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UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 12 (INTEGRATION)

1. For each of the following, compute the volume of revolution obtained when the region between the given curve and the x-axis is rotated about the x-axis.

(a) $y = x^3$ from $x = 0$ to $x = 2$

(b) $x^2 + y^2 = 1$

(c) $x^2 - y^2 = 16$ from $x = 4$ to $x = 8$

[Only tackle the next one if you have found the other three easy and fancy a challenge.]

(d) $x = t - \sin t$, $y = 1 - \cos t$ from $t = 0$ to $t = 2\pi$.

[The first step is to write $\displaystyle\int y^2\,dx$ as $\displaystyle\int y^2 \frac{dx}{dt}\,dt$. This will leave you with an integral all in terms of t. Use the binomial theorem to expand the integrand you get. This should leave you facing a combination involving $\cos t$, $\cos^2 t$ and $\cos^3 t$. Only the third of these is new: deal with it by writing it as $\cos t(1 - \sin^2 t)$ and making the substitution $u = \sin t$, or by using integration by parts.]

2. Find the lengths of each of the following curves:

(a) $y = x\sqrt{x}$ from $x = 0$ to $x = 5$.

(b) $y = \cosh x$ from $x = -1$ to $x = 1$.

[The next is not as bad as it looks.]

(c) $x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t$ between $t = 0$ and $t = \pi/3$.

[The next is hard; same instructions as for 1(d).]

(d) $x = 2\cos t + \cos 2t$, $y = 2\sin t + \sin 2t$ between $t = 0$ and $t = \pi$.

[In this one you should reach a point where you have a $(1 + \cos t)$ term sitting inside a square root. When you do use the trig formula $\cos 2\theta = 2\cos^2\theta - 1$ with $\theta = \frac{1}{2}t$ which will enable you to get rid of the square root. As long as the square root is there you will probably not be able to handle the integral; once it is gone you should be alright.]

3. For each of the following sections of graph calculate the area of the surface created when section is rotated about the $x-$axis:

(a) $y = x + 2$ from $x = 0$ to $x = 4$.

(b) $y^2 = 16x$ from $x = 0$ to $x = 5$.

(c) $y = x^3$ from $x = 0$ to $x = 2$.

4. For each of the following regions calculate the area and then the coordinates of the centroid (center of gravity):

(a) The region in the first quadrant bounded by the x-axis, the y-axis and the circle $x^2 + y^2 = 4$.

(b) The region in the first quadrant bounded by the x-axis, the y-axis and the ellipse $4x^2 + y^2 = 4$.

(c) The region bounded by the y-axis, the line $y = 4$ and the curve $y = x^2$.

(d) The region which lies in the first quadrant and is bounded by the line $y = x$ and the curve $y = x^3$.

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SOLUTIONS TO EXERCISES 12 (INTEGRATION)

1. The second of these requires a little thought and the fourth that you don't panic, but the other two are very easy.

(a) $y^2 = x^6$. So all we have to do is integrate x^6, plug in the two limits and multiply by π. Answer: $128\pi/7$.

(b) The small snag here is that I haven't told you the limits. However, you should have recognised the equation as that of the circle which has radius 1 and centre the origin. This meets the x-axis at $x = -1$ and $x = 1$, and so these have to be the limits. The rest is easy. $y^2 = 1 - x^2$, and so the volume is $\int_{-1}^{1}(1 - x^2)\,dx$, which comes out as $4\pi/3$. (The volume swept out is, of course, a sphere of radius 1. What we have just done is calculate the volume of such a sphere. Had the radius been r rather than 1, the volume would have been $4\pi r^3/3$. Verify this.)

(c) $y^2 = x^2 - 16$, a function you should have no difficulty integrating. Final answer: $256\pi/3$.

(d) $y^2 = (1 - \cos t)^2$, and so this integral will be much more easily handled if we let t be the variable of integration rather than x.

$$\int_{t=0}^{t=2\pi} y^2\,dx \;=\; \int_{t=0}^{t=2\pi} y^2\,\frac{dx}{dt}\,dt \;=\; \int_0^{2\pi}(1-\cos t)^2(1-\cos t)\,dt$$

$(1 - \cos t)^3 = 1 - 3\cos t + 3\cos^2 t - \cos^3 t$. Writing $\cos^3 t$ as $\cos t(1 - \sin^2 t)$ and using the fact that $\cos 2t = 2\cos^2 t - 1$ turns this into $1 - 4\cos t + \frac{3}{2}(\cos(2t) + 1) + \cos t \sin^2 t$. Integrating this lot gives $t - 4\sin t + \frac{3}{4}\sin(2t) + 3t/2 + \frac{1}{3}\sin^3 t$, most of which vanishes when you put $t = 0$ and $t = 2\pi$. Final answer $5\pi^2$ (5π from the integral as set down, and π from the fact that the volume is the integral of πy^2 rather than just y^2).

2. These start very easy but don't end that way. If you did all of them you can feel quite pleased with yourself.

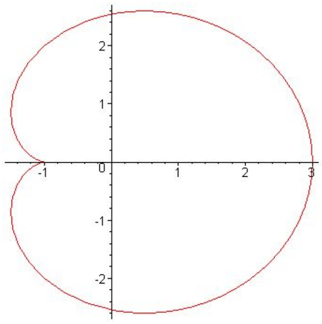
(a) dy/dx is $\frac{3}{2}\sqrt{x}$, and so the integral we have to deal with is $\int_0^5 \sqrt{1 + 9x/4}\,dx$.

$$\int_0^5 \sqrt{1+\frac{9x}{4}} \;=\; \frac{8}{27}\left[\left(1+\frac{9x}{4}\right)^{\frac{3}{2}}\right]_0^5 \;=\; \frac{8}{27}\left[\left(\frac{7}{2}\right)^3 - 1\right] \;=\; \frac{335}{27}$$

(b) The derivative of $\cosh x$ is $\sinh x$, $1 + \sinh^2 x$ is $\cosh^2 x$, and so, after taking the square root, we are left with the simple task of integrating $\cosh x$. The integral is $\sinh x$, and after putting the limits we arrive at $e - e^{-1}$

(c) This looks a lot fiercer than it is. Clearly we are better to work in terms of the parameter t. So calculate the derivative of both x and y with respect to t, square both, add, take the square root of the sum and then integrate. The derivative of x with respect to t simplifies down to $\sec t - \cos t$, which squares to $\sec^2 t - 2 + \cos^2 t$. The square of the derivative of y with respect to t is just $\sin^2 t$. So inside the square root we get $\sec^2 t - 1$, and this is $\tan^2 t$. So the integral we have to deal with is $\int \tan t\,dt$ with limits $t = 0$ and $t = \pi/3$. The integral of $\tan t$ is $-\ln(\cos t)$ or, if you prefer, $\ln(\sec t)$. Final answer $\ln 2$.
(This is an odd-looking equation for a curve, but it is a curve that occurs naturally. If you have an object on the end of a rope and walk in a straight line pulling it, and if the object doesn't start immediately behind you but off to one side, then this is the slightly curved path that it will follow.)

(d) This is the equation of the heart-shaped curve known as the cardioid. Again we do the entire calculation in terms of t. Differentiating both x and y with respect to t, squaring and adding gives us $(2\sin t + 2\sin 2t)^2 + (2\cos t + 2\cos 2t)^2$ which equals $8 + 8(\cos 2t\cos t + \sin 2t\sin t)$. Now use your trig formulas. $\cos 2t\cos t + \sin 2t\sin t$ equals $\cos(2t - t)$, which is just $\cos t$. So inside our square root we have $8(1 + \cos t)$. Another trig formula, this time the one for $\cos 2\theta$ with $\theta = t/2$ turns $(1 + \cos t)$ into $2\cos^2(t/2)$. So our integrand is the (non-negative) square root of $16\cos^2(t/2)$, and for the given range this is $4\cos(t/2)$ (since $\cos(t/2) \geqslant 0$ for $0 \leqslant t \leqslant \pi$). The hard work is now over. The indefinite integral is $8\sin(t/2)$ and putting in the limits gives us the answer 8. (The length of the full curve is double this: I only asked for the section above the x-axis.) For your interest, the cardoid is drawn below:



3. **(a)** $(1+(dy/dx)^2$ is 2, and so the integral we have to evaluate is the integral from 0 to 4 of $2\pi(x+2)\sqrt{2}$. The answer is $32\pi\sqrt{2}$.

(b) $y = 4\sqrt{x}$ and so $dy/dx = 2/\sqrt{x}$. Consequently the integrand we have to tackle is $2\pi 4\sqrt{x}\sqrt{1 + 4x^{-1}}$, which simplifies to $8\pi\sqrt{x+4}$. An indefinite integral for this is $16\pi/3\,(x + 4)^{3/2}$, and when we put in the limits we get $304\pi/3$.

(c) The integrand turns out to be $2\pi x^3\sqrt{1 + 9x^4}$, which looks formidable until you realise that the substitution $u = 1 + 9x^4$ renders it harmless. $''du = 36x^3\,dx''$, and so we find ourselves having to integrate nothing more complicated than $\sqrt{u}$. The final answer for the indefinite integral is $(2\pi/54)\,(1+ 9x^4)^{3/2}$, and putting in the limits produces $(\pi/27)\,(145^{3/2} - 1)$.

4. **(a)** This is a circle of radius 2, the area of the full circle is 4π, and so the area of the quadrant is π.

$$M_x = \int_0^2 \frac{1}{2}y^2\,dx = \int_0^2 \frac{1}{2}(4 - x^2)\,dx = \left[2x - \frac{x^3}{6}\right] = 4 - \frac{8}{6} = \frac{8}{3}$$

So $\bar{y} = 8/(3\pi)$. The whole figure is clearly symmetrical in x and y, and so this is $\bar{x}$ also.

(b) To find the area we integrate $\sqrt{4 - 4x^2}$ from $x = 0$ to $x = 1$. Use the substitution $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$ and the indefinite integral turns into $\int 2\cos^2\theta\,d\theta$. This is one you have met before. Use the fact that $2\cos^2\theta = \cos 2\theta + 1$, and the integral comes out as $\frac{1}{2}\sin 2\theta + \theta$. Now consider the limits: When $x = 0$ $\theta = 0$, and when $x = 1$ $\theta = \pi/2$. So the area is $\pi/2$. For M_x we have to evaluate the integral of $y^2/2$ between $x = 0$ and $x = 1$. This is easy and gives an answer 4/3. The integral for M_y is not much harder: to integrate $2x\sqrt{1 - x^2}$ you just put $u = 1 - x^2$, and you should get an answer of 2/3. So the centroid has coordinates $(4/(3\pi), 8/(3\pi))$.

(c) The area is the difference between the area of the region under the line $y = 4$ from $x = 0$ to $x = 2$ and the area under the curve for the same values of x. This is $8 - 8/3$, which is 16/3. The integrals for M_x and M_y are completely straightforward and produce answers of 64/5 and 4, respectively. So the centroid is $(3/4, 12/5)$.

(d) The area is the integral between $x = 0$ and $x = 1$ of $x - x^3$. This is 1/4. For the moments one approach is to take as the basic rectangular element the vertical one with width δx and height running from $y = x^3$ at the bottom to $y = x$ at the top. This basic rectangle has as its centre of gravity the point whose x-coordinate is x and whose y-coordinate is $(x + x^3)/2$. So for M_y we have to integrate $x(x - x^3)$ and for M_x we have to integrate $(x - x^3)(x + x^3)/2$. The respective answers are 2/15 and 2/21, which gives as centroid the point $(8/15, 8/21)$.