Author	AP/SD/AS
Date	17.1.013
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UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 10 (INTEGRATION)

1. Find the points where the line $y = x + 2$ meets the curve $y = x^3 - 4x^2 + 4x + 2$ and then find the total area of the region enclosed by the line and the curve.

2. The graphs $y = 6x - x^2$ and $y = x^2 - 2x$ are both parabolas. In each case find the critical point of the graph and the points where the graph crosses the x-axis. Then find the points where the two curves meet. Using just these pieces of information sketch the two graphs on the one diagram. Calculate the area of the region enclosed by the two curves.

3. The curve given by the equation $y^2 = x^2 - x^4$ looks like a horizontal figure of eight, with the crossing point at the origin. Find the total area of the region enclosed by the curve.

4. If you see the derivative of a function, $g'(x)$ say, as a factor of an integrand then it is usually a good idea to put to try the substitution $u = g(x)$. If you see a function of $ax + b$ in the integrand then $u = ax + b$ is a good substitution (indeed if you see a function of any function of x then putting u equal to the function of x is worth a shot). Using these guidelines consider the following four integrals:

$$\int (2x-4)^5\,dx \qquad \int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx, \qquad \int \frac{\sec^2 x}{1+\tan x}\,dx, \qquad \int \frac{x^2}{1+x^3}\,dx\ .$$

(*With the first, we have a function of* $2x - 4$ *so try* $u = 2x - 4$ *(this is probably better than 'try it and see' method). In the second note that* $1/\sqrt{x}$ *is (modulo a constant factor) the derivative of* $\sqrt{x}$ *so try* $u = \sqrt{x}$. *In the third, note that* $\sec^2 x$ *is the derivative of* $\tan x$ *so try* $u = \tan x$. *The fourth doesn't announce itself quite so loudly, but* x^2 *is 'almost' the derivative of* $1 + x^3$, *and it would be much nicer to have plain, simple* u *on the bottom line rather than the somewhat frightening* $1 + x^3$*; so try* $u = 1 + x^3$.)

5. Integrate the following functions:

$$e^{3x+1}\,, \qquad x\cos(x^2+1)\,, \qquad \frac{2x}{x^2+4}\,, \qquad x\sqrt{x^2-1}\,, \qquad \frac{\sqrt{\ln x}}{x}\ .$$

6. When the integrand is a mixture of $\sin x$ and $\cos x$, substitutions which often work are $u = \sin x$ or $u = \cos x$. If one of the functions occurs to an *odd* power, put u equal to the other one. If both do, either substitution should work; if neither do, you'll probably have to try something else.

$$\sin^2 x\cos x\,, \qquad \frac{\sin x}{\cos^4 x}\,, \qquad \sin^5 x\cos^3 x$$

(*In the last of these you still have some work to do apart from the substitution. The key to doing it is to make use of the fact that* $\sin^2 x = 1 - \cos^2 x$.)

7. Integrands involving the square roots of quadratic expressions are both common and awkward but they can be done by substitution. Try:

$$\int \sqrt{4-x^2}\,dx\,, \qquad \int \sqrt{x^2-4}\,dx\,, \qquad \int \sqrt{x^2+4x+5}\,dx$$

(*For the first put* $x = 2\sin\theta$, *for the second put* $x = 2\cosh\theta$. *For the third 'completing the square' gives* $x^2 + 4x + 5 = (x+2)^2 + 1$, *now put* $x = \cosh\theta - 2$.)

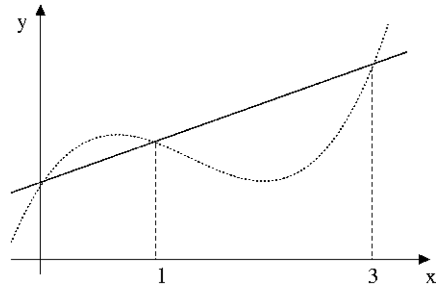
University of Aberdeen Department of Mathematical Sciences	Author Date Code	AP/SD/AS 17.1.013

EG1504 Engineering Mathematics 1
Solutions to Exercises 10 (Integration)

1. The intersections come when $x^3 - 4x^2 + 4x + 2 = x + 2$, i.e. when $x^3 - 4x^2 + 3x = 0$.

$x^3 - 4x^2 + 3x$ factorises as $x(x^2 - 4x + 3)$ and then as $x(x-1)(x-3)$. So the three values of x which give intersections are $x = 0$, $x = 1$ and $x = 3$.

When $x = 0$ the curve has slope 4. This is steeper than the line, which has slope 1. It is recommended you draw a sketch. Note that the curve is above the line in the interval from 0 to 1 and below the line in the interval from 1 to 3 This means that from $x = 0$ to $x = 1$ the area required is the area under the curve minus that under the line and from $x = 1$ to $x = 3$ it is the area under the line minus that under the curve.



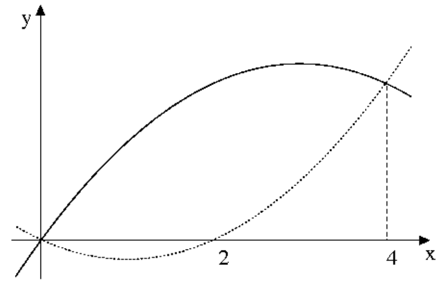
$$\begin{aligned}
\text{First Section} &= \int_0^1 (x^3 - 4x^2 + 4x + 2) - (x + 2)\, dx \\
&= \int_0^1 (x^3 - 4x^2 + 3x)\, dx \\
&= \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1 = \frac{5}{12}
\end{aligned}$$

$$\begin{aligned}
\text{Second Section} &= \int_1^3 (-x^3 + 4x^2 - 3x)\, dx \\
&= \left[-\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2}\right]_1^3 = \frac{8}{3}
\end{aligned}$$

$$\text{Total Area} = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

2. A graph crosses the x-axis when $y = 0$. With the first graph this is when $x = 0$ and when $x = 6$. The critical points for this graph occur when $dy/dx = 6 - 2x = 0$, i.e. when $x = 3$. So there is just the one critical point, it occurs at $(3, 9)$ and it is a local maximum ($d^2y/dx^2 = -2$).

The second graph crosses the x-axis when $x = 0$ and when $x = 2$. It has just one critical point, which is at $(1, -1)$ and is a local minimum. The two curves cross when $6x - x^2 = x^2 - 2x$, i.e. when $2x^2 = 8x$, i.e. when $x = 0$ and when $x = 4$. One (though not in this case the quickest) way to deal with this area is in three sections:



1. the area of region between $y = 6x - x^2$, $x = 0$ and $x = 2$ and x-axis
2. the area of region between $y = x^2 - 2x$, $x = 0$ and $x = 2$ and x-axis

3. the area of region between $y = 6x - x^2$, $y = x^2 - 2x$, $x = 2$ and $x = 4$

Taking account of the part regions above and below the x-axis we see

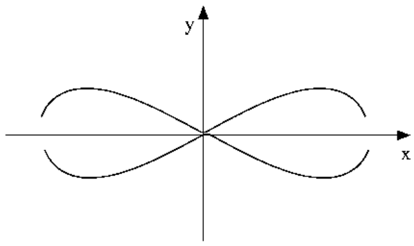
$$\begin{aligned}
\text{Area} &= \int_0^2 (6x - x^2)\,dx - \int_0^2 (x^2 - 2x)\,dx + \int_2^4 (6x - x^2)\,dx - \int_2^4 (x^2 - 2x)\,dx \\
&= \int_0^2 (6x - x^2)\,dx - \int_0^2 (x^2 - 2x)\,dx + \int_2^4 (8x - 2x^2)\,dx \\
&= \left[3x^2 - \frac{x^3}{3}\right]_0^2 - \left[\frac{x^3}{3} - x^2\right]_0^2 + \left[4x^2 - \frac{2x^3}{3}\right]_2^4 \\
&= \frac{64}{3}
\end{aligned}$$

If you look at the first line of this equation and use the properties of the definite integral, you will find that the expression on right-hand side simplifies down to

$$\int_0^4 ((6x - x^2) - (x^2 - 2x))\,dx = \int_0^4 (8x - 2x^2)\,dx\ .$$

We remark that $6x - x^2 \geqslant x^2 - 2x$ for $0 \leqslant x \leqslant 4$ (see lecture notes).

3. The curve is symmetrical about both axes, and so the total area is four times that in the first quadrant.



The curve crosses the positive part of the x-axis when $x = 0$ and when $x = 1$. So these are the limits.

Therefore the area is $\displaystyle\int_0^1 y\,dx = \int_0^1 x\sqrt{1 - x^2}\,dx$.

To deal with this use the substitution $u = 1 - x^2$. Then $''du = -2xdx''$. We also have that $u = 1$ when $x = 0$ and $u = 0$ when $x = 1$ have

$$\text{Area in first quadrant} = -\frac{1}{2}\int_{x=0}^{x=1} \sqrt{u}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=0} = \frac{1}{3}$$

So the total area is 4/3.

4. $\dfrac{1}{12}(2x - 4)^6$, $\quad -2\cos(\sqrt{x})$, $\quad \ln(1 + \tan x)$, $\quad \frac{1}{3}\ln(1 + x^3)$

5. Put $u = 3x + 1$. Then $''du = 3dx''$, and the integral becomes $\dfrac{1}{3}\displaystyle\int u\,du$. Ans: $\dfrac{1}{3}e^{3x+1}$.

Put $u = x^2 + 1$. Then $''du = 2xdx''$, and the integral becomes $\dfrac{1}{2}\displaystyle\int \cos u\,du$. Ans: $\dfrac{1}{2}\sin(x^2 + 1)$.

Put $u = x^2 + 4$. Then $''du = 2xdx''$, and the integral becomes $\displaystyle\int \frac{du}{u}$. Ans: $\ln(x^2 + 4)$.

Put $u = x^2 - 1$. Then $''du = 2xdx''$, and the integral becomes $\dfrac{1}{2}\displaystyle\int \sqrt{u}\,du$. Ans: $\dfrac{1}{3}(x^2 - 1)^{3/2}$.

Put $u = \ln x$. Then $''du = \dfrac{1}{x}dx''$, and the integral becomes $\displaystyle\int \sqrt{u}\,du$. Ans: $\dfrac{2}{3}(\ln x)^{3/2}$.

6. The answers are:

$$\frac{1}{3}\sin^3 x\,, \qquad \frac{1}{3\cos^3 x}\,, \qquad \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x$$

Put $u = \sin x$ in the first one and $u = \cos x$ in the second to get $\int u^2\,du$ and $-\int u^{-4}\,du$, respectively. In the last one, write $\cos^3 x$ as $\cos x \cos^2 x = \cos x(1 - \sin^2 x)$. Now the substitution $u = \sin x$ gives $''du = \cos x\,dx''$, and the integral turns into $\int (u^5 - u^7)du$.

7. For the first: $''dx = 2\cos\theta\,d\theta''$ and $\sqrt{4-x^2}$ becomes $2\cos\theta$. So the integral becomes $\int 4\cos^2\theta\,d\theta$. To handle this you need to use the trig formula for $\cos 2\theta$. This tells you that $2\cos^2\theta = \cos 2\theta + 1$ and so the integral can be written as $\int(2\cos 2\theta + 2)\,d\theta$. We can now write down the answer: it is $\sin 2\theta + 2\theta$, and all that then remains is to express the answer in terms of the original variable, x. To do this first write $\sin 2\theta$ as $2\sin\theta\cos\theta$, and then note that $\cos\theta = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$ We also have that $\theta = \arcsin(x/2)$. Substituting this lot into our answer turns it into $\frac{1}{2}x\sqrt{4-x^2} + 2\arcsin(x/2)$. And it would be a useful differentiate and simplify exercise were you to check my answer by differentiating it and seeing if you get back to $\sqrt{4-x^2}$.

With the second one, if $x > 2$ then the substitution $x = 2\cosh\theta$ (we may assume $\theta > 0$) produces the integral $\int 4(\sinh\theta)^2\,d\theta$ (since $\sinh\theta > 0$), which equals $\int(e^{2\theta} - 2 + e^{-2\theta})\,d\theta$. This integrates as $\frac{1}{2}\left(e^{2\theta} - e^{-2\theta}\right) - 2\theta = 2\sinh\theta\cosh\theta - 2\theta$. To write this as a function of x is possible but not easy. [The interesting part is expressing θ as a function of x. We have

$$2\cosh\theta = x \quad \text{so} \quad e^{\theta} + e^{-\theta} = x$$

and hence

$$e^{2\theta} - xe^{\theta} + 1 = 0$$

and hence

$$e^{\theta} = \frac{x \pm \sqrt{x^2-4}}{2}\,.$$

The question now is '+' or '−'? Think about it !!!!]

The last one is similar.