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EG1504 Engineering Mathematics 1
Exercises 11 (Integration)

1. Integration by parts is the one to go for when the integrand is the product of two functions and there is no obvious substitution to try. Your suspicion that this is the method to use in a particular case will be strengthened if one part of the integral is something you would rather differentiate than integrate. So, for example, in the first of this set $\ln x$ is a function easier to differentiate than integrate, so try parts with $\ln x$ as the "u" and x^2 as the "dv/dx". In the second one we are faced with an integrand which would be a lot easier to handle if it were just $\sqrt{x+1}$ rather than x times this. So put it into the sort of form you want by using parts with "u" equal to x. In the third one we don't seem to have a product to work with. However, we do have a function that we'd sooner differentiate than integrate. So force the integrand to be a product by writing it as $1.\arcsin x$ and then taking $\arcsin x$ as the "u" and 1 as the "dv/dx". The fourth one is clearly a product, but using integration by parts doesn't seem to advance us forward. However, if, having integrated by parts once, you then integrate by parts again and then stare hard at what you get, you should find that you have cracked it. (This necessity to integrate by parts twice rather than once is not uncommon when you have $\sin x$ or $\cos x$ to deal with.)

Integrate the following functions:

$$x^2 \ln x, \qquad x\sqrt{x+1}, \qquad \arcsin x, \qquad e^x \sin 2x\ .$$

2. Integration by parts again, but this time without hints:

$$xe^{3x}, \qquad x^2 e^{2x}, \qquad x^2 \sin x, \qquad \sin x \sin 3x$$

3. Integrate the following functions:

$$x\exp(x^2), \qquad xe^x, \qquad \frac{1}{x^2\sqrt{1+x^2}}, \qquad \sin^4 x \cos x, \qquad \ln x$$

$$\frac{\sin\theta}{\cos\theta}, \qquad \frac{x}{1+x}, \qquad \frac{1}{(3x+5)^2}, \qquad \frac{\sin x}{\sqrt{1-\cos x}}$$

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EG1504 ENGINEERING MATHEMATICS 1
SOLUTIONS TO EXERCISES 11 (INTEGRATION)

1. Answers: $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3$, $\frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2}$.

With the third one the method indicated gives you a subsidiary integral $\int x/\sqrt{1-x^2}$ which integrates to give $\sqrt{1-x^2}$. This leads to a final answer for the original question of $x \arcsin x + \sqrt{1-x^2}$.

Let $I(x) = \displaystyle\int e^x \sin 2x\,dx$. Integrating by parts once, with e^x as the part being integrated and $\sin 2x$ as the part being differentiated gives

$$I(x) = e^x \sin 2x - \int 2e^x \cos 2x\,dx$$

Now repeat the process on the subsidiary integral, with the same basic choices as before, e^x as the part to be integrated. You get

$$\int 2e^x \cos x\,dx = 2e^x \cos 2x + \int 4e^x \sin 2x\,dx = 2e^x \cos 2x + 4I(x)\ .$$

Substitute this into the first formula and you get

$$I(x) = e^x \sin 2x - (2e^x \cos 2x + 4I(x))$$

which rearranges to give

$$5I(x) = e^x \sin 2x - 2e^x \cos 2x$$

dividing through by 5 gives us $I(x)$.

2. Answers: $\dfrac{1}{3}xe^{3x} - \dfrac{1}{9}e^{3x}$, $\dfrac{1}{2}x^2e^{2x} - \dfrac{1}{2}xe^{2x} + \dfrac{1}{4}e^{2x}$.

The third is another case of going through the routine twice:

$$\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx = -x^2\cos x + 2x\sin x + 2\cos x\ .$$

And so is the fourth. With this one it doesn't matter too much which of $\sin x$ and $\sin 3x$ you take as the part to be differentiated. For ease of typing I shall take $\sin 3x$:

$$\begin{aligned}
\int \sin x \sin 3x\,dx &= -\cos x \sin 3x + \int 3\cos x \cos 3x\,dx \\
&= -\cos x \sin 3x + 3\left[\sin x \cos 3x + \int 3 \sin x \sin 3x\,dx\right] \\
&= -\cos x \sin 3x + 3\sin x \cos 3x + 9\int \sin x \sin 3x\,dx \\
&= -\cos x \sin 3x + 3\sin x \cos 3x + 9\int \sin x \sin 3x\,dx
\end{aligned}$$

This rearranges as $\displaystyle\int \sin x \sin 3x\,dx = \tfrac{1}{8}\big(\cos x \sin 3x - 3\sin x \cos 3x\big)\ .$

3. Put $u = x^2$. This turns the integral into $\dfrac{1}{2}\displaystyle\int e^u\,du$, and leads to an answer of $\frac{1}{2}\exp(x^2)$.

Use parts with x as the bit to be differentiated. You should end up with the answer $xe^x - e^x$.

To get rid of the square root we make use of the trig formula $\sec^2\theta = 1 + \tan^2\theta$. Let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$. This gives us an integrand with $\sec^2\theta$ on the top line and $\tan^2\theta\sec\theta$ on the bottom. This cancels down to $\sin^{-2}\theta\cos\theta$. Now put $u = \sin\theta$ to get $\int u^{-2}\,du$. This is $-u^{-1}$, which equals $1/\sin\theta$. Since $\tan\theta = x$, $\sin\theta = x/\sqrt{1+x^2}$. (Draw a right angled triangle with two smallest sides x and 1 to see where this comes from.) So the final answer is $-\sqrt{1+x^2}/x$.

This is easier. Put $u = \sin x$. Then $du = \cos x\,dx$ and the integral turns into $\int u^4\,du$. This is $\frac{1}{5}u^5$, giving a final answer of $\frac{1}{5}\sin^5 x$.

This is like the arcsin one done in the question on integration by parts: you have to think of the integrand as $1.\ln x$ and then adopt the $\ln x$ as the bit to be differentiated. The clue comes from the fact that ln is a function you can differentiate easily and so you arrange the break-up so that that is what you get to do to it. Once you have decided to do this the integral becomes easy. The answer is $x\ln(x) - x$.

Put $u = \cos\theta$. Then the integral turns into $\int(-u^{-1})\,du$, and this comes out as $-\ln u$. So the answer is $-\ln(\cos\theta)$.

Two ways to tackle this: The first is to put $u = x + 1$. Then $du = dx$ and $x = u - 1$. This turns the integral into $\int(1 - u^{-1})\,du$, giving an intermediate answer of $u - \ln u$ and a final answer of $x + 1 - \ln(x+1)$. The alternative is to begin with some algebra. Write x as $(x+1) - 1$ on the top line of the integrand. This enables us to break the integrand up as $1 - (x+1)^{-1}$, which can be integrated directly to get $x - \ln(x+1)$. Note that the two answers differ by an additive constant, but that this does not matter as indefinite integrals are only specified to within an additive constant. Both answers differentiate to give the integrand and that is what determines correctness for integrals, not differences in appearance.

Put $u = 3x + 5$. Then $du = 3dx$. The substitution turns the integral into $\int \frac{1}{3}u^{-2}\,du$, which produces the answer $-\frac{1}{3}u^{-1}$. So the final answer is $-\frac{1}{3}(3x+5)^{-1}$.

For the last one try $u = 1 - \cos x$. $du = \sin x\,dx$, and the integral is transformed into $\int(1/\sqrt{u})\,du$, giving a final answer $2\sqrt{1-\cos x}$.