Chapter 16: Particle Swarm Optimization

Computational Intelligence: Second Edition

Contents

Introduction

- Particle swarm optimization (PSO):
  - developed by Kennedy & Eberhart [3, 7],
  - first published in 1995, and
  - with an exponential increase in the number of publications since then.

Introduction

- Particle swarm optimization (PSO):
    - developed by Kennedy & Eberhart [3, 7],
    - first published in 1995, and
    - with an exponential increase in the number of publications since then.
- What is PSO?
    - a simple, computationally efficient optimization method
    - population-based, stochastic search
    - based on a social-psychological model of social influence and social learning [8]
    - individuals follow a very simple behavior: emulate the success of neighboring individuals
    - emergent behavior: discovery of optimal regions in high dimensional search spaces

Introduction: What are the origins of PSO?

- In the work of Reynolds on “boids” [12]: Flocking is an emergent behavior which arises from the interaction of simple rules:
  - Collision avoidance
  - Velocity matching
  - Flock centering
- The work of Heppner and Grenander on using a “rooster” as an attractor [5]
- Simplified social model of determining nearest neighbors and velocity matching

Origins of PSO (cont)

- Initial objective: to simulate the graceful, unpredictable choreography of collision-proof birds in a flock
- At each iteration, each individual determines its nearest neighbor and replaces its velocity with that of its neighbor
- Resulted in synchronous movement of the flock
- Random adjustments to velocities prevented individuals to settle too quickly on an unchanging direction
- Adding roosters as attractors:
  - personal best
  - neighborhood best
  - $\longrightarrow$ particle swarm optimization

Overview of basic PSO

- What are the main components?
  - A swarm of particles
  - Each particle represents a candidate solution
  - Elements of a particle represent parameters to be optimized

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- What are the main components?
  - A swarm of particles
  - Each particle represents a candidate solution
  - Elements of a particle represent parameters to be optimized
- The search process:
  - Position updates

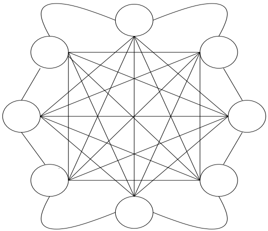
    $$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1)$$

    where

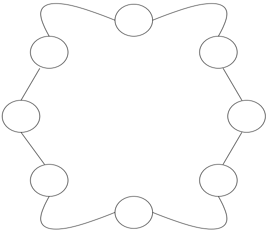
    $$\mathbf{x}_{ij}(0) \sim U(x_{min,j}, x_{max,j})$$
  - Velocity
    - drives the optimization process
    - step size
    - reflects experiential knowledge and socially exchanged information

Social network structures (Figure 16.4):

- Social interaction based on neighborhoods
- First used network structures: star and ring topologies



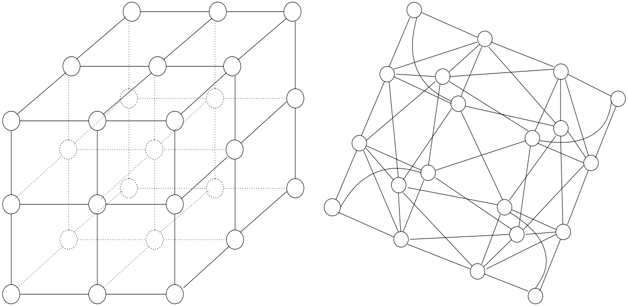
(a) Star



(b) Ring

Figure: Social Network Structures

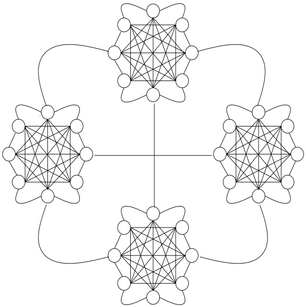
Other social structures (Figure 16.4):



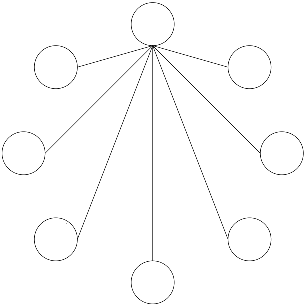
(a) Von Neumann (b) Pyramid

Figure: Advanced Social Network Structures

Other social structures (cont...)



(a) Clusters



(b) Wheel

Figure: Advanced Social Network Structures

Global Best (gbest) PSO

- Uses the star social network
- Velocity update per dimension:

$$\begin{aligned} v_{ij}(t+1) &= v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] \\ &\quad + c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)] \end{aligned}$$

- $v_{ij}(0) = 0$ (usually, but can be random)
- c_1, c_2 are positive acceleration coefficients
- $r_{1j}(t), r_{2j}(t) \sim U(0,1)$

Global Best PSO (cont)

- $\mathbf{y}_i(t)$ is the personal best position calculated as (assuming minimization):

$$\mathbf{y}_i(t+1) = \begin{cases} \mathbf{y}_i(t) & \text{if } f(\mathbf{x}_i(t+1)) \geq f(\mathbf{y}_i(t)) \\ \mathbf{x}_i(t+1) & \text{if } f(\mathbf{x}_i(t+1)) < f(\mathbf{y}_i(t)) \end{cases}$$

Global Best PSO (cont)

- $\hat{\mathbf{y}}(t)$ is the global best position calculated as

$$\begin{aligned}\hat{\mathbf{y}}(t) &\in \{\mathbf{y}_0(t), \ldots, \mathbf{y}_{n_s}(t)\} | f(\hat{\mathbf{y}}(t)) \\ &= \min\{f(\mathbf{y}_0(t)), \ldots, f(\mathbf{y}_{n_s}(t))\}\end{aligned}$$

or

$$\begin{aligned}\hat{\mathbf{y}}(t) &\in \{\mathbf{x}_0(t), \ldots, \mathbf{x}_{n_s}(t)\} | f(\hat{\mathbf{y}}(t)) \\ &= \min\{f(\mathbf{x}_0(t)), \ldots, f(\mathbf{x}_{n_s}(t))\}\end{aligned}$$

where n_s is the number of particles in the swarm

gbest PSO Algorithm

Create and initialize an n_x-dimensional swarm, S;
repeat
    for *each particle* $i = 1, \ldots, S.n_s$ **do**
        if $f(S.\mathbf{x}_i) < f(S.\mathbf{y}_i)$ **then**
            $S.\mathbf{y}_i = S.\mathbf{x}_i$;
        end
        if $f(S.\mathbf{y}_i) < f(S.\hat{\mathbf{y}})$ **then**
            $S.\hat{\mathbf{y}} = S.\mathbf{y}_i$;
        end
    end
    for *each particle* $i = 1, \ldots, S.n_s$ **do**
        update the velocity and then the position;
    end
until *stopping condition is true* ;

Local Best (lbest) PSO

- Uses the ring social network

$$\begin{aligned} v_{ij}(t+1) &= v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] \\ &\quad + c_2 r_{2j}(t)[\hat{y}_{ij}(t) - x_{ij}(t)] \end{aligned}$$

- $\hat{\mathbf{y}}_i$ is the neighborhood best, defined as:

$$\hat{\mathbf{y}}_i(t+1) \in \{\mathcal{N}_i | f(\hat{\mathbf{y}}_i(t+1)) = \min\{f(\mathbf{x})\}, \ \forall \mathbf{x} \in \mathcal{N}_i\}$$

with the neighborhood defined as

$$\begin{aligned} \mathcal{N}_i &= \{\mathbf{y}_{i-n_{\mathcal{N}_i}}(t), \mathbf{y}_{i-n_{\mathcal{N}_i}+1}(t), \ldots, \\ &\quad \mathbf{y}_{i-1}(t), \mathbf{y}_i(t), \mathbf{y}_{i+1}(t), \ldots, \mathbf{y}_{i+n_{\mathcal{N}_i}}(t)\} \end{aligned}$$

where $n_{\mathcal{N}_i}$ is the neighborhood size

Local Best PSO (cont...)

- Neighborhoods:
  - Neighborhoods are based on particle indices, not spatial information
  - Neighborhoods overlap to facilitate information exchange
- The *lbest* PSO algorithm?
- *gbest* PSO vs *lbest* PSO:
  - Speed of convergence
  - Susceptibility to local minima?

Velocity Components

- Previous velocity, $\mathbf{v}_i(t)$
  - inertia component
  - memory of previous flight direction
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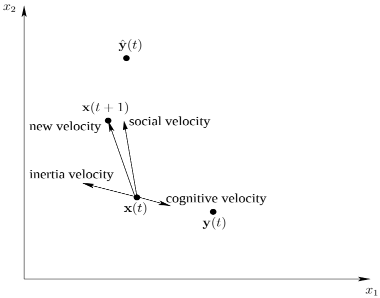
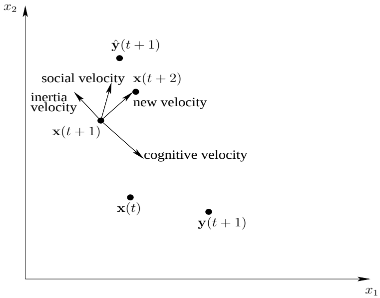
  - quantifies performance relative to past performances
  - memory of previous best position
  - nostalgia
- Social component, $c_2\mathbf{r}_2(\hat{\mathbf{y}}_i - \mathbf{x}_i)$
  - quantifies performance relative to neighbors
  - envy

Geometric Illustration Figure 16.1



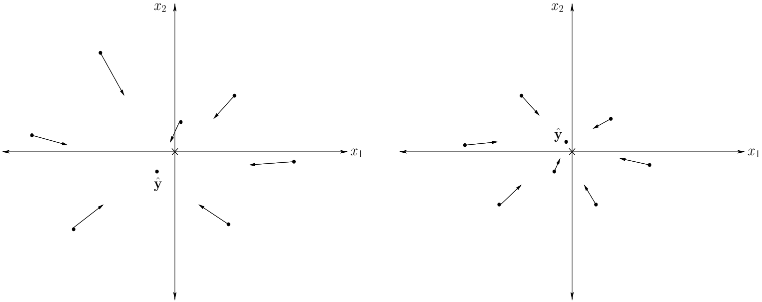
(a) Time Step t



(b) Time Step $t+1$

Figure: Geometrical Illustration of Velocity and Position Updates for a Single Two-Dimensional Particle

Cumulative Effect of Position Updates (Figure 16.2)



(a) Time Step t (b) Time Step $t+1$

Figure: Multi-particle *gbest* PSO Illustration

Stopping Conditions

- Terminate when a maximum number of iterations, or FEs, has been exceeded
- Terminate when an acceptable solution has been found, i.e. when (assuming minimization)

$$f(\mathbf{x}_i) \leq |f(\mathbf{x}^*) - \epsilon|$$

- Terminate when no improvement is observed over a number of iterations

Stopping Conditions (cont)

- Terminate when the normalized swarm radius is close to zero, i.e.

$$R_{norm} = \frac{R_{max}}{\text{diameter}(S(0))} \approx 0$$

where

$$R_{max} = ||\mathbf{x}_m - \hat{\mathbf{y}}||, \quad m = 1, \ldots, n_s$$

with

$$||\mathbf{x}_m - \hat{\mathbf{y}}|| \geq ||\mathbf{x}_i - \hat{\mathbf{y}}||, \quad \forall i = 1, \ldots, n_s$$

- Terminate when the objective function slope is approximately zero, i.e.

$$f^{'}(t) = \frac{f(\hat{\mathbf{y}}(t)) - f(\hat{\mathbf{y}}(t-1))}{f(\hat{\mathbf{y}}(t))} \approx 0$$

Basic PSO Variations: Introduction

- Main objectives of these variations are to improve
  - Convergence speed
  - Quality of solutions
  - Ability to converge

Exploration-Exploitation Tradeoff

- **Exploration**: the ability to explore regions of the search space
- **Exploitation**: the ability to concentrate the search around a promising area to refine a candidate solution

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- **Exploitation**: the ability to concentrate the search around a promising area to refine a candidate solution
- c_1 vs c_2 and the influence on the exploration–exploitation tradeoff

$$\begin{aligned} v_{ij}(t+1) &= v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] \\ &\quad + c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)] \end{aligned}$$

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  - Limit step sizes

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- Optimal value is problem-dependent

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- Controls global exploration of particles
- Optimal value is problem-dependent
- Does not confine the positions, only the step sizes

Velocity Clamping Problem (Figure 16.5)

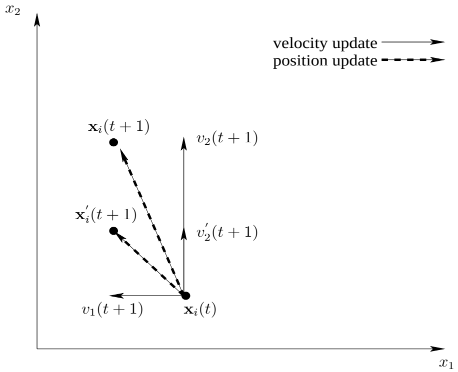

Figure: Effects of Velocity Clamping

More Velocity Clamping Problems

- What happens when, for a dimension,

$$x_{max,j} - x_{min,j} << V_{max}?$$

- What happens if all velocities are equal to the maximum velocity?
  - Particles search on the boundaries of the hypercube,

$$[\mathbf{x}_i(t) - \mathbf{V}_{max}, \mathbf{x}_i(t) + \mathbf{V}_{max}]$$

Velocity Clamping:

- Dynamically changing V_{max} when *gbest* does not improve over τ iterations [13]

$$V_{max,j}(t+1) = \begin{cases} \beta V_{max,j}(t) & \text{if } f(\hat{\mathbf{y}}(t)) \geq f(\hat{\mathbf{y}}(t-t^{'})) \\ & \forall\ t^{'} = 1, \ldots, \tau \\ V_{max,j}(t) & \text{otherwise} \end{cases}$$

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- Exponentially decaying V_{max} [4]

$$V_{max,j}(t+1) = (1 - (t/n_t)^{\alpha}) V_{max,j}(t)$$

Inertia Weight[15]

- Developed to control exploration and exploitation
- Controls the momentum
- Velocity update changes to

$$\begin{aligned} v_{ij}(t+1) &= wv_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] \\ &\quad + c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)] \end{aligned}$$

- For $w \geq 1$
  - velocities increase over time and swarm diverges
  - particles fail to change direction towards more promising regions
- For $0 < w < 1$
  - particles decelerate
  - convergence also dependent on values of c_1 and c_2

Inertia Weight (cont...)

- Exploration–exploitation trade-off
  - large values – favor exploration
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Inertia Weight (cont...)

- Exploration–exploitation trade-off
  - large values – favor exploration
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- Best value is problem-dependent
- Dynamically changing inertia weights
  - $w \sim N(0.72, \sigma)$
  - linear decreasing [17]

$$w(t) = (w(0) - w(n_t))\frac{(n_t - t)}{n_t} + w(n_t)$$

  - non-linear decreasing [23]

$$w(t+1) = \alpha w(t^{'})$$

with $w(t) = 1.4$

Constriction Coefficient [2]

- Developed to ensure convergence to a stable point without the need for velocity clamping

$$\begin{aligned} v_{ij}(t+1) &= \chi[v_{ij}(t) + \phi_1(y_{ij}(t) - x_{ij}(t)) \\ &\quad + \phi_2(\hat{y}_j(t) - x_{ij}(t))] \end{aligned}$$

where

$$\chi = \frac{2\kappa}{|2 - \phi - \sqrt{\phi(\phi - 4)}|}$$

with

$$\phi = \phi_1 + \phi_2$$

$$\phi_1 = c_1 r_1, \quad \phi_2 = c_2 r_2$$

Constriction Coefficient (cont...)

- If $\phi \geq 4$ and $\kappa \in [0, 1]$, then the swarm is guaranteed to converge to a *stable point*
- $\chi \in [0, 1]$
- κ controls exploration–exploitation
  $\kappa \approx 0$: fast convergence, exploitation
  $\kappa \approx 1$: slow convergence, exploration
- Effectively equivalent to inertia weight for specific χ:
  $w = \chi, \phi_1 = \chi c_1 r_1$ and $\phi_2 = \chi c_2 r_2$

Synchronous vs asynchronous updates

- Synchronous:
  - Personal best and neighborhood bests updated separately from position and velocity vectors
  - slower feedback
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Synchronous vs asynchronous updates

- Synchronous:
  - Personal best and neighborhood bests updated separately from position and velocity vectors
  - slower feedback
  - better for *gbest*
- Asynchronous:
  - new best positions updated after each particle position update
  - immediate feedback about best regions of the search space
  - better for *lbest*

Velocity Models

- *Cognition-only* model:

$$v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)(y_{ij}(t) - x_{ij}(t))$$

  - particles are independent hill-climbers
  - local search by each particle
- *Social-only* model:

$$v_{ij}(t+1) = v_{ij}(t) + c_2 r_{2j}(t)(\hat{y}_j(t) - x_{ij}(t))$$

  - swarm is one stochastic hill-climber
- *Selfless* model:
  - Social model, but *nbest* solution chosen from neighbors only
  - Particle itself is not allowed to become the *nbest*

Basic Parameters

- Swarm size, n_s
- Particle dimension, n_x
- Neighborhood size, $n_{\mathcal{N}_i}$
- Number of iterations, n_t
- Inertia weight, w
- Acceleration coefficients, c_1 and c_2

Acceleration Coefficients

- $c_1 = c_2 = 0$?
- $c_1 > 0, c_2 = 0$: cognition-only model
- $c_1 = 0, c_2 > 0$: social-only model
- $c_1 = c_2 > 0$:
  - particles are attracted towards the average of $\mathbf{y}_i$ and $\hat{\mathbf{y}}_i$
- $c_2 > c_1$:
  - more beneficial for unimodal problems

Acceleration Coefficients (cont)

- $c_1 < c_2$:
  - more beneficial for multimodal problems
- Low c_1 and c_2:
  - smooth particle trajectories
- High c_1 and c_2:
  - more acceleration, abrupt movements

Acceleration Coefficients (cont)

- $c_1 < c_2$:
  - more beneficial for multimodal problems
- Low c_1 and c_2:
  - smooth particle trajectories
- High c_1 and c_2:
  - more acceleration, abrupt movements
- Adaptive acceleration coefficients

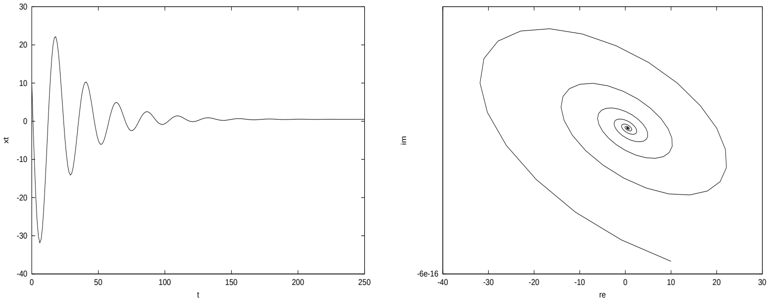
$$c_1(t) = (c_{1,min} - c_{1,max})\frac{t}{n_t} + c_{1,max}$$

$$c_2(t) = (c_{2,max} - c_{2,min})\frac{t}{n_t} + c_{2,min}$$

Simplified Particle Trajectories [19, 22]

- No stochastic component
- Single, one-dimensional particle
- Using w
- Personal best and global best are fixed: $y = 1.0, \hat{y} = 0.0$
- $\phi_1 = c_1 r_1$ and $\phi_2 = c_2 r_2$

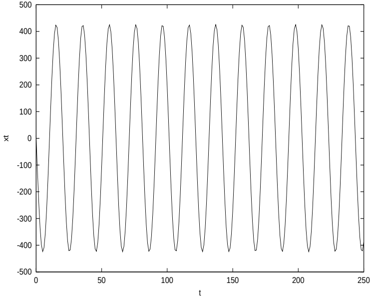
Example trajectories:



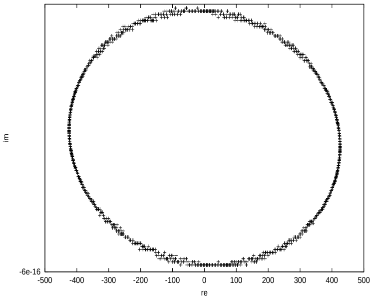
(a) Time domain (b) Phase space

Figure: Convergent Trajectory for Simplified System, with $w = 0.5$ and $\phi_1 = \phi_2 = 1.4$

Example trajectories:



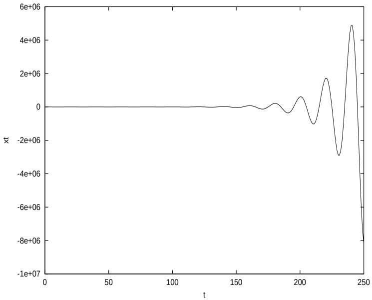
(a) Time domain



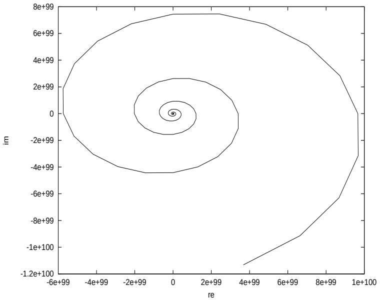
(b) Phase space

Figure: Cyclic Trajectory for Simplified System, with $w = 1.0$ and $\phi_1 = \phi_2 = 1.999$

Example trajectories:



(a) Time domain



(b) Phase space

Figure: Divergent Trajectory for Simplified System, with $w = 0.7$ and $\phi_1 = \phi_2 = 1.9$

Convergence Conditions:

What do we mean by the term convergence?

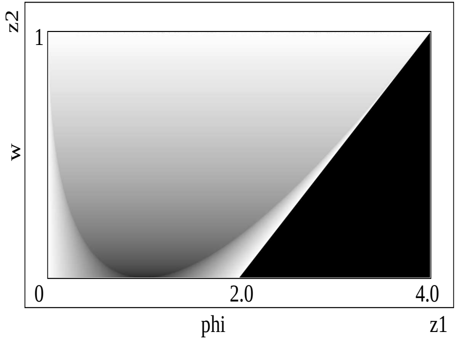


Figure: Convergence Map for Values of w and $\phi = \phi_1 + \phi_2$

Convergence Conditions (cont)

- Convergence behavior is sensitive to the values of w and c_1, c_2
- Theoreticall derived heuristics for setting these values:
  - Constriction coefficient
  - The trajectory of a particle converges if

$$1 > w > \frac{1}{2}(\phi_1 + \phi_2) - 1 \geq 0$$

  - Since $\phi_1 = c_1 U(0,1)$ and $\phi_2 = c_2 U(0,1)$, the acceleration coefficients, c_1 and c_2 serve as upper bounds of ϕ_1 and ϕ_2
  - So, particles converge if

$$1 > w > \frac{1}{2}(c_1 + c_2) - 1 \geq 0$$

Convergence Conditions (cont)

It can happen that, for stochastic ϕ_1 and ϕ_2 and a w that violates the condition above, the swarm may still converge

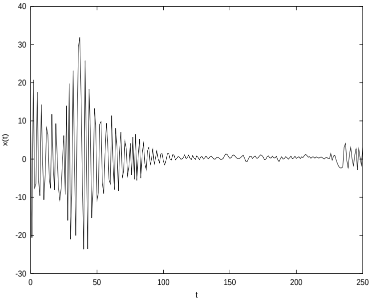


Figure: Stochastic Particle Trajectory for $w = 0.9$ and $c_1 = c_2 = 2.0$

Guaranteed Convergence PSO

- For the basic PSO, what happens when

$$\mathbf{x}_i = \mathbf{y}_i = \hat{\mathbf{y}}$$

and if this condition persists for a number of iterations?

Guaranteed Convergence PSO

- For the basic PSO, what happens when

$$\mathbf{x}_i = \mathbf{y}_i = \hat{\mathbf{y}}$$

  and if this condition persists for a number of iterations?
- The problem:

$$w\mathbf{v}_i \rightarrow 0$$

  which leads to stagnation, where particles have converged on the best position found by the swarm

Guaranteed Convergence PSO (cont)

- How can we address this problem?

Guaranteed Convergence PSO (cont)

- How can we address this problem?
- Force the global best position to change when

$$\mathbf{x}_i = \mathbf{y}_i = \hat{\mathbf{y}}$$

This is what happens in the GCPSO, where the global best particle is forced to search in a bounding box around the current position for a better position

GCPSO Equations

- The position update of the global best particle changes to

$$x_{\tau j}(t+1) = \hat{y}_j(t) + wv_{\tau j}(t) + \rho(t)(1 - 2r_2(t))$$

  where τ is the index of the global best particle
- The velocity update of the global best particle then has to change to

$$v_{\tau j}(t+1) = -x_{\tau j}(t) + \hat{y}_j(t) + wv_{\tau j}(t) + \rho(t)(1 - 2r_{2j}(t))$$

  where where $\rho(t)$ is a scaling factor

GCPSO Equations (cont)

- The scaling factor is defined as

$$\rho(t+1) = \begin{cases} 2\rho(t) & \text{if } \#successes(t) > \epsilon_s \\ 0.5\rho(t) & \text{if } \#failures(t) > \epsilon_f \\ \rho(t) & \text{otherwise} \end{cases}$$

  where $\#successes$ and $\#failures$ respectively denote the number of consecutive successes and failures

- A failure is defined as $f(\hat{\mathbf{y}}(t)) \leq f(\hat{\mathbf{y}}(t+1))$

Social-Based Particle Swarm Optimization

- Spatial neighborhoods
- Fitness-Based spatial neighborhoods
- Growing neighborhoods
- Hypercube structure
- Fully informed PSO
- Barebones PSO

Spatial Social Networks: Algorithm 16.4

- Neighborhoods usually formed on the basis of particle indices
- Based on spatial neighborhoods:

Calculate the Euclidean distance $\mathcal{E}(\mathbf{x}_{i_1}, \mathbf{x}_{i_2}), \forall i_1, i_2 = 1, \ldots, n_s$;
$S = \{i : i = 1, \ldots, n_s\}$;
for $i = 1, \ldots, n_s$ **do**
    $S^{'} = S$;
    for $i^{'} = 1, \ldots, n_{\mathcal{N}_{i^{'}}}$ **do**
        $\mathcal{N}_i = \mathcal{N}_i \cup \{\mathbf{x}_{i^{''}} : \mathcal{E}(\mathbf{x}_i, \mathbf{x}_{i^{''}}) = \min\{\mathcal{E}(\mathbf{x}_i, \mathbf{x}_{i^{'''}}), \forall \mathbf{x}_{i^{'''}} \in S^{'}\}$;
        $S^{'} = S^{'} \setminus \{\mathbf{x}_{i^{''}}\}$;
    end
end

Fitness-Based Spatial Neighborhoods

- The neighborhood of particle i is defined as the $n_{\mathcal{N}}$ particles with the smallest value of

$$\mathcal{E}(\mathbf{x}_i, \mathbf{x}_{i'}) \times f(\mathbf{x}_{i'})$$

  where $\mathcal{E}(\mathbf{x}_i, \mathbf{x}_{i'})$ is the Euclidean distance between the particles
- Overlapping neighborhoods are allowed

Growing Neighborhoods

- Start with a ring topology having smallest connectivity, and grow towards a star topology
- Particle position $\mathbf{x}_{i_2}(t)$ is added to the neighborhood of particle position $\mathbf{x}_{i_1}(t)$ if

$$\frac{||\mathbf{x}_{i_1}(t) - \mathbf{x}_{i_2}(t)||_2}{d_{max}} < \epsilon$$

where d_{max} is the largest distance between any two particles

$$\epsilon = \frac{3t + 0.6n_t}{n_t}$$

with n_t the maximum number of iterations.

Hypercube Structure

- Neighborhood structure for binary-valued problems
- Particles are defined as neighbors if the Hamming distance between the bit representation of their indices is one
- Total number of particles must be a power of two, where particles have indices from 0 to $2^{n_{\mathcal{N}}} - 1$
- Hypercube has the properties [1]:
  - Each neighborhood has exactly $n_{\mathcal{N}}$ particles.
  - The maximum distance between any two particles is exactly $n_{\mathcal{N}}$.
  - If particles i_1 and i_2 are neighbors, then i_1 and i_2 will have no other neighbors in common.

Fully Informed PSO

- Velocity equation is changed such that each particle is influenced by the successes of all its neighbors, and not on the performance of only one individual
- Each particle in the neighborhood, $\mathcal{N}_i$, of particle i is regarded equally:

$$\mathbf{v}_i(t+1) = \chi\left(\mathbf{v}_i(t) + \sum_{m=1}^{n_{\mathcal{N}_i}} \frac{\mathbf{r}(t)(\mathbf{y}_m(t) - \mathbf{x}_i(t))}{n_{\mathcal{N}_i}}\right)$$

where $n_{\mathcal{N}_i} = |\mathcal{N}_i|$, and $\mathbf{r}(t) \sim U(0, c_1 + c_2)^{n_x}$

Fully Informed PSO (cont)

- A weight is assigned to the contribution of each particle based on the performance of that particle:

$$\mathbf{v}_i(t+1) = \chi\left(\mathbf{v}_i(t) + \frac{\sum_{m=1}^{n_{\mathcal{N}_i}}\left(\frac{\phi_m \mathbf{p}_m(t)}{f(\mathbf{x}_m(t))}\right)}{\sum_{m=1}^{n_{\mathcal{N}_i}}\left(\frac{\phi_m}{f(\mathbf{x}_m(t))}\right)}\right)$$

where $\phi_m \sim U(0, \frac{c_1+c_2}{n_{\mathcal{N}_i}})$, and

$$\mathbf{p}_m(t) = \frac{\phi_1 \mathbf{y}_m(t) + \phi_2 \hat{\mathbf{y}}_m(t)}{\phi_1 + \phi_2}$$

Fully Informed PSO (cont)

- Advantages:
  - More information is used to decide on the best direction to seach
  - Influence of a particle on the step size is proportional to the particle's fitness
- Disadvantages
  - Influences of multiple particles may cancel each other
  - What happens when all particles are positioned symmetrically around the particle being updated?

Barebones PSO

- Formal proofs [18, 19, 22] have shown that each particle converges to a point that is a weighted average between the personal best and neighborhood best positions:

$$\frac{y_{ij}(t) + \hat{y}_{ij}(t)}{2}$$

- This behavior supports Kennedy's proposal to replace the entire velocity by

$$v_{ij}(t+1) \sim N\left(\frac{y_{ij}(t) + \hat{y}_{ij}(t)}{2}, \sigma\right)$$

where

$$\sigma = |y_{ij}(t) - \hat{y}_{ij}(t)|$$

Barebones PSO (cont)

- The position update is simply

$$x_{ij}(t+1) = v_{ij}(t+1)$$

- Alternative formulation:

$$v_{ij}(t+1) = \begin{cases} y_{ij}(t) & \text{if } U(0,1) < 0.5 \\ N(\frac{y_{ij}(t)+\hat{y}_{ij}(t)}{2}, \sigma) & \text{otherwise} \end{cases}$$

- There is a 50% chance that the j-th dimension of the particle dimension changes to the corresponding personal best position

Hybrid Algorithms

- Selection-based PSO
- Reproduction in PSO
- Mutation in PSO
- Differential evolution based PSO

Selection-Based PSO: Algorithm 16.5

Calculate the fitness of all particles;
for *each particle* $i = 1, \ldots, n_s$ **do**
  Randomly select n_{ts} particles;
  Score the performance of particle i against the n_{ts} randomly selected particles;
end
Sort the swarm based on performance scores;
Replace the worst half of the swarm with the top half, without changing the personal best positions;

- What are the problems with this?
- Any advantages?

Global Best Spawning: Algorithm 16.6

if $\hat{\mathbf{y}}(t)$ *is in a potential minimum* **then**
  repeat
    $\hat{\mathbf{y}} = \hat{\mathbf{y}}(t)$;
    for *NumberOfSpawns=1 to 10* **do**
      for $a = 1$ *to NumberOfSpawns* **do**
        $\hat{\mathbf{y}}_a = \hat{\mathbf{y}}(t) + N(0, \sigma)$;
        if $f(\hat{\mathbf{y}}_a) < f(\hat{\mathbf{y}})$ **then**
          $\hat{\mathbf{y}} = \hat{\mathbf{y}}_a$;
        end
      end
    end
  until $f(\hat{\mathbf{y}}) \geq f(\hat{\mathbf{y}}(t))$;
  $\hat{\mathbf{y}}(t) = \hat{\mathbf{y}}$;
end

Using Arithmetic Crossover

- An arithmetic crossover operator to produce offspring from two randomly selected particles [10, 11]:

$$\begin{aligned}
\mathbf{x}_{i_1}(t+1) &= \mathbf{r}(t)\mathbf{x}_{i_1}(t) + (\mathbf{1} - \mathbf{r}(t))\mathbf{x}_{i_2}(t) \\
\mathbf{x}_{i_2}(t+1) &= \mathbf{r}(t)\mathbf{x}_{i_2}(t) + (\mathbf{1} - \mathbf{r}(t))\mathbf{x}_{i_1}(t)
\end{aligned}$$

  with the corresponding velocities,

$$\begin{aligned}
\mathbf{v}_{i_1}(t+1) &= \frac{\mathbf{v}_{i_1}(t) + \mathbf{v}_{i_2}(t)}{||\mathbf{v}_{i_1}(t) + \mathbf{v}_{i_2}(t)||}||\mathbf{v}_{i_1}(t)|| \\
\mathbf{v}_{i_2}(t+1) &= \frac{\mathbf{v}_{i_1}(t) + \mathbf{v}_{i_2}(t)}{||\mathbf{v}_{i_1}(t) + \mathbf{v}_{i_2}(t)||}||\mathbf{v}_{i_2}(t)||
\end{aligned}$$

  where $\mathbf{r}_1(t) \sim U(0,1)^{n_x}$

Using Arithmetic Crossover (cont)

- Personal best position of an offspring is initialized to its current position:

$$\mathbf{y}_{i_1}(t+1) = \mathbf{x}_{i_1}(t+1)$$

- Particles are selected for breeding at a user-specified breeding probability
- Random selection of parents prevents the best particles from dominating the breeding process
- Breeding process is done for each iteration after the velocity and position updates have been done

Using Arithmetic Crossover (cont)

- Disadvantage:
  - Parent particles are replaced even if the offspring is worse off in fitness
  - If $f(\mathbf{x}_{i_1}(t+1)) > f(\mathbf{x}_{i_1}(t))$ (assuming a minimization problem), replacement of the personal best with $\mathbf{x}_{i_1}(t+1)$ loses important information about previous personal best positions
- Solutions:
  - Replace parent with its offspring only if fitness of offspring is better than that of the parent.

Mutation PSO

- Mutate the global best position:

$$\hat{\mathbf{y}}(t+1) = \hat{\mathbf{y}}^{'}(t+1) + \eta^{'}\mathbf{N}(0,1)$$

  where $\hat{\mathbf{y}}^{'}(t+1)$ represents the unmutated global best position, and $\eta^{'}$ is referred to as a learning parameter
- Mutate the componets of position vectors: For each j, if $U(0,1) < P_m$, then component $x_{ij}^{'}(t+1)$ is mutated using [6]

$$x_{ij}(t+1) = x_{ij}^{'}(t+1) + N(0,\sigma)x_{ij}^{'}(t+1)$$

  where

$$\sigma = 0.1(x_{max,j} - x_{min,j})$$

Mutation PSO (cont)

- Each x_{ij} can have its own deviation:

$$\sigma_{ij}(t) = \sigma_{ij}(t-1)\, e^{\tau' N(0,1) + \tau N_j(0,1)}$$

with

$$\begin{aligned} \tau' &= \frac{1}{\sqrt{2\sqrt{n_x}}} \\ \tau &= \frac{1}{\sqrt{2n_x}} \end{aligned}$$

Mutation PSO (cont)

- Secrest and Lamont [14] adjust particle positions as follows

$$\mathbf{x}_i(t+1) = \begin{cases} \mathbf{y}_i(t) + \mathbf{v}_i(t+1) & \text{if } U(0,1) > c_1 \\ \hat{\mathbf{y}}(t) + \mathbf{v}_i(t+1) & \text{otherwise} \end{cases}$$

where

$$\mathbf{v}_i(t+1) = |\mathbf{v}_i(t+1)|\mathbf{r}_\theta$$

$\mathbf{r}_\theta$ is a random vector with magnitude of one and angle uniformly distributed from 0 to 2π and

$$\mathbf{v}_i(t+1)| = \begin{cases} N(0,(1-c_2)||\mathbf{y}_i(t) - \hat{\mathbf{y}}(t)||_2) & \text{if } U(0,1) > c_1 \\ N(0, c_2||\mathbf{y}_i(t) - \hat{\mathbf{y}}(t)||_2) & \text{otherwise} \end{cases}$$

Differential Evolution PSO

- After the normal velocity and position updates,
  - Select $\mathbf{x}_1(t) \neq \mathbf{x}_2(t) \neq \mathbf{x}_3(t)$
  - Compute offspring:

$$x'_{ij}(t+1) = \begin{cases} x_{1j}(t) + \beta(x_{2j}(t) - x_{3j}(t)) & \text{if } U(0,1) \leq P_c \\ & \text{or } j = U(1, n_x) \\ x_{ij}(t) & \text{otherwise} \end{cases}$$

  where $P_c \in (0,1)$ is the probability of crossover, and $\beta > 0$ is a scaling factor
  - Replace position of the particle if the offspring is better, i.e. $\mathbf{x}_i(t+1) = \mathbf{x}'_i(t+1)$ only if $f(\mathbf{x}'_i(t+1)) < f(\mathbf{x}_i(t))$, otherwise $\mathbf{x}_i(t+1) = \mathbf{x}_i(t)$

Differential Evolution PSO (cont)

- Apply DE crossover on personal best positions only [24]:

$$y'_{ij}(t+1) = \begin{cases} \hat{y}_{ij}(t) + \delta_j & \text{if } U(0,1) < P_c \text{ and } j = U(1, n_x) \\ y_{ij}(t) & \text{otherwise} \end{cases}$$

  where δ is the general difference vector defined as,

$$\delta_j = \frac{y_{1j}(t) - y_{2j}(t)}{2}$$

  and $y_{1j}(t)$ and $y_{2j}(t)$ randomly selected personal best positions
- $y_{ij}(t+1)$ is set to $y'_{ij}(t+1)$ only if the new personal best has a better fitness

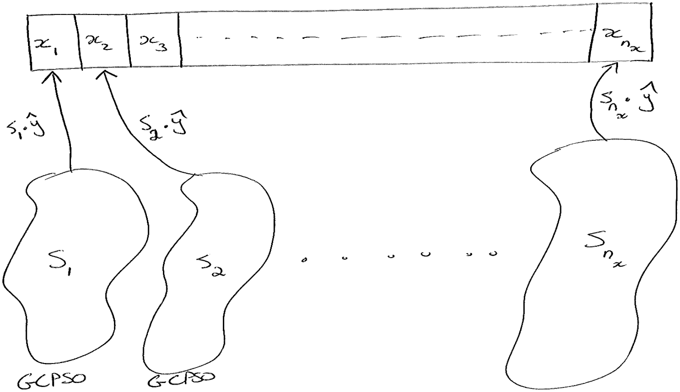
Sub-Swarm Based PSO

- Cooperative Split PSO
- Hybrid Cooperative Split PSO
- Predator-Prey PSO
- Life-cycle PSO
- Attractive and Repulsive PSO

Cooperative Split PSO

- Each particle is split into K separate parts of smaller dimension [19, 20, 21]
- Each part is then optimized using a separate sub-swarm
- If $K = n_x$, each dimension is optimized by a separate sub-swarm, using any PSO algorithm
- What are the issues?
  - Problem if there are strong dependencies among variables
  - How should the fitness of sub-swarm particles be evaluated?

Cooperative Split PSO: Fitness Evaluation



Cooperative Split PSO: Algorithm 16.8

$K_1 = n_x \bmod K$ and $K_2 = K - (n_x \bmod K)$;
Initialize K_1 $\lceil n_x/K \rceil$-dimensional and K_2 $\lfloor n_x/K \rfloor$-dimensional swarms;
repeat
  for *each sub-swarm* $S_k, k = 1, \ldots, K$ **do**
    for *each particle* $i = 1, \ldots, S_k.n_s$ **do**
      if $f(\mathbf{b}(k, S_k.\mathbf{x}_i)) < f(\mathbf{b}(k, S_k.\mathbf{y}_i))$ **then**
        $S_k.\mathbf{y}_i = S_k.\mathbf{x}_i$;
      end
      if $f(\mathbf{b}(k, S_k.\mathbf{y}_i)) < f(\mathbf{b}(k, S_k.\hat{\mathbf{y}}))$ **then**
        $S_k.\hat{\mathbf{y}} = S_k.\mathbf{y}_i$;
      end
    end
    Apply velocity and position updates;
  end
until *stopping condition is true* ;

Cooperative Split PSO (cont)

- Advantages:
  - Instead of solving one large dimensional problem, several smaller dimensional problems are now solved
  - Fitness function is evaluated after each subpart of the context vector is updated, allowing a finer-grained search
  - Improved accuracies have been obtained for many optimization problems

Hybrid Cooperative Split PSO

- Hybrid Cooperative Split PSO:
  - Have subswarm and a main swarm
  - Main swarm solves the complete problem
  - After one iteration of the cooperative algoritihm, replace a randomly selected particle from the main swarm with the context vector
  - Randomly selected particle's personal best should not be a neighborhood best
  - After a GCPSO update of the main swarm, replace a randomly selected particle from each subswarm with the corresponding element in the global best position of the main swarm
  - Randomly selected subswarm particle's personal best should not be the global best for that particle.

Predator-Prey PSO

- Competition introduced to balance exploration–exploitation
- Uses a second swarm of predator particles:
  - Prey particles scatter (explore) by being repelled by the presence of predator particles
  - Silva *et al.* [16] use only one predator to pursue the global best prey particle
  - The velocity update for the predator particle is defined as

$$\mathbf{v}_p(t+1) = \mathbf{r}(\hat{\mathbf{y}}(t) - \mathbf{x}_p(t))$$

    where $\mathbf{v}_p$ and $\mathbf{x}_p$ are respectively the velocity and position vectors of the predator particle, p
    $\mathbf{r} \sim U(0, V_{max,p})^{n_x}$
  - $V_{max,p}$ controls the speed at which the predator catches the best prey

Predator-Prey PSO (cont)

- The prey particles update their velocity using

$$\begin{aligned} v_{ij}(t+1) &= wv_{ij}(t) + c_1 r_{1j}(t)(y_{ij}(t) - x_{ij}(t)) \\ &+ c_2 r_{2j}(t)(\hat{y}_j(t) - x_{ij}(t)) \\ &\quad +c_3 r_{3j}(t) D(d) \end{aligned}$$

  d is Euclidean distance between prey, i, and the predator $r_{3j}(t) \sim U(0,1)$, and $D(d) = \alpha\, \mathrm{e}^{-\beta d}$
- $D(d)$ quantifies the influence that the predator has on the prey, growing exponentially with proximity
- Position update of prey particles: If $U(0,1) < P_f$, then the prey velocity update above is used, otherwise the normal velocity update is used

Life-Cycle PSO

- Life-cylce PSO is used to change the behavior of individuals [9, 10]
- An individual can be in any of three phases:
  - a PSO particle,
  - a GA individual, or
  - a stochastic hill-climber
- All individuals start as PSO particles
- If an individual does not show an acceptable improvement in fitness, it changes to the next life-cycle
- First start with PSO, then GA, then hill-climbing, to have more exploration initially, moving to more explotation

Attractive and Repulsive PSO (ARPSO)

- A single swarm is used, which switch between two phases depending on swarm diversity
- Diversity is measured using

$$\text{diversity}(S(t)) = \frac{1}{n_s} \sum_{i=1}^{n_s} \sqrt{\sum_{j=1}^{n_x} (x_{ij}(t) - \overline{x}_j(t))^2}$$

where $\overline{x}_j(t)$ is the average of the j-th dimension over all particles, i.e.

$$\overline{x}_j(t) = \frac{\sum_{i=1}^{n_s} x_{ij}(t)}{n_s}$$

Attractive and Repulsive PSO (cont)

- If diversity$(S(t)) > \varphi_{min}$, then switch to attraction phase
- Otherwise, swarm switches to repulsion phase until a threshold diversity, φ_{max} is reached
- Attraction phase uses basic velocity update
- Repulsion phase changes velocity update to

$$v_{ij}(t+1) = wv_{ij}(t) - c_1 r_{1j}(t)(y_{ij}(t) - x_{ij}(t)) - c_2 r_{2j}(t)(\hat{y}_{ij}(t) - x_{ij}(t))$$

Multi-Start PSO

- Major problems with the basic PSO is lack of diversity when particles start to converge to the same point
- Multi-start methods have as their main objective to increase diversity, whereby larger parts of the search space are explored
- This is done by continually inject randomness, or chaos, into the swarm
- Note that continual injection of random positions will cause the swarm never to reach an equilibrium state
- Methods should reduce chaos over time to ensure convergence

Craziness PSO

- *Craziness* is the process of randomly initializing some particles [7]
- What are the issues in reinitialization?
  - What should be randomized?
  - When should randomization occur?
  - How should it be done?
  - Which members of the swarm will be affected?
  - What should be done with personal best positions of affected particles?

Repelling Methods

- The focus of repelling methods is to improve exploration abilities
- Charged PSO:
  - Changes the velocity equation by adding a particle acceleration, $\mathbf{a}_i$:

$$\begin{aligned} v_{ij}(t+1) &= wv_{ij}(t) + c_1 r_1(t)[y_{ij}(t) - x_{ij}(t)] \\ &+ c_2 r_2(t)[\hat{y}_j(t) - x_{ij}(t)] \\ &+ a_{ij}(t) \end{aligned}$$

Charged PSO (cont)

- The acceleration determines the magnitude of inter-particle repulsion:

$$\mathbf{a}_i(t) = \sum_{l=1, i \neq l}^{n_s} \mathbf{a}_{il}(t)$$

- The repulsion force between particles i and l defined as

$$\mathbf{a}_{il}(t) = \begin{cases} \left(\frac{Q_i Q_l((\mathbf{x}_i(t) - \mathbf{x}_l(t))}{||\mathbf{x}_i(t) - \mathbf{x}_l(t)||^3}\right) & \text{if } R_c \leq ||\mathbf{x}_i(t) - \mathbf{x}_l(t)|| \leq R_p \\ \left(\frac{Q_i Q_l(\mathbf{x}_i(t) - \mathbf{x}_l(t))}{R_c^2 ||\mathbf{x}_i(t) - \mathbf{x}_l(t)||}\right) & \text{if } ||\mathbf{x}_i(t) - \mathbf{x}_l(t)|| < R_c \\ 0 & \text{if } ||\mathbf{x}_i(t) - \mathbf{x}_l(t)|| > R_p \end{cases}$$

where Q_i is the charged magnitude of particle i

Charged PSO (cont)

- Neutral particles have a zero charged magnitude, i.e. $Q_i = 0$
- Inter-particle repulsion occurs only when the separation between two particles is within the range $[R_c, R_p]$
  - R_c is the core radius
  - R_p is the perception limit
- The smaller the separation, the larger the repulsion between the corresponding particles
- The acceleration, $\mathbf{a}_i(t)$, is determined for each particle before the velocity update

Binary PSO

- PSO was originally developed for continuous-valued search spaces
- Binary PSO was developed for binary-valued domains
- Particles represent positions in binary space

$$\mathbf{x}_i \in \mathbb{B}^{n_x}, \; x_{ij} \in \{0, 1\}$$

- Changes in a particle's position then basically implies a mutation of bits, by flipping a bit from one value to the other

Binary PSO

- Interpretation of velocity vector:
  - Velocity may be described by the number of bits that change per iteration, which is the Hamming distance between $\mathbf{x}_i(t)$ and $\mathbf{x}_i(t+1)$, denoted by $\mathcal{H}(\mathbf{x}_i(t), \mathbf{x}_i(t+1))$
  - If $\mathcal{H}(\mathbf{x}_i(t), \mathbf{x}_i(t+1)) = 0$, zero bits are flipped and the particle does not move; $||\mathbf{v}_i(t)|| = 0$
  - $||\mathbf{v}_i(t)|| = n_x$ is the maximum velocity, meaning that all bits are flipped

Binary PSO

- Interpretation of velocity for a single dimension:
  - Velocity is used to calculate a probability of a bit flip:

$$v'_{ij}(t) = \text{sig}(v_{ij}(t)) = \frac{1}{1 + e^{-v_{ij}(t)}}$$

  - Position update changes to

$$x_{ij}(t+1) = \begin{cases} 1 & \text{if } r_{3j}(t) < \text{sig}(v_{ij}(t+1)) \\ 0 & \text{otherwise} \end{cases}$$

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