Advances in Portfolio Selection Under Discrete Choice Constraints: A Mixed–Integer Programming Approach and Heuristics

by

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Abstract

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Over the last year or so, we have witnessed the global effects and repercussions related to the field of finance. Supposed blue chip stocks and well-established companies have folded and filed for bankruptcy, an event that might have thought to been absurd two years ago. In addition, finance and investment science has grown over the past few decades to include a plethora of investment options and regulations. Now more than ever, developments in the field are carefully examined and researched by potential investors. This thesis involves an investigation and quantitative analysis of key money management problems. The primary area of interest is *Portfolio Selection*, where we develop advanced financial models that are designed for investment problems of the 21^{st} century.

Portfolio selection is the process involved in making large investment decisions to generate a collection of assets. Over the years the selection process has evolved dramatically. Current portfolio problems involve a complex, yet realistic set of managing constraints that are coupled to general historic risk and return models. We identify three well-known portfolio problems and add an array of practical managing constraints that form three different types of *Mixed-Integer Programs*. The product is advanced mathematical models related to risk-return portfolios, index tracking portfolios, and an integrated stock-bond portfolio selection model. The numerous sources of uncertainty are captured in a *Stochastic Programming* framework, and *Goal Programming* techniques are used to facilitate various portfolio goals. The designs require the consideration of modelling elements and variables with respect to problem solvability. We minimize trade-offs in modelling and solvability issues found in the literature by developing problem specific algorithms. The algorithms are tailored to each portfolio design and involve decompositions and heuristics that improve solution speed and quality. The result is the generation of portfolios that have intriguing financial outcomes and perform well with respect to the market.

Portfolio selection is as dynamic and complex as the recent economic situation. In this thesis we present and further develop the mathematical concepts related to portfolio construction. We investigate the key financial problems mentioned above, and through quantitative financial modelling and computational implementations we introduce current approaches and advancements in field of *Portfolio Optimization*.

Dedication

To my parents, Jim and Menka Stoyan.

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Chapter 1

Introduction and Thesis Outline

1.1 Introduction

On September 6^{th} 2007, the following headlines were in the business section of three well-know Canadian newspapers:

"Forzani has Blowout Quarter"

THE GLOBE AND MAIL

"Forzani Net Income Increases, Declares Dividend Starting in Q4"

THE NATIONAL POST

"Booming Forzani Launching Dividend"

THE TORONTO STAR

The Forzani Group Ltd. is a sporting goods retailer that trades on the Toronto Stock Exchange (TSX). Only one year later, on September 4^{th} 2008, the following was printed:

"Weather, Economic Woes Hurt Forzani's Bottom Line"

THE GLOBE AND MAIL

"Sporting Goods Retailer Posts 72% Profit Drop"

THE NATIONAL POST

"Forzani's Three-Month Profit Dives 72%"

THE TORONTO STAR

Such activities pose difficult problems for investors. The Forzani Group is just one example of the uncertainties present in the stock market. When one considers thousands of companies like this, the problem of selecting a profitable stock only becomes more complex. Furthermore, selecting a set of stocks to invest in while considering various decision rules or portfolio qualities, is an even greater problem to solve. *Portfolio Selection* is the process that involves making large investment decisions based on a criteria or array of policies in constructing a set of investing constituents. Currently, there exist many companies and well-known corporations that design various funds to engage in a variety of security and monetary investment strategies. As many have witnessed over the last year or so, even the best portfolio managers can run into difficulties. On September 16^{th} 2008, The Wall Street Journal had the following title concerning three top investment firms:

> "Lehman Files for Bankruptcy, Merrill Sold, AIG Seeks Cash"
>
> THE WALL STREET JOURNAL

Regardless of the reason these investment firms fell, they exemplify that top money managers can make poor investment decisions and hence, portfolio selection is a complex and dynamic field. Every financial corporation and company has their own sophisticated techniques and unique methods to handle investment problems, however, uncertainties in the market may remain undetected even by supposed experts.

In this thesis we present the concept of portfolio selection from a quantitative perspective and design advanced modelling and solution methods. In the next chapters we address the events above and attempt to answer many of the questions involved in investment decisions through mathematical discussions, financial models, computational implementations, and the introduction of recent developments in *Portfolio Optimization*.

1.1.1 Financial Situation

The first decade of the 21^{st} century has taken financial analysts and monetary consultants on a very active and at times, revolutionary journey. Many investment institutions have ended up in a world where new financial techniques and managing methods are welcomed, which might have not been the case a few years ago. Since the development of monetary means of exchange, financial endeavours have always been a part of large corporations, small companies, and simply the average persons life. However, due to the activity in recent financial markets, more and more investors are questioning their funds, increasing their activity with portfolio managers, and are highly cautions with any monetary endeavours. The crash in present markets is proving to have an effect on all sectors, as they are intertwined and attached to many economic ventures.

Since financial markets are the arena to which all types of investing instruments are bought, sold, and traded, the current economic situation is having a great effect on the prosperity of many nations. This, however, is not unique; over the years erratic market behaviour has lead to numerous economic eras. In the Roaring 20s, stock market activity was at an all time high as a diverse set of investors from analysts to steel workers increased their fortunes by investing in various securities. At that time it was almost impossible to loose money in the stock market and the result was a large rise in market price averages. Eventually the market crashed and caused the beginning of the Great Depression in the 1930s. This rise and fall of financial markets can be compared to what has occurred in 2008, however, such events are not always the case. Depending on the financial environment, not all market crashes cause economic collapses. In 1987, due to the financial response from monetary officials, the market crash in the late 1980s provided grounds for financial and economic prosperity in the early 90s. The current financial situation seems to have historic similarities with respect to both: the Roaring

20s and the 1980s. At the start of the 21^{st} century we have experienced great economic prosperity and financial growth. However, many question whether 2009 is the beginning of a depression, as in the 1930s, or if the financial crash we are experiencing will lead to economic developments and monetary growth, as in the 1990s. What is different about the current financial situation is that markets are enriched with new types of investments, regulations, and highly qualified global investors that utilize a broad range of analytical techniques, computer algorithms, and other sophisticated tools to generate portfolios that can shift market results. Such uncertainties provide academics with the forum for scientific developments in the area of portfolio selection and financial engineering. We begin this document by outlining the key ingredients and noble advancements in the field of investment theory and finance. Then we motivate the ideas behind state-of-the-art portfolio models and present the contributions of this dissertation to the discipline of financial engineering and optimization.

1.2 Investors and the Financial Market

When companies need capital to expand and purchase additional assets necessary for growth, they often offer some form of investing instruments to the public. Typically, the instruments are in the form of stocks and bonds, but they may also involve other types of derivatives. Companies that issue stock shares rarely deal directly with investors that buy or sell stocks, instead they are traded with other investors that have the opposite idea with respect to share value. Establishing the price of the stock then becomes the issue, as the trade price may be far from the value that was initially set by the company issuing the stock. In fact, there are many factors that may affect stock price in financial markets, even simple economic speculation can cause shifts in share prices. This is predominantly the area where financial analysts reside; whereby they exploit various means and develop different methods that attempt to take advantage of price shifts.

Classically, Burton Malkiel (1999) defines two major trains of thought when it comes to financial analysts. The first involves the Firm-Foundation Theory, in which an investor believes that every stock has an implicit price called the intrinsic value. The idea is that over some period of time a stock will always converge to its intrinsic value. Hence, when a given stock market price falls below its intrinsic value, then it exhibits a favourable investment opportunity; and vice versa. The other train of thought Malkiel believes to be shared by money managers is the Castle-in-the-Air Theory. The idea behind the Castle-in-the-Air Theory is that investments are based on what the average opinion is likely to be, rather than personal insights. For example, if you are one out of one hundred judges at a dog show and receive a prize if the dog you select turns out to be the winner, then the Castle-in-the-Air Theory states that to win the prize (pick the winner) you should select the dog that you think most people are likely to favour and not a dog based on your own tastes. This idea can be carried over to stock market investments. Hence, one would invest in stocks that they predict will grasp the average interest of future market investors.

There are many additional varieties of the two decision-making ideologies we mention. In this dissertation, we present investment ideologies based on quantitative models. Regardless of the approach, most support the opinion that markets are not necessarily efficient. Basically, the Efficient Market Hypothesis (EMH) states that financial markets reflect all known information about the prices of traded assets, regardless of the investment type. Hence, the EMH asserts that it is impossible to consistently outperform the market because all information is available to any investor and through mere luck wealth is gained or lost. Although money managers and financial analysts might disagree, we will save such discussion for later and first investigate the different market instruments available for investors.

1.2.1 Investment Instruments

There are a number of different investment instruments in financial markets, in this section we describe the ones that are highly marketable and most commonly used by investors. Common stock or equities is one of the most traded investment instruments by money managers. Equities represent ownership shares in a corporation and thus, shareholders are subject to financial benefits or losses a corporation may endure. Such corporations are run by managers that are selected by a board of directors. The board of directors is elected by shareholders, of which the board has the responsibility of ensuring that management runs the corporation in the best interest of the investors. There also exists common stock that does not provide shareholders with voting privileges, which is sometimes referred to as restricted shares. Other than the restriction on voting rights, restricted shares are equivalent to standard equities and reap the same financial benefits or losses. The final important characteristic of common stock is what occurs with respect to shareholders if a corporation goes bankrupt. Equities have two features with respect to this issue, namely: limited liability and residual claim. Limited liability states that in the event of corporation bankruptcy, creditors are not subject to the personal assets of the shareholder, and the greatest amount a shareholder can lose is their original investment. Residual claim states that in the event of a liquidation, shareholders are the last to claim any of the assets and/or income from a corporation [Bodie *et al.*, 2005]. Examples of claimants that are subject to a corporation's assets/income before shareholders are: tax authorities, suppliers, employees, and bondholders.

Bonds are another common investing instrument that can be issued by corporations or by governments at any level; i.e. federal, provincial, or municipal. As with stocks, corporations offer corporate bonds for the purpose of raising capital necessary for growth. Corporate bonds vary in quality which primarily depend on the strength of the corporation issuing the bond. Bonds enable firms to borrow money from the public with the agreement that the initial bond amount will be repaid with interest (coupon rate) at a future date known as the maturity date. Bond interest or the coupon rate may be paid in installments throughout the duration of the bond or given all at once upon maturity. The latter is a called zero coupon bond and simply refers to a bond without coupons, of which the bond is sold to investors at a reduced price with the agreement that the issuer will repay the initial bond price at maturity. There are a number of additional options that are available with corporate bonds. Callable bonds give the issuer the option to repurchase the bond at a stipulated call price. Retractable bonds give the holder the option to redeem the bond before maturity and extendible bonds allow for extensions of the maturity date. Convertible bonds give the holder the right and option to convert each bond into a specified number of stock shares. These describe the most common bond options, however, for more information the reader may refer to [Luenberger, 1998]. Depending on the state of a financial market, corporate bonds may exhibit varying degrees of price fluctuations with respect to the initial price and coupon rates. Typically, government bonds are less variable, however, they do not offer the options and rates of corporate bonds.

Finally, mortgage-backed securities and preferred stock are two types of market instruments that possess features of both stocks and bonds. With the introduction of variable rate mortgages, banks were able to offer loans with interest rates that vary with a measure of the current market. Such mortgages may offer lower rates of interest, but involve risks attached to market interest rates via a mortgage-backed security. A mortgage-backed security is a claim to ownership in a pool of mortgages of which owners receive cash inflows as the loans are paid off. Preferred stock, on the other hand, behaves like a bond in the sense that it pays a holder a fixed amount every year and in the event of bankruptcy, preferred stockholders will be payed out before common stockholders. However, holders are limited by not having voting privileges, the shares can be callable from the issuing corporation, convertible into common stock, and additional terms are possible by the issuer. Other types of money market instruments include treasury bills, commercial paper, eurodollars, income trusts, derivatives such as options and futures, etc. The focus of this document primarily concerns the money instruments described above, for additional information on investment options one may consult [Luenberger, 1998]. In addition, let it be clear that stocks and most bonds exhibit varying degrees of price uncertainty, which is associated with the issuer, the market, and the options and features related to them. Bond and stock market indices are a reflection of the average market return and represent a measure of such variabilities in price. In the next section we will describe various market indices and their role in financial markets.

1.2.2 Market Indices

Market indices are a collection of one type of investment instrument (i.e. stocks or bonds) that provide a performance measure for investors. Indices also allow individuals to compare foreign markets and evaluate the risks associated with specific portfolios. There are many different types of indices, each of which measure or elicit particular characteristics of the market they pertain to. Typically, indices are composed of top performing market constituents and an important property involves the number and type of issuing constituents, or different companies they contain. Another important property is that although there are different methods of weighting the constituents of an index, generally the exact weighting values are not released to the public. Finally, a third property of indices is that upon the replacement or rebalancing of constituents within an index, transaction costs are not incurred; whereas this is not the case for investors.

Bond and stock markets contain the most highly publicized and well-known indices. The Toronto Stock Exchange (TSX) contains Canada's most recognized stock market indicator, known as the S&P/TSX Composite Index. The Composite Index contains over 220 of the largest securities in the TSX – in terms of market value, and is based on a broad set of companies [TSX, 2009]. The TSX also contains a number of sector specific indices and smaller indices such as the TSX 60; which is composed of 60 securities. The S&P/TSX Composite Index is a market-value-weighted index, also called a capitalization-weighted index, where constituents are weighted in proportion to their market capitalization. This is typical practice for indices and is used in other markets such as the S&P 500 from the New York Stock Exchange (NYSE), the Wilshire 5000 Equity Index from the NYSE and the AMerican stock EXchange (AMEX), the Nasdaq Composite Index from the National Association of Securities Dealers Automated Quotations (NASDAQ), the Hang Seng Index from the Hong Kong Stock Exchange (HKSE), and others. The Dow Jones Industrial Average (DJIA) on the other hand, uses a price-weighted index that is generated by adding security prices and dividing by a certain number; at one time this number was equal to the total stocks used in the index. The price-weighted average for indices implies that high price constituents are given more weight. Another relevant type of index weighting is an equally-weighted index, which is self-explanatory and is composed of securities with equivalent weights.

Over the years, due to the strong average performance of particular indices many money managers and investment companies have developed index funds that are based on yielding returns similar to given market indices. Although such funds are designed to replicate index values, performance issues with regards to the weighting of constituents and the costs associated with rebalancing are just a few of the difficulties posed to fund managers. As mentioned above, indices are not subject to transaction costs. Hence, in order to minimize costs associated with rebalancing fund managers may try to design portfolios with far less constituents than the S&P/TSX Composite Index, S&P 500, or Wilshire 5000. Although this may not guarantee a reduction in transaction costs in the worst case it develops funds that are easier to manage, which is another important portfolio characteristic. Opposed to considering over 5000 stocks on a daily basis, as with the Wilshire 5000, fund managers aim to capture similar portfolio results using far less constituents. We will discuss issues related to index funds and further motivate the concept of index tracking in Chapter 4.

As with stock market indices, bond indices provide a measure of performance with respect to bond markets. There are many different types of bond indices that offer unique information on particular aspects of bond market performance. Bond indices can be based on a broad spectrum of categories such as their credit rating, maturity, or composition (i.e. government bonds, corporate bonds, mortgage-back securities, etc.). The most well-known Canadian bond index is the Scotia Capital Bond Index, see [Scotia Capital Bond Index, 2009]. In the Unites States of America (USA): the Salomon Brothers, Merrill Lynch, and the Lehman Brothers comprise the main indices; the latter two have recently been acquired by banks. Typically, bond indices differ from stocks in that they are computed monthly, contain a large number of constituents or issuers, and the true rates of return may be unreliable due to bond trades not being up-to-date. More information on bond indices and their use will be presented in Chapter 5.

1.3 Research Objective and Contribution

The goal of this research is to illustrate the effects current advancements in engineering and mathematical programming can have towards developments in finance and investment science. There are a large number of complex financial problems that are typically solved using elementary methods in practice; such as the ones mentioned in Subsection 1.2. In the past, the field of finance and engineering may not have been believed to be related. Designing financial models generally involves some type of selection criteria and other computationally intensive constraints that pose problems with regards to solvability and may not be tractable from an engineering standpoint. To overcome such problems, some of the first financial engineering developments made modelling assumptions that the financial community may have thought to be unrealistic. In other cases, where unrealistic assumptions were not made, solution optimality was an issue or the investment results were not practical from a money managers prospective, as possible with the Mean-Variance Optimization model of [Markowitz, 1952]. However, with recent mathematical and computational developments, the progress of financial engineering is becoming more prevalent and applicable. In this dissertation we present three well-known financial problems that are yet to be modelled realistically and solved efficiently. We design and outline advanced mathematical framework that allows each financial problem to be captured effectively. The designs are accompanied by model specific algorithms that overcome computational complexities that have prevented previous attempts. The three financial models we investigate are:

- A risk versus return portfolio model that contains practical constraints limiting the size of the portfolio, the buying/selling of assets, trading, and buy-in/holding thresholds;

- A two-stage index tracking portfolio that includes a number of financial constraints and an advanced Stochastic Programming (SP) approach to consider index and security price uncertainties;

- A bond and stock portfolio selection model that integrates uncertainties in security price and portfolio goals in a large-scale stochastic-goal programming portfolio model.

The three investigations listed above are yet to be investigated using the realistic and sophisticated modelling techniques we present, in addition to the promising solution strategies. As mentioned, the novelty of creating a model that is financially practical and mathematically solvable is not necessarily common to the field. Our contribution involves developing:

- Realistic financial models that comprise of a practical set of portfolio managing constraints;
- A method to capture uncertainties related to security price and associated model risks or goals;
- Highly specific algorithms that are designed to generate viable solutions to the large-scale financial models constructed above.

Hence, this dissertation provides new solution methods to solve state-of-the-art financial problems and presents intuitive results with respect to the developing field of portfolio optimization.

1.4 Thesis Outline

The thesis is organized as follow: in Chapter 2 we provide an overview of investment theory, present relevant financial optimization models, and give a brief literature review on the topic of portfolio selection. In Chapter 3 we add various portfolio characteristics to a well-known financial optimization model and develop an algorithm and approximation method to enhance solvability. In Chapter 4 we present the modelling framework and computational results of an index tracking portfolio that captures various sources of problem uncertainty through a SP with recourse approach. In Chapter 5 we design a bond and stock portfolio model that has the capacity to safeguard investments during times of poor economic development using a number of portfolio goals. The mathematical framework is composed of a stochastic and goal programming problem that requires a highly specific algorithm. Chapters 3–5 provide in-sample results that contain current financial data, which posses numerous economic instabilities. Finally, in Chapter 6 we conclude on our findings and discuss future research opportunities. Below we provide a brief overview that highlights the discussions specific to each chapter.

Chapter 2. Portfolio Theory

In this chapter we provide a brief history of the ideologies and models involved with portfolio selection. Quantitative portfolio selection initiated in 1952 with the introduction of Harry Markowitz's Mean Variance Optimization (MVO) model. Since his original research many methods have been developed and improved upon to handle current financial complexities. We present such models and from a literature review of current portfolio models, we also define important characteristics related to the designs.

Chapter 3. Mean–Absolute Deviation with Discrete Choice

We present a detailed analysis of an approximation to the Markowitz Mean Variance Optimization (MVO) model. Although the MVO portfolio may be used by portfolio managers in practice, improvements to various areas of the design are necessary for current financial markets. We describe an approximation that is equivalent to the MVO under certain assumptions and is more tractable with regards to solvability. Using the approximation, the portfolio design has the capacity to produce optimal results for instances where related literature has great difficulties. We also present the results for large-scale implementations, which require the development of a model specific algorithm.

Chapter 4. Stochastic Programming Approach to Index Tracking

In this chapter we consider the problem of tracking a target portfolio or index under uncertainty. Due to an embedded NP-hard subproblem, many of the current index tracking models only consider a small number of important portfolio elements such as transaction costs, number of securities to hold, rebalancing, etc. In this chapter we present a tracking portfolio model that includes a comprehensive set of real-world portfolio elements and considers market uncertainties. An index tracking model is defined in a Stochastic Mixed-Integer Programming (SMIP) framework. Due to the size and complexity of the stochastic problem, the SMIP model is decomposed into subproblems and an iterative algorithm is developed that exploits the decomposition. A two-stage SMIP is solved and the results are compared with actual index values. We also provide single-scenario dynamic comparisons to illustrate the performance and strengths of the method.

Chapter 5. Portfolio Selection

In this chapter, we investigate the problem of portfolio selection when security and bond investments are considered. From the past it has been shown that bonds have an inverse relationship with that of securities and due to the current economic crisis, such an analysis is of great significance. The portfolio model is designed to uphold a number of important portfolio elements and objectives while considering various uncertainties present in the market. The resulting formulation is one of the first financial modelling applications involving a Stochastic-Goal Mixed-Integer Programming (SGMIP) approach with recourse. The large-scale financial problem is solved using a model-specific algorithm and we present encouraging results with respect to market performance.

Chapter 6. Conclusion and Future Research

Finally, we conclude on the findings of the three models we describe in Chapters 3–5. The results that we present in this document illustrate the strengths of current optimization methods towards portfolio selection. We mention future research directions with respect to the fields of finance, optimization, and portfolio selection. We also provide discussions on alternative modelling techniques and algorithm ideas that can be used to enhance and continue developments of such research.

Chapter 2

Portfolio Theory

2.1 Portfolio Management and Efficient Markets

It should be evident that portfolio managers employ various techniques or combinations of methods to make investment decisions, regardless of the size of the portfolio. Such methods become especially important for large accounts, where an increase of one percent in return may amount to millions of dollars. For this reason, money managers develop unique portfolio policies and financial techniques that provide insight towards generating market profits. Competition between fund investors almost negates the fact that easily implemented security evaluation techniques are used, otherwise such investments patters would be reflected in stock prices and may be leveraged by investors. In the simplest sense, a portfolio managers evaluation technique consists of the fundamental analysis of the corporations issuing the securities. Fundamental analysis involves an investigation of the corporation issuing a given security to determine its value. It may use a number of different evaluation factors to price securities, such as a firms earnings, dividend prospects, expected interest rate, past earnings, balance sheets, or some of the techniques mentioned in Section 1.2. Many fund managers participate in active investment portfolio strategies. An active investment strategy is one in which portfolio managers seek higher than average market returns using various investment techniques to manage portfolio activity. In contrast, a passive investment strategy does not attempt to outperform the market, instead investors try to develop diversified portfolios that represent the market. Typically, passive investment managers design long-term portfolios that support buy-and-hold type positions and are not aimed at making a "quick buck," as with active investment strategies. An underlying ideology for passive investment managers might be that markets are efficient, which is formally defined in the efficient market hypothesis.

The Efficient Market Hypothesis (EMH) in the most general form asserts that security prices reflect all information that can be derived by market examination. Hence, financial gains can not be made through technical analysis or some type of insider information. The EMH contains three different versions that differ in strength. The first is the weakest form of the EMH, which states that information from market trading, trading volume, past prices, and interest is captured in security prices. Also, if there exists a reliable investment signal then investors will have already exploited the signal and it will lose its value as it becomes widely known, e.g. the security price will increase. The second form of the EMH is slightly stronger in the sense that it assumes all the ideologies of the first with the addition that information regarding the future prospects of a corporation are reflected in the stock price. Hence, a corporation's profit forecasts, balance sheets, accounting information, etc. are captured in the value of the security. Finally, the strongest form of the EMH adds to the second version by including that security prices are reflective of all information relevant to the corporation issuing them, which includes insider information or information known by firm employees. Although few might argue that insider information is captured in a securities price, there exist a number of laws and regulations that attempt to prevent trading on such information.

Fundamentally, if one believes that markets are truly efficient, then carefully selecting a set of securities to invest in does not make sense and one can simply use a random selection process to choose portfolio investments. Technical analysts would argue differently, in that security prices possess predictable patterns and through careful evaluation one can take advantage of price shifts. To date, there exists a number of publications involving various methods to price and evaluate future security values [Brown, 1828; Chen *et al.*, 2003; Edwards and Mangee, 2001; Lux, 2001; Malkiel, 1999; Osborne, 1959; Vandewalle *et al.*, 2001]. In the next section we present some of the key ideas behind pricing stocks and bonds, and then go on to illustrate fundamental investment and portfolio selection models.

2.1.1 Security Value and Investment Theory

Over the past 60 years or so there has been numerous mathematical attempts and methods aimed at predicting the price of securities. In one of the most elementary cases, the probability of a stock being a certain price can be described as a Markov process. A Markov process states that only the present value of a variable is relevant or necessary to predict the probability of future values, making all historical prices irrelevant. Given a sequence of random variables $\phi_1, \phi_2, ...$, a Markov process is formally defined as any stochastic process where

$$P(\phi_{n+1} = \Phi | \phi_n = \Phi_n, ..., \phi_1 = \Phi_1) \quad = \quad P(\phi_{n+1} = \Phi | \phi_n = \Phi_n). \qquad (2.1)$$

Historically, stock prices have shown to possess similar properties illustrated in equation (2.1), which states that prices represent random events where only today's stock price affects the probability of tomorrow's. From this simple idea, a number of stochastic principles have been adopted to define future security values. One extension of equation (2.1) is a stochastic process known as a martingale. A martingale is a process where

$\mathrm{E}[|\phi_n|] < \infty \; \forall \; n$ and

$$\mathrm{E}[\phi_{n+1}|\phi_1, \phi_2, ..., \phi_n] = \phi_n. \tag{2.2}$$

If ϕ_n represents security price, then (2.2) implies that given past security prices, the expectation of the future price is equal to its last price.

One of the most commonly used methods to predict stock price behaviour comes from developments that describe Brownian motions. A Brownian motion defines the random movement of particles in a liquid or gas. Given that the price movement of a stock is random, then the discrete-time financial version of the model is as follows:

$$\frac{\delta\phi}{\phi} = \mu\delta t + \sigma\epsilon\sqrt{\delta t}, \tag{2.3}$$

where $\delta\phi$ is the change in stock return over a small time interval δt, and ϵ is a random variable that typically has a normal distribution. In addition, μ is the expected rate of return and σ is the standard deviation. Thus, the solution to (2.3) provides the return of a given security over the time period δt. The equation is made up of two intuitive parts, one that gives the expected value of a stock and another that adds a random element to the equation. Specifically, $\mu\delta t$ provides the expected value of the return over the prescribed time period and $\sigma\epsilon\sqrt{\delta t}$ captures the stochastic elements present in the return. Brownian motion approximations typically perform well in many financial markets, which can be attributed to the combination of the mean and random element of the approximation. There are a number of publications that use Brownian motions to predict security prices and apply different variations of them towards specific financial problems, e.g. [Aldabe *et al.*, 1998; Osborne, 1959; Rogers, 1997; Ross, 2003, 1996]. In addition, other than directly deriving security approximations, technical analysts have created many alternative measures that account for properties such as the confidence in a stock price and/or the corporations issuing them. One of such measures is the beta value of a given security. The beta value or beta coefficient defines the market risks associated with a particular security. Generally, high beta values equate to risky stocks and vice versa. As one may expect, similar corporations will have comparable beta values. To find the beta vale of a particular security a benchmark is necessary, which is usually the market index. Given that the current security return is ϕ and the benchmark is defined as I_M, then the beta value (β) is computed by simply:

$$\beta = \frac{\mathrm{Cov}(\phi, I_M)}{\sigma_M^2}, \tag{2.4}$$

where

$$\sigma_M^2 = \mathrm{Var}(I_M). \tag{2.5}$$

In equation (2.4), $\mathrm{Cov}(\cdot\,,\cdot\,)$ involves deriving the covariance of two elements over a given time window. Also, from equation (2.4) it should be clear that beta coefficients provide values of risk based on the benchmark I_M. Therefore, if a particular security's price movement is greater than the prescribed benchmark, a value of $\beta > 1$ is given. This constitutes a volatile or risky security, whereas $\beta < 1$ is treated as an investment that involves a low amount of risk. For more information on β values one may consult [Bodie *et al.*, 2005]. Using such a value, money managers can also examine the risk of the portfolio they generate. To calculate the risk of a portfolio, one multiplies the β value of a specific stock by the portfolio weight invested in the stock, and then sums together these values for each security in the portfolio. Hence, the risk of the portfolio is simply the weighted average of the individual security β values. In the next section we will investigate other well-known procedures that are used to generate and evaluate portfolio risk; however, one may recognize that even a simple idea can develop into a portfolio managers selection criteria.

There also are technical methods designed for pricing bonds that deal with issues regarding defaulting and variable rates of interest. Defaulting and the interest rates of bonds are typically linked to credit risk and the company that is issuing the bond. In addition, there may be variability in bond structure that can affect the amount of interest collected, such as maturity or other attributes mentioned in Subsection 1.2.1. In the simplest case, say we invest in an n-year zero-coupon bond, or a bond where the interest plus the principal is paid by the issuer at maturity. Then, given a principal of C dollars and a risk-free rate of interest k, the present value (PV), or the amount the bond is worth today, is equal to

$$PV = \frac{C}{(1+k)^n} \,. \tag{2.6}$$

If the rate of interest is not constant, then (2.6) becomes

$$PV = \frac{C}{\prod_{i=1}^{n}(1+k_i)} \,, \tag{2.7}$$

where $k_i \,\forall\; i = 1, ..., n$ is the annual interest rate for year i. However, many of the interest rates observed in the market are not risk-free and usually bonds provide coupons (interest payments) periodically. For such bonds, the price of the bond is calculated as the present value of all payments that will be collected by bondholders. Given a bond that is compounded annually and provides a coupon at the rate of c per year, then the present value is

$$PV = \sum_{i=1}^{n-1} \frac{cC}{(1+k_i)^i} + \frac{C(1+c)}{(1+k_n)^n} \,. \tag{2.8}$$

After finding PV, one can use the solution to derive other characteristics of bonds such as the bond yield, which requires solving equation (2.8) when the interest rate is constant, i.e. $k_i = k_{(i+1)} \,\forall\; i = 1, ..., n-1$. In addition, one may solve for c when $PV = C$ (the principal) to find the par yield, which defines the coupon rate that a bond gives to equal its principal value at maturity. There are a number of other issues that can be investigated with bonds, one can refer to [Hull, 1998] for a complete description. The most interesting, and the area where theoretically rich bond pricing models are formulated involve those with uncertain rates of interest. A common technique to price such bonds is to simply take the expected rate of interest each year. Hence, (2.8) becomes

$$PV = \sum_{i=1}^{n-1} \frac{cC}{(1+\mathrm{E}[k_i])^i} + \frac{C(1+c)}{(1+\mathrm{E}[k_n])^n} \tag{2.9}$$

when uncertainties in bond interest rates exist. Later in this document we will investigate portfolios that contain such bonds, specifically ones with variable rates of interest. Typically, these bonds offer the possibility of high interest rates or greatest revenue, yet are associated with various risks. For such investments, due to the risks related to defaulting and interest rates, bond ratings are defined that classify bonds and provide a measure that evaluates various bond risks based on a set of particular factors. Bond ratings such as the Dominion Bond Rating Service (DBRS) and the Standard & Poor's (S&P) Bond Rating, rate bonds from high quality (AAA) to low quality (D). The DBRS measures bond default risk in Canada and classifies bonds into 10 categories; from greatest to least they are: AAA, AA, A, BBB, BB, B, CCC, CC, C, D. This information can then be used to evaluate the risks associated with defaulting and interest rate returns on bond portfolios; similarly to what is done with β values in stock models. We later present a bond-stock portfolio that uses both measures to minimize investment risk and maximize portfolio return, but first we will describe the fundamental models that initiated such discussions.

2.2 Portfolio Optimization

In order to manage the risks and uncertainties associated with the investments mentioned in the previous subsection, financial analysts have developed various mathematical techniques and models aimed at maximizing revenue or minimizing risk when generating portfolios. Say an investor uses individual security β values defined in equation (2.4) of

Subsection 2.1.1 to minimize the risk of his or her portfolio. The investor is still left with the problem of selecting the number of securities to include in the portfolio and the amount to invest in each security. If one just considers the number of securities to include in the portfolio, finding the optimal number leads to an optimization problem and the issue of portfolio diversification arises. Diversification is the process of designing a portfolio that contains a wide variety of investments such that poor financial developments in one sector of the market do not affect portfolio returns. In other words, one who designs a diverse portfolio would avoid risks associated with "putting all your eggs in one basket." One way to capture portfolio diversification and minimize risk is by using variance. Since variance measures the difference between a quantity and the average value, minimizing variance restricts outliers or quantities that do not perform with the status quo. If there is a number of quantities, covariance describes how these quantities are correlated or the amount the variables change together. Thus, given that the quantity relates to stock performance, designing a diverse portfolio implies minimizing variance (covariance between stocks) or reducing the number of securities that perform similarly. If we let r_i be the return rate for security i, μ_i be the mean, and define x_i as the portfolio weight invested in security i, then the portfolio variance for n securities is

$$\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j (r_i - \mu_i)(r_j - \mu_j)\right]. \tag{2.10}$$

By taking the minimum of equation (2.10) with respect to the weights x_i, the number of securities that behave similarly in the portfolio are minimized. Hence, this imposes a diverse investments strategy and therefore reduces the risk of the portfolio. In 1952 Harry Markowitz used the idea of minimizing variance to design portfolios that reduce risks associated with diversification. He defined a very well-known financial model called the Mean-Variance Optimization (MVO) model, which is aimed at minimizing variance while maintaining a given portfolio return. There are different versions of the MVO portfolio problem, the most general is the following:

$$\begin{aligned}
\min \quad & \frac{1}{2}\,\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j (r_i - \mu_i)(r_j - \mu_j)\right] && (2.11)\\
\text{s.t.} \quad & \sum_{i}^{n} \mu_i x_i \geq R && (2.12)\\
& \sum_{i}^{n} x_i = 1 && (2.13)\\
& x_i \geq 0 \qquad\qquad \forall\, i = 1, ..., n. && (2.14)
\end{aligned}$$

In (2.11)–(2.14) above, the objective function minimizes covariance (variance between two different quantities) subject to equation (2.12), which requires the average portfolio return to be greater than or equal to R. Equations (2.13) and (2.14) are simply portfolio bookkeeping constraints that ensure securities are only bought and their weight sums to one. Solutions to the system of equations in (2.11)–(2.14) for different return values R produce what is known as the efficient frontier. The efficient frontier is a risk versus return plot that is defined by the optimized portfolios of equations (2.11)–(2.14). Figure 2.1 illustrates the efficient frontier, which is clearly outlined by the outermost line of optimal portfolios that define the minimum risk or standard deviation (square root of variance) for a given return value. The minimum-variance point, shown at the leftmost tip of the curve in Figure 2.1, is the point at which lower portfolio returns give higher risk values, and therefore are unwise or illegitimate investments. Thus, solutions to the MVO model provide financial advisors with portfolios that exhibit the least risk for a given expected return. Although this model is very intuitive it does have drawbacks and trade-offs, which we will discuss in Chapter 3.

Over a decade later Sharpe, Lintner and Mossin developed the Capital Asset Pricing Model (CAPM) in three separate publications [Lintner, 1965; Mossin, 1966; Sharpe, 1964]. CAPM was primarily a method to price the expected return of securities based on

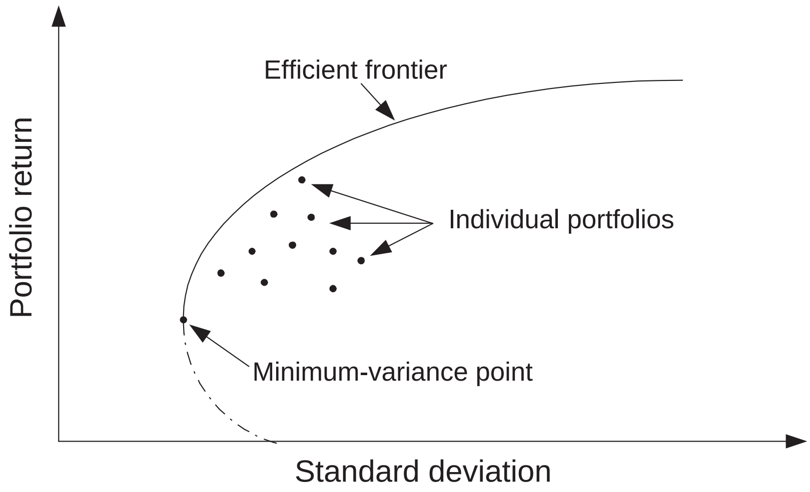

Figure 2.1: Efficiency frontier for the MVO model.

the β values shown in equation (2.4). However, CAPM offered additional insight towards theoretical portfolio theory and the model proposed by Markowitz. In the most basic sense, the CAPM investment strategy implies that investors hold the market portfolio. The reason being is that if every investor was to use the MVO portfolio strategy, and by some force they all have the same estimates, then everyone would buy the same portfolio. Thus, this would be equal to the market portfolio because as prices adjust to meet supply and demand eventually the shifts would reach an equilibrium and drive market efficiency. Therefore, this implies that investors only need to hold the market portfolio as other MVO portfolio optimizers will cause security prices to move to their proper value. Given the size of current financial markets it may be difficult to hold such a portfolio, however, various index funds have been developed that are designed to replicate the market portfolio. In Subsection 1.2.2 we have outlined the characteristics of different indices and their value with regards to investments. Although managers can only generate index funds, they are constructed to follow the characteristics and return values of well-known indices. In Chapter 4 we address problems associated with designing index funds and provide an efficient index tracking model.

Since the original works of the Markowiz MVO and the Sharpe, Lintner and Mossin CAPM, numerous publications and developments have been made in the area of portfolio optimization and financial engineering. As mentioned in section 1.3, portfolio modelers face the problem of designing funds that capture practical managing characteristics and are computationally tractable. In the next section we outline the key contributions made to the field of financial optimization and the important literature associated with the developments presented in this document.

2.2.1 Brief Literature and Model Review

In this dissertation we present complex portfolio selection models that are solved using advanced model specific algorithms. Over the years, money managers have outlined a set of characteristics or financial elements that should be considered in selecting the assets of a portfolio. Publications such as [Lintner, 1965; Markowitz, 1952; Mossin, 1966; Sharpe, 1964] started off initial investigations with respect to risk, return, diversification, and other general portfolio measures. Then literature on the topic went in two main directions: either they extended to investigations involving broader areas such as bonds and derivatives, or became more specified involving models with portfolio rich constraints. Although it is difficult to account for every constraint or investment instrument, in the next chapters we investigate current portfolio issues and important managing characteristics. One important characteristic amongst such models is the number of assets to hold in the portfolio. Portfolio managers are averse to holding a large array of assets in a portfolio since such funds are difficult to manage, track, and can lead to high costs associated with rebalancing and general portfolio maintenance. This poses a very difficult problem for most fund managers as both the CAPM and MVO models produce results that prescribe a wide array of investments. Thus, various publications have addressed the problem of limiting the number of securities to include in a portfolio, which forms a discrete choice constraint. If we investigate a simple index fund, Coleman *et al.* (2006) present an optimization model that tracks an index and limits the number of securities used in the portfolio, commonly referred to as the names-to-hold constraint. Mathematically the problem is defined as follows:

$$\min \quad \sum_{i=1}^{n} TE(x_i) \tag{2.15}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} g(x_i) \leq G \tag{2.16}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{2.17}$$

$$g(x_i) \in \mathbb{B} \qquad \forall\, i = 1, ..., n \tag{2.18}$$

$$x_i \geq 0 \qquad \forall\, i = 1, ..., n, \tag{2.19}$$

where $TE(x_i)$ is a tracking error function, $g(x_i)$ is a variable that defines whether a given security is used in the portfolio, G is an upper bound on the number of securities to hold, and $\mathbb{B}$ refers to the set of binary variables. The tracking error function in (2.15) attempts to capture the return of the index it is trying to replicate and can take on many different forms. Equation (2.16) contains the names-to-hold constraint that is responsible for limiting the number of securities used in the portfolio. Coleman *et al.* define (2.15)–(2.19) as an NP-hard problem, and thus including one simple money managing characteristic in the design of a portfolio can lead to a very difficult mathematical problem. They then go on to add a relaxation to (2.15)–(2.19) and develop an algorithm to generate solutions; one can refer to [Coleman *et al.*, 2006] for more information. The authors of [Crama and Schyns, 2003; Jobst *et al.*, 2005] also consider models with the names-to-hold constraint, where the portfolio element is applied to the Markowitz MVO problem. In addition, they add other characteristics related to buying, selling, trading, and max/min security buy-in values. With regards to the MVO portfolio selection model, Adcock and Meade (1994), Chang *et al.* (2000), Kellerer *et al.* (2000), Lin and Wang (2002), Liu *et al.* (2003), and Xia *et al.* (2000) offer alternative insights towards various practical constraints and computational aspects in comparison to the original problem. Later in this document, we present an index tracking portfolio that appeals more towards passive investors. In addition to Coleman *et al.*'s (2006) index tracking model described above, the authors of [Beasley *et al.*, 2003; Bertsimas *et al.*, 1999; Gaivoronski *et al.*, 2005; Ji *et al.*, 2005] present well-known index tracking designs that consider a number of different portfolio attributes.

Finally, we investigate broader investment areas involving stock and bond portfolio selection models that are well suited for the financial instabilities present in the current economy. The authors of [Konno and Kobayashi, 1997] propose one of the first stock-bond MVO models, however, there has also been other portfolio designs involving integrated investment approaches. Literature from Gaivoronski and De Lange (2000), Sodhi (2005), Tapaloglou *et al.* (2008), and Zenios (1995) provide a mixture of selection models that involve a variety of investing instruments with different portfolio goals. In addition to the publications we have mentioned, this document also includes an investigation on the computational issues posed by portfolio problems and issues regarding particular portfolio characteristics. Literature in this area is highly emphasized on the mathematical and computational sides of the problem. Authors endure the obstacle of deciding the number of crucial portfolio elements to include in a model versus problem solvability. Sodhi (2005) reports that models using Stochastic Programming (SP) with recourse have two major obstacles: (a) having to choose from a plethora of modelling choices while being consistent with each other or financial theory, and (b) solving the models, which involve a large number of variables due to scenarios that grow exponentially over time. Gaivoronski and

De Lange (2000) believe that (a) and (b) are dependent on each other, in that increasing the degree of flexibility in the decision process leads to increasing the number of decision variables, which then leads to trade-offs due to computational limitations. As shown in (2.15)–(2.19), even the inclusion of one portfolio element can cause great computational complexities. Hence, for the sake of problem complexity and sensitivity we save further discussions of literature as they relate to their respective sections.

Chapter 3

Mean–Absolute Deviation with Discrete Choice

3.1 Introduction to Mean–Absolute Deviation

In 1952 Harry Markowitz designed a portfolio selection model that revolutionized finance and lead to numerous future developments in financial theory. He later received a noble prize for his original work on Mean-Variance Optimization (MVO) in 1990 [Markowitz, 1952]. As shown in Section 2.2, in its basic form the MVO model minimizes risk while achieving a given level of expected return. Risk is captured in a covariance matrix that may also be defined in a multi-objective function, where an optimal portfolio is based on the expected covariance and the expected return. Today, the MVO model is widely accepted by the financial field. Many investment companies rely on MVO portfolio designs that are enhanced by additional measures based on various financial characteristics. Typically, financial agencies develop MVO selection models with added attributes that interest or concern the firm and capture realistic trading constraints, such as holding a low number of securities or reducing transactions. The addition of these attributes, however, increase the complexity of the problem and often lead to optimality issues. Since Markowitz's original idea, a number of investigations on simply the sensitivity of the MVO model have been made. The authors of [Hanoch and Levy, 1969] show that the MVO model produces a valid efficiency frontier when the return distributions are Gaussian normal. In [Kallberg and Ziemba, 1983], the authors investigate various multi-objective functions and show that parameter values associated with the function have a small effect on the optimal basis. Chopra and Ziemba (1993) provide an investigation that illustrates that the greatest threat or highest level of sensitivity in the MVO portfolio model lies in the values of the expected return.

In addition to MVO sensitivity concerns, although the model is very intuitive, it possesses various practical and computational issues. From the practical side, the MVO model in its purest conception omits a number of real-world portfolio characteristics such as portfolio size, trading constraints, buy-in thresholds, etc. From the computational side, in its basic form the model is a Quadratic Program (QP), which becomes difficult to solve if real-world characteristics involving integer constraints are included. Hence, over the years a number of publications have addressed practical and computational issues of the model. Crama and Schyns (2003) add buying, selling, trading, portfolio size, and floor and ceiling constraints to the MVO model, which form a Quadratic Mixed-Integer Program (QMIP). The QMIP is solved using Simulated Annealing (SA), where the worst efficient frontier is produced when all constraints are included in the model versus only having one or two. Jobst *et al.* (2001) investigate MVO models where portfolio size, buy-in thresholds, and roundlot constraints are considered separately. The resulting NP-hard QMIP is solved using two different heuristics, of which they demonstrate that even the addition of one portfolio characteristic requires an algorithm to produce solutions. The authors of [Chang *et al.*, 2000] also confirm this result generating heuristics for

MVO models that limit portfolio size. In addition, Adcock and Meade (1994), Lin and Wang (2002), and Liu *et al.* (2003) investigate Markowitz QP models while considering transaction costs. Thus, adding various characteristics to the MVO model is essential for current financial applications, however, the resulting QP is further constrained and contains computational complexities.

In 1991, Konno and Yamazaki show that under multivariate normal return distributions the MVO model has an equivalent linear form in a model they define as the Mean-Absolute Deviation (MAD). The linear MAD model possesses less computational issues than its QP counterpart. In this chapter, we investigate the addition of practical portfolio constraints to the MAD model and develop a heuristic that pushes the computational envelope far beyond MVO designs. We develop portfolio selection models based on the constraints outlined in Crama and Schyns [2003] and Jobst *et al.* [2001]. Hence, we add constraints on portfolio size, buying, selling, trading, and floor and ceiling bounds to the MAD model and report on the performance improvements from the MVO designs in the publications mentioned above. A comparison of the different model investigations from the literature is shown in Table 3.1. Although the computational complexities of the MAD model are much simpler, under practical conditions the returns are not multivariate normal distributions and the model becomes an approximation to the MVO. However, under these conditions we illustrate that the MAD approximation performs well and the approach overcomes the computational complexities posed by the MVO QMIP. In addition, if the L1-norm is used to describe the risk measure (covariance matrix in the MVO model), then we have an equivalent representation. Mansini and Speranza (2005) consider a mean semi-deviation model that is equal to one-half of the MAD model, where transaction costs and roundlots are included in the design. They partition the problem into subproblems and use a heuristic to obtain solutions. The model is an extension of

Kellerer *et al.* [2000], where they investigate a similar Mixed-Integer Program (MIP) with fixed costs and transaction lots. The model we propose involves a larger array of practical constraints and performs comparable to what is shown in Mansini and Speranza [2005]. We are the first to solve the MAD model with the prescribed practical constraints, and design a novel heuristic that improves solution efficiency and performs very well with regards to the efficient frontier when the problem is solved to optimality. Hence, this

Constraints and Optimality	Jobst *et al.* (2001)	Crama and Schyns (2003)	Mansini and Speranza (2005)	MAD Constraint Model
Portfolio Size	✓	✓		✓
Buy-in Constraints	✓	✓		✓
Buying Constraints		✓		✓
Selling Constraints		✓		✓
Trading Constraints		✓	✓	✓
Optimality obtained			✓	✓

Table 3.1: Constraint and optimality comparison of MVO models versus MAD models.

analysis provides a strong approximation to the MVO model with additional portfolio constraints that are much more tractable than the QMIP found in related literature. In addition, we design a heuristic for the linear MIP that is influenced by the results in Jobst *et al.* (2003), yet has better solution time and performs very similarly to relaxed portfolio versions where optimality was satisfied.

In this chapter we begin by comparing the MAD model to the MVO model in Section 3.2. Then we add the constraints mentioned above individually, until all are considered in a large portfolio selection model. In Section 3.3, we present a heuristic that is capable of considering problems with a significant number of variables while producing solutions with fast CPU time. Finally, in the last section we highlight the results of the algorithm and practical constraints included in the MAD model.

3.2 Model and Comparisons

We begin by defining the basic MVO problem in [Markowitz, 1952]. Although the general form is outlined in Section 2.2, we explicitly define the QP so that it can be compared to the equivalent MAD model shown in [Konno and Yamazaki, 1991]. Therefore, the portfolio selection MVO problem is the following:

$$\min \quad \sum_{i}^{n} \sum_{j}^{n} x_i \, Q_{ij} \, x_j \tag{3.1}$$

$$\text{s.t.} \quad \sum_{i}^{n} \mu_i x_i \geq R \tag{3.2}$$

$$\sum_{i}^{n} x_i = 1 \tag{3.3}$$

$$x_i \geq 0 \qquad \forall \, i = 1, ..., n, \tag{3.4}$$

where x_i is the decision variable for the amount invested in security $i = 1, ..., n$, μ_i is the mean return of security i, n is the total number of securities, R is the expected portfolio return, and $Q_{ij} = \text{cov}(r_i, r_j) = (1/T) \sum_t^T (r_{it} - \overline{r}_i)(r_{jt} - \overline{r}_j)$ is the covariance matrix for the return rate r_i of security i over a total of T time-stages. As mentioned in the introduction, if the rates of return (r_i) have a multivariate normal distribution, then Konno and Yamazaki show that (3.1)–(3.4) is equivalent to the following MAD model:

$$\min \quad \sum_{t=1}^{T} y_t + z_t \tag{3.5}$$

$$\text{s.t.} \quad y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \qquad \forall \, t = 1, ..., T \tag{3.6}$$

$$\sum_{i=1}^{n} \mu_i x_i \geq R \tag{3.7}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{3.8}$$

$$y_t \geq 0, z_t \geq 0 \qquad \forall \, t = 1, ..., T, \tag{3.9}$$

where constraint (3.6), y_t and z_t allow for the linear transformation. As mentioned, under practical situations the distribution of r_i may not be multivariate normally distributed. This can lead to sources of error when comparing MAD and MVO portfolio values; however, there are also imperfections of the MVO model under practical situations. We will discuss these issues below and show that the MAD model has greater advantages with respect to practical modelling issues, optimality, and computational efficiencies.

As the general MVO model shown in (3.1)–(3.4) produce very dense portfolios, various attempts to reduce portfolio size or the number of security investments have been made. A cardinality constraint or names-to-hold constraint (see Subsection 2.2.1) that limits the number of security investments manages problems where portfolios contain many small security weights. In [Jobst *et al.*, 2001], a cardinality constraint is added to (3.1)–(3.4) such that

$$\begin{aligned}
\min \quad & \sum_{i}^{n} \sum_{j}^{n} x_i \, Q_{ij} \, x_j && && (3.10)\\
\text{s.t.} \quad & \sum_{i}^{n} \mu_i x_i \geq R && && (3.11)\\
& \sum_{i}^{n} x_i = 1 && && (3.12)\\
& \sum_{i}^{n} g_i = G && && (3.13)\\
& l_i g_i \leq x_i \leq u_i g_i && \forall\, i = 1, ..., n && (3.14)\\
& g_i \in \mathbb{B} && \forall\, i = 1, ..., n, && (3.15)
\end{aligned}$$

where g_i is a binary variable that is equal to one if $x_i > 0$, and zero otherwise. Also, G is the number of different securities to be held in the portfolio, and l_i/u_i are lower/upper buy-in constraints (also referred to as floor/ceiling constraints); respectively. By adding the same constraint to (3.5)–(3.9) the MAD Linear Program (LP) becomes:

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} y_t + z_t & & \qquad (3.16)\\
\text{s.t.} \quad & y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i & \forall\, t = 1, ..., T & \qquad (3.17)\\
& \sum_{i=1}^{n} \mu_i x_i \geq R & & \qquad (3.18)\\
& \sum_{i=1}^{n} x_i = 1 & & \qquad (3.19)\\
& \sum_{i=1}^{n} g_i \leq G & & \qquad (3.20)\\
& l_i g_i \leq x_i \leq u_i g_i & \forall\, i = 1, ..., n & \qquad (3.21)\\
& y_t \geq 0, z_t \geq 0 & \forall\, t = 1, ..., T & \qquad (3.22)\\
& g_i \in \mathbb{B} & \forall\, i = 1, ..., n, & \qquad (3.23)
\end{aligned}$$

One may note that (3.20) is an inequality constraint in contrast to (3.13). The portfolios we investigate contain a large array of investments and the problem is to reduce the number of portfolio constituents, as was the problem in [Jobst *et al.*, 2005]. Thus, the inequality constraint in (3.20) satisfies the requirement of reducing portfolio size and also complies with this managerial characteristic; hence, the portfolio problem is not changed. Following the same data procedures in [Jobst *et al.*, 2001], we randomly selected 30 stocks from the S&P TSX 60 Composite Index for 60 monthly returns (June 2002 to May 2007). In Figure 3.1 we present the MVO efficiency frontier over this data set for problem (3.1)–(3.4) and in Figure 3.2 we present the MAD efficiency frontier for the model in equations (3.5)–(3.9). As one can see in Figures 3.1 and 3.2, the efficient frontiers are almost identical; Figure 3.3 is comparison of the two graphs together. The y-axes expected return values are kept consistent for all illustrations in this document, which were set to range from $3\% - 14\%$ and increased by 0.2% intervals. From Figure 3.3, the MAD efficient frontier has a slightly steeper slope and greater standard deviation as the expected return increases in comparison to the MVO results. Table 3.2 explicitly shows the difference in return values for a prescribed standard deviation or risk. The

Risk (σ)	MVO % Return	MAD % Return	% Difference
0.06	4.2	3.8	0.2
0.10	6.0	5.4	0.4
0.14	7.4	6.6	0.6
0.18	8.8	7.6	1.0
0.22	9.8	8.4	1.2
0.26	11.0	9.4	1.4
0.30	12.0	10.2	1.6

Table 3.2: MAD verses MVO model return comparison for a given amount of risk.

solution time (CPU time) was faster for the MAD model, however, more significate CPU time differences between the two models come into effect when cardinality constraints are added; shown in equations (3.13) and (3.20). In addition, the covariance matrix and commercial quadratic optimization solvers such as CPLEX require that the hessian is positive semidefinite, which poses a problem for the MVO model when using real data; see [Fletcher, 1985; Higham, 2002] for more information on this issue.

For the MVO cardinality model, Jobst *et al.* (2001) report that a heuristic is necessary to obtain a solution, which is exactly what we find when trying to implement (3.10)–(3.15) using the TSX data set mentioned above. On the other hand, the MAD cardinality model in (3.16)–(3.23) solved the problem to optimality, where following [Jobst *et al.*, 2001] we precisely set $G = 4$. The efficiency frontier for equations (3.16)–(3.23) is shown in Figure 3.4. The MAD cardinality model has similarities to the frontier illustrated in Figure 3.2, but the steep downward slope from 0–0.1 units is comparable to the quadratic function

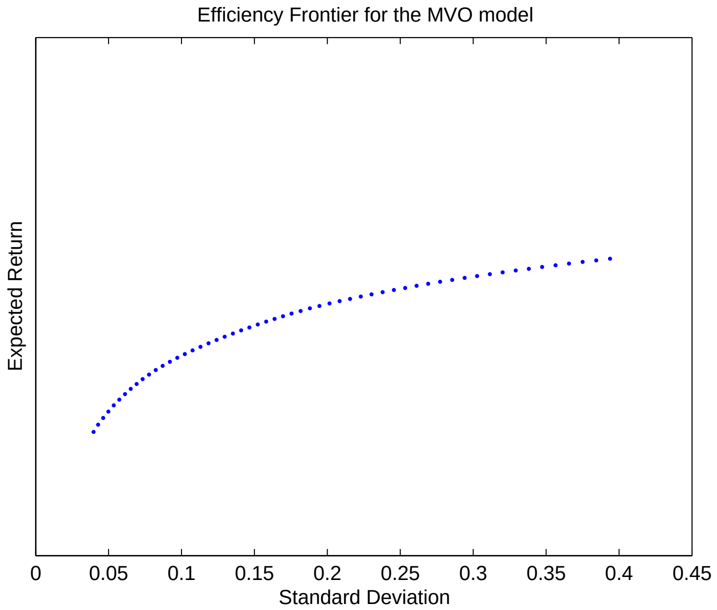

Figure 3.1: Efficiency frontier for the MVO model.

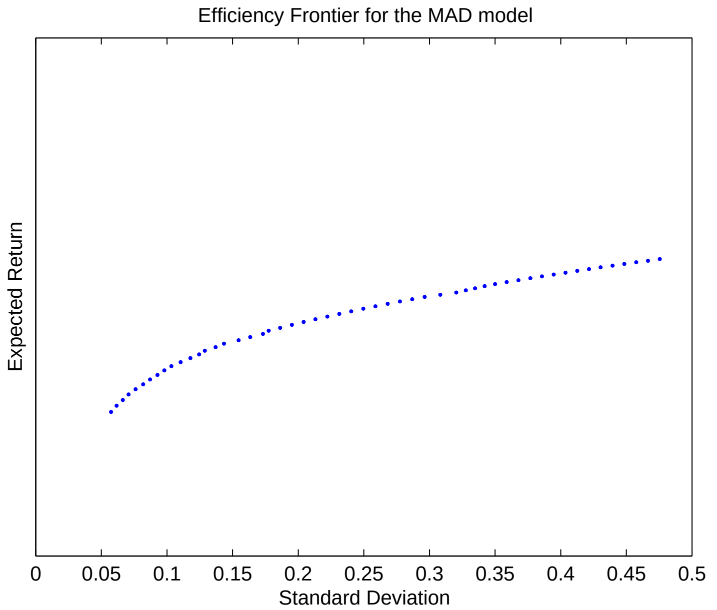

Figure 3.2: Efficiency frontier for the MAD model.

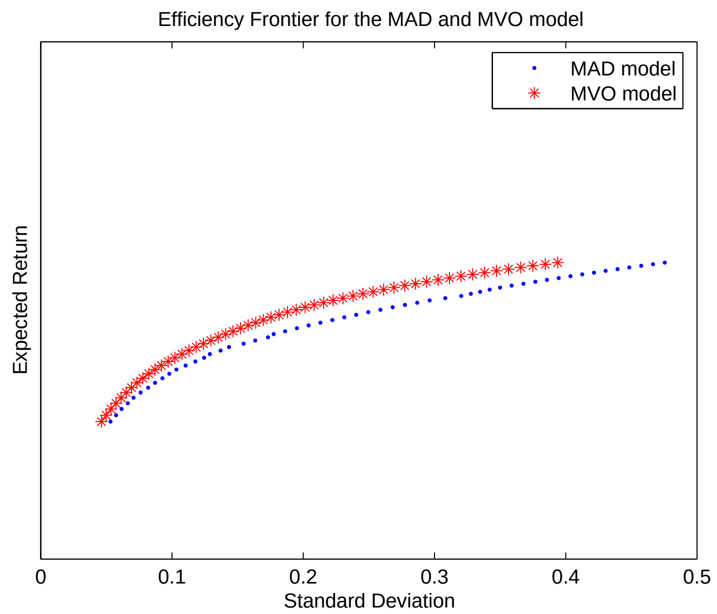

Figure 3.3: Comparison of the efficient frontier for the MAD and MVO model.

of the MVO model in Figure 3.1. The subtle discontinuities in the efficient frontier of Figure 3.4 are due to the addition of the cardinality constraint, which were first observed in the MVO cardinality model of [Chang *et al.*, 2000] and later shown in [Crama and Schyns, 2003; Jobst *et al.*, 2001]. When using the reoptimization heuristic in [Jobst *et al.*, 2001] on the same size problem, we obtain a solution CPU time of 12.00s (seconds); Jobst *et al.* report that it takes 10.00s on their CPU and data set and when using their integer restart heuristic it takes 57.55s. Without addressing the quality of the portfolios generated by the heuristics, the MAD cardinality model is solved to optimality using CPLEX directly (no heuristic necessary) with an average CPU time of 0.078125s, which equates to a 99.22% increase in speed. Table 3.3 illustrates the CPU time improvements made by the MAD model, where $\sim$ indicates a linear extrapolation on the MVO CPU times for instances when the MVO results were not possible. Next, we pushed the MAD cardinality model to its computational limits by dramatically increasing the number of

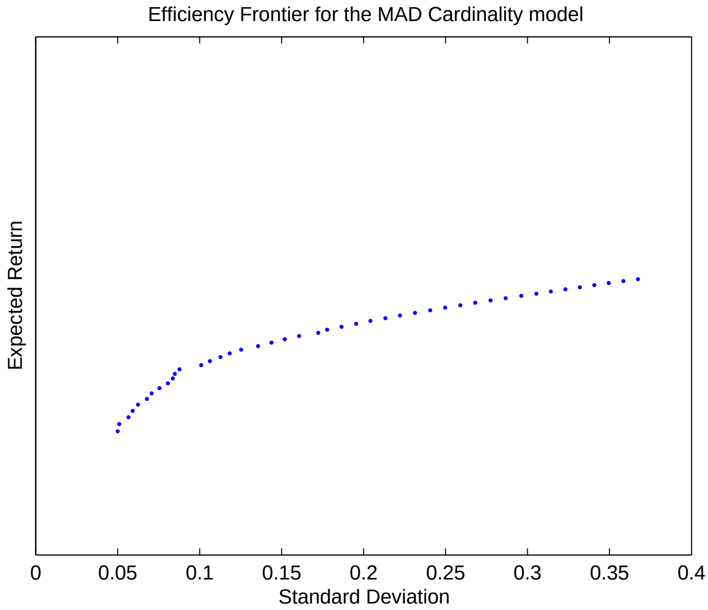

Figure 3.4: Efficiency frontier for the MAD cardinality model with $G = 4$.

securities and time-stages (monthly returns). In order to account for over 120 time-stages and keep the covariance matrix Q dense, we found that a maximum of 853 TSX securities could be used over that time period. Figure 3.5 presents the efficiency frontier when 100 time-stages are used with the portfolio size $G = 35$. In Figure 3.6, 120 time-stages are used with $G = 75$; both results involved 853 securities. Note that an algorithm was not necessary for the results shown in Figures 3.5 or 3.6. It took CPLEX an average of

Problem Size	MVO CPU time (s)	MAD CPU time (s)	% Speed up
30	10.00	0.08	99.22
100	44.93	0.28	99.37
300	$\sim$ 144.73	0.36	99.75
500	$\sim$ 244.53	0.48	99.80
853	$\sim$ 370.78	0.67	99.82

Table 3.3: CPU speed up difference between the MAD and MVO model.

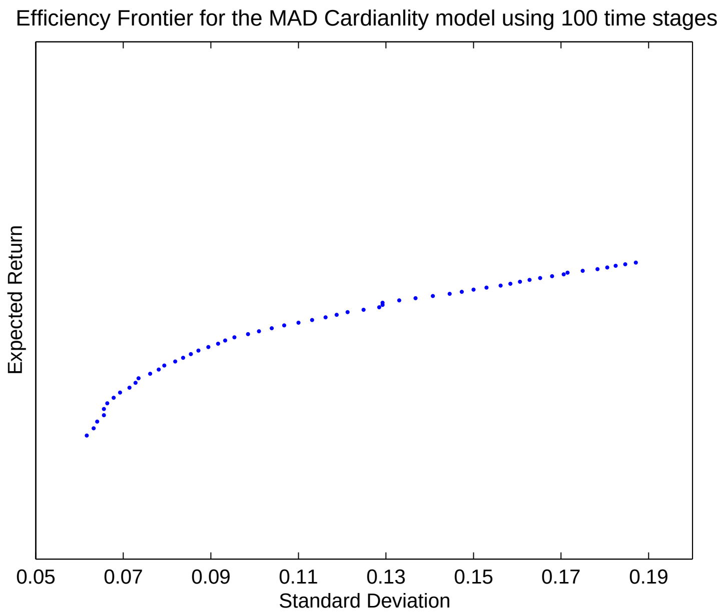

Figure 3.5: Efficiency frontier for the MAD Cardinality model with $G = 35$ using 100 time-stages and 853 securities.

0.671875s to compute the values for the 100 time-stage results and 1.046875s to compute the values for the 120 time-stage results. In comparison to the first four graphs of Figures 3.1–3.4, the slopes are similar, however the standard deviation is lower for higher expected return values, which is primarily due to the larger data set that is used for these results. As in Figure 3.4, the slopes of the efficient frontier in Figures 3.5 and 3.6 are similar to the quadratic function in the MVO of Figure 3.1. Hence, the MAD model under cardinality constraints produce efficient frontiers that have quadratic function qualities that are more similar to Markowitz MVO models than when the constraints are not included. In order to decrease the size of the portfolio (i.e. G value) further than what is presented in Figures 3.5 and 3.6 an algorithm is necessary, which we discuss in the next section.

Next we took the MAD cardinality model a step further by enhancing the practi-

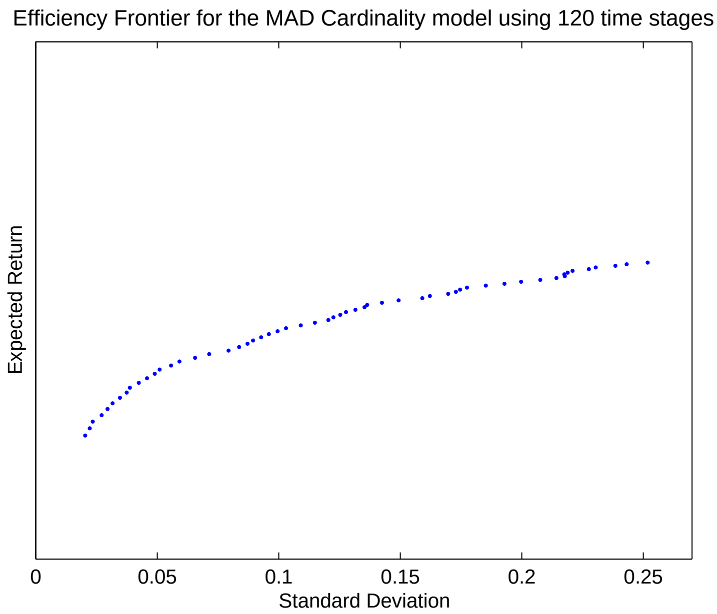

Figure 3.6: Efficiency frontier for the MAD Cardinality model with $G = 75$ using 120 time-stages and 853 securities.

cal constraints, as done in [Crama and Schyns, 2003]. In addition to the cardinality constraint, Crama and Schyns add buying, selling, and trading constraints to the MVO model. The QMIP in [Crama and Schyns, 2003] with the additional practical constraints is the following:

$$\begin{aligned}
\min \quad & \sum_{i}^{n}\sum_{j}^{n} x_i Q_{ij} x_j && && (3.24)\\
\text{s.t.} \quad & \sum_{i}^{n} \mu_i x_i = R && && (3.25)\\
& \sum_{i}^{n} x_i = 1 && && (3.26)\\
& \max(x_i - x_i^0, 0) \leq \overline{B}_i && \forall\, i = 1, ..., n && (3.27)\\
& \max(x_i^0 - x_i, 0) \leq \overline{S}_i && \forall\, i = 1, ..., n && (3.28)\\
& x_i = x_i^0 \text{ or } x_i \geq (x_i^0 + \underline{B}_i) g_i \text{ or } x_i \leq (x_i^0 - \underline{S}_i) g_i && \forall\, i = 1, ..., n && (3.29)\\
& \sum_{i}^{n} g_i \leq G && && (3.30)\\
& l_i g_i \leq x_i \leq u_i g_i && \forall\, i = 1, ..., n && (3.31)\\
& g_i \in \mathbb{B} && \forall\, i = 1, ..., n, && (3.32)
\end{aligned}$$

where x_i^0 is the weight of security i in the initial portfolio, $\overline{B}_i$ and $\overline{S}_i$ denote the maximum purchase/sale of security i, and $\underline{B}_i$ and $\underline{S}_i$ denote the minimum purchase/sale of security i; respectively. In the MVO model of (3.24)–(3.32), constraint (3.27) bounds the number of securities that may be purchased, constraint (3.28) bounds the number of securities that may be sold, constraint (3.29) bounds the number of transactions, constraint (3.30) limits portfolio size, and constraint (3.31) bounds the buy-in minimum/maximum amount invested in any security. Each of the additional constraints are important elements for portfolio managers and just as in [Jobst *et al.*, 2001] a heuristic is necessary to solve the QMIP presented in (3.24)–(3.32). Crama and Schyns (2003) provide a few techniques that may be used to solve the large model, but most of the results they present involve solving a constraint subset of (3.24)–(3.32). To solve the large QMIP a Simulated Annealing (SA) algorithm is implemented, however, they only provide a brief discussion on the results of the MVO efficient frontier when all constraints are considered. Adding the same constraints to the MAD model gives:

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{T} y_t + z_t && && (3.33)\\
\text{s.t.} \quad & y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i && \forall\, t = 1, ..., T && (3.34)\\
& \sum_{i=1}^{n} \mu_i x_i \geq R && && (3.35)\\
& \sum_{i=1}^{n} x_i = 1 && && (3.36)\\
& \max(x_i - x_i^0, 0) \leq \overline{B}_i && \forall\, i = 1, ..., n && (3.37)\\
& \max(x_i^0 - x_i, 0) \leq \overline{S}_i && \forall\, i = 1, ..., n && (3.38)\\
& x_i = x_i^0 \text{ or } x_i \geq (x_i^0 + \underline{B}_i) g_i \text{ or } x_i \leq (x_i^0 - \underline{S}_i) g_i && \forall\, i = 1, ..., n && (3.39)\\
& \sum_{i=1}^{n} g_i \leq G && && (3.40)\\
& l_i g_i \leq x_i \leq u_i g_i && \forall\, i = 1, ..., n && (3.41)\\
& y_t \geq 0, z_t \geq 0 && \forall\, t = 1, ..., T && (3.42)\\
& g_i \in \mathbb{B} && \forall\, i = 1, ..., n, && (3.43)
\end{aligned}
$$

For the sake of comparison, we run similar constraint subsets (3.37)–(3.41) of the MAD model to what is done in [Crama and Schyns, 2003], then implemented the whole model (3.33)–(3.43) when all constraints are considered. Figures 3.7–3.10 provide the results of the MAD efficiency frontier with the cardinality constraint included in the model and individually adding constraints (3.37)–(3.39). Then we consider the case when all constraints are implemented in Figure 3.10. To keep the graphs similar to what is shown earlier in Figures 3.1– 3.4 and [Jobst *et al.*, 2001], we use 60 securities over 60 time periods (June 2002 to May 2007) from the TSX 60 Composite Index and set $G = 15$. Crama and Schyns (2003) have results with 3 – 151 securities, and for lack of space we chose a security data set that lies between this range and can be compared to [Jobst *et al.*, 2001]. Figure 3.7 is an illustration of the efficient frontier when the trading constraint

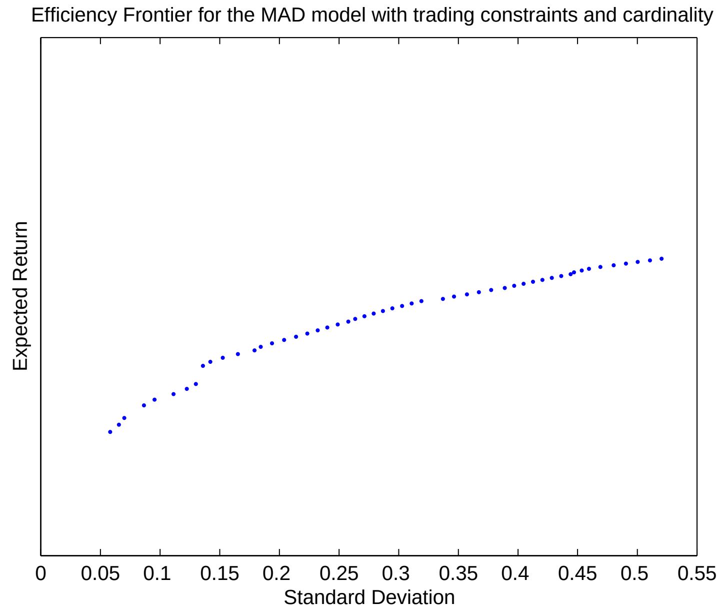

Figure 3.7: Efficiency frontier for the MAD model with trading constraints.

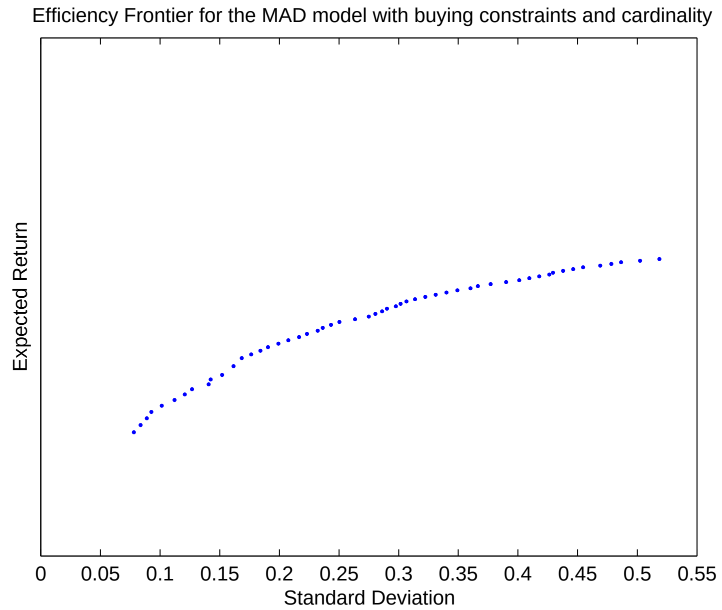

Figure 3.8: Efficiency frontier for the MAD model with buying constraints.

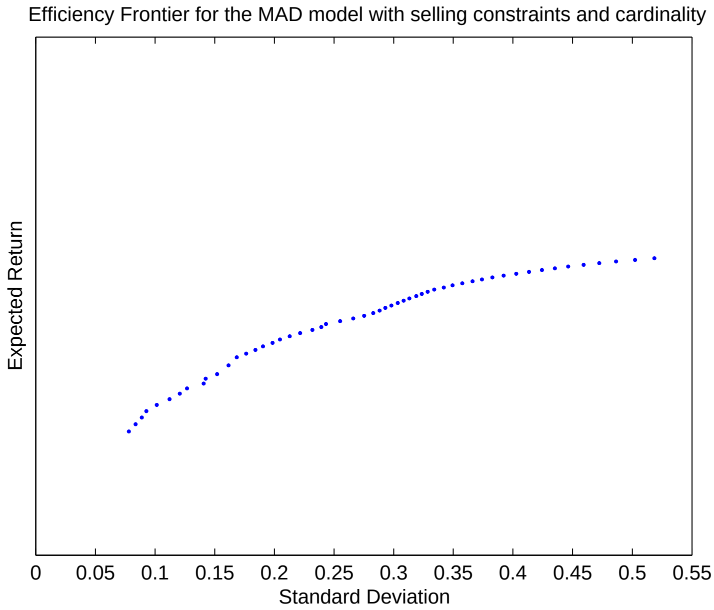

Figure 3.9: Efficiency frontier for the MAD model with selling constraints.

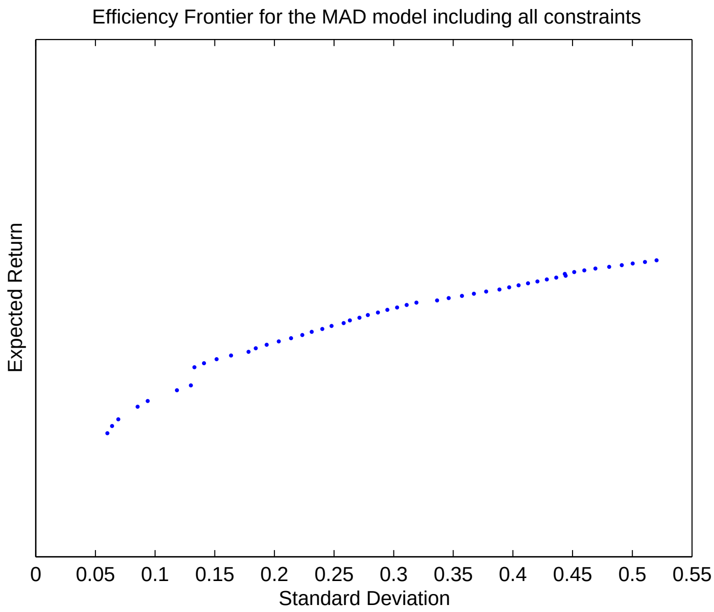

Figure 3.10: Efficiency frontier for the MAD model with all constraints, (3.33)–(3.43).

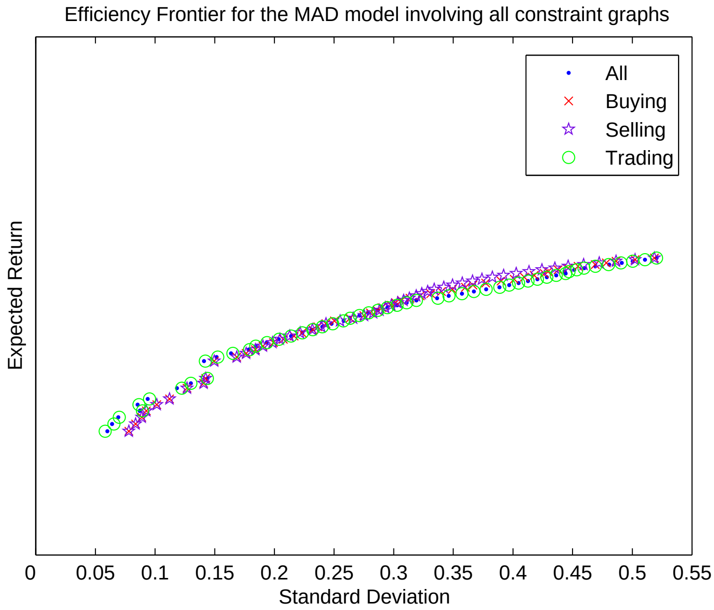

Figure 3.11: Efficiency frontier for the MAD model with Figures (3.7)–(3.10) imposed on each other.

(3.39) is included. As one may observe, the standard deviation has increased slightly in comparison to Figure 3.4, where only the cardinality constraint is used. In Figure 3.8, the buying constraint (3.37) was imposed on the model. Similar to Figure 3.7, constraining the amount of buying produces portfolio risk (standard deviation) issues as the expected return increases. Adding the selling constraint (3.38) produces the efficiency frontier shown in Figure 3.9. This portfolio has a comparable efficient frontier to Figure 3.8, which one would expect since it is performs a similar task. Finally, Figure 3.10 provides the efficient frontier when the MAD model was solved including all of the constraints presented in (3.33)–(3.43). Figure 3.10 is the product of running the constraints in Figures 3.7–3.9 collectively, and Figure 3.11 is an illustration of the plots imposed on each other. As shown in Figure 3.11, the cardinality constraint results have great similarities to the plot containing all constraints, which illustrates that cardinality has a large bearing on the problem. With respect to the MVO presented in Crama and Syhyns (2003), the model that considered all constraints posed the greatest threat to solvability and produced the worst efficiency frontier in their experimental results section. The full MAD model (3.33)–(3.43) was solved without any heuristic, which is necessary in the MVO model of [Crama and Schyns, 2003]. The average solution time was 0.140625s, Crama and Syhyns (2003) do not provide solution times for their complete model. However, they do provide solutions times for their constraint subproblem runs using their SA algorithm, which is dependent on the number of moves defined and the number of securities used; nevertheless, the CPU time ranged from 0.15s – 7h (hours) and 58m (minutes).

From the results and comparisons above, one should be convinced that the MAD model produces respective portfolio results without the expensive solvability issues common to its quadratic counterpart (MVO). The MAD results are solved in seconds without using any heuristic, whereas the same can not be said about the QMIPs in [Crama and Schyns, 2003; Jobst *et al.*, 2001]. The cardinality and additional portfolio constraints used in both papers do not pose the same computational difficulties when applied to the MAD model, even when variable counts are increased. The portfolio size constraint is one of the most important characteristics, as [Crama and Schyns, 2003; Jobst *et al.*, 2001] illustrate; however, it is also one of the most difficult to satisfy. In the next section we push the MAD variable count past the computational limits shown in Figures 3.5 and 3.6 with the introduction of a model specific algorithm.

3.3 MAD Implementation

In the previous section, it was shown that the MAD cardinality model outperforms the MVO with regards to practical constraints and computational complexities. Figures 3.5 and 3.6, however, contain results where the MAD model develops computational limitations. In order to lower the value of G and/or increase the number of decision variables a heuristic is necessary. In [Jobst *et al.*, 2001], two types of heuristics are designed and investigated, namely an integer restart and re-optimization heuristic. The integer restart heuristic is based on using previous integer solutions and relaxing the expected return constraint to obtain portfolio results. One of the limitations of the heuristic, other than the obvious optimality issues, is that the efficiency frontier may be incomplete due to relaxations associated with the expected return values. The other heuristic presented in [Jobst *et al.*, 2001], namely the re-optimization heuristic, is fairly simple. First, the MVO model without the cardinality constraint and buy-in lower bound is solved. Then, only the G securities with the largest weights are used to solve the full MVO cardinality model with the buy-in lower bound. For the re-optimization heuristic, optimality issues are even greater than in the integer restart since a large amount of information is disregarded before solving the model the second time. Nevertheless, of the two heuristics proposed by Jobst *et al.* (2001) the re-optimization heuristic seems to be the most promising with respect to CPU time, and produces a complete efficiency frontier.

For the MAD cardinality problem, we design a heuristic that first solves a subproblem of (3.16)–(3.23) and then uses an algorithm to enforce the cardinality constraint. The basics of the algorithm are as follows: first, (i) solve the MAD model without cardinality or buy-in lower bound constraints. Next, (ii) add the cardinality constraint and a relaxation to G for any $x_i > 0$ in (i); and minimize the relaxation in an iterative procedure. Then, (iii) if the cardinality number is still not satisfied but reduced in comparison to step (i), add the lower bound constraint and solve; otherwise impose the cardinality constraint using $x_i > 0$ in (ii) and an iteration to reduce the basis to the desired size. Finally, (iv) solve the MAD model with the buy-in lower bound thereby satisfying all constraints. In each of the steps (i)–(iv) mentioned above, subproblems of (3.16)–(3.23) are solved until the last step where all constraints are considered. The first subproblem for the algorithm to solve is shown below

$$\begin{aligned}
\text{MAD–LP:} \qquad & \min \quad \sum_{t=1}^{T} y_t + z_t && && (3.44)\\
& \text{s.t.} \quad y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i && \forall\, t = 1, ..., T && (3.45)\\
& \sum_{i=1}^{n} \mu_i x_i \geq R && && (3.46)\\
& \sum_{i=1}^{n} x_i = 1 && && (3.47)\\
& 0 \leq x_i \leq u_i && \forall\, i = 1, ..., n && (3.48)\\
& y_t \geq 0, z_t \geq 0 && \forall\, t = 1, ..., T, && (3.49)
\end{aligned}$$

where the cardinality constraint and buy-in lower bound is removed. The second sub-problem takes the optimal basis from MAD–LP and solves the same problem with a relaxation or penalty variable ξ added to the cardinality constraint and penalty parameter ς in the objective. If the optimal basis ($x_i^\star > 0$) has been reduced in the solution to (3.44)–(3.49), we let x_i be the elements $x_i^\star > 0$ in (3.44)–(3.49) for $i = 1, ..., n$ and solve the following problem

$$\begin{aligned}
\text{MAD–CP:} \qquad & \min \quad \sum_{t=1}^{T} y_t + z_t + \varsigma\, \xi && && (3.50)\\
& \text{s.t.} \quad y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i && \forall\, t = 1, ..., T && (3.51)\\
& \sum_{i=1}^{n} \mu_i x_i \geq R && && (3.52)\\
& \sum_{i=1}^{n} x_i = 1 && && (3.53)\\
& \sum_{i=1}^{n} g_i \leq G + \xi && && (3.54)\\
& 0 \leq x_i \leq u_i && \forall\, i = 1, ..., n && (3.55)\\
& y_t \geq 0, z_t \geq 0 && \forall\, t = 1, ..., T && (3.56)\\
& g_i \in \mathbb{B} && \forall\, i = 1, ..., n. && (3.57)
\end{aligned}$$

If $\xi = 0$ then the cardinality constraint is met and we are satisfied, otherwise we increase the value of ς in an iterative procedure such that the algorithm tries to force $\xi = 0$. If the cardinality constraint is not met, but the number is reduced in comparison to MAD–LP, then using the basis in (3.50)–(3.57) we solve

$$\begin{aligned} \text{MAD–LC:} \qquad & \min \quad (3.44) && && (3.58)\\ & \text{s.t.} \quad (3.45) - (3.47),\ (3.49) && && (3.59)\\ & \qquad\ \ l_i g_i \leq x_i \leq u_i g_i && \forall\, i = 1, ..., n && (3.60) \end{aligned}$$

in anticipation of further reducing the cardinality number. If the cardinality number is reduced we repeat the last two steps - solving MAD–CP then MAD–LC; otherwise we enforce the cardinality constraint by solving MAD–LP using the current basis and removing the lowest weight x_i. After the cardinality number is met, then the solution to

$$\begin{aligned} \text{MAD–LB:} \qquad & \min \quad (3.44) && && (3.61)\\ & \text{s.t.} \quad (3.45) - (3.47),\ (3.49) && && (3.62)\\ & \qquad\ \ l_i \leq x_i \leq u_i && \forall\, i = 1, ..., n && (3.63) \end{aligned}$$

ensures all constraints in the MAD Cardinality model (3.16)–(3.23) are met. The pseudocode for the proposed heuristic is as follows:

`1. Initialize:`

Set model parameters R, G, μ_i, l_i, u_i $\forall\, i = 1, ..., n$ and

computational parameters $\varsigma = 0, h = 0,\ \tau > 0,\ \Delta > 0,\ H > 0.$

`2. LP solution:`

Solve MAD–LP (3.44)–(3.49)

(i) if $\sum g_i \leq G \quad \Rightarrow$ solve MAD–LB for $x_i^\star > 0 \Rightarrow$ `Terminate.`

(ii) otherwise, set $\varsigma = \tau$, $\widehat{G} = \sum g_i$ and

$\widehat{x} \subseteq \{x^\star : x_i^\star > 0\ \forall\, i = 1, ..., n\}$, `go to 3.`

`3. Cardinality and Penalty:`

Solve MAD–CP (3.50)–(3.57) using $\widehat{x}$ from `2` (ii)

(i) if $\xi = 0 \quad \Rightarrow$ solve MAD–LB for $x_i^\star > 0 \Rightarrow$ `Terminate.`

(ii) else if $\sum g_i \leq \widehat{G}$ and $h < H$, set $\overline{G} = \sum g_i$ and

$\overline{x} \subseteq \{x : x_i > 0\ \forall\, i = 1, ..., n\}$, `go to 4.`

(iii) else if $h < H$,

set $\tau = \tau + \Delta$ and $h = h + 1$, `go to 3.`

(iv) otherwise, set $\overline{G} = \sum g_i$ and

$\overline{x} \subseteq \{x : x_i > 0\ \forall\, i = 1, ..., n\}$, `go to 4.`

`4. Re-solve:`

(i) if $\overline{G} < \widehat{G}$, solve MAD–LC (3.58)–(3.60) using $\overline{x}$ from `3` (ii) or (iv)

(a) if $\sum g_i \leq G \quad \Rightarrow \quad$ `Terminate.`

(b) else if $\sum g_i < \overline{G}$, set $\widehat{G} = \sum g_i$ and

$\widehat{x} \subseteq \{x^\star : x_i^\star > 0\ \forall\, i = 1, ..., n\}$, `go to 3.`

(c) otherwise, `go to 5.`

(ii) otherwise, `go to 5.`

`5. Impose Cardinality:`

Solve MAD–LP (3.44)–(3.49) using $\overline{x}$ from `3` (ii) or (iv)

(i) set $\widetilde{x} = \{x - \min(x_i,\ \forall\, i = 1, ..., n)\}$ and $\widetilde{G} = \sum g_i$

(a) if $\widetilde{G} \leq G$, `go to 5` (ii).

(b) otherwise, `go to 5` using $\widetilde{x}$ from `5` (i)

(ii) Solve MAD–LB (3.61)–(3.63) using $\widetilde{x} \quad \Rightarrow \quad$ `Terminate.`

In the algorithm above, τ is the initial value of the penalty parameter when MAD–

CP is initiated and Δ is the amount τ is increased every time the step is repeated. H defines the number of times the algorithm will undergo the penalty adjustment procedure, or try to push the cardinality penalty variable ξ to be equal to zero. In step 2, the algorithm solves MAD–LP and keeps the optimal basis to be the starting point for step 3, where the cardinality constraint, penalty variables, and penalty parameters are added. Given the cardinality number is not satisfied, an iterative penalty procedure tries to force the cardinality constraint to be met by making the penalty variable (which relaxes this constraint) very expensive. This is achieved by increasing the value ς in the objective function. After repeating this H times, if $\xi \neq 0$ and the cardinality number is reduced in comparison to step 2, then step 4 requires that the current optimal basis is re–solved using MAD–LC, where the lower bound is imposed. If MAD–LC further reduces the cardinality number then the steps are repeated. Otherwise, in the worst case scenario, step 5 imposes the cardinality constraint by solving MAD–LP using the current basis and subtracting the minimum x_i investment weight from the portfolio until equations (3.16)–(3.23) are satisfied. An overview of the algorithm is shown in Figure 3.12. Step 4 was added to the implementation because under some test runs the basis was reduced using the lower bound buy-in constraint (3.60), and this problem solved in milliseconds; which further justified its inclusion. In step 5, MAD–LB is solved last since the lower bound constraint l_i may cause more than one portfolio weight x_i to have the prescribed lower bound value; as was the case with our test runs. For such an instance, deciding the basic variable to remove from the set of lower bounds becomes a problem. Solving MAD–LP gives one (or possibly a few) minimum x_i weights, which is (are) less than or equal to the corresponding basis values in MAD–LB. Thus, after each iterative solution of MAD–LP, one variable in the optimal basis can be removed until $n \leq G$. This argument is shown in Lemma 1, of which the proof is trivial (but for the sake of completeness is shown in Appendix A).

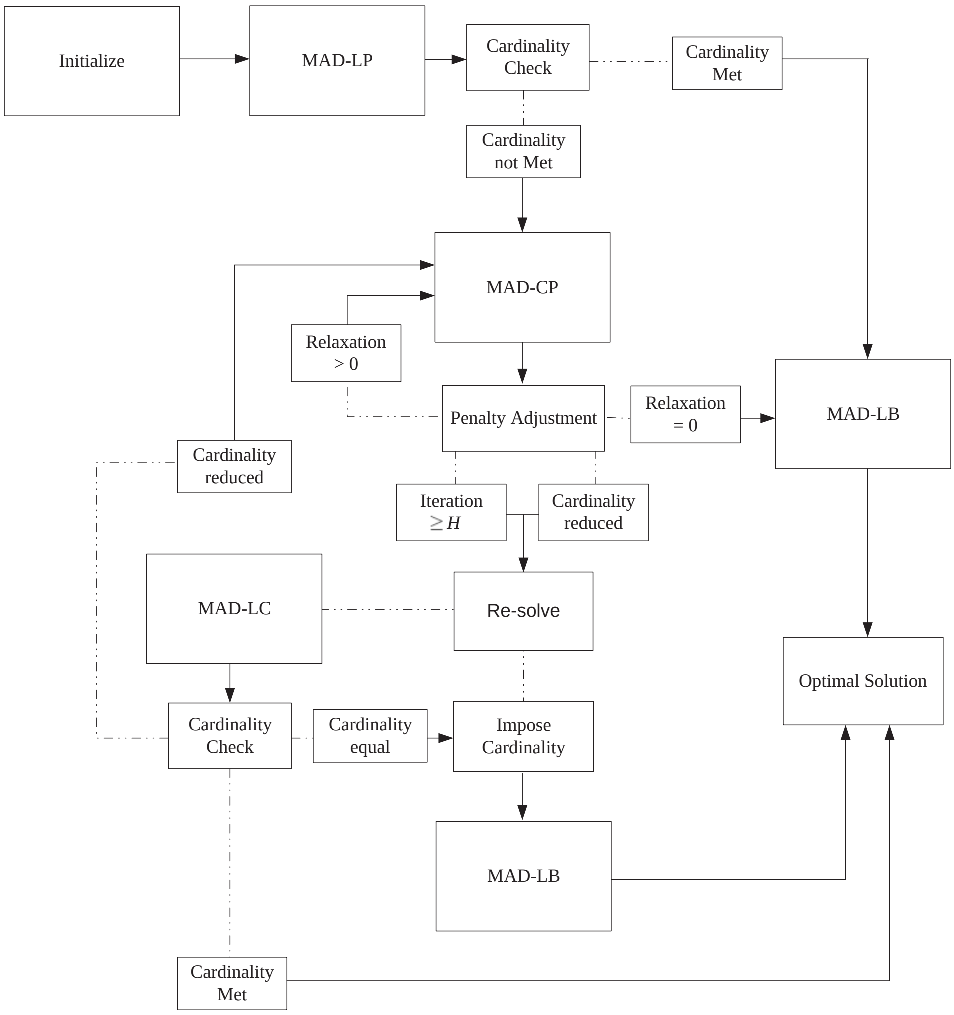

Figure 3.12: MAD Cardinality algorithm overview.

Lemma 1 Given the following problems have a unique solution:

$$
\begin{array}{lllll}
\min & c^\top x & & \min & c^\top \overline{x} \\
& Ax \geq b \quad (P) & \text{and} & & A\overline{x} \geq b \quad (\overline{P}) \\
& x \geq 0 & & & \overline{x} \geq l
\end{array}
$$

if $l > 0$, then the values of the optimal basis $x^\star \leq \overline{x}^\star$.

The algorithm in Figure 3.12 tries to satisfy the MAD cardinality model in (3.16)–(3.23) without sacrificing optimality. Each subproblem aims to reduce the size of the basis until the cardinality constraint is met. In the worst case, the algorithm enforces the cardinality constraint by iteratively removing the smallest weight in the basis. This is performed in step 5, where removing one weight at a time (or a few, given there are equivalent optimal basis values) the algorithm terminates after the cardinality number is reached. If the values of l_i and/or G are small, this step can be sped up; otherwise, it may give no improvement with respect to CPU time and not be necessary. In such a case, step 5 is replaced with the following two steps:

`5. Impose Cardinality` *Large Step*:

Solve MAD–LC (3.58)–(3.60) using $\overline{x}$ from `4` (i)(c) or (ii) (via `3` (ii) or (iv))

(i) set $\widetilde{x} = \{x - (x_i = l_i \cup x_i = 0, \; \forall\, i = 1, ..., n)\}$ and $\widetilde{G}^i = \sum g_i$

(a) if $\widetilde{G}^i = G \quad \Rightarrow$ solve MAD–LB for $x_i^\star > 0 \Rightarrow$ `Terminate.`

(b) else if $\widetilde{G}^i = \widetilde{G}^{i-1}$ or $\widetilde{G}^i = \overline{G}$, `go to 6.`

(c) else if $\widetilde{G} > G$, `go to 5` using $\widetilde{x}$ from `5` (i).

(d) otherwise, `go to 6.`

`6. Impose Cardinality` *Small Step*:

Solve MAD–LP (3.44)–(3.49) using $\{\widetilde{x} + (x_i = l_i \cup x_i = 0, \; \forall\, i = 1, ..., n)\}$ from `5` (i)

(i) set $\breve{x} = \{x - \min(x_i, \; \forall\, i = 1, ..., n)\}$ and $\breve{G} = \sum g_i$

(a) if, $\breve{G} \leq G$, `go to 6` (ii).

(b) otherwise, `go to 6` using $\breve{x}$ from `6` (i)

(ii) Solve MAD–LB (3.61)–(3.63) using $\breve{x} \quad \Rightarrow \quad$ `Terminate.`

The *Large Step* removes all the lower bound weights as long as this set of weights does not make the cardinality number less than G. If so, the *Small Step* takes over and removes one weight at a time until the cardinality number is reached. Hence, if l_i and/or G are fairly small, then this speeds up step 5 (`Impose Cardinality`) of the initial algorithm. Otherwise, large l_i values will take too many weights x_i away and/or large G values will only need to remove a small number of weights, making *Large Step* 5 above ineffective. In Table 3.4, we provide the results with respect to CPU time when using the initial heuristic (or normal step) and the fast step. From Table 3.4, the average CPU solution time was 6.938542s and 12.465625s for the fast and normal step algorithm; respectfully. In [Jobst *et al.*, 2001], a maximum of 225 securities were solved using 60 time-stages, which took a time of 18345.56s and 280.92s using their integer restart and re-optimization heuristic; respectfully. Thus, if one was to ignore the 628 security difference, in the best case the algorithm above still outperforms [Jobst *et al.*, 2001] by 97.53%. Figure 3.13 depicts the efficient frontier using 120 time-stages and 853 securities. In the previous section,

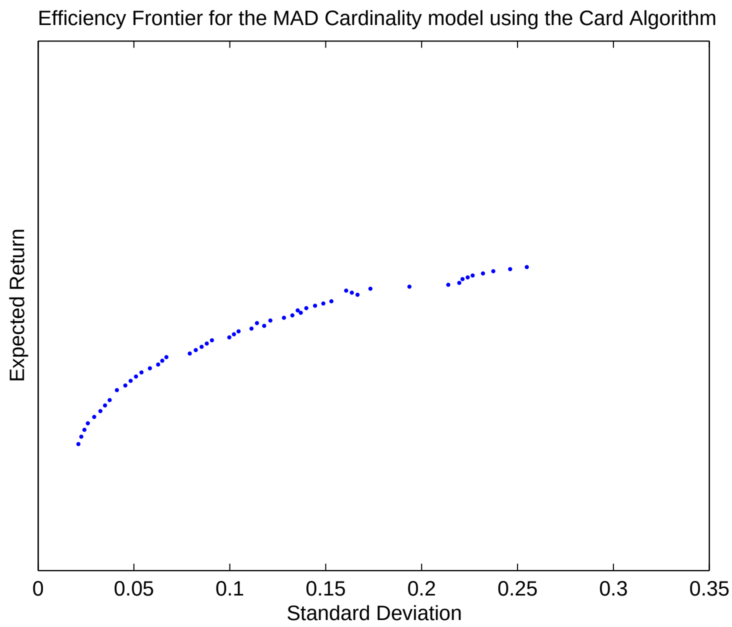

Figure 3.13: Efficiency frontier for the MAD Cardinality model using 120 time-stages and 853 securities, where $G = 60$.

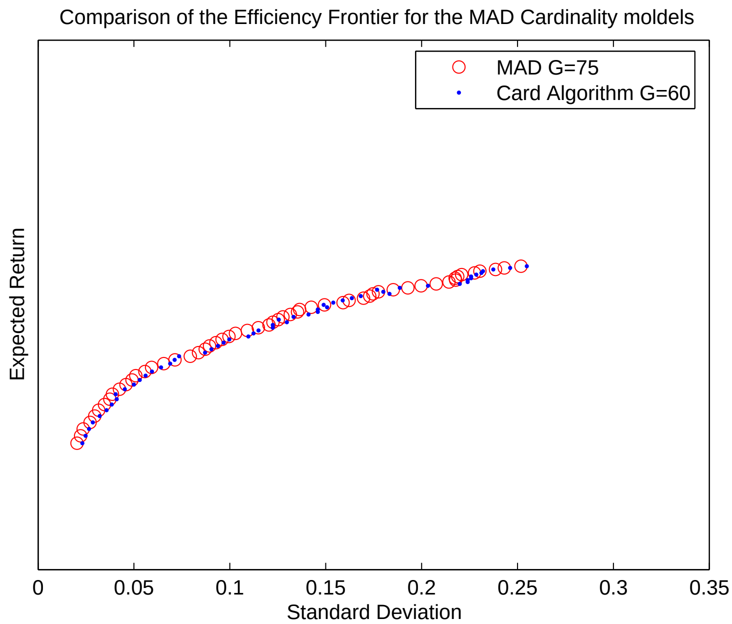

Figure 3.14: Efficiency frontier comparison of the results presented in Figure 3.6 ($G = 75$) of Section 3.2 and the MAD cardinality algorithm shown in Figure 3.13 ($G = 60$).

Index	Number of Securities / Time-Stages	Cardinality Number	Solution Time (s): *Fast Step*	*Normal Step*
S&P TSX	853/100	25	8.453125	8.453125
	853/120	56	3.375000	4.312500
		58	4.015625	13.515625
		59	9.484375	12.265625
		59	3.546875	13.687500
		60	13.390625	23.140625
			5.000000	8.062500
			6.359375	17.046875
			5.421875	6.609375
			4.437500	5.968750
			6.031250	8.890625
			4.156250	14.000000
			5.500000	6.203125
			4.312500	6.796875
	853/138	70	20.593750	38.031250

Table 3.4: CPU time for the initial heuristic (normal step) and fast step.

MAD Cardinality solved the problem to optimality using $G = 75$ securities. Using the proposed heuristic we have reduced this number to at least $G = 60$, and the results are remarkably similar to Figure 3.6 with respect to the shape of the efficiency frontier and the standard deviation values; as shown in Figure 3.14. Another interesting result from our computational runs is that if one solves MAD–LP and then takes the lowest values out to meet the cardinality constraint, as done in the re-optimization heuristic of [Jobst *et al.*, 2001], then the optimal basis ($x_i^\star$) is not, or very unlikely, to be the same as our proposed method. This is because the heuristic in Jobst *et al.* (2001) heavily reduces the size of the basis in one step, where it is possible to produce infeasible or inefficient portfolios. We iteratively reduce the problem size in an attempt to minimize deviations from optimality. In any case, from the results of Figure 3.6 and Table 3.4, we have constructed a heuristic that satisfies the cardinality constraint, has fast CPU time, and performs well with respect to the efficient frontier.

3.4 Result Discussions

We present various portfolio additions to the original Markowitz mean-variance model that has the capacity to handle financial problems of the 21^{st} century. In the past, literature on this topic has been limited by solution time and quality due to the complex computational requirements of the MVO problem. We illustrate that the linear approximation defined in [Konno and Yamazaki, 1991] can be used to overcome MVO limitations and is well suited for additional modelling constraints. Following current publications on MVO developments, we show that the MAD model can solve the same portfolio managing problems considered in [Crama and Schyns, 2003; Jobst *et al.*, 2001] without the use of a model specific algorithm. In fact, the graphs of 3.4–3.10 produce valid efficient frontiers, are solved in less time, and can handle problems with more variables than what is shown in [Crama and Schyns, 2003; Jobst *et al.*, 2001]. A valid efficient frontier implies that for a prescribed return the MAD portfolio model generates efficient standard deviation values or risk, with respect to the MVO model.

In Jobst *et al.* (2001) the authors develop a fairly simple heuristic to solve a MVO portfolio model with additional constraints. Although the MAD model does not require an algorithm to solve the same problem to optimality, we next addressed the limitations of the design by constructing an algorithm to solve much greater instances; over 65 times larger. From Table 3.4, the algorithm proved to solve the large set of portfolio decisions in under 2/3 of a minute. The algorithm uses a decomposition strategy that involves solving a complex sequence of subproblems to generate optimal solutions that minimize instances where decision variables are negated. The algorithm performance results with respect to the efficient frontier are very strong for lower portfolio return values, especially when considering the number of variables and constraints used in the problem. In fact, only for high expected return values does the graph in Figure 3.13 become somewhat distorted, which is most probably due to the cardinality constraint and not the algorithm; as this characteristic is also prevalent in the graph of Figure 3.6 where no algorithm is used. In terms of future research associated with such investigations, additional modelling techniques and constraints can be added to the MAD portfolio design, as well as different types of algorithmic approaches. An area of recent interest involves methods that integrate security price uncertainties to the problem. In the next chapter we present a Stochastic Programming (SP) method that has the capacity to include such characteristics and addresses other issues of the portfolio design.

Chapter 4

Stochastic Programming and Index Tracking

4.1 Introduction to the Index Tracking Problem

Index investments have received a lot of attention in the past decade. The problem of replicating or tracking the index has grasped the interest of more than just portfolio managers. Index portfolios retain a passive investment strategy that is aimed at long term wealth acquisition; in contrast to active portfolio strategies that aim to have high equity returns over short time periods. As outlined in Subsection 1.2.2 and motivated in Section 2.2, index funds are designed to reproduce (track) a market performance measure (such as the S&P/TSX composite index) and typically involve long term investments. The generation of index funds stems from MVO risk management ideas presented in the last chapter and CAPM strategies that imply holding a market portfolio. Such investments are suitable for pension funds, insurance premiums, corporate financial obligations, or simply an individuals savings. In fact, many active portfolio mangers simply cannot outperform the market, leaving index funds as a strong investment alternative. Also, corporations carry the burden of supplying their employees with retirement packages, maternity leaves, and so forth, making long term portfolio investments an attractive option that can generate additional income for a company. Recently, corporate pension plans in the USA are reported to invest a quarter of their equity holdings in index portfolios [Gaivoronski *et al.*, 2005]. Index portfolios have also received similar popularity from private investors and portfolio managers.

The framework for designing index funds comes from index tracking – the problem of defining a set of security investments that track (reproduce) the performance of an index (or target value). The index tracking problem is the main focus of this chapter. We develop an index tracking model that captures a comprehensive set of practical constraints along with a decomposition algorithm to compute optimal (or near optimal) solutions. Important elements that are used to construct an index tracking portfolio include:

1. Minimize a tracking error value;
2. Diversity or exposure to similar economic sectors to that of the index;
3. Hold a small number of securities;
4. Minimize transaction costs;
5. Include uncertainty in the value of future security distributions;
6. Posses a portfolio managing or rebalancing strategy at future time decisions.

In reference to the first point, tracking error is expressed as the relative difference between the designed portfolio and index value. Although various attempts to construct index tracking portfolios have been made, as described in Subsection 2.2.1 solving some tracking models are NP-hard [Coleman *et al.*, 2006]. Thus, there exists a trade-off between creating a solvable tracking model and including as many of the elements listed above in the portfolio as possible. In [Gaivoronski *et al.*, 2005], the authors consider items 1, 3, 5, and 6 in their portfolio design. They use two different approximations for generating security return values, where a historic trajectory approximation proves to produce better results with regards to tracking error. As is common to many tracking portfolio models, Gaivoronski *et al.* report that tracking error increases as the number of time periods increase. In addition, since the model does not account for transaction costs their portfolio value may be consistently lower than what is anticipated. Coleman *et al.* (2006) include items 1, 3, and 5 in their portfolio model whereby they use a continuous approximation to express the number of securities in the portfolio. The approximation is relaxed after adding it to the objective function with an expression for tracking error. Although they are able to keep the portfolio size relatively small, their model does not consider other important elements such as sector exposure, transaction costs, and a rebalancing strategy. Bertsimas *et al.* (1999) also design a basic tracking portfolio model that captures most of the key portfolio components in a mixed-integer multi-objective optimization problem. In their model, future security values are purely based on expected return and every portfolio element has a user-specified parameter attached to it, which is derived from market simulations. The authors of [Yao *et al.*, 2006] track an index using a few assets under a stochastic linear quadratic control framework, where they are able to avoid integer variable selection criterions by first choosing a subset of the assets to which they will invest in. Finally, Beasley *et al.* (2003) initially account for items 1–6 in their original portfolio model; however, after adding an index tracking heuristic, particular characteristics become more relevant to their design. The final deterministic portfolio model includes the prediction of future security distributions embedded in a heuristic whose index difference is minimized by a large, continuous, multi-objective function. They report portfolio values that are almost equal to or less than the index, hence their evolutionary (genetic) heuristic has a very unique approach.

The index tracking model we propose accounts for items one to six and involves a technique for generating discrete future security distributions. In order to account for all the portfolio elements involved in the problem, mixed-integer optimization is necessary. In addition, to facilitate randomness found in the stock market, we use a SP technique to capture any discrepancies that may occur within future security distributions. Thus, we formulate an index tracking portfolio model subject to uncertainty through SMIP with recourse. Using the SMIP approach we capture many possible future market realizations (recourse decisions) through scenarios, in which all scenarios have present information incorporated in them. Hence, in the second-stage (or higher), if one of the many possible market realizations occur (stochastic scenarios in the problem) the model has accounted for these decisions and has the capacity to consider future decisions based on the correct realizations. Although similar stochastic techniques have been used for other types of portfolio construction [Escudero *et al.*, 2007; Ji *et al.*, 2005; Zenios *et al.*, 1998], to the best of our knowledge, we are the first to develop an index tracking portfolio model that includes a comprehensive set of real-world portfolio elements and is solved via SMIP. Other stochastic modelling techniques, such as Robust optimization, do not account for all recourse decisions, which has great relevance to this financial application. We combine index tracking with stochastic integer optimization to form an index tracking SMIP problem.

The combination of the integer and stochastic requirements of the problem make it highly challenging with respect to solvability and computation. For this reason, we also develop a decomposition strategy that successfully solves the demanding optimization issues of the problem, in which the algorithm may be parallelized to produce even better CPU times. Finally, we detail the scenario generation techniques used in the SMIP that produce the final portfolio results. Although we have increased the complexity of the problem with respect to most of the literature on index tracking, i.e. [Beasley *et al.*, 2003; Bertsimas *et al.*, 1999; Coleman *et al.*, 2006; Gaivoronski *et al.*, 2005; Ji *et al.*, 2005], the method allows for more precise portfolio construction and strong index tracking results. In conclusion, this analysis includes two major contributions: an index tracking model that combines security price uncertainty with a comprehensive set of real-world portfolio elements, and a problem specific decomposition algorithm tailored to solve the SMIP. With respect to the literature on this topic, the mixed-integer modelling approach and accompanying algorithmic strategy are novel contributions to the field.

In this chapter, we begin with a two-stage portfolio formulation in Section 4.2, where we first develop a deterministic, dynamic mixed-integer tracking portfolio model. Next, we add uncertainty to the system of equations that produces a stochastic mixed-integer portfolio model. In Section 4.3 we deal with the problem's implementation issues. In particular, we provide the details of the algorithm strategy and illustrate how the scenarios are generated. Finally, in Section 4.4 the method's index tracking results are compared to the S&P/TSX composite monthly total return index. We also include the value of uncertainty by reporting the stochastic versus deterministic mixed-integer programming results and follow with a conclusion of the approach.

4.2 SMIP Model Formulation

First, we define and describe how the portfolio elements listed on page 63 form the two-stage index tracking mixed-integer optimization problem. Then we add scenarios to capture uncertainties in index and security prices, which defines the SMIP.

4.2.1 Two–Stage Index Tracking Portfolio

We begin by outlining how to capture the six critical elements listed on page 63 that are necessary for designing an index tracking portfolio. The portfolio model will account for:

(a) A net portfolio value that is equivalent to or representative of the index by minimizing their difference;

(b) Exposure to similar economic sectors to that of the index, or ensure that the portfolio is sector diversified;

(c) Hold a small number of securities using an upper bound;

(d) Minimize the number of transactions;

(e) Derivation of future security distributions that involves uncertainty via stochastic scenarios;

(f) Rebalancing strategy that adapts to portfolio cash flows and future decisions.

Items (a) and (b) ensure that the tracking portfolio's composition is parallel to that of the index. Item (b) also acts as a safeguard towards poor sector development, which can be due to normal market fluctuations or some natural disaster, terrorist activity, etc. Item (c) is necessary due to financial limitations faced by many portfolio managers. Although an index is usually composed of many securities, portfolio managers aim to replicate the index with as few securities as possible. In addition, as the index is not subject to transaction costs, a portfolio with a low number of securities can reduce such costs. If this does not occur, item (d) requires that the number of transactions are kept to a minimum, which in-turn limits transaction costs. Item (e) can have a large impact on the problem, as the choice of an unrepresentative future security distribution can lead to a poorly constructed portfolio. This is the area of the problem that adds uncertainty, which is captured by the SP approach. By introducing different market scenarios, we integrate uncertainty into the indexing problem. Since we do not consider riskless investments, the model does not contain any arbitrage opportunities. Finally, item (f) deals with rebalancing issues such as decision rules on cash flow and other portfolio managing policies. Including items (a)–(f) in the portfolio model produces a very mathematically demanding problem, however, as mentioned in the introduction, these characteristics are essential to producing high performance index funds.

By mathematically expressing each of the characteristics mentioned on page 67, we design the index tracking portfolio problem. To begin, let x_i be the number of units of security i that are purchased in the first time period, at time $t = 0$, where there is a total of n securities. Let y_i^t be the number of units of security i that are invested at time $t > 0$, where there is a total of m time periods and set $T = m - 1$. We consider a two-stage SMIP, however, the framework can also be extended to multi-stage developments. For lack of space we omit writing the detailed extensions and new scenarios necessary for multi-stage models; the reader may simply note that the design is capable of such developments. Thus, $i = 1, 2, ..., n$, $t = 0, ..., T$, $x_i \in \mathbb{I}$, and $y_i^t \in \mathbb{I}$. In addition, x_i is a first-stage decision variable and y_i^t is a second-stage decision variable (y_i^t also accounts for higher stages in multi-stage models). The integer component of this variable provides precise portfolio element modelling and allows us to identify the exact number of security units to purchase or sell at specific time-stages. In many papers, [Coleman *et al.*, 2006; Gaivoronski *et al.*, 2005; Ji *et al.*, 2005; Zenios *et al.*, 1998], x_i and y_i^t are expressed as real variables that represent the fraction of the portfolio that is invested in security i. Under this framework, if these decision variables are set to be real numbers additional constraints are necessary and as it turns out, the model becomes very sensitive towards such constraints. Also, setting these decision variables to be real numbers may permit portfolio errors as there is a high possibility for round-offs where fractional security investments cannot represent whole units, and thus, the final portfolio composition is not precisely represented. Furthermore, the optimal basis for an integer optimization problem may be very different than that of the equivalent linear problem. In any case, using our model construction and algorithm, we are able to handle the complexity issues posed by these variables, which provide more practical results. Further comments on this issue will be presented below and the reader may refer to [Wolsey, 1998] for additional information on integer programming.

Next, we define ϕ_i^0 as the unit price of security i at the portfolio start time $t = 0$, and ϕ_i^t as the predicted unit price of security i at time $t > 0$; thus $\phi_i^t \in \mathbb{R}$. Also, we let I^0 be the index or target value at the portfolio start time and I^t be the predicted index value at time $t > 0$, where $I^t \in \mathbb{R}$ and is given in monetary terms. Thus, ϕ_i^0 and I^0 are known parameters, whereas ϕ_i^t and I^t are unknown for $t > 0$. By forcing the net portfolio value to be equal to or finitely close to the index value, item (a) is captured by minimizing the following term:

$$|\sum_{i=1}^{n} \phi_i^0 x_i - I^0| + \sum_{t=1}^{T} |\sum_{i=1}^{n} \phi_i^t y_i^t - I^t|. \tag{4.1}$$

One may note that equation (4.1) can be modelled as a linear objective, which reduces computational complexities and the reason higher order methods were not used. We incorporate the sector exposure element by first introducing the indicator $Q(i, s)$, which defines the sector s a security belongs to. Hence, we set

$$Q(i,s) = \begin{cases} 1, & \text{if security } i \text{ is in sector } s; \\ 0, & \text{otherwise;} \end{cases} \tag{4.2}$$

where $s = 1, 2, ..., S$ and S defines the total number of sectors. By letting f_s^t be the fraction of the portfolio invested in sector s at time t, we can select the portfolio such that it contains similar sector exposure to that of the index, or at least ensure that it is sector diversified. Since f_s^t is a fractional value, we have that

$$\sum_{s=1}^{S} f_s^t = 1 \qquad \forall\, t = 0, ..., T, \tag{4.3}$$

and $f_s^t \geq 0$. In addition, f_s^0 is a first-stage parameter, whereas for $t > 0$ f_s^t is a second-stage parameter. Therefore, the sector exposure element is captured by the following constraint:

$$\sum_{i=1}^{n} Q(i,s)\phi_i^t y_i^t = f_s^t I^t + \xi_s^t \qquad \forall\, s = 1, ..., S,\ t = 0, ..., T, \tag{4.4}$$

where $\xi_s^t \in \mathbb{R}$ is a sector relaxation variable that corresponds to each f_s^t value we define. As the composition of the index in each sector is unknown, we permit this part of the model to be slightly flexible. Essentially, ξ_s^t allows the objective to find a more suitable value for f_s^t if a feasible solution cannot be generated under the given variable assignments. Note that for $t = 0$, x_i will be in place of y_i^t, as it is shown explicitly in the model under equation (4.16) on page 73.

Next we introduce the binary variable g_i^t, responsible for controlling the number of securities to hold in the portfolio. We define

$$g_i^t = \begin{cases} 1, & \text{if security } i \text{ is used in the portfolio at} \\ & \text{time } t \text{ (i.e. if } x_i \text{ or } y_i^t > 0); \\ 0, & \text{otherwise;} \end{cases} \tag{4.5}$$

where $g_i^t \in \mathbb{B}$. In order to set the variable g_i^t and keep the model linear, a large scalar value C is introduced and used in the following constraint:

$$g_i^t \leq y_i^t \qquad \forall\, i = 1, ..., n,\ t = 1, ..., T \tag{4.6}$$

$$y_i^t \leq \mathrm{C}\, g_i^t \qquad \forall\, i = 1, ..., n,\ t = 1, ..., T, \tag{4.7}$$

where x_i is used in place of y_i^t when $t = 0$. One may verify that equations (4.6) and (4.7) set g_i^t to meet the binary definitions in (4.5). We then set a parameter, G^t, to be an upper bound on the number of securities to hold in the portfolio. Hence, the constraint corresponding to limiting the number of securities to hold in the portfolio is as follows:

$$\sum_{i=1}^{n} g_i^t \leq G^t \qquad \forall\, t = 0, ..., T. \tag{4.8}$$

Since an inequality sign is used above, we can remove constraint (4.6) and the problem is not affected. Finally, to minimize the number of transactions between time periods, we define

$$h_i^1 = |y_i^1 - x_i| \qquad \forall\, i = 1, ..., n \tag{4.9}$$

and for $t > 1$ we have

$$h_i^t = |y_i^t - y_i^{t-1}| \qquad \forall\, i = 1, ..., n,\ t = 2, ..., T, \tag{4.10}$$

where $h_i^0 = 0\ \forall\ i = 1, ..., n$. Thus, h_i^t represents the number of units of a security that are bought or sold between time periods. This entity will be minimized in the objective to reduce the number of transactions, which therefore decreases transaction costs. Also, embedded in (4.1)–(4.10) is a rebalancing strategy. At optimality, we define a portfolio that rebalance's according to: (a) closeness to the index, (b) representative investments across all sectors, (c) does not exceed a desirable portfolio size, and (d) minimizes the number of securities bought/sold. Next, we add cash balancing constraints to ensure that the portfolio manages its cash flows and does not invest with insufficient funds. The cash balancing constraints are as follows:

$$B^1 = \sum_{i=1}^{n} \phi_i^1 x_i - \sum_{i=1}^{n} \phi_i^1 y_i^1 \tag{4.11}$$

$$B^t = \sum_{i=1}^{n} \phi_i^t y_i^{t-1} - \sum_{i=1}^{n} \phi_i^t y_i^t + B^{t-1} \quad \forall\, t = 2, ..., T \tag{4.12}$$

$$B^t \geq 0 \quad \forall\, t = 1, ..., T\,. \tag{4.13}$$

From above, if $B^1 = 0$, then all equity generated from first-period investments must be re-invested in the second-period. Whereas, $B^1 \geq 0$ allows for equity surpluses to be used for portfolio maintenance or stored for later time periods. The same would follow for multi-stage problems. Also note that the initial portfolio investment is set to be the same value as the index. This is captured in equation (4.1), where we take

$$\min \left\{ \left| \sum_{i=1}^{n} \phi_i^0 x_i - I^0 \right| \right\} \tag{4.14}$$

and I^0 is known. The portfolio is geared towards tracking the index by minimizing trading and is set up to accomplish this by the positive/negative cash flows of its investments, not by exercising $B^t > 0$. Furthermore, one may note that many of the decisions listed on page 67 are based on the predicted value of each security ϕ_i^t and the index I^t. Different values of ϕ_i^t and I^t for $t > 0$ can cause the portfolio to invest in one direction rather than another. This leads to the issue of market randomness, which is continuously investigated by a variety of sources, e.g. [Gaivoronski *et al.*, 2005; Malkiel, 1999]. In Subsection 4.2.4 we introduce stochastic elements to the portfolio model that account for various degrees of security price randomness and other uncertainties, but first we present the deterministic model.

4.2.2 Deterministic Dynamic Portfolio Model

We illustrate the deterministic dynamic mixed-integer programming model as it will be compared to the stochastic model in Section 4.4. Before presenting the Deterministic Equivalent Program (DEP), we define the set of securities as $\widehat{\Upsilon} := \{i : i = 1, ..., n\}$, the set of sectors as $\widehat{S} := \{s : s = 1, ..., S\}$, and the set of time-stages as $\widehat{T} := \{t : t = 0, ..., T\}$, $\overline{T} := \{t : t \in \widehat{T}, t \neq 0\}$, and $\widetilde{T} := \{t : t \in \widehat{T}, t \neq \{0, 1\}\}$. Also, we introduce an objective weighting parameter $\mu_k \geq 0 \ \forall \ k = 1, 2, 3$ that manages the relaxation and transaction cost functions effect on the index tracking objective. Therefore, the dynamic mixed-integer DEP model becomes:

$$\min\ \mu_1 \left| \left(\sum_{i=1}^{n} \phi_i^0 x_i \right) - I^0 \right| + \mu_1 \sum_{t=1}^{T} \left| \left(\sum_{i=1}^{n} \phi_i^t y_i^t \right) - I^t \right| \tag{4.15}$$

$$+ \mu_2 \sum_{t=0}^{T} \sum_{s=1}^{S} \left| \xi_s^t \right| + \mu_3 \sum_{t=1}^{T} \sum_{i=1}^{n} h_i^t$$

$$\text{s.t.}\quad \sum_{i=1}^{n} Q(i,s)\phi_i^0 x_i = f_s^0 I^0 + \xi_s^0 \qquad \forall\, s \in \widehat{S} \tag{4.16}$$

$$\sum_{i=1}^{n} Q(i,s)\phi_i^t y_i^t = f_s^t I^t + \xi_s^t \qquad \forall\, s \in \widehat{S},\ t \in \overline{T} \tag{4.17}$$

$$\sum_{i=1}^{n} g_i^t \leq G^t \qquad \forall\, t \in \widehat{T} \tag{4.18}$$

$$x_i \leq \mathrm{C}\, g_i^0 \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.19}$$

$$y_i^t \leq \mathrm{C}\, g_i^t \qquad \forall\, i \in \widehat{\Upsilon},\ t \in \overline{T} \tag{4.20}$$

$$h_i^1 = |y_i^1 - x_i| \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.21}$$

$$h_i^t = |y_i^t - y_i^{t-1}| \qquad \forall\, i \in \widehat{\Upsilon},\ t \in \widetilde{T} \tag{4.22}$$

$$B^1 = \sum_{i=1}^{n} \phi_i^1 x_i - \sum_{i=1}^{n} \phi_i^1 y_i^1 \tag{4.23}$$

$$B^t = \sum_{i=1}^{n} \phi_i^t y_i^{t-1} - \sum_{i=1}^{n} \phi_i^t y_i^t + B^{t-1} \qquad \forall\, t \in \widetilde{T} \tag{4.24}$$

$$B^t \geq 0,\ B^t \in \mathbb{R} \qquad \forall\, t \in \overline{T} \tag{4.25}$$

$$x_i \geq 0,\ x_i \in \mathbb{I} \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.26}$$

$$y_i^t \geq 0,\ y_i^t \in \mathbb{I} \qquad \forall\, i \in \widehat{\Upsilon},\ t \in \overline{T} \tag{4.27}$$

$$g_i^t \in \mathbb{B} \qquad \forall\, i \in \widehat{\Upsilon},\ t \in \widehat{T} \tag{4.28}$$

$$\xi_s^t \in \mathbb{R} \qquad \forall\, s \in \widehat{S},\ t \in \widehat{T}, \tag{4.29}$$

where constraints (4.19)–(4.20) set the binary variable g_i^t, and C is a sufficiently large constant such that $\forall\ x_i$ and $y_i^t <$ C. Also, short positions are not permitted as $x_i,\ y_i^t \geq 0$ and μ_1 was heavily weighted with respect to μ_2 and μ_3. For more information on parameter weighting the reader may refer to [Chankong and Haimes, 1983]. The optimization problem listed above accounts for a total of $2(m \times n) + (m \times S) + (m - 1) \times (n + 1)$ decision variables, where ϕ_i^t, I^t, f_s^t, $Q(i, s)$, and G^t are parameters.

4.2.3 Stochastic Optimization Problem

Before developing the SMIP portfolio framework, we provide a brief discussion of a classical SP problem. Hence, we present the following two-stage Stochastic Mixed-Integer problem:

$$\begin{aligned} \min \quad & \mathrm{c}^\top x + \mathrm{E}[\min q(\omega)^\top y(\omega)] && (4.30) \\ \text{s.t.} \quad & \mathrm{A}x = \mathrm{b} && (4.31) \\ & T(\omega)x + Wy(\omega) = h(\omega) && (4.32) \\ & x \geq 0,\ y(\omega) \geq 0, && (4.33) \end{aligned}$$

where x is a first-stage decision vector, while A, b, and c are the corresponding first–stage parameters. Given that there are $\omega \in \Omega$ random events, $y(\omega)$ is the second-stage decision variable, and $q(\omega)$, $T(\omega)$, W, and $h(\omega)$ are the corresponding second-stage parameters. Also, since the problem is mixed-integer, the decision variables x and $y(\omega)$ contain restrictions that require some (or all if it were pure integer) of the variables to be integer or binary, hence $x \subseteq \mathbb{I}$, $y(\omega) \subseteq \mathbb{I}$. Finally, $\mathrm{E}[\cdot]$ denotes the expectation with respect to the random event ω.

The first-stage problem involved with solving (4.30)–(4.33) is as follows:

$$\begin{aligned} \min \quad & \mathrm{c}^\top x && (4.34) \\ \text{s.t.} \quad & \mathrm{A}x = \mathrm{b} && (4.35) \\ & x \geq 0, && (4.36) \end{aligned}$$

where the parameters A, b, and c are all known. The solution to (4.30)–(4.33) is not as simple as solving (4.34)–(4.36) and then going on to the second-stage, as these problems involve time-stage interlinking constraints. In the second-stage, any feasible solution must consider uncertain elements involved in the second time-stage plus feasibility issues with constraints that cover both time-stages, known as non-anticipativity constraints.

For any realization ω, if we define

$$Q(x) = \mathrm{E}[\min_{y}\{q(\omega)^{\top} y | Wy = h(\omega) - T(\omega)x, y \geq 0\}], \tag{4.37}$$

then the Deterministic Equivalent Program (DEP) becomes

$$\min \quad \mathrm{c}^{\top} x + Q(x) \tag{4.38}$$

$$\text{s.t.} \quad \mathrm{A}x = \mathrm{b} \tag{4.39}$$

$$x \geq 0, \tag{4.40}$$

where the stochastic program unravels to a nonlinear or linear program (depending on the objective and constraints) when the second-stage function is known. Thus, finding an optimal value(s) to (4.30)–(4.33) requires the decision variables of (4.34)–(4.36) to be considered with those in the second-stage. For more information on SP the reader may refer to [Birge and Louveaux, 1997].

4.2.4 Stochastic Mixed–Integer Portfolio Model

The introduction of stochastic elements provides a means for investors to capitalize on favorable financial opportunities and/or avoid unfavorable ones. Since the portfolio model in (4.15)–(4.29) is influenced by the prediction of I^t and ϕ_i^t, we introduce uncertainty to the model by creating stochastic scenarios to compensate for instances when the predicted values are too aggressive or simply incorrect. Although there are several publications [Chen *et al.*, 2003; Masolivera *et al.*, 2000; Osborne, 1959] that involve highly sophisticated methods of generating ϕ_i^t, it is difficult to account for all market fluctuation and most methods fail to represent the market as a whole. In addition, the developed methods commonly emit larger deviations from the actual market values as the number of time periods increase. For this reason, we generate a number of stochastic scenarios that the market may experience, which may follow any one, or a combination of the different strategies presented in [Beasley *et al.*, 2003; Chen *et al.*, 2003; Gaivoronski and Stella, 2003; Ji *et al.*, 2005; Masolivera *et al.*, 2000; Osborne, 1959; Zenios *et al.*, 1998]; or may be developed by the user. This information is then included in the variables related to I^t and ϕ_i^t, additional variables and constraints are introduced to compensate for the new information, and finally, everything is integrated into the existing framework to form the large SMIP.

We define a total of L possible scenarios that the financial market may follow or shift to. We then add this component to the expected security values $\phi_i^t \; \forall \; t > 0$ which, depending on the user, may be generated similarly to what is done in [Beasley *et al.*, 2003; Gaivoronski and Stella, 2003; Ji *et al.*, 2005; Zenios *et al.*, 1998]; or what we present in Subsection 4.3.2 below. Thus, for scenarios $\ell = 1, ..., L$, ϕ_i^t subsequently becomes

$$\phi_{i\ell}^t \quad \forall \; t > 0 \tag{4.41}$$

and the second-stage decision variables $y_{i\ell}^t$ and $g_{i\ell}^t$ undergo the same transformation to account for the scenario additions, whereby $y_{i\ell}^t$ is structured according to Figure 4.1. As mentioned in Subsection 4.2.1, the model can be extended to multi-stage developments. For the multi-stage design, scenario index ℓ also becomes a subscript of t, which accounts for the additional scenario evolutions that a variable has undergone. Thus, for higher time-stages $t_{\ell,\ell,\ell,...}$ is stored for each scenario evolution, as shown in Figure 4.1. In addition, variable r is used to account for any new scenarios that are introduced in the multi-stage design, one can refer to [Birge and Louveaux, 1997; Dantzig and Infanger, 1993] for an example of such extensions. Since, we consider a two-stage problem we omit these indices from the notation; however, the reader should note that they can be incorporated to the existing framework. Under this design, scenarios provide the model with

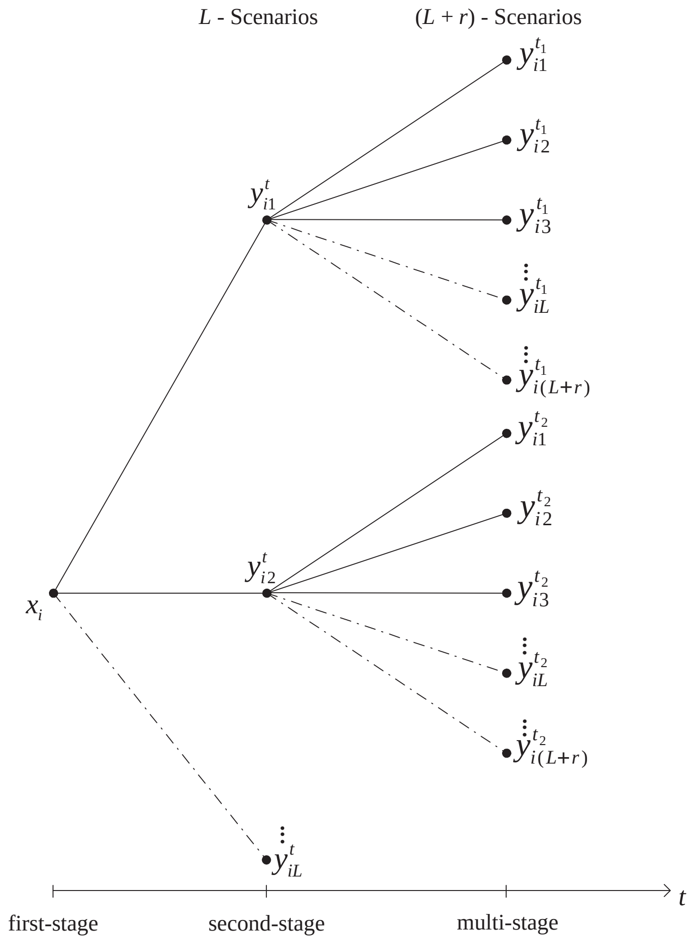

Figure 4.1: Evolution of L portfolio scenarios with respect to security investments, where r accounts for new scenarios introduced in the multi-stage problem.

the capacity to account for market shifts such as: favorable or poor sector development, company privatization, economic recessions, natural disasters, etc. Hence, each scenario can support different possible expected security values with the original ϕ_i^t embedded in the system. Next we use the same concept to introduce scenarios to the index values. Index values are subject to similar market fluctuation and shifts to that of securities. Thus, for $t > 0$ we add $\ell = 1, ..., L$ scenarios to the predicted index values, which is captured by the term I^t_ℓ. In addition, scenarios have to be introduced to the variables that represent the fraction of the index in each sector and the relaxation variables for the sector constraint. Hence, for $t > 0$ we have the following notation additions:

$$f^t_{s\ell}\,,\ \xi^t_{s\ell}\,\forall\, t > 0. \tag{4.42}$$

Finally, the function that minimizes transaction costs and the cash balancing constraints also obtain a scenario index. Hence, for each scenario $\ell = 1, ..., L$ we have

$$h^1_{i\ell} = |y^1_{i\ell} - x_i|,\ \forall\, i \in \widehat{\Upsilon} \tag{4.43}$$

and for higher time periods ($t > 1$)

$$h^t_{i\ell} = |y^t_{i\ell} - y^{t-1}_{i\ell}|,\ \forall\, i \in \widehat{\Upsilon}. \tag{4.44}$$

For the cash balancing constraints, the following must be satisfied for each scenario ℓ:

$$B^1_\ell = \sum_{i=1}^n \phi^1_i x_i - \sum_{i=1}^n \phi^1_{i\ell}\, y^1_{i\ell} \tag{4.45}$$

$$B^t_\ell = \sum_{i=1}^n \phi^t_{i\ell}\, y^{t-1}_{i\ell} - \sum_{i=1}^n \phi^t_{i\ell}\, y^t_{i\ell} + B^{t-1}_\ell, \quad \forall\, t \in \widetilde{T}. \tag{4.46}$$

Thus, for the two-stage problem the decision variable count has increased by a factor of approximately $5L$; as one can deduce from the scenario tree in Figure 4.1.

Before stating the two-stage stochastic portfolio model, we begin by outlining the first-stage parameters and decision variables. In the first-stage f^0_s and G^0 are parameters, whereas x_i, ξ^0_s, g^0_i are decision variables. In addition, the index I^0 and security values ϕ^0_i are all known first-stage values. For $t > 0$, we add the second-stage variables to the problem that include scenarios $\ell = 1, ..., L$. Finally, we attach the probability of each scenario p_ℓ to the objective function, where $\sum_{\ell=1}^L p_\ell = 1$ and $p_\ell \geq 0$. After defining the set of scenarios to be $\widehat{L} := \{\ell : \ell = 1, ..., L\}$, the two-stage stochastic mixed-integer index tracking DEP is:

$$\begin{aligned}\min\ \ & \mu_1 \left| \left(\sum_{i=1}^{n} \phi_i^0 x_i \right) - I^0 \right| + \mu_1 \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell \left| \left(\sum_{i=1}^{n} \phi_{i\ell}^t \, y_{i\ell}^t \right) - I_\ell^t \right| \\ & + \mu_2 \sum_{s=1}^{S} \left| \xi_s^0 \right| + \mu_2 \sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{s=1}^{S} \left| \xi_{s\ell}^t \right| + \mu_3 \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{\ell=1}^{L} h_{i\ell}^t \end{aligned} \tag{4.47}$$

$$\text{s.t.} \sum_{i=1}^{n} Q(i,s)\phi_i^0 x_i = f_s^0 I^0 + \xi_s^0 \qquad \forall\, s \in \widehat{S} \tag{4.48}$$

$$\sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^t \, y_{i\ell}^t = f_{s\ell}^t I_\ell^t + \xi_{s\ell}^t \qquad \forall\, s \in \widehat{S},\ \ell \in \widehat{L},\ t \in \overline{T} \tag{4.49}$$

$$\sum_{i=1}^{n} g_i^0 \leq G^0 \tag{4.50}$$

$$\sum_{i=1}^{n} g_{i\ell}^t \leq G^t \qquad \forall\, \ell \in \widehat{L},\ t \in \overline{T} \tag{4.51}$$

$$x_i \leq \mathrm{C}\, g_i^0 \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.52}$$

$$y_{i\ell}^t \leq \mathrm{C}\, g_{i\ell}^t \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \overline{T} \tag{4.53}$$

$$h_{i\ell}^1 = |y_{i\ell}^1 - x_i| \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.54}$$

$$h_{i\ell}^t = |y_{i\ell}^t - y_{i\ell}^{t-1}| \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \widetilde{T} \tag{4.55}$$

$$B_\ell^1 = \sum_{i=1}^{n} \phi_i^1 x_i - \sum_{i=1}^{n} \phi_{i\ell}^1 y_{i\ell}^1 \qquad \forall\, \ell \in \widehat{L} \tag{4.56}$$

$$B_\ell^t = \sum_{i=1}^{n} \phi_{i\ell}^t y_{i\ell}^{t-1} - \sum_{i=1}^{n} \phi_{i\ell}^t y_{i\ell}^t + B_\ell^{t-1} \qquad \forall\, \ell \in \widehat{L},\ t \in \widetilde{T} \tag{4.57}$$

$$B_\ell^t \geq 0,\ B_\ell^t \in \mathbb{R} \qquad \forall\, \ell \in \widehat{L},\ t \in \overline{T} \tag{4.58}$$

$$x_i \geq 0,\ x_i \in \mathbb{I} \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.59}$$

$$y_{i\ell}^t \geq 0,\ y_{i\ell}^t \in \mathbb{I} \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \overline{T} \tag{4.60}$$

$$g_i^0,\ g_{i\ell}^t \in \mathbb{B} \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \overline{T} \tag{4.61}$$

$$\xi_s^0,\ \xi_{s\ell}^t \in \mathbb{R} \qquad \forall\, s \in \widehat{S},\ \ell \in \widehat{L},\ t \in \overline{T}. \tag{4.62}$$

One should note that equation (4.48) is a first-stage sector constraint, whereas equation (4.49) accounts for second-stage sector constraints; as does (4.50) and (4.51), respectively.

In the objective function (4.47), the first-stage variables are evident where $t = 0$. In addition, the non-anticipativity constraints are implicitly captured by decisions on x_i and $y_{i\ell}^t$. The optimization problem above accounts for a total of $\sum_{k=1}^{m} L^{k-1} \times n + 2(m-1) \times n \times L + (S+1) \times (m-1) \times L + S + n$ decision variables and thus, for a small increase in the number of time-stages or scenarios, the problem can grow to be very large. In the next section we introduce an implementation strategy to handle the large SMIP problem presented in (4.47)–(4.62).

4.3 Implementation Issues

The model developed in the previous section captures a demanding set of important real-world portfolio elements. The SMIP also has an embedded NP-hard subproblem; see [Coleman *et al.*, 2006]. In this section, we detail the computational issues behind solving the large SMIP designed in Subsection 4.2.4 and outline the structure that is used for generating the various market scenarios.

4.3.1 Algorithm Design

Due to the large number of variables and the nature of the stochastic model, a decomposition algorithm is designed to accommodate the complexity of the resulting portfolio presented in (4.47)–(4.62). Unlike common stochastic decomposition algorithms that divide a problem based on time-stages or scenarios, we separate the problem into a number of subproblems by using two constraints in the model. As one might infer from the literature on stochastic algorithms, e.g. [Carøe and Schultz, 1999; Nürnberg and Römisch, 2002], enforcing non-anticipativity constraints between decompositions may cause computational challenges towards solvability and CPU time. We avoid such complexities by implicitly including the non-anticipativity constraints within the subproblems. The algorithm operates by separating the model into subproblems based on sector constraints (4.48) and (4.49). The sector based decomposition is accompanied by a relaxation, where after each subproblem is solved the global optimum is evaluated. Then, depending on specific conditions, the optimum is either accepted or a penalty adjustment is made to the relaxation and the problem is re-solved. This is equivalent to solving the large problem, however, the decomposition algorithm provides faster CPU time, requires less memory, and is potentially capable of handling problems with a larger number of variables (as the subproblem SMIPs are generally easier to solve).

The sector based decomposition exploits the specific structure of the problem. In order to do so we first define that such a decomposition is viable given that a problem is *MIP-separable*; a concept from separable programming [Winston, 1994] and allows for geographical decompositions similar to what is shown in [Dentcheva et al., 2004].

Definition 4.3.1. Given the problem

$$\min_{x,\, y} \left\{ \mathrm{c}^\top x + \mathrm{d}^\top y : \mathrm{A}x + \mathrm{B}y = \mathrm{b}, (x,\, y) \geq 0, (x,\, y) \in \mathbb{I} \right\}. \tag{4.63}$$

If b can be divided and/or duplicated such that there exists at least one $x_i \in x$ and/or $y_i \in y$ where $\mathrm{c}_i x_i + \mathrm{d}_i y_i$ is only dependent on $\mathrm{A}_i x_i = \mathrm{b}_i$ and/or $\mathrm{B}_i y_i = \mathrm{b}_i$, and independent with respect to all other variables and constraints in (4.63), then problem (4.63) is said to be *MIP-separable*.

Thus, given $z^\star = (4.63)$, then from *Definition 4.3.1* it follows that if (4.63) is *MIP-separable* there exists a bounded set of $1 \leq n < \infty$ separable subproblems such that

$$z^\star = \sum_{i=0}^{n} z_i^\star, \tag{4.64}$$

where $z_i^\star = \min\{\mathrm{c}_i^\top x_i + \mathrm{d}_i^\top y_i\}$ is the optimal solution of subproblem i. The general strategy of the algorithm is to partition the SMIP in (4.47)–(4.62) into smaller subproblems that are identical in form. To do so the problem must be *MIP-separable*, of which (4.47)–(4.62) is *MIP-separable* based on the sector constraints, where

$$\sum_{i=1}^{n} Q(i,s)\phi_i^0 x_i = f_s^0 I^0 + \xi_s^0 \tag{4.65}$$

$$\sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^t y_{i\ell}^t = f_{s\ell}^t I_\ell^t + \xi_{s\ell}^t, \tag{4.66}$$

allows for a sector based decomposition. Firstly, $Q(i,s)$ is a binary indicator that is equal to one only if x_i or y_i^t belong to sector s. Secondly, $f_s^0 I^0$ and $f_{s\ell}^t I_\ell^t$ represent fractions of the right hand side that are solely applicable to variables x_i and y_i^t corresponding to each sector. Thus, this constraint only applies to variables in a given sector s. Also, relaxations ξ_s^0 and $\xi_{s\ell}^t$ make the problem equivalent to solving the minimum of the same set of equations with (4.65) and (4.66) in the objective function. The idea behind the sector adjustment using (4.65)–(4.66) is shown below:

$$\begin{array}{rl} \min & \mathrm{c}^\top x + |\varepsilon| \\ \text{s.t.} & \mathrm{A}_1 x = \mathrm{b}_1 \\ & \mathrm{A}_2 x = \mathrm{b}_2 + \varepsilon \\ & x \geq 0, \varepsilon \in \mathbb{R} \end{array} \equiv \begin{array}{rl} \min & \mathrm{c}^\top x + |\mathrm{A}_2 x - \mathrm{b}_2| \\ \text{s.t.} & \mathrm{A}_1 x = \mathrm{b}_1 \\ & x \geq 0, \end{array} \tag{4.67}$$

where A_1, A_2 are matrices and b_1, b_2, c are vectors. In addition, constraints (4.52)–(4.55) are independent with respect to all other variables and can be separated amongst the sector subproblems. This produces the following $s = 1, 2, ..., S$ sector decomposed SMIP subproblems:

$$\begin{aligned} \min \quad & \mu_1 \left| \left(\sum_{i=1}^{n} Q(i,s)\phi_i^0 x_i \right) - f_s^0 I^0 \right| + \varrho_s^0 \left|\gamma_s^0\right| + \sum_{t=1}^{T}\sum_{\ell=1}^{L} \varrho_s^t \left|\gamma_{s\ell}^t\right| \\ & + \mu_1 \sum_{t=1}^{T}\sum_{\ell=1}^{L} p_\ell \left| \left(\sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^t y_{i\ell}^t \right) - f_{s\ell}^t I_\ell^t \right| + \mu_3 \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{\ell=1}^{L} h_{i\ell}^t \end{aligned} \tag{4.68}$$

$$\text{s.t } \sum_{i=1}^{n} g_i^0 \leq G_s^0 + \gamma_s^0, \tag{4.69}$$

$$\sum_{i=1}^{n} g_{i\ell}^t \leq G_s^t + \gamma_{s\ell}^t \qquad \forall\, \ell \in \widehat{L},\ t \in \overline{T} \tag{4.70}$$

$$x_i \leq \mathrm{C}\, g_i^0 \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.71}$$

$$y_{i\ell}^t \leq \mathrm{C}\, g_{i\ell}^t \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \overline{T} \tag{4.72}$$

$$h_{i\ell}^1 = |y_{i\ell}^1 - x_i| Q(i,s) \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.73}$$

$$h_{i\ell}^t = |y_{i\ell}^t - y_{i\ell}^{t-1}| Q(i,s) \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \widetilde{T} \tag{4.74}$$

$$B_{s\ell}^1 = \sum_{i=1}^{n} Q(i,s)\phi_i^1 x_i - \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^1 y_{i\ell}^1 \qquad \forall\, \ell \in \widehat{L} \tag{4.75}$$

$$B_{s\ell}^t = \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^t y_{i\ell}^{t-1} - \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^t y_{i\ell}^t + B_{s\ell}^{t-1} \qquad \forall\, \ell \in \widehat{L},\ t \in \widetilde{T} \tag{4.76}$$

$$B_{s\ell}^t \geq 0,\ B_{s\ell}^t \in \mathbb{R} \qquad \forall\, \ell \in \widehat{L},\ t \in \overline{T} \tag{4.77}$$

$$x_i \geq 0,\ x_i \in \mathbb{I} \qquad \forall\, i \in \widehat{\Upsilon} \tag{4.78}$$

$$y_{i\ell}^t \geq 0,\ y_{i\ell}^t \in \mathbb{I} \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \overline{T} \tag{4.79}$$

$$g_i^0,\ g_{i\ell}^t \in \mathbb{B} \qquad \forall\, i \in \widehat{\Upsilon},\ \ell \in \widehat{L},\ t \in \widehat{T} \tag{4.80}$$

$$\gamma_s^0,\ \gamma_{s\ell}^t \in \mathbb{R} \qquad \forall\, \ell \in \widehat{L},\ t \in \overline{T}. \tag{4.81}$$

The only constraints that are not independent with respect to the sector subproblems are those having to do with the bound on the number of securities to hold in the portfolio (G^t) and the cash balancing constraint (B_ℓ^t). Thus, to satisfy the decomposition a sector index is added to $B_{s\ell}^t$ and G_s^t, as shown above. The cash balancing constraint is maintained within each sector by setting

$$B_\ell^t = B_{1\ell}^t + B_{2\ell}^t + \cdots + B_{S\ell}^t\,. \tag{4.82}$$

Also, the parameter that defines the upper bound on the number of securities to hold in the portfolio is sector specific such that

$$G^t = G_1^t + G_2^t + \cdots + G_S^t. \tag{4.83}$$

Sector parameters $B^t_{s\ell}$ and G^t_s are determined using the fraction of the index invested in each sector, $f^t_{s\ell}$. As shown in (4.69)–(4.70), penalty variables γ^0_s and $\gamma^t_{s\ell}$ are added to the names-to-hold constraints, which are accompanied by penalty parameters ϱ^0_s and ϱ^t_s in the objective function. As with the MAD model of Chapter 3, this is the most difficult constraint in the problem to satisfy and forms the basis of the decomposition algorithm. The idea behind the algorithm is to first find a solution using relaxed constraints (4.69)–(4.70), and then add a cost associated to violations of this constraint (using the penalty parameters ϱ^0_s and ϱ^t_s) until the relaxation is equal to zero and the problem is solved to optimality. This step is similar to *Step 3* of the algorithm proposed in Section 3.3. Because of the positive results, the iteration will be integrated into the decomposition algorithm proposed in this section. To do so, first a master problem is generated by adding penalty variables and parameters to the SMIP in (4.47)–(4.62) as follows:

$$\min \quad (4.47) + \sum_{s=1}^{S} \varrho^0_s \left|\gamma^0_s\right| + \sum_{t=1}^{T}\sum_{s=1}^{S}\sum_{\ell=1}^{L} \varrho^t_s \left|\gamma^t_{s\ell}\right| \tag{4.84}$$

$$\text{s.t} \quad \sum_{i=1}^{n} g^0_i \leq \sum_{s=1}^{S} G^0_s + \sum_{s=1}^{S} \gamma^0_s, \tag{4.85}$$

$$\sum_{i=1}^{n} g^t_{i\ell} \leq \sum_{s=1}^{S} G^t_s + \sum_{s=1}^{S} \gamma^t_{s\ell} \qquad \forall\, \ell \in \widehat{L},\ t \in \overline{T} \tag{4.86}$$

$$\gamma^0_s,\, \gamma^t_{s\ell} \ \in\ \mathbb{R} \qquad \forall\, \ell \in \widehat{L},\ s \in \widehat{S},\ t \in \overline{T} \tag{4.87}$$

$$(4.48), (4.49), (4.52) - (4.62). \tag{4.88}$$

The master problem in (4.84)–(4.88) sets the sector distribution of the cash balancing constraint (4.82) and the number of securities (4.83), and then undergoes a sector decomposition to produce the $s = 1, ..., S$ subproblems shown in (4.68)–(4.81). If the penalty variables in (4.84)–(4.88) become sufficiently large then [Blair and Jeroslow, 1981] have shown that the master problem is equivalent to the SMIP presented in (4.47)–(4.62); we provide the details to their Theorem on page 89. Finally, the algorithm evaluates the global optimum and based on the value of the penalty variables, either the current optimal solution is accepted or an iterative penalty adjustment is performed and the problem is re-solved. The fundamentals of the algorithm are shown in Figure 4.2.

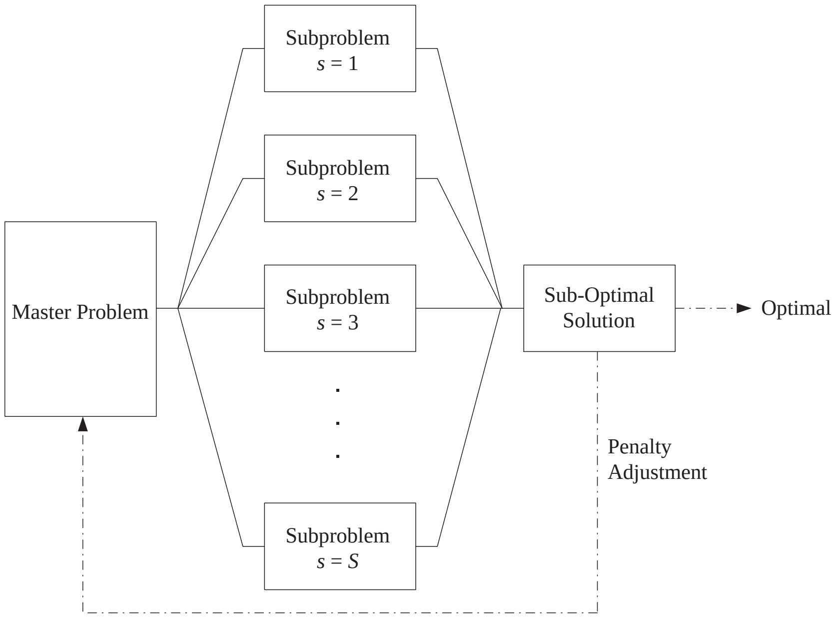

Figure 4.2: The basic functionality of the decomposition algorithm.

Coleman *et al.* (2006) have pointed out that constraints (4.50) and (4.51) present the greatest computational challenge to the problem. As shown above, these constraints are relaxed in (4.69) and (4.70) by penalty variables γ_s^0 and $\gamma_{s\ell}^t$. The algorithm then uses penalty parameters ϱ_s^0 and ϱ_s^t in an iterative procedure that minimizes the cost associated with violations to constraints (4.50) and (4.51). Hence, the iterative penalty procedure or adjustment is geared towards minimizing the size of the relaxation while obtaining an optimal solution in a reasonable amount of time and using the minimal amount of memory necessary. If a Lagrangian decomposition is implemented, time and memory allocation become an issue primarily due to the additional dual problems necessary to be solved between iterations. At each iteration the penalty adjustment step considers the value of penalty variables

$$\gamma_s^0 \text{ and } \gamma_{s\ell}^t \,.$$

Then, if the size of the penalty variables is large, the penalty parameters

$$\varrho_s^0 \text{ and } \varrho_{s\ell}^t$$

are adjusted by increasing in value, which causes penalty variables to become expensive and may be reduced in the next iteration; depending on the amount the penalty parameters are increased. The main steps of the algorithm are as follows (next page):

`1. Initialize:`

a) Set model parameters: $\phi_{i\ell}^t$, I_ℓ^t, $Q(i,s)$, $f_{s\ell}^t$, G^t, p_ℓ, μ_k, and C.

b) Set penalty adjustment parameters: $\hat{r}^1, \hat{r}^2, r^1, r^2, k, z, d, \tau$,

and H, where $\text{Optimal}_0 = z$,

$$\varrho_s^0 = \hat{r}^1,$$

$$\varrho_{s\ell}^t = \hat{r}^2.$$

`2. Decompose Master SMIP problem:`

Separate (4.84)–(4.88) into Subproblems and adjust sector

parameters G_s^t and $B_{s\ell}^t$.

`3. Solve SMIP Subproblems:`

For $s = 1, ..., S$ solve the SMIP in equations (4.68)–(4.81), where

$\text{Optimal}_k = \sum_1^S(\text{Optimal solution for SMIP subproblem } s)$.

`4. Penalty Adjustment:`

if $\gamma_s^0, \gamma_{s\ell}^t = 0$

$\hookrightarrow \text{Optimal} = \text{Optimal}_k \quad \Rrightarrow$ Terminate.

else if $\text{Optimal}_k < \text{Optimal}_{k-1} - \tau$

$\hookrightarrow \text{Optimal} = \text{Optimal}_k \quad \Rrightarrow$ Terminate.

else

for $\gamma_s^0 > 0$ set $\varrho_s^0 = r^1 + \varrho_s^0$

$\gamma_{s\ell}^t > 0$ set $\varrho_{s\ell}^t = r^2 + \varrho_{s\ell}^t$

`5. Update:`

if $d < H$ set

$k = k + 1$

$d = d + 1$

$\hookrightarrow$ `Go to 2.` Solve Optimal_{k+1} with updated $\varrho_{s\ell}^t$ and ϱ_s^0.

else

$\hookrightarrow \text{Optimal} = \text{Optimal}_k \quad \Rrightarrow$ Terminate.

For the iteration above, z, $\hat{r}^1$, $\hat{r}^2$, r^1 and r^2 are initialization, where in practice $z = 0$ and penalty adjustment parameters r^1 and r^2 are set to be relatively large such that optimality is preserved [Blair and Jeroslow, 1981]. As with the algorithm of Chapter 3,

H is a parameter that defines the number of times the algorithm will search for better penalty variables, given they are not all zero. Finally, τ is the tolerance that the objective value should decrease with respect to the previous value before accepting the solution.

Some aspects of the penalty adjustment and general algorithm are worth addressing. Although the penalty adjustment step is similar to a portion of what is presented in Section 3.3, there is a model specific decomposition and much larger set of integer components and requirements present in the SMIP model. First, it should be evident that the algorithm finds the optimal solution if the penalty variables are all zero. Second, if one or more of the penalty variables are greater than zero, then we increase their respective penalty parameters by r^1, r^2; given $d < H$. Thus, the penalty parameter amount of increase between iterations is set by r^1, r^2, which can be linear values or more aggressive quadratic functions. As mentioned, this will cause the corresponding penalty variables to be expensive, and in the next iteration these variables may decrease to zero. If not, we repeat the process until we obtain the first case where the objective value is smaller than the previous minus a tolerance τ, which implies that at least one of the penalty variables is less than what it was originally. There is, however, the case where a smaller optimal solution is never found, in this situation we stop after searching H times. For such an instance, if we are not satisfied with the present solution, we may re-initialize the values of ϱ_s^0 and $\varrho_{s\ell}^t$ such that they are fractions of $\hat{r}^1$, $\hat{r}^2$, and re-solve. Re-setting r^1, r^2 to be a smaller step size may also be beneficial. This will produce a solution(s) with a larger violation to the problem's feasible region, however, the violation will only be with regards to the names-to-hold constraints, which is a managing characteristic. Although the method of dealing with the penalty variables is truly a heuristic, in practice H was never an issue. In addition, one may also note that as $\varrho_{s\ell}^t$ becomes sufficiently large the optimal solution in (4.68)–(4.81) is equivalent to the SMIP in (4.47)–(4.62). Blair and

Jeroslow (1981) prove this result in a theorem of their paper whereby they fundamentally state that if A, B, b, c, and d in (4.63) are rational, and assuming problem (4.63) on page 81 is bounded below in value (i.e. has an optimal solution), then if ρ becomes sufficiently large for

$$\min_{x,\,y} \{\mathrm{c}^\top x + \mathrm{d}^\top y + \rho|\mathrm{A}x + \mathrm{B}y - \mathrm{b}|, (x,\, y) \geq 0, (x,\, y) \in \mathbb{I}\} \tag{4.89}$$

the optimal solution(s) of (4.89) is equal to the optimal solution(s) of (4.63). This result is formally illustrated in Appendix A. As shown in the computations leading up to (4.68)–(4.81), we have performed the same operations as in (4.89) with the exception that we do not move all constraints into the objective function. Hence, $\{(4.68)–(4.81)\} \subset \{(4.89)\}$ and the theorem from [Blair and Jeroslow, 1981] holds.

The key step in the algorithm is step 2, where the master problem is separated into smaller subproblems. The subproblems have less variables and less constraints making them easier to solve and requiring less memory. Also, we avoid algorithmic complexities posed by non-anticipativity constraints by implicitly including them within each subproblem. One should note that this sector decomposition strategy is very specific and sensitive to the model presented in (4.47)–(4.62). It preformed best when the variable count of the subproblems was less than approximately 3500 variables. Hence, applying this algorithm to other models requires that the problem is first *MIP-separable*, and second, that one can generate small enough subproblems. Also, one may note that the penalty adjustment is of practical nature. It allows the solver to find an optimal solution in a reasonable amount of time, while using a minimal amount of memory, and only if necessary, it violates the constraints that define the number of names-to-hold in the portfolio. Coleman *et al.* (2006) demonstrate that a relaxation is necessary to solve a similar problem that contains less constraints and has a less complex objective function than the current system. In [Ruszczyński, 2005] various decomposition strategies are presented that entail cutting plane methods, nested and regularized decompositions, trust regions, and augmented Lagrangian methods. Due to the nature of our problem, the model specific decomposition is a strong approach since the subproblem reduction in size does not affect the optimization algorithm. The strategy of the decomposition algorithm in this analysis is to decompose the problem without having to relax non-anticipativity constraints. Instead a penalty-based approach is used to relax constraints that results in a natural partition into subproblems that are interpreted as sector subproblems. Each subproblem is a SMIP in its own right, which includes non-anticipativity constraints and is solved using the CPLEX MIP solver. Thus, the strategy will be especially effective when each subproblem is not too difficult to solve. In fact, Schultz (2003) lists model specific algorithm designs as further issues to consider in a paper that highlights many of the issues behind SMIP models. In [Sen, 2005] a stagewise (recourse-directive) decomposition for a two-stage SMIP and a two-scenario decomposition for a multi-stage SMIP are presented. Although both algorithm strategies are intuitive, using the proposed decomposition strategy we avoid dealing with non-anticipativity constraints that are commonly associated with high costs to CPU time; one can also refer to [Carøe and Schultz, 1999] for more information on non-anticipativity decompositions. Dentcheva and Römisch (2004) perform a very similar decomposition strategy to ours, in what they refer to as a geographical decomposition. They use a Lagrangian dual approach that provides a value for upper and lower bounds on their final solution. The approach improves the quality of the final solution, however, they are exposed to costs with respect to CPU time. In another paper – [Nürnberg and Römisch, 2002], the authors perform a similar decomposition on a power generation system and use a Lagrangian heuristic opposed to the penalty method shown above. In comparison to the proposed algorithm, the Lagrangian method provides upper and lower bound solutions to the problem, but at a cost of CPU time (primarily due to solving the additional dual problem between iterations).

Sample Problem	1	2	3	4	5
Time difference (h)	+ 1.18	+ 4.42	+ 3.45	+1.62	+3.71

Table 4.1: Difference in CPU time for each sample problem, where the + indicates the hours the decomposition penalty algorithm finished before the Lagrangian heuristic.

The proposed penalty method will have faster CPU times, but does not provide information on solution bounds. However, as mentioned above, if the penalty value becomes sufficiently large the solution is guaranteed to be optimal; see [Blair and Jeroslow, 1981]. With that said, in the next section (Section 4.4) the results provide strong solutions. As mentioned, the only violation, if any, is in the names-to-hold constraint. Hence, for our portfolio problem, the algorithm is definitely more practical with regards to CPU time and is able to capture many characteristics of financial portfolio models.

Using sample problems from Section 4.4, an algorithm similar to Nürnberg and Römisch's (2002) Lagrangian heuristic was run until comparable optimal values were obtained or the Lagrangian multipliers were finitely close in value between steps in the algorithm. With regards to CPU time, the penalty method was consistently faster than the Lagrangian heuristic. From Table 4.1, we show the difference in CPU time (hours) between the proposed algorithm and the Lagrangian approach using five sample problems, where the + indicates the additional time taken by the Lagrangian heuristic. One of the main reasons the penalty algorithm was able to outperform the Lagrangian heuristic was because of the highly specific decomposition strategy. We separate the SMIP into smaller subproblems where each subproblem is able to capture all model requirements. The Lagrangian heuristic uses a subproblem breakdown similar to [Carøe and Schultz, 1999], where the most challenging part of the algorithm is dealing with the non-anticipativity constraints and capturing $h^t_{i\ell}$. Nürnberg and Römisch (2002) report to solve larger scale problems than the ones tested, hence, their algorithm may have a greater impact on such problems. As mentioned, the Lagrangian provides an optimal upper and lower bound to the problem whereas the penalty algorithm requires sufficiently large penalty values to guarantee optimality.

4.3.2 Scenario Generation for the Index Problem

Extensive research has evolved for generating scenarios for stochastic programs, e.g. [Dupacova *et al.*, 2001; Høyland and Wallace, 2001; Pflug, 2001]. We use a reasonable method to generate scenarios for the model by employing historical market data. The motivation comes from the notion that if market behaviour and the index are representative for a specific historic time interval, then one might expect that this behaviour pattern will continue in the near future. Hence, as index values are generally increasing functions (depending on the market), one can assume that they may not deviate far from some general functional path at future time periods. Gaivoronski *et al.* (2005) use the same rationale for their portfolio selection model, whereby historic market predictions produce encouraging results for future events. Likewise, we investigate past index values and produce scenarios that are consistent with an approximation of that function plus some worst and best case historical deviations. Since we consider both, the best and worst case of past market results, one would argue that our method of generating scenarios involves a low amount of risk. We develop two types of scenarios: one for the index values $\Omega^{(\cdot)}$ and the other for the individual security values $\Psi_i^{(\cdot)}$, $\forall\, i \in \widehat{\Upsilon}$.

To begin, first we generate a trajectory of the index based on historical data, which is defined as Ω^{A}. The trajectory Ω^{A} is defined $\forall\, t \in \overline{T}$, however, for simplicity we will omit writing additional subscripts t and note that Ω^{A} covers all time periods. Then, given k

past monthly/yearly index values, for $t = \hat{t} - k$ we let

$$\omega^t = \frac{I^{t+1} - I^t}{I^t} + 1, \tag{4.90}$$

where $\hat{t}$ is the current year and I^t is the index value at time t. Then the scenario lower and upper bound deviations, Ω^{D} and Ω^{U} respectfully, become

$$\Omega^{\mathrm{D}} \quad = \quad \min\{\omega^t - \hat{q}, \forall\, t \in [\hat{t} - k, \hat{t}\,]\} \tag{4.91}$$

and

$$\Omega^{\mathrm{U}} \quad = \quad \max\{\omega^t + \hat{q}, \forall\, t \in [\hat{t} - k, \hat{t}\,]\}, \tag{4.92}$$

where $0 \leq \hat{q} \leq \frac{1}{2}$ is a user-defined scenario safeguard that may account for volatility, some type of market fluctuation, or be equal to zero. Note that for the case where $\hat{q} > 0$, the scenario generation assumes that the volatility or the market fluctuations are symmetric. The smallest problem we design consists of 3 scenarios: Ω^{A}, Ω^{D}, and Ω^{U}. Therefore, if for example $L = 3$, the index approximations will correspond to:

$$I_1^t = \Omega^{\mathrm{A}},\ I_2^t = \Omega^{\mathrm{D}}\Omega^{\mathrm{A}},\ \text{and}\ I_3^t = \Omega^{\mathrm{U}}\Omega^{\mathrm{A}} \tag{4.93}$$

$\forall\ t \in \overline{T}$. We follow the scenario tree from [Høyland and Wallace, 2001]. Any additional scenarios are generated by taking symmetric differences between $(\Omega^{\mathrm{D}}, \Omega^{\mathrm{A}})$ and $(\Omega^{\mathrm{A}}, \Omega^{\mathrm{U}})$, and if there is an odd number of additional scenarios the weighted average of the k historical index values between $(\Omega^{\mathrm{D}}, \Omega^{\mathrm{U}})$ is used for the remaining scenario. Hence, the multiplicative procedure found in (4.93) is used with $\eta\, \Omega^{\{\mathrm{D},\mathrm{U}\}}\, \forall\, \eta \in \mathbb{Q}$ for additional scenarios. One may refer to Figure 4.3 for an illustrative example of a two-stage problem where 3 scenarios evolve to 5 scenarios with regards to index approximations. Furthermore, as the large problem is decomposed into subproblems based on the sectors, the scenarios may be sensitive to each sector. Hence, the scenarios $\{\Omega^{\mathrm{D}}, \Omega^{\mathrm{A}}, \Omega^{\mathrm{U}}\}$ can be geared towards sectors $s = 1, ..., S$, which would result in scenarios $\{\Omega_s^{\mathrm{D}}, \Omega_s^{\mathrm{A}}, \Omega_s^{\mathrm{U}}\}$ for each

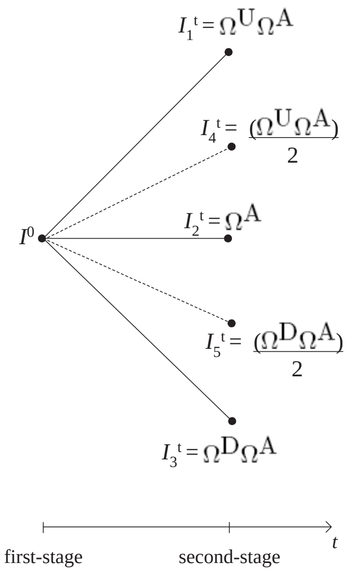

Figure 4.3: Index scenario development for the two-stage SMIP, where $L = 5$.

sector s.

For the generation of future security distributions we use a reliable approximation method accompanied by the largest upper and lower bound index deviations, Ω^{U} and Ω^{D}. Therefore, one may use a security approximation method such as the ones found in [Chen *et al.*, 2003; Masolivera *et al.*, 2000; Osborne, 1959]. After a security approximation method is set, we let Ψ_i^{A} be equal to that approximation for security i, $\forall\, i \in \widehat{\Upsilon}$. As with Ω^{A}, Ψ_i^{A} is defined $\forall\, t \in \overline{T}$, but for simplicity we will omit writing additional subscripts t and note that Ψ_i^{A} covers all time periods. Next, we follow (4.93) to generate the security distributions of the other scenarios with Ψ_i^{A} in place of Ω^{A}. Thus, using the three scenario example above, the scenario distribution for security i will correspond to

$$\phi_{i1}^t = \Psi_i^{\mathrm{A}},\ \phi_{i2}^t = \Omega^{\mathrm{D}}\Psi_i^{\mathrm{A}},\ \text{and}\ \phi_{i3}^t = \Omega^{\mathrm{U}}\Psi_i^{\mathrm{A}}, \tag{4.94}$$

$\forall\, i \in \widehat{\Upsilon}$ and $t \in \overline{T}$. Again, if any additional scenarios are generated, they follow the same multiplicative procedure as mentioned in the index case. In conclusion, one may note that as the number of scenarios involved in the problem increase so do the number of variables. In the illustration of Subsection 4.2.4, for a two-stage problem the decision variable increase is by a factor of approximately $5L$, where L is the number of total scenarios. This can have many repercussions with regards to modelling, implementation, and problem solvability. Thus, this issue must be considered when determining the number of scenarios used in the SMIP.

4.4 Portfolio Results using the TSX

We solve the problem presented in (4.47)–(4.62) over two-stages using the monthly returns of the Toronto Stock Exchange (TSX). The stochastic model accounts for every security in the TSX from December 2004 to December 2005, which amounts to a total of approximately 1,150 securities for each time period [TSX, 2009]. The S&P/TSX composite monthly total return index was set as the target portfolio. The TSX consists of 9 general sectors ranging from energy to communication & Media. The model accounted for 8 sectors ($S = 8$) as we merged consumer discretionary and consumer staples together due to their market similarities and size; however, one could easily keep them separate. Finally, we designed three market scenarios using the framework presented in Subsection 4.3.2, where an approximation developed by [Osborne, 1959] was used to define Ψ_i^{A}; a similar approximation technique is used in [Jobst *et al.*, 2005]. Also, a higher weight p_ℓ was imposed on scenario Ω^{A} than that of Ω^{U} and Ω^{D}. This was because Ω^{U} and Ω^{D} were the market "best" and "worst" case scenarios, refer to Subsection 4.3.2. With this in place, the SMIP problem we consider involves 12,685 decision variables, 12,650 of which are integer variables. The problem was solved using the algorithm described in Subsection 4.3.1 along with CPLEX 9.1 on a Pentium 4, 2.4 GHz CPU. In some cases, memory limitations became an issue, however, we have reported the best found solution when that was a factor. The algorithm improved memory allocation and CPU time when compared to using CPLEX 9.1 alone.

We present three results: the SMIP portfolio performance with respect to the actual S&P/TSX composite monthly index values, the value of the stochastic portfolio model in comparison to the corresponding deterministic dynamic model, and the portfolio index tracking performance as the number of scenarios increase. To begin, we provide the SMIP portfolio results when compared to the S&P/TSX index. We solved the portfolio model given in equations (4.47)–(4.62) for each month from December 2004 to December 2005 to compute the two-stage portfolio values over those years. Figure 4.4 provides the first-stage portfolio values when compared to the S&P/TSX index. Figure 4.5 contains the second-stage portfolio values compared to the S&P/TSX index when the best case scenarios were taken and Figure 4.6 contains the second-stage portfolio values compared to the S&P/TSX index when the worst case scenarios were taken. Figures 4.4–4.6 contain in-sample results. One may also note that the portfolios derived in the Figures posses each of the six elements they were designed to uphold. In some cases it may have been difficult to match the index values, however, the portfolio is less susceptible to market fluctuations, contains a low number of names, and performs minimal transactions. The average number of names held was 127, with a range from 74 to 180. Note that although the algorithms penalty variables caused this range, in practice, an index portfolio with under 150 names is acceptable and only on 2 occasions did the number of portfolio names exceed this value. Nevertheless, this is a trade-off of our solution approach, although the penalty is minimized by the algorithm in some cases it is not possible to find a feasible solution and stay within a strict bound on the number of names-to-hold in the portfolio. The average number of trades was 6, with a range of 0 to 35. Figure 4.7(a) depicts the best case tracking portfolio, where in the second-stage the portfolio came within 0.0341% of the index. The worst case tracking portfolio is depicted in Figure 4.7(b), where in

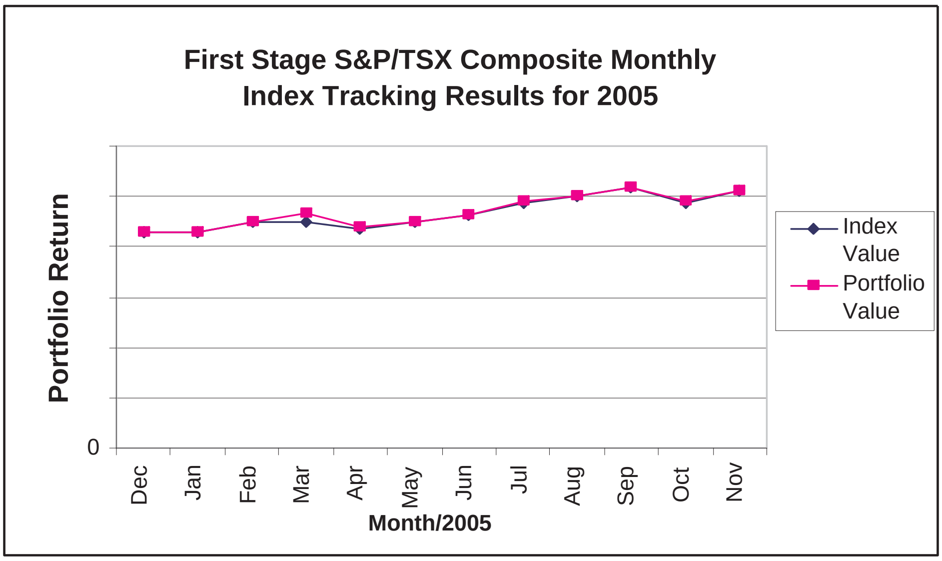

Figure 4.4: First-stage index tracking results in comparison to the S&P/TSX composite index.

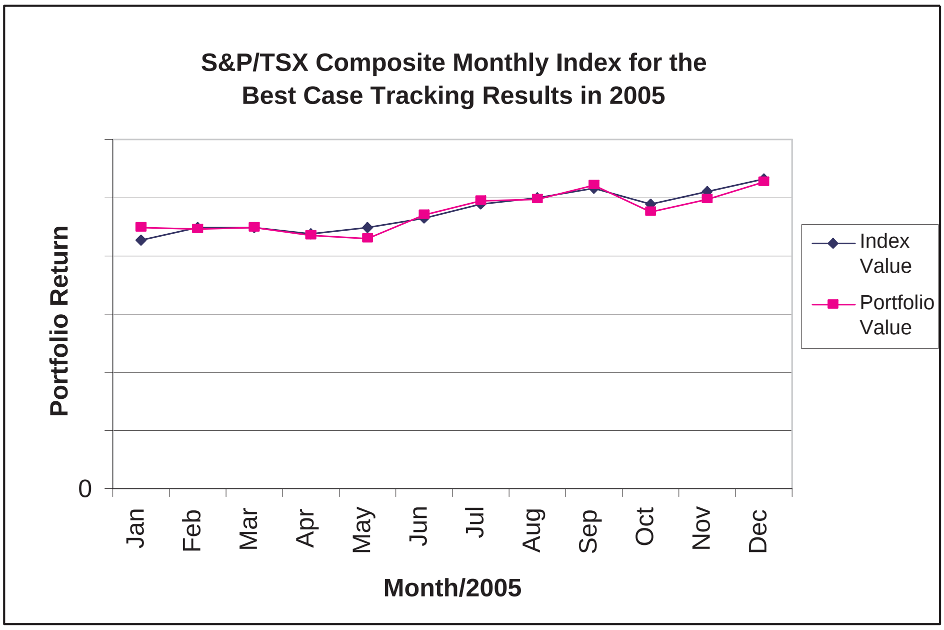

Figure 4.5: Second-stage index tracking results for the best case scenario in comparison to the S&P/TSX composite index.

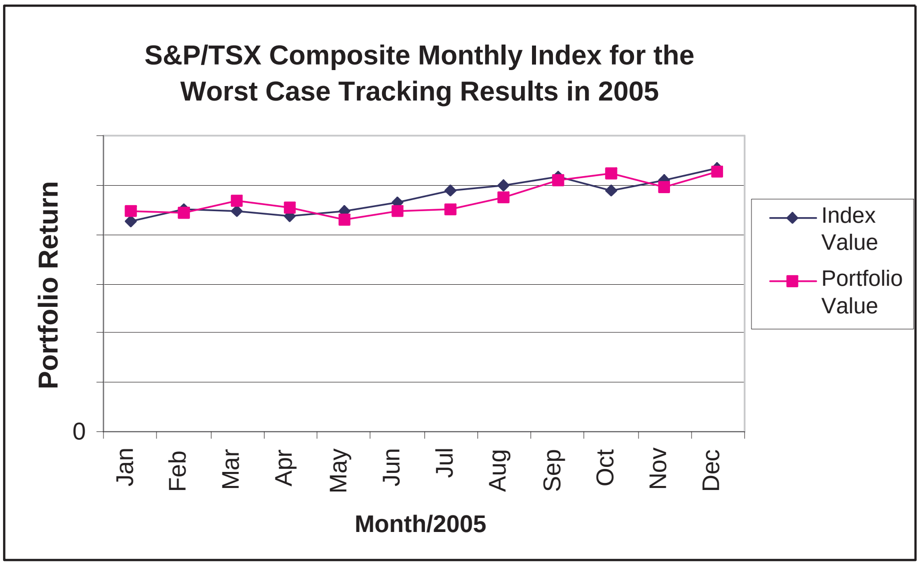

Figure 4.6: Second-stage index tracking results for the worst case scenario in comparison to the S&P/TSX composite index.

the second-stage the portfolio rose to 6.7651% above the index. The portfolio size in February–March, 2005 (Figure 4.7(a)) was 107 names and made 0 trades. The portfolio size in September–October, 2005 (Figure 4.7(b)) was 100 names and made 2 trades. The poor results in Figure 4.7(b) are due to the scenario for which an increasing index value was approximated, however, in reality the index value decreased during that time period. The decreasing index value was captured in the best case scenario and these results are shown in Figure 4.5.

With respect to the modelling approach, Figure 4.8 illustrates the results when a deterministic dynamic portfolio model is used to track the index versus that of the stochastic model. We present the months of January–February and August–September, 2005 for comparison. These were the two months where the dynamic portfolio preformed best when compared to the stochastic version. The portfolio model presented in (4.15)–(4.29) was used for the dynamic portfolio. The best dynamic portfolio performance was found

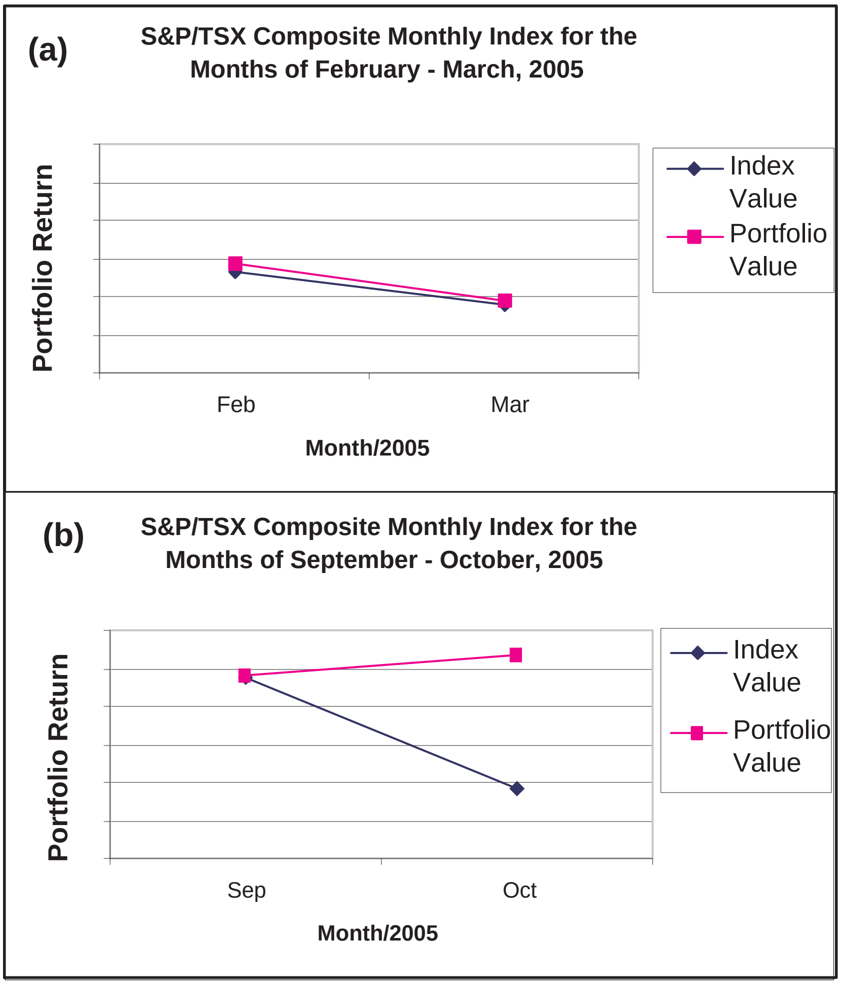

Figure 4.7: (a) The best case index tracking portfolio and (b) the worst case index tracking portfolio in 2005.

in the months of January–February, the results are show in Figure 4.8(a). The dynamic portfolio value came within 1.2907% of the index in the second-stage, however, the worst case stochastic portfolio was only 0.9628% away from the index. Also, the dynamic portfolio held 164 names versus 123 names in the stochastic portfolio case. Both the dynamic and stochastic portfolio models did not make any trades during the observed months. In Figure 4.8(b) we present the dynamic portfolio results for the months of August–September. For these months, the dynamic portfolio value came within 2.2252% of the index in the second-stage. Again, however, the worst case stochastic portfolio was only 1.5486% away from the index value. In addition, for these months the dynamic portfolio held 141 names versus 125 names in the stochastic portfolio case. The stochastic portfolio did make 13 trades in the worse case (0 trades for the best case), whereas the dynamic portfolio did not make any trades during the observed months. The positive stochastic portfolio results in both examples are due to the model considering a number of uncertain future events (scenarios) in index and security prices while making investment decisions.

Finally, we investigated the implications of including more scenarios in the model. Scenarios were added to the SMIP in (4.47)–(4.62) using the scenario generation technique presented in Subsection 4.3.2. Figure 4.9(a) provides the 7 scenario index tracking results compared to the S&P/TSX composite index and Figure 4.9(b) shows the same comparison when 11 scenarios are used. In Figure 4.9 we present the results for the months of September–October, as these months produced the worst tracking values for the 3 scenario test results. For the second-stage best case results, the 7 scenario model came within 0.3541% of the index and the 11 scenario model improved on this value with being only 0.1713% away from the index. For the best case 3 scenario model this value was 2.3356% for the same months. The portfolio worst case scenario values did not improve as significantly, this is primarily due to the method used in increasing the number

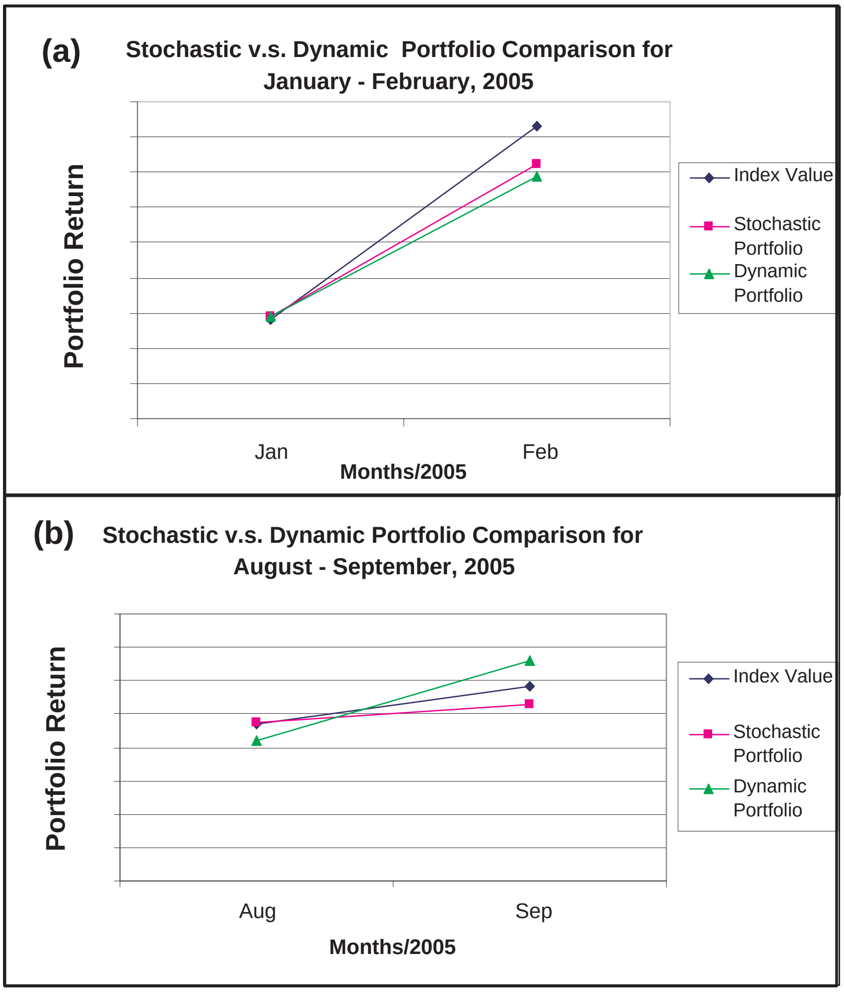

Figure 4.8: Dynamic and Stochastic portfolio comparisons with the S&P/TSX composite index for the months of (a) January–February, 2005 and (b) August–September, 2005.

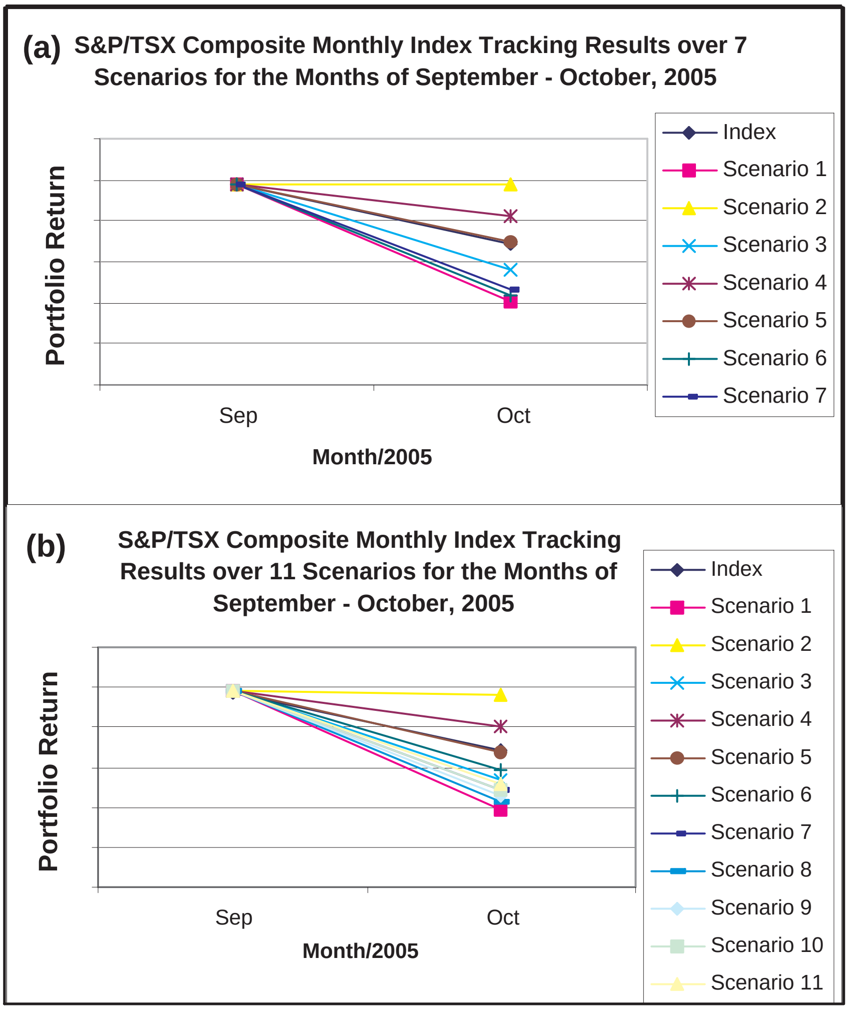

Figure 4.9: (a) 7 scenario portfolio results and (b) 11 scenario portfolio results over September-October, 2005.

of scenarios; refer to Subsection 4.3.2. Nonetheless, as the number of scenarios increased slight improvements to the worst case portfolio values are possible (approximately 1.15% for the given months). We highlight the results for the worst case 3 scenario months, September–October, however, over all comparisons as the number of scenarios were added to the model, the value of the best case portfolio became closer to the index. Hence, the 3 scenario index tracking results in Figures 4.4–4.6 can only be improved upon by adding scenarios.

To investigate the algorithms ability further, we pushed the number of scenarios to their computational limits with the results shown in Figure 4.10. As the number of scenarios become large, portfolio solutions group together and provide very similar portfolio values at a cost of higher CPU times. One can observe that the 25 scenario results in Figure 4.10(a) are in similar groupings to that of the 11 scenario results in Figure 4.9(b). Within the groupings, the optimal basis results for the different portfolio solutions are comparable. However, as the number of scenarios become extremely large, shown in Figure 4.10(b), the groupings are less prevalent and the numerous scenario possibilities span the region approximated by $(\Omega^{\mathrm{D}}, \Omega^{\mathrm{U}})$. In doing so, the best case scenario results in Figure 4.10(b) are improved and come within 0.0246% of the index, which is closer than the 11 scenario results mentioned above. However, for extremely large scenario models, due to the size of the subproblems the decomposition strategy does not perform well and algorithm CPU time is increased. Typically, CPU time rises to approximately double the amount for the 41 scenario results and memory issues arise for any scenarios higher than that. If CPU time is a concern, then 11 scenario models provide results where the best case portfolio solutions are improved in a reasonable amount of time and the optimal basis for the different scenarios are unique. The 41 scenario models provide the strongest best case portfolio results, but at the highest cost of CPU time and can have a similar optimal basis for various scenarios. Nonetheless, in any of the higher scenario results, Figures 4.9–4.10, the worst case portfolio values are similar to the what is shown in the 3 scenario models of Figures 4.4–4.6. Hence, as illustrated in Figures 4.4–4.6, strong portfolio results are captured with fast CPU times that can only be improved upon using the scenario generation method presented in Subsection 4.3.2.

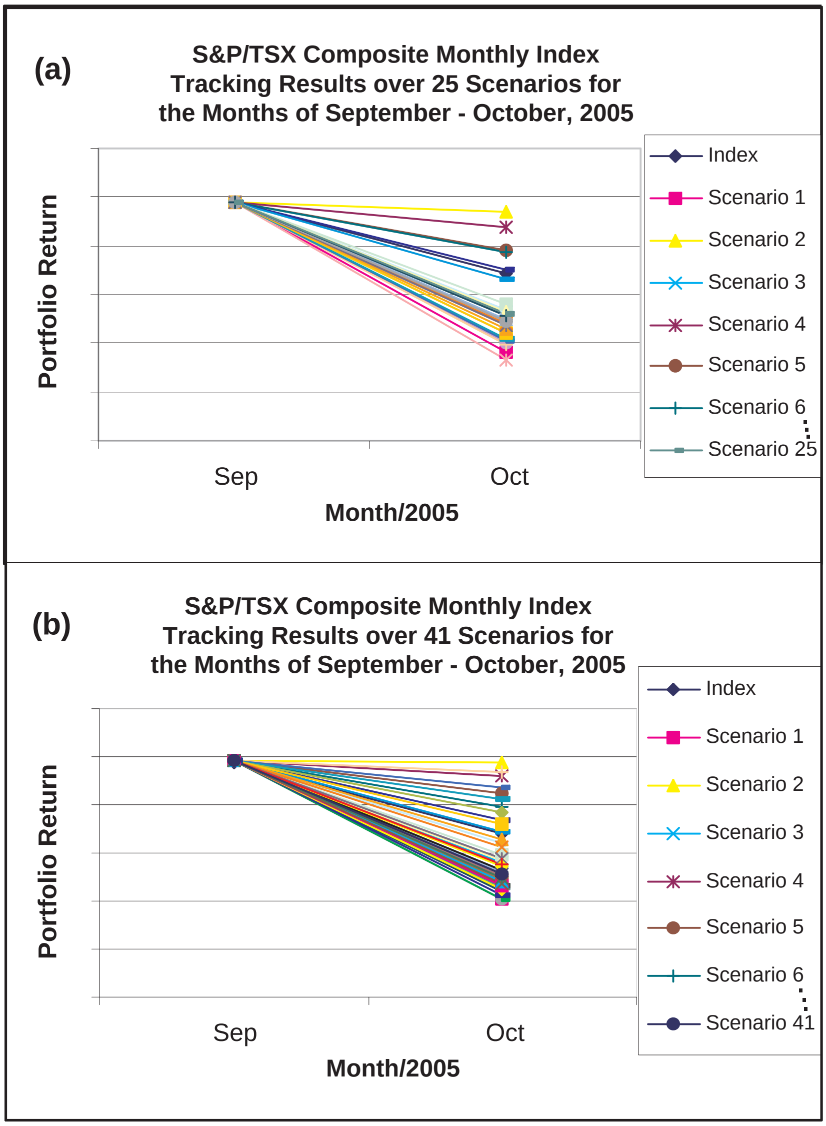

Figure 4.10: (a) 25 scenario portfolio results and (b) 41 scenario portfolio results over September-October, 2005.

4.5 Result and Model Discussions

We provide an advanced optimization approach to formulating an index tracking portfolio model. The SMIP development produced very respectable results while capturing the different characteristics of an index tracking portfolio. From Section 4.4, we have obtained competitive tracking values that we feel would meet most portfolio managers' requirements. The portfolio results generated have done so while upholding the six realistic and practical elements we mentioned in Section 4.1. Also, we have shown that under our framework stochastic portfolio modelling can outperform a dynamic model and this value can be improved upon by increasing the number of scenarios. One of the reasons that the stochastic portfolio was stronger than the deterministic dynamic model was because the non-anticipativity constraints forced the portfolio to make investments while simultaneously considering many future uncertain scenario events. As our contribution was primarily focused on the formulation of index portfolios, we applied a well-known method of generating individual security return values. If other highly sophisticated methods of generating security return values were used, then it is possible that even stronger portfolio results may be obtained.

Although there are many sources for producing poor index tracking results, from Section 4.4 we trust that the approach minimizes such instances and encapsulates market direction. The only trade-off of the solution approach is with respect to the number of names-to-hold constraint. For some instances, this constraint had to be relaxed to find a feasible solution; however, in such cases the relaxation was minimized by the algorithm. One may also note, the algorithm developed in Subsection 4.3.1 can be parallelized. In which case, if the number of processors was greater than or equal to the number of subproblems, then CPU time would be equal to the time it takes to run the slowest subproblem. Thus, running a parallel code can have an even greater impact with regards to our implementation strategy. We will continue such research discussions and future directions in the conclusion of Chapter 6. With the index tracking portfolio in place, a Goal Programming (GP) extension to develop an index-based portfolio selection model that involves a larger array of investing instruments can now be introduced; which we present in the next chapter.

Chapter 5

Goal Programming and Portfolio Selection

5.1 Introduction to the Current Portfolio Problem

Recent economic turmoil and market uncertainties have created a world of financial problems and prompted investors to guard their capital across the globe. In fact, it is one of the first times in history when economies have been affected worldwide. Banks and other financial institutions add to the complexity of the problem by introducing new investment products and derivatives. This has caused many money managers to become careful financial planners and develop safe, structured portfolios as their attempts to outperform the market frequently fail. In Chapter 1 we mention historic times where similar events have occurred. The early 1980s is an instance where North America fell into a recession due to comparable causes. At that time there was volatility across all markets, even bond prices fluctuated dramatically. Currently, bond investments seem to be reasonable financial options for many money managers due to their steady cash flow. Bonds differ from securities primarily in that they have fixed maturities and bond prices are based on the future stream of interest rates, as shown in Subsection 2.1.1. Typically, bonds have a degree of uncertainty that posses an inverse relationship with that of securities. Due to the status of the economy, security-bond relationships, price movements of historical recessions, and recent investment trends, an investigation involving a security-bond portfolio selection model seems intuitive and of great relevance. From the past, there exists compelling evidence that long-term investments are one of the best financial strategies. In this chapter, we extend the portfolio design from Chapter 4 to develop a portfolio selection model that pursues a long-term strategy while considering security and bond investments. As we have presented earlier, such portfolios are difficult to construct and are computationally demanding. A modeler faces the trade-off between trying to capture numerous realistic portfolio elements and solving the problem to optimality. Issues that have plagued this document, such as the number of time-stages, type of portfolio elements, number of portfolio elements, and abundant uncertainties involved in the problem, require great consideration.

As we mentioned earlier, in 1952 Harry Markowitz introduced one of the first portfolio selection models. Shown in Chapter 3, the MVO generates portfolios that are limited by a single period approach, has implementation issues, and as it turns out risk and covariance were not equivalent measures. Since his initial works, many papers have been published on various portfolio selection models, but few that involve a combination of security and bond investments. Some may argue that bonds and securities have different properties and should be treated independently; however, from the past we see that both investments have degrees of volatility that may be inversely related. Several years after Markowitz's publication, Konno and Kobayashi (1997) proposed a short-term security-bond MVO model that was one of the first attempts at investigating such an integrated portfolio design. The model is subject to similar critiques to that of Markowitz, however, their results show that security-bond portfolios have promising results and further investigations are valid. With regards to publications that involve an assortment of instruments, Asset and Liability Management (ALM) is an area that contains model rich applications involving complex investing instruments. For example, [Zenios, 1995] and [Escudero *et al.*, 2007] provide ALM models for mortgage-backed securities. Unlike portfolio selection, however, their asset and liability models have the additional requirements of meeting financial obligations and other design-specific constraints. We present a portfolio selection model that considers various investment options and is composed of a set of practical portfolio elements.

From the literature on portfolio selection, there exists a variety of publications that consider portfolio elements over one type of investment. In [Tapaloglou *et al.*, 2008], the authors minimize the conditional value-at-risk of portfolio losses while considering various portfolio rebalancing and managing elements. As mentioned in Chapter 4, Bertsimas *et al.* (1999) define important elements of an index-tracking portfolio and produce a multi-objective mixed-integer programming model. The portfolio model we construct involves similar financial elements considered in [Bertsimas *et al.*, 1999; Markowitz, 1952; Zenios, 1995; Tapaloglou *et al.*, 2008], however, the design incorporates security and bond investments. In addition, the uncertainty associated with recent financial markets is captured in an underlying SMIP approach with recourse. From last chapter and as pointed out by Sodhi (2005), SP models with recourse have the obstacle of: (a) having to choose from a number of modelling characteristics or mathematical attributes, and (b) solving models that involve a large amount of variables; due to scenarios and variables introduced in (a). In [Gaivoronski and De Lange, 2000], the authors define (a) and (b) to be related, and hence increasing the degree of flexibility in the decision process lead to increasing the number of decision variables. This then leads to computational trade-offs and other limitations. Hence, rather than limiting the number of variables in their stochastic model, they define a number of decision rules to reduce the number of stochastic scenarios in the large decision tree, known as fix-mix decision rules. Mulvey and Vladimirou (1992) were one of the first to provide an approach to a large-scale financial planning problem involving a multiscenario stochastic program. In an attempt to reduce solution time they take advantage of their model's structure, which leads to the introduction of many complicating constraints. The large-scale financial approach we propose will be similar to that of [Mulvey and Vladimirou, 1992] in that we design a model-specific algorithm to provide viable solutions. The algorithm, in the simplest sense, is composed of a decomposition strategy and iterative penalty procedure. However, to handle the numerous portfolio elements included in the design and overcome SP obstacles mentioned by [Sodhi, 2005] or limitations in [Gaivoronski and De Lange, 2000], a Goal Programming (GP) approach is incorporated in the SMIP model. This requires more finesse with respect to algorithm design and, to the best of our knowledge, is one of the first financial models of its nature. The inclusion of GP to the SMIP has two main advantages with respect to large-scale portfolio design. Firstly, from the practical side, GP allows the model to capture different portfolio objectives in a manner that maximizes optimal settings. Secondly, GP permits relaxations of the large-scale optimization problem (if necessary), which has great relevance with respect to minimizing computational issues. These advantages, however, come with the cost of dimensionality, as GP increases the number of continuous variables in the problem. To facilitate this trade-off, we design a model specific algorithm that minimizes such issues through a decomposition strategy.

To date, there has been a small number of financial publications involving Stochastic-Goal Programming (SGP). The authors of [Ji *et al.*, 2005] investigate one of the most relevant financial models to what we propose using a stochastic-goal linear programming approach. Their portfolio design involves a rebalancing strategy over risky assets and a risk-free asset, where portfolio goals are set to match a target value. The only goal in the problem involves a target wealth constraint in which they include an investigation on the size of the relaxation parameters associated with this goal. The portfolio design is composed of a few portfolio characteristics, where they avoid complexities posed by integer variables. Ballestero (2005) provides another publication that considers a financial SGP problem. He presents the stochastic-goal programming framework to structure a mean-variance optimization model, where computational investigations and portfolio results are not provided. In [Muhlemann *et al.*, 1978], the authors develop a multi-objective stochastic programming model that considers two portfolio goals. Although the problem size is small, it is one of the first portfolio investigations involving uncertainty and multiple objectives. Aouni *et al.* (2005) introduce a SGP model characterized by imperfect information where the goal values associated with the objectives are considered by the decision maker's preference. They solve the problem through Chance Constrained Programming (CCP), which is a method to solve GP models and was developed by [Charnes and Cooper, 1959]. The authors of [Ben Abdelaziz *et al.*, 2007] also use CCP to solve a multi-objective SP using a method they call chance constrained compromise programming. In the sections below, we present a security-bond portfolio selection model that uses a Stochastic-Goal Mixed-Integer Programming (SGMIP) approach to manage portfolio objectives and numerous sources of uncertainty. We develop a model that considers many practical portfolio constraints and additional financial characteristics that have not been included in previous publications. This increases the complexity of the problem, however, solvability issues are addressed though a model specific algorithm. Although Goal Programming was originally proposed by Charnes and Cooper in 1957 (later formalized in the text [Charnes and Cooper, 1961]), stochastic-goal programming investigations arose some time later. SGP methods have been generalized and examined in various publications [Heras and Aguado, 1999; Sengupta, 1979; Van Hop, 2007]. Some SGP applications can be found in production, management, and industrial problems, such as [Al-Zahrani and Ahmad, 2004; Ballestero, 2000; Bravo and Gonzalez, 2009], however, there are limited publications involving financial models. Hence, the financial SGMIP approach we present in this chapter is an innovative contribution to the field.

The chapter is organized as follows: in Section 5.2 we define the SGMIP portfolio model, the subproblems considered by the GP, and the scenario generation procedure. In Section 5.3, we describe the algorithm used to solve the SGMIP and include additional implementation issues such as the initial solution strategy. Finally, in Section 5.4 we present the results and performance of the SGMIP with respect to the current economic market, and include a discussion of our findings in Section 5.5.

5.2 Stochastic–Goal Programming Model

In this section we outline the elements involved with the portfolio selection model and present the final SGMIP portfolio design. The combination of GP and SMIP requires additional model specifications, which are also introduced in this section.

5.2.1 Goal Programming Problem

Before defining the SGMIP portfolio problem we will outline the formulation behind the GP aspect of the design; for the SP elements one can refer to Subsection 4.2.3 on page 74. GP is an intuitive technique used to manage multi-objective optimization problems. The modelling stems from a problems design in which each of the objectives, or goals, are fundamentally solved separately and then combined in a large problem that attempts to meet each one. Specifically, the solutions to each of the objective subproblems become

Attributes	Objective 1	Objective 2	Objective 3	Constraints
Attribute 1	c_1^1	c_1^2	c_1^3	a_1
Attribute 2	c_2^1	c_2^2	c_2^3	a_2
Upper bound				b

Table 5.1: GP example with multi-objective parameters

a constraint in a large optimization problem that are attached to relaxation variables, which are minimized in the objective function. The general formulation of a GP problem is the following:

$$\min \quad (\varrho^+)^\top \gamma^+ + (\varrho^-)^\top \gamma^- \tag{5.1}$$

$$\text{s.t.} \quad f(x) + \gamma^+ - \gamma^- = z \tag{5.2}$$

$$\mathrm{A}x = \mathrm{b} \tag{5.3}$$

$$x \geq 0 \tag{5.4}$$

$$\gamma^+, \gamma^- \geq 0, \tag{5.5}$$

where $x = x_1, ..., x_n$ is the vector of decision variables, $\varrho^+, \varrho^- \in \mathbb{R}^m$ are goal weighting parameters, $f(x) = \sum_{j=1}^n c_{ij} x_j \;\; \forall \; i = 1, ..., m$ are the goals associated with objective i, and $\gamma^+, \gamma^- \in \mathbb{R}^m$ are the positive and negative deviations from goals $z \in \mathbb{R}^m$; respectively. To illustrate the approach, say that we are given the simple multi-objective problem shown in Table 5.1. As shown, the example consists of 2 attributes, 2 constraints, an upper bound, and 3 objectives. If we let x_1 and x_2 be the decision variable for attributes 1 and 2 respectively, then the optimization problem for objective 1 only, is the following:

$$z_1^\star = \max \quad \mathrm{c}_1^1 x_1 + \mathrm{c}_2^1 x_2 \tag{5.6}$$

$$\text{s.t.} \quad \mathrm{a}_1 x_1 + \mathrm{a}_2 x_2 \leq \mathrm{b} \tag{5.7}$$

$$x_1, x_2 \geq 0. \tag{5.8}$$

Similarly, if one were to only consider the second objective, the problem becomes:

$$z_2^\star = \max \quad \mathrm{c}_1^2 x_1 + \mathrm{c}_2^2 x_2 \tag{5.9}$$

$$\text{s.t.} \quad \mathrm{a}_1 x_1 + \mathrm{a}_2 x_2 \leq \mathrm{b} \tag{5.10}$$

$$x_1, x_2 \geq 0 \tag{5.11}$$

and the third objective solves the same problem as in (5.6)–(5.8) and (5.9)–(5.11) using the third column of Table 5.1, which provides the optimal value $z_3^\star$. Next, $z_1^\star$, $z_2^\star$, and $z_3^\star$ are used with their respective objective functions as constraints in a larger problem that also introduces relaxation variables and parameters. Given the relaxation variables for each objective is $\gamma_i \ \forall \ i = 1, 2, 3$ and the parameters are ϱ_i, then the GP problem is the following:

$$\min \quad \varrho_1\gamma_1 + \varrho_2\gamma_2 + \varrho_3\gamma_3 \tag{5.12}$$

$$\text{s.t.} \quad \mathrm{c}_1^1 x_1 + \mathrm{c}_2^1 x_2 = z_1^\star - \gamma_1 \tag{5.13}$$

$$\mathrm{c}_1^2 x_1 + \mathrm{c}_2^2 x_2 = z_2^\star - \gamma_2 \tag{5.14}$$

$$\mathrm{c}_1^3 x_1 + \mathrm{c}_2^3 x_2 = z_3^\star - \gamma_3 \tag{5.15}$$

$$\mathrm{a}_1 x_1 + \mathrm{a}_2 x_2 \leq \mathrm{b} \tag{5.16}$$

$$x_1, x_2 \geq 0 \tag{5.17}$$

$$\gamma_1, \gamma_2, \gamma_3 \geq 0. \tag{5.18}$$

The constraints from each subproblem are included in (5.16)–(5.17) above. Equations (5.13)–(5.15) are aimed at meeting each of the subproblem objectives, where if necessary the value is relaxed using γ_i. The values of ϱ_i can then be used to reduce instances where γ_i are very large or small with respect to the other relaxation variables. Also note that since the subproblems (i.e. (5.6)–(5.8) and (5.9)–(5.11)) produce maximization values, then $z_1^\star$, $z_2^\star$, and $z_3^\star$ are the upper bounds of constraints (5.13)–(5.15), and hence, the associated relaxation variables only need to be subtracted from these values. Therefore, the GP formulation in (5.12)–(5.18) above is designed such that each of the individual objectives are satisfied unless relaxations are absolutely necessary, in which case any deviations from the optimal subproblem solutions are minimized. For more information on GP the reader may refer to [Barichard *et al.*, 2009; Charnes and Cooper, 1961; Tanino *et al.*, 2003].

GP is very practical with respect to financial problems and in the next sections we will show how the modelling can be used to minimize various portfolio risks and money managing characteristics. Before doing so, we outline the integration of SP and GP in developing the SGMIP. Following the notation of Subsection 4.2.3 on page 74, given there are $\omega \in \Omega$ random events and $y(\omega)$ is the second stage decision variable, then the general Stochastic-Goal Programming (SGP) problem is:

$$\min \quad (\varrho^+)^\top \gamma^+ + (\varrho^-)^\top \gamma^- + (\varsigma^+)^\top \delta^+ + (\varsigma^-)^\top \delta^- \tag{5.19}$$

$$\text{s.t.} \quad f(x) + \gamma^+ - \gamma^- = z \tag{5.20}$$

$$Q(y) + \delta^+ - \delta^- = z(w) \tag{5.21}$$

$$\mathrm{A}x = \mathrm{b} \tag{5.22}$$

$$x \geq 0 \tag{5.23}$$

$$\gamma^+, \gamma^- \geq 0 \tag{5.24}$$

$$\delta^+, \delta^- \geq 0 \tag{5.25}$$

where $\varsigma^+, \varsigma^- \in \mathbb{R}^p$ are goal weighting parameters, $Q(y) = \mathrm{E}[\min\{q_\nu(\omega)^\top y(\omega) | W(\omega) = h(\omega) - T(\omega)x,\ y(\omega) \geq 0\}]\ \forall\ \nu = 1, ..., p$ are the goals associated with the second-stage objective ν, and $\delta^+, \delta^- \in \mathbb{R}^p$ are the positive and negative deviations from second-stage goals $z(\omega) \in \mathbb{R}^p$; respectively. Now say we were to add a second-stage and uncertainty to the problem in (5.12)–(5.18). Then for any realization ω we define

$$Q_\nu(x_1, x_2) = \mathrm{E}[\min_y\{q_\nu(\omega)^\top y | Wy = h(\omega) - T(\omega)(x_1, x_2), y \geq 0\}] \tag{5.26}$$

and given $p = 3$, the Stochastic-Goal Deterministic Equivalent Program (SGDEP) becomes:

$$\min \quad \sum_{i=1}^{3} \varrho_i \gamma_i + \sum_{\nu=1}^{3} \varsigma_\nu^+ \delta_\nu^+ + \varsigma_\nu^- \delta_\nu^- \tag{5.27}$$

$$\text{s.t.} \quad \mathrm{c}_1^i x_1 + \mathrm{c}_2^i x_2 = z_i^\star - \gamma_i \qquad \forall\, i = 1, 2, 3 \tag{5.28}$$

$$Q_\nu(x_1, x_2) + \delta_\nu^+ - \delta_\nu^- = z(\omega)_\nu \qquad \forall\, \nu = 1, 2, 3 \tag{5.29}$$

$$\mathrm{a}_1 x_1 + \mathrm{a}_2 x_2 \leq \mathrm{b} \tag{5.30}$$

$$x_1, x_2 \geq 0 \tag{5.31}$$

$$\gamma_1, \gamma_2, \gamma_3 \geq 0 \tag{5.32}$$

$$\delta_1, \delta_2, \delta_3 \geq 0. \tag{5.33}$$

Thus, (5.27)–(5.33) has the added objective of optimizing relaxations associated with goals ν while satisfying the second-stage constraints in (5.26). The solution to (5.27)–(5.33) provides the optimal value for the GP relaxation values, in which case the optimal solution for the different objectives can be derived from the goals $z_i^\star$, $z(\omega)_\nu$. Although the SGDEP has increased the complexity of solving a SP and GP independently, the SP solution can be generated easily and weights can be adjusted for each portfolio goal. The authors of [Aouni *et al.*, 2005; Ji *et al.*, 2005; Sengupta, 1979] provide investigations of the parameters and relaxations involved with SGP problems. As mentioned in Section 5.1, GP was developed by [Charnes and Cooper, 1961] and further investigated by [Ijiri, 1964]. Heras and Aguado (1999) provide the framework and generalizations for SGP and [Contini, 1968] present one of the first stochastic approaches to GP, which involves a QP formulation. In our developments below we further constrain the problem to involve integer decision variables in the SGMIP, which is shown in the next section.

5.2.2 SGMIP Model Formulation

To begin, we describe the decision variables related to the various assets in the portfolio, namely: securities, bonds, and riskless investments. We will maintain similar variable definitions to what was used in Chapter 3 and 4, however, some will change slightly to meet the requirements of this design. Thus, we define x_i to be the fraction of the portfolio invested in security i that is purchased in the first-stage, where there is a total of n securities. We let $y_{i\ell}^t$ be the fraction of the portfolio invested in security i that is invested in the second-stage ($t > 0$), where there is a total of m time periods; and again for simplicity we set $T = m-1$. In addition, the model captures future market uncertainty by expressing possible future economic outcomes as scenarios, where $\ell = 1, 2, ..., L$ represents the scenario and L is the total number of scenarios used in the model. One can refer to Subsection 4.2.4 for more information on why ℓ is added to particular variables. In this section we save such discussions as it has already been outlined in Chapter 4. Hence, $i = 1, 2, ..., n$, $t = 0, 1, ..., T$, and $\ell = 1, 2, ..., L$, where $x_i \in \mathbb{R}$, and $y_{i\ell}^t \in \mathbb{R}$. Later, in Subsection 5.2.4 we discuss how to generate viable future scenarios and their effect on the final solution. The unit price of security i at time $t = 0, 1, ..., m$ under scenario ℓ will be captured by $\phi_{i\ell}^t$, where the price is known at $t = 0$ and note that there is only one scenario in the first-stage. Next, we let $z_{j\ell}^t$ be the fraction of the portfolio invested in bond j to purchase at time t under scenario ℓ, hence $z_{j\ell}^t \in \mathbb{R}$. We include a total of h different types of bonds in the model, $j = 1, 2, ..., h$, with their respective coupons and maturities embed. In addition, we set $h_j^\star$ to indicate the time to maturity for each bond j. The price of bond j at time t under scenario ℓ will be represented by $\varphi_{j\ell}^t$ and its respective return at maturity will be stored in $\psi_{j\ell}^t$. Since there is only one scenario in the first-stage, $\ell = 1$ for $z_{j\ell}^0$ and $\psi_{j\ell}^0$, and at time $t = 0$ we will omit writing ℓ. Next, we define B_ℓ^t to be the amount invested in a riskless or near–riskless investment that incurs a small cost of ς percent over a specified time period. Hence, $B_\ell^t \in \mathbb{R}$ represents the near-riskless investment at time t under scenario ℓ, where there is only one scenario in the first-stage ($\ell = 1$ for B^0_ℓ). Also, we define $\widehat{B}$ as the initial wealth of the portfolio. Finally, the model we design may also be applied and extended to ALM problems, in which case F^T_{obg} will represent the liabilities involved in the model that require the portfolio to meet a terminal financial obligation. The financial obligation may be any number of entities ranging from pension fund payment, customer investment pay-outs, or insurance claims. This equates to maximizing the following equation

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{i=1}^{n}\phi^1_{i\ell}\,x_i \;&+\; \sum_{t=1}^{T}\sum_{\ell=1}^{L}\sum_{i=1}^{n}p_\ell\,\phi^{t+1}_{i\ell}y^t_{i\ell} + \sum_{t=1}^{T}\sum_{\ell=1}^{L}\sum_{j=1}^{h}p_\ell\,\psi^t_{j\ell}\,z^{t-h^\star_j}_{j\ell} \\ &+\; \sum_{t=1}^{T}\sum_{\ell=1}^{L}p_\ell\,\varsigma B^t_\ell + \varsigma B^0 - F^T_{\text{obg}},\end{aligned} \tag{5.34}$$

where p_ℓ denotes the probability of a scenario realization, and hence, $\sum_{\ell=1}^{L} p_\ell = 1$ and $p_\ell \geq 0$. Since, we are interested in a portfolio selection design we will let $F^T_{\text{obg}} = 0$ for the remainder of this chapter; however, the reader should note that the model can be extended for an ALM framework given $F^T_{\text{obg}} > 0$.

In addition to maximizing equation (5.34), the portfolio model is designed to take a passive investment strategy while upholding practical portfolio elements from various types of investments. The important portfolio elements the model will consider are as follows:

(i) Minimize risk;

(ii) Minimize portfolio management costs;

(iii) Maximize liquidity;

(iv) Stay within a tolerance of a market performance measure;

(v) Hold a small number of investments.

The reasons for including elements (i) and (ii) should be obvious. Element (iii) is included because we are constructing a long-term portfolio investment, where bonds and securities having a small number of issuing shares may have liquidity issues. Hence, if portfolio rebalancing is necessary, maximizing liquidity favors such an event. Element (iv) ensures that the model is not deviating from a market index or performance measure, which is a common passive investment strategy. Finally, element (v) is present in all portfolio designs thus far and is of great practical value, which is a requirement of portfolio managers. As will be presented in this section, these elements define a multi-objective SMIP in which a GP approach is applied to satisfy the objectives. Using the three different investment options defined above, we will now describe how elements (i)–(v) are captured in the portfolio selection model.

We begin by defining the elements that pertain to each investment type followed by the elements that span multiple types of investments. Some constraints are similar to what is shown in Chapter 4, however, because of the introduction of new decision variables and old ones that have changed slightly, we explicitly define all constraints. First we describe how the model captures security investments related to elements (i)–(v). This pertains to variables associated with security investments x_i and $y^t_{i\ell}$. The model aims to keep portfolio managing fees to a minimum, which involves minimizing transaction costs and therefore minimizing the number of transactions between time periods. Thus, we define $\varpi^t_{i\ell}$ to be such that

$$\varpi^t_{i\ell} = |y^t_{i\ell} - y^{t-1}_{i\ell}| \qquad \forall\, i = 1, ..., n,\ t = 2, ..., T,\ \ell = 1, ..., L, \tag{5.35}$$

and for $t = 1$

$$\varpi^1_{i\ell} = |y^1_{i\ell} - x_i| \qquad \forall\, i = 1, ..., n,\ \ell = 1, ..., L; \tag{5.36}$$

where $\varpi^0_{i\ell} = 0\ \forall\, i, \ell$. Thus, $\varpi^t_{i\ell}$ represents the fraction of a security that is bought or sold between time periods. This entity will be minimized in the objective function to reduce the number of transactions and therefore keep portfolio costs to a minimum. Secondly, our portfolio will be sector diversified. Diversity will reduce the risk of the portfolio model by acting as a safeguard towards poor sector development, which can be a result of a number of events. As in Section 4.2, we incorporate the sector exposure element by first introducing the variable $Q(i, s)$, which defines the sector s security i belongs to. Hence, we set

$$Q(i, s) \quad = \quad \begin{cases} 1, & \text{if security } i \text{ is in sector } s; \\ 0, & \text{otherwise}; \end{cases} \tag{5.37}$$

where there is a total of S sectors and $Q(i, s) \in \mathbb{B}$. By letting f_s^t be the fraction of the portfolio that is invested in sector s at time t, we can set our portfolio to have the appropriate sector diversification. Since f_s^t is a fractional value, we have that

$$\sum_{s=1}^{S} f_s^t = 1 \qquad \forall\, t = 0, ..., T, \tag{5.38}$$

and $f_s^t \in [0, 1]$. In addition, as defined earlier f_s^0 is a first-stage parameter, whereas for $t > 0$ f_s^t is a second-stage parameter. The sector exposure element is the same as in the model of Chapter 4, which forms the following constraint:

$$\sum_{i=1}^{n} Q(i, s)\phi_{i\ell}^t y_{i\ell}^t = f_s^t \sum_{i=1}^{n} \phi_{i\ell}^t y_{i\ell}^t + \xi_{s\ell}^t \quad \forall\, s = 1, ..., S,\ t = 1, ..., T,\ \ell = 1, ..., L, \tag{5.39}$$

where $\xi_{s\ell}^t$ is a sector penalty variable that corresponds to f_s^t under any scenario $\ell = 1, ..., L$. Again, $\xi_{s\ell}^t$ permits the model to find a more suitable value for f_s^t if a feasible solution cannot be generated under the given variable assignments. Note, this constraint will also be applicable for x_i in place of $y_{i\ell}^t$, as it is shown explicitly in equation (5.63) below. Finally, we bound the number of different securities and bonds the portfolio uses. Thus, we define

$$g_{i\ell}^t = \begin{cases} 1, & \text{if security } i \text{ is used in the portfolio at time} \\ & t \text{ under scenario } \ell \text{ (i.e. if } x_i, y_{i\ell}^t > 0); \\ 0, & \text{otherwise;} \end{cases} \tag{5.40}$$

where $g_{i\ell}^t \in \mathbb{B}$ and in the first-stage there is only one scenario ($\ell = 1$ for $g_{i\ell}^0$). As earlier, G^t is the upper bound on the number of securities to hold in the portfolio and the cardinality constraint corresponding to limiting the securities to hold in the portfolio is:

$$\sum_{i=1}^{n} g_{i\ell}^t \leq G^t \qquad \forall\, t = 0, ..., T,\ \ell = 1, ..., L. \tag{5.41}$$

The same follows for the number of different bonds to use in the portfolio. Hence, we bound the number of bonds in the portfolio by letting

$$\tilde{g}_{j\ell}^t = \begin{cases} 1, & \text{if bond } j \text{ is used in the portfolio at time} \\ & t \text{ under scenario } \ell \text{ (i.e. if } z_{j\ell}^t > 0); \\ 0, & \text{otherwise;} \end{cases} \tag{5.42}$$

and impose the following constraint:

$$\sum_{j=1}^{h} \tilde{g}_{j\ell}^t \leq \widetilde{G}^t \qquad \forall\, t = 0, ..., T,\ \ell = 1, ..., L, \tag{5.43}$$

where $\widetilde{G}^t$ is the upper bound on the number of bonds to hold in the portfolio and $\tilde{g}_{j\ell}^t \in \mathbb{B}$ with $\ell = 1$ at $t = 0$. Using these elements, plus the addition of a performance measure and various portfolio goals, we capture the long-term investment characteristics of the portfolio selection model.

Next we introduce the portfolio elements that span over various investments. First we define a performance measure that the model can not drastically outperform. The reason for including such a characteristic is to ensure that the portfolio follows a passive investment strategy. Such a strategy implies that investments that behave similar to a market index minimize portfolio risk, and vice versa. In order to do so, we set R_ℓ^t to be a maximum benchmark that the portfolio must not outperform under scenario ℓ and at time t. The portfolio benchmark R_ℓ^t may be the market index, however, this assumption is only fair if the original portfolio investment is at least equal to the present index value. The value of R_ℓ^t is derived in a separate subproblem that will be described in the next section. The constraint in the model is as follows:

$$\sum_{i=1}^{n} \phi_{i\ell}^{t+1} y_{i\ell}^{t} \leq R_\ell^t + \chi_\ell^t \qquad \forall\, t = 0, ..., T,\ \ell = 1, ..., L, \tag{5.44}$$

where $\chi_\ell^t \geq 0$ is a relaxation that satisfies our GP approach, $\chi_\ell^t \in \mathbb{R}$, and $\ell = 1$ for R_ℓ^0. Also, in the first-stage (5.44) will be applied to x_i, where only one scenario is considered. As well, we only constrain the performance of our security investments, which allows the portfolio to invest in bonds when stock market investments are not favorable. Historically, it has been shown that securities and bonds tend to have an inverse relationship with respect to their index values. Hence, when coupled with other portfolio goals, such a constraint exploits this characteristic. This will be shown in the results section (Section 5.4), where for instances when the performance measure representing the market index is decreasing, this constraint together with the objective function forces the model to consider bond investments. Next, we consider the risk associated with individual security and bond investments. First we describe how security risk is minimized. The *beta coefficient*, common to financial literature [Bodie *et al.*, 2005] and defined in Chapter 2, is a risk measure that relates a single securities risk relative to the overall stock market. The risk associated with the stock market is defined in an index portfolio I_M^t at time t, i.e. the S&P TSX Composite Index. Hence, β_i corresponds to the beta value of security i as follows:

$$\beta_i \;=\; \frac{\mathrm{Cov}(\hat{\phi}_i, I_M)}{\sigma_M^2}, \tag{5.45}$$

where

$$\begin{aligned} \mathrm{Cov}(\hat{\phi}_i, I_M) &= \frac{1}{h_p}\sum_{t=-h_p}^{0}(\hat{\phi}_i^t - \bar{\phi}_i)(I_M^t - \overline{I}_M), &(5.46)\\ \sigma_M^2 &= \mathrm{Var}(I_M), &(5.47)\end{aligned}$$

and $\hat{\phi}_i$ refers to h_p known historical price movements of security i, and $\bar{\phi}_i$ and $\overline{I}_M$ are the arithmetic mean of $\hat{\phi}_i^t$ and I_M^t from time $t = -h_p, ..., 0$, respectively. Therefore, if a securities price movement is greater than the market index, then $\beta_i > 1$, which constitutes a more volatile or risky security; and vice versa. For more information on β_i values one may consult [Bodie *et al.*, 2005]. Considering the risk associated with securities in the portfolio, the following constraint is added

$$\sum_{i=1}^{n}\beta_i g_i^0 \leq \beta^\star + \delta^0\,, \tag{5.48}$$

where δ^0 is a penalty variable for the optimal $\beta^\star$ value. The penalty variable δ^0 is multiplied by a penalty parameter and minimized in the objective function, shown later in equation (5.57). The optimal value $\beta^\star$ is solved in a separate subproblem described in the next section, where the objective function is similar to

$$\beta^\star = \min\{\sum_{i=1}^{n}\beta_i g_i^0\}. \tag{5.49}$$

Since β_i is calculated using historical price values, $\beta^\star$ provides the optimum value based on market history. For higher time periods ($t > 0$) we add uncertainty to the optimal risk value by including scenarios. Thus, optimal security risk values become $\beta_\ell^\star$ for $t > 0$, with $\beta_{i\ell}^t$ indicating the risk associated with individual securities as shown in the following constraint:

$$\sum_{i=1}^{n}\beta_{i\ell}^t\, g_{i\ell}^t \leq \beta_\ell^\star + \delta_\ell^t \qquad \forall\, t = 0, ..., T,\, \ell = 1, ..., L. \tag{5.50}$$

Therefore, the uncertainty of future risk values for securities is dependent on the scenarios at higher time periods. Note that in the first-stage we omit writing $\ell = 1$ and $t = 0$, and also the penalty variable δ_ℓ^t is added for each time and scenario in (5.50). In addition to security risk, there is also risk associated with bond investments. Although bonds are generally less risky than securities, their rate of return may be variable, the issuer may default on the obligation, or the bond may be callable and offer less interest. For this reason there exists bond ratings, i.e. the Dominion Bond Rating Service (DBRS) and the Standard & Poor's (S&P) Bond Rating, that rate bonds from high (AAA) to low (D) quality, which was described in Chapter 2 on page 16. This information is converted to a numerical value where α_j defines the quality or inverse risk of bond j and $0 \leq \alpha_j \leq 1$. After solving a similar subproblem to (5.49) but taking $\alpha^\star$ as a maximum value, the following constraint is added to the portfolio model:

$$\sum_{j=1}^{h} \alpha_j \tilde{g}_j^0 \geq \alpha^\star - \tilde{\delta}^0, \tag{5.51}$$

where $\tilde{\delta}^0$ is a penalty variable for the optimal $\alpha^\star$ value. As with security risk, at higher time periods $\alpha_\ell^\star$ and $\alpha_{i\ell}^t$ will be employed to facilitate future uncertainties with respect to bond ratings. Hence, for $t > 0$ we have the following constraint

$$\sum_{j=1}^{h} \alpha_{j\ell}^t \, \tilde{g}_{j\ell}^t \geq \alpha_\ell^\star - \tilde{\delta}_\ell^t \qquad \forall \, t = 1, ..., T, \; \ell = 1, ..., L. \tag{5.52}$$

Similar to δ^t, the penalty variable $\tilde{\delta}^t$ is accompanied by a penalty parameter and minimized in the objective function, shown in (5.57) on page 126. The details of the subproblem that generates $\alpha^\star$, $\alpha_\ell^\star$, and $\alpha_{i\ell}^t$ will be presented in Section 5.2.3.

The last element to be considered in the portfolio selection model is liquidity. Liquidity is present in each type of investment, where typically securities are the most liquid. Within each investing instrument there also exists different levels of liquidity. For instance, a security that has over one million shares is considered to be more liquid than one that has a thousand. For this reason we define a liquidity value for each security and bond. Let $\Lambda(i,t,\ell)$ provide the liquidity value for security i at time t under scenario ℓ and $\tilde{\Lambda}(j,t,\ell)$ be the liquidity of bond j at time t under scenario ℓ. At time $t=0$ there will only be one scenario and we will omit writing ℓ. As we aim to invest in instruments with high liquidity, we solve for the liquidity optimal value $\Lambda^\star_\ell$ under each scenario in a similar method to what is shown in (5.52). Thus, we include

$$\sum_{i=1}^{n} \Lambda(i,t,\ell) g^t_{i\ell} + \sum_{j=1}^{h} \tilde{\Lambda}(j,t,\ell)\tilde{g}^t_{i\ell} \geq \Lambda^\star_\ell - \lambda^t \qquad \forall\, t=0,...,T,\ \ell=1,...,L, \quad (5.53)$$

as a constraint in our problem, where $\lambda^t \geq 0$ is a penalty variable that is minimized in the objective function and accompanied by a penalty parameter. Also, in the first-stage we will omit writing $\ell=1$ with respect to the optimal liquidity value $\Lambda^\star$, as shown later in equation (5.73). The specifics of the liquidity subproblem, and how $\Lambda(j,t,\ell)$ and $\tilde{\Lambda}(j,t,\ell)$ are generated is discussed in the next section.

Finally, we add portfolio accounting constraints to the model as follows:

$$\widehat{B} = \sum_{i=1}^{n} \phi^0_i x_i + \sum_{j=1}^{h} \varphi^0_j z^0_j + B^0 \quad (5.54)$$

$$B^1_\ell = \sum_{i=1}^{n} \phi^1_{i\ell} x_i - \sum_{i=1}^{n} \phi^1_{i\ell} y^1_{i\ell} + \sum_{j=1}^{h} \psi^1_{j\ell} z_j^{1-h^\star_j} - \sum_{j=1}^{h} \varphi^1_{j\ell} z^1_j - \sum_{i}^{n} \tau^1_\ell \varpi^1_{i\ell} + \varsigma B^0 \quad (5.55)$$

$$B^t_\ell = \sum_{i=1}^{n} \phi^t_{i\ell} y^{t-1}_{i\ell} - \sum_{i=1}^{n} \phi^t_{i\ell} y^t_{i\ell} + \sum_{j=1}^{h} \psi^t_{j\ell} z_{j\ell}^{t-h^\star_j} - \sum_{j=1}^{h} \varphi^t_{j\ell} z^t_{j\ell} - \sum_{i}^{n} \tau^t_\ell \varpi^t_{i\ell} + \varsigma B^{t-1}_\ell \quad (5.56)$$

$\forall\, t=1,...,T,\ \ell=1,...,L$, and where τ^t_ℓ defines the relative cost of a security transaction. Thus, (5.54)–(5.56) ensures that all portfolio wealth is being invested at each time period, including dividends. Note that, as mentioned earlier $\widehat{B}$ is the initial portfolio wealth and B^t_ℓ is the amount invested in the riskless investment at time t under scenario ℓ. Additionally, upper bounds are added to security and bond decision variables, where d_i and $\widetilde{d}_j$ are the maximum fractions of the portfolio to be invested in security i or bond j; respectively.

Above we have outlined the important portfolio elements that account for maximizing portfolio return while considering three different types of investments and including a passive strategy that facilitates portfolio size, diversity, risk, liquidity, transaction costs, and a performance measure. The SP model will capture the uncertainty of future security prices and bond interest rates through the use of scenarios. Further comments about scenario generation and definitions will be discussed in Section 5.2.4. A rebalancing strategy is also implicity embedded in the portfolio elements listed above. Portfolio equity is invested in a manner that maximizes performance goals and abides by elements (i)–(v). The weight of each of the portfolio goals is the last element to be included in the objective function of the portfolio selection model. We add the GP parameter μ_k to each of the seven portfolio goals described above, where $\sum_{k=1}^{7} \mu_k = 1$ and $\mu_k \geq 0$. After defining the set of securities $\Upsilon := \{i : i \in [1, n]\}$, the set of bonds $\Xi := \{j : j \in [1, h]\}$, the set of sectors $\widehat{S} := \{s : s \in [1, S]\}$, the set of scenarios $\Omega := \{\ell : \ell \in [1, L]\}$, and the set of time periods $\widehat{T} := \{t : t \in [0, T]\}$, $\overline{T} := \{t : t \in [1, T]\}$, and $\widetilde{T} := \{t : t \in [2, T]\}$, the SGMIP model with recourse becomes:

$$\begin{aligned}
\min \quad & - \mu_1 \left(\sum_{\ell=1}^{L} \sum_{i=1}^{n} \phi_{i\ell}^{1}\, x_i + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{i=1}^{n} p_\ell\, \phi_{i\ell}^{t+1} y_{i\ell}^{t} + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{j=1}^{h} p_\ell\, \psi_{j\ell}^{t}\, z_{j\ell}^{t-h_j^\star} \right. \\
& \qquad \left. + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell\, \varsigma B_\ell^t + \varsigma B^0 \right) + \mu_2 \left(\sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{i=1}^{n} p_\ell\, \varpi_{i\ell}^{t} \right) \\
& + \mu_3 \left(\sum_{s=1}^{S} |\xi_s^0| + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{s=1}^{S} p_\ell\, |\xi_{s\ell}^{t}| \right) + \mu_4 \left(\delta^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell\, \delta_\ell^t \right) \\
& + \mu_5 \left(\tilde{\delta}^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell\, \tilde{\delta}_\ell^t \right) + \mu_6 \left(\lambda^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell\, \lambda_\ell^t \right) + \mu_7 \left(\chi^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \chi_\ell^t \right)
\end{aligned} \tag{5.57}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \phi_i^0 x_i + \sum_{j=1}^{h} \varphi_j^0 z_j^0 + B^0 = \widehat{B} \tag{5.58}$$

$$\begin{matrix} \sum_{i=1}^{n} \phi_{i\ell}^1 x_i - \sum_{i=1}^{n} \phi_{i\ell}^1 y_{i\ell}^1 + \sum_{j=1}^{h} \psi_{j\ell}^1 z_j^{1-h_j^\star} \\ - \sum_{j=1}^{h} \varphi_{j\ell}^1 z_j^1 - \tau_\ell^1 \varpi_{i\ell}^1 + \varsigma B^0 \end{matrix} = B_\ell^1 \qquad \forall\, \ell \in \Omega \tag{5.59}$$

$$\begin{matrix} \sum_{i=1}^{n} \phi_{i\ell}^t y_{i\ell}^{t-1} - \sum_{i=1}^{n} \phi_{i\ell}^t y_{i\ell}^t + \sum_{j=1}^{h} \psi_{j\ell}^t z_{j\ell}^{t-h_j^\star} \\ - \sum_{j=1}^{h} \varphi_{j\ell}^t z_{j\ell}^t - \tau_\ell^t \varpi_{i\ell}^t + \varsigma B_\ell^{t-1} \end{matrix} = B_\ell^t \quad \forall\, \ell \in \Omega,\, t \in \widetilde{T} \tag{5.60}$$

$$\sum_{i=1}^{n} \phi_{i\ell}^1 x_i \leq R^0 + \chi^0 \tag{5.61}$$

$$\sum_{i=1}^{n} \phi_{i\ell}^{t+1} y_{i\ell}^t \leq R_\ell^t + \chi_\ell^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.62}$$

$$\sum_{i=1}^{n} Q(i,s) \phi_i^0 x_i = f_s^0 \sum_{i=1}^{n} \phi_i^0 x_i + \xi_s^0 \qquad \forall\, s \in \widehat{S} \tag{5.63}$$

$$\sum_{i=1}^{n} Q(i,s) \phi_{i\ell}^t y_{i\ell}^t = f_s^t \sum_{i=1}^{n} \phi_{i\ell}^t y_{i\ell}^t + \xi_{s\ell}^t \qquad \forall\, \ell \in \Omega,\ s \in \widehat{S},\ t \in \overline{T} \tag{5.64}$$

$$\sum_{i=1}^{n} g_i^0 \leq G^0 \tag{5.65}$$

$$\sum_{i=1}^{n} g_{i\ell}^t \leq G^t \qquad \forall\, \ell \in \Omega,\ t \in \overline{T} \tag{5.66}$$

$$\sum_{j=1}^{h} \widetilde{g}_j^0 \leq \widetilde{G}^0 \tag{5.67}$$

$$\sum_{j=1}^{h} \widetilde{g}_{j\ell}^t \leq \widetilde{G}^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.68}$$

$$\sum_{i=1}^{n} \beta_i g_i^0 \leq \beta^\star + \delta^0 \tag{5.69}$$

$$\sum_{i=1}^{n} \beta_{i\ell}^t\, g_{i\ell}^t \leq \beta_\ell^\star + \delta_\ell^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.70}$$

$$\sum_{j=1}^{h} \alpha_j \tilde{g}_j^0 \geq \alpha^\star - \tilde{\delta}^0 \tag{5.71}$$

$$\sum_{j=1}^{h} \alpha_{j\ell}^t\, \tilde{g}_{j\ell}^t \geq \alpha_\ell^\star - \tilde{\delta}_\ell^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.72}$$

$$\sum_{i=1}^{n} \Lambda(i,0)g_i^0 + \sum_{j=1}^{h} \tilde{\Lambda}(j,0)\tilde{g}_i^0 \geq \Lambda^\star - \lambda^0 \tag{5.73}$$

$$\sum_{i=1}^{n} \Lambda(i,t,\ell)g_{i\ell}^t + \sum_{j=1}^{h} \tilde{\Lambda}(j,t,\ell)\tilde{g}_{j\ell}^t \geq \Lambda_\ell^\star - \lambda_\ell^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.74}$$

$$\varpi_{i\ell}^1 = |y_{i\ell}^1 - x_i| \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega \tag{5.75}$$

$$\varpi_{i\ell}^t = |y_{i\ell}^t - y_{i\ell}^{t-1}| \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega,\, t \in \widetilde{T} \tag{5.76}$$

$$x_i \leq \mathrm{C}\, g_i^0 \qquad \forall\, i \in \Upsilon \tag{5.77}$$

$$y_{i\ell}^t \leq \mathrm{C}\, g_{i\ell}^t \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.78}$$

$$z_j^0 \leq \mathrm{C}\, g_j^0 \qquad \forall\, j \in \Xi \tag{5.79}$$

$$z_{j\ell}^t \leq \mathrm{C}\, \widetilde{g}_{j\ell}^t \qquad \forall\, j \in \Xi,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.80}$$

$$x_i \leq d_i \qquad \forall\, i \in \Upsilon \tag{5.81}$$

$$y_{i\ell}^t \leq d_i \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.82}$$

$$z_j^0 \leq \widetilde{d}_j \qquad \forall\, j \in \Xi \tag{5.83}$$

$$z_{j\ell}^t \leq \widetilde{d}_j \qquad \forall\, j \in \Xi,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.84}$$

$$B^0 \geq 0, B^0 \in \mathbb{R} \tag{5.85}$$

$$B_\ell^t \geq 0, B_\ell^t \in \mathbb{R} \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.86}$$

$$x_i \geq 0,\ x_i \in \mathbb{R} \qquad \forall\, i \in \Upsilon \tag{5.87}$$

$$y_{i\ell}^t \geq 0,\ y_{i\ell}^t \in \mathbb{R} \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.88}$$

$$z_j^0 \geq 0,\ z_j^0 \in \mathbb{R} \qquad \forall\, j \in \Xi \tag{5.89}$$

$$z_{j\ell}^t \geq 0,\ z_{j\ell}^t \in \mathbb{R} \qquad \forall\, j \in \Xi,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.90}$$

$$\varpi_{i\ell}^t \geq 0,\ \varpi_{i\ell}^t \in \mathbb{R} \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.91}$$

$$\delta^0 \geq 0,\ \delta^0 \in \mathbb{R} \tag{5.92}$$

$$\delta_\ell^t \geq 0,\ \delta_\ell^t \in \mathbb{R} \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.93}$$

$$\tilde{\delta}^0 \geq 0,\ \tilde{\delta}^0 \in \mathbb{R} \tag{5.94}$$

$$\tilde{\delta}^t_\ell \geq 0,\ \tilde{\delta}^t_\ell \in \mathbb{R} \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.95}$$

$$\lambda^0 \geq 0,\ \lambda^0 \in \mathbb{R} \tag{5.96}$$

$$\lambda^t_\ell \geq 0,\ \lambda^t_\ell \in \mathbb{R} \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.97}$$

$$g^0_i \in \mathbb{B}, \widetilde{g}^0_j \in \mathbb{B} \qquad \forall\, i \in \Upsilon,\, j \in \Xi \tag{5.98}$$

$$g^t_{i\ell} \in \mathbb{B}, \widetilde{g}^t_{j\ell} \in \mathbb{B} \qquad \forall\, i \in \Upsilon,\, j \in \Xi,\, \ell \in \Omega,\, t \in \overline{T} \tag{5.99}$$

$$\xi^0_s \in \mathbb{R}, \xi^t_{s\ell} \in \mathbb{R} \qquad \forall\, s \in \widehat{S},\, \ell \in \Omega,\, t \in \overline{T}, \tag{5.100}$$

where equations (5.77)–(5.80) set up the binary decision variables, C is a large constant, and d_i and $\widetilde{d}_j$ are upper bounds on the amount invested in a security or bond; respectively. The model above involves a number of key portfolio characteristics with an underlying passive investment strategy, which accounts for $\sum_{t=1}^{T} L^t \times (4 + 3n + 2h + S) + 2n + 2h + S + 4$ decision variables. In the next section we define the subproblems involved with risk $(\beta^t_{i\ell}, \alpha^t_{i\ell})$, liquidity $(\Lambda(i, t, \ell))$, the performance measure (R^t_ℓ), and the details of our scenario generation (Ω).

5.2.3 Additional Model Specifications

The portfolio design in (5.57)–(5.100) is constructed with a number of different portfolio goals to consider. In this section we outline the subproblems used to generate the values of the portfolio goals followed by a discussion on the scenarios designed to capture the various degrees of uncertainty present in the model. Thus, in order to define the optimal values $\beta^\star$, $\alpha^\star$, and $\Lambda^\star$ presented in (5.49), (5.51), and (5.53) three portfolio goal subproblems are solved. We first address the subproblem associated with the risk of individual securities. As mentioned in the previous section, β_i corresponds to the beta value of security i based on h_p historical price movements, where

$$\beta_i = \frac{\mathrm{Cov}(\phi_i, I_M)}{\sigma_M^2}. \tag{5.101}$$

Using the first-stage variables corresponding to securities, the value of $\beta^\star$ is computed in the following subproblem:

$$\min \ \tau \sum_{i=1}^{n} \beta_i g_i^0 + (1-\tau) \left| \sum_{i=1}^{n} \phi_i^0 x_i - \widehat{B}_\beta \right| \tag{5.102}$$

$$\text{s.t.} \ \sum_{i=1}^{n} g_i^0 \leq G^0 \tag{5.103}$$

$$x_i \leq \mathrm{C}\, g_i^0 \qquad \forall\, i \in \Upsilon \tag{5.104}$$

$$x_i \geq 0,\ x_i \in \mathbb{I} \qquad \forall\, i \in \Upsilon \tag{5.105}$$

$$g_i^0 \in \mathbb{B} \qquad \forall\, i \in \Upsilon, \tag{5.106}$$

where $0 < \tau < 1$ and $\widehat{B}_\beta$ is the approximation of initial funds used for security investments. Therefore, $\beta^\star = \sum_{i=1}^{n} \beta_i (g_i^0)^*$, where $(g_i^0)^*$ is the optimal basis from the solution to (5.102)–(5.106). The cardinality constraint (5.103) is added to the subproblem since the model in (5.57)–(5.100) limits the number of names in the portfolio, and hence, the approximation is relative. Also, the L1-norm in equation (5.102) can be modelled as a linear objective, which reduces computational complexities and the reason higher order equations were not used. The same reasoning and approach is used in equation (5.107) and (5.112) below. One should also note that since β_i is based on historical price movements, $\beta^\star$ provides the best known risk value for time $t = 0$. SP scenarios are then used to facilitate future uncertain market positions and variability with respect the value of $\beta_\ell^\star$. The generation of scenarios for security risk values $\beta_\ell^\star$ and $\beta_{i\ell}^t$ will be discussed in the next subsection. The second subproblem requires solving $\alpha^\star$ for the SGMIP, which defines the risk associated with individual bonds. The subproblem is similar to (5.102)–(5.106), however, since α_j defines the quality of bond j the objective function is maximized. Thus, given $\widehat{B}_\alpha$ is the portion of initial funds used for bond investments, the subproblem becomes

$$\min \quad -\tau \sum_{j=1}^{h} \alpha_j \tilde{g}_j^0 + (1-\tau) \Big| \sum_{j=1}^{h} \varphi_j^0 z_j^0 - \widehat{B}_\alpha \Big| \tag{5.107}$$

$$\text{s.t.} \quad \sum_{i=1}^{h} \tilde{g}_j^0 \leq \widetilde{G}^0 \tag{5.108}$$

$$z_j^0 \leq \mathrm{C}\, \tilde{g}_j^0 \qquad \forall\, j \in \Xi \tag{5.109}$$

$$z_j^0 \geq 0,\; z_j^0 \in \mathbb{I} \qquad \forall\, j \in \Xi \tag{5.110}$$

$$\tilde{g}_j^0 \in \mathbb{B} \qquad \forall\, j \in \Xi, \tag{5.111}$$

where $\alpha^\star = \sum_{j=1}^{h} \alpha_j (\tilde{g}_j^0)^*$ and $(\tilde{g}_j^0)^*$ is the optimal basis from the solution to (5.107)–(5.111). As with $\beta_\ell^\star$, scenarios are added to $\alpha_\ell^\star$ to capture future uncertainties with respect to bond risk. The scenarios generated for $\alpha_\ell^\star$ and $\alpha_{j\ell}^t$ are shown in the next subsection.

For the derivation of the optimal liquidity value $\Lambda^\star$, securities and bonds are first separated into liquidity classes, then a function is introduced to approximate individual liquidity values within the classes. We define K liquidity classes and assign $\kappa_q \in [0,1]\ \forall\ q = 1, ..., K$, where securities having over 1 million shares obtain the largest κ_q–class value and long term bonds obtain the lowest value. Then, using an approach similar to [Bertsimas *et al.*, 1999], we define a piecewise liquidity function $f_v(t)$ for each security ($v = i,\ \forall\ i \in \Upsilon$) or bond ($v = j,\ \forall\ j \in \Xi$) within each κ_q–class, as shown in Figure 5.1. The combination of liquidity classes and the piecewise liquidity function provides a value $\Lambda(i,t) = \kappa_q f_i(t)$ for securities and $\tilde{\Lambda}(j,t) = \kappa_q f_j(t)$ for bonds. Thus, for the first-stage

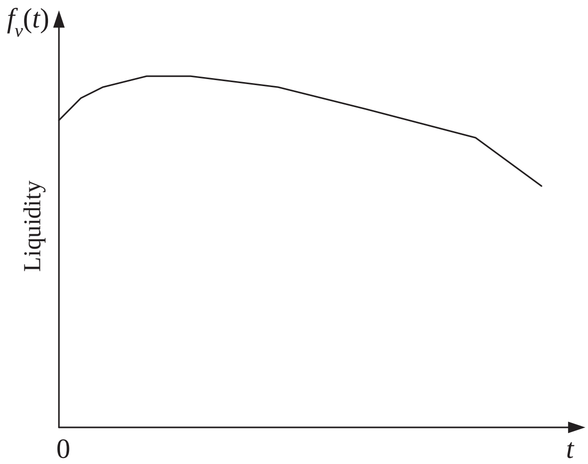

Figure 5.1: Example of a liquidity function $\Lambda(i,t)$ for security i.

variables, the liquidity subproblem is as follows:

$$\min \quad -\tau\left(\sum_{i=1}^{n}\Lambda(i,0)g_i^0+\sum_{j=1}^{h}\tilde{\Lambda}(j,0)\tilde{g}_j^0\right) \tag{5.112}$$

$$+(1-\tau)\left|\sum_{i=1}^{n}\phi_i^0 x_i+\sum_{j=1}^{h}\varphi_j^0 z_j^0-(\widehat{B}_\beta+\widehat{B}_\alpha)\right|$$

$$\text{s.t.} \quad \sum_{i=1}^{n} g_i^0 \le G^0 \tag{5.113}$$

$$\sum_{i=1}^{h}\tilde{g}_j^0 \le \widetilde{G}^0 \tag{5.114}$$

$$x_i \le \mathrm{C}\, g_i^0 \qquad \forall\, i\in\Upsilon \tag{5.115}$$

$$z_j^0 \le \mathrm{C}\,\tilde{g}_j^0 \qquad \forall\, j\in\Xi \tag{5.116}$$

$$x_i \ge 0,\ x_i\in\mathbb{I} \qquad \forall\, i\in\Upsilon \tag{5.117}$$

$$z_j^0 \ge 0,\ z_j^0\in\mathbb{I} \qquad \forall\, j\in\Xi \tag{5.118}$$

$$g_i^0\in\mathbb{B} \qquad \forall\, i\in\Upsilon \tag{5.119}$$

$$\tilde{g}_j^0\in\mathbb{B} \qquad \forall\, j\in\Xi, \tag{5.120}$$

where $\Lambda^\star=\sum_{i=1}^{n}\Lambda(i,0)(g_i^0)^*+\sum_{j=1}^{h}\tilde{\Lambda}(j,0)(\tilde{g}_j^0)^*$, and $(g_i^0)^*$ and $(\tilde{g}_i^0)^*$ are the optimal values from equations (5.112)–(5.120). For higher time periods ($t > 0$) scenarios are added to liquidity values $\Lambda_\ell^\star$, $\Lambda(i,t,\ell)$ and $\tilde{\Lambda}(j,t,\ell)$ to account for the uncertainty of future markets, as will be discussed in the next subsection.

5.2.4 Scenario Generation for the SGMIP Portfolio

The portfolio model presented in section 5.2.2 contains various degrees of uncertainty to which we account for using a SMIP design with recourse. We now address how scenarios are constructed to capture future uncertainties within the model. There are two major sources of uncertainty with regards to the portfolio model, namely: the uncertainty of future security and bond values, and the uncertainty of the portfolio goals at higher time periods. In order to facilitate the different uncertainties present in the model we partition the set of scenarios $\Omega = \{\Omega^s, \Omega^b, \Omega^\beta, \Omega^\alpha, \Omega^\Lambda, \Omega^R\}$, where Ω^s accounts for security price scenarios, Ω^b accounts for bond price scenarios, Ω^β accounts for security risk scenarios, Ω^α accounts for bond risk scenarios, Ω^Λ accounts for liquidity scenarios, and Ω^R accounts for performance measure scenarios. Also, we will define the total number of scenarios with respect to each $\Omega^{(\cdot)}$ partition, hence $L^{(\cdot)}$ will define the total number of scenarios for element $(\cdot)$. We begin by addressing the uncertainty present in securities and bonds. The scenarios constructed for securities will follow what is presented in Chapter 4 and be extended for bonds. We will briefly outline the scenario generation for securities, for more information one can refer to Subsection 4.3.2. The set of scenarios related to securities Ω^s contains a total of L^s scenarios, where $\ell^s = 1, 2, ..., L^s$. The market index is used as the basis towards constructing future scenarios. The motivation comes from the notion that an index represents or at least is a reflection of security price trends for a given financial market, thus scenarios based on a market index is a strong indicator of future price movements. Hence, we investigate historical index values and produce scenarios that are consistent with an approximation of that function plus some worst and best case deviations. Thus, setting I^t to be the market index value at time t and taking h_I historical time periods, the worst case scenario ℓ^s_{W} is

$$\ell^s_{\mathrm{w}} = \min\left\{\frac{2I^{t+1} - I^t}{I^t}\right\} + \lambda \qquad \forall\, t = h_I, ..., 0. \tag{5.121}$$

Equation (5.121) defines the lowest value the index decreased with respect to the previous time-stage over the historical time horizon. Also, $\lambda \in \mathbb{R}$ allows for small scenario adjustments, where typically $\lambda = 0$. Similarly, the best case scenario ℓ^s_{B} is

$$\ell^s_{\mathrm{B}} = \max\left\{\frac{2I^{t+1} - I^t}{I^t}\right\} + \lambda \qquad \forall\, t = h_I, ..., 0, \tag{5.122}$$

which represents the greatest value the index has risen to with respect to the previous time period. Given the scenario ℓ^s_{I} is the index approximation, then other scenarios will be generated within $(\ell^s_{\mathrm{W}}, \ell^s_{\mathrm{I}}) \bigcup (\ell^s_{\mathrm{I}}, \ell^s_{\mathrm{B}})$. Thus, at the very least $L^s = 3$, which includes the three scenarios mentioned $\{\ell^s_{\mathrm{W}}, \ell^s_{\mathrm{I}}, \ell^s_{\mathrm{B}}\}$ and when added to the original index value evolve according to the illustration in Figure 5.2.

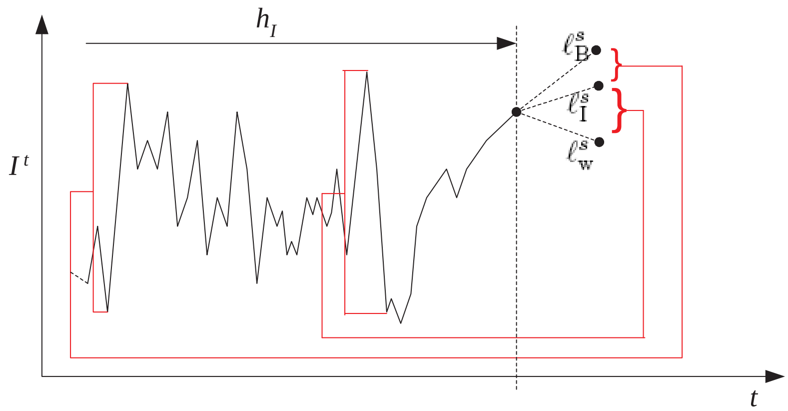

Figure 5.2: $L^s = 3$ security scenario generations using market index values over the historical time window h_I.

The scenarios for bond investments are generated using the same motivation as securities, with the exception that the bond index is used for I^t. For the set of scenarios related to bond investments Ω^b, a total of L^b scenarios are considered where $\ell^b = 1, 2, ..., L^b$. Then a bond index approximation ℓ_I^b is generated using h_I historical time periods and using bond index values for equations (5.121) and (5.122), ℓ_{W}^b and ℓ_{B}^b are derived; respectively. Given $L^b = 3$, then the bond index scenarios will be calculated similarly to what is shown in Figure 5.2 and additional scenarios ($L^b > 3$) will be constructed from the set $(\ell_{\mathrm{W}}^b, \ell_{\mathrm{I}}^b) \bigcup (\ell_{\mathrm{I}}^b, \ell_{\mathrm{B}}^b)$. In both cases, additional scenarios are generated by taking symmetric differences between $(\ell_{\mathrm{W}}^s, \ell_{\mathrm{I}}^s)$ and $(\ell_{\mathrm{I}}^s, \ell_{\mathrm{B}}^s)$ for securities, and $(\ell_{\mathrm{W}}^b, \ell_{\mathrm{I}}^b)$ and $(\ell_{\mathrm{I}}^b, \ell_{\mathrm{B}}^b)$ for bonds. If there is an odd number of additional scenarios the weighted average of the historical values between $(\ell_{\mathrm{W}}^s, \ell_{\mathrm{B}}^s)$ and $(\ell_{\mathrm{W}}^b, \ell_{\mathrm{B}}^b)$ are used. The approach exploits historical index values to generate scenarios based on the notion that indices are a reflection of their respective market trends. Hence, if an index value is increasing, as should be the direction of the average shares within that market. This may not be true for all shares, however, using the other goals of the portfolio model presented in (5.57)–(5.100), the design should be able to make this distinction.

The last source of uncertainty in the model is the uncertainty of portfolio goals at higher time periods. In the first-stage ($t = 0$) all values are known, thus the variable and parameter definitions are set with certainty. However, as the time periods progress the values of R_ℓ^t, $\beta^\star$, $\alpha^\star$, and $\Lambda^\star$ may change. We begin by addressing the performance measure R_ℓ^t. As described in (5.44), R_ℓ^t acts as a safeguard to ensure the model is following a passive investment strategy. In the first-stage R^0 is set to be a benchmark similar to a market index, given the initial investment $\widehat{B}$ is greater than or equal to that amount. If not, then R_ℓ^t is simply a ratio of $\widehat{B}$ and the desired benchmark. At higher time periods R_ℓ^t is set to behave similar to the percent increase or decrease with respect to the desired benchmark. Thus, one of the scenarios becomes the best approximation to the benchmark, which was chosen to be the market index in the results of Section

5.4. The other scenarios are generated similarly to what is shown in Figure 5.2, using (5.121)–(5.122) with ℓ_{w}^{R} and ℓ_{B}^{R} in place of ℓ_{w}^{s} and ℓ_{B}^{s}; respectively. Hence, there is a total of L^{R} scenarios for the performance measure, where scenarios generated for $L^{R} > 3$ are taken from the set $(\ell_{\mathrm{W}}^{R}, \ell_{\mathrm{I}}^{R}) \bigcup (\ell_{\mathrm{I}}^{R}, \ell_{\mathrm{B}}^{R})$; as described in Ω^{s} and Ω^{b}. For the sets Ω^{β} and Ω^{α} related to risk goals, we generate scenarios that are based on a combination of their respective benchmarks (securities or bonds) and their historic risk values. For security risk: $\ell^{\beta} = 1, 2, ..., L^{\beta}$ with a total of L^{β} scenarios, and for bond risk: $\ell^{\alpha} = 1, 2, ..., L^{\alpha}$ with a total of L^{α} scenarios, the same procedure is used as in equations (5.121)–(5.122). Thus, for ℓ^{β} scenarios are based on the product of what is shown in (5.121)–(5.122) and the initial $\beta^{\star}$ value. For ℓ^{α}, the same is done as in ℓ^{β} with the exception that a bond benchmark is used in combination with $\alpha^{\star}$.

Finally, for Ω^{Λ} the scenarios $\ell^{\Lambda} = 1, 2, ..., L^{\Lambda}$ (with a total of L^{Λ}) are approximated using the value of $\lambda^{\star}$ and the security scenarios in Ω^{s}. The security benchmark is used because as index values increase, one would expect that a higher degree of market activity is exhibited, and thus higher degrees of liquidity are present. The opposite would be true if a decrease in market values were exhibited. To conclude, using any set of scenarios $\Omega = \{\Omega^{s}, \Omega^{b}, \Omega^{\beta}, \Omega^{\alpha}, \Omega^{\Lambda}, \Omega^{R}\}$, the evolution of the scenario tree would behave as depicted in Figure 5.3. Thus, in this section we have addressed a number of scenario definitions used to capture the various uncertainties present in the model. This is achieved by designing several realizations that future portfolio investments and goals may become. By devising many proficient possible outcomes on the price of securities and bonds, we capture a number of different economic trends. Although the exact future economic direction may be impossible to encapsulate, it may span over a couple of possible future scenarios. In this case, the worst instance of these outcomes would represent a lower bound on the optimal portfolio value, and thus, uncertainty may ultimately be captured.

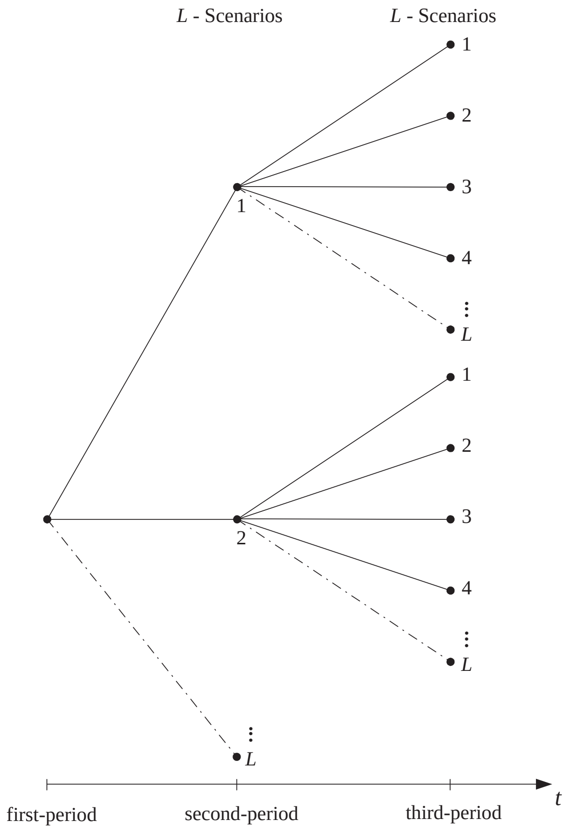

Figure 5.3: Evolution tree for a total of L scenarios and 3 time periods.

5.3 Implementation and Algorithm Design

In order to solve the SGMIP presented in (5.57)–(5.100) a model specific heuristic is designed. The algorithm capitalizes on the structure of the problem by exploiting particular constraints and relaxing others. The relaxations are minimized in an iterative procedure that does not affect the portfolio goals mentioned in Section 5.2.3. The algorithm uses some of the techniques presented in Section 4.3.1, but also involves additional measures to facilitate bonds and portfolio goals. Basically, the algorithm operates by first decomposing the model into security and bond subproblems. The subproblems are then solved and used as a warm start to solve the master problem. The master problem then checks an optimality criteria and either provides the optimal solution or resolves the model by adjusting initializations. An outline of the algorithm is shown in figure 5.4. As illus-

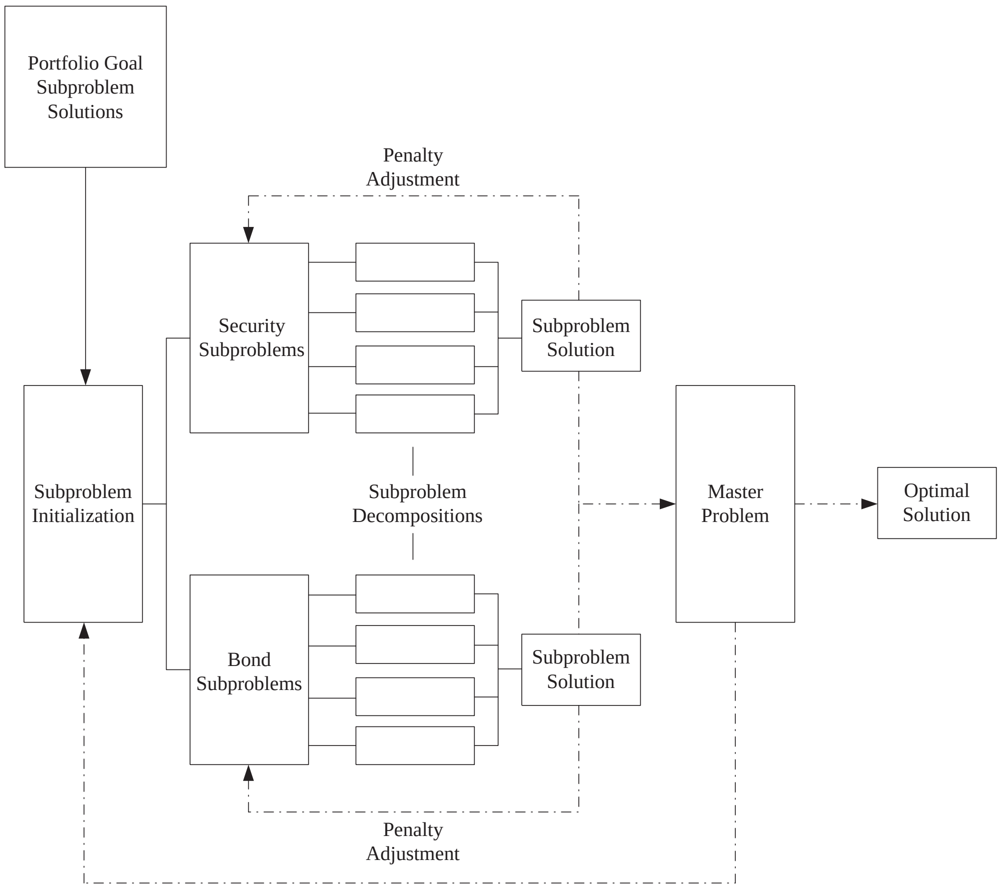

Figure 5.4: Outline of the SGMIP algorithm.

trated, the solution method begins by initializing the portfolio goals presented in Section 5.2.3 and other parameters associated with the model. Then the problem is decomposed geographically into security and bond subproblems, which can be further broken down into smaller problems if necessary. The decompositions are accompanied by relaxations that are minimized in an iterative penalty adjustment step, which perform essentially the same steps as in Figure 4.2 (page 85) and will be described in further detail below. After the subproblems are solved, the optimal basis and parameter settings are sent to the master problem. The master problem contains the same relaxations as in the subproblem decompositions and uses the minimized relaxation values, parameter settings, and optimal basis to produce solutions to the large SGMIP presented in (5.57)–(5.100).

One of the strengths of the algorithm is in the geographical or model specific decomposition into security and bond subproblems. Due to the nature of the problem, the algorithm can solve each of the subproblems separately in a way that does not affect or change the large SGMIP. In addition, each of the security or bond subproblems can be further decomposed into smaller *Subproblem Decompositions* based on what is shown in Subsection 4.3.1 on page 80. For the additional decompositions, subscripts are added to parameters associated with security and bond variables. Taking the security subproblem, further decompositions are based on sectors. Hence, subscript s is added to security parameters where there are a total of S subproblems and from Section 5.2.2 we have $\widehat{\mathrm{S}} := \{s : s \in [1, S]\}$. Similarly, subscript b is added to bond parameters and given there are J bond decompositions we let $\widehat{J} := \{b : b \in [1, J]\}$. Bond decompositions are based on the sector the corporation issuing the bond belong to, which is analogous to the security approach. Another strength of the algorithm is that relaxations are added to difficult constraints, which produces solutions that are iteratively minimized until optimality, or near-optimality is obtained. The difficult constraints we are referring to are the names-to-hold constraints (5.65)–(5.68), which have been show to cause NP-hard issues [Coleman *et al.*, 2006] and is one of the focuses of this document. Therefore, the master problem with relaxations are:

$$
\begin{aligned}
\min \quad & (5.57) + \varrho^0\gamma^0 + \widetilde{\varrho}^0\widetilde{\gamma}^0 + \sum_{t=1}^{T}\sum_{\ell=1}^{L}(\varrho_\ell^t\gamma_\ell^t + \widetilde{\varrho}_\ell^t\widetilde{\gamma}_\ell^t) && (5.123)\\
\text{s.t.} \quad & (5.58) - (5.64) && (5.124)\\
& \sum_{i=1}^{n} g_i^0 \le G^0 + \gamma^0 && (5.125)\\
& \sum_{i=1}^{n} g_{i\ell}^t \le G^t + \gamma_\ell^t \qquad \forall\, \ell \in \Omega,\ t \in \overline{T} && (5.126)\\
& \sum_{j=1}^{h} \widetilde{g}_j^0 \le \widetilde{G}^t + \widetilde{\gamma}^0 && (5.127)\\
& \sum_{j=1}^{h} \widetilde{g}_\ell^t \le \widetilde{G}^t + \widetilde{\gamma}_\ell^t \qquad \forall\, \ell \in \Omega, t \in \overline{T} && (5.128)\\
& \gamma^0 \ge 0,\ \gamma^0 \in \mathbb{R} && (5.129)\\
& \widetilde{\gamma}^0 \ge 0,\ \widetilde{\gamma}^0 \in \mathbb{R} && (5.130)\\
& \gamma_\ell^t \ge 0,\ \gamma_\ell^t \in \mathbb{R} \qquad \forall\, \ell \in \Omega,\ t \in \overline{T} && (5.131)\\
& \widetilde{\gamma}_\ell^t \ge 0,\ \widetilde{\gamma}_\ell^t \in \mathbb{R} \qquad \forall\, \ell \in \Omega, t \in \overline{T} && (5.132)\\
& (5.69) - (5.100), && (5.133)
\end{aligned}
$$

where γ_ℓ^t and ϱ_ℓ^t are security penalty variables and parameters, and $\widetilde{\gamma}_\ell^t$ and $\widetilde{\varrho}_\ell^t$ are bond penalty variables and parameters; respectfully. As shown in Figure 5.4, the problem is first divided into large security and bond subproblems that can be further decomposed. Hence, the sector subscript s is added security parameters G_s and G_s^t, and a relaxation γ_s and $\gamma_{s\ell}^t$ is added to constraints (5.125)–(5.126); respectfully. Similarly, the subscript b is added to $\widetilde{G}_b$ and $\widetilde{G}_b^t$, with relaxations $\widetilde{\gamma}_b$ and $\widetilde{\gamma}_{b\ell}^t$ added to (5.127)–(5.128); respectfully. As in earlier approaches, the algorithm acts to minimize relaxations or constraint violations by minimizing them in the objective function accompanied by penalty parameters ϱ_s^t for securities and $\widetilde{\varrho}_b^t$ for bonds. Thus, in the master problem, the security and bond values are collected such that

$$\begin{aligned}
G^0 &= G_1^0 + G_2^0 + \cdots + G_S^0 && (5.134)\\
\widetilde{G}^0 &= \widetilde{G}_1^0 + \widetilde{G}_2^0 + \cdots + \widetilde{G}_J^0 && (5.135)\\
G^t &= G_1^t + G_2^t + \cdots + G_S^t \qquad \forall\, t \in \overline{T} && (5.136)\\
\widetilde{G}^t &= \widetilde{G}_1^t + \widetilde{G}_2^t + \cdots + \widetilde{G}_J^t \qquad \forall\, t \in \overline{T} && (5.137)
\end{aligned}$$

and similarly the penalty parameters become

$$\begin{aligned}
\varrho^0 &= \varrho_1^0 + \varrho_2^0 + \cdots + \varrho_S^0 && (5.138)\\
\widetilde{\varrho}^0 &= \widetilde{\varrho}_1^0 + \widetilde{\varrho}_2^0 + \cdots + \widetilde{\varrho}_J^0 && (5.139)\\
\varrho_\ell^t &= \varrho_{1\ell}^t + \varrho_{2\ell}^t + \cdots + \varrho_{J\ell}^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} && (5.140)\\
\widetilde{\varrho}_\ell^t &= \widetilde{\varrho}_{1\ell}^t + \widetilde{\varrho}_{2\ell}^t + \cdots + \widetilde{\varrho}_{J\ell}^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T}. && (5.141)
\end{aligned}$$

In the simplest case, the SGMIP is decomposed into one security and one bond subproblem with $S = 1$ and $J = 1$ from (5.134)–(5.137) and (5.138)–(5.141) above. However, for instances where $S > 1$ and/or $J > 1$ additional parameters are necessary to accommodate the other portfolio goals. Taking the security subproblem, we add subscript s to sector values for the initial wealth $\widehat{B}_s$, near-riskless investment B_s^0 and $B_{s\ell}^t$, sector index values R_s^0 and $R_{s\ell}^t$, and index relaxations χ_s^0 and $\chi_{s\ell}^t$ for each of the S decompositions. Each upholds equalities equivalent to (5.134)–(5.141), where for example: the portfolio benchmark has a value for each sector decomposition such that

$$\begin{aligned}
R^0 &= R_1^0 + R_2^0 + \cdots + R_S^0 && (5.142)\\
R^t &= R_1^t + R_2^t + \cdots + R_S^t \qquad \forall\, t \in \overline{T}. && (5.143)
\end{aligned}$$

Therefore, the security subproblem for $s = 1, ..., S$ is

$$\begin{aligned}\min \quad & -\mu_1 \left(\sum_{\ell=1}^{L} \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^{1}\, x_i + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{i=1}^{n} p_\ell\, Q(i,s)\phi_{i\ell}^{t+1} y_{i\ell}^{t} \right. \\ & \qquad \left. + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell\, \varsigma B_{s\ell}^{t} + \varsigma B_s^0 \right) + \mu_2 \left(\sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{i=1}^{n} p_\ell\, Q(i,s)\varpi_{i\ell}^{t} \right) \\ & \quad + \mu_4 \left(\delta_s^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_{s\ell}\, \delta_{s\ell}^{t} \right) + \mu_7 \left(\chi_s^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \chi_{s\ell}^{t} \right) \\ & \quad + \left(\varrho_s^0 \gamma_s^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \varrho_{s\ell}^{t} \gamma_{s\ell}^{t} \right)\end{aligned} \tag{5.144}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} Q(i,s)\phi_i^0 x_i + B_s^0 = \widehat{B}_s \tag{5.145}$$

$$\begin{matrix} \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^{1} x_i - \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^{1} y_{i\ell}^{1} \\ -Q(i,s)\tau_\ell^1 \varpi_{i\ell}^{1} + \varsigma B_s^0 \end{matrix} = B_{s\ell}^{1} \qquad \forall\, \ell \in \Omega \tag{5.146}$$

$$\begin{matrix} \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^{t} y_{i\ell}^{t-1} - \sum_{i=1}^{n} Q(i,s)\phi_{i\ell}^{t} y_{i\ell}^{t} \\ -Q(i,s)\tau_\ell^t \varpi_{i\ell}^{t} + \varsigma B_{s\ell}^{t-1} \end{matrix} = B_{s\ell}^{t} \qquad \forall\, \ell \in \Omega,\, t \in \widetilde{T} \tag{5.147}$$

$$\sum_{i=1}^{n} \phi_{i\ell}^{1} Q(i,s) x_i \le R_s^0 + \chi_s^0 \tag{5.148}$$

$$\sum_{i=1}^{n} \phi_{i\ell}^{t+1} Q(i,s) y_{i\ell}^{t} \le R_{s\ell}^{t} + \chi_{s\ell}^{t} \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.149}$$

$$\sum_{i=1}^{n} Q(i,s) g_i^0 \le G_s^0 + \gamma_s^0 \tag{5.150}$$

$$\sum_{i=1}^{n} Q(i,s) g_{i\ell}^{t} \le G_s^t + \gamma_{s\ell}^{t} \qquad \forall\, \ell \in \Omega,\ t \in \overline{T} \tag{5.151}$$

$$\sum_{i=1}^{n} Q(i,s)\beta_i g_i^0 \le \beta^\star + \delta_s^0 \tag{5.152}$$

$$\sum_{i=1}^{n} Q(i,s)\beta_{i\ell}^{t}\, g_{i\ell}^{t} \le \beta_\ell^\star + \delta_{s\ell}^{t} \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.153}$$

$$\varpi_{i\ell}^{1} = Q(i,s)|y_{i\ell}^{1} - x_i| \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega \tag{5.154}$$

$$\varpi_{i\ell}^t = Q(i,s)|y_{i\ell}^t - y_{i\ell}^{t-1}| \qquad \forall\, i \in \Upsilon,\, \ell \in \Omega,\, t \in \widetilde{T} \tag{5.155}$$

$$(5.77) - (5.99) \tag{5.156}$$

$$\gamma_s^0 \geq 0,\, \gamma_s^0 \in \mathbb{R} \tag{5.157}$$

$$\gamma_{s\ell}^t \geq 0,\, \gamma_{s\ell}^t \in \mathbb{R} \qquad \forall\, \ell \in \Omega,\; t \in \overline{T}, \tag{5.158}$$

where $Q(i,s)$ is the sector indicator variable described in Subsection 5.2.2. Also, the last term in parentheses of the objective function (5.144) does not have a weight attached to it because it will be used by the algorithm to meet the names-to-hold constraint; as is the case in the objective of (5.160) below. Hence, to decompose the bond subproblems we introduce a similar indicator variable $\widehat{Q}(j,b)$ and define

$$\widehat{Q}(j,b) \quad = \quad \begin{cases} 1, & \text{if bond } j \text{ is in subproblem } b; \\ 0, & \text{otherwise.} \end{cases} \tag{5.159}$$

Then we define the subproblems b by assigning the sector the corporations issuing the bonds belong to. In addition, bond subproblem values for the initial wealth $\widehat{B}_b$, and the near-riskless investment B_b^0 and $B_{b\ell}^t$ are partitioned in the same manner as what is shown in (5.142)–(5.143), where S is replaced by J decompositions. Therefore, the bond subproblem for $b = 1, ..., J$ becomes:

$$\begin{aligned} \min \quad & -\mu_1 \left(\sum_{t=1}^{T} \sum_{\ell=1}^{L} \sum_{j=1}^{h} p_\ell\, \widehat{Q}(i,b) \psi_{j\ell}^t\, z_{j\ell}^{t-h_j^\star} + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell\, \varsigma B_{b\ell}^t + \varsigma B_b^0 \right) \\ & + \mu_5 \left(\tilde{\delta}_b^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} p_\ell\, \tilde{\delta}_{b\ell}^t \right) + \left(\widetilde{\varrho}_b^0 \widetilde{\gamma}_b^0 + \sum_{t=1}^{T} \sum_{\ell=1}^{L} \widetilde{\varrho}_b^0 \widetilde{\gamma}_{b\ell}^t \right) \end{aligned} \tag{5.160}$$

$$\text{s.t.} \quad \sum_{j=1}^{h} \widehat{Q}(i,b) \varphi_j^0 z_j^0 + B_b^0 = \widehat{B}_b \tag{5.161}$$

$$\sum_{j=1}^{h} \widehat{Q}(i,b) \psi_{j\ell}^1 z_j^{1-h_j^\star} - \sum_{j=1}^{h} \widehat{Q}(i,b) \varphi_{j\ell}^1 z_j^1 + \varsigma B_b^0 = B_{b\ell}^1 \qquad \forall\, \ell \in \Omega \tag{5.162}$$

$$\sum_{j=1}^{h} \widehat{Q}(i,b) \psi_{j\ell}^t z_{j\ell}^{t-h_j^\star} - \sum_{j=1}^{h} \widehat{Q}(i,b) \varphi_{j\ell}^t z_{j\ell}^t + \varsigma B_{b\ell}^{t-1} = B_{b\ell}^t \qquad \forall\, \ell \in \Omega,\, t \in \widetilde{T} \tag{5.163}$$

$$\sum_{j=1}^{h} \widehat{Q}(i,b)\widetilde{g}_j^0 \leq \widetilde{G}_b^t + \widetilde{\gamma}_b^0 \tag{5.164}$$

$$\sum_{j=1}^{h} \widehat{Q}(i,b)\widetilde{g}_{j\ell}^t \leq \widetilde{G}_b^t + \widetilde{\gamma}_{b\ell}^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.165}$$

$$\sum_{j=1}^{h} \widehat{Q}(i,b)\alpha_j \tilde{g}_j^0 \geq \alpha^\star - \tilde{\delta}_b^0 \tag{5.166}$$

$$\sum_{j=1}^{h} \widehat{Q}(i,b)\alpha_{j\ell}^t\, \tilde{g}_{j\ell}^t \geq \alpha_\ell^\star - \tilde{\delta}_{b\ell}^t \qquad \forall\, \ell \in \Omega,\, t \in \overline{T} \tag{5.167}$$

$$(5.77) - (5.99) \tag{5.168}$$

$$\widetilde{\gamma}_b^0 \geq 0,\, \widetilde{\gamma}_b^0 \in \mathbb{R} \tag{5.169}$$

$$\widetilde{\gamma}_{b\ell}^t \geq 0,\, \widetilde{\gamma}_{b\ell}^t \in \mathbb{R} \qquad \forall\, \ell \in \Omega,\, t \in \overline{T}, \tag{5.170}$$

In both (5.144)–(5.158) and (5.160)–(5.170) the model specific decompositions take advantage of the problems structure and other than partitioning parameters and variables, the subproblem configurations are equivalent to the large SGMIP in (5.57)–(5.100). In addition, the penalty parameters that are used in each subproblem act to minimize constraint relaxations in a penalty adjustment step that is similar to what is done in Subsection 4.3.1. Specifically, relaxations or penalty variables $\gamma_s, \gamma_{s\ell}^t, \widetilde{\gamma}_b$ and $\widetilde{\gamma}_{s\ell}^t$ are minimized by using parameters $\varrho_s, \varrho_{s\ell}^t, \widetilde{\varrho}_b$ and $\widetilde{\varrho}_{s\ell}^t$. The adjustment functions by first allowing the algorithm to find an initial solution using small penalty parameter values, and therefore large violations to constraints (5.65)–(5.68). Then, the penalty parameter values are iteratively increased, causing constraint violations to be expensive and thus minimizing the size of the relaxations. The subproblem solutions are then sent to the master problem to be used as an initial solution, which adds liquidity constraints (5.73)–(5.74) before solving. The master problem then evaluates the optimal solution by checking that the portfolio goals fall within a specific criteria (i.e. (5.142)–(5.143) and similar constraints hold) and either accepts the optimal solution or increases the size of the parameters associated with the variables that fall below the criteria; and then resolves. The main steps in the algorithm are as follows:

`0. Portfolio Goal Solutions:`

a) Solve $\beta^\star$ subproblem (5.102)–(5.106).

b) Solve $\alpha^\star$ subproblem (5.107)–(5.111).

c) Solve $\Lambda^\star$ subproblem (5.112)–(5.120).

`1. Initialize:`

a) Set model parameters: $\phi^t_{i\ell}$, $\psi^t_{j\ell}$, $\widehat{B}$, $Q(i,s)$, $\widehat{Q}(i,b)$, $f^t_{s\ell}$, G^t, $\widetilde{G}^t$, $\beta^t_{i\ell}$, $\alpha^t_{j\ell}$, $\Lambda(i,t,\ell)$, $\tilde{\Lambda}(j,t,\ell)$, τ^t_ℓ, p_ℓ, μ_k, ς, and C.

b) Set portfolio goals from subproblem solutions: R^t_ℓ, $\beta^\star_\ell$, $\alpha^\star_\ell$, and $\Lambda^\star_\ell$

c) Set goal and penalty adjustment parameters: $\hat{r}^1, \hat{r}^2, \widetilde{r}^1, \widetilde{r}^2, r^1, r^2, \bar{r}, k, z,$ τ_1, τ_2, and counters d_1, d_2, H_1, H_2, where $\text{OptS}_0 = \text{OptB}_0 = z$,

$$\varrho^0_s = \hat{r}^1 \text{ and } \widetilde{\varrho}^0_b = \widetilde{r}^1,$$
$$\varrho^t_{s\ell} = \hat{r}^2 \text{ and } \widetilde{\varrho}^t_{b\ell} = \widetilde{r}^2.$$

`2. Decompose SGMIP into subproblems:`

Partition (5.57)–(5.100) into subproblems (5.144)–(5.158) and (5.160)–(5.170), and adjust parameters to subscript s and b; respectively.

`3. Solve SGMIP Subproblems:`

a) For $s = 1, ..., S$ solve security subproblems (5.144)–(5.158), where $\text{OptS}_k = \sum_1^S(\text{Optimal solution for security subproblem } s)$.

b) For $b = 1, ..., J$ solve bond subproblems (5.160)–(5.170), where $\text{OptB}_k = \sum_1^J(\text{Optimal solution for bond subproblem } b)$.

`I. Penalty Adjustment:`

a) if $\gamma^0_s, \gamma^t_{s\ell} = 0$

$\hookrightarrow \text{OptS} = \text{OptS}_k \quad \Rrightarrow$ Terminate (`I`.a)– go to `Step 4`.

else if $\text{OptS}_k < \text{OptS}_{k-1} - \tau_1$

$\hookrightarrow \text{OptS} = \text{OptS}_k \quad \Rrightarrow$ Terminate (`I`.a)– go to `Step 4`.

else

for $\gamma^0_s > 0$ set $\varrho^0_s = r^1 + \varrho^0_s$

$\gamma^t_{s\ell} > 0$ set $\varrho^t_{s\ell} = r^2 + \varrho^t_{s\ell}$

b) if $\widetilde{\gamma}_b^0, \widetilde{\gamma}_{b\ell}^t = 0$

$\hookrightarrow$ OptB = OptB$_k$ $\Rrightarrow$ Terminate (`I.`b)– go to `Step 4`.

else if OptB$_k$ < OptB$_{k-1} - \tau_1$

$\hookrightarrow$ OptB = OptB$_k$ $\Rrightarrow$ Terminate (`I.`b)– go to `Step 4`.

else

for $\widetilde{\gamma}_s^0 > 0$ set $\widetilde{\varrho}_b^0 = r^1 + \widetilde{\varrho}_b^0$

$\widetilde{\gamma}_{s\ell}^t > 0$ set $\widetilde{\varrho}_{s\ell}^t = r^2 + \widetilde{\varrho}_{b\ell}^t$

`II. Update and Resolve Subproblems:`

if $d < H_1$ set

$k = k + 1$

$d_1 = d_1 + 1$

$\hookrightarrow$ `Go to Step 3.` Solve OptS$_{k+1}$ and OptB$_{k+1}$ with updated $\varrho_{s\ell}^t, \varrho_s^0, \widetilde{\varrho}_{b\ell}^t$, and $\widetilde{\varrho}_b^0$.

else

$\hookrightarrow$ OptS = OptS$_k$ and

OptB = OptB$_k$ $\Rrightarrow$ Terminate (`II.`)– go to `Step 4`.

`4. Solve Master Problem:`

a) From `Step 3` set parameters ϱ^t and $\widetilde{\varrho}^t$, and set the optimal basis from OptS and OptB as the initial solution.

b) Solve the SGMIP in (5.123)–(5.133).

c) if $d_2 < H_2$ set

$d_2 = d_2 + 1$

i) if δ_ℓ^t, $\widetilde{\delta}_\ell^t$, λ_ℓ^t, $\chi_\ell^t < \tau_2$

$\Rrightarrow$ Terminate – `Optimal Solution`.

else

for δ_ℓ^t, $\widetilde{\delta}_\ell^t$, λ_ℓ^t, $\chi_\ell^t \geq \tau_2$

$\hookrightarrow (\mu_4, \mu_5, \mu_6, \mu_7) = \bar{r}(\mu_4, \mu_5, \mu_6, \mu_7)$ – go to `Step 2`.

else

$\Rrightarrow$ Terminate – `Optimal Solution`.

For the algorithm above, parameters in `Step 1.`c) are used to minimize the relaxations presented in (5.123)–(5.133) and ensure penalty goals are met. As one can deduce,

Sample Problem	Time difference (h)
1	+ 4.21
2	+ 4.97
3	+ 3.62
4	+ 4.65
5	+ 3.42
6	+ 4.62
7	+ 8.03
8	+ 2.10
9	+ 7.68
10	+ 4.91

Table 5.2: Difference in CPU time for each sample problem, where the + indicates the hours the decomposition algorithm finished before CPLEX.

$\hat{r}^1, \hat{r}^2, \widetilde{r}^1$, and $\widetilde{r}^2$ are penalty parameter initializations, whereas r^1, r^2, and $\bar{r}$ account for penalty and goal parameter amplifications. Also, z is the optimal solution initialization, which is typically set to zero, and d_1 and d_2 are simply counters that ensure the algorithm does not exceed H_1 or H_2 iterations. Finally, τ_1 is the tolerance that the objective value should decrease with respect to the previous value before accepting a solution and τ_2 is the maximum value that a portfolio goal cannot exceed.

Some aspects of the algorithm presented in this section are similar to the implementation of Chapter 4, however, the design involves additional measures and is more complex than its predecessor. We can now address such issues. As mentioned earlier, the optimal solution is found if the penalty variables are all equal to zero. Otherwise, the penalty adjustment step works to find an optimal value to the subproblems by minimizing the size of the penalty variables. For more information on the penalty adjustment (`Step 3.I.`) one can refer to Section 4.3.1, as it preforms the same procedure for bonds and stocks. After obtaining subproblem solutions, the optimal basis and penalty parameters are sent to the master problem (5.123)–(5.133). Due to the initial solution generated by the optimal basis and parameters solved in subproblems (5.144)–(5.158) and (5.160)–(5.170), `Step 4.`(b) solves relatively quickly. Then, `Step 4.`(c) is used as a portfolio goal check to ensure portfolio goals are not disregarded. Finally, portfolio optimality depends on the size of the relaxation variables. The larger the penalty parameters – the less constraint violations will occur, however, this comes with the cost of CPU time. Furthermore, since the subproblems can be solved much faster than solving the large master problem, obtaining the optimal value of the penalty parameters and performing the penalty adjustment step in the smaller subproblems is much more efficient than the converse. In Table 5.2 we present the CPU time difference between running the prescribed algorithm versus CPLEX, where the sample problems were taken from the portfolio results we provide in the next section. CPLEX was implemented using a default termination criteria, in which after a number of iterations variables are rounded to produce integer results; if necessary. As shown in Table 5.2, in all cases the algorithm we develop is able to outperform CPLEX with respect to the SGMIP model.

Finally, the algorithm presented in this section evaluates the goal parameters and relaxations in `Step 4.(i)`. The optimal values for each of the portfolio goals were derived in Subsection 5.2.3, where the portfolio goal subproblems contained models that were less constrained than the large SGMIP. Hence, it is not possible to improve these values in the further constrained problem of (5.57)–(5.100). The authors of [Ji *et al.*, 2005] provide a relevant financial example where the size of the relaxation parameters associated with portfolio goals have various effects on the problem. We set a tolerance for large

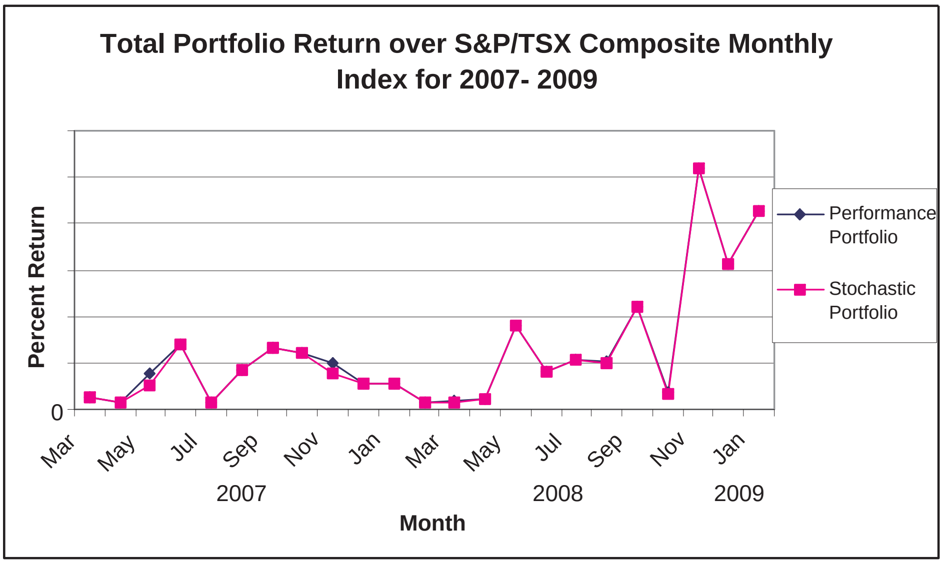

Figure 5.5: Worst case total portfolio results versus the known performance portfolio with respect to percent return over the S&P/TSX Composite Index.

deviations involving such relaxations following [Ji *et al.*, 2005] in `Step 4.(i).` For more information on goal parameter settings in a stochastic-goal program, one can also refer to [Sengupta, 1979; Van Hop, 2007]. In the next section we present the performance results of the algorithm and parameter settings in relation to portfolio return and capturing the modelling characteristics described in Section 5.2.

5.4 SGMIP Results

We solve the two-stage SGMIP problem presented in (5.57)–(5.100) using the monthly returns of the Toronto Stock Exchange (TSX) and Canadian bonds and convertibles. The stochastic model accounts for every security in the TSX from January 2007 to January 2009, which amounts to approximately 2000 securities for each time period depending on the month [TSX, 2009]. Also, we consider all Canadian bonds and convertibles over the same time period, which accounts for approximately 3000 variables. The S&P/TSX

composite monthly total return index was set as the portfolio benchmark R^0. In addition, we designed three market scenarios using the framework presented in Subsection 5.2.4 and used sector subproblem decompositions mentioned in Section 4.4. With this in place, the SGMIP problem we consider involves over 45,175 decision variables. The problem was solved using the algorithm described in Section 5.3 on a Pentium 4, 2.4 GHz CPU using CPLEX 9.1. For some instances where memory limitations became an issue, we report the best found solution. When compared to using CPLEX alone, the algorithm we designed improved memory allocation and CPU time.

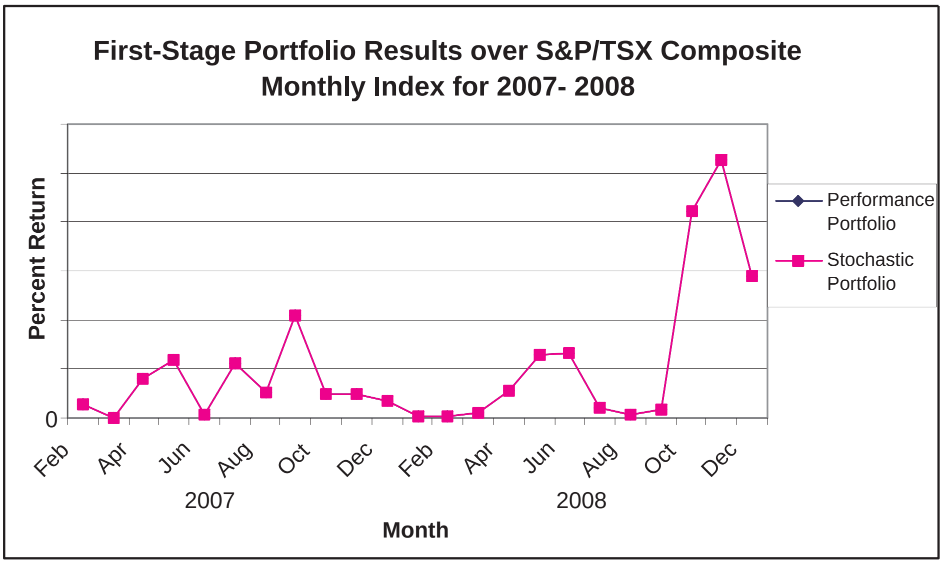

Figure 5.6: First-stage portfolio results versus the known performance portfolio with respect to percent return over the S&P/TSX Composite Index.

We present in-sample SGMIP portfolio results in comparison to the S&P/TSX composite monthly index. Portfolio managers value a portfolio based on their results with respect to a well-known benchmark. In this section we will display the portfolio percent return over the S&P/TSX composite index and also compare the stochastic portfolio to a performance portfolio, in which an equivalent single-scenario dynamic portfolio is

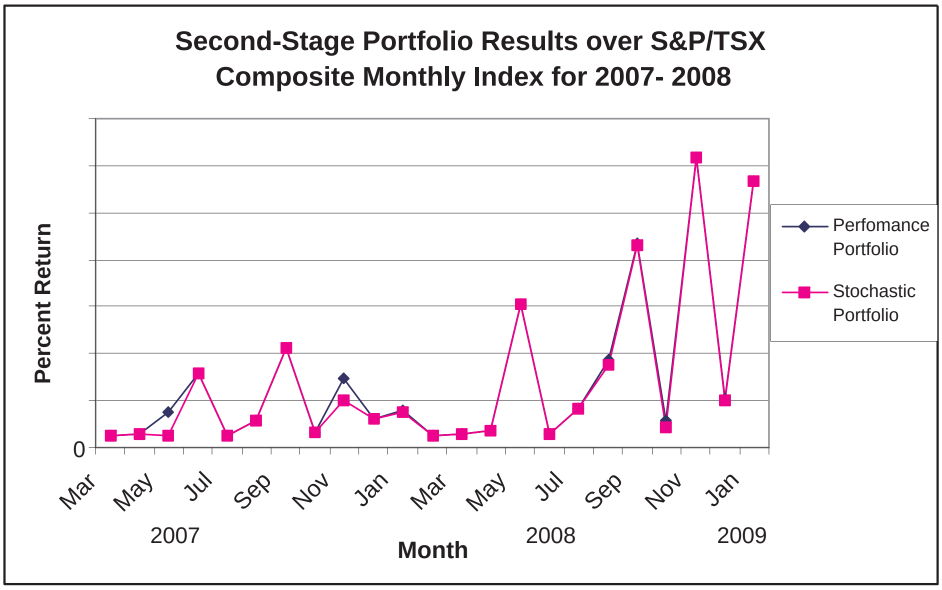

Figure 5.7: Second-stage portfolio results versus the known performance portfolio with respect to percent return over the S&P/TSX Composite Index.

solved allowing CPLEX to run fully. The CPU time results are shown in Table 5.2 of the previous section. Here we provide the quality of the portfolio solutions when CPLEX is run for the additional times presented earlier, where the algorithm in Section 5.3 is not implemented. In Figure 5.5 we show the portfolio return results over the S&P/TSX composite index and a comparison with a performance portfolio that is run from 2.10–8.03h longer on CPLEX. In all cases the stochastic portfolio model is able to outperform the index, especially when the index starts to drop heavily from November 2008 onward. The worst case stochastic portfolio results are presented in Figure 5.5, where even in the worst case the model behaves similarly to the performance portfolio. In many time periods the stochastic portfolio results are parallel to the performance portfolio, however, there are some instances where running CPLEX longer does not allow the stochastic portfolio to make as high gains. In the worst case, the stochastic portfolio total return is 3.2010% less than the performance portfolio. Also, the average value of $G^t + \widetilde{G}^t$ was 75, which defines the upper bound on the names-to-hold constraint that the algorithm in Section 5.3 initially relaxes to obtain a solution. The model on average makes 1 trade per month to rebalance portfolio investment weights. An interesting result of the portfolio model is that in the first-stage there are almost no differences in the investment results between the stochastic and performance portfolio, as shown in Figure 5.6. Hence, the extra gains that the performance portfolio makes in the second-stage seem to be associated with issues relating to algorithm speed-up and uncertainty. In Figure 5.7, the second-stage results are shown with respect to the performance portfolio and the S&P/TSX composite index. In the second-stage the worst case results occur in May 2007, where the performance portfolio outperforms the stochastic version by 6.5242%. For some of the months this value can be improved upon, which is illustrated in Figure 5.8 where we present the best and worst case scenario results. The stochastic portfolio return in

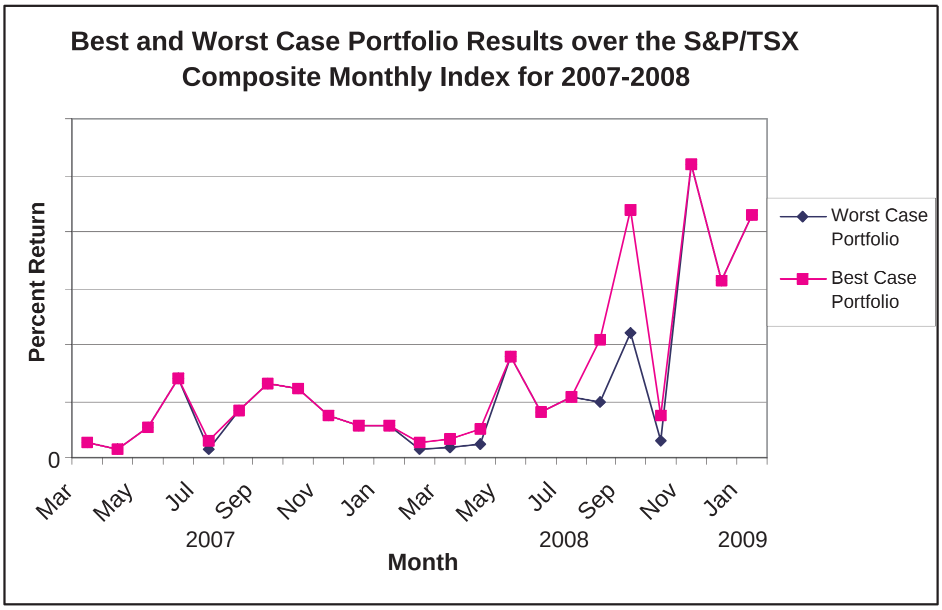

Figure 5.8: Best versus Worst case portfolio results with respect to percent return over the S&P/TSX Composite Index.

Figures 5.5–5.7 can increase by an average of 1.2496–1.6214% if we consider the best case results. In Figure 5.8, the month of September 2008 had the largest deviation with respect to the best versus worst case portfolio return, with a difference 8.3597%. Using the results from Section 4.4, improvements with respect to portfolio return can be made if more scenarios are included in the problem. The model was then run considering 11 scenarios, as this proved to be a valid index from the results in Chapter 4. For the 11 scenario problem, the worst case scenario results were almost equivalent to the 3 scenario problem, which is again due to the prescribed scenario generation method. However, the best case results for the 11 scenario problem had a slight improvement of 2.1557% with respect to the portfolio return. The addition of extra scenarios come with the cost of over 2.5 $\times$ the CPU time it takes to run the 3 scenario problem; as the portfolio problem in (5.57)–(5.100) greatly increases in size with every scenario and subproblem decompositions become too large. Nevertheless, the 3 scenario model is still able to outperform the S&P/TSX composite index and capture the portfolio elements listed in Section 5.2. The most interesting element of the stochastic portfolio is shown in Figure 5.9, where we illustrate the percentage of the portfolio that is invested in bonds. Initially, in Section 5.1 we mentioned that typically bonds and stocks have an inverse relationship with regards to investment returns. In other words, when stock prices rise bond interest is not as high, and vice versa. Figure 5.9 provides the proportion of the portfolio that is invested in bonds and includes a plot of the S&P/TSX composite index that has been reduced for illustrative purposes. As one may observe, when the S&P/TSX composite index value begins to decrease in September 2008, portfolio bond investments increase. This is a factor that we intended on capturing by adding the numerous portfolio elements present in the SGMIP model of (5.57)–(5.100). Thus, given the large portfolio design in Section 5.2, the algorithm and stochastic-goal programming approach perform well with respect to the S&P/TSX composite index and the factors that were intended to be captured in the design.

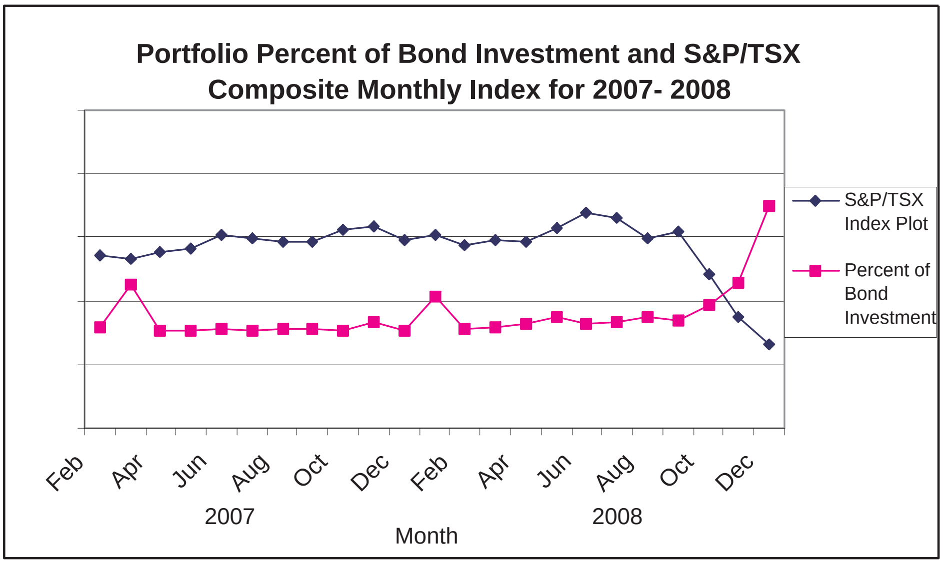

Figure 5.9: Percent of the portfolio that is invested in bonds and a comparison with that to the plot of the S&P/TSX Composite Index.

5.5 Portfolio Discussions

We present a complex portfolio selection problem that is able to capture financial instabilities present in current markets and is composed of a number of portfolio managing characteristics. In addition, we provide the details of a model specific algorithm that is designed to improve solution time and quality. From the results section, the dynamic of including stock and bond investments allows the portfolio to utilize different investment instruments in order to outperform the market index. This element is clearly illustrated in Figure 5.5 of Section 5.4. In comparison to the original work of Konno and Kobayashi (1997), the portfolio presented in Section 5.2 contains a number of additional managing characteristics that capture price uncertainty and associated risks. Figure 5.9 displays the capabilities of the model with respect to bond and stock investment weights during different financial periods. As mentioned in Section 5.1, bonds and stocks typically have an inverse relationship. Hence, when financial markets are strong, investing in stocks are favoured; and vice versa. Although this constraint is not implicitly defined in the design, through the modelling of portfolio goals and managing characteristics, the desired portfolio attribute can been seen when investigating portfolio constituent weighting in Figure 5.9.

The SGMIP is the main contributing factor for the strong portfolio results shown in Section 5.4. The modelling allows for the combination of uncertainty and risk in an intuitive design that focuses on a number of practical portfolio constraints. Using historical price movements to generate scenarios for securities and the addition of portfolio goals to minimize risks that may not be caught by price uncertainties alone, the portfolio results presented in Section 5.4 were obtained. Thus, the portfolio results are a combination of the SP aspect of uncertainty and the risk management utilization of GP. The portfolio algorithm also contributed to solution quality and was able to do so while minimizing CPU time. Similar to the previous chapter, we have designed a passive portfolio model that is aimed at minimizing market risks, but in this case we are more geared towards yielding a profit. The model has the capacity to handle additional portfolio managing characteristics, however, the algorithm would also need to facilitate such changes. In its current state, we consider the most practical financial characteristics, which also turned out to be some of the most complex. Also, by adding a few constraints to the design an ALM extension is possible; as mentioned in Subsection 5.2.2. Under the current SGMIP, the algorithm has been pushed to its limits with respect to the number of variables and solvability. Further discussions of future research topics and directions of the model will be presented in Chapter 6. In conclusion, the SGMIP portfolio selection model and algorithm presented in this chapter provide strong portfolio results for unstable markets and offer a convincing report that bond-stock portfolios are efficient.

Chapter 6

Conclusion and Future Research

6.1 Conclusion

We present three mathematical approaches to well-known financial problems that possess numerous sources of complexity. From the financial perspective, each of the models contain different portfolio goals and requirements, however, they involve similar money managing constraints. The quantitative designs answer, or at least provide great insight to the portfolio selection questions posed at the start of this document. The financial characteristics are constructed using practical constraints that minimize many assumptions used in other publications and have the capacity to consider future uncertainties. The addition of recourse scenarios to encapsulate uncertainties with respect to security price and risk is one aspect that differentiates our approach from previous attempts. The algorithms and mathematical models presented in Chapters 3 – 5 are the first of their nature. As mentioned in Chapter 1, many financial approaches involve complex assumptions or are aimed at highly specific aspects of money management so that solutions to mathematical portfolio designs can be simplified. We present portfolio models that are focused on direct financial representations and instead, develop highly specific algorithms to deal with computational treats posed by additions in modelling complexities. Hence, this thesis provides the framework and a detailed analysis of three instances where mathematical programming and financial modelling have been efficiently combined to produce effective designs. The contribution of this document involves the design and implementation of the following:

- Financial models that are composed of a comprehensive set of portfolio managing characteristics;
- A method to capture numerous financial uncertainties involved with security price, risk, and other portfolio goals;
- Highly specific algorithms designed to solve a demanding set of requirements found in the financial models constructed above.

The results presented in Chapters 3, 4, and 5 are direct investigations of the contributions above and support future approaches. In Chapter 3 we investigated models that contain a number of practical managing constraints added to the Markowitz risk verses return portfolio. Limiting the size of the portfolio proved to be the most interesting yet difficult constraint to include in the design. Based on the results of Jobst *et al.* [2001], we implemented an algorithm that is able to handle problems more than 65 times larger and produce respective efficient frontiers. In Chapter 4 we introduced a stochastic programming index tracking portfolio. The design involved a set of realistic index tracking constraints that used a SMIP modelling structure to facilitate future uncertainties related to security prices and index values. The algorithm generated to solve the model was specifically aimed at model structure and satisfying the names-to-hold constraint. Finally, in Chapter 5 a bond and stock portfolio selection model was presented that integrates uncertainties in security price and portfolio goals in a SGMIP design. The large MIP possessed a number of subproblems that were sequentially solved then combined in an algorithm that produced competitive results with respect to various benchmarks. The material presented in each chapter are the first of their nature and novel to the field. We provide solutions to a topic that has received a great amount of attention recently.

6.1.1 Model Comparisons

In each of the chapters we have concluded on the specific models and commented on the results of individual sections, yet how these models relate to each other is another important aspect of this document. The overall aim of the three investigations is to construct long-term investment portfolios that are well-suited for current financial markets and contain practical managing constraints. The MAD model with additional constraints is very typical of what is used by many financial companies to manage long-term portfolios. Although the MAD model is very practical from an industrial standpoint, it has some limitations. Firstly, the single-period approach limits the models capacity to deal with issues related to future portfolio rebalancing, transaction costs, future risks, and so forth. This is an aspect that the other models both consider, namely: the SMIP of Chapter 4 and the SGMIP of Chapter 5. Secondly, since the MAD model is generated from the MVO design, it is subject to many of the same financial criticisms. One of such critiques is related to whether or not portfolio risk is captured effectively using a covariance matrix. Another concerns the models sensitivity to the design parameters. For instance, Chopra and Ziemba (1993) state that the expected return has a strong effect on the solution. The SMIP and SGMIP handle such issues in a different manner than the MAD model. Taking the idea of portfolio risk, the SMIP and SGMIP account for this characteristic using various constraints and objectives. For example, through the use of the sector constraint a degree of portfolio diversity can be achieved, which is interpreted as being risk-adverse. In addition, the aim or objective of the portfolios are attributed to reducing risk. In the index tracking model, the objective of following or replicating a market index is known to be risk-adverse. Also, in the stock-bond SGMIP portfolio a number of risk measures are added to the model, one of which includes the index value and others are related to portfolio β and α values. Although such methods do not follow the traditional MVO risk measure, they perform well financially and are not subject to the same sensitivity issues of the Markowitz design. A final limitation of the MAD model is that it does not consider future uncertainties. The SMIP and SGMIP encapsulate modelling uncertainties by including future scenarios in the SP part of the design. This aspect of the model also contributes to capturing portfolio risk as future asset values are unknown.

Although the SMIP and SGMIP models overcome various limitation of the MAD model by including a number portfolio constraints, the designs pose greater difficulty with respect to generating optimal solutions; which was not the case for most instances of the MAD model. With regards to computational complexity, the SGMIP was the largest and most difficult problem to solve. In addition to the cardinality constraint, common to all models, the SGMIP contained many portfolio constraints to accommodate the different investing instruments, which came with a cost to CPU time. The SGMIP performance results, however, produced the most reliable investment figures. It possessed a two-stage stochastic approach to facilitate uncertainty issues omitted from the MAD model and contained different investment instruments and accompanying constraints to perform well under times of poor index results. This would have been an issue in the SMIP index tracking model. Although true index investors would ride out poor economic index returns, the SGMIP does so while investing in more profitable alternative instruments. In any case, each model fulfills its respective design. The MAD model facilitates managing designs used in practice, the index tracking SMIP generates reliable portfolios for index investors, and the SGMIP provides portfolio selection models for various types of investments that are well-suited for times of economic instability.

Further investigations will provide more insight towards these approaches and the results may improve or broaden the scope of this document. In the next section we highlight future directions of the research that may extend such studies and continue developments in the field of financial engineering and portfolio optimization.

6.2 Future Research

With respect to future research directions, below we discuss extensions from the models we have presented and then follow with other implementations involving further modelling and algorithm designs.

6.2.1 Mean–Variance Portfolio

In terms of future research associated with MVO investigations, additional modelling techniques and constraints can be added to the MAD portfolio design. As shown in Chapter 5, there exists a number of portfolio goals, risk measures, and managing characteristics that can be added to the design. Also, the SMIP method used in Chapter 4 can be applied to the design to generate a two-stage portfolio extension of the MAD model, which integrates security price uncertainties to the problem. Another area of interest consists of different types of algorithmic approaches, e.g. the simulated annealing method used in [Crama and Schyns, 2003]. There exists a number of heuristics that have been applied to various MIP problems. The combination of an ad hoc algorithm, such as the one presented in Chapter 3 and additional heuristics may improve CPU time and optimality results. We will discuss more on algorithmic approaches in Subsection 6.2.4.

6.2.2 Index Tracking Portfolio

With respect to the future developments of the index tracking model presented in Chapter 4, a multi-stage implementation that includes regime switching and/or scenario fix-mix seems promising. The two-stage approach performs very well and attempts that consider longer time durations would be the next direction, however, such models require additional time-stages and scenario generation methods. In addition, as the model and algorithm combination were designed for smaller markets such as the TSX, additional testing on larger markets may produce interesting results and further validate the design. With the index tracking portfolio in place, a SMIP index-based Asset and Liability Management (ALM) model might be another direction that may be explored.

The only trade-off of the SMIP solution approach is with respect to the names-to-hold constraint. For some instances, it was necessary to relax this constraint in order to find a feasible solution; however, the relaxation and such occurrences were minimized by the algorithm. Further investigations on such instances may improve optimal values and the algorithm design. Also, the algorithm developed in Subsection 4.3.1 can be parallelized. In which case, if we were to run a MPI (Message Passage Interface) code on the algorithm, where the number of processors are greater than or equal to the number of subproblems, then the CPU time would be equal to the time it takes to run the slowest subproblem. Thus, running a parallel code can have an even greater impact with regards to our implementation strategy.

6.2.3 Current Portfolio Selection

The SGMIP portfolio presented in Chapter 5 provided solutions to current portfolio problems. In terms of future work, the model can be enhanced with the addition of other portfolio managing characteristics, which would probably be in a more specific area of finance. We integrated the most common and some of the most complex portfolio elements into the design, however, there are less common portfolio elements that may be applicable. As mentioned in Section 5.2.2, with the inclusion of a few portfolio constraints the design can be extended to an ALM model. Also, the portfolio can cover a longer time period than what was shown in Section 5.4, in which case additional discussions on scenarios and scenario generation would be warranted. Finally, expanding the size of the problem by adding constraints and scenarios would mean that the algorithm would also have to be tailored to facilitate such additions. In its current state, we have pushed the algorithm to its limits with regards to the number of variables and solvability. Adding additional constraints and scenarios would present an even larger problem for algorithm design and solvability. Although we have developed a sensible algorithm, as mentioned in Subsection 6.2.1, additional experiments with various heuristics can be combined with the model specific solution strategy we present to possibly further improve computational results.

6.2.4 Modelling and Algorithmic Approaches

With respect to future modelling designs, *Robust Optimization* (RO) is an applicable method that may be integrated or used independently to model the problems considered in this document. Although RO generally involves introducing additional relaxations to the problem, the method may offer further insights or validations with regards to the optimal solutions derived in this document. The combination of the Stochastic, Goal, and Mixed Integer Programming financial approach of Chapter 5 is, to the best of our knowledge, one of the first of its nature. Introducing RO techniques to the design may have future modelling benefits. Both, the SMIP and SGMIP problem requirements are more demanding than what is typical for robust applications, however, using alternative approaches might offer different information about the designs.

We have developed three model specific algorithms that involve sensible solution strategies and produce viable approaches to a set of NP-hard problems. The algorithm designs take advantage of both, the model structure and the solver approach. All three problems consider some variation of MIPs, and with respect to future algorithm approaches there lies two major directions. The first, and more concrete, involves designing integrated MIP decompositions and algorithms. Although we have tested our method with the most well-known MIP algorithm techniques (when similar models existed), it is possible that combining such approaches might generate improved results. For example, methods that bounce between cutting-planes, Lagrangian decompositions, Benders method, column generations, etc. might offer additional insights. The second direction involves more of an abstract approach, which can be found with most heuristics. There may be an intuitive method to find a strong initial solution or address optimality issues that would improve results. Heuristics such as simulated annealing, genetic algorithms, and tabu searches are just a few examples of what has been developed thus far. Applying such methods to the designs presented in Chapters 3–5 and taking advantage of model decompositions may enhance results and CPU time. Finally, combinations of the two major directions we mention are additional areas for investigation with regards to future algorithmic approaches of the models we have presented.

As with any area of science and engineering, future directions and potential interests are the origin of novel discoveries and key developments in the field. Albert Einstein said "The important thing is not to stop questioning. Curiosity has its own reason for existing. One cannot help but be in awe when he contemplates the mysteries of eternity, of life, of the marvellous structure of reality." The mathematical framework and developments presented in this dissertation keeps us questioning; and provides a basis and foundation for future developments in the field of financial engineering and portfolio optimization.

Appendix A

Outstanding comments from Chapters 3–5

For the sake of completeness, we will prove LEMMA 1 on page 53.

LEMMA 1

Given the following problems have a unique solution:

$$\begin{array}{llll} \min & \mathrm{c}^\top x & & \\ & \mathrm{A}x \geq \mathrm{b} & (P) & \\ & x \geq 0 & & \end{array} \quad \text{and} \quad \begin{array}{llll} \min & \mathrm{c}^\top \overline{x} & & \\ & \mathrm{A}\overline{x} \geq \mathrm{b} & (\overline{P}) & \\ & \overline{x} \geq l & & \end{array}$$

if $l > 0$, then the values of the optimal basis $x^\star \leq \overline{x}^\star$.

Proof of LEMMA 1

Given that $l > 0$, and $x^\star$ and $\overline{x}^\star$ are the optimal basis to solutions of (P) and $(\overline{P})$; respectively. By contradiction, if $x^\star > \overline{x}^\star$ then $x^\star = 0$ is a violation since $l > 0$ and therefore, $x^\star \leq \overline{x}^\star$. $\square$

Next, we provide the details to the argument on page 89, which uses the results from [Blair and Jeroslow, 1981, 1977].

THEOREM 1

For the general mixed-integer program, given there is a finite set of variables in the program

$$\min \quad cx + dy \tag{A.1}$$

$$\text{s.t.} \quad Ax + By = b \tag{A.2}$$

$$x \geq 0, x \in \mathbb{I} \tag{A.3}$$

$$y \geq 0, \tag{A.4}$$

and suppose that A, B, c, d, and b are rational, and that (A.1)–(A.4) is consistent with a finite value. Also assume that (A.1)–(A.4) is bounded below in value (i.e. (A.1)–(A.4) has an optimal solution). Then for

$$\min \quad cx + dy + \varrho \left\| Ax + By - b \right\| \tag{A.5}$$

$$x \geq 0, x \in \mathbb{I} \tag{A.6}$$

$$y \geq 0, \tag{A.7}$$

when ϱ is sufficiently large, the optimal solution(s) to (A.5)–(A.7) are exactly the optimal solution(s) to (A.1)–(A.4).

To prove THEOREM 1 we first state the following Corollary and Lemma.

COROLLARY 1

For the mixed-integer program in (A.1)–(A.4), if A, B, c, d, and b are rational and

(A.1)–(A.4) is consistent and has a finite value. Then for

$$\begin{aligned} \inf \quad & cx + dy + \varrho \, \|Ax + By - b\| && \text{(A.8)} \\ & x \geq 0, x \in \mathbb{I} && \text{(A.9)} \\ & y \geq 0, && \text{(A.10)} \end{aligned}$$

when ϱ is sufficiently large, the optimal solution(s) to (A.8)–(A.10) are exactly the optimal solution(s) to (A.1)–(A.4). Specifically, the value to (A.8)–(A.10) is that of (A.1)–(A.4) for ϱ large.

Lemma 2

Let $G(z)$ denote the value of (A.1)–(A.4) with b replaced by z, then there is a $\varrho' > 0$ such that

$$G(z) \geq G(b) - \varrho' \|z - b\| \quad \forall \ z. \tag{A.11}$$

Proof of Corollary 1

From Lemma 2, given $\varrho > \varrho'$ (i.e. ϱ is sufficiently large) and $Ax + By \neq b$ then

$$\begin{aligned} cx + dy + \varrho \, \|Ax + By - b\| &\geq G(Ax + By) + \varrho \|Ax + By - b\| && \text{(A.12)} \\ &> G(Ax + By) + \varrho' \|Ax + By - b\| && \text{(A.13)} \\ &\geq G(b) = cx^\star + dy^\star, && \text{(A.14)} \end{aligned}$$

since $G(b)$ is the optimal value of (A.1)–(A.4) by definition. Thus, when ϱ is sufficiently large, the optimal solution(s) to (A.8)–(A.10) are equal to (A.1)–(A.4). □

For the proof of Lemma 2 the reader can refer to [Blair and Jeroslow, 1981] and Theorem 2.1 in [Blair and Jeroslow, 1981].

Proof of THEOREM 1

From COROLLARY 1 we have that for ϱ sufficiently large, the solution(s) to

$$\min\{cx + dy : Ax + By = b\} \;=\; \inf\{cx + dy + \varrho\|Ax + By - b\|\}. \tag{A.15}$$

Therefore, given that (A.1)–(A.4) is bounded below in value, then $\inf\{\cdot\} = \min\{\cdot\}$ and the optimal solution(s) to (A.5)–(A.7) are equal to the solution(s) of (A.1)–(A.4). □

A result of THEOREM 1 is that the penalty parameter ϱ can be unbounded. One may refer to [Blair and Jeroslow, 1981], where they consider the following problem for instances when $b \geq 0$:

$$\min \quad y \tag{A.16}$$

$$\text{s.t.} \quad x + y = b \tag{A.17}$$

$$x \geq 0, x \in \mathbb{I} \tag{A.18}$$

$$y \geq 0. \tag{A.19}$$

The solution to (A.16)–(A.19) is simply equal to the fractional part of b. Now, given that $b = 1 - \epsilon$ for a small $\epsilon > 0$, then (A.8)–(A.10) requires that

$$\inf \quad y + \varrho\,\|x + y - (1 - \epsilon)\| \tag{A.20}$$

$$x \geq 0, x \in \mathbb{I} \tag{A.21}$$

$$y \geq 0, \tag{A.22}$$

and thus,

$$\inf\{y + \varrho\|x + y - (1 - \epsilon)\|\} \;\geq\; 1 - \epsilon. \tag{A.23}$$

If $x = 1$ and $y = 0$, then from (A.23) we have

$$\varrho\|\epsilon\| \;\geq\; 1 - \epsilon \tag{A.24}$$

$$\varrho \;\geq\; \frac{1 - \epsilon}{\epsilon} \tag{A.25}$$

and as $\epsilon \to 0^+$, $\varrho \to \infty$. For more discussions on this topic see [Blair and Jeroslow, 1981, 1977]. In the analysis of Chapters 3–5, the penalty step is based on THEOREM 1 in that penalty parameters (ϱ) are aimed to be sufficiently large such that optimal solution(s) are obtained using the proposed algorithms in Sections 3.3, 4.3, and 5.3.

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