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UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 7 (DIFFERENTIATION)

1. Show that the maximum and minimum values achieved by

$$f(x) = x^2 - 3x + 2$$

on the interval $0 \leqslant x \leqslant 4$ are 6 and $-1/4$.

2. Show that the maximum and minimum values achieved by

$$f(x) = x^3 - 6x^2 + 9x + 6$$

on the interval $-1 \leqslant x \leqslant 4$ are 10 and -10.

3. Let $f(x) = (x-1)^2(x+2)^3$. Find the three critical points of f. Show that one is a local maximsing point, one a local minimising point and one is a point of inflexion.

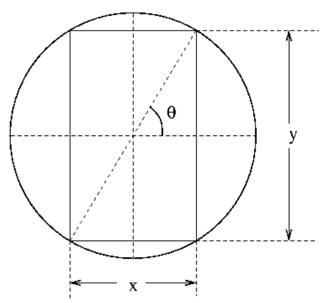
4. Let $y = x(x^2-5)^2$. Find the maximum and minimum values that y can take on the interval $-2 \leqslant x \leqslant 3$.

5. You are given 100 metres of fencing, with which you are to make a rectangular enclosure. What should the dimensions be if the area enclosed is to be as large as possible?

[Let the enclosure have length x and width y. Write down the length of the perimeter in terms of x and y. Equate this to 100 and thereby get a formula for y in terms of x. Next write down the area, A, of the enclosure in terms of x and y. Substitute for y to get a formula for A in terms of x. Then differentiate and find the maximum value that A can take. And the endpoints? Well, clearly x can't be less than 0 or bigger than 50.]

6. The stiffness, S, of a rectangular beam is proportional to xy^3, where x is the width and y is the depth of the beam.

Find the dimensions of the stiffest beam that can be cut from a cylindrical log of diameter d.
[Work in terms of the angle θ. $S = \lambda xy^3$ for some constant λ. Get x and y in terms of θ. Substitute to get S as a function of θ, and then differentiate with respect to θ.]



7. What is the maximum value achieved by $f(x) = xe^{-x}$ for $x \geqslant 0$?

8. What is the minimum value achieved by $f(x) = \dfrac{2}{x} + \ln x$ for $x > 0$?

9. The speed v of a wave in deep water is given by

$$v = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}}$$

where T is the surface tension, ρ the water density, λ the wavelength of the wave and g is acceleration due to gravity.

Show that as λ varies the least speed is given by $\left(\dfrac{4gT}{\rho}\right)^{\frac{1}{4}}$.

[The variables in this question are v and λ; everything else is constant. Differentiate v with respect to λ.]

10. An electrical circuit, with inductance L, capacitance C and resistance R in series, has a current with amplitude

$$I = V\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-\frac{1}{2}},$$

where V is the voltage and $\dfrac{\omega}{2\pi}$ is the frequency of the voltage. Show that I is at a maximum at a frequency $\dfrac{1}{2\pi\sqrt{LC}}$.

[This time the only variables are I and ω.]

11. A snail, who can travel at 2 metres per hour (running), lives at a point S which is 11 metres from a straight highway. The nearest point on the highway is O. He wishes to travel to the point D, which is 100 metres down the highway from O. A friendly tortoise has agreed to pick him up at some point P on the highway. The tortoise has a cruising speed of 20 metres per hour. Where should P be if the total travelling time is to be minimised?

[Let x be the distance from O to P. Begin by working out the travelling time T in terms of the variable x.]

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SOLUTIONS TO EXERCISES 7 (DIFFERENTIATION)

1. Differentiating the function we get $f'(x) = 2x - 3$. This will be zero when $x = 3/2$. (The second derivative of $f(x)$ is the constant 2. So $x = 3/2$ is a local minimsing point). Note that $f(2/3) = -1/4$. Now consider the two end points. $f(0) = 2$ and $f(4) = 6$. Putting all these facts together we see that the maximum value of the function on this range is 6, achieved at $x = 4$, and the minimum value is $-1/4$, achieved at $x = 3/2$.

2. This time $f'(x) = 3x^2 - 12x + 9$. Solving the quadratic $f(x) = 0$ we get $x = 1$ and $x = 3$. (The second derivative is $f''(x) = 6x - 12$. So $f''(1) = -6$ and $f''(3) = 6$ so we have a local minimum at $x = 3$ and a local maximum turning point at $x = 1$.) Note that $f(1) = 10$ and $f(3) = 6$ The values at the end points are $f(-1) = -10$ and $f(4) = 10$. So the extreme values in the interval are 10 and -10. The maximum is achieved at 1 (critical point) and 4 (endpoint). The minimum is achieved at -1 (endpoint).

3. Differentiate and get

$$f'(x) = 2(x-1)(x+2)^3 + 3(x-1)^2(x+2)^2$$

At this point your school training probably tells you to multiply out. This is not a wrong thing to do, but it is a dumb thing to do. You are going to be looking for the solutions to $f'(x) = 0$. These will be much easier to find if you have $f'(x)$ in factorised form. Multiplying out will not only take a fair amount of time and give you lots of scope for arithmetical error; it will leave you with the problem of factorising a quartic, and that is hard. The expression you have for $f'(x)$ is already in partially factorised form. So make use of this fact. There are common factors of $(x-1)$ and $(x+2)^2$ staring at you. Don't disregard them. Instead take them out as common factors. This gives

$$f'(x) = (x-1)(x+2)^2[2(x+2) + 3(x-1)] = (x-1)(x+2)^2(5x+1)$$

From which we can see that $f'(x) = 0$ when $x = 1, -2, -1/5$, these are the critical points.

Looking at the signs of the three factors (note that $(x+2)$ is squared) we get:

(a) if $x < -2$ then $f'(x) > 0$;

(b) if $-2 < x < -1/5$ then $f'(x) > 0$;

(c) if $-1/5 < x < 1$ then $f'(x) < 0$;

(d) if $x > 1$ then $f'(x) > 0$.

So we see that there is a local maximum at $x = -1/5$, a local minimum at $x = 1$ and a point of inflexion at $x = -2$.

4. Differentiate y by treating it as the product of x and $(x^2-5)^2$ and by using the chain rule when you come to differentiate $(x^2-5)^2$. You should get

$$\frac{dy}{dx} = 1.(x^2-5)^2 + x.2(x^2-5).2x = (x^2-5)^2 + 4x^2(x^2-5) = (x^2-5)[(x^2-5) + 4x^2]$$

so

$$\frac{dy}{dx} = 5(x^2-5)(x^2-1)$$

Thus $y' = 0$ when $x = \pm\sqrt{5}$ and when $x = \pm 1$. We therefore have what seems to be four critical points. However $x = -\sqrt{5}$ is not in the interval $-2 \le x \le 3$, and so can be ignored. Now calculate y for each of the three critical points that are within range and also at the two endpoints. You get

$$y|_{x=-2} = -2, \quad y|_{x=-1} = -16, \quad y|_{x=1} = 16, \quad y|_{x=\sqrt{5}} = 0, \quad y|_{x=3} = 48$$

So the maximum is 48 (achieved at the endpoint $x = 3$) and the minimum is -16 (achieved at the critical value $x = -1$).

5. The perimeter has length $2x + 2y$. This is to equal 100, and so $y = 50 - x$. The standard area formula for a rectangle tells us that $A = xy$, and substituting turns this into $A = 50x - x^2$. The derivative of A with respect to x is $50 - 2x$, and this is zero when $x = 25$. So there is just the one critical point. Calculating A for $x = 0$, $x = 25$, and $x = 50$, we find that the maximum occurs at $x = 25$. When $x = 25$, y is also 25. So the maximum area is achieved by a square enclosure of side 25 metres.

6. The question doesn't actually say that you cut right to the edge of the log, but common sense makes it clear that you should, and the diagram in the question did this step for you. We have $x = d\cos\theta$ and $y = d\sin\theta$. So

$$S = \lambda d^4 \cos\theta \sin^3\theta \qquad (0 \leqslant \theta \leqslant \pi/2)$$

$$\begin{aligned}\frac{dS}{d\theta} &= \lambda d^4(-\sin^4\theta + 3\cos^2\theta\sin^2\theta) \\ &= \lambda d^4 \sin^2\theta(3\cos^2\theta - \sin^2\theta) \\ &= \lambda d^4 \sin^2\theta(3 - 4\sin^2\theta)\end{aligned}$$

Now $\theta = 0$ or $\theta = \pi/2$ are certainly not the required answer. It is easy to see that the maximum comes when $3 = 4\sin^2\theta$. So

$$\sin\theta = \frac{\sqrt{3}}{2} \qquad \cos\theta = \frac{1}{2}$$

Hence

$$x = \frac{d}{2}, \qquad y = \frac{d\sqrt{3}}{2}.$$

7. The derivative is $f'(x) = (1 - x)e^{-x}$ and the second derivative is $f''(x) = (x - 2)e^{-x}$.

e^{-x} is always positive. So $f(0) = 0$ and $f(x) > 0$ for $x > 0$. We have $f'(x) > 0$ for $0 < x < 1$ and we have $f'(x) < 0$ for $x > 1$.

Thus $f(x)$ is increasing for $0 < x < 1$ and decreasing for $x > 1$.

So the maximum value is e^{-1} and it comes when $x = 1$.

8. The derivative is

$$\begin{aligned}f'(x) &= \frac{-2}{x^2} + \frac{1}{x} \\ &= \frac{x-2}{x^2}\end{aligned}$$

This is zero when $x = 2$. When $0 < x < 2$ the derivative is negative and when $x > 2$ it is positive. So the slope goes from negative to positive as x increases past 2. This means that we have a local minimum at $x = 2$, and the fact that it is the only stationary point shows that it is a global minimising point.

[Since $f(x)$ is strictly decreasing for $0 < x < 2$ we have $f(2) < f(x)$ for $0 < x < 2$.
Since $f(x)$ is strictly increasing for $x > 2$ we have $f(2) < f(x)$ for $x > 2$.
Thus $f(2) \leqslant f(x)$ for all $x > 0$.]

The minimum value is $1 + \ln 2$.

9. Don't be put off by all the symbols! The only variable is λ , we are just thinking of v as a function of λ. Differentiating we get

$$\frac{dv}{d\lambda} = \frac{1}{2v}\left(\frac{g}{2\pi} - \frac{2\pi T}{\rho\lambda^2}\right) .$$

You can now easily check that we get our minimum when

$$\frac{g}{2\pi} = \frac{2\pi T}{\rho\lambda^2} \qquad \text{i.e.} \qquad \lambda = 2\pi\sqrt{\frac{T}{\rho g}} .$$

Plug this back into the formula for v to get the answer.

10. Differentiate I with respect to ω. This calls for careful use of the chain rule. $I = Vt^{-\frac{1}{2}}$ where $t = R^2 + u^2$ and $u = \omega L - \dfrac{1}{\omega C}$. So

$$\frac{dI}{d\omega} = \frac{dI}{dt}.\frac{dt}{du}.\frac{du}{d\omega} = -\frac{1}{2}V\left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-\frac{3}{2}} .2\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right) .$$

Since L and C are both positive, the only way this can be zero is for $\omega L - \dfrac{1}{\omega C}$ to be zero. When this happens $\omega = \dfrac{1}{\sqrt{LC}}$, and substituting this into the expression for the frequency gives the answer.

11. $SP = \sqrt{x^2 + 121}$, and so the time to reach the highway is this distance divided by 2. The distance from P to D is $100 - x$, and so the time for this section is $100 - x$ divided by 20. Therefore T is the sum of these two times. So

$$T = \frac{\sqrt{x^2+121}}{2} + \frac{100-x}{20} \qquad \text{and} \qquad \frac{dT}{dx} = \frac{x}{2\sqrt{x^2+121}} - \frac{1}{20} .$$

Equating the derivative to zero, and rearranging we get $10x = \sqrt{x^2 + 121}$. Square both sides to get $100x^2 = x^2 + 121$, from which we have $99x^2 = 121$ and hence $9x^2 = 11$. So $x = \sqrt{11}/3$. The extreme possibilities for x are 0 and 100. Calculate T for each of these values of x, and you will find that the smallest value of T is given by $x = \sqrt{11}/3$.