Contents

Chapter 5

More Functions

5.1 Introduction

In this chapter some important functions will be introduced. Properties of these functions will be established, derivatives computed and associated problems investigated.

A large part of this chapter will be devoted to *inverse* functions. To introduce the ideas let us go back to an example from Chapter 3 (Example 3.1.4). This problem was concerned with the angle of elevation θ from an observer to a rocket which had been launched vertically from a point some distance from the observer. It was established that

$$\tan\theta = \frac{t^2}{4} \tag{5.1}$$

where t was the time from launch. The problem concerned the rate of change of θ with respect to t. Ideally we would have liked to write θ as a function of t but, at that stage, did not know how to get the function and we avoided this problem by using implicit differentiation.

First, rather than ask for θ as a function of t, we ask the simpler question:

What is θ when $t = 6$?

The answer in words could be '*a* number whose tangent is 9' ($6^2/4 = 9$). Part of the graph $x = \tan\theta$ is given in Figure 5.1. The figure indicates that there are several such numbers (three are indicated in the figure but there are infinitely many (one in every interval of length π)).

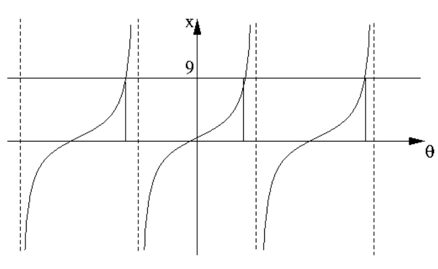

Figure 5.1: Graph $x = \tan\theta$.

There is exactly one number whose tangent is 9 which lies in the interval between $-\pi/2$ and $\pi/2$ and, since in the rocket problem we know that θ lies between 0 and $\pi/2$, this must be our required value of θ. Thus more precisely the answer in words could be '*the* number between 0 and $\pi/2$ whose tangent is 9'. This is an answer of sorts, but it is not really the answer we would want.

However we note, from the graph, that given any number x there is a *unique* number between $-\pi/2$ and $\pi/2$ whose tangent is x, and so we can define a function, which is denoted by 'arctan', by

$$\arctan(x) = \theta \quad \text{where } \theta \text{ is } \textit{the} \text{ number between } -\pi/2 \text{ and } \pi/2 \text{ with } \tan\theta = x.$$

Unfortunately this function cannot be expressed in terms of our existing repertoire of functions, so we have to derive its properties from first principles.

Going back to the rocket problem, from equation (5.1), we deduce

$$\theta = \arctan\Big(\frac{t^2}{4}\Big)$$

and thus, now, we have θ as a function of t. The function 'arctan' is called the *inverse* function to 'tan' on the interval $\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$.

The word 'inverse' is *not* used in the sense of 'one over'.

Let f be a (real) function. We can think of f as a process which takes as input a number a and outputs the number b where $b = f(a)$. Informally an inverse function to f reverses this process it takes as input a number b and outputs a number a such that $f(a) = b$.

Consider the function, call it f, whose graph is given in Figure 5.2.

To read off the value of f at a given number a: locate a on the x-axis, go directly up to the graph and then move across to the y-axis to get $b = f(a)$.

To reverse the process: given a number b (for this function b must be positive), locate b on the y-axis, go directly across to the graph and then move down to the x-axis to get a such that $f(a) = b$.

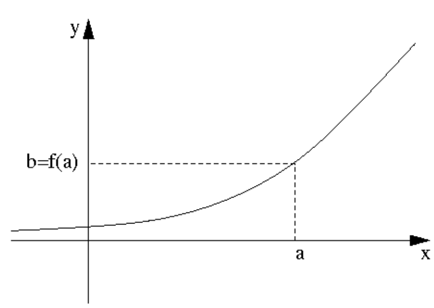

Figure 5.2: Graph $y = f(x)$.

In this case for any number b with $b > 0$ we see from the graph that there is a unique number a such that $f(a) = b$ and so we can define a function, g say, whose value at b is the number a. g is called the *inverse* to f (on $\mathbb{R}$), often denoted by f^{-1}.

5.2 Inverse Trigonometric Functions

5.2.1 Inverse to the function 'sin'

For each number b with $-1 \leq b \leq 1$ there are infinitely many numbers a with $\sin a = b$. In order to define an inverse function we must choose exactly one of them for each such b. If we consider the graph $x = \sin\theta$ we see for each b with $-1 \leq b \leq 1$ there is exactly one number a in the interval from $-\pi/2$ to $\pi/2$ with $\sin a = b$ (see Figure 5.3).

Thus if $-1 \leq x \leq 1$ we define

$$\arcsin(x) = \theta \quad \text{where } \theta \text{ is } \textit{the} \text{ number between } -\pi/2 \text{ and } \pi/2 \text{ with } \sin\theta = x.$$

Note that from the definition

$$\sin(\arcsin(x)) = x \qquad \text{for } -1 \leq x \leq 1 \tag{5.2}$$

and

$$\arcsin(\sin(\theta)) = \theta \qquad \text{for } -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2} \tag{5.3}$$

The function 'arcsin' is the inverse to 'sin' on the interval from $-\pi/2$ to $\pi/2$.

As a technical point, note that if $x > 1$ or $x < -1$ then there does not exist a number whose sine is x. Thus $\arcsin x$ is only defined for $-1 \leq x \leq 1$ and hence the domain of arcsin is the closed interval $[-1, 1]$. In Figure 5.3, $a = \arcsin b$.

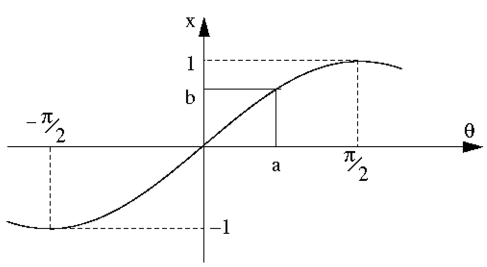

Figure 5.3: Graph $x = \sin\theta$.

□

We often just write $\arcsin x$ for $\arcsin(x)$.

5.2.2 Inverse to the function 'cos'

The function 'arccos' is defined in a similar way. If we consider the graph $x = \cos\theta$ we see that for each number b with $-1 \leq b \leq 1$ there is exactly one number a in the interval from 0 to π with $\sin a = b$ (see Figure 5.4).

Thus if $-1 \leq x \leq 1$ we define

$$\arccos(x) = \theta \quad \text{where } \theta \text{ is } \textit{the} \text{ number between 0 and } \pi \text{ with } \cos\theta = x,$$

the domain of ‘arccos’ is again the closed interval $[-1, 1]$.

The function ‘arccos’ is the inverse to ‘cos’ on the interval 0 to π.

In Figure 5.4, $a = \arccos(b)$.

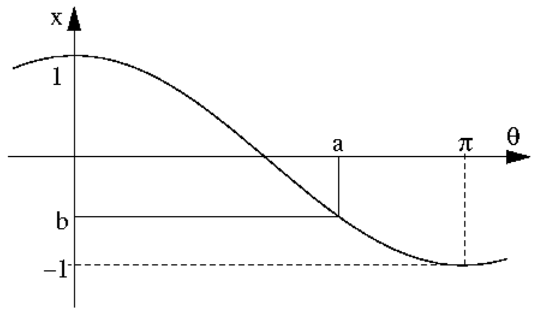

Figure 5.4: Graph $x = \cos\theta$.

Note that (similar to (5.2) and (5.3))

$$\cos(\arccos(x)) = x \qquad \text{for } -1 \le x \le 1 \tag{5.4}$$

and

$$\arccos(\cos(\theta)) = \theta \qquad \text{for } 0 \le \theta \le \pi \tag{5.5}$$

Often we just write $\arccos x$ for $\arccos(x)$.

5.2.3 Inverse to the function ‘tan’

The function, ‘arctan’, was mentioned in the introduction. For completeness we repeat the definition:. For any number x:

$$\arctan(x) = \theta \quad \text{where } \theta \text{ is } \textit{the} \text{ number between } -\pi/2 \text{ and } \pi/2 \text{ with } \tan\theta = x.$$

The domain of ‘arctan’ is the whole of $\mathbb{R}$. The function ‘arctan’ is the inverse to ‘tan’ on the interval $-\pi/2$ to $\pi/2$.

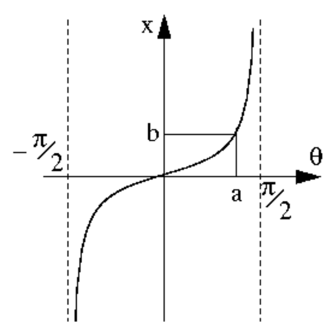

Figure 5.5: Graph $x = \tan\theta$.

In Figure 5.5, $a = \arctan b$. From the definition

$$\tan(\arctan(x)) = x \qquad \text{for any number } x \tag{5.6}$$

and

$$\arctan(\tan(\theta)) = \theta \qquad \left(-\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\right) \tag{5.7}$$

Often we just write $\arctan x$ for $\arctan(x)$.

5.2.4 Calculations

You can find particular values of inverse trig functions from your calculator.

The functions 'arcsin', 'arccos' and 'arctan' are usually written on calculators as '$\sin^{-1}$', '$\cos^{-1}$' and '$\tan^{-1}$' respectively. (As we mentioned earlier, the inverse to a function f is often written as f^{-1}.)

[This can be confusing (because $1/(\sin x)$ is sometimes written as $\sin^{-1} x$ which is certainly not the same as $\arcsin x$). To avoid confusion we will write always write $\operatorname{cosec} x$, $\sec x$ and $\cot x$ for $1/(\sin x)$, $1/(\cos x)$ and $1/(\tan x)$ respectively so that when we write $\sin^{-1}$, etc we mean the inverse functions.]

With your calculator set to radians the functions '$\sin x$, $\cos x$ and $\tan x$' will give the sine, cosine and tangent of an angle of x radians and for instance '$\tan^{-1}(x)$ on your calculator will give you the the angle in *radians* between $-\pi/2$ and $\pi/2$ whose tangent is x.

5.2.1 Exercise. What does your calculator give for $\arcsin(0.23)$, $\arccos(-0.23)$, $\arccos(1.23)$ and $\arctan(-4.38)$? □

Suppose that you are given b $(-1 \le b \le 1)$ and you want to find a such that $\sin(a) = b$. One possibility is $\arcsin(b)$ but we know there are infinitely many other possibilities and $\arcsin(b)$ may not be the the number required by the problem at hand. Similar situations occur with 'tan' and 'cos'.

Two examples were discussed in lectures.

The following results are useful.

- Let $\theta = \arcsin(b)$. Then $\sin(a) = b$ if and only if $a = n\pi + (-1)^n\theta$ for some integer n.
- Let $\theta = \arccos(b)$. Then $\cos(a) = b$ if and only if $a = 2n\pi \pm \theta$ for some integer n.
- Let $\theta = \arctan(b)$. Then $\cos(a) = b$ if and only if $a = n\pi + \theta$ for some integer n.

5.2.5 Graphs of inverse trigonometric functions

The graphs of the inverse trigonometric functions, discussed in previous sections, are given in Figures 5.6, 5.7 and 5.8.

Observe that if you take the graph $x = \sin\theta$ and rotate it by an angle π about the line $x = \theta$ then you will see the graph of the 'arcsin'. Similarly for 'cos' and 'tan'. Why ?

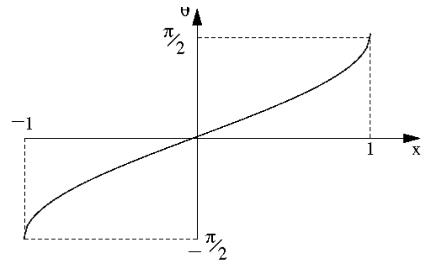

Figure 5.6: Graph $\theta = \arcsin x$.

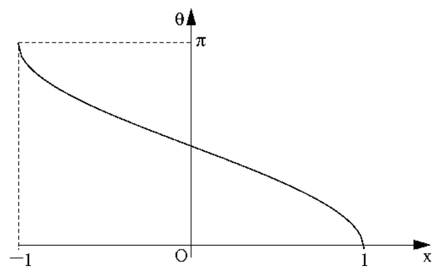

Figure 5.7: Graph $\theta = \arccos x$.

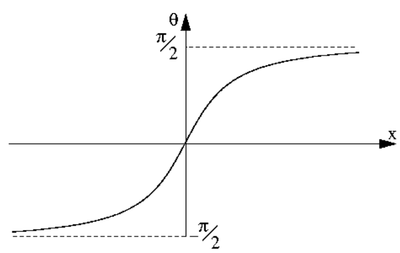

Figure 5.8: Graph $\theta = \arctan x$.

5.3 Derivatives of the Inverse Trigonometric Functions

[The presentation of results in this section looks different to the presentation in the lectures. However the difference is caused only by a change in notation. We use 'Leibniz's notation here.]

5.3.1 Derivative of 'arctan'

We begin with the function arctan, because this is the one with fewest complications. Let

$$\theta = \arctan x$$

Then by (5.6)

$$\tan \theta = x$$

Differentiating both sides with respect to x using the chain rule

$$\sec^2\theta\,\frac{d\theta}{dx} = 1$$

and so

$$\frac{d\theta}{dx} = \frac{1}{\sec^2\theta}$$

This gives us an expression for $d\theta/dx$, but not in a form we can use very easily. We want $d\theta/dx$ as a function of x (what we have got is $d\theta/dx$ as a function of θ). We use the result that

$$\sec^2\theta = 1 + \tan^2\theta$$

and so

$$\sec^2\theta = 1 + \tan^2\theta = 1 + x^2\ .$$

Substituting this into our formula

$$\frac{d\theta}{dx} = \frac{1}{1+x^2}$$

which is normally presented as

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

or in Newtonian notation

$$\arctan'(x) = \frac{1}{1+x^2}. \tag{5.8}$$

This can now be added to our list of standard derivatives.

5.3.2 Derivative of 'arcsin'

Now suppose that

$$\theta = \arcsin x \quad (-1 < x < 1).$$

Then by 5.2

$$\sin\theta = x.$$

Differentiating both sides with respect to x

$$\cos\theta\,\frac{d\theta}{dx} = 1$$

and so

$$\frac{d\theta}{dx} = \frac{1}{\cos\theta}.$$

But

$$\cos^2\theta = 1 - \sin^2\theta = 1 - x^2\ .$$

Since $-\pi/2 < \theta < \pi/2$ (from the defintion of 'arcsin'), we have $\cos\theta > 0$ thus

$$\cos\theta = +\sqrt{1-x^2}$$

(positive square root). Substituting this into our formula

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

which is normally presented as

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$$

---

or

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1). \tag{5.9}$$

This can now be added to our list of standard derivatives. □

5.3.3 Derivative of 'arccos'

Finally suppose that $\theta = \arccos x \quad (-1 < x < 1)$. Then by 5.4

$$\cos\theta = x.$$

Differentiating both sides with respect to x

$$-\sin\theta \frac{d\theta}{dx} = 1$$

and so

$$\frac{d\theta}{dx} = -\frac{1}{\sin\theta}.$$

But

$$\sin^2\theta = 1 - \cos^2\theta = 1 - x^2 \ .$$

Note that $0 < \theta < \pi$ by the defintion of 'arccos', and so $\sin\theta > 0$ thus

$$\sin\theta = +\sqrt{1-x^2}$$

(positive square root). Substituting this into our formula

$$\frac{d\theta}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

which is normally presented as

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$$

---

or

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1). \tag{5.10}$$

Another addition to our list of standard derivatives.

5.4 Exponential and Logarithm

5.4.1 Exponential function

Let a be a *positive* real number. Consider the function f given by

$$f(x) = a^x.$$

(Since a^x is defined for any real number x, the domain of f is $\mathbb{R}$.) To calculate the derivative of f at x we need to investigate what happens to the Newton Quotient of f at x, that is

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h}-a^x}{h}$$

as h tends to 0. By the laws of indices (reviewed in Chapter 1 (Revision))

$$\frac{a^{x+h}-a^x}{h} = \frac{a^x a^h - a^x}{h} = \frac{a^h-1}{h}\,.\,a^x.$$

Thus to work out the derivative from first principles we must investigate what happens the quotient

$$\frac{a^h-1}{h} \quad \text{as } h \text{ tends to zero.}$$

This is not an easy problem.

Using a calculator to evaluate the quotient $(a^h-1)/h$ for very small values of h and should convine you that the quotient tends to a limit as h tends to 0 and

when $a = 1.0$, the limit is 0 (no calculation required);
when $a = 1.5$, the limit is approximately 0.405;
when $a = 2.0$, the limit is approximately 0.693;
when $a = 2.5$, the limit is approximately 0.916;
when $a = 3.0$, the limit is approximately 1.098;
when $a = 3.5$, the limit is approximately 1.253;

and so on. Infact the quotient does indeed tend to a limit as h tends to 0. The limit depends on a, we will denote it by $C(a)$. Thus

$$\frac{a^h-1}{h} \to C(a) \quad \text{as} \quad h \to 0\ ,$$

so

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h}-a^x}{h} = \frac{a^h-1}{h}\,.\,a^x \to C(a).a^x \quad \text{as} \quad h \to 0$$

and hence f is differentiable at x with derivative

$$f'(x) = C(a)\,a^x. \tag{5.11}$$

for any number x. From the experiment with the calculator it seems there may be a value of a, between 2.5 and 3.0, for which the limit $C(a)$ is 1. Indeed this is the case. There is a (unique) number, it is denoted by e, that has the property

$$C(e) = \lim_{h\to 0}\frac{e^h-1}{h} = 1. \tag{5.12}$$

It turns out that $e = 2.718281...$ to 6 decimal places. This number, perhaps surprisingly, turns out to be the fourth most important number for engineers (after 0, 1 and π).

The exponential function, denoted by exp, is the function given by

$$\exp(x) = e^x$$

This is an extemely important function. All engineers need to study it carefully, they must learn and understand its properties. First of all, from the laws of indices (discussed in Section 1), note that

$$e^{x+y} = e^x\, e^y, \quad \left(e^x\right)^y = e^{xy}$$

which, in terms of the exponential function can be written

$$\exp(x+y) = \exp(x)\ \exp(y), \quad \left(\exp(x)\right)^y = \exp(xy) \ .$$

Also note that $\exp(0) = 1$ and $\exp(1) = e$. From (5.11) and (5.12) we see that exp is differentiable at all points of $\mathbb{R}$ and

$$\exp'(x) = \exp(x)$$

This is often written as

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

(It follows that all the higher order derivatives of exp exist and $\exp^{(n)}(x) = \exp(x)$.)

5.4.2 Natural logarithms

Since $\exp'(x) = e^x > 0$ for all values of x it follows that the exponential function is strictly increasing. The graph of the exponential function is given in Figure 5.9

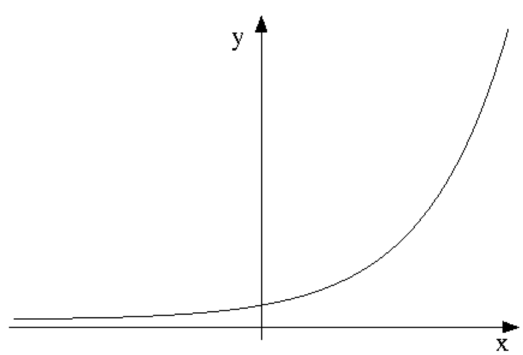

Figure 5.9: Graph $y = \exp(x)$.

From the graph we see that for each number $y > 0$ there is a unique number x such that $\exp(x) = y$. The number x is called the *natural logarithm* of y and denoted by $\ln(y)$.

Note that $\ln(y)$ is defined only for *positive* numbers y.

The function 'ln' is the inverse to 'exp' on $\mathbb{R}$. The number $\ln(y)$ is sometimes called the logarithm of y to the base e and written as

$$\ln(y) = \log_e(y).$$

[In fact we can consider for each $a > 0$ logaithms to the base a. Formerly for $y > 0$

$$\log_a(y) = x \quad \text{where } x \text{ is the number for which } a^x = y.$$

In practice only logarithms to the base 2, e and 10 are used. We will restrict our attention to logarithms to the base e, that is natural logarithms.]

From the definition

$$\exp(\ln(y)) = y, \qquad (y > 0) \tag{5.13}$$

$$\ln(\exp(x)) = x, \qquad (\text{for any number } x), \tag{5.14}$$

which can be written

$$e^{\ln(y)} = y \qquad (y > 0). \tag{5.15}$$

$$\ln(e^x) = x; \qquad (\text{for any number } x), \tag{5.16}$$

Natural logarithms have some nice properities, *which should be memorised*:

(i) $\ln(ab) = \ln(a) + \ln(b)$ for all positive numbers a and b;
(ii) $\ln(a/b) = \ln(a) - \ln(b)$ for all positive numbers a and b;
(iii) $\ln(a^x) = x\ln(a)$ for all numbers x and for all positive numbers a.

[To see (i): from the laws of indices (see Section 1) and (5.15):

$$e^{\ln(a)+\ln(b)} = e^{\ln(a)}\, e^{\ln(b)} = ab$$

Thus by (5.14) and above

$$\ln(a) + \ln(b) = \ln\left(e^{\ln(a)+\ln(b)}\right) = \ln ab.$$

To see (ii):

$$\ln(a) = \ln(b.(a/b)) = \ln(b) + \ln(a/b)$$

by (i). To see (iii):

$$a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)} \ ,$$

hence

$$\ln(a^x) = \ln\left(e^{x\ln(a)}\right) = x\ln(a)$$

again from the laws of indices and (5.14).]

Next we calculate the derivative of 'ln'. From (5.13)

$$\exp(\ln(y)) = y \quad (y > 0).$$

Differentiating with respect to y (and using the chain rule):

$$\exp'(\ln(y))\,\ln'(y) = 1 \quad (y > 0).$$

Therefore

$$\ln'(y) = \frac{1}{\exp'(\ln(y))} = \frac{1}{\exp(\ln(y))} = \frac{1}{y} \quad (y > 0).$$

Changing the variable y to x, this can be written as

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad (x > 0).$$

Note that 'ln' is strictly increasing. The graph is given in Figure 5.10. □

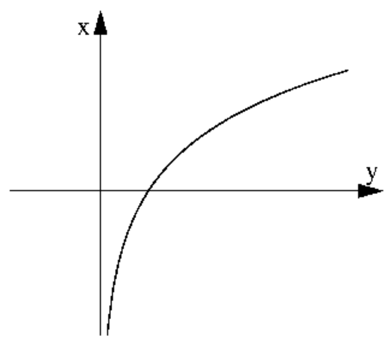

Figure 5.10: Graph $x = \ln(y)$.

Let $f(x) = a^x$ where a is a positive number (constant). We can use our results on 'exp' and 'ln' to calculate the derivative of f. We know by (5.15) that

$$a = e^{\ln(a)}, \quad \text{and so} \quad a^x = \left(e^{ln(a)}\right)^x = e^{\ln(a)x}$$

(the last equailty following from the laws of indices). Using the chain rule

$$f'(x) = e^{\ln(a)x}.\ln(a) = \ln(a).a^x$$

which is usually presented as

$$\frac{d}{dx}(a^x) = \ln(a).a^x.$$

It follows that

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)} \quad (x > 0).$$

It is left as an exercise to verify this last result.

5.4.3 Scientific importance of the exponential function

Whenever we have a situation concerning growth or decay, ‘exp’ is likely to put in an appearance. The reason for this is to be found in the relation between the function and its derivative. It is common in growth/decay situations for the rate of change of a variable at an instant to be proportional to the the value of the variable at that instant.

5.4.1 Example. Consider a population of animals or plants. Without predators to keep things in check, the colony will grow at a rate proportional to its current size, the rate of proportionality k being dependent on the birth and death rate. So we have

$$\frac{dP}{dt} = kP$$

where P is the size of the population and t is time. □

5.4.2 Example. (Newton’s Law of Cooling) A body heats up or cools down at a rate proportional to the temperature difference between it and its surroundings. So

$$\frac{d\theta}{dt} = k\theta$$

where θ is the temperature difference and k a constant. □

5.4.3 Example. Substances dissolve (e.g. sugar in water) at a rate proportional to the current mass of undissolved material. So we have the same sort of law as in the other two. □

5.4.4 Example. Radio-active decay follows the same pattern. The rate of decay of a mass of radio-active material is proportional to the mass present, that is

$$\frac{dm}{dt} = k\,m$$

where m is the mass of material and k is a constant. □

Suppose that x depends on time t and

$$\frac{dx}{dt} = kx \tag{5.17}$$

where k is a constant. Let $x = Ae^{kt}$ where A is a constant. Then, by the chain rule,

$$\frac{dx}{dt} = Ae^{kt}\,.\,k = kx.$$

so that $x = Ae^{kt}$ satisfies (5.17). Infact (as shown in the lectures) that if (5.17) holds then x *must* be given by $x = Ae^{kt}$ for some constant A.

5.4.5 Example. A steel ingot is cooling down to room temperature (which is presumed constant at 15°C). After 3 hours its temperature is 150°C and after 10 hours it is 30°C. What is the temperature as a function of time?

Solution. Let θ the temperature difference in °C. Let t be the time in hours from the instant the ingot begins to cool. (θ depends on t.) We know that there is a constant k such that

$$\frac{d\theta}{dt} = k\,\theta.$$

and so

$$\theta = Ae^{kt} \qquad \text{for some } A.$$

[We need to calculate both A and k from the information given.]

From the given data:

$$135 = Ae^{3k}, \qquad 15 = Ae^{10k}. \tag{5.18}$$

Dividing

$$9 = e^{-7k}$$

Therefore

$$\ln 9 = -7k \quad \text{and so} \quad k = -\frac{1}{7}\ln 9$$

This gives k, and putting the value into (5.18) gives $A = 135\,e^{\frac{3}{7}\ln(9)} = 135 \times 9^{\frac{3}{7}}$. □

5.5 Hyperbolic Functions

These are certain combinations of e^x and e^{-x} which occur often enough to justify giving them names of their own.

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}, \qquad \tanh x = \frac{\cosh x}{\sinh x}$$

A justification for giving them similar names to trigonometric functions is that they have similar properties. In particular similar formulae to the trig formulae hold. For instance

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$
$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

which can easily be verified by replacing the 'cosh's' and 'sinh's' by their expressions in terms of powers of e into the right hand sides of the formulae above and multiplying out, then checking what's left over equals the left hand sides. These formulae are very similar to the corresponding additive formulae for 'cos' and 'sin'. Putting $y = -x$ in the first formula gives

$$\cosh^2 x - \sinh^2 x = 1$$

again very similar to the corresponding trig formula but note the minus sign.

Formulae for derivatives are:

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x}$$

again similar to trig results (but no minus sign!!!)

5.6 Inverse Functions

The functions ‘arccos, arcsin, arctan, ln’ are examples of ‘inverse’ functions. Some of the results that hold for them apply more generally.

Let f be a real function.

If $f'(x) > 0$ whenever x is in some interval I then we know that f is strictly increasing on I.

If $f'(x) < 0$ whenever x is in some interval I then we know that f is strictly decreasing on I.

Figure 5.11: Graphs of a function increasing and another decreasing on an interval I.

In both cases it can be proved that the values $f(x)$ for x in I form an interval, J say, moreover for each number y in J there is a *unique* number x in I with $f(x) = y$ and so we can define a function g by

$$g(y) = x \qquad (y \text{ in } J).$$

The function g is called the inverse of f on the interval I. Clearly, from the definition of g we see that

$$g(f(x)) = x \qquad (x \text{ in } I)$$

and

$$f(g(y)) = y \qquad (y \text{ in } J).$$

By the chain rule we obtain

$$f'(g(y)).g'(y) = 1 \qquad (y \text{ in } J)$$

and hence

$$g'(y) = \frac{1}{f'(g(y))} \qquad (y \text{ in } J).$$

In Leibniz's notation, if $y = f(x)$ and $x = g(y)$ then we have

$$\frac{dx}{dy} = g'(y) = \frac{1}{f'(g(y))} = \frac{1}{\dfrac{dy}{dx}}$$

so we get the more memorable result

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \quad \text{where } \frac{dy}{dx} \text{ is evaluated at } x = g(y) \quad .$$

This result can be used to calculate some important derivatives. For instance we already know that

$$\frac{d}{dx}\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\,x^{\frac{1}{n}-1} \qquad (x > 0) \quad .$$

It was shown in the lectures that this result can be justified by taking $f(x) = x^n$ and considering the inverse to f on the interval $(0, \infty)$. ■