Contents

Chapter 1

Revision

1.1 Real Numbers

The numbers you learnt about at school are called *real* numbers (as opposed to complex numbers which you will learn about in the non-calculus part of this course). You will recall that real numbers can be expressed as decimals. For example:

$$\begin{aligned} 5 &= 5.0 \\ -\frac{3}{4} &= -0.75 \\ \frac{1}{3} &= 0.3333... \\ \sqrt{2} &= 1.4142... \\ \pi &= 3.14159... \end{aligned}$$

(the three dots indicate that the sequence of digits continues indefinitely).

At school you learnt to add and multiply real numbers, to subtract a real number from another real number and to divide a real number by a *non-zero* real number (although now you probably rely on your calculator to do these jobs for you). For real numbers a and b: the sum of a and b is denoted by $a + b$, the product of a and b is denoted by $a \times b$, $a.b$ or just ab and a divided by b is denoted by $a \div b$, a/b or $\frac{a}{b}$.

The set of all real numbers is denoted by $\mathbb{R}$. There are some important subsets. For instance the set of all positive integers (natural numbers) denoted by $\mathbb{N}$, the set of all integers denoted by $\mathbb{Z}$ and the set of rational numbers (that is set of all numbers which can be expressed in the form m/n where m and n are integers with $n \neq 0$) denoted by $\mathbb{Q}$. Thus

$$\begin{aligned} \mathbb{N} &= \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\ \ldots\} \\ \mathbb{Z} &= \{\ldots -5,\ \ -4,\ \ -3,\ \ -2,\ \ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5\ \ldots\} \\ \mathbb{Q} &= \{m/n\ :\ m \in \mathbb{Z},\ n \in \mathbb{Z} \text{ and } n \neq 0\} \end{aligned}$$

The basic laws of arithmetic are:

- $a + b = b + a$ for any real numbers a and b;
- $(a + b) + c = a + (b + c)$ for any real numbers a, b and c;
- there is a real number called zero, denoted by 0, such that $a + 0 = a$ for any real number a;
- for each real number a there is a real number denoted by $-a$, called the additive inverse of a, such that $a + (-a) = 0$;
- $ab = ba$ for all real numbers a and b;
- $(ab)c = a(bc)$ for all real numbers a, b and c;
- there is a real number called one, denoted by 1, such that $a.1 = a$ for all real numbers a
- for each *non-zero* real number a there is a real number denoted by a^{-1}, called the multiplicative inverse of a, such that $a\,a^{-1} = 1$;
- $a(b + c) = ab + ac$ for any real numbers a, b and c;

You may not have seen the laws written out so formally before. However you will be familiar with them because you used them whenever you did a calculation involving real numbers! They are the basic rules. All other arithmetic properties of real numbers can be deduced from these laws. Recall that $a - b = a + (-b)$ and $a/b = a\,b^{-1}$ if $b \neq 0$. □

In secondary school you were introduced to equations. You learnt how to solve equations of the form

$$ax + b = 0,$$

for the unknown x, where a and b are real numbers with $a \neq 0$. This was easy, the solution is $x = -b/a$. The reason it was easy is because you learnt about division. The next type of equation that we need to solve are of the form

$$ax^2 + bx + c = 0,$$

where a, b and c are real numbers with $a \neq 0$. There are problems. For instance consider the case

$$x^2 - 2 = 0.$$

From the equation we deduce that $x^2 = 2$ so that x must be a number whose square is 2. The ancient Greeks worried a lot about this because Pythagoras (of right-angled triangle fame) showed there was no number that satisfied this equation. To be more precise he showed there was no *rational* number which satisfied this equation. Unfortunately Pythagoras thought that every number was a rational number and did not know about non-rational numbers. After Pythagoras's day the set of all real numbers were discovered. The real numbers consist of the rational numbers and *irrational* (that is non-rational) numbers. It is now known that the equation above has two solutions, namely the irrational numbers denoted by $\sqrt{2}$ and $-\sqrt{2}$ (as a decimal $\sqrt{2}$ is approximately 1.142 (or 1142/1000)).

The quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has a real solution if $b^2 \geq 4ac$. This begs the question: 'Can we solve $ax^2 + bx + c = 0$ if $b^2 < 4ac$?' The answer is yes provided we extend our number system to *complex* numbers. You will learn about complex numbers in the non-calculus part of this course.

In the calculus part of the course 'number' will mean real number. $\square$

1.2 Powers

Let a be a real number.

If n is a *positive* integer n, then a^n (we say 'a to the *power* of n') is just the product of n a's. That is

$$a^n = a.\,a.\,a.\,\ldots.a \quad (n \text{ terms}).$$

Counting gives the following rules, known as *laws of indices*:

$$a^m a^n = a^{m+n}$$

$$(a^m)^n = (a^n)^m = a^{mn}$$

which are valid for all *positive* integers m and n.

We define

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n} \text{ if } n \text{ is positive.}$$

With these definitions it is easy to show that the laws of indices hold for *any* integers m and n (not just positive integers).

As an example note, from the laws, that

$$2^3.\,2^{-1} = 2^{3+(-1)} = 2^2 = 4.$$

This can be checked directly $2^3.\,2^{-1} = 8.\,(1/2) = 4$. $\square$

Let a be a *positive* number.

For any positive integer n we define

$$a^{\frac{1}{n}} \text{ to be the } \textit{positive } n\text{-th root of } a$$

(that is is the positive number whose n-th power is a). For example

$$a^{\frac{1}{2}} = \sqrt{a} \qquad (\sqrt{a} \text{ is always taken to be the } \textit{positive} \text{ square root}).$$

[The restriction to positive a is to avoid problems with things like the square roots of negative numbers.]

For any integer m and any *positive* integer n, we define

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m.$$

For example

$$a^{-\frac{1}{2}} = a^{\frac{-1}{2}} = \left(a^{\frac{1}{2}}\right)^{-1} = \frac{1}{\sqrt{a}}.$$

It turns out that

$$\left(a^m\right)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m,$$

both are equal to $a^{\frac{m}{n}}$. Sometimes one is easier to compute than the other:

$$9^{\frac{3}{2}} = \left(9^{\frac{1}{2}}\right)^3 = 3^3 = 27 \quad \text{is easier than} \quad 9^{\frac{3}{2}} = \left(9^3\right)^{\frac{1}{2}} = \sqrt{729}\ (= 27).$$

The discussion above shows that if a is positive then a^x is defined for any *rational* number x and one can check that the laws of indices (as displayed below) hold for all rational numbers x and y.

Indeed if a is *positive* then it is possible to define a^x for any real number x and establish the following laws of indices.

For any positive number a and for all real numbers x and y:

$$a^x a^y = a^{x+y}$$

$$(a^x)^y = (a^y)^x = a^{xy}$$

(Indeed the definitions are motivated by the desire that the laws should hold in general.) Another important law of indices, not mentioned so far, is given below.

For any positive numbers a and b and for all real numbers x:

$$(ab)^x = a^x b^x$$

(This is easily checked in the case that x is rational.)

Students of EG1006 are expected to know and be able to use the laws of indices discussed above.

Actual values of powers of numbers can be obtained from simple scientific calculators. For instance

$$0.25^{0.13} = 0.835087919$$

which is accurate to 8 decimal places. (A good enough approximation for most purposes !!) Your calculator manual will explain how to obtain powers of numbers, you should be familiar with the procedure.

1.3 Algebraic Manipulation

It is assumed that you have experience in manipulating expressions using the laws of arithmetic and indices, writing the expressions in equivalent but more convenient ways.

1.3.1 Example. Simplify the expression $x^{2/3}x^{-3/2}$.

Solution. By the first of the laws of indices

$$x^{\frac{2}{3}}x^{-\frac{3}{2}} = x^{\frac{2}{3}-\frac{3}{2}} = x^{-\frac{5}{6}}.$$

$\square$

Some general advice:

- Use brackets. (Often students write $a + b \cdot c + d$ for $(a + b)(c + d)$, try to avoid this. Without the brackets $a+b\cdot c+d$ can be confused with $(a+b)c+d$, $a+bc+d$ or $a + b(c + d)$.)
- Remember 'putting over a common denominator' we get:

  $$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$$

  Note that, in general,

  $$\frac{a}{b} + \frac{c}{d} \quad \text{is NOT equal to} \quad \frac{a + c}{b + d}\ ,$$

  this is mentioned because many students seem to think they are equal!!!
- Judged by their exam scripts some students think that $(a + b)^2$ equals $a^2 + b^2$ or that $\sqrt{a + b}$ equals $\sqrt{a} + \sqrt{b}$. You are reminded that these results are NOT true.

1.3.2 Exercise. Simplify the following expression $\dfrac{1}{x+1} - \dfrac{1}{x-1}$.

Solution. Put over a common denominator:

$$\frac{1}{x+1} - \frac{1}{x-1} = \frac{(x-1)-(x+1)}{(x-1)(x+1)} = \frac{-2}{(x-1)(x+1)} = \frac{2}{(1-x^2)}.$$

$\square$

Try some exercises:

1.3.3 Exercise. Simplify the following expression: $(x^{-1}\sqrt{y})^{3/2}$. $\square$

1.3.4 Exercise. Simplify the following expression:

$$\frac{1+x}{1-x} - \frac{1-x}{1+x}$$

$\square$

As a general rule one should never multiply out brackets unless you have to because mathematical expressions are usually easier to deal with if you know their factors. However occasionally multiplying out can simplify expressions. Try the following examples.

1.3.5 Exercise. Multiply out the brackets in the following expressions and simplify the results:

$$x^{1/3}(x^{2/3} + y^{1/3}), \quad (x^{1/2} + x^{-1/2})(x^{1/2} - x^{-1/2})$$
$$(x + 1)(x - 2), \quad (2 + x)^2(1 - x), \quad (1 - x)(1 - y)(1 - z).$$

$\square$

1.4 Summation Notation

The Greek capital letter Σ is used to indicate a summation. The terms that come after the Σ describe the form of the terms to be added together, and the decoration top and bottom of the Σ specifies the range of numbers to be summed.

For instance suppose that there is a number a_k defined for each integer k from the integer m and to the integer n then the sum of all these numbers is denoted by

$$\sum_{k=m}^{n} a_k \quad \text{(the sum from } k = m \text{ to } k = n \text{ of the } a_k\text{)}.$$

we sometimes write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

the expresion on the right being called the expansion of the sum. k is called the index of summation. There is nothing special about the symbol k, we can replace it by any convenient symbol, so for instance

$$\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_j.$$

The way in which the range of the sum is described can vary, for instance we could write

$$\sum_{m \leq k \leq n} a_k$$

for the above sum. Some examples:

1.4.1 Example. Write out $\displaystyle\sum_{k=1}^{6} k^2$ explicitly. (The sum could be written as $\displaystyle\sum_{1 \leq k \leq 6} k^2$.)

Solution. The expansion is $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$ — the sum from $k = 1$ to $k = 6$ of all terms of the form k^2. □

1.4.2 Example. Write out $\displaystyle\sum_{r=0}^{7} (2r + 1)^3$ explicitly.

Solution. The given sum is the sum from $r = 0$ to $r = 7$ of all terms of the form $(2r + 1)^3$ and so is $1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3$. □

1.4.3 Example. Write out the expansion of $\displaystyle\sum_{i=0}^{n} x^i$.

Solution. The expansion is $1 + x + x^2 + x^3 + \cdots + x^n$. (It is understood that n is a positive integer and x represents a real number (so $x^0 = 1$ and $x^1 = x$)). □

In more advanced work you can get 'double sums'. Here there will be two indices rather than one, and a summation range is given for each.

1.4.4 Example. Write out $\sum_{r=1}^{3}\big(\sum_{s=1}^{4} rs^2\big)$ explicitly.

Solution. Here we have the sum of all terms of the form rs^2 for the given ranges of r and s. To expand it, first fix r at its lowest value and let s run over the range described; then move r to the next value and repeat; and so on. With this example the expansion is

$$(1.1^2 + 1.2^2 + 1.3^2 + 1.4^2) + (2.1^2 + 2.2^2 + 2.3^2 + 2.4^2) + (3.1^2 + 3.2^2 + 3.3^2 + 3.4^2).$$

□

More generally one can consider $\sum_{i=1}^{m}\big(\sum_{j=1}^{n} a_{ij}\big)$. If we arrange the a_{ij}'s into the rectangular array:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array}$$

(m rows and n columns, a_{ij} in i-th row and j-th column) then we see that $\sum_{j=1}^{n} a_{ij}$ is just the sum of numbers in the i-th row and so the double sum is the sum of all the row sums and thus the sum of all the numbers in the array.

1.4.5 Exercise. Explain why $\sum_{i=1}^{m}\big(\sum_{j=1}^{n} a_{ij}\big) = \sum_{j=1}^{n}\big(\sum_{i=1}^{m} a_{ij}\big)$ (so we can 'change the order of summation'). □

With double sums the range descriptions can get quite fancy. In each case one has to be careful to make sure exactly what is meant.

1.5 The Binomial Theorem

1.5.1 Binomial coefficients

The Binomial Theorem is an important theorem, useful across a wide range of mathematics. It gives you a fast method of expanding powers of sums. If for example you are faced with an expression such as $(x+y)^6$, you can write down the expansion without going through the long process of multiplying out. It gives, almost immediately, that

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

The theorem enables you to calculate all the terms of this expansion very easily. To describe it in detail we need more notation.

For any positive integer n, *factorial* n, written as $n!$, denotes the product of all the integers from 1 up to n. For example

$$
\begin{aligned}
1! &= 1 \\
2! &= 1 \times 2 = 2 \\
3! &= 1 \times 2 \times 3 = 6 \\
4! &= 1 \times 2 \times 3 \times 4 = 24 \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 = 120
\end{aligned}
$$

and so on.

It is convenient also to have a meaning for $0!$. We define $0!$ to be 1. There is no real mystery here. It just turns out to be a good idea. With this convention formulae can be written in a more uniform way with fewer special cases.

1.5.1 Theorem. *(Binomial Theorem) Given any two numbers x and y and any positive integer n, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k}$ is shorthand for $\dfrac{n!}{k!\,(n-k)!}$. □

The numbers $\binom{n}{k}$ are known as *binomial coefficients*. Here are some properties which can help you calculate them:

- remember that it is understood that $0! = 1$.
- the binomial expansion is symmetrical, with $\binom{n}{k} = \binom{n}{n-k}$; in other words, the coefficient of $x^{n-k}y^k$ is the same as that of $x^k y^{n-k}$;
- if k is positive then after some cancellation we get

  $$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} ,$$

  in particular (taking $k = n$) $\binom{n}{n} = 1$;

  if $k = 0$, then answer is 1 for all n;
- the expression $\binom{n}{k}$ is the number of ways of choosing k objects from a set of n objects, and so it occurs a lot in books on Statistics, where it is often referred to as "n choose k";
- the binomial theorem generalises to the situation where n is a number other than a positive integer, but this will be treated later in the course.

1.5.2 Example. Use the binomial theorem to expand $(x+y)^5$.

Solution. We first put $n = 5$ in the theorem and write out the summation in full.

$$(x+y)^5 = \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5.$$

We now start calculating the binomial coefficients

$$\binom{5}{0} = 1, \qquad \binom{5}{1} = \frac{5}{1!} = 5, \qquad \binom{5}{2} = \frac{5.4}{2.1} = 10.$$

Symmetry gives the other coefficients. Hence

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

□

1.5.2 Pascal's Triangle

This is an alternative way of working out binomial coefficients. It is quite efficient when n is small, but not so good when it isn't.

$$\begin{array}{r|ccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Within the triangle each number is the sum of the two immediately above it. The n-th row gives the binomial coefficients for $(x+y)^n$. The drawback is that in order to get at a row you have to calculate all its predecessors.

1.5.3 Using the Binomial Theorem

We now show how the theorem can help in situations which on first sight don't seem to be a good fit for it. This is characteristic of a good theorem.

There is nothing special about the symbols x and y used in the statement of the binomial theorem, they can be replaced by any numbers or any expressions representing numbers.

1.5.3 Example. Expand $\left(x + \dfrac{1}{x}\right)^5$.

Solution. Write down the expansion for $(x+y)^5$ and then put $y = 1/x$. □

1.5.4 Example. Expand $\left(x^2 - \dfrac{1}{x}\right)^6$.

Solution. Write down the expansion for $(a+b)^6$ and then put $a = x^2$ and $b = -\dfrac{1}{x}$. □

1.5.5 Example. Expand $(1 + x + x^2)^4$.

Solution. Think of this as $(a + b)^4$ where $a = 1 + x$ and $b = x^2$. Expand, substitute, and then use the Binomial Theorem again to deal with the powers of $(1 + x)$. □

1.5.6 Example. What is the coefficient of x^5 in $\left(x^2 + \dfrac{2}{x}\right)^{10}$?

Solution. This time it is not necessary to write down the full expansion. The general term in the expansion of $(a + b)^{10}$ is $\dbinom{10}{k} a^{10-k} b^k$. Replacing a by x^2 and b by $\dfrac{2}{x}$ turns this into $\dbinom{10}{k} x^{20-2k} 2^k x^{-k}$, i.e. into $2^k \dbinom{10}{k} x^{20-3k}$.

Now work out which k you want. We have $20 - 3k = 5$ when $k = 5$. So the required coefficient is the one with $k = 5$. So the required coefficient is

$$2^5 \binom{10}{5} = 32.\frac{10.9.8.7.6}{5.4.3.2.1} = 32.9.4.7 = 8064$$

□

1.6 Coordinate Geometry

1.6.1 Cartesian Coordinates

A *directed* line, often referred to as an *axis*, consists of a line together with a direction along the line (usually indicated by an arrow).

Let A and B be two distinct points on a *directed* line. The length of the line segment between A and B is denoted by $|AB|$. It is assumed that the length is measured in appropriate units, *unless otherwise is said it will be assumed that the units are metres.* The length is a *positive* number.

The direction from A to B is either the same as the given direction of the line or opposite to it. We define the *signed* distance *from A to B* to be $|AB|$ if the direction from A to B is the *same* as the direction of the line and it is $-|AB|$ if the direction from A to B is *opposite* to the direction of the line.

Let O be fixed point on the directed line ('O' is for origin).

The *Cartesian coordinate* of a point P on the line with respect to the chosen origin O is the *signed* distance from O to P.

The origin O divides the directed line into two half-lines called the *negative* and *positive* axis as in Figure 1.1. The negative half line contains points whose coordinates

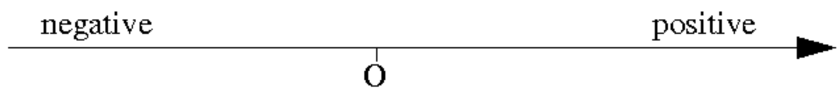


Figure 1.1: Directed Line

are negative, similarly the positive half line contains points whose coordinates are positive.

Often we refer to the *point* a, meaning of course the point with coordinate a (formally we *identify* a number a with the point with Cartesian coordinate a with respect to O).

Next we discuss Cartesian co-ordinates of points in a *plane*.

To define Cartesian co-ordinates in a plane two mutually perpendicular lines in the plane are chosen. (Good choices can simplify the problem at hand.) The point at which the lines meet is called the *origin* and is usually denoted by O. Next a direction on both lines is chosen. We call the two directed lines the x-axis and the y-axis and refer to the plane as the (x, y)-plane.

For a point P in the plane, let A and B be the feet of the perpendiculars from P to the x-axis and y-axis respectively (see Figure 1.2).

Let a=*signed* distance from O to A along the x-axis
and b=*signed* distance from O to B along the y-axis,

The numbers a and b are called the Cartesian coordinates of P relative to the chosen axes, a is called the x-coordinate of P and b is called the y-coordinate of P. The Cartesian coordinates of P are usually written as an ordered pair (a, b) (in that order − the x-coordinate is given first).

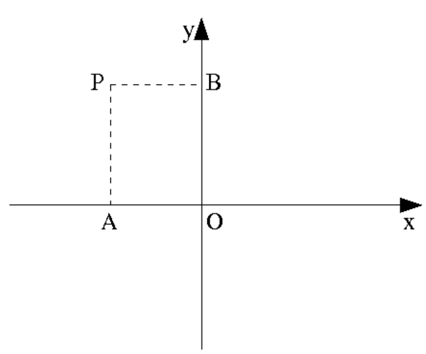


Figure 1.2: Cartesian Coordinates

(If P is the point indicated in Figure (1.2) then a is negative and b is positive.)

We can think of x and y as variables which assign coordinates to points in the plane. Thus we could refer to the point P as the point with $x = a$ and $y = b$.

Once the coordinate system has been chosen, it is standard practice to identify points with their coordinate pairs. We refer to the 'point' $(a,\ b)$ meaning, of course, the point with coordinates $x = a$ and $y = b$.

Rather than use x and y for Cartesian coordinates we could use any symbols that take our fancy. For instance we could use u and v in which case we would refer to the $(u,\ v)$-plane.

The most common way of specifying a point in a plane is by Cartesian coordinates, however there are lots of other coordinate systems in use as well, particularly in specialised applications. For instance polar co-ordinates wbich are discussed in section 1.8.2

1.6.2 Distance between points

By Pythagoras' theorem the distance d between the two points (x_1, y_1) and (x_2, y_2) in the (x, y)-plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.1}$$

For example, the distance between the points $(1, 2)$ and $(5, 7)$ is

$$\sqrt{(5-1)^2 + (7-2)^2} = \sqrt{4^2 + 5^2} = \sqrt{41}.$$

The distance between the points $(-2, -5)$ and $(-3, 7)$ is

$$\sqrt{(-3-(-2))^2 + (7-(-5))^2} = \sqrt{(-1)^2 + 12^2} = \sqrt{145}$$

It is important to realise that the formula for distance between points holds for any Cartesian coordinate system in the plane.

1.6.3 Equation of a line

Consider a line in the (x, y)-plane. Let $(x_1,\ y_1)$ and $(x_2,\ y_2)$ be two *distinct* points on the line and let

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

It turns out that this number m is the same for any two distinct points on the line, it is called the *slope* or *gradient* of the line. It turns out that $m = \tan\theta$ where θ is the angle the line makes with the positive x-axis.

The slope is defined for any line other than those lines that are perpendicular to the x-axis. We do not give a slope for those lines.

The following results should be well known.

- Two lines with the same slope are parallel.
- The condition for two lines of slopes m_1 and m_2 to be *perpendicular* is that

$$m_1 m_2 = -1 \tag{1.2}$$

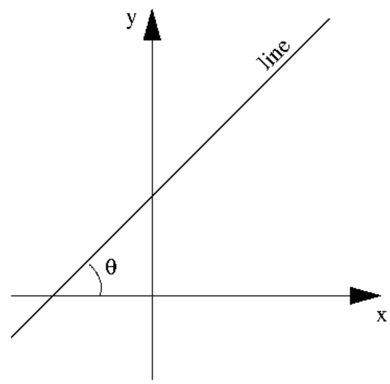


Figure 1.3: The slope of a line.

- An equation of a straight line having slope m is of the form

$$y = mx + c \tag{1.3}$$

- The constant c tells us where the line meets the y-axis.
- The last equation can be expressed in different useful forms.

  (i) An equation of the straight line through the point (a, b) with slope m is

$$y - b = m(x - a) \tag{1.4}$$

  This is clear. The coefficient of x is m, so the line has slope m. When $x = a$ the RHS is zero, so $y = b$. Therefore the line goes through (a, b).

  (ii) If $x_1 \neq x_2$ then an equation of the straight line through (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) \tag{1.5}$$

- The above results do not apply to lines perpendicular to the x-axis. An equation of such a line is of the form

$$x = \text{constant} \tag{1.6}$$

- The most general equation of a straight line is

$$ax + by + c = 0 \tag{1.7}$$

  where a and b are constants not both zero. (If $b \neq 0$ then we can re-arrange the equation as $y = -\frac{a}{b}x - \frac{c}{b}$ which is of the form $y = mx + c$ as before. If, on the other hand, $b = 0$ then the equation becomes

$$x = -\frac{c}{a} = \text{constant}$$

  which covers the case of lines perpendicular to the x-axis.)

1.6.1 Example. Find an equation of the straight line through $(1, 2)$ with slope -1.

Solution. Using the above formula the equation is $y - 2 = -1(x - 1)$ or, tidying up, $y = 3 - x$. □

1.6.2 Example. Find an equation of the straight line through $(4, 5)$ which is perpendicular to the line $y = 2x - 3$.

Solution. The second line has slope 2. So, by the formula $m_1 m_2 = -1$ the required slope of our line is $-1/2$. So an equation of the line is $y - 5 = -\frac{1}{2}(x - 4)$
or $x + 2y = 14$ □

1.6.3 Example. What is the equation of the straight line through the two points $(1, 3)$ and $(-3, 8)$?

Solution. We can do this by the earlier formula, but it is probably easier to do it in two stages. First, the slope of the line must be

$$m = \frac{8 - 3}{-3 - 1} = -\frac{5}{4}$$

So an equation is $y - 3 = -\frac{5}{4}(x - 1)$ or $5x + 4y = 17$. □

1.6.4 Finding the intersection of two lines

Suppose two lines intersect at the point. Then the the point must lie on both lines, so its coordinates must satisfy equations of both lines. We treat the equations of the two lines as a pair of simultaneous equations which we solve to get the coordinates of the point of intersection.

1.6.4 Example. Find the point of intersection of the line which passes through $(1, 1)$ and $(5, -1)$ and the line which passes through $(2, 1)$ and $(3, -3)$.

Solution. Line one has equation $y - 1 = \left(\frac{-1 - 1}{5 - 1}\right)(x - 1)$ i.e. $2y + x = 3$.

Line two has equation $y - 1 = \left(\frac{-3 - 1}{3 - 2}\right)(x - 2)$ i.e. $y + 4x = 9$.

Solving these two equations simultaneously gives us the point $(\frac{15}{7}, \frac{3}{7})$.i

[If you cannot rember how to solve these equations:

$$\begin{aligned} 2y + x &= 3 \\ y + 4x &= 9 \ . \end{aligned}$$

Muliply the second equation by 2:

$$\begin{aligned} 2y + x &= 3 \\ 2y + 8x &= 18 \ . \end{aligned}$$

Subtract:

$$\begin{aligned} 2y + x &= 3 \\ -7x &- 15 \end{aligned}$$

So $x = 15/7$ and $y = (3 - 15/7)/2 = 3/7$.] $\square$

1.6.5 Example. Where do the two lines $y = 3x - 2$ and $y = 5x + 7$ meet?

Solution. The point (x, y) where they meet must lie on both lines, so x and y must satisfy both equations. So we are looking to solve the two simultaneous equations

$$y = 3x - 2 \quad \text{and} \quad y = 5x + 7$$

Well, in that case, $3x - 2 = 5x + 7$, so $2x = -9$ and $x = -9/2$. Putting this back in one of the equations we get $y = -27/2 - 2 = -31/2$. $\square$

Some exercises:

1.6.6 Exercise. Find an equation of the straight line passing through the point $(-3, 5)$ and having slope 2.

1.6.7 Exercise. Find an equation of the line passing through the points $(-4, 2)$ and $(3, 8)$. Find equations of the lines through $(5, 5)$ parallel and perpendicular to this line.

1.6.8 Exercise. Where do the lines $y = 4x - 2$ and $y = 1 - 3x$ meet? Where does the line $y = 5x - 6$ meet the graph $y = x^2$?

1.6.9 Exercise. Let l_1 be the line of slope 1 through $(1, 0)$ and let l_2 be the line of slope 2 through $(2, 0)$. Where do they meet and what is the area of the triangle formed by l_1, l_2 and the x-axis? $\square$

1.6.5 Circles

The circle in the (x, y)-plane with centre A and radius r is the set of points in a plane that are at distance r from A. By our formula for the distance between two points, we can say that the distance between (x, y) and (a, b) is r if and only if

$$(x - a)^2 + (y - b)^2 = r^2$$

This is therefore an equation of the circle of radius r having centre at the point (a, b).

We get a simple special case if the centre happens to be at the origin. Then the equation becomes

$$x^2 + y^2 = r^2$$

An equation of the form

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the equation of a circle if $a^2 + b^2 - c > 0$'. To see this, note that 'completing the square' gives

$$x^2 + 2ax = (x + a)^2 - a^2 \quad \text{and} \quad y^2 + 2by = (y + b)^2 - b^2.$$

Hence the equation becomes

$$(x + a)^2 + (y + b)^2 = a^2 + b^2 - c$$

This is an equation of the circle with centre at $(-a, -b)$, case and radius $r = \sqrt{a^2 + b^2 - c}$ *provided* that the RHS of the equation is positive.

The tangent to a circle at a point P on the circle is the straight line through P that just touches the circle at P. By the geometry of the circle this line is perpendicular to the radius AP (A is the centre of the circle). So we can write down the equation of the tangent: we know it goes through P and we know its slope from the formula $m_1 m_2 = -1$.

1.6.10 Example. Find an equation of the tangent to the circle

$$x^2 + y^2 - 2x - 4y = 20$$

at the point (4, 6) on the circle.

Solution. The center of the cicle is (1, 2) and so the radius to the point (4, 6) has slope 4/3. The slope of the tangent at (4, 6) is $-3/4$ and so has equation $4y+3x = 36$. □

Some exercises:

1.6.11 Exercise. Find an equation of the circle with centre $(1, 2)$ and radius 3. Find the two points where this circle meets the x-axis and the two points where this circle meets the y-axis. What is the area of the quadrilateral formed by these four points?

1.6.12 Exercise. What are the centre and the radius of the circle with equation

$$x^2 + y^2 - 2x + 4y = 4 \quad ?$$

□

1.7 Angles and Trigonometry

1.7.1 Degrees and radians

Engineers normally measure the size of an angle in degrees or radians. Degrees are easy and familiar. There are 360 degrees in a single rotation. A right angle is therefore 90°. Radians are less familiar, but you must get used to them because in this course we will deal almost exclusively with radians. There are strong mathematical reasons for doing this.

The size in *radians* of the angle subtended at the centre of a circle by an arc of the circle is given by

$$\theta = \frac{l}{r},$$

where l is the length of the arc and r is the radius of the circle (one radian is the size of angle subtended by a arc of length r).

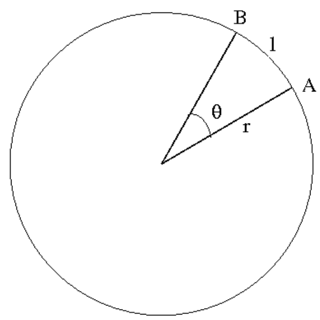


Figure 1.4: Size of an angle in radians

This means that the size of a single revolution is 2π radians because the length of the circumference of a circle of radius r is $2\pi r$.

Outside of engineering and mathematics the use of degrees is almost universal, because the numbers are nicer. Degrees are also the older system, being in use by Greek mathematicians in the second century BC, with throwbacks from there to the Babylonian astronomers and mathematicians with their base 60 number system. Radians, along with the trigonometric functions such as sine and cosine, were introduced by Indian mathematicians in the sixth century AD. The numbers involved are more awkward, but the system gives a neater relationship between angle and arc length on the circle, and this makes for tidier formulae elsewhere. This is especially true of calculus formulae and so in this context radians are preferred by everyone.

When you are doing calculus you should *always* use radians to measure angles. A lot of the formulas for standard derivatives and integrals are false if you don't.

Most calculators wake up in degree mode. You must switch to radians before doing any calculation for this course which involves a trig function. If you don't, you will get wrong answers, because the formulas you are using are built on radian measurement.

We have the following obvious conversions:

$$360^\circ = 2\pi \text{ radians} \quad 180^\circ = \pi \text{ radians} \quad 90^\circ = \tfrac{1}{2}\pi \text{ radians}$$

$$45^\circ = \frac{1}{4}\pi \text{ radians} \quad 60^\circ = \frac{1}{3}\pi \text{ radians} \quad 30^\circ = \frac{1}{6}\pi \text{ radians}$$

Angles can be specified as either *clockwise* or *anticlockwise*.

Some care must be exercised. The angle between two lines in plane seen as anticlockwise from one side of a plane but will appear clockwise from the other side of the plane. Hence before discussing 'clockwise and anticlockwise' we must specify the side from which the plane is viewed.

Once the viewpoint is chosen, a real number θ is understood to represent an *anticlockwise* angle if $\theta > 0$ and a *clockwise* angle if $\theta < 0$, of size $|\theta|$ radians in both cases. *This convention will be used throughout this course.*

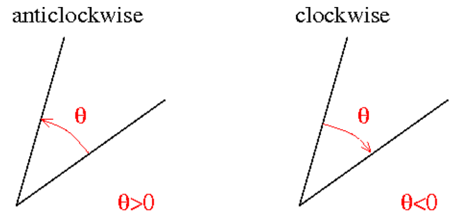


Figure 1.5: Anticlockwise and Clockwise

1.7.2 Trigonometric Functions

You should have met and be familiar with the functions sine, cosine and tangent. It is essential, as engineers, that you are familiar with these functions and their properties. They are usually introduced using a right-angled triangle. For the triangle shown in Figure 1.6

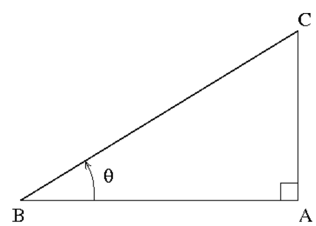


Figure 1.6: A labelled triangle

$$\sin\theta = \frac{|CA|}{|CB|}, \qquad \cos\theta = \frac{|AB|}{|CB|}, \qquad \tan\theta = \frac{|CA|}{|AB|}$$

(θ is the size of the angle measured in *radians*). It is emphasised that, in this course, $\sin\theta$ means the sine of an angle θ *radians*, which is somewhat different to the sine of an angle of θ *degrees*. Similar remarks for $\cos\theta$. Remember in this course we use *radians*, not *degrees*.

The ratios above only give the values of $\sin\theta$ and $\cos\theta$ for $0 < \theta < \pi/2$. In fact $\sin\theta$ and $\cos\theta$ are defined for any value of θ. $\square$

Perhaps the easiest way to understand to the definitions of sine and cosine is by considering rotations in the $(x,\ y)$-plane.

(First we fix our viewpoint.) The (x, y)-plane is viewed so that direction of the x-axis is to the right and the y-axis is 'upwards' as in the Figure 1.7 (this is possible from exactly one side of the plane).

Consider the circle of radius 1 with centre at the origin in the (x, y)-plane and let OX be the radius along the *positive* x-axis as in Figure 1.7.

For any number θ rotate the radius OX through an angle θ radians about O and suppose that X ends up at the point P, then

$$\cos\theta = x\text{-coordinate of } P, \qquad \sin\theta = y\text{-coordinate of } P.$$

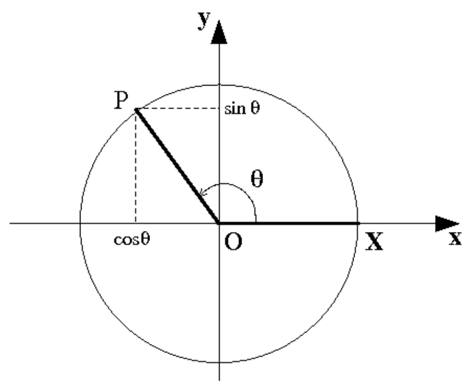


Figure 1.7: Circle with radius 1.

If θ is positive we rotate OX anticlockwise, if θ is negative we rotate OX clockwise as seen from our chosen viewpoint.

Moreover $\tan\theta = \sin\theta/\cos\theta$ for any θ for which $\cos\theta \neq 0$.

We can draw some immediate and important conclusions from Figure 1.7.

- For any number θ, $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$.
- Turning OP through a full rotation (in either direction) brings us back to where we started. So the trig functions are *periodic* with period 2π:

$$\cos(\theta + 2\pi) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta$$

- Turning through half a rotation takes us from (x, y) to the opposite point $(-x, -y)$. So

$$\sin(\theta + \pi) = -\sin\theta \qquad \cos(\theta + \pi) = -\cos\theta$$

- Turning through $\pi/2$ (a quarter rotation) has a more complicated effect. If you think about it you will see that the effect is to send (x, y) to $(-y, x)$. So

$$\cos(\theta + \tfrac{1}{2}\pi) = -\sin\theta \qquad \sin(\theta + \tfrac{1}{2}\pi) = \cos\theta$$

- Turning through a negative angle is like rotating through the corresponding positive angle and then reflecting in the x-axis. i.e. if a turn of OX through negative angle θ gets us to (x, y) then a turn through the corresponding positive angle will take us to $(x, -y)$. So

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

It is also worth knowing that

$$\sin(\pi/2 - \theta) = \cos\theta \qquad \cos(\pi/2 - \theta) = \sin\theta$$

Because the point P lies on the circle so its distance from the origin is 1, from we obtain the very important relationship

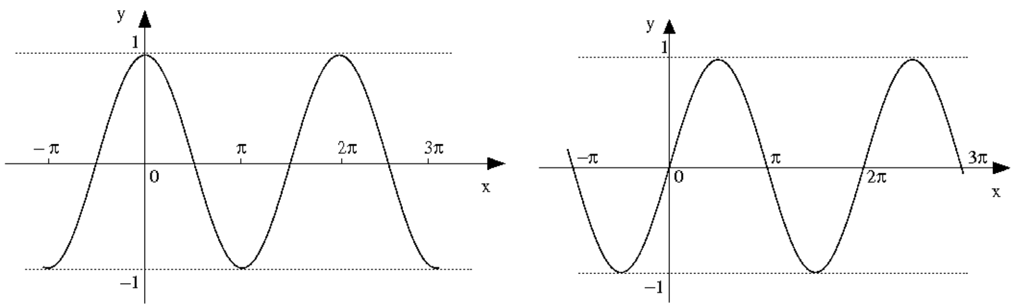
$$\sin^2\theta + \cos^2\theta = 1 \tag{1.8}$$

(see Section 1.6.5) and from this we get

$$1 + \tan^2\theta = \sec^2\theta.$$

□

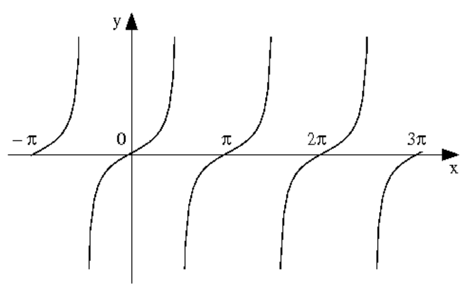
The graphs of the trig functions, sine and cosine, are sketched in Figure 1.8.



$y = \cos x$ $\quad$ $y = \sin x$

Figure 1.8: Graphs of Cosine and Sine

The graph of the function tangent is sketched in Figure 1.9



$y = \tan x$

Figure 1.9: Graph of the Tangent function

□

There are important *addition formulas* for the trig functions:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \tag{1.9}$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b) \tag{1.10}$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{1.11}$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) \tag{1.12}$$

and from these we get the special cases

$$\begin{aligned}
\sin(2a) &= 2\sin(a)\cos(a) && (1.13)\\
\cos(2a) &= \cos^2(a) - \sin^2(a) && (1.14)\\
&= 2\cos^2(a) - 1 && (1.15)\\
&= 1 - 2\sin^2(a) && (1.16)
\end{aligned}$$

We also wiil also meet:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Some exercises:

1.7.1 Exercise. Starting from the trig formulas (1.9–1.12) show that

$$\sin 2x = 2\sin x\cos x, \qquad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x),$$

1.7.2 Exercise. Do these in the same way.

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\tan x + \tan y}{1 - \tan x\tan y}, \qquad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \operatorname{cosec}^2 x.$$

1.7.3 Exercise. Use the fact that you know the values of the trig functions at π and $\pi/2$ to show that

$$\sin(\pi - x) = \sin x, \qquad \cos(\pi - x) = -\cos(x), \qquad \sin(\pi/2 - x) = \cos(x),$$

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x+2\pi) = \cos x$$

□

The values of the trig functions at certain numbers are worth knowing. They are given in Table 1.1.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0	1	0	0
π	-1	0	0
$\pi/2$	0	1	-
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$
$\pi/6$	$\sqrt{3}/2$	$1/2$	$1/\sqrt{3}$

Table 1.1: Some values of trigonometric functions.

Some notational remarks are in order. Formally 'sin', 'cos' and 'tan' are functions. Thus, for instance, the value of 'sin' at x should be written as $\sin(x)$ (note the brackets). Similarly for the other trigonometric functions. However, if no confusion results then it is convention to omit the brackets and, for instance, write just $\sin x$ as we did earlier in this section. However brackets are sometimes necessary. For instance in the case $\sin(x+y)$, without the brackets becomes $\sin x + y$, which could be mistaken for $\sin(x) + y$. When in doubt always use the brackets!!!

Another convention is that $(\sin x)^2$ (meaning of course $\sin(x) \times \sin(x)$) is usually written as $\sin^2 x$. More generally $\sin^n x$ means $\sin x$ to the power n. This convention can lead to confusion as we will see later!!!

1.8 Other Topics

1.8.1 Curves

Often a curve in the (x, y)-plane is given by an equation involving x and y. For example, as we have seen

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the equation of a circle. But there are other ways of describing such a curve.

If a curve is traced out by a moving point, then you could give the x-coordinate and the y-coordinate of the point 'at time t', so you end up with both x and y as functions of t.

In general one can describe a curve by giving the x-coordinates and y-coordinates of points on the curve as functions of some variable. Suppose that

$$x = f(u), \qquad y = g(u). \tag{1.17}$$

As u increases the point $(f(u), g(u))$ traces out a curve in the (x, y)-plane. (1.17) is called a parameterisation of the curve, u is called a parameter.

1.8.1 Example. The curve traced out by a point on the circumference of a wheel that is rolling along in a straight line is given (with respect to suitable coordinates) by

$$x = a(u - \sin u)\,, \qquad y = a(1 - \cos u)$$

where u is the parameter and a is the radius of the wheel. (In fact u is the angle of rotation of the wheel.) □

1.8.2 Example. The circle centre (a, b) and radius r can be described parametrically by

$$x = a + r\cos\theta\,, \qquad y = b + r\sin\theta\,.$$

In this case θ is the parameter. □

This is often the simplest way to deal with complicated curves, and is clearly the approach to adopt if you want to plot a curve using a computer.

1.8.2 Polar Co-ordinates

So far we have used Cartesian Coordinates to describe points in a plane. There are lots of other coordinates in use as well, particularly in specialised applications, for instance polar coordinates. Polar coordinates are especially useful in situations where information is most conveniently expressed in terms of distance from the origin.

Let P be a point in the (x, y)-plane, other than the origin.

The *polar coordinates* of P are (r, θ) where $r = |OP| > 0$ and θ is the angle swept out from the positive x-axis to the radius OP.

By convention, we take the angle range to be $-\pi < \theta \leq \pi$. (other people use $0 \leq \theta < 2\pi$).

Note that we do not give polar coordinates for the origin, because the angle does not make sense there.

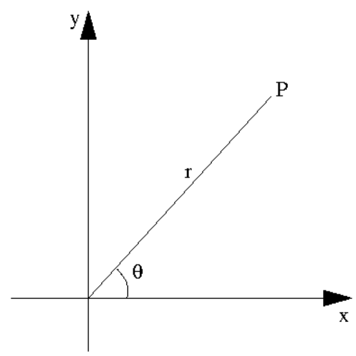


Figure 1.10: Polar co-ordinates (r, θ) of a point.

Now consider the problem of converting between Cartesian and polar coordinates. One way round is easy:

$$x = r\cos\theta \qquad y = r\sin\theta$$

(these are really just the definitions of $\sin\theta$ and $\cos\theta$.)

The other way round is more tricky and needs some understanding of inverse trig functions. First of all, without any difficulty,

$$r = \sqrt{x^2 + y^2}$$

The real problem is to get θ in terms of x and y. Dividing the above equations gives

$$\tan\theta = \frac{y}{x}$$

So we are tempted to write

$$\theta = \arctan(\frac{y}{x})$$

but this could be wrong.

First of all, we have to be careful about the case $x = 0$ (which corresponds to the y-axis) because we don't want to divide by zero.

In the case $x = 0$ inspection of Figure 1.10 tells us that if $y > 0$ then $\theta = \pi/2$ and if $y < 0$ then $\theta = -\pi/2$.

Our problems don't end there. Recall that 'arctan' always gives values in the interval from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$ so it is not good enough in itself to solve our problem. The obvious trouble is that, for example, $(1, 1)$ and $(-1, -1)$ give the same value for y/x even though the 'angles swept out from the origin' area different ($\pi/4$ and $-3\pi/4$). We have to take account of the quadrant in which the point lies.

The correct conversion rules are these, which you should check through as an exercise.

- if $x > 0$ then $\theta = \arctan(y/x)$ (in this case the point (x, y) is in the first or fourth quadrant);
- if $x < 0$ and $y \geq 0$ then $\theta = \arctan(y/x) + \pi$ (in this case the point (x, y) is in the second quadrant);
- if $x < 0$ and $y < 0$ then $\theta = \arctan(y/x) - \pi$ (in this case the point (x, y) is in the third quadrant).

The following diagrams indicate θ in the case the the point (x, y) is in the second quadrant ($x < 0$, $y > 0$) and in the third quadrant ($x < 0$, $y < 0$):

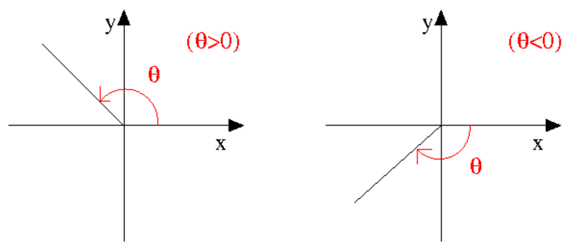


Figure 1.11: Polar co-ordinate θ.

1.8.3 Example. For instance the point $(-1, -1)$ has polar coordinates given by

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2};$$

$$\theta = \arctan\left(\frac{-1}{-1}\right) - \pi = \arctan 1 - \pi = \frac{\pi}{4} - \pi = -\tfrac{3}{4}\pi$$

since $(-1, -1)$ is in the third quadrant. $\square$