



## Two-sided coherent risk measures and their application in realistic portfolio optimization <sup>☆</sup>

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### ABSTRACT

By using a different derivation scheme, a new class of two-sided coherent risk measures is constructed in this paper. Different from existing coherent risk measures, both positive and negative deviations from the expected return are considered in the new measure simultaneously but differently. This innovation makes it easy to reasonably describe and control the asymmetry and fat-tail characteristics of the loss distribution and to properly reflect the investor's risk attitude. With its easy computation of the new risk measure, a realistic portfolio selection model is established by taking into account typical market frictions such as taxes, transaction costs, and value constraints. Empirical results demonstrate that our new portfolio selection model can not only suitably reflect the impact of different trading constraints, but find more robust optimal portfolios, which are better than the optimal portfolio obtained under the conditional value-at-risk measure in terms of diversification and typical performance ratios.

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### 1. Introduction

One of the basic problems of applied finance is the optimal selection of stocks, with the conflicting objectives of maximizing future return and minimizing investment risk. The first systematic treatment of this dilemma is the mean–variance (MV) approach proposed by Markowitz (1952). This approach has limited generality since the MV model will only lead to optimal decisions if utility functions are quadratic or if investment returns are jointly elliptically distributed. Quadratic utility is unlikely because it implies increasing absolute risk aversion over the whole domain. The assumption of elliptically distributed returns sounds unrealistic since it rules out possible asymmetry in return distributions. It is nowadays a stylised fact that the distributions of many financial return series are non-normal, with skewness and/or leptokurtosis ('fat tails') pervasive. The well-accepted method for measuring future return is to use the mean of the random return distribution. Nonetheless, the

best way to measure the investment risk has not been found even after many different measures having been proposed. Aimed at characterizing an investor's preferences, it is recently examined in Rachev et al. (2008) about desirable properties of an ideal risk measure.

The MV model treats both the above and the below expected returns equally. Nevertheless, there is ample evidence that agents often treat losses and gains asymmetrically (see, for example, Kahneman et al., 1990). The concept of the downside risk apparently has considerable impact on the investor's viewpoint regarding the risk. A general family of below-target risk measures, named the lower partial moment (LPM), was proposed by Bawa (1975) and Fishburn (1977). A comprehensive accumulation of knowledge on the concept of downside risk can be found in Sortino and Satchell (2001). Especially, in the last 10 years, there has been a great momentum in research on quantile-based risk measures because of the introduction of the value-at-risk (VaR) (Morgan, 1996). Unfortunately, VaR has undesirable properties such as lack of sub-additivity; when calculated using scenarios, VaR is non-convex, non-smooth as a function of investment positions and is difficult to optimize (Mausser and Rosen, 1991).

Aimed at providing a comprehensive theory that can relate and compare different risk measure approaches, an important step toward consistent measures of risk was made in a series of papers

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(Artzner et al., 1997, 1999; Delbaen, 2002), which introduced the notion of a coherent risk measure. It is easy to demonstrate that VaR does not provide coherency, neither do existing LPM measures. Extensive researches have been aroused since the appearance of this new definition. Typical measures of this class include the expected shortfall (ES) defined in Acerbi and Tasche (2002) and the conditional value-at-risk (CVaR) developed in Rockafellar and Uryasev (2002). The most general theoretical result about this kind of measures is a complete space of coherent risk measures spectrally generated in Acerbi (2002). The relationship among moment-based, distortion and spectral risk measures is recently investigated in Adam et al. (in press).

These new risk measures alleviate the problems inherent in VaR by considering losses beyond the VaR level. Nevertheless, considered in these measures is only the linear probability weighted combination of losses beyond VaR. As we know, it is necessary to consider different orders of moments of a non-normal distribution in order to describe it comprehensively. Therefore, it should be better to consider the probability weighted combination of higher order losses below some critical value. In respect of risk measurement, the expectation of suitable order of lower tail losses is useful for describing the investor's degree of risk aversion, taking higher order losses is also helpful for controlling fat tails in the loss distribution (Chen and Wang, 2007). The only four papers, which mentioned the application of higher moments in the coherent risk measure construction, are Delbaen (2002), Fischer (2001, 2003) and Tasche (2002). It is briefly discussed in an example in Delbaen (2002) about the possible use of higher moments for defining coherent risk measures. Given in Fischer (2001, 2003) are some examples of coherent risk measures defined by directly combining the negative mean and weighted one-sided lower moments. It is also mentioned in Remark 3.8 in Tasche (2002) about the incorporation of higher moment effects in ES. In these theoretical oriented papers, the authors did not discuss the application of those risk measures to portfolio optimization.

In all the above mentioned LPM measures and coherent risk measures, the attention is put on either the demand side or the offer side. Only one-sided, that is, the lower part distribution information is used in the associated risk measure. Nevertheless, from the game theory point of view, the investor should take into account the specific action of the seller of a stock when he/she wants to buy that stock. This means that the attitudes towards potential losses from both demand and offer sides have to be taken into account simultaneously. In other words, to suitably measure investment risk and thereafter to make robust investment decision, a more general “two-sided” risk measure should be adopted. Meanwhile, as an unwanted side-effect of the strict separation between downside risk and upside potential, data are thrown away in the estimation process of LPM and quantile-based measures. This partial use of data renders those measures much more susceptible to estimation risk than their full domain analogue. Therefore, from the practical application point of view, it is also necessary for us to use some kind of two-sided or full domain risk measure. Preferably, such a measure will satisfy coherency conditions.

The only “two-sided” risk measure we have seen so far is that in Hamza and Janssen (1998), where the risk is measured through the convex combination of semivariances of the portfolio rate of return. Although the degree of risk aversion can be reflected by selecting the combination coefficient, the corresponding adjustment is only linear. This measure is still symmetrical in the sense that semivariances are used for measuring both below and above deviations from the mean. These two facts cause it difficult to represent the different roles of demand and offer sides and to control the fat-tail phenomenon. Last but not least, this risk measure is not a coherent measure.

Instead of measuring the overall seriousness of possible losses associated with random payoffs as that in coherent risk measures, general deviation measures are recently introduced in Rockafellar

et al. (2006) to measure the pure uncertainty in the random payoff. This kind of measures need not be symmetric between “ups” and “downs”. It is shown that lower range bounded deviation measures correspond one-to-one with coherent, strictly expectation bounded risk measures. Many interesting examples of deviation measures are derived from variants of CVaR and from basic error functionals in Rockafellar et al. (2006). Moreover, Rockafellar et al. (2007) lately demonstrated the existence of equilibrium in a financial market with investors using a diversity of deviation measures.

For real applications, what we mainly concern should be the practicality of the proposed risk measure for finding robust and efficient portfolios. Just recently, Quaranta and Zaffaroni (in press) applied the robust optimization technique to the minimization of the CVaR and the corresponding portfolio selection problem. Except for this, most existing theoretical papers about coherent risk measure (such as Acerbi, 2002; Fischer, 2003; Rockafellar et al., 2006) did not consider the application of the proposed risk measure for making optimal investment decision, needless to say the realistic portfolio selection problem with multiple market frictions taken into account simultaneously.

Bearing in mind the above limitations in current risk measures, a new class of two-sided coherent risk measures is constructed in this paper by properly combining the downside and the upside of the random payoff. As an application-oriented research and along a new derivation way, our new measure can be regarded as the improvement on preliminary coherent risk measures in Acerbi (2002), Fischer (2001, 2003) and as a variant of deviation measures in Rockafellar et al. (2006). Compared with existing risk measures, our new measure has the following advantages: the whole domain loss distribution information is utilized, which makes the new measure superior for finding robust (with respect to both trading sides) and stable (with respect to the estimation error) investment decisions; by suitably selecting the convex combination coefficient and the order of the norm of the downside random loss, it is easy for our new measure to reflect the investor's risk attitude and to control the asymmetry and fat tails of the loss distribution. Most importantly, it is easy to compute the new measure's value and to apply it to find the optimal portfolio. We will demonstrate these points by establishing a portfolio selection model with multiple market friction constraints and applying it to find robust optimal portfolios in real application.

## 2. The new risk measure and its properties

We consider a one-period framework, which means that we have the current date 0 and a future date  $D$ . No trading is possible between 0 and  $D$ . Therefore, “risk” here can be represented by the random payoff  $X$  (that is, the random profit if  $X \geq 0$ , or loss if  $X < 0$ ), defined on a probability space  $(\Omega, \mathcal{F}, P)$ , of some assets or portfolios at  $D$ . Measuring risk is then equivalent to establishing a correspondence  $\rho$  between the space of  $X$  and the set of real numbers. To avoid tediously mathematical treatment, we assume that  $X$  is an  $p$ -integrable random variable.  $X$  can thus be treated as an element of the space  $L^p(\Omega, \mathcal{F}, P)$  for  $1 \leq p \leq \infty$ .

In what follows, the risk measure  $\rho(X)$  is considered to be the minimum extra cash added to  $X$  that makes the position acceptable for the holder (Artzner et al., 1999). In this sense, a positive  $\rho(X)$  means that the investor has to add  $\rho(X)$  amount of extra cash to ensure the acceptance of his future position, a negative  $\rho(X)$  means that the investor can withdraw  $-\rho(X)$  amount of money without affecting the acceptance of his future position. As usual, we define  $\|X\|_p = (E_Q|X|^p)^{1/p}$ ,  $\|X\|_\infty = \text{ess.sup}\{|X|\}$ . Let  $X^-$  denote  $\max(-X, 0)$ , and  $X^+$  denote  $(-X)^-$ . Lastly, define  $\sigma_p^\pm(X) = \|(X - E_Q[X])^\pm\|_p$ .

For reasons demonstrated in the last section, we propose here a new kind of two-sided risk measures that takes into account both sides of the loss distribution. Relative to the expected value  $E_Q[X]$ , the random variable  $(X - E_Q[X])^-$  is the “downside” of  $X$ , which

may well be more crucial to an application than the corresponding “upside” random variable  $(X - E_Q[X])^+$ . Due to this and the monotonically increasing property of  $\|\cdot\|_p$  with respect to  $p$ , the convex combination of the 1-norm of the upside random variable  $(X - E_Q[X])^+$  and the  $p$ -norm ( $p \geq 1$ ) of the downside random variable  $(X - E_Q[X])^-$  is used to define the new risk measure as follows.

**Definition 2.1.** Given  $p \in [1, \infty)$ ,  $0 \leq a \leq 1$ , the new risk measure  $\rho_{a,p} : L^p(Q) \rightarrow \mathbf{R}$  is determined by

$$\begin{aligned} \rho_{a,p}(X) &= a\sigma_1^+(X) + (1-a)\sigma_p^-(X) - E_Q[X] \\ &= a\|(X - E_Q[X])^+\|_1 + (1-a)\|(X - E_Q[X])^-\|_p - E_Q[X], \quad (1) \end{aligned}$$

$\rho_{a,p}(X)$  improves one-sided risk measures in Delbaen (2002), Fischer (2001, 2003) and Tasche (2002) by considering not only the lower partial moment  $\sigma_p^-(X)$ , but also the upper partial moment  $\sigma_1^+(X)$ .  $\rho_{a,p}(X)$  also extends the risk measure in Hamza and Janssen (1998) since, instead of adopting both semivariances, it uses the 1-norm of  $(X - E_Q[X])^+$  and the  $p$ -norm of  $(X - E_Q[X])^-$ . Our new measure looks like the deviation measures in Rockafellar et al. (2006) in that the positive deviation and negative deviation from the mean are treated asymmetrically. Nevertheless, the deviation measures in Theorem 3 in Rockafellar et al. (2006) were constructed by first combining positive and negative deviations, and then taking the  $p$ -norm of the weighted combination. Derived along a different way,  $\rho_{a,p}(X)$  is generated by first taking the 1-norm of the positive deviation and the  $p$ -norm of the negative deviation, respectively, and then taking the convex combination of these two norms. Our new risk measures are also different from related coherent, strictly expectation bounded risk measures given in Rockafellar et al. (2006) as  $\|(X - E_Q[X])^+\|_1$  and  $-E_Q[X]$  are simultaneously included in (1). Two other functions of  $-E_Q[X]$  in (1) are: it ensures that  $\rho_{a,p}(X)$  has good properties such as coherency; it allows us to provide a reasonable explanation for (1). Hence,  $E_Q[X]$  in (1) is not a benchmark, and  $\rho_{a,p}(X)$  is not a performance measure, it differs from those performance measures in Biglova et al. (2004), Farinelli and Tibiletti (2008) and related papers.

Moreover, since  $E((X - E_Q[X])^+) = E((X - E_Q[X])^-)$ , when the investor minimizes  $\|(X - E_Q[X])^+\|_1 = E((X - E_Q[X])^+)$ , he/she actually minimizes his/her dispersion from the mean. This is the main difference between one-sided coherent risk measures and our two-sided risk measure in (1). The most important advantage of  $\rho_{a,p}(X)$  is its practical value.  $\rho_{a,p}$  can be used to find more realistic and robust optimal portfolios than those obtained under CVaR, as will be shown in Section 4. Here, we can illustrate the flexibility and practicality of  $\rho_{a,p}$  as follows.

Formula (1) depends on two parameters  $p$  and  $a$ , which can be used to flexibly express different attitudes towards risk.  $a$  is a ‘global’ factor linearly adjusting the balance between good volatility and bad volatility, it can also reflect risk neutrality or preference by setting  $p = 1$  and  $a = 0.5$  or  $p = 1$  and  $0.5 < a \leq 1$ , respectively;  $p$  is a ‘local’ factor nonlinearly controlling investment risk, the greater the  $p$ , the more risk-aversion the investor; the pervasive skewness and/or leptokurtosis of the loss distribution can be accounted by choosing suitable  $p$ . The choice of  $p$  and  $a$  thus relies on the agent’s attitude towards risk and the payoff data property. In reality, one can first select  $a$ , the value of  $p$  is then properly selected to subtly control the loss risk and extreme events.

Since  $\|(X - E_Q[X])^+\|_1 = \|(X - E_Q[X])^-\|_1$ ,  $\rho_{a,p}(X)$  can be treated as the convex combination of coherent risk measures  $\|(X - E_Q[X])^-\|_1 - E_Q[X]$  and  $\|(X - E_Q[X])^-\|_p - E_Q[X]$  mentioned in Fischer (2003) and Rockafellar et al. (2006). Because the convex combination of coherent risk measures is still a coherent risk measure, we have:

**Theorem 2.1.** For any  $p : 1 \leq p \leq \infty$  and  $a \in [0, 1]$ , the risk measure  $\rho_{a,p}$  defined by (1) is a coherent risk measure.

Furthermore, by utilizing Lemma 4.2 in Fischer (2003) and Proposition 2.2 in Acerbi (2002), it is easy to prove the following theorem:

**Theorem 2.2.** For any  $p_i \geq 1, 0 \leq a_i \leq 1, \lambda_i \in (0, 1), i = 1, \dots, \infty$ , with  $\sum_{i=1}^{\infty} \lambda_i = 1$ , the measure

$$\rho_{a,p}(X) = \sum_{i=1}^{\infty} \lambda_i \rho_{a_i, p_i} = -E_Q[X] + \sum_{i=1}^{\infty} \lambda_i a_i \sigma_1^+ + \sum_{i=1}^{\infty} \lambda_i (1 - a_i) \sigma_{p_i}^-$$

is a coherent risk measure on  $L^q(Q)$ , here  $q = \sup\{p_i | p_i \geq 1, i = 1, \dots, \infty\}$ .

Theorem 2.2 tells us that the “generalized” convex combination of multiple  $\rho_{a,p}$ s can also be adopted as the risk measure. Through this, advantages of different  $\rho_{a,p}$ s can be utilized simultaneously. For instance, by taking the convex combination of  $\rho_{a_1,2}$  ( $0 \leq a_1 \leq 1$ ) and  $\rho_{a_2,\infty}$  ( $0 \leq a_2 \leq 1$ ), we get the following coherent risk measure

$$\rho(X) = -E_Q[X] + (\lambda_1 a_1 + \lambda_2 a_2) \sigma_1^+ + \lambda_1 (1 - a_1) \sigma_2^- + \lambda_2 (1 - a_2) \sigma_{\infty}^-,$$

here  $0 < \lambda_1, \lambda_2 < 1, \lambda_1 + \lambda_2 = 1$ . This risk measure possesses properties of absolute deviation (AD) (Konno and Yamazaki, 1991), semivariance and maximum loss (Cai et al., 2000) risk measures.

The next theorem shows the behavior of the new risk measure with respect to two parameters  $a$  and  $p$  included in its definition.<sup>1</sup>

**Theorem 2.3.** The coherent risk measure  $\rho_{a,p}$  is non-decreasing with respect to  $p$ , and is non-increasing with respect to  $a$ , respectively.

The monotonic property of  $\rho_{a,p}$  with respect to  $p/a$  can be utilized to reflect the investor’s attitude towards risk. Concretely, the non-decreasing property of  $\rho_{a,p}$  with respect to  $p$  means that: the greater the  $p$ , the larger the  $\rho_{a,p}$ , the investor adopting this risk measure would treat  $X$  riskier than the investor adopting  $\rho_{a,p}$  with a smaller  $p$ . Therefore,  $\rho_{a,p}$  with large  $p$  should be connected with the strongly risk-averse investor. On the other hand, the non-increasing property of  $\rho_{a,p}$  with respect to  $a$  means that, for fixed  $p$ ,  $\rho_{a,p}$  with large  $a$  should be connected with the less risk-averse investor. When  $a = 1$ ,  $\rho_{a,p}$  essentially collapses to the absolute deviation measure; when  $a = 0$ ,  $\rho_{a,p}$  reduces to the risk measure in Fischer (2003).

Taking AD and CVaR as examples, we now discuss the relationship among  $\rho_{a,p}$  and other typical risk measures. Concretely,<sup>2</sup>

**Theorem 2.4.** The risk measures  $\rho_{a,p}$  and AD satisfy

$$\rho_{a,p}(X) + E_Q[X] \geq \text{AD}(X)/2, \quad (2)$$

where the equality holds if and only if  $p = 1$ . Meanwhile, the following inequality holds for  $\rho_{a,1}$  and CVaR:

$$\rho_{a,1}(X) < \text{CVaR}_{\gamma}(X), \quad (3)$$

here  $0 \leq \gamma \leq \alpha < 1$ ,  $\alpha$  is selected such that  $E_Q[X] \in [x_{\alpha}, x^{\alpha}]$ , with  $x_{\alpha}$  and  $x^{\alpha}$  being the lower and upper  $\alpha$ -quantiles, respectively.  $\text{CVaR}_{\gamma}$  denotes the CVaR value at the confidence level  $\gamma$ . Furthermore,

$$\text{CVaR}_{\alpha}(X) - \rho_{a,1}(X) = (1/\alpha - 1)E_Q[(X - E_Q[X])^-].$$

### 3. The realistic portfolio selection model under new risk measure

The application of our new risk measure for optimal and robust investment decision making will be demonstrated in this section by establishing the associated portfolio selection model. In order

<sup>1</sup> The detailed proof is available upon request from the authors.

<sup>2</sup> Due to the space limitation, we do not include here the concrete proof, which can be provided upon request.

to achieve a greater realism in our problem modelling, several market frictions will be taken into account simultaneously. Suppose that there exist  $n$  risky assets and one riskless asset available for investment. To make it easier to follow our exposition, we first put together all the notations that will appear hereafter, as follows:

- $x_i^0$  the initial holding of the  $i$ th risky asset ( $1 \leq i \leq n$ ) or the riskless asset ( $i = n + 1$ );
- $x_i$  the final amount that the investor will invest in the  $i$ th ( $1 \leq i \leq n$ ) risky asset or the riskless asset ( $i = n + 1$ );
- $r_i$  the holding period rate of return on risky asset  $i$  ( $i = 1, \dots, n$ ), equal to the ratio of the price of the asset at the end of the investment period to its current price, and minus 1;
- $r_{n+1}$  the constant holding period rate of return on the riskless asset;
- $\hat{r}_i = E[r_i]$  the expected rate of return on risky asset  $i$  ( $i = 1, \dots, n$ );
- $d_i$  the dividend yield on risky asset  $i$  ( $i = 1, \dots, n$ ), equal to the monetary dividend divided by its current price;
- $t_g$  the marginal capital gains tax rate for the investor;
- $t_0$  the marginal ordinary income tax rate for the investor;
- $d'_i$  ( $d''_i$ ) the proportional transaction cost ratio for buying (selling) the security  $i$  ( $i = 1, \dots, n$ );
- $p_i$  ( $q_i$ ) the unit price for buying (selling) the risky asset  $i$  ( $1 \leq i \leq n$ );
- $p_{n+1} = q_{n+1}$  the unit price of the riskless asset;
- $C$  the total capital for investment.

With the above notations, the actual buying and selling amount for the  $i$ th risky asset will be given by  $x'_i = (x_i - x_i^0)^+$  and  $x''_i = (x_i - x_i^0)^-$ , respectively. The net investment in risky asset  $i$  can be expressed as  $p_i x'_i - q_i x''_i$ ,  $i = 1, \dots, n$ , here  $p_i x'_i$  is the corresponding purchase and  $q_i x''_i$  is the corresponding sale. The net investment in the riskless asset is  $p_{n+1}(x_{n+1} - x_{n+1}^0)$ . The total capital gain on the portfolio  $x = (x_1, \dots, x_n, x_{n+1})$  is  $\sum_{i=1}^n r_i(p_i x'_i - q_i x''_i)$ , and the total ordinary income on the portfolio is

$$\sum_{i=1}^n d_i(p_i x'_i - q_i x''_i) + r_{n+1}p_{n+1}(x_{n+1} - x_{n+1}^0).$$

Transaction cost is an important factor for an investor to take into consideration in portfolio selection. We assume that transaction cost is a V-shaped function of the difference between an existing portfolio and a new one. The total transaction cost is then  $\sum_{i=1}^n (d'_i p_i x'_i + d''_i q_i x''_i)$ .

As what happens in reality, it is assumed that there are taxes on both capital gain and ordinary income, it is also assumed that dividends and transaction costs on risky assets are paid at the end of the investment period and are known with certainty at the beginning. Hence, the net cash flow after excluding taxes and transaction costs on the portfolio is

$$\begin{aligned} g(x) &= (1 - t_g) \sum_{i=1}^n r_i(p_i x'_i - q_i x''_i) \\ &\quad + (1 - t_0) \left[ \sum_{i=1}^n d_i(p_i x'_i - q_i x''_i) + r_{n+1}p_{n+1}(x_{n+1} - x_{n+1}^0) \right] \\ &\quad - \sum_{i=1}^n (d'_i p_i x'_i + d''_i q_i x''_i) \\ &= \sum_{i=1}^n \hat{R}_i(p_i x'_i - q_i x''_i) + \hat{R}_{n+1}p_{n+1}(x_{n+1} - x_{n+1}^0) \\ &\quad - \sum_{i=1}^n (d'_i p_i x'_i + d''_i q_i x''_i), \end{aligned}$$

where  $\hat{R}_i \equiv (1 - t_g)r_i + (1 - t_0)d_i$  is the after-tax rate of return with dividend reinvested on risky asset  $i$ ,  $i = 1, \dots, n$ , and  $\hat{R}_{n+1} \equiv (1 - t_0)r_{n+1}$  is the after-tax rate of return on the riskless asset. The

expected net cash flow after excluding taxes and transaction costs on the portfolio is thus

$$\begin{aligned} G(x) &\equiv E[g(x)] \\ &= \sum_{i=1}^n \hat{R}_i(p_i x'_i - q_i x''_i) + \hat{R}_{n+1}p_{n+1}(x_{n+1} - x_{n+1}^0) \\ &\quad - \sum_{i=1}^n (d'_i p_i x'_i + d''_i q_i x''_i), \end{aligned}$$

here  $\hat{R}_i \equiv E[\hat{R}_i] = (1 - t_g)\hat{r}_i + (1 - t_0)d_i$  is the expected after-tax rate of return with dividend reinvested on risky asset  $i$ ,  $i = 1, \dots, n$ .

According to the definition of the new risk measure  $\rho_{a,p}$ , the risk of random  $g(x)$  can be expressed as

$$\begin{aligned} \rho_{a,p}(g(x)) &= a \left\| \left( \sum_{i=1}^n (R_i - \hat{R}_i)(p_i x'_i - q_i x''_i) \right)^+ \right\|_1 \\ &\quad + (1 - a) \left\| \left( \sum_{i=1}^n (R_i - \hat{R}_i)(p_i x'_i - q_i x''_i) \right)^- \right\|_p - G(x) \\ &= (1 - t_g) \left\{ a \left\| \left( \sum_{i=1}^n (r_i - \hat{r}_i)(p_i x'_i - q_i x''_i) \right)^+ \right\|_1 \right. \\ &\quad \left. + (1 - a) \left\| \left( \sum_{i=1}^n (r_i - \hat{r}_i)(p_i x'_i - q_i x''_i) \right)^- \right\|_p \right\} - G(x). \end{aligned}$$

Except for taxes and transaction costs, other typical constraints should also be considered so that real characteristics of the practical investment environment can be properly modelled. Concretely, the constraint on the capital budget can be written as

$$\sum_{i=1}^n [(1 + d'_i)p_i x'_i + (-1 + d''_i)q_i x''_i] \leq C.$$

Lower and upper bound (denoted as  $\underline{x}_i$  and  $\bar{x}_i$ , respectively) constraints on the holdings of risky assets are simply

$$\underline{x}_i \leq x_i \leq \bar{x}_i, \quad i = 1, \dots, n,$$

which are imposed partly because of institutional restrictions and partly because of deficiencies in the estimation of investment returns.

The value constraint is introduced to avoid security  $i$  to constitute more than a given percent,  $v_i$ , of the total portfolio value. This kind of constraints can be described as

$$q_i x_i \leq v_i \sum_{j=1}^{n+1} q_j x_j, \quad i = 1, \dots, n + 1.$$

Considering real situations in many stock markets, we rule out the possibility of short sales. That is,  $x_j \geq 0$  for  $1 \leq j \leq n$ . The investor's objective is to minimize the investment risk while ensuring certain amount of net cash at the end of the investment period. With all the above practical constraints, the optimal investment strategy can be found by solving the following comprehensive portfolio selection model

$$\begin{aligned} \min \quad & \rho_{a,p}(g(x)) \\ \text{s.t.} \quad & E(g(x)) \geq e, \\ & \sum_{i=1}^n [(1 + d'_i)p_i x'_i + (-1 + d''_i)q_i x''_i] \leq C, \\ & q_i x_i \leq v_i \sum_{j=1}^{n+1} q_j x_j, \quad i = 1, \dots, n + 1, \\ & x'_i = (x_i - x_i^0)^+, \quad i = 1, \dots, n, \\ & x''_i = (x_i - x_i^0)^-, \quad i = 1, \dots, n, \\ & 0 \leq \underline{x}_i \leq x_i \leq \bar{x}_i, \quad i = 1, \dots, n + 1, \end{aligned} \tag{4}$$

where  $e$  is the demanded minimal net cash value.

As the complete distribution of  $r_i$  is rarely known in reality, the expected rate of return  $\hat{r}_i$  is very often estimated by the average of a set of data  $\{r_{it}, t = 1, \dots, T\}$ , here  $r_{it}$  can be either the observed historical sample or the forecasted rate of return at time  $t$  for security  $i$ . In other words, we set  $\hat{r}_i = \sum_{t=1}^T r_{it}/T$ . The portfolio risk  $\rho_{a,p}(g(x))$  can then be estimated as

$$\hat{\rho}_{a,p}(x) = (1 - t_g) \left\{ a \left[ \sum_{t=1}^T u_t / T \right] + (1 - a) \left[ \sum_{t=1}^T v_t^p / T \right]^{1/p} \right\} - \left\{ \sum_{i=1}^n ((1 - t_g) \sum_{t=1}^T r_{it} / T + (1 - t_0) d_i) (p_i x'_i - q_i x''_i) + (1 - t_0) r_{n+1} p_{n+1} (x_{n+1} - x_{n+1}^0) - \sum_{i=1}^n (d'_i p_i x'_i + d''_i q_i x''_i) \right\},$$

here  $u_t$  and  $v_t$  satisfy

$$u_t - v_t = \sum_{i=1}^n (r_{it} - \hat{r}_i) (p_i x'_i - q_i x''_i), \quad t = 1, \dots, T, \\ u_t v_t = 0, \quad u_t \geq 0, \quad v_t \geq 0, \quad t = 1, \dots, T.$$

With the above expression for the risk, problem (4) can be transformed into the following realistic portfolio optimization problem:

$$\min \quad (1 - t_g) \left\{ a \left[ \sum_{t=1}^T u_t / T \right] + (1 - a) \left[ \sum_{t=1}^T v_t^p / T \right]^{1/p} \right\} - \left\{ \sum_{i=1}^n ((1 - t_g) \sum_{t=1}^T r_{it} / T + (1 - t_0) d_i) (p_i x'_i - q_i x''_i) + (1 - t_0) r_{n+1} p_{n+1} (x_{n+1} - x_{n+1}^0) - \sum_{i=1}^n (d'_i p_i x'_i + d''_i q_i x''_i) \right\} \quad (5)$$

$$\text{s.t.} \quad \sum_{i=1}^n ((1 - t_g) \sum_{t=1}^T r_{it} / T + (1 - t_0) d_i) (p_i x'_i - q_i x''_i) + (1 - t_0) r_{n+1} p_{n+1} (x_{n+1} - x_{n+1}^0) - \sum_{i=1}^n (d'_i p_i x'_i + d''_i q_i x''_i) \geq e, \quad (6)$$

$$u_t - v_t = \sum_{i=1}^n (r_{it} - \hat{r}_i) (p_i x'_i - q_i x''_i), \quad t = 1, \dots, T, \quad (7)$$

$$\sum_{i=1}^n [(1 + d'_i) p_i x'_i + (-1 + d''_i) q_i x''_i] \leq C, \quad (8)$$

$$q_i x_i \leq v_i \sum_{j=1}^{n+1} q_j x_j, \quad i = 1, \dots, n+1, \quad (9)$$

$$x'_i = (x_i - x_i^0)^+, \quad i = 1, \dots, n, \quad (10)$$

$$x''_i = (x_i - x_i^0)^-, \quad i = 1, \dots, n, \quad (11)$$

$$0 \leq x_i \leq \bar{x}_i, \quad i = 1, \dots, n, \quad (12)$$

$$u_t v_t = 0, \quad v_t \geq 0, \quad u_t \geq 0, \quad t = 1, \dots, T. \quad (13)$$

Due to constraints (10), (11) and (13), this problem is a nonlinearly constrained non-smooth optimization problem and is not easy to solve for large  $n$  and/or  $T$ . Fortunately, when  $p_i = q_i$  ( $1 \leq i \leq n$ ), it is easy to prove that the problem (5)–(13) is equivalent to the following linearly constrained optimization problem<sup>3</sup>:

$$\min \quad (1 - t_g) \left\{ a \left[ \sum_{t=1}^T u_t / T \right] + (1 - a) \left[ \sum_{t=1}^T v_t^p / T \right]^{1/p} \right\} - \left\{ \sum_{i=1}^n ((1 - t_g) \sum_{t=1}^T r_{it} / T + (1 - t_0) d_i) p_i (x'_i - x''_i) + (1 - t_0) r_{n+1} p_{n+1} (x_{n+1} - x_{n+1}^0) - \sum_{i=1}^n p_i (d'_i x'_i + d''_i x''_i) \right\} \quad (14)$$

$$\text{s.t.} \quad \sum_{i=1}^n ((1 - t_g) \sum_{t=1}^T r_{it} / T + (1 - t_0) d_i) p_i (x'_i - x''_i) + (1 - t_0) r_{n+1} p_{n+1} (x_{n+1} - x_{n+1}^0) - \sum_{i=1}^n p_i (d'_i x'_i + d''_i x''_i) \geq e, \quad (15)$$

$$u_t - v_t = \sum_{i=1}^n (r_{it} - \hat{r}_i) p_i (x'_i - x''_i), \quad t = 1, \dots, T, \quad (16)$$

$$\sum_{i=1}^n p_i [(1 + d'_i) x'_i + (-1 + d''_i) x''_i] \leq C, \quad (17)$$

$$p_i x_i \leq v_i \sum_{j=1}^{n+1} p_j x_j, \quad i = 1, \dots, n+1, \quad (18)$$

$$x_i = x'_i - x''_i + x_i^0, \quad i = 1, \dots, n, \quad (19)$$

$$0 \leq x_i \leq \bar{x}_i, \quad i = 1, \dots, n, \quad (20)$$

$$u_t \geq 0, \quad v_t \geq 0, \quad t = 1, \dots, T, \quad (21)$$

$$x'_i \geq 0, \quad x''_i \geq 0, \quad i = 1, \dots, n. \quad (22)$$

The coherency of  $\rho_{a,p}(g(x))$  means that the objective function in problem (14) is a convex function, and problem (14)–(22) is thus a linearly constrained convex programming problem. Many efficient algorithms can be used to solve this problem.

#### 4. The empirical study about the portfolio selection model under new risk measure

The optimal portfolio selection under model (14)–(22) will be investigated empirically in this section by using trade data from the Chinese stock markets.<sup>4</sup> Without the riskless asset, 200 stocks are randomly selected from all the A-share stocks in Chinese stock markets, daily data in 239 trading days from July 3, 2000 to June 29, 2001 are used to determine coefficient values in problem (14)–(22). Specifically,  $r_{it}$  is chosen as the daily rate of return of security  $i$  on day  $t$ ,  $d_i$  is the daily dividend yield of stock  $i$ , both  $p_i$  and  $q_i$  are set to the daily close price of risky stock  $i$  on June 29, 2001. Without loss of generality, it is assumed that the investor only holds cash at the beginning of the investment period, i.e.,  $x_i^0 = 0$  ( $1 \leq i \leq n$ ), we set  $t_g = t_0 = 0$  and  $C = 1,000,000$ . To demonstrate the actual effect of the proposed new risk measure on optimal investment decision making, results obtained under our new measure will be compared with the corresponding results obtained under CVaR.

The impact of different risk measures and different market frictions on the optimal portfolio selection will be examined in succession from the following perspectives: the diversification of the optimal portfolio, its return value, its risk magnitude and the performance of the optimal portfolio. The portfolio diversification will be examined by the number of stocks actually contained in the optimal portfolio and the Herfindahl index (H-index) of concentration (Clarke and Davies, 1983). The lower the H-index, the better the portfolio's diversification. In order to compare the performance of various optimal portfolios from different points of view, we calculated many performance measures, such as the famous Sharpe (the return/standard deviation, R/Std) ratio, the return/ $\rho_{a,p}$  (R/Risk)

<sup>4</sup> The experiment data comes from the China Stock Market & Accounting Research Database, which is compiled by the GTA Information Technology Company, Shenzhen, China.

<sup>3</sup> The detailed proof can be provided upon request.



ratio, the return/CVaR (R/CVaR) ratio, as well as the newly introduced one-sided variability ratios: the Rachev generalized (G-Rachev) ratio and the Farinelli–Tibiletti (F–T) ratio. The G-Rachev ratio (Biglova et al., 2004) is defined as the ratio between the power CVaR of the opposite of the excess return at a given confidence level and the power CVaR of the excess return at another confidence level; The F–T ratio (Farinelli and Tibiletti, 2008) is defined as the ratio between the weighted favorable events and the unfavorable ones.<sup>5</sup> As all these ratios are the reward-to-risk indexes, the higher the ratio, the more efficient the corresponding optimal portfolio.

First, we consider the influence of different risk measures on the optimal portfolio. Considering the practical situation of Chinese stock markets, we set  $d'_i = d''_i = 0.35\%$ , and  $v_i = 20\%$  for all the chosen stocks,  $e = 0.1\%$ . Explained in the following are characteristics of optimal portfolios obtained under risk measures  $\rho_{0.5,1}$ ,  $\rho_{0.5,2}$ ,  $\rho_{0.5,5}$  and  $\text{CVaR}_{0.05}$ , respectively.<sup>6</sup>

The optimal portfolios have the same return value equaling to  $C \cdot e$ . From  $\rho_{0.5,1}$  to  $\rho_{0.5,2}$ , and to  $\rho_{0.5,5}$ , the risk value of the corresponding optimal portfolio is monotonically increasing, which is consistent with the conclusion of Theorem 2.3. Thus, compared with  $\rho_{0.5,1}$  ( $\rho_{0.5,2}$ ),  $\rho_{0.5,2}$  ( $\rho_{0.5,5}$ ) is more suitable for conservative investors. The R/Risk ratio monotonically decreases from  $\rho_{0.5,1}$  to  $\rho_{0.5,2}$ , to  $\rho_{0.5,5}$  and to  $\text{CVaR}_{0.05}$ . To further show the advantage of our new risk measure, we examine optimal portfolios' performances under four common performance measures R/Std, R/CVaR, G-Rachev and F–T. Naturally, the  $\text{CVaR}_{0.05}$  optimal portfolio has the largest R/CVaR ratio. For three other ratios, we always have, from the largest one to the smallest one, the  $\rho_{0.5,5}$  optimal portfolio > the  $\rho_{0.5,2}$  optimal portfolio > the  $\text{CVaR}_{0.05}$  optimal portfolio > the  $\rho_{0.5,1}$  optimal portfolio. The last inequality is due to that, as one can see from Theorem 2.4,  $\rho_{0.5,1}$  is equivalent to the absolute deviation measure. These performance measures show that the performance of the optimal portfolio obtained under our new risk measure is better than that under CVaR. The larger the  $p$ , the more significant the performance improvement. There are two reasons for this result: one is that, while keeping the advantages of CVaR, our two-sided risk measure considers both below-mean losses and above-mean earnings simultaneously but differently; the other is that the parameter  $p$  can be used to control the fat-tail phenomenon and to flexibly reflect the degree of the investor's risk aversion.

With regard to the optimal portfolio's configuration, both the number of stocks actually included in the optimal portfolio and the H-index value tell us that our new risk measure is better than CVaR in terms of the diversification effect. The larger the  $p$ , the better the diversification.

Next, we investigate how different proportional transaction cost ratios could affect the diversification of the optimal portfolio and its performance. We fix  $v_i (1 \leq i \leq n)$  at 20%,  $e = 0.1\%$ , and the transaction cost ratio  $d'_i = d''_i \equiv d$  is always increased from 0.05% to 0.35%, with the stepsize being 0.05%. Except for risk measures with  $a = 0.5$  and  $p = 1, 2$  and 5, respectively, the impact of  $a$  in  $\rho_{a,p}$  will also be examined by fixing  $p$  to 2 and increasing  $a$  from 0 to 1, with the stepsize being 0.25.

When  $d$  increases from 0.05% to 0.35%, the return of every optimal portfolio always equals to the required minimal return value, but the value of risk monotonically increases. This results in that the R/Risk ratio decreases. The value of other performance mea-

asures also uniformly decreases with the increase of  $d$ . Therefore, the performance of the optimal portfolio deteriorates when  $d$  increases. Meanwhile, as  $d$  increases from 0.05% to 0.35%, the number of stocks really contained in the optimal portfolio monotonically decreases, and the value of the corresponding H-index constantly increases. Consequently, the diversification of the optimal portfolio degenerates. These results are rather reasonable. The investment cost is a monotonically increase function of  $d$ , therefore, in order to ensure the required investment return and to control risk, the investor has to reduce the number of stocks really invested and to select those stocks with relatively higher return, thus higher risk, than otherwise. The mathematical reason for above trends is that the feasible region of problem (14)–(22) monotonically shrinks when  $d$  increases. Our results also confirm the existing empirical conclusion that the portfolio performance and diversification often decreases with the increase of transaction cost.

As for the influence of different  $p$ s and  $a$ s, we have observed that: for  $\rho_{0.5,p}$  optimal portfolios, the larger the  $p$ , the faster the increasing (decreasing) speed of the H-index (of any considered performance ratio). This illustrates that the more risk-averse the investor, the more sensitive the optimal portfolio's performance and diversification. For the risk-averse investor, the large transaction cost ratio would accelerate the deterioration of his/her optimal portfolio's efficiency. For  $\rho_{a,2}$  optimal portfolios, we have  $\rho_{0.2} > \rho_{0.25,2} > \rho_{0.5,2} > \rho_{0.75,2} > \rho_{1,2}$  for almost all  $d$ s, which is consistent with the conclusion in Theorem 2.3; the larger the  $a$ , the slower the increasing (decreasing) speed of the H-index (of any examined performance ratio). This can be explained by the definition of  $\rho_{a,p}$ : the smaller the  $a$ , the bigger the weight on the lower  $p$ th partial moment, which corresponds to the more risk-averse investor. Therefore, just like the above conclusion for  $p$ , the small  $a$  would cause the optimal portfolio's diversification and performance becoming more sensitive to the increase of  $d$ . These observations tell us that, in order to find a better diversified portfolio with good performance in practice, one need to choose a proper combination of parameters  $a$  and  $p$  with regard to the specific  $d$ .

Last but not least, for any examined  $d$ , the optimal portfolio determined under any one of seven new risk measures is always better than the optimal portfolio gotten under  $\text{CVaR}_{0.05}$  because, compared with the  $\text{CVaR}_{0.05}$  optimal portfolio, the optimal portfolio under  $\rho_{a,p}$  is better diversified in terms of the number of included stocks and the H-index value, and the ratio value of the  $\rho_{a,p}$  optimal portfolio under each of five performance measures is almost always higher than the corresponding ratio value under  $\text{CVaR}_{0.05}$ .

Thirdly investigated is the impact of the proportion factor  $v = v_i (1 \leq i \leq n)$  in value constraints on the optimal portfolio's characteristics. Here we set  $d_i = 0.35\% (1 \leq i \leq n)$ ,  $e = 0.1\%$ , and let  $v$  to change from 0.2 to 0.6, with the stepsize being 0.1. From results obtained under  $\rho_{0.5,1}$ ,  $\rho_{0.5,5}$ ,  $\rho_{0.5,2}$ ,  $\rho_{0.25,2}$ ,  $\rho_{0.75,2}$  and  $\text{CVaR}_{0.05}$ , the following common phenomena have been observed: for chosen risk measure, although all the optimal portfolios have the same return value, their risk values monotonically decrease with the increase of  $v$ . This results in that the R/Risk ratio uniformly increases with respect to  $v$ . The value of each of four performance measures is also monotonically increasing with respect to  $v$ . These variations imply that the performance of the optimal portfolio could be improved if  $v$  is enlarged. Meanwhile, with the increase of  $v$ , the optimal portfolio is getting more and more concentrated since the number of selected stocks monotonically decreases and the corresponding H-index constantly increases. These observations are consistent with the meaning of the value constraint: when  $v$  increases, the proportion of total capital one can invest in any specific stock is enlarged, and the diversification requirement weakens. As a result, the investor could concentrate the

<sup>5</sup> We also calculated the return/AD ratio, the return/minimax ratio, the return/Gini ratio, the return/VaR ratio, and the Rachev ratio. The overall conclusion for the return/AD ratio and the return/Gini ratio (the return/minimax ratio and the return/VaR ratio, the Rachev ratio) is similar to that gotten under the Sharpe ratio (the return/CVaR ratio, the Rachev generalized ratio, respectively), and is thus omitted here.

<sup>6</sup> Due to the space limitation, we do not report, here and in the following, characteristics of various optimal portfolios, which can be provided upon request.

investment on a few stocks with high return and good performance, this would in return improve the overall performance of the final optimal portfolio. The mathematical explanation for the above conclusion is that, for problem (14)–(22), the feasible region corresponding to a given  $\bar{v}$  encompasses the feasible region corresponding to any smaller  $0 < v < \bar{v}$  as its subset.

As for the concrete influence of different parameter ( $p$  and  $a$ ) values, we have basically the same conclusion as that for the transaction cost situation, which can be similarly explained and is thus omitted.

What is more important, for any given  $v$ , the performance ratio of the optimal portfolio determined under our new risk measure is almost surely bigger than that of the corresponding optimal portfolio obtained under  $\text{CVaR}_{0.05}$  in terms of five examined performance measures. Therefore, when the value constraint is considered, our new risk measure is better than  $\text{CVaR}$  in helping investors to find more efficient investment strategy.

Finally, the impact of different target rates of return  $e$  on the optimal portfolio selection is investigated with  $v_i = 20\%$ ,  $d_i = 0.0$  ( $1 \leq i \leq n$ ). With the increase of  $e$ , both the return and risk of the optimal portfolio are monotonically increasing. Nonetheless, as the risk value increases faster than that of the return value, the R/Risk ratio is monotonically decreasing. The same variation pattern holds for values of other performance measures. These conclusions confirm the well-known fact that portfolios with higher returns are almost surely accompanied by higher risks. Mathematically, these changes can be explained by the fact that the feasible region of problem (14)–(22) monotonically shrinks as  $e$  increases. An obvious financial explanation is that: to achieve the required high return target, the investor tends to concentrate his/her investment on a few stocks with high return, thus large risk. For this reason, the diversification of the optimal portfolio determined under any adopted risk measure monotonically decreases with the increase of  $e$ , this can be deduced from the uniform increase of the H-index value and the basically monotonic decrease of the number of stocks included in the optimal portfolio.

Other conclusions we can derive are similar to what we have found from previous groups of results. Their explanations are thus omitted due to the space limitation.

## 5. Conclusion

By asymmetrically considering both negative and positive deviations from the expected return, a new class of two-parameter coherent risk measures  $\rho_{a,p}$  has been proposed in this paper. While keeping advantages of existing LPM measures and coherent risk measures, the proposed measure avoids shortcomings of these risk measures.

Based on the proposed risk measure, a realistic portfolio selection model has also been established in this paper by taking into account typical trading frictions. What is more important, detailed empirical results show that, as long as values of  $a$  and  $p$  in  $\rho_{a,p}$  are properly chosen, the established portfolio optimization model is very useful and flexible for determining robustly optimal investment strategy. Our new risk measure and the portfolio selection model can reasonably reflect the influence of different market constraints, and can be used to find optimal portfolios with specific characteristics such as the return-risk pattern. In all the empirical tests, the optimal portfolio determined under our new risk measure is almost always better than the corresponding optimal portfolio obtained under the currently popular risk measure  $\text{CVaR}$  in terms of the diversification degree and typical performance mea-

sures. Consequently, our new risk measure is superior and more suitable for the optimization of choices in financial management.

In practice, securities can never be acquired in continuous quantity, but in multiples of a minimum transaction unit. One natural extension to our portfolio optimization model is then to take this kind of minimum trading size constraints into account, which will change model (5)–(13) or (14)–(22) into a mixed-integer nonlinear optimization problem. The efficient method to solve the resulting mixed-integer programming problem and the associated empirical study are left for future research.

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