UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

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EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 3 (DIFFERENTIATION)

1. Find the derivatives of the functions defined by:

$$\begin{aligned} f(x) &= x^2 + 3x + 1 \\ g(t) &= t^3 + t^2 + 2t \\ h(x) &= x^3 - 3x^{-2} \\ j(\phi) &= 3.5\phi^4 - 4\phi^6 \\ x(y) &= 2y^{3.5} + 3.2y^{1.25} \\ p(t) &= \frac{2}{\sqrt{t}} - (\sqrt{t})^3 \\ q(t) &= t^n + t^{2n} - nt^{-n} \quad \text{(n a constant).} \end{aligned}$$

2. To show that the derivative of $\tan x$ with respect to x is $\sec^2 x$, the quotient rule together with knowledge of the derivatives of $\sin x$ and $\cos x$ were used. Use the same approach to show that

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x\,.$$

Show that

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x\,.$$

[Recall that $\sec x = \dfrac{1}{\cos x}$, $\operatorname{cosec} x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$.]

3. Use the product and quotient rules to differentiate the functions defined below:

$$f(x) = (x^2+1)(x^3+1) \qquad g(x) = (x^2+2x)(3x+1) \qquad h(x) = (\sin x + \cos x)(\sin x + 3\cos x)$$

$$F(t) = \frac{t+1}{t-1} \qquad G(t) = \frac{2t+3}{4t-1} \qquad H(x) = \frac{x^2+x+1}{x+2}$$

$$\kappa(u) = \frac{u^3+6}{u^2+2} \qquad \alpha(t) = \frac{t}{t^n+1} \qquad \beta(\theta) = \frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}$$

$$p(\lambda) = \frac{\sin\lambda\cos\lambda}{\sin\lambda + \cos\lambda} \qquad r(t) = \frac{\sin t + t}{\cos t - t} \qquad s(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$$

4. Let a, b and c be (real) functions. By using the product rule twice, show that:

$$(abc)' = a'bc + ab'c + abc'\,.$$

Evaluate $\dfrac{d}{dx}\big((1+x^2)(1-x)^2(1+x)^4\big)$.

5. Use the chain rule to differentiate the following with respect to x:

$$(3x+1)^5 \qquad \sin 5x \qquad \sqrt{1+x^2} \qquad \sin(1+x^2)$$

$$\tan(2x+1) \qquad \sin(\cos x) \qquad \frac{(3x+2)^5}{x} \qquad 2(\sin x+\cos x)^{\frac{1}{2}}$$

$$(x^3+1)^{\frac{1}{3}} \qquad (x^3+2x^2+1)^{-3} \qquad \tan(1+x^2) \qquad \tan(\tan x)$$

and the following with respect to t:

$$2(\sin t+\cos 2t)^2 \qquad \sin^2(4t^3+1) \qquad \sin\left(\frac{1+t}{1-t}\right).$$

6. Find $f'(x)$ in the cases:

$$f(x)=x+\frac{1}{1+x} \qquad f(x)=\tan^5 x \qquad f(x)=\frac{1+x+x^2}{(1-x)^2}$$

$$f(x)=\sin(\sin(\sin x)) \qquad f(x)=\operatorname{cosec}^2 x \qquad f(x)=\tan(\sqrt{x^2-1})$$

$$f(x)=x^{\frac{1}{n}} \qquad f(x)=\frac{x^{\frac{1}{3}}-x^{-\frac{1}{3}}}{x^{\frac{2}{3}}} \qquad f(x)=\frac{1}{\sqrt{1+x}+\sqrt{1-x}}$$

It is good algebra practice to try to simplify each result as much as possible - but don't try too hard, some of them don't really simplify at all!

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EG1504 Engineering Mathematics 1
Solutions to Exercises 3 (Differentiation)

1. The answers are:

$$
\begin{aligned}
f'(x) &= 2x+3\\
g'(t) &= 3t^2+2t+2\\
h'(x) &= 3x^2+6x^{-3}\\
j'(\phi) &= 14\phi^3-24\phi^5\\
x'(y) &= 7x^{2.5}+4y^{0.25}\\
p'(t) &= -t^{-\frac{3}{2}}-\frac{3}{2}\sqrt{t}\\
q'(t) &= nt^{n-1}+2nt^{2n-1}+n^2t^{-n-1}
\end{aligned}
$$

2. By the quotient rule.

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x\sin x-\cos x\cos x}{\sin^2 x} = -\operatorname{cosec}^2 x\,.$$

By the chain rule

$$
\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}\left((\cos x)^{-1}\right) = (-1)(\cos x)^{-2}(-\sin x) = \sec x\tan x\\
\frac{d}{dx}(\operatorname{cosec} x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = (-1)(\sin x)^{-2}\cos x = -\operatorname{cosec} x\cot x\,.
\end{aligned}
$$

3. The answers are:

$$
\begin{aligned}
f'(x) &= 2x(x^3+1)+(x^2+1)3x^2 = 5x^4+3x^2+2x\\
g'(x) &= (2x+2)(3x+1)+(x^2+2x)3 = 9x^2+14x+2\\
h'(x) &= (\cos x-\sin x)(\sin x+3\cos x)+(\sin x+\cos x)(\cos x-3\sin x)\\
&= -2\sin 2x+4\cos 2x\\
F'(t) &= \frac{1(t-1)-(t+1)1}{(t-1)^2} = \frac{-2}{(t-1)^2}\\
G'(t) &= \frac{2(4t-1)-4(2t+3)}{(4t-1)^2} = \frac{-14}{(4t-1)^2}\\
H'(x) &= \frac{(2x+1)(x+2)-(x^2+x+1)1}{(x+2)^2} = \frac{x^2+4x+1}{(x+2)^2}\\
\kappa'(u) &= \frac{3u^2(u^2+2)-2u(u^3+6)}{(u^2+2)^2} = \frac{u^4+6u^2-12u}{(u^2+2)^2}\\
\alpha'(t) &= \frac{1.(t^n+1)-nt^{n-1}t}{(t^n+1)^2} = \frac{(1-n)t^n+1}{(t^n+1)^2}\\
\beta'(\theta) &= \frac{(\cos\theta-\sin\theta)(\sin\theta-\cos\theta)-(\sin\theta+\cos\theta)(\cos\theta+\sin\theta)}{(\sin\theta-\cos\theta)^2} = \frac{-2}{(\sin\theta-\cos\theta)^2}
\end{aligned}
$$

$$p'(\lambda) = \frac{(\sin\lambda + \cos\lambda)(\cos^2\lambda - \sin^2\lambda) - \sin\lambda\cos\lambda(\cos\lambda - \sin\lambda)}{(\sin\lambda + \cos\lambda)^2} = \frac{\cos^3\lambda - \sin^3\lambda}{1 + \sin 2\lambda}$$
$$r'(t) = \frac{(\cos t - t)(\cos t + 1) - (\sin t + t)(-\sin t - 1)}{(\cos t - t)^2} = \frac{1 + t\sin t - t\cos t + \sin t + \cos t}{(\cos t - t)^2}$$
$$s'(x) = \left(\frac{\sqrt{x}+1}{2\sqrt{x}} - \frac{\sqrt{x}-1}{2\sqrt{x}}\right)(\sqrt{x}+1)^{-2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,.$$

4. Think of this in two stages. First we differentiate $a(bc)$, as the product of a and bc.

$$(a(bc))' = a'(bc) + a(bc)'$$

Now use the product rule again for $(bc)'$:

$$(abc)' = a'bc + a(b'c + bc') = a'bc + ab'c + abc'.$$

In Leibniz's notation:

$$\frac{d}{dx}\big(a(x)b(x)c(x)\big) = \frac{d}{dx}(a(x))\,b(x)c(x) + a(x)\,\frac{d}{dx}(b(x))\,c(x) + a(x)b(x)\,\frac{d}{dx}(c(x))\,.$$

[The formula is very easy to remember. Make sure that you are clear about the approach, which is to take things in stages. And it doesn't just apply here: whenever you have a complicated maths problem to tackle it is a good idea to try and tame it by breaking it down into manageable steps.]

Put $a(x) = (1+x^2)$, $b(x) = (1-x)^2$ and $c(x) = (1+x)^4$. Then $a'(x) = 2x$, $b'(x) = -2(1-x)$ and $c'(x) = 4(1+x)^3$. We therefore have

$$\begin{aligned}
\frac{d}{dx}\big(a(x)b(x)c(x)\big) &= 2x(1-x)^2(1+x)^4 + (1+x^2)(-2)(1-x)(1+x)^4 + (1+x^2)(1-x)^2 4(1+x)^3\\
&= 2(1-x)(1+x)^3\left[x(1-x)(1+x) - (1+x^2)(1+x) + 2(1+x^2)(1-x)\right]\\
&= 2(1-x)(1+x)^3\left[x(1-x^2) - (1+x+x^2+x^3) + 2(1-x+x^2-x^3)\right]\\
&= 2(1-x)(1+x)^3\left[x - x^3 - 1 - x - x^2 - x^3 + 2 - 2x + 2x^2 - 2x^3\right]\\
&= 2(1-x)(1+x)^3(1 - 2x + x^2 - 4x^3)\,.
\end{aligned}$$

5. Answers:

$$5(3x+1)^4.3 = 15(3x+1)^4 \qquad 5\cos 5x \qquad \frac{x}{\sqrt{1+x^2}} \qquad 2x\cos(1+x^2)$$

$$2\sec^2(2x{+}1) \qquad (-\sin x)\cos(\cos x) \qquad \frac{15(3x+2)^4.x - 1.(3x+2)^5}{x^2} \qquad 2(\frac{1}{2})(\sin x{+}\cos x)^{-1/2}(\cos x{-}\sin x)$$

$$\frac{x^2}{(1+x^3)^{\frac{2}{3}}} \qquad -3(x^3+2x^2+1)^{-4}(3x^2+4x) \qquad 2x\sec^2(1+x^2) \qquad \sec^2(\tan x)\sec^2 x\,.$$

$$4(\sin t + \cos 2t)(\cos t - 2\sin 2t) \qquad 24t^2\sin(4t^3+1)\cos(4t^3+1) \qquad \frac{2}{(1-t)^2}\cos\left(\frac{1+t}{1-t}\right)\,.$$

The penultimate one is possibly the most complicated. To tackle it break it up into a *chain* as follows:

$$\begin{aligned}
\text{Put}\quad & y = \sin^2(4t^3+1) \quad \text{for convenience, then}\\
& y = u^2 \quad \text{where } u = \sin(4t^3+1)\\
& u = \sin v \quad \text{where} \quad v = 4t^3+1.
\end{aligned}$$

Now the chain rule says that:

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \\ &= (2u)(\cos v)(12t^2) \\ &= (2\sin(4t^3+1))(\cos(4t^3+1))(12t^2)\,.\end{aligned}$$

6. The results are, in order:

$$f'(x) = 1 - \frac{1}{(1+x)^2} \qquad f'(x) = 5\tan^4 x \sec^2 x \qquad f'(x) = \frac{3(1+x)}{(1-x)^3}$$

$$f'(x) = \cos x \cos(\sin x)\cos(\sin\sin x) \qquad f'(x) = -\frac{2\cos x}{\sin^3 x}$$

$$f'(x) = \frac{x\sec^2\sqrt{x^2-1}}{\sqrt{x^2-1}} \qquad f'(x) = \frac{1}{n}x^{\frac{1}{n}-1} \qquad f'(x) = -\frac{1}{3}x^{-\frac{4}{3}} + x^{-2}$$

$$f'(x) = -\left(\sqrt{1+x}+\sqrt{1-x}\right)^{-2}\left(\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}}\right)$$