}{\partial n_{g_0}}\big(\Delta_{g_0}(u\psi)\big) - \frac{\partial}{\partial n_{g_0}}(u\psi)\Delta_{g_0}(u\psi)\, d\sigma + \int_{B^4} (\Delta_{g_0}\psi u)^2\, dV_{g_0} \\
&\quad = \mathcal{I}_1(g_0)[\psi u] - \mathcal{I}_2(g_0)[\psi u] + \int_{B^4} (\Delta_{g_0}\psi u)^2\, dV_{g_0},
\end{aligned}$$

and since $\int_{B^4}(\Delta_{g_0}\psi u)^2\, dV_{g_0}$ is the Paneitz energy of ψu with respect to g_0, we are interested in computing the difference

$$\mathcal{I}_1(g^*)[u]-\mathcal{I}_2(g^*)[u]-\big(\mathcal{I}_1(g_0)[\psi u]-\mathcal{I}_2(g_0)[\psi u]\big).$$

We do this not for general u but for $u=U_f$ satisfying (2.15)–(2.17). We start with the following lemma.

LEMMA 3.2
For u satisfying (2.15)–(2.17) we have the identity

$$\begin{aligned}\mathcal{I}_1(g^*)[u]-\mathcal{I}_1(g_0)[\psi u]=\frac{d-4}{2}\oint_{S^{d-1}} f\Big(\frac{\partial^2}{\partial n_{g_0}^2}u+\tilde{\Delta} f\Big)\, d\sigma\\ -d\frac{(d-4)^2}{4}\oint_{S^{d-1}} f^2\, d\sigma.\end{aligned}$$

Proof
On S^{d-1} we have $n_{g^*}=n_{g_0}$, and n_{g_0} can be identified with the normal direction $\frac{\partial}{\partial r}$. Since in addition $\mathcal{I}_1(g^*)[u]=0$, we start by noting

$$\begin{aligned}\mathcal{I}_1(g^*,u)-\mathcal{I}_1(g_0)[\psi u]&=-\mathcal{I}_1(g_0)[\psi u]=-\oint_{S^{d-1}}\frac{\partial}{\partial n_{g_0}}(\psi u)\Delta_{g_0}(\psi u)\, d\sigma\\ &=\frac{\partial}{\partial n_{g_0}}(\psi u)\Delta_{g_0}(u\psi)\big|_{\partial B^4}\\ &=f\psi'(1)\psi(1)\Big(\frac{\partial^2}{\partial r^2}u\big|_{r=1}+\tilde{\Delta}u\Big)+u^2\psi'(\Delta_{g_0}\psi)|_{r=1}\\ &=-\frac{d-4}{2}u\Big(\frac{\partial^2}{\partial r^2}u\big|_{r=1}+\tilde{\Delta}u\Big)+d\frac{(d-4)^2}{4}u^2\\ &=-\frac{d-4}{2}f\Big(\frac{\partial^2}{\partial n_{g_0}^2}u+\tilde{\Delta} f\Big)+d\frac{(d-4)^2}{4}f^2,\end{aligned}$$

from which the result follows. □

We also have the following result.

LEMMA 3.3
For any u satisfying (3.7) and (3.8), the following identity holds:

$$\mathcal{I}_2(g^*)[u]-\mathcal{I}_2(g_0)[\psi u]=\frac{d}{2}\oint_{S^{d-1}} f\tilde{\Delta} f\, d\sigma+\frac{d-4}{2}\oint_{S^{d-1}} f\frac{\partial^2}{\partial n_{g_0}^2}u\, d\sigma.$$

Proof
Since $\sqrt{|g^*|} = \psi^{\frac{2d}{d-4}}\sqrt{|g_0|}$ and $(g^*)^{-1}\sqrt{|g^*|} = \psi^{\frac{2(d-2)}{d-4}}\sqrt{|g_0|}g_0^{-1}$, we have

$$\begin{aligned}\Delta_{g^*}u &= \frac{1}{\sqrt{|g^*|}}\partial_i\big(\sqrt{|g^*|}g^{ij}\partial_j u\big)\\ &= \frac{2(d-2)}{d-4}\psi^{-\frac{d}{d-4}}\langle\nabla\psi,\nabla u\rangle_{g_0} + \psi^{-\frac{4}{d-4}}\Delta_{g_0}u,\end{aligned}$$

and since ψ is radial, we obtain

$$\Delta_{g^*}u = \frac{2(d-2)}{d-4}\psi^{-\frac{d}{d-4}}\psi'\frac{\partial u}{\partial r} + \psi^{-\frac{4}{d-4}}\Delta_{g_0}u.$$

Since the normal direction n_{g_0} coincides with the radial direction $\frac{\partial}{\partial r}$ we have

$$\frac{\partial}{\partial n_{g^*}}\Delta_{g^*}u = \psi^{-\frac{2}{d-4}}\frac{\partial}{\partial r}\Big(\frac{2(d-2)}{d-4}\psi^{-\frac{d}{d-4}}\psi'\frac{\partial u}{\partial r} + \psi^{-\frac{4}{d-4}}\Delta_{g_0}u\Big).$$

Since u satisfies a Neumann boundary condition we have

$$\frac{\partial}{\partial n_{g^*}}\Delta_{g^*}u\Big|_{r=1} = \Big(\frac{2(d-2)}{d-4}\psi'\frac{\partial^2 u}{\partial r^2} - \frac{4}{d-4}\psi'\Delta_{g_0}u + \frac{\partial}{\partial r}\Delta_{g_0}u\Big)\Big|_{r=1},$$

and writing

$$\Delta_{g_0}u = \frac{\partial^2}{\partial r^2}u + \frac{d-1}{r}\frac{\partial}{\partial r}u + r^{-2}\tilde{\Delta}u,$$

we obtain

$$\Delta_{g_0}u|_{r=1} = \frac{\partial^2}{\partial r^2}u\Big|_{r=1} + \tilde{\Delta}u|_{r=1}$$

and

$$\begin{aligned}\frac{\partial}{\partial n_{g^*}}\Delta_{g^*}u &= 2\psi'|_{r=1}\frac{\partial^2}{\partial r^2}u\Big|_{r=1} + \psi'|_{r=1}\tilde{\Delta}u + \frac{\partial}{\partial r}\Delta_{g_0}u\Big|_{r=1}\\ &= -(d-4)\frac{\partial^2}{\partial r^2}u\Big|_{r=1} + 2\tilde{\Delta}u|_{r=1} + \frac{\partial}{\partial r}\Delta_{g_0}u\Big|_{r=1}.\end{aligned}$$

Observe, on the other hand, that

$$\Delta_{g_0}(\psi u) = u\Delta_{g_0}\psi + 2\langle\nabla u,\nabla\psi\rangle_{g_0} + \psi\Delta_{g_0}u \tag{3.9}$$

and

$$2\langle\nabla u,\nabla\psi\rangle_{g_0} = 2\psi'\frac{\partial}{\partial r}u.$$

Since $\Delta_{g_0}\psi$ is a constant and u satisfies a Neumann boundary condition we obtain

$$\partial_{n_{g_0}}\big(\Delta_{g_0}(\psi u)\big)u = \frac{\partial}{\partial r}\Big(2\psi' \frac{\partial}{\partial r}u + \psi \Delta_{g_0} u\Big)\Big|_{r=1} \tag{3.10}$$

$$= 3\psi' \frac{\partial^2 u}{\partial r^2}\Big|_{r=1} + \psi' \tilde{\Delta} u|_{r=1} + \frac{\partial}{\partial r}\Delta_{g_0} u\Big|_{r=1} \tag{3.11}$$

$$= -\frac{3}{2}(d-4)\frac{\partial^2 u}{\partial r^2}\Big|_{r=1} - \frac{d-4}{2}\tilde{\Delta} u\Big|_{r=1} + \frac{\partial}{\partial r}\Delta_{g_0} u\Big|_{r=1}. \tag{3.12}$$

We have obtained

$$\begin{aligned}
&\oint_{S^{d-1}} \frac{\partial}{\partial n_{g^*}}(\Delta_{g^*} u)\, d\sigma - \oint_{S^{d-1}} \frac{\partial}{\partial n_{g_0}}(\Delta_{g_0}\psi u)\, d\sigma \\
&\quad = \oint_{S^{d-1}} u\Big(-(d-4)\frac{\partial^2}{\partial r^2}u\Big|_{r=1} + 2\tilde{\Delta} u|_{r=1} + \frac{\partial}{\partial r}\Delta_{g_0} u\Big|_{r=1}\Big)\, d\sigma \\
&\qquad - \oint_{S^{d-1}} u\Big(-\frac{3}{2}(d-4)\frac{\partial^2 u}{\partial r^2}\Big|_{r=1} - \frac{d-4}{2}\tilde{\Delta} u|_{r=1} + \frac{\partial}{\partial r}\Delta_{g_0} u\Big|_{r=1}\Big)\, d\sigma \\
&\quad = \frac{d}{2}\oint_{S^{d-1}} f\tilde{\Delta} f\, d\sigma + \frac{d-4}{2}\oint_{S^{d-1}} u \frac{\partial^2}{\partial r^2}u\Big|_{r=1}\, d\sigma \\
&\quad = \frac{d}{2}\oint_{S^{d-1}} f\tilde{\Delta} f\, d\sigma + \frac{d-4}{2}\oint_{S^{d-1}} f \frac{\partial^2}{\partial n_{g_0}^2}u\, d\sigma.
\end{aligned}$$

□

We have shown the following result.

COROLLARY 3.4
For u satisfying (3.7) and (3.8), the following identity holds:

$$\begin{aligned}
&\mathcal{I}_1(g^*)[u] - \mathcal{I}_2(g^*)[u] - \big(\mathcal{I}_1(g_0)[\psi u] - \mathcal{I}_2(g_0)[\psi u]\big) \\
&\quad = -2\oint_{S^{d-1}} f\tilde{\Delta} f\, d\sigma - \frac{d(d-4)^2}{4}\oint_{S^{d-1}} f^2\, d\sigma.
\end{aligned}$$

COROLLARY 3.5
The energy identity (2.14) is equivalent to

$$\mathcal{I}_1(g_0)[\psi U_f] - \mathcal{I}_2(g_0)[\psi U_f]$$

$$= 2\oint_{S^{d-1}} f P_3 f\, d\sigma - (d-4)\oint_{S^{d-1}} Q_3 f^2\, d\sigma \tag{3.13}$$

$$-2\oint_{S^{d-1}} |\tilde{\nabla} f|^2\, d\sigma + \frac{d(d-4)^2}{4}\oint_{S^{d-1}} f^2\, d\sigma, \tag{3.14}$$

where Q_3 is given by (2.6).

4. Proof of Theorem A

We first assert that there is a class of pseudodifferential operators of fractional order $2\gamma \leq d-1$ that are intrinsically defined on S^{d-1} with the conformal covariant property. The formula was explicitly computed by Branson [3]. In the special case when $2\gamma = 3$, we denote the operator as $\mathscr{P}_3$. The formula of $\mathscr{P}_3$ is given as

$$\mathscr{P}_3 = (B-1)B(B+1), \tag{4.1}$$

where $B = \sqrt{-\tilde{\Delta} + (\frac{d-2}{2})^2}$. Note that B (and therefore $\mathscr{P}_3$) is completely determined on spherical harmonics by

$$B\xi^k = \Big(k + \frac{d-2}{2}\Big)\xi^k,$$

where ξ^k is a spherical harmonic of order k. Observe also that for the rough Laplacian $\tilde{\Delta}$ with respect to the round metric $g_{S^{d-1}}$ we have

$$\tilde{\Delta}\xi^k = -k(k+d-2)\xi^k.$$

The operator $\mathscr{P}_3$ plays a role in the formulation of the following sharp Moser–Trudinger and Sobolev inequalities proved in [1]. Before stating the results from [1, Theorems 1, 5], we adopt a convention about normalized measures intended to simplify the proof of Lemma 4.4 and Theorem A.

CONVENTION 4.1

We use the following convention for normalized measures.

- *On S^{d-1}, $d\sigma$ is the unnormalized Lebesgue measure with respect to the spherical metric $g_{S^{d-1}}$. We use $d\xi$ to denote the normalized measure*

$$d\xi = \frac{dV_{g_{S^{d-1}}}}{|S^{d-1}|} = \frac{d\sigma}{\frac{2\pi^{d/2}}{\Gamma(d/2)}}, \tag{4.2}$$

  where $|S^{d-1}|$ is the volume of S^{d-1} with respect to the metric $g_{S^{d-1}}$.
- *dx denotes the Lebesgue measure on B^d with respect to the Euclidean metric, and $d\mu$ is the measure*

$$d\mu = \frac{dx}{|B^d|}, \tag{4.3}$$

  where $|B^d|$ is the volume of the d-ball B^d with respect to the Euclidean metric. Note that $d|B^d| = |S^{d-1}|$.

We observe that, from the definition of the normalized measures $d\xi$ and $d\mu$ in (4.2) and (4.3), respectively, the measure $|S^{d-1}|^{-1}\, dV_{g_0} = \frac{dx^2}{|S^{d-1}|}$ is such that

$$\big(|S^{d-1}|^{-1}\, dV_{g_0}\big)|_{S^{d-1}} = d\xi|_{\partial B^4}$$

and also

$$\frac{dV_{g_0}}{|S^{d-1}|} = \frac{|B^d|}{|S^{d-1}|}\, d\mu = \frac{1}{d}\, d\mu. \tag{4.4}$$

We now state the main inequalities in [1] that we are interested in.

THEOREM 4.2 (see [1])
Let $f \in C^\infty(S^{d-1})$, and let $\mathcal{P}_3$ be the operator defined on S^{d-1} by (4.1).
(a) *If $d = 4$ and $\varphi \in C^\infty(S^3)$, then*

$$\log\Big(\oint_{S^3} e^{\varphi-\bar{\varphi}}\, d\xi\Big) \le \frac{1}{2(3)!}\oint_{S^3} \varphi \mathcal{P}_3 \varphi\, d\xi.$$

(b) *If $d > 4$ and $f \in C^\infty(S^{d-1})$, then*

$$a_d\Big(\oint_{S^{d-1}} |f|^{\frac{2(d-1)}{(d-4)}}\, d\xi\Big)^{\frac{2(d-1)}{(d-4)}} \le \oint_{S^{d-1}} f\mathcal{P}_3 f\, d\xi,$$

where $a_d = \frac{d(d-2)(d-4)}{8}$.

The operator $\mathcal{P}_3$ given above is closely related to the scattering operator P_3 mentioned in Proposition 2.1. In fact, Branson proved the following.

THEOREM 4.3 (see [2])
In the model space (B^d, S^{d-1}, g_0) for $d > 4$ and for any g in the same conformal class as g_0, we have
(1) $P_3 = \mathcal{P}_3$,
(2) $Q_3(g) = \frac{2}{d-4}\mathcal{P}_3(1) = \frac{d(d-2)}{4}$.

We now start the proof of Theorem A by first justifying the boundary condition (1.7) imposed in the statement of the theorem.

LEMMA 4.4
A function u satisfies (3.7) and (3.8) if and only if the function $v = \psi u$ is in $\mathcal{G}_f$, where $\mathcal{G}_f$ denotes the class of functions

$$\mathcal{G}_f = \Big\{v \in C^\infty(B^d) : v = f \text{ on } S^{d-1},\ \text{and } \frac{\partial v}{\partial n_{g_0}} = -\frac{d-4}{2} f \text{ on } S^{d-1}\Big\}.$$

Proof

This follows directly from the properties of the function ψ, which equals 1 and whose Neumann derivative equals $-\frac{d-4}{2}$ on the boundary S^{d-1} of the Euclidean ball B^d. □

Proof of Theorem A

We start by noting that, since $Q_3(g^*) = \frac{d(d-2)}{4}$, we can rewrite the energy identity (3.13) as

$$\oint_{S^{d-1}} f P_3 f \, d\sigma = \frac{1}{2}\mathcal{I}_1(g_0)[\psi U_f] - \frac{1}{2}\mathcal{I}_2(g_0)[\psi U_f] + \oint_{S^{d-1}} |\tilde{\nabla} f|^2 \, d\sigma + \frac{d(d-4)}{4}\oint_{S^{d-1}} |f|^2 \, d\sigma,$$

and by Theorem 4.3 we see that

$$\oint_{S^{d-1}} f P_3 f \, d\sigma = \oint_{S^{d-1}} f \mathcal{P}_3 f \, d\sigma. \tag{4.5}$$

On the other hand, if we set $v_f = \psi U_f$, we see that v_f is a biharmonic extension of f to B^4 and v_f is in $\mathcal{G}_f$. Since v_f is biharmonic, we have

$$\begin{aligned}\frac{1}{2}\mathcal{I}_1(g_0)[\psi U_f] - \frac{1}{2}\mathcal{I}_2(g_0)[\psi U_f] &= \frac{1}{2}\mathcal{I}_1(g_0)[v_f] - \frac{1}{2}\mathcal{I}_2(g_0)[v_f] \\ &= \frac{1}{2}\int_{B^4} (\Delta_{g_0} v_f)^2 \, dx.\end{aligned}$$

From Theorem 4.2(b), (4.5), and (4.4) we obtain

$$\begin{aligned} a_d\Big(\oint_{S^{d-1}} |f|^{\frac{2(d-1)}{d-4}} \, d\xi\Big)^{\frac{d-4}{2(d-1)}} &\le \oint_{S^{d-1}} f P_3 f \, d\xi \\ &= \frac{1}{2d}\int_{B^4} (\Delta_{g_0} v_f)^2 \, d\mu + \oint_{S^{d-1}} |\tilde{\nabla} f|^2 \, d\xi \\ &\quad + \frac{d(d-4)}{4}\oint_{S^{d-1}} f^2 \, d\xi. \end{aligned} \tag{4.6}$$

Thus, for a general function $v \in \mathcal{G}_f$, it is clear that $\int_{B^4} (\Delta_{g_0} v)^2 \, d\mu$ is minimized by biharmonic functions in $\mathcal{G}_f$, which yields the desired inequality. The assertion on the functions for which equality in (1.8) is attained follows from an observation made in [1], where it is proved that equality in Theorem 4.2(b) is attained for functions of the form $f(\xi) = c|1 - \langle z_0, \xi\rangle|^{\frac{4-d}{4}}$ where c is a constant. We remark that, for $v \in \mathcal{G}_f$, inequality (4.6) is the same as (1.8) in the statement of Theorem A after changing the notation $d\xi$, $d\mu$ back to $d\sigma$, dx. □

5. Proof of Theorem B

Recall that, when $d = 4$, the metric g^* is given by $g^* = e^{2\rho} g_0$ where $\rho(x) = \frac{1-|x|^2}{2}$, and, therefore, for the Paneitz operator P_4 we have

$$(P_4)_{g^*} = e^{-4\rho}(\Delta_{g_0})^2. \tag{5.1}$$

Fixing a function $\varphi \in C^\infty(S^3)$, recall that we have defined the class C_φ given by

$$C_\varphi = \Big\{ w \in C^\infty(B^4) : w|_{S^3} = \varphi, \frac{\partial}{\partial n} w|_{S^3} = 0 \Big\}.$$

Assuming that w_φ is a biharmonic function in C_φ, we first recall that the energy identity (2.14) in the case $d = 4$ is of the form

$$E_4(g^*)[w_\varphi] = 2 \oint_{S^3} \varphi P_3 \varphi \, d\sigma. \tag{5.2}$$

Using the identities (3.5) and (3.6) together with the proof of Lemma 3.1 we reach the conclusion that (5.2) is equivalent to

$$-\mathcal{I}_2(g^*)[w_f] = 2 \oint_{S^3} \varphi P_3 \varphi \, d\sigma. \tag{5.3}$$

We now compute the difference $\mathcal{I}_2(g_*)[w_\varphi] - \mathcal{I}_2(g_0)[w_\varphi]$. We start by noting that for any w in C_φ we have

$$\begin{aligned} \frac{\partial}{\partial n_{g^*}} (\Delta_{g^*} w)\big|_{S^3} &= \frac{\partial}{\partial n_{g_0}} (\Delta_{g_0} w)\big|_{S^3} + 2\Delta_{g_0} w - 2 \frac{\partial^2}{\partial n_{g_0}^2} w\big|_{S^3} \\ &= \frac{\partial}{\partial n_{g_0}} (\Delta_{g_0} w)\big|_{S^3} + 2\tilde{\Delta}\varphi. \end{aligned}$$

Thus, (5.3) takes the form

$$-\mathcal{I}_2(g_0)[w_\varphi] - 2 \oint_{S^3} \varphi \tilde{\Delta} \varphi \, d\sigma = 2 \oint_{S^3} \varphi P_3 \varphi \, d\sigma. \tag{5.4}$$

Integrating by parts we obtain

$$\int_{B^4} (\Delta_{g_0} w_\varphi)^2 \, dV_{g_0} + 2 \oint_{S^{d-1}} |\tilde{\nabla} \varphi|^2 \, d\sigma = 2 \oint_{S^{d-1}} \varphi P_3 \varphi \, d\sigma. \tag{5.5}$$

We now prove Theorem B.

Proof of Theorem B

Normalizing the measure $d\sigma$ to obtain $d\xi$ and assuming that $\bar{\varphi} = 0$ we see from Theorem 4.2(a) that

$$\log\Big(\oint_{S^3} e^{3\varphi}\, d\xi\Big) \le \frac{9}{2(3)!}\oint_{S^3} \varphi P_3\varphi\, d\xi$$
$$= \frac{1}{4(4!)}\int_{B^4} (\Delta_{g_0} w_\varphi)^2\, d\mu + \frac{9}{2(3)!}\oint_{S^3} |\tilde{\nabla}\varphi|^2\, d\xi.$$

Clearly w_φ minimizes $\int_{B^4}(\Delta_{g_0} w_\varphi)^2\, d\mu$ in the class C_φ, and this yields the desired inequality. As in the proof of Theorem A, the assertion about the functions for which equality is attained in (1.10) follows from the form of the functions for which equality is achieved in Theorem 4.2(a). As proved by Beckner [1], all of these functions are of the form $\varphi = -\log|1-\langle z_0, \xi\rangle| + c$, where c is a constant, $\xi \in S^3$, and $|z_0| < 1$ is fixed. □

6. Proof of Theorem C

We start by describing the term $\mathbf{I}_2[g, w]$ appearing in (1.13) explicitly. However, before writing out the formula proved in [8], we need to describe important boundary analogues of the Paneitz operator and of Q-curvature for compact manifolds with boundary in dimension 4. Consider a manifold with boundary $(M^4, \partial M, g)$, and let $\tilde{g}$ be the restriction of g to ∂M. Let n be the unit *outward* normal, and let us use Greek letters to denote directions tangent to the boundary, that is, $g_{\alpha\beta} = \tilde{g}_{\alpha\beta}$. Observe that with this convention the second fundamental form is given by

$$\mathrm{II}_{\alpha\beta} = \frac{1}{2}\frac{\partial}{\partial n} g_{\alpha\beta}, \tag{6.1}$$

and let us use H to denote the mean curvature, that is, $H = g^{\alpha\beta}\mathrm{II}_{\alpha\beta}$. The boundary operator P_3^b defined in [8] is

$$P_3^b w = -\frac{1}{2}\frac{\partial}{\partial n}\Delta_g w - \tilde{\Delta}\frac{\partial}{\partial n} w - \frac{2}{3} H\tilde{\Delta} w + \mathrm{II}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta w$$
$$+ \frac{1}{3}\tilde{\nabla} H\cdot\tilde{\nabla} w + (F - 2J)\frac{\partial}{\partial n} w, \tag{6.2}$$

where $\tilde{\Delta}, \tilde{\nabla}$ are the boundary counterparts of Δ_g, ∇. Note that this time the function w is defined in all of $\overline{B}^4$. The convention that we use for the Laplacian is that $\Delta_g w$ stands for

$$\Delta_g w = \nabla_n\nabla_n w + g^{\alpha\beta}\nabla_\alpha\nabla_\beta w,$$

where $\nabla_n\nabla_n w$ means two covariant derivatives with respect to the metric g in the normal direction n. The terms F and J and (6.2) are used to denote

- $F = \mathrm{Ric}_{nn}$,
- $J = \frac{R_g}{6}|_{\partial M}$, that is, the restriction to the boundary of the ambient scalar curvature R_g.

We also have a boundary curvature T_3 given by

$$T_3 = -\frac{1}{2}\frac{\partial}{\partial n}J + JH - \langle G, \mathrm{II}\rangle + \frac{1}{9}H^3 - \frac{1}{3}\mathrm{tr}_{\tilde{g}}(\mathrm{II}^3) + \frac{1}{3}\tilde{\Delta}H, \tag{6.3}$$

where

- $G^{\alpha}_{\beta} = R^{\alpha}_{n\beta n}$,
- $\langle G, \mathrm{II}\rangle = R_{\alpha n\beta n}\mathrm{II}_{\alpha\beta}$,
- $\mathrm{II}^3_{\alpha\beta} = \mathrm{II}_{\alpha\gamma}\mathrm{II}^{\gamma\delta}\mathrm{II}_{\delta\beta}$.

For the model case $(B^4, S^3, ds^2_{\mathbb{R}^4})$ we easily see that

$$\mathrm{II} = g_{S^3}, \tag{6.4}$$

$$H = 3, \tag{6.5}$$

and $\langle G, \mathrm{II}\rangle = F = J = 0$, so that if n is the outer normal to S^3 with respect to $g_0 = ds^2$, then

$$P^b_3 = -\frac{1}{2}\frac{\partial}{\partial n}\Delta_{g_0} - \tilde{\Delta}\frac{\partial}{\partial n} - \tilde{\Delta} \tag{6.6}$$

and

$$T_3 = \frac{1}{9}H^3 - \frac{1}{3}\mathrm{tr}(\mathrm{II}^3) = 2. \tag{6.7}$$

The formula proved by Chang and Qing [8] is

$$\mathbf{I}_2[g, w] = b_2[g_0, w] + \frac{1}{12}D[g_0, w], \tag{6.8}$$

where

$$\begin{aligned} b_2[g, w] = \frac{1}{4}\int_{B^4} w(P_4)_g w\, dV_g + \frac{1}{2}\int_{B^4} wQ_g\, dV_g \\ + \frac{1}{2}\int_{S^3} wP^b_3 w\, d\sigma + \int_{S^3} wT_3\, d\sigma \end{aligned} \tag{6.9}$$

and[1]

$$\begin{aligned} D[g, w] = -\oint_{S^3}\Big(-R_g - \frac{1}{3}H^2 + 3F\Big)\frac{\partial}{\partial n}w\, d\sigma \\ - \oint_{S^3} H\frac{\partial^2}{\partial n^2}w\, d\sigma - \oint_{S^3} H\tilde{\Delta}w\, d\sigma. \end{aligned} \tag{6.10}$$

[1]Note that we are using a different sign convention than the one used by Chang and Qing (see [8], [9]), since they consider the inner unit normal and not the outer normal.

We have used in (6.9) the same notation employed in [8]. For the model case (B^4, S^3, g_0), a simple computation shows that

$$D[g_0, w] = 3\oint_{S^3}\Big(\frac{\partial}{\partial n}w - \frac{\partial^2}{\partial n^2}w\Big). \tag{6.11}$$

It is well known that the Paneitz operator $(P_4)_g$ and Q-curvature $(Q_4)_g$ have conformal covariance properties, but it turns out that P_3^b and T_3 are also conformally covariant. More precisely, letting $g_\tau = e^{2\tau}g$, we have the transformation laws

(1) $(P_3^b)_{g_\tau} = e^{-3\tau}(P_3^b)_g$,

(2) $(P_3^b)_g\tau + (T_3)_g = (T_3)_{g_\tau}e^{3\tau}$.

Applying the conformal covariance properties above to the pair of metrics g_0 and $g^* = e^{2\rho}g_0$, we easily obtain the following.

LEMMA 6.1

The metric g^* has the following properties:

(1) $(Q_4)_{g^*} = (Q_4)_{g_0} = 0$,

(2) $(T_3)_{g^*} = (T_3)_{g_0} = 2$.

Hence,

$$b_2[g^*, w] = b_2[g_0, w], \tag{6.12}$$

for all $w \in C^\infty(B^4)$.

It was pointed out by Chang and Qing [9] that, on the model space $(B^4, S^3, |dx|^2)$ where again $g_0 = |dx|^2$ is the Euclidean metric, there is a close relationship between the operator P_3^b and Beckner's operator $\mathscr{P}_3$. In fact, we have the following result.

LEMMA 6.2 ([9, Lemma 3.3, Corollary 3.1])

Suppose that w satisfies

$$\Delta_{g_0}^2 w = 0 \quad \text{on } B^4, \tag{6.13}$$

$$w|_{S^3} = \varphi. \tag{6.14}$$

Then

$$P_3^b w|_{S^3} = \mathscr{P}_3\varphi.$$

Remark 6.3

We point out that Lemma 6.2 holds regardless of the value of the normal derivative of w at S^3.

As above, we are interested in the class C_φ of functions w defined on B^4 satisfying $w|_{S^3} = f$ and $\frac{\partial}{\partial n} w|_{S^3} = 0$. In addition to conformal invariance, the functional $\mathbf{I}_2$ satisfies the following property for the metrics g_0 and $g^* = e^{2\rho} g_0$.

LEMMA 6.4
For any function w in C_φ and $g^* = e^{2\rho} g_0$, where again $\rho = \frac{(1-|x|^2)}{2}$, we have

$$\mathbf{I}_2[g^*, w] = \mathbf{I}_2[g_0, w + \rho].$$

Proof
Since (B^4, g^*) has a totally geodesic boundary and $w \in C_\varphi$, we observe that $D[g^*, w] = 0$. It then follows from Lemma 6.1 that

$$\mathbf{I}_2[g^*, w] = b_2[g^*, w] = b_2[g_0, w]. \tag{6.15}$$

From the expression for $D[g_0, w]$ in (6.11) we have

$$\begin{aligned} \mathbf{I}_2[g_0, w + \rho] &= b_2[g_0, w + \rho] + \frac{1}{12} D[g_0, w] \\ &= b_2[g_0, w + \rho] + \frac{1}{4} \oint_{S^3} (w + \rho)_n \, d\sigma - \frac{1}{4} \oint_{S^3} (w + \rho)_{nn} \, d\sigma. \end{aligned} \tag{6.16}$$

We now compute all terms involved in the expression of $b_2[g_0, w + \rho]$. The function ρ satisfies the following equations with respect to the flat metric:

$$\begin{aligned} (P_4)_{g_0} \rho &= 0, \\ P_3^b \rho &= 0, \end{aligned}$$

and from these identities we observe that

$$\begin{aligned} \int_{B^4} (w + \rho)(P_4)_{g_0}(w + \rho) \, dV_{g_0} = \int_{B^4} & w (P_4)_{g_0} w \, dV_{g_0} \\ &+ \int_{B^4} \rho (P_4)_{g_0} w \, dV_{g_0}, \end{aligned} \tag{6.17}$$

$$\oint_{S^3} (w + \rho)(P_3^b)_{g_0}(w + \rho) \, d\sigma = \oint_{S^3} w (P_3^b)_{g_0} w \, d\sigma. \tag{6.18}$$

Our goal now is to compute the term $\int_{B^4} \rho (P_4)_{g_0} w \, dV_{g_0}$. Integrating by parts we have

$$\begin{aligned} &\int_{B^4} \rho (P_4)_{g_0} w \, dV_{g_0} \\ &\quad = \int_{B^4} \rho (\Delta_{g_0}^2) w \, dV_{g_0} \end{aligned}$$

$$= \int_{B^4} (\Delta_{g_0}\rho)(\Delta_{g_0}w)\, dV_{g_0} + \oint_{S^3} \rho \frac{\partial}{\partial n}(\Delta_{g_0}w)\, d\sigma - \oint_{S^3} \frac{\partial \rho}{\partial n} \Delta_{g_0} w\, d\sigma$$
$$= -4 \int_{B^4} \Delta_{g_0} w\, dV_{g_0} + \oint_{S^3} \Delta_{g_0} w\, d\sigma$$
$$= \oint_{S^3} -4\frac{\partial w}{\partial n} + \Delta_{g_0} w\, d\sigma. \tag{6.19}$$

Using (6.19) together with the expression $\Delta_{g_0}|_{S^3} = w_{nn} + 3w_n + \tilde{\Delta} w$ we conclude that

$$\int_{B^4} w (P_4)_{g_0} w\, dV_{g_0} = \oint_{S^3} (w_{nn} - w_n)\, d\sigma. \tag{6.20}$$

Combining (6.17), (6.18), Lemma 6.1, and (6.20) we obtain

$$b_2[g_0, w+\rho] = b_2[g_0, w] + \frac{1}{4}\oint_{S^3} (w_n - w_{nn})\, d\sigma, \tag{6.21}$$

and combining (6.21) with (6.15) and (6.16) we conclude that

$$\mathbf{I}_2[g^*, w] = \mathbf{I}_2[g_0, w+\rho],$$

for all $w \in C_\varphi$ as needed. □

We now prove Theorem C.

Proof of Theorem of C
We start by proving part (1). Since $w \in C_\varphi$, we have from (6.12) that

$$\mathbf{I}_2[g^*, w] = b_2[g_0, w].$$

Recall that for the model space (B^4, S^3, g_0) and for $w \in C_\varphi$ we have

$$(P_4)_{g_0} w = \Delta^2_{g_0} w, \tag{6.22}$$
$$(P_3^b)_{g_0} w = -\frac{1}{2}\frac{\partial}{\partial n}\Delta_{g_0} w - \tilde{\Delta}\varphi, \tag{6.23}$$
$$(T_3)_{g_0} = 2, \tag{6.24}$$
$$(Q_4)_{g_0} = 0, \tag{6.25}$$

and a simple computation shows that

$$\mathbf{I}_2[g^*, w] = b_2[g_0, w] = \frac{1}{4}\int_{B^4} (\Delta_{g_0} w)^2\, dV_{g_0} + \frac{1}{2}\oint_{S^3} |\tilde{\nabla}\varphi|^2\, d\sigma + 2\oint_{S^3} \varphi\, d\sigma.$$

From Theorem B we have the lower bound

$$\begin{aligned}\mathbf{I}_2[g^*, w] &= \frac{1}{4}\int_{B^4}(\Delta_{g_0} w)^2\, dV_{g_0} + \frac{1}{2}\oint_{S^3}|\tilde{\nabla}\varphi|^2\, d\sigma + 2\oint_{S^3}\varphi\, d\sigma \\ &\geq \frac{4\pi^2}{3}\log\Big(\frac{1}{2\pi^2}\oint_{S^3} e^{3(\varphi-\bar{\varphi})}\, d\sigma\Big) + 2\oint_{S^3}\varphi\, d\sigma,\end{aligned}$$

and since $\oint_{S^3} e^f\, d\sigma = |S^3| = 2\pi^2$ we obtain

$$\mathbf{I}_2[g^*, w] \geq -4\pi^2\bar{\varphi} + 2\oint_{S^3}\varphi\, d\sigma = -2\oint_{S^3}\varphi\, d\sigma + 2\oint_{S^3}\varphi\, d\sigma = 0.$$

Now, if $\mathbf{I}_2[g^*, w] = 0$, we have

$$\frac{1}{4}\int_{B^4}(\Delta_{g_0} w)^2\, dV_{g_0} + \frac{1}{2}\oint_{S^3}|\tilde{\nabla}\varphi|^2\, d\sigma = \frac{4\pi^2}{3}\log\Big(\frac{1}{2\pi^2}\oint_{S^3} e^{3(\varphi-\bar{\varphi})}\, d\sigma\Big),$$

and from Theorem B it follows that $\varphi(\xi) = -\log|1 - \langle z_0, \xi\rangle|1 - \langle z_0, \xi\rangle| + c$, where $z_0 \in B^4$ is fixed, $\xi \in S^3$, c is a constant, and w is a biharmonic extension of φ in C_φ.

Part (2) follows from combining part (1) with Lemma 6.4. ☐

In a forthcoming joint work of the authors with Jeffrey Case, we give some further applications of the lower bound in Theorem C and also extend the study to extremals for the log-determinant functional (1.13).

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