Chapter 1

Algebraic Graph Theory

- Linear algebra
- Group theory (Cayley graphs, Dynkin diagrams)

1.1 Eigenvalues

Definition 1.1 (Adjacency matrix). Let $G = (V, E)$ be a finite graph. The *adjacency matrix* $A(G) = \{0, 1\}^{V\times V}$ is defined by

$$A(G)_{v,w} = \begin{cases} 1 & \text{if } v \sim w \\ 0 & \text{otherwise} \end{cases}$$

Recall the characteristic polynomial of a square matrix M over $\mathbb{C}$

$$\phi_M(t) \coloneqq \det(tI - M).$$

Question: Is a graph discussed in the lecture (and homework) always simple?

Answer: Yes. (According to the instructor).

Definition 1.2 (Characteristic polynomial). Let $G = (V, E)$ be a finite graph. The *Characteristic polynomial* is defined by

$$\phi_G(t) \coloneqq \phi_{A(G)}(t),$$

and call the zeros (with multiplicitier) the eigenvalues of G.

Definition 1.3 (Spectrum). The *spectrum* of G is the multiset of its eigenvalues.

Example 1.1 (Spectrum).

$$A(G) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\phi_G(t) = \det \begin{pmatrix} t & -1 \\ -1 & t \end{pmatrix} = t^2 - 1$$

$$\text{spectrum} : \{1, -1\}$$

Remark. Eigenvalues provide information about the connectivity of the graph.

Theorem 1.1. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric, then A* orthogonally diagonalizable *over $\mathbb{R}$, i.e., there exists an orthonormal basis*

$$v^{(1)}, v^{(2)}, \cdots, v^{(n)} \in \mathbb{R}^n$$

of eigenvectors of A corresponding to real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$, we have

$$A = \sum_{i=1}^{n} \lambda_i v^{(i)} v^{(i)T}$$

Therefore the eigenvalues of any graph G are all real and we'll denote then

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$

where $n = |V(G)|$.

Question: What is the intuition for

$$A = \sum_{i=1}^{n} \lambda_i v^{(i)} v^{(i)T}?$$

Answer: The fact that the sum equals A we can just check explicitly: if we denote the sum by S, we compute

$$Sv^{(i)} = \left(\sum_{j=1}^{n} \lambda_j v^{(j)} v^{(j)T}\right) v^{(i)} = \sum_{j=1}^{n} \lambda_j v^{(j)} \left(v^{(j)T} v^{(i)}\right) = \lambda_i v^{(i)} \quad \forall i = 1, \cdots, n,$$

and so $S = A$. Remember that $x^T y$ is just the inner product of vectors x and y in $\mathbb{R}^n$. Here we use distributive law and the property of orthonormal basis that $v^{(i)T} v^{(j)} = v^{(j)T} v^{(i)} = \mathbb{I}_{\{i=j\}}$.

Theorem 1.2 (Perron-Forbenius)**.** *If a matrix $A \in \mathbb{R}^{n\times n}$ has nonnegative entries, then the spectral radius of A (i.e., the maximum magnitude over all complex eigenvalues of A) is an eigenvalue of A, corresponding to an eigenvector in $\mathbb{R}^n_{\geq 0}$.*

Therefore for any graph G, $\lambda_1(G)$ is the spectral radius and corresponds to an eigenvector with nonnegative entries. Perron-Forbenius also implies if G is connected, then $\lambda_1(G)$ has multiplicity 1.

Definition 1.4 (disjoint union)**.** If $G = (V, E)$, $G' = (V', E')$ are graphs, their *disjoint union* is the graph

$$G \sqcup G' = (V \sqcup V', E \sqcup E')$$

and

$$A(G \sqcup G) = \begin{bmatrix} A(G) & 0 \\ 0 & A(G') \end{bmatrix}$$

Remark. Spectrum doesn't detect if the graph is connected.

Example 1.2.

$$\begin{aligned} G &: V = \{1,2\}, E = \{\{1,2\}\} \\ \text{spectrum} &: \{1,-1\} \\ G \sqcup G &: V = \{1,2,3,4\}, E = \{\{1,2\},\{3,4\}\} \\ \text{spectrum} &: \{1,1,-1,-1\} \end{aligned}$$

Remark. Note that the spectrum of $G \sqcup G'$ is the multiset union of the spectra of G and G'.

1.2 Regular Graphs

Definition 1.5. A graph $G = (V, E)$ is called *k-regular* if every vertex has degree $k \in \mathbb{N}$.

Remark. G is regular if and only if $\bar{e}$ is an eigenvector, where $\bar{e} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^n$.

$$A(G) \cdot \bar{e} = \deg_G(V)$$

Proposition 1.1. *Let $G = (V, E)$, then*

$$\mathop{\mathbb{E}}_{v \in V(G)} \deg_G(v) \overset{(i)}{\leq} \lambda_1(G) \overset{(ii)}{\leq} \max_{v \in V(G)} \deg_G(v),$$

where the expectation of the degree *is defined to be*

$$\mathop{\mathbb{E}}_{v \in V(G)} \deg_G(v) \coloneqq \frac{1}{n} \sum_{v \in V(G)} \deg_G(v).$$

We have equality in (i) if and only if G is regular. If G is connected, equality holds in (ii) if and only if G is regular.

Proof. (i) For any symmetric $A \in \mathbb{R}^{n \times n}$,

$$\lambda_{\max}(A) \geq x^T A x, \quad \forall x \in \mathbb{R}^n \text{ with } ||x|| = 1$$

with equality if and only if x is an eigenvector of A with eigenvalue $\lambda_{\max}(A)$.

$$\begin{aligned} A &= \sum_j \lambda_j x^{(j)} x^{(j)T} \\ x &= \sum_j c_j x^{(j)} \\ \sum_j c_j^2 &= 1 \\ x^T A x &= \sum_j \lambda_j c_j^2 \end{aligned}$$

Question: The first equation is from Theorem 1.1, but where do the following equations come from?

Answer: The second equation comes from the fact that a vector in $\mathbb{R}^n$ can be represented as the linear combination of an orthonormal basis. The third equation comes from the constraint $||x|| = 1$, where the norm can be written as $\sqrt{x^T x} = 1$. The fourth one comes from calculation

$$Ax = \sum_j \lambda_j x^{(j)} x^{(j)T} \sum_i c_i x^{(i)} = \sum_j c_j \lambda_j x^{(j)}$$

so that

$$x^T Ax = \sum_i c_i x^{(i)T} \sum_j c_j \lambda_j x^{(j)} = \sum_i c_i^2 \lambda_i$$

Here we use distributive law and the property of orthonormal basis that $x^{(i)T} x^{(j)} = x^{(j)T} x^{(i)} = \mathbb{I}_{\{i=j\}}$.

Take $A = A(G), x = \frac{1}{\sqrt{n}}\bar{e}$, then

$$x^T Ax = \frac{1}{n} \underbrace{\bar{e}^T A\bar{e}}_{\text{sum of entries}} = \text{average degree}$$

(ii) Let's assume $V(G) = [n] := \{1, 2, \cdots, n\}$. Recall that by Perron-Forbenius, $\lambda_1(G)$ corresponds to an eigenvector $x \in \mathbb{R}^n_{\geq 0}$. Then for any $i \in [n]$,

$$\lambda_1(G)x_i = \Big(A(G)x\Big)_i = \sum_{j=1}^{n} A(G)_{i,j} x_j = \sum_{j \sim i} x_j.$$

Now take $i \in [n]$ so that x_i is maximum. Then

$$\lambda_1(G)x_i = \sum_{j\sim i} x_j \leq \sum_{j\sim i} x_i = \deg_G(v_i)x_i \leq \left(\max_{v\in V(G)} \deg_G(v)\right) x_i \qquad (*)$$

So

$$\lambda_1(G) \leq \max_{v\in V(G)} \deg_G(v).$$

Suppose equality holds everywhere in (ii) and G is connected, then equality holds everywhere in (*). So $x_j = x_i$ for all $j \sim i$. Applying the same argument to $j \sim i$, then $x_h = x_j = x_i$ for any $h \sim j$, *etc.* Since G is connected, x is a multiple of $\bar{e}$, *i.e.*, $\bar{e}$ is an eigenvector of G.

□

Corollary 1.1. *The complete graph K_n is the only graph on n vertices with eigenvalue $n-1$. In particular, K_n is uniquely determined by its spectrum.*

Remark. Not all graphs can be recovered by their spectrum, *i.e.*, the spectrum is not a faithful (in that graph can be uniquely determined) graph invariant (in that spectrum depends on graph only up to isomorphism).

Example 1.3. Spectrum does not tell connectivity. For example, the following two graphs (Figure 1.1) have the same spectrum $(-2, 2, 0, 0, 0)$.

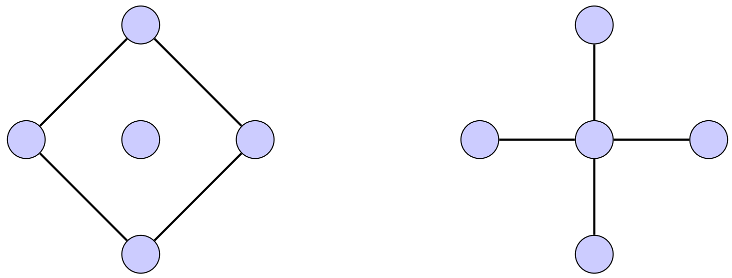


Figure 1.1: Two graphs with the same spectrum

1.3 Bipartite Graphs

Recall graph is bipartite with biparts x, y if and only if

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

Lemma 1.1. *Let B be an $n \times n$ matrix such that $B^T B$ has nonzero eigenvalues $\lambda_1, \cdots, \lambda_r$ (with multiplicities), then the nonzero eigenvalues of*

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$$

are precisely $\sqrt{\lambda_1}, -\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_r}, -\sqrt{\lambda_r}$ (with multiplicities).

Remark. The eigenvalues of $B^T B$ are real (from symmetry) and nonnegative since if $B^T B x = \lambda x$ (we assume $||x|| = 1$)

$$\lambda = \lambda x^T x = x^T(\lambda x) = x^T(B^T B x) = (Bx)^T(Bx) = \langle Bx, Bx\rangle \geq 0.$$

Proof. Note that

$$\begin{bmatrix} tI_m & -B \\ -B^T & tI_n \end{bmatrix} \begin{bmatrix} I_m & B \\ 0 & tI_n \end{bmatrix} = \begin{bmatrix} tI_m & 0 \\ -B^T & t^2 I_n - B^T B \end{bmatrix}.$$

Take determinant gives

$$\phi_A(t)t^n = t^m \phi_{B^T B}(t^2)$$

$\square$

Question: Why does the matrix on RHS have the determinant $t^m \phi_{B^T B}(t)$? Particularly, why is

$$\det\left(t^2 I_n - B^T B\right) = \phi_{B^T B}\left(t^2\right)?$$

Answer: By definition, $\phi_{B^T B}(x) = \det(xI_n - B^T B)$. Therefore, $\phi_{B^T B}(t^2) = \det(t^2 I_n - B^T B)$

Example 1.4. Let $K_{m,n}$ be the complete bipartite graph with biparts of size m and n and all possible edges between them, then

$$A(K_{m,n}) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

where B is an $m \times n$ all-one matrix. We have

$$\underbrace{B^T B}_{n \times n} = mJ_n$$

where J_n is the $n \times n$ all-one matrix. Recall that J_n has exactly one nonzero eigenvalues namely n, so the eigenvalues of $K_{m,n}$ are $\sqrt{mn}, -\sqrt{mn}$ and 0 (multiplicity $m + n - 2$).

Example 1.5. Let C_{2n} be a bipartite graph with adjacency matrix

$$A(C_{2n}) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

where $B \in \mathbb{R}^{n \times n}$ *e.g.*

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{5 \times 5}$$

We can verify that

$$B^T B = 2I_n + \underbrace{A(C_n)}_{\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}}$$

Therefore if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of C_n, then the eigenvalues of C_{2n} are

$$\pm\sqrt{2 + \lambda_k}, \quad k = 1, \cdots, n$$

This agrees with our early calculations

$$\lambda_k = 2\cos\left(\frac{2\pi k}{n}\right)$$

with identity

$$1 + \cos(2\theta) = 2\cos^2\theta$$

Corollary 1.2. *If G is bipartite graph on n vertices then $\lambda_n(G) = -\lambda_1(G)$.*

Remark. A converse holds for connected graphs.

Proposition 1.2. *If G is connected on n vertices and $\lambda_n(G) = -\lambda_1(G)$, then G is bipartite.*

Remark. It's another consequence of the Perron-Frobenius theorem. We don't prove this.

1.4 Cartesian Products

Definition 1.6 (Cartesian product)**.** Let G and H be graphs, the *Cartesian product* $G \square H$ has vertex set $V(G) \times V(H)$ with edges of the forms

- $(v, w) \sim (v', w)$ where $v \sim v'$ in G and $w \in V(H)$
- $(v, w) \sim (v, w')$ where $w \sim w'$ in H and $v \in V(G)$

Example 1.6. $P_m \square P_n$ is the $m \times n$ rectangular lattice. $P_2 \square P_2 \square P_2$ is the 1-skeleton of cube. In general, $\underbrace{P_2 \square \cdots \square P_2}_{n \text{ copies}}$ is the 1-skeleton of the n-dimensional hypercube $[0,1]^n \subset \mathbb{R}^n$

Remark. Cartesian product is associative.

Definition 1.7 (Tensor product)**.** Let $\mathbb{R}^m \otimes \mathbb{R}^n$ denote the tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$ which we will identify with the vector space of matrices $\mathbb{R}^{m \times n}$. Let $e^{(i)}$ denote the unit vector (i^{th} entry is one and zeros elsewhere) and define the standard basis of $\mathbb{R}^{m \times n}$:

$$\left\{ e^{(i)} e^{(j)T} = \begin{array}{c c} & j \\ i & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, 1 \le i \le m, 1 \le j \le n \right\}$$

For $A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{n \times n}$, let $A \otimes B$ denote the endomorphism of $\mathbb{R}^{m \times n}$ given by

$$(A \otimes B)(M) \coloneqq AMB^T \quad \text{for } M \in \mathbb{R}^{m \times n}.$$

Lemma 1.2. *Let two matrices $A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{n \times n}$, then*

$$(A \otimes B)_{(i,j),(k,l)} = A_{i,k} B_{j,l}, \quad (1 \le i, k \le m, 1 \le j, l \le n)$$

Proof. Recall that the entries of a matrix $M \in \mathbb{R}^{d \times d}$ are characterized by

$$M e^{(j)} = \sum_{i=1}^{d} M_{i,j} e^{(i)} \quad \text{for } 1 \le j \le d.$$

$$\begin{aligned} (A \otimes B)\left(e^{(k)} e^{(l)T}\right) &= A\left(e^{(k)} e^{(l)T}\right) B^T \\ &= \left(A e^{(k)}\right) \left(B e^{(l)}\right)^T \\ &= \left(\sum_{i=1}^{m} A_{i,k} e^{(i)}\right) \left(\sum_{j=1}^{n} B_{j,l} e^{(j)}\right)^T \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,k} B_{j,l} e^{(i)} e^{(j)T} \end{aligned}$$

□

Corollary 1.3. *Let G and H be graphs with basis of eigenvectors $x^{(1)}, \cdots, x^{(m)} \in \mathbb{R}^m$ and $y^{(1)}, \cdots, y^{(n)} \in \mathbb{R}^n$ corresponding to eigenvalues $\lambda_1, \cdots, \lambda_m$ and $\mu_1, \cdots, \mu_n$, then*

$$A(G \square H) = A(G) \otimes I_n + I_m \otimes A(H)$$

and $G \square H$ has eigenvectors $x^{(i)} y^{(j)T}$ corresponding to eigenvalues $\lambda_i + \mu_j$ for $1 \leq i \leq m, 1 \leq j \leq n$.

Remark. Note that $\left\{x^{(i)} y^{(j)T}\right\}$ forms a basis of $\mathbb{R}^{m \times n}$ since they are the unit matrices (matrices with a 1 and 0's elsewhere) under the basis $x^{(1)}, \cdots, x^{(m)}$ of $\mathbb{R}^m$ and the basis dual to $y^{(1)}, \cdots, y^{(n)} \in \mathbb{R}^n$. (If $y^{(1)}, \cdots, y^{(n)} \in \mathbb{R}^n$ are orthonormal, it's self-dual).

1.5 Enumerating Walks

Definition 1.8 (Walk)**.** Let G be a graph. A *walk* in G is a sequence of vertoices $(v_0, \cdots, v_l)$ such that $v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_l$. We call l the *length* of the walk. (A *path* is a walk with no vertices repeated except possibly $v_0 = v_l$.) We call a walk *closed* if $v_0 = v_l$ (a closed path is called cycle). We will consider *marked closed walks*, *i.e.*, closed walks whose starting vertex is distinguished (and an occurance of that vertex within the walk if it appears multiple times).

Example 1.7. Triangle has 6 marked closed walks:

$$\begin{array}{cc} 1\,2\,3 & 1\,3\,2 \\ 2\,1\,3 & 2\,3\,1 \\ 3\,1\,2 & 3\,2\,1 \end{array}$$

Lemma 1.3. *Let G be a graph and $l \in \mathbb{N}$. For any $v, w \in V(G)$, the number of walks of length l from v to w in G equals*

$$\left(A(G)^l\right)_{v,w}, \quad (A(G)^0 = I)$$

Proof.

$$\begin{aligned} \left(A(G)^l\right)_{v,w} &= \sum_{u_1, \cdots, u_{l-1} \in V(G)} A(G)_{v,u_1} A(G)_{u_1,u_2} \cdots A(G)_{u_{l-2},u_{l-1}} A(G)_{u_{l-1},w} \\ &= \sum_{u_1, \cdots, u_{l-1} \in V(G)} 1_{v \sim u_1 \sim u_2 \sim \cdots \sim u_{l-1} \sim w} \end{aligned}$$

□

Corollary 1.4. *Let G be a graph on n vertices with adjacency matrix A and $l \in \mathbb{N}$.*

(i) the number of marked closed walks in G of length l equals

$$\sum_{v \in V(G)} \left(A^l\right)_{v,v} = \operatorname{tr}\left(A^l\right) = \sum_{j=1}^{n} \lambda_j^l(G)$$

(ii) *the total number of walks in G of length l equals*

$$\sum_{v,w\in V(G)} \left(A^l\right)_{v,w} = \textit{sum of entries of } A^l$$

Example 1.8. Let $G = K_n$ the complete graph on n vertices. Recall that the eigenvalues of G are $n-1, -1, -1, \cdots, -1$. Hence, the number of marked closed walks of length l is

$$(n-1)^l + (n-1)(-1)^l$$

Now let's find the total number of walks of length l. Recall $A(G) = J_n - I_n$. Therefore by binomial theorem

$$(J_n - I_n)^l = \sum_{d=0}^{l} \binom{l}{d} (-1)^{l-d} J_n^d$$

By $J_n^2 = nJ_n$ anmd induction,

$$J_n^d = n^{d-1} J_n, \quad \forall d \geq 1$$

So the sum of the entries of J_n^d is $n^{d-1}n^2 = n^{d+1}$ for all $d \in \mathbb{N}$. Then the sum of entries of $A(G)^l$, (*i.e.*, number of walks of length l) is

$$\sum_{d=0}^{l} \binom{l}{d} (-1)^{l-d} n^{d+1} = n(n-1)^l$$

Finally, note that by symmetry $\left(A(G)^l\right)_{v,w}$ is the same for all distinct $v, w \in V(G)$.

$$\begin{aligned}\left(A(G)^l\right)_{v,w} &= \frac{\text{sum of entries of } A(G)^l - \text{sum of diagonal entries of } A(G)^l}{\text{\# of off-diagonal entries}} \\ &= \frac{n(n-1)^l - \left[(n-1)^l + (n-1)(-1)^l\right]}{n^2 - n} = \frac{(n-1)^l - (-1)^l}{n}\end{aligned}$$

Corollary 1.5. *The number of triangles in a graph G is*

$$\frac{1}{6} A(G)^3$$

The coefficient $\frac{1}{6}$ comes from the fact that a triangle has 6 (marked closed) walks of length 3.

Lemma 1.4. *Let $M \in \mathbb{R}^{n\times n}$. The sum of the entries of M is*

$$\bar{e}^T M \bar{e} = \operatorname{tr}(MJ_n) = \det(I_n + MJ_n) - 1$$

where $\bar{e} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^n$.

Proof. We can check explicitly that $\bar{e}^T M \bar{e}$ is the sum of entries.

(i)

$$\underbrace{\bar{e}^T M \bar{e}}_{1\times 1} = \operatorname{tr}\left(\bar{e}^T M \bar{e}\right) = \operatorname{tr}\left(M \bar{e} \bar{e}^T\right) = \operatorname{tr}(MJ_n)$$

(ii) Note that $\mathrm{rank}(MJ_n) \leq 1$. For any matrix $N \in \mathbb{R}^{n\times n}$ of rank at most 1, we have the *Sherman-Morrison identity*

$$\det(I_n + N) = 1 + \mathrm{tr}(N)$$

This is true since N has at most one nonzero eigenvalue λ and both sides equal to $1+\lambda$.

□

1.6 Walk Generating Functions

Definition 1.9 (ordinary generating function)**.** The *ordinary generating function* (ogf) of the sequence $(a_l)_{l\in\mathbb{N}}$ is the formal power series

$$\sum_{l=0}^{\infty} a_l t^l$$

It is called ordinary to distinguish from the *exponential generating function*

$$\sum_{l=0}^{\infty} \frac{a_l}{l!} t^l$$

Recall the geometric series formula

$$\sum_{l=0}^{\infty} t^l = \frac{1}{1-t}$$

We can think of $\frac{1}{1-t}$ as a rational function of t, or as the inverse of the formal power series $1-t$. More generally,

$$\sum_{l=0}^{\infty} (tA)^l = (I_n - tA)^{-1}$$

Definition 1.10. Let G be a graph of n vertices with adjacency matrix A.

(i) For $v, w \in V(G)$, let $W^G_{v,w}(t)$ denote the ordinary generating function for number of walks from v to w with respect to lenth.

$$W^G_{v,w}(t) = \sum_{l=0}^{\infty} (A^l)_{v,w} t^l = \left((I_n - tA)^{-1}\right)_{v,w}$$

(ii) Let $W^G_{\text{closed}}(t)$ denote the ordinary generating function for marked closed walks in G with respect to lenth.

$$W^G_{\text{closed}}(t) = \sum_{l=0}^{\infty} \mathrm{tr}(A^l) t^l = \mathrm{tr}\left((I_n - tA)^{-1}\right)$$

(iii) Let $W^G_{\text{all}}(t)$ denote the ordinary generating function for all walks in G with respect to lenth.

$$W^G_{\text{all}}(t) = \sum_{l=0}^{\infty} \underbrace{\operatorname{tr}(A^l J_n)}_{\text{sum of entries of } A^l} t^l = \operatorname{tr}\left((I_n - tA)^{-1} J_n\right)$$

Example 1.9. Recall

$$\operatorname{tr}\left(A(K_n)^l\right) = (n-1)^l + (n-1)(-1)^l$$

Therefore,

$$\begin{aligned} W^{K_n}_{\text{closed}}(t) &= \sum_{l=0}^{\infty} \left[(n-1)^l + (-1)^l(n-1)\right] t^l \\ &= \frac{1}{1-(n-1)t} + (n-1)\frac{1}{1-t} \end{aligned}$$

Theorem 1.3. *Let G be a graph with n vertices.*

$$W^G_{closed}(t) = \operatorname{tr}\left((I_n - tA(G))^{-1}\right) = \sum_{j=1}^{n} \frac{1}{1 - t\lambda_j(G)} = \frac{\phi'_G\left(\frac{1}{t}\right)}{t\phi_G\left(\frac{1}{t}\right)}$$

Proof. The eigenvalues of $A(G)$ are $\lambda_1, \lambda_2, \cdots, \lambda_n$, so the eigenvalues of $(I_n - tA(G))^{-1}$ are

$$(1 - t\lambda_1(G))^{-1}, \cdots, (1 - t\lambda_n(G))^{-1}$$

Thus,

$$\operatorname{tr}\left((I_n - tA(G))^{-1}\right) = (1 - t\lambda_1(G))^{-1} + \cdots + (1 - t\lambda_n(G))^{-1}$$

On the other hand,

$$\begin{aligned} \frac{\phi'_G\left(\frac{1}{t}\right)}{t\phi_G\left(\frac{1}{t}\right)} &= \frac{1}{t} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=\frac{1}{t}} \log \phi_G(s) \\ &= \frac{1}{t} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=\frac{1}{t}} \log[(s - \lambda_1(G)) \cdots (s - \lambda_n(G))] \\ &= \frac{1}{t} \left.\left(\frac{1}{s - \lambda_1(G)} + \cdots + \frac{1}{s - \lambda_n(G)} \right)\right|_{s=\frac{1}{t}} \\ &= \frac{1}{1 - t\lambda_1(G)} + \cdots + \frac{1}{1 - t\lambda_n(G)} \end{aligned}$$

Alternative proof using the identity

$$\operatorname{tr}(\log(A)) = \log(\det(A))$$

as functions on matrices. □

Example 1.10. Let $G = K_n$, then

$$\phi_{K_n}(t) = (t-(n-1))(t+1)^{n-1}$$
$$\phi'_{K_n}(t) = (t+1)^{n-2}(tn - n^2 + 2n)$$

Then

$$W^{K_n}_{\text{closed}}(t) = \frac{\left(1+\frac{1}{t}\right)^{n-2}\left(\frac{n}{t} - n^2 + 2n\right)}{t\left(\frac{1}{t}-(n-1)\right)\left(\frac{1}{t}+1\right)}$$

1.7 Asymptotic Behavior

Theorem 1.4. *Let G be a connected non-bipartite graph with largest eigenvalue λ_1 corresponding to an eigenvector x with $||x|| = 1$. (recall from the Perron-Frobenius theorem that $|\lambda_j(G)| < \lambda_1$ for all $j > 1$), then*

$$\lim_{l\to\infty} \frac{A(G)^l}{\lambda_1^l} = xx^T$$

In particular, for all $v, w \in G$

$$A(G)^l_{v,w} \sim \lambda_1^l x_v x_w$$

Proof. Let's assume that $V(G) = [n]$. Let $x^{(1)}, x^{(2)}, \cdots, x^{(n)} \in \mathbb{R}^n$ be an orthonormal basis of eigenvectors of $A(G)$ corresponding to eigenvalues $\lambda, \lambda_2, \cdots, \lambda_n$. Recall that

$$A(G) = \sum_{j=1}^{n} \lambda_j x^{(j)} x^{(j)T}$$

Since $x^{(i)T}x^{(j)} = \delta_{ij}$, we get

$$A(G)^l = \sum_{j=1}^{n} \lambda_j^l x^{(j)} x^{(j)T}$$

Therefore,

$$\frac{A(G)^l}{\lambda_1^l} = xx^T + \sum_{j=2}^{n} \left(\frac{\lambda_j}{\lambda_1}\right)^l x^{(j)} x^{(j)T}$$

Since $|\lambda_j| < \lambda_1$ for $j > 1$, we get

$$\lim_{l\to\infty} \frac{A(G)^l}{\lambda_1^l} = xx^T$$

□

Remark. 1. Connected: λ_1 has multiplicity 1.

2. Non-bipartite: $-\lambda_1$ not an eigenvalue.
For connected bipartite graphs, we have the following

$$A(G)^l_{v,w} = \begin{cases} 0 & l \not\equiv d(v,w) \mod 2 \\ \sim 2\lambda_1^l x_v x_w & l \equiv d(v,w) \mod 2 \end{cases}$$

Example 1.11. Consider $G = P_n$, which is connected and bipartite. Recall the eigenvalues of G are

$$\lambda_k = 2\cos\left(\frac{\pi k}{n+1}\right) \quad k = 1, \cdots, n$$

We can check that an eigenvector $x \in \mathbb{R}^n$ with $||x|| = 1$ corresponding to $\lambda_1 = 2\cos\left(\frac{\pi}{n+1}\right)$ is given by

$$x_j = \sqrt{\frac{2}{n+1}} \sin\left(\frac{\pi j}{n+1}\right)$$

(entries bigger if closer to the middle of the path). An eigenvector $y \in \mathbb{R}^n$ with $||y|| = 1$ corresponding to $-\lambda_1$ is given by

$$y_j = (-1)^{j-1} x_j \quad j = 1, \cdots, n$$

Hence, as $l \to \infty$, we have

$$\left(A(G)^l\right)_{ij} = \begin{cases} 0 & l \not\equiv |i-j| \mod 2 \\ \sim 2\left(2\cos\left(\frac{\pi}{n+1}\right)^l\right)\left(\frac{2}{n+1}\right)\sin\left(\frac{\pi i}{n+1}\right)\sin\left(\frac{\pi j}{n+1}\right) & l \equiv |i-j| \mod 2 \end{cases}$$

Why does $||x|| = 1$?

$$\sum_{j=1}^{n} \sin^2\left(\frac{\pi j}{n+1}\right) = \frac{1}{2}\sum_{j=0}^{2n+1} \sin^2\left(\frac{\pi j}{n+1}\right)$$

Recall

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Letting $\rho = \frac{2\pi i}{2n+2}$,

$$\sin\left(\frac{j\pi}{n+1}\right) = \frac{\rho^j - \rho^{-j}}{2i}$$

Thus,

$$\begin{aligned} \frac{1}{2}\sum_{j=0}^{2n+1} \sin^2\left(\frac{\pi j}{n+1}\right) &= \frac{1}{2}\sum_{j=0}^{2n+1}\left(\frac{\rho^j - \rho^{-j}}{2i}\right) \\ &= \frac{1}{2}\sum_{j=0}^{2n+1}\left(\frac{\rho^{2j} - 2 + \rho^{-2j}}{-4}\right) \\ &= -\frac{1}{8}\left(\underbrace{\sum_{j=0}^{2n+1}\rho^{2j}}_{=0} + \underbrace{\sum_{j=0}^{2n+1} -2}_{=-2(2n+2)} + \underbrace{\sum_{j=0}^{2n+1}\rho^{-2j}}_{=0}\right) \\ &= \frac{n+1}{2} \end{aligned}$$

1.8 Graph Homomorphisms

Definition 1.11. Let G and H be graphs. We say that $\phi : V(G) \to V(H)$ is a homomorphism if

$$\forall v, w \in V(G), \quad v \sim_G w \implies \phi(v) \sim_H \phi(w).$$

We call ϕ an isomorphism if ϕ is a bijection and its inverse ϕ^{-1} is also a homomorphism. If ϕ is a bijection, it is an isomorphism if and only if

$$\forall v, w \in V(G), \quad v \sim_G w \iff \phi(v) \sim_H \phi(w)$$

Definition 1.12 (Automorphism)**.** An automorphism of G is an isomorphism from G to G.
Note: $(Aut(G), \circ)$ forms a group.

Let $\mathfrak{S}(V)$ denote the symmetric group of all permutations (i.e. bijections) of the set V (We also let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[n] = \{1, 2, ..., n\}$). $Aut(G)$ is a subgroup of $\mathfrak{S}(V(G))$.

Example 1.12. $\mathrm{Aut(K_n)} = \mathfrak{S}_n$, $\mathrm{Aut(C_n)} = D_n$(the dihedral group of order $2n$ of rotations and reflections of a regular n-gon).

Theorem 1.5 (Lie Theory)**.** *Miss this part...*

Example 1.13. $n \geq 2$, P_n has 2 automorphism (the identity map and the map that reflects the path about its center).

Definition 1.13 (m-coloring)**.** m-coloring of a graph G is a labeling of its vertices by a color $\varphi(V) \in [m]$. An m-coloring is called **proper** if no two adjacent vertices are given the same color, i.e.,φ is proper if and only if φ is homomorphism form G to K_m.

Definition 1.14. A permutation matrix is a square $\{0, 1\}$ matrix with exactly one 1 in every row and column. For each permutation $\pi \in \mathfrak{S}_n$. Let $P(\pi)$ denote the $n \times n$ permutation matrix where j^{th} column has a 1 in row $\pi(j)$ for all $j \in [n]$.

Example 1.14. Let $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Then

$$P(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $P(\pi)P(\sigma) = P(\pi\sigma)$. Let P_n be the set of all $n \times n$ permutation matrices. Then it's easy to verify that $P_n \leq \mathrm{GL}_n(\mathbb{R})$ and $P_n \cong \mathfrak{S}_n$.

Definition 1.15. Define the sign or signature $\mathrm{sgn}(\pi)$ of $\pi \in \mathfrak{S}_n$ by $\mathrm{sgn}(\pi) = \det(P(\pi))$.

Lemma 1.5. *Let $\pi \in \mathfrak{S}_n$, then*

(i) If π can be represented as a product of m transpositions, then $\mathrm{sgn}(\pi) = (-1)^m$.

(ii) $\operatorname{sgn}(\pi) = (-1)^{\# i<j:\pi(j)>\pi(i)}$.

Proof. (i) Using multiplicativity of the determinant, it suffices to prove that every transposition has sign -1. This is true since the permutation matrix of a transposition is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

up to a permutation of the basis vectors.

(ii) Check that

(a) $\operatorname{sgn}(\mathrm{id}) = 1$.

(b) Since every transposition has sign -1, note that if there is inversion $((i,j) : i < j, \pi(j) > \pi(i))$ in a permutation, then there must exist an inversion between 2 consecutive indices (e.g. $\pi = 3124$ contains inversion (1,2) and (1,3)). Now it is enough to note that one could reverse such an inversion using transposition. So the total number of transpositions needed is the number of inversions.

□

Example 1.15. $\operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = 1$

Proposition 1.3. *Let G and H be graphs on the vertex set $[n]$ and $\pi \in \mathfrak{S}_n$. Then π is an isomorphism from G to H if and only if*

$$A(G) = P(\pi)^{-1} A(H) P(\pi).$$

Proof. Note that $(P(\pi)^{-1}A(H)P(\pi))_{i,j} = A(H)_{\pi(i),\pi(j)}$. Therefore we have an equivalence chain,

$$\begin{aligned} & A(G) = P(\pi^{-1})A(H)P(\pi) \\ \iff & A(G)_{i,j} = A(H)_{\pi(i),\pi(j)} \\ \iff & (i \sim j \iff \pi(i) \sim \pi(j)) \\ \iff & \pi \text{ is an isomorphism} \end{aligned}$$

□

Corollary 1.6. *The automorphisms of a graph G correspond to permutation matrices P satisfying $A(G) = P^{-1}A(G)P$.*

Theorem 1.6. *Suppose that the eigenvalues of G are all distinct. Then every automorphism of G has order at most 2.*

Proof. Let $A = A(G) \in \mathbb{R}^{n\times n}$. Every automorphism of G corresponds to a permutation matrix P with $AP = PA$. We must show that $P^2 = I_n$.

Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be the eigenvalues of A with eigenbasis $x^{(1)}, \dots, x^{(n)} \in \mathbb{R}^n$. Then for $i \in [n]$, we have

$$AP = PA \implies APx^{(i)} = PAx^{(i)} \implies A(Px^{(i)}) = \lambda_i Px^{(i)} \implies Px^{(i)} = \mu_i x^{(i)}$$

Since $P^{n!} = I_n$, $(\mu_i)^{n!} = 1$. Thus $\mu_i = \pm 1$. We get $P^2 x^{(i)} = \mu_i^2 x^{(i)} = x^{(i)}$ for all $i \in [n]$. Since $\{x^{(i)}\}$ forms a basis of $\mathbb{R}^{\ltimes}$, P^2 is I_n. ☐

Remark. in fact, almost all graphs have a trivial automorphism group, i.e.

$$\lim_{n\to\infty} \frac{\#\text{graphs } G \text{ with } V(G) = [n] \text{ and } |\mathrm{Aut}(G)| = 1}{\#\text{graphs } G \text{ with } V(G) = [n]} = 1.$$

See section 2.3 of Godsil-Royle for a proof.

Remark (Graph Isomorphic Problem).

- Babai & Luks(1983) $2^{O(\sqrt{n \log n})}$
- Babai(2017) $2^{O((\log n)^c)}$

1.9 Expander Graphs

Vaguely speaking, an expander graph is a sparse graph (*i.e.*, it has few edges) which is nonetheless highly connected. Expander graphs have applications to

- constructing error-correcting codes
- derandomize algorithms
- serve methods in number theory
- hyperbolic manifolds

Definition 1.16 (edge expansion ratio)**.** Lety G be a graph. For $S \subseteq V(G)$ denote the set of edges with one endpoint in S and one endpoint not in S by ∂S. Define the edge expansion ratio

$$h(G) := \min_{\substack{S \subseteq V(G) \\ |S| \le |V(G)|/2}} \frac{|\partial S|}{|S|}$$

Definition 1.17. A sequence of k-regular graphs $(G_n)_{n\in\mathbb{N}}$ is called a family of expander graphs if

$$|V(G_0)| < |V(G_1)| < \cdots$$

and there exists $\varepsilon > 0$ such that $h(G_n) \ge 0, \forall n \in \mathbb{N}$.

Remark. It's relatively easy to show that such families exist by probabilistic arguments, but it's relatively difficult to construct them explicitly.

Theorem 1.7. *Suppose G is connected and k-regular so $\lambda_1(G) = k$, then*

$$\frac{k - \lambda_2(G)}{2} \leq h(G) \leq \sqrt{2k(k - \lambda_2(G))}$$

Therefore a family $(G_n)_{n\in\mathbb{N}}$ of k-regular graphs with $|V(G_0)| < |V(G_1)| < \cdots$ forms a family of expander graphs if and only if there exists $\varepsilon' > 0$ such that

$$k = \lambda_2(G_n) \geq \varepsilon' \quad \forall n \in \mathbb{N}$$

The spectral gap $k - \lambda_2(G)$ or sometimes $\min\{|k - \lambda_2 G|, k - |\lambda_n(G)|\}$ measures how rapidly a random walk on G mixes.

Chapter 2

Tilings, Spanning Trees and Electrical Networks

- motivation (and some results) come from Physics, Chemistry and Computer Science.

2.1 Tilings and Perfect Matchings

We'll prove the following theorems.

Theorem 2.1 (Kasteleyn (1961))**.** *Let $m, n \in \mathbb{N}$ be such that mn is even. The number of domino tilings of an $m \times n$ board equals*

$$T(m,n) = \prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right)\right)^{\frac{1}{4}}$$

Example 2.1. $\pi(2,3)$ has 3 tilings.

Figure 2.1: Three tilings of $\pi(2,3)$

Remark. If m, n both odd, there's no domino tilings of an $m \times n$ board.

Definition 2.1 (Perfect matching)**.** A *perfect matching* of a graph G is a subset of its edges which meets every vertex exactly once.

Remark.

- Only graphs with even number of vertices can have a perfect matching.
- Note that domino tiling of an $m \times n$ board correspond to perfect matchings of the $m \times n$ grid $P_m \square P_n$.

Example 2.2. K_4 has 3 perfect matchings. Each of them are rotated from anther.

We will present a general method for conducting perfect matchings of certain graphs due to Kasteleyn, called Pfaffian method.

Figure 2.2: Three perfect matchings of K_4

2.2 Skew-Symmetric Matrices

Definition 2.2 (skew-symmetric matrix)**.** A matrix $A \in \mathbb{R}^{n\times n}$ is called *skew symmetric* if

$$A^T = -A.$$

Remark. Note that for skew-symmetric A,

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$$

Therefore if n is odd, $\det(A) = 0$. However, if n is even, $\det(A)$ is a perfect square in the entries of A.

Example 2.3.

$$\det \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = (af - be + cd)^2 = \text{Pfaffian}^2$$

Definition 2.3 (Pfaffian)**.** Let $A \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric. We define the *Pfaffian* of A

$$\mathrm{pf}(A) := \sum_{\substack{\text{perfect matching of } K_{2n} \\ M=\{\{i_1,j_1\},\cdots,\{i_n,j_n\}\}}} \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_n & j_n \end{pmatrix} \prod_{r=1}^{n} A_{i_r,j_r}$$

Example 2.4. Let $A \in \mathbb{R}^{4\times 4}$ be skew-symmetric, then

$$\begin{aligned} \mathrm{pf}(A) &= \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} A_{1,2}A_{3,4} \\ &+ \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} A_{1,3}A_{2,4} \\ &+ \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} A_{1,4}A_{2,3} \\ &= A_{1,2}A_{3,4} - A_{1,3}A_{2,4} + A_{1,4}A_{2,3} \end{aligned}$$

Remark. Why is the Pfaffian well-defined?

(a) the M-term does not change if we swap the order of a given edge

$$\{i_r, j_r\} \longleftrightarrow \{j_r, i_r\}$$

- $\operatorname{sgn}\begin{pmatrix} \cdots & 2r-1 & 2r & \cdots \\ \cdots & i_r & j_r & \cdots \end{pmatrix} = -\operatorname{sgn}\begin{pmatrix} \cdots & 2r-1 & 2r & \cdots \\ \cdots & j_r & i_r & \cdots \end{pmatrix}$

- $A_{i_r,j_r} = -A_{j_r,i_r}$

(b) the M-term does not change if we swap two consequtive edges

$$\{i_r, j_r\} \longleftrightarrow \{i_{r+1}, j_{r+1}\}$$

- $\operatorname{sgn}\begin{pmatrix} \cdots & 2r-1 & 2r & 2r+1 & 2r+2 & \cdots \\ \cdots & i_r & j_r & i_{r+1} & j_{r+1} & \cdots \end{pmatrix}$
- unchanged

  Question: This part is not clear enough.

So we can compute

$$\operatorname{pf}(A) = \underbrace{\frac{1}{2^n n!}}_{\substack{\text{there are } \frac{(2n)!}{2^n n!} \\ \text{perfect matchings}}} \sum_{\pi \in \mathfrak{S}_{2n}} \operatorname{sgn}(\pi) \prod_{i=1}^{2n} A_{\pi(2i-1),\pi(2i)}$$

Theorem 2.2 (Cayley 1849)**.** *For any skew-symmetric $A \in \mathbb{R}^{2n \times 2n}$,*

$$\det(A) = \operatorname{pf}(A)^2$$

Theorem 2.3. *Every planar graph has a Pfaffian orientation.*

Theorem 2.4 (EDIT: Kuratowski)**.** *A graph is planar if and only if there is no subdivision isomorphic to $K_{3,3}$ or K_5.*

Theorem 2.5. *A graph has a Pfaffian orientation if and only if no $K_{3,3}$.*

Theorem 2.6. *The number of pilings using dominos on an $m \times n$ board is*

$$T(m,n) = \prod_{j=1}^{m} \prod_{k=1}^{n} (4\cos^2(\frac{\pi j}{m+1}) + 4\cos^2(\frac{\pi k}{n+1}))^{\frac{1}{4}}.$$

Proof. Any two domino tilings are related by a sequence of flips. Any two perfect matchings

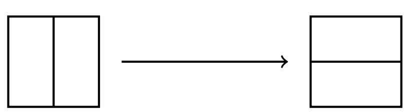


Figure 2.3: Flip

contribute to the same sgn. Thus the number of perfect matchings of $P_m \square P_n$ is $|Pf(\vec{A}(D))| = \sqrt{|\det(\vec{A}(G))|}$. By rescaling you get $|\det(A)| = |\det(A'')|$, where

$$A'' = I_m \otimes A(P_n) - i(A(P_m) \otimes I_n).$$

Recall that the eigenvalues of P_n are $2\cos(\frac{\pi k}{n+1}), k \in [n]$. Therefore, eigenvalues of A'' are

$$2\cos(\frac{\pi k}{n+1}) - i2\cos(\frac{\pi j}{m+1}).$$

Then we have the number of perfect matchings

$$\begin{aligned}
&|Pf(\vec{A}(D))| = \sqrt{|\det(A)|} \\
=&|\det(A'')|^{\frac{1}{2}} \\
=&\prod_{j=1}^{m}\prod_{k=1}^{n}|4\cos^2(\frac{\pi k}{n+1}) + 4\cos^2(\frac{\pi k}{n+1})|^{1/4}
\end{aligned}$$

□

2.2.1 Asymptotic of $T(m,n)$

We have

$$\begin{aligned}
\frac{\log(T(m,n))}{mn} &= \frac{1}{mn}\log\prod_{j=1}^{m}\prod_{k=1}^{n}(4\cos^2(\frac{\pi j}{m+1}) + 4\cos^2(\frac{\pi k}{n+1}))^{\frac{1}{4}} \\
&= \frac{1}{4\pi^2}(\frac{\pi}{m})(\frac{\pi}{n})\sum_{j=1}^{m}\sum_{k=1}^{n}\log(4\cos^2(\frac{\pi j}{m+1}) + 4\cos^2(\frac{\pi k}{n+1})) \\
\xrightarrow{m,n\to\infty} &= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\log(4\cos^2 x + 4\cos^2 y)\mathrm{d}x\mathrm{d}y \\
&= \frac{G}{\pi}
\end{aligned}$$

where $G = \dfrac{1}{4\pi}\int_0^{\pi}\int_0^{\pi}\log(4\cos^2 x + 4\cos^2 y)\mathrm{d}x\mathrm{d}y = 1 - \dfrac{1}{3^2} + \dfrac{1}{5^2} + ...$

(Open Problem) is G irrational?

$\log T(m,n) \approx G_{m,n}$. So $T(m,n) \approx \mathrm{e}^{G_{m,n}/\pi}$

2.3 Rhombic tilings

Theorem 2.7 (Mac Mahon (1916))**.**

Theorem 2.8 (Lindstorm, Gessel, Viennot)**.**

Theorem 2.9.

$$N(a,b,c) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

Proof. We will apply Jacobi's Identity:

$$\det(A)\det(A_{[2,n-1][2,n-1]}) = \det(A_{[2,n][2,n]})\det(A_{[1,n-1][1,n-1]}) - \det(A_{[2,n][1,n-1]})\det(A_{[1,n-1][2,n]})$$

for any $n \times n$ matrix A. This is a special case of general determinant identities called Grassmann-pliicker relations (or Schouter(???) identities in physics). In our case, we have

$$N(a,b,c)N(a,b,c-2) = N(a,b,c-1)N(a,b,c-1) - N(a-1,b+1,c-1)N(a+1,b-1,c-1),$$

where $N(a,b,0) = 1, N(a,b,c) = \binom{a+b}{a}$. Now just need to check that

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

satisfies the same recursion. □

There is a direct way to relate determinants to enumerating perfect matchings.

2.4 EDIT: Aztec Diamond

The n^{th} Aztec diamond is a symmetric board with rows of length $2, 4, 6, \ldots, 2n, 2n, \ldots, 6, 4, 2$.

Theorem 2.10. *The n^{th} Aztec diamond has $2^{\binom{n+1}{2}}$ domino tilings.*

2.5 Spanning Trees

Theorem 2.11. *Let $m, n \in \mathbb{N}$ be odd. The number of domino tilings of an $m \times n$ board with one corner square removed is*

$$\prod_{j=1}^{\lfloor \frac{m}{2} \rfloor}\prod_{k=1}^{\lfloor \frac{n}{2} \rfloor}(4\cos^2(\frac{\pi j}{m+1}) + 4\cos^2(\frac{\pi k}{n+1}))$$

To prove this theorem we first construct a bijection from domino tilings to spanning trees of a grid graph. Then we will count spanning trees using the matrix-tree theorem.

Definition 2.4. A spanning subgraph H of G is a graph $V(H) = V(G)$ and $E(H) \subset E(G)$. If H is a tree, we call H a spanning tree of G.

Theorem 2.12. *let $m, n \in \mathbb{Z}_{>0}$, define a map φ from the set of domino tilings of a $(2m-1)(2n-1)$ board whose north east corner is removed to the set of spanning trees of $P_m \square P_n$ as follows:*

Embed $P_m \square P_n$ inside the (2m-1)(2n-1) board. Given a domino tiling M, let $\varphi(M)$ be the spanning tree whose edges cross only one edge of M. This φ is a bijection.

Proof. First we prove φ is well-defined. This can be seen from the fact that

1. the edges of $\varphi(M)$ correspond to the vertices of $P_m \square P_n$ except for the northeast corner vertex. So $\varphi(M)$ has $mn - 1$ edges,

2. $\varphi(M)$ is acyclic, otherwise, the path would enclose odd number of squares, which cannot be filled with dominoes.

Then we must construct the unique domino tiling M with $\varphi(M) = T$. Root T at the northeast corner vertex and direct all edges toward the root. This allows us to fill in the dominoes along the tree.

Now we need to fill in the remainder. The remaining squares can be divided into several connected component $H_1, \dots, H_l$ (of the dual graph of the board). Note that each H_i is acyclic, otherwise H_i would divide T into ≥ 2 connected components. So, the all H_i are trees.

Similarly, each H_i intersects the boundary of the boundary of the board exactly once. Therefore each H_i has a unique root. By the same method we could tile all the area. □

Theorem 2.13 (Propp)**.**

2.6 Kirchoff's matrix-tree theorem

Lemma 2.1 (Laplacian eigenvalues of Cartesian products)**.**

2.6.1 Matrix Tree Theorem

Theorem 2.14 (Matrix-Tree Theorem)**.** *The number of spanning trees of a graph G equals*

$$\begin{aligned} &\det(L(G)_{V(G)\setminus\{v\},V(G)\setminus\{v\}}) \text{ any } v \text{ in } V(G) \\ =&\frac{1}{n}\mu_1...\mu_{n-1} \quad 0, \mu_1, ..., \mu_{n-1} \text{ are Laplacian eigenvalue.} \end{aligned}$$

Proof. (This is the second part) We need to show that

$$\det(L(G)_{V(G)\setminus\{v\},V(G)\setminus\{v\}}) \text{ any } v \text{ in } V(G) = \frac{1}{n}\mu_1...\mu_{n-1}$$

.

Lemma 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix where eigenvalues are $0, \mu_1, \mu_2, ..., \mu_{n-1}$. Then $\mu_1\mu_2...\mu_{n-1} = \sum_{i=1}^{n} \det(A_{[n]\setminus\{i\},[n]\setminus\{i\}})$*

Proof. Note that

$$\begin{aligned} \Phi_A(t) &= t(t-\mu_1)...(t-\mu_{n-1}) \\ [t]\Phi_A(t) &= (-1)^{n-1}\mu_1...\mu_{n-1} \\ \text{on the other hand, } [t]\Phi_A(t) &= [t]\det(tI_n - A) \\ &= \sum_{i=1}^{n} \det((-A)_{[n]\setminus\{i\},[n]\setminus\{i\}}) \\ &= (-1)^{n-1}\sum_{i=1}^{n} \det((A)_{[n]\setminus\{i\},[n]\setminus\{i\}}) \end{aligned}$$

□

Apply the lemma to $A = L(G)$. □

2.6.2 Directed Matrix Tree Theorem

Definition 2.5. Let D be a directed graph, the *Laplacian* of D $L(D) \in \mathbb{R}^{V(D)\times V(D)}$ is defined by

$$L(D)_{v,w} = \begin{cases} \text{outdeg}(v) - (\text{loops at } v) & \text{if } v = w \\ -(\text{edges } v \to w) neqw \end{cases}$$

Remark.

- In our definition, we allow D with loop and multiple edges.
- Note that $L(D)\vec{e} = 0$.

Definition 2.6. A rooted (directed) graph is a (directed) graph with a distinguished vertex, root. A rooted in tree is the (unique) orientation of an undirected rooted tree such that every edge points toward the root.

Theorem 2.15 (Tutle(1948)Directed matrix tree theorem)**.** *Let D be a directed graph and $v \in V(D)$. Then the number of spanning tree rooted at v is*

$$\det(L(D)_{V(D)\setminus\{v\},V(D)\setminus\{v\}})$$

Proof. (Later...) □

Remark. The directed matrix-tree theorem implies Kirchhoff's matrix-tree theorem:

Given G, let D be the directed graph with $V(D) = V(G)$ and edges $v \leftrightarrows w$ for every $\{v, w\}$ of G.

2.7 Eulerian Cycles

Definition 2.7. Let $d, n \in \mathbb{Z}_{>0}$. A d-ary sequence of length n is a sequence of length n on the alphabet $\{0, 1, \ldots, d-1\}$. A d-ary de Bruijn sequence of order n is a sequence $(a_1, \ldots, a_{d^n}) \in \{0, 1, \ldots, d-1\}^{d^n}$ such that the subsequences $(a_j, a_{j+1}, \ldots, a_{j+n-1})$ (n-modular) are all the d-ary sequences of length n (each appearing exactly once.

Example 2.5. A binary de Bruijn sequence of order 3 is

$$00010111 \text{ and } 00011101$$

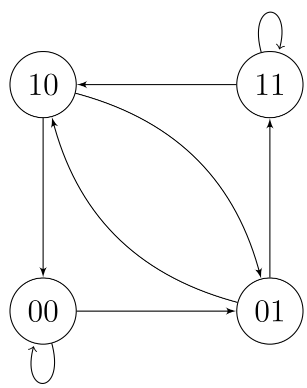
Why do de Bruijn sequences exist?
How many of them are there?
How do we construct them?
Applications:

1. Position determination along a circular hallway.
2. Magic: Ask for the suits of any n consecutive cards in a deck of size 2^n, etc.

Let us show that de Bruijin sequences exist, for every d, n.

Definition 2.8. Define the de Bruijn graph $D_{d,n}$ as the directed graph with vertex set $\{0,\dots,d-1\}^{n-1}$ with edges $(a_1,a_2,\dots,a_{n-1}) \to (a_2,\dots,a_{n-1},a_n)$ for all $(a_1,\dots,a_n) \in \{0,\dots,d-1\}^n$.

Example 2.6. $D_{2,3}$:



Key observations: d-ary de Bruijn sequences of order n correspond to Eulerian cycles of $D_{d,n}$, i.e. directed cycles which use every edge (exactly once).

Theorem 2.16 (Euler)**.** *An undirected graph (without isolated vertices) is Euclian if and only if every vertex has even degree.*

Theorem 2.17 (Good (1947))**.**

Theorem 2.18 ("BEST", van Aardenne-Ehrenfest, de Bruijn (1951), Smith, Tulk (1941))**.**

2.8 de Bruijn sequence

Chapter 3

Generating Functions

3.1 Catalan numbers

Definition 3.1. Let $n \in \mathbb{N}$. A Dyck path of order n is a path in the plane from $(0,0)$ to $(2n,0)$, moving by steps up(1,1) and down(1,-1) and never passing below the x-axis.

Let $\mathcal{C}_n$ denote the set of Dyck paths of order n. We call $C_n = |\mathbb{C}_n|$, the n^{th} Catalan number. e.g. $C_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14$.

Let's prove the classical formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

Lemma 3.1 (Catalan recurrence). *The Catalan sequence $(C_n)_{n=0}^{\infty}$ is characterized by the recurrence relation*

$$C_n = \sum_{j=0}^{n-1} C_j C_{n-1-j}, \qquad C_0 = 1$$

Proof. For $n \neq 1$, we construct a bijection

$$\phi : \mathcal{C}_n \to \bigsqcup_{j=0}^{n-1} \mathcal{C}_j \times \mathcal{C}_{n-1-j}$$

Given a Dyck path P, let $(2(j+1),0)$ be the first point where the path p touches the x-axis after $(0,0)$. Decompose P to two Dyck paths- one is from $(1,1)$ to $(2j+1,1)$ (p_1), the other is from $(2(j+1),0)$ to $(2n,0)$ (p_2). Note that $p_1 \in \mathcal{C}_j$, $p_2 \in \mathcal{C}_{n-1-j}$.

Set $\phi(p) = (p_1, p_2)$. Observe that ϕ has an inverse (by combining p_1 and p_2 in the reverse way). Therefore ϕ is a bijection. $\square$

Define the Catalan ordinary generating function

$$c(x) = \sum_{n\geq 0} C_n x^n.$$

Lemma 3.2.

$$c(x) = xc(x)^2 + 1.$$

Proof. Let us compare coefficients of x^n on both sides, for all $n \in \mathbb{N}$. If $n = 0$, both are 1. Now suppose $n \geq 1$.

$$\begin{aligned}
&[x^n](xc(x)^2 + 1)\\
=&[x^{n-1}]c(x)^2\\
=&[x^{n-1}](\sum_i C_i x^i)(\sum_j C_j x^j)\\
=&\sum_{j=0}^{n-1} C_{n-1-j}C_j\\
=&[x^n]c(x)
\end{aligned}$$

□

Theorem 3.1. *For $n \in \mathbb{N}$, $C_n = \frac{1}{n+1}\binom{2n}{n}$*

Proof. We use the equation $xc(x)^2 - x(x) + 1 = 0$, so $c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. Since $C(0) = 1$, so

$$\begin{aligned}
c(x) &= \frac{1 - \sqrt{1-4x}}{2x}\\
&= \frac{1 - \sum_{j \geq 0} \binom{\frac{1}{2}}{j} (-4x)^j}{2x}\\
&= \sum_{j \geq 1} \frac{(2j-2)!}{j!(j-1)!} x^{j-1}
\end{aligned}$$

Thus $C_n = [x^n]c(x) = \frac{(2n)!}{(n+1)!n!}$. □

3.1.1 Lagrange implicit function theorem

A systematical way to prove that $C(x) = xC(x)^2 + 1$ is use Lagrange implicit function theorem.

Theorem 3.2 (Lagrange implicit function theorem)**.** *If $A(x) = x\phi(A(x))$, where ϕ is formal power series, then*

$$[x^n]A(x) = \frac{1}{n}[t^{n-1}]\phi(t)^n.$$

Proof. Let $\phi(t) = (t+1)^2$

$$\begin{aligned} A(x) &= c(x) - 1 \\ c(x) &= x(c(x))^2 + 1 \\ \Rightarrow A(x) &= x(A(x)+1)^2 \\ C_n = [x^n]c(x) &= [x^n]A(x) \\ &= \frac{1}{n}[t^{n-1}]((t+1)^2)^n \\ &= \frac{1}{n}\binom{2n}{n-1}. \end{aligned}$$

□

Remark. $D(x) = xD(x)^k + 1$ giving $[x^n]D(x) = \frac{1}{n}\binom{kn}{n-1} (n \geq 1)$.

3.1.2 Combinatorial proof

- Rotate Dyck path by 45°, rescale by $\frac{1}{\sqrt{2}}$. We then have $n \times n$ rectangle weakly above main diagonal.
- Add a right step at the end, we then have $n \times (n+1)$ rectangle.
- Given any up-right lattice path in an $n \times (n+1)$ rectangle, we can continue it periodically to form a path of slope $\frac{n}{n+1}$. If we start reading at any $2n+1$ points in a given period, we get distinct path since $\gcd(n, n+1) = 1$.
- There is a unique path among these 2n+1 path which stays above the main diagonal. Take a line of slope $\frac{n}{n+1}$ and move it up until it touches our infinite path.
- So $C_n = \frac{1}{2n+1}\binom{2n+1}{n}$.

3.2 Catalan objects

3.3 Formal power series

Definition 3.2 (formal power series)**.** Let $\mathbb{C}[[x]]$ denote the ring of formal power series (in one variable). Its elements are denoted

$$A = A(x) = \sum_{n \geq 0} a_n x^n \qquad (a_n \in \mathbb{C})$$

(Though we use function notation and intuition, this is just a sequence $(a_n, a_1, \cdots)$.)

- $\mathbb{C}[[x]]$: ring of formal power series
$$\sum_{n\geq 0} a_n x^n$$
addition, multiplication are as for analytic functions, also a notion of composition.
- $A \in \mathbb{C}[[x]]$ is invertible if and only if $A(0) \neq 0$.
- $A \in \mathbb{C}[[x]]$ with $A(0) = 0$ has a compositional inverse if and only if $[x^1]\, A(x) \neq 0$.

3.4 Formal Laurent series

Definition 3.3 (formal Laurent series)**.** Let $\mathbb{C}((x))$ denote the field of formal Laurent series (in one variable), *i.e.*, power series of the form

$$\sum_{n\geq k} a_n x^n \qquad (a_n \in \mathbb{C})$$

for some $k \in \mathbb{Z}$ with the expected addition and multiplication. Note htat indeed every nonzero formal Laurent series f has an inverse: we can write

$$f = x^k \cdot g$$

where $g \in \mathbb{C}[[x]]$ is a formal power series with $g(0) \neq 0$. We know $g^{-1} \in \mathbb{C}[[x]]$, so

$$f^{-1} = x^{-k} \cdot g^{-1} \in \mathbb{C}((x))$$

That is, $\mathbb{C}((x))$ is the field of fractions of $\mathbb{C}[[x]]$.

3.4.1 Derivatives

Definition 3.4 (derivative)**.** For

$$f = \sum_{n\geq k} f_n x^n \in \mathbb{C}((x)),$$

define its derivative

$$f' \coloneqq \sum_{n\geq k} n f_n x^{n-1} \in \mathbb{C}((x)),$$

The key observation for proving Lagrange's theorem is that

$$\left[x^{-1}\right] f' = 0 \quad \forall f \in \mathbb{C}((x))$$

Let's see to recover the usual properties of the derivative. For example, the product rule

$$(fg)' = f'g + fg'$$

Since this equation is linear in f and g, it suffices to prove it where $f = x^l$ and $g = x^m$. Then it reduces to

$$\left(x^{l+m}\right)' = x^l \left(x^m\right)' + \left(x^l\right)' x^m$$

where we can verify explicitly. Applying the product rule $n-1$ times $n \geq 1$ we obtain

$$(f^n)' = nf^{n-1}f'$$

for positive n. If n is negative, we obtain the power rule from the equation

$$0 = \left(f^n f^{-n}\right)' = f^n \left(f^{-n}\right)' + (f^n)' f^{-n}$$

This proves the power for all $n \in \mathbb{Z}$. Note that this immediately implies the chain rule

$$(f(A(x)))' = f'(A(x))A'(x) \qquad \forall A \in \mathbb{C}[[x]], A(0) \neq 0, f \in \mathbb{C}((x))$$

This is because chain rule is linear in f and it reduces to the power rule when $f = x^n$.

3.5 Lagrange's theorem

Theorem 3.3 (Lagrange implicit function theorem/Lagrange inversion formula)**.** *Suppose that $\phi \in \mathbb{C}[[x]]$ with $\phi(0) \neq 0$. then there exists a unique $A \in \mathbb{C}[[x]]$ with*

$$A(x) = x\phi(A(x))$$

for any $f \in \mathbb{C}((x))$ and $n \neq 0$,

$$[x^n]\, f(A(x)) = \frac{1}{n}\left[x^{n-1}\right] f'(x)\phi^n(x)$$

If $f(x) = x$, this states

$$[x^n]\, A(x) = \frac{1}{n}\left[x^{n-1}\right] \phi^n(x)$$

To prove this theorem, we need the following lemma.

Lemma 3.3 (Residue composition)**.** *Let $A \in \mathbb{C}[[x]]$, where*

$$A(x) = \sum_{n \geq k} a_n x^n \quad \textit{where } k \geq 1 \textit{ and } a_k \neq 0$$

then for any $f \in \mathbb{C}((x))$,

$$\left[x^{-1}\right] f(x) = \frac{1}{k}\left[x^{-1}\right] f(A(x)) \cdot A'(x)$$

Proof of lemma. Since the equation is linear in f, it suffices to prove it when $f = x^n, n \in \mathbb{Z}$. We want to prove

$$\frac{1}{k}\left[x^{-1}\right] A^n(x) \cdot A'(x) \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

- If $n \neq -1$, then

$$\left[x^{-1}\right] A^n(x) \cdot A'(x) = \left[x^{-1}\right] \frac{\left(A^{n+1}(x)\right)'}{n+1} = 0$$

- If $n = -1$, then write

$$A(x) = x^k \cdot B(x) \quad \text{where } B \in \mathbb{C}[[x]], B(0) \neq 0$$

Then $B^{-1}(x) \in \mathbb{C}[[x]]$. We have

$$\begin{aligned}
\left[x^{-1}\right] A^{-1}(x) \cdot A'(x) &= \left[x^{-1}\right] \left(x^k B(x)\right)^{-1} \left(x^k B(x)\right)' \\
&= \left[x^{-1}\right] x^{-k} B^{-1}(x) \left[x^k B'(x) + k x^{k-1} B(x)\right] \\
&= \left[x^{-1}\right] (\underbrace{B^{-1}(x) B'(x)}_{\in \mathbb{C}[[x]]} + k x^{-1}) = 0 + k = k
\end{aligned}$$

$\square$

Proof of Lagrange theorem. Since $\phi(0) \neq 0$, $\phi^{-1} \in \mathbb{C}[[x]]$. Let

$$B(x) \coloneqq x \phi^{-1}(x) \in \mathbb{C}[[x]]$$

Note that the equation $A(x) = x\phi(A(x))$ is just $B(A(x)) = x$. So the unique A is the compositional inverse of B. Moreover, $A(B(x)) = 0$. For $n \neq 0$,

$$\begin{aligned}
[x^n]\, f(A(x)) &= \left[x^{-1}\right] x^{-n-1} f(A(x)) \\
&= \frac{1}{1} \left[x^{-1}\right] B^{-n-1}(x) f(\underbrace{A(B(x))}_{x}) B'(x) && \text{(Lemma 3.3 with } k = 1\text{)} \\
&= \left[x^{-1}\right] f(x) \cdot \frac{\left(B^{-n}(x)\right)'}{-n}
\end{aligned}$$

Note that $[x^{-1}]\, p(x) q(x) = 0 \implies [x^{-1}]\, p(x) q'(x) = -[x^{-1}]\, p'(x) q(x)$ Then,

$$\begin{aligned}
\left[x^{-1}\right] f(x) \cdot \frac{\left(B^{-n}(x)\right)'}{-n} &= \frac{1}{n} \left[x^{-1}\right] f'(x) \left(x \phi^{-1}(x)\right)^{-n} \\
&= \frac{1}{n} \left[x^{-1}\right] f'(x) x^{-n} \phi^n(x) \\
&= \frac{1}{n} \left[x^{n-1}\right] f'(x) \phi^n(x)
\end{aligned}$$

$\square$

Example 3.1 (special series)**.** Consider the formal power series with $x \in \mathbb{C}$

$$\mathcal{E}(x) \coloneqq \sum_{n \geq 0} \frac{x^n}{n!}$$

$$\lambda(x) \coloneqq \sum_{n \geq 1} \frac{x^n}{n}$$

$$B_\alpha(x) \coloneqq \sum_{n \geq 0} \binom{\alpha}{n} x^n$$

We can show that $\mathcal{E}, \lambda, B_\alpha$ satisfy the usual properties of e^x, $-\log(1-x)$ and $(1+x)^\alpha$, *e.g.*,

$$\begin{aligned}\mathcal{E}(x+y) &= \mathcal{E}(x)\mathcal{E}(y)\\ \mathcal{E}'(x) &= \mathcal{E}(x)\\ \lambda(1-\mathcal{E}(-x)) &= x\\ B_{\alpha+\beta}(x) &= B_\alpha(x)B_\beta(x)\end{aligned}$$

So we will just denote these by

$$e^x \quad -\log(1-x) \quad (1+x)^\alpha$$

We can prove these identies directly, or use the following shortcut: if f, g analytic power series and there exists $r > 0$ such that

$$f(z) = g(z) \qquad |z| < r$$

then

$$[x^n]\, f(x) = [x^n]\, g(x) \qquad \forall n \in \mathbb{N}$$

This follows by taking the derivative of $f(z) = g(z)$ n times and set $z = 0$. (And this works in multivariables.)

3.5.1 Applications of Lagrange's Theorem

Theorem 3.4 (Cayley's Formula). *There are n^{n-2} trees and n^{n-1} rooted trees on the vector set $[n]$. We can use Lagrange theorem to prove.*

Proposition 3.1. *Let a_n denote the number of rooted trees on $[n]$,*

$$A(x) = \sum_{n\geq 1} \frac{a_n}{n!} x^n \in \mathbb{C}[[x]](\textit{exponential generating function}),$$

then $A(x) = xe^{A(x)}$.

Proof. For $n \geq 1$, we have $[x^n]xe^{A(x)} = [x^{n-1}] \sum_{m\geq 0} \dfrac{A^m}{m!}$.

$$\begin{aligned} n = 1 &\Rightarrow \frac{a_1}{1!}\\ n \geq 2 &\Rightarrow \text{RHS} = \sum_{m=1}^{n-1} \frac{1}{m!} \sum_{i_1..i_m \geq 1, i_1+..+i_m = n-1} \frac{a_{i_1}}{i_1!} \cdots \frac{a_{i_m}}{i_m!}\end{aligned}$$

We need to show that this equals $\dfrac{a_n}{n!}$, i.e.,

$$a_n = \sum_{1}^{n-1} \frac{n}{m!} \sum_{i_1..i_m \geq 1, i_1+..+i_m = n-1} \binom{n-1}{i_1, \cdots, i_m} a_{i_1} \ldots a_{i_m} (*)$$

Interpretation:

- $n = \#$ vertices
- $m = \#$ neighbors of r
- $m! =\#$ ways to order the smaller rooted trees
- $i_1 ... i_m = \#$ vertices of smaller trees
- $\binom{n-1}{i_1, \cdots, i_m} = \#$ ways to choose the vertex sets of the smaller rooted trees

So we have $A(x) = x\phi(x)$, where $\phi(x) = e^x$. By Lagrange's theorem $n \neq 0$,

$$\begin{aligned}
[x^n]A(x) &= \frac{1}{n}[x^{n-1}]\phi(x)^n \\
\frac{a_n}{n!} &= \frac{1}{n}[x^{n-1}]e^{nx} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!}
\end{aligned}$$

So $a_n = n^{n-1}$. □

Theorem 3.5 (Abel's binomial theorem)**.** *For $r \in \mathbb{C}$, we have (for $n \geq 1$)*

$$\begin{aligned}
[x^n]e^{rA(x)} &= \frac{1}{n}[x^{n-1}](e^{rx})'(e^x)^n \\
&= \frac{r}{n}[x^{n-1}]e^{rx}e^{nx} \\
&= \frac{r}{n}\frac{(r+n)^{n-1}}{(n-1)!} = \frac{r(r+n)^{n-1}}{n!} \\
[x^0]e^{rA(x)} &= 1
\end{aligned}$$

So

$$e^{rA(x)} = \sum_{n \geq 0} \frac{r(r+n)^{n-1}}{n!}x^n.$$

Taking $[x^n]$ in the identity, $e^{(\alpha+\beta)A(x)} = e^{\alpha A(x)}e^{\beta A(x)} (\alpha, \beta \in \mathbb{C})$.
We obtain

$$\frac{(\alpha+\beta)(\alpha+\beta+n)^{n-1}}{n!} = \sum_{i,j \geq 0, i+j=n} \frac{\alpha(\alpha+i)^{i-1}}{i!}\frac{\beta(\beta+n-i)^{i-1}}{i!}$$

$$\Rightarrow (\alpha+\beta)(\alpha+\beta+n)^{n-1} = \sum_{i=1}^{n} \binom{n}{i} \alpha(\alpha+i)^{i-1}\beta(\beta+n-i)^{n-i-1}.$$

Theorem 3.6 (negative binomial)**.**

$$(1-x)^{-\alpha} = \sum_{n \geq 0} \binom{\alpha+n-1}{n} x^n \quad \alpha \in \mathbb{C}$$

Proof. exercise. □

Example 3.2 (quintic equation)**.** Consider the quintic equation

$$t^5 - t + \frac{1}{2} = 0$$

This is an example of an unsolvable quintic, *i.e.*, we cannot find all solutions $t \in \mathbb{C}$ by radicals. However, we can solve for (one possible) t as a power series. Let's consider more generally,

$$t^5 - t + x = 0$$

and regard t as a formal power series in $\mathbb{C}[[x]]$. The corresponding quintic is unsolvable for almost all value of x. We have

$$t\left(t^4 - 1\right) = -x \implies t = x\left(1 - t^4\right)^{-1}$$

i.e.,

$$t = x\phi(t) \quad \text{where } \phi(x) = \left(1 - x^4\right)^{-1}$$

By Lagrange theorem for $n \geq 1$,.

$$[x^n]\, t = \frac{1}{n}\left[x^{n-1}\right]\left(1 - x^4\right)^{-n}$$

By Theorem 3.6,

$$[x^n]\, t = \frac{1}{n}\left[x^{n-1}\right]\sum_{i\geq 0}\binom{n+i-1}{i}x^{4i} = \begin{cases} \frac{1}{4i+1} & \text{if } n-1 = 4i \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$t = \sum_{i\geq 0}\frac{1}{4i+1}\binom{5i}{i}x^{4i+1}$$

As an analytic power series, it has radius of convergence $r = 4 \times 5^{-\frac{5}{4}} \approx 0.53499$. So when $x = \frac{1}{2} < 0.53$, $t \approx 0.5506$.

3.6 Series multisection