UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

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EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 6 (DIFFERENTIATION)

1. When a circle of radius r rolls along the x-axis, a point on its circumference traces out a curve which is given parametrically by

$$x = r(t - \sin t), \qquad y = r(1 - \cos t)\ .$$

Find dy/dx as a function of t. What is the equation of the tangent line to the curve at the point $t = \pi/3$?

Just using common sense and the description of the curve in terms of the rolling circle, sketch what you think the curve will look like. Then look at the parametric equations for the curve and answer the following questions:

(i) For what values of t is $y = 0$?

(ii) How far apart are these points where the curve touches the x-axis?

(iii) What is the greatest value that y can take?

(iv) Why are the answers to (ii) and (iii) the ones that common sense would have you expect?

2. Use your calculator to find the values of β (if they exist) for which $0 \leqslant \beta \leqslant 2\pi$ and $\cos\beta = -0.2$.

3. Find the derivatives of the following functions:

$$f(x) = \arcsin(x^2 - 2), \qquad g(x) = \arccos(1 + \sin x), \qquad h(x) = (\arcsin x + \arccos x)^5,$$

$$k(x) = (\arctan x)^{-3}, \qquad l(x) = \tan^{-1}\sqrt{x^2 - 1}\ .$$

For what values of x, if any, are the above functions defined? (For instance it does not make sense to talk about $\arcsin(2)$ or $\arccos(-3)$.)

4. Simplify the following expressions:

$$\exp(\ln x + \ln y), \qquad \ln(e^x e^y)\ .$$

5. Differentiate the following with respect to x:

$$\ln(1 + x), \qquad \exp(1 + \ln x), \qquad \ln(1 + e^x), \qquad \exp(5x + 2), \qquad \frac{\sinh x}{\cosh x} \quad (= \tanh x),$$

$$\exp(1 + \sin x), \qquad \ln\left(\left(\frac{1-x}{1+x}\right)^{\frac{3}{2}}\right) i, \qquad e^x \sin 2x, \qquad \ln(\sin x), \qquad \ln(\ln x), \qquad \exp(e^x)\ .$$

6. Show that $p = Ae^{2t} + Be^{3t}$ satisfies the equation

$$\ddot{p} - 5\dot{p} + 6p = 0$$

for any constants A and B. (Dots and double dots denote first and second derivatives with respect to t.)

7. By the chain rule,

$$\frac{d}{dx}(\ln(f(x)) = \frac{1}{f(x)} \cdot \frac{d}{dx}(f(x)) \ .$$

Find the derivatives of f by first taking logs (assuming that this is allowed) in the cases:

$$\text{(i)} \quad f(x) = \left(\frac{1+x^2}{1-x^2}\right)^{\frac{2}{3}}, \qquad \text{(ii)} \quad f(x) = \frac{(1+x)^{\frac{5}{7}}(1+x^2)^{\frac{3}{5}}}{(1-x)^{\frac{3}{2}}(1-x^2)} \ .$$

8. Let a be a positive constant. By taking logs of both sides of the equation and then differentiating, show that the derivative of

$$f(x) = a^x \quad \text{is} \quad f'(x) = \ln(a)a^x \ .$$

9. Show that if $p = Ae^{-kx}\sin(\omega x + \phi)$ (where A, k, ω, ϕ are constants) then

$$\frac{d^2p}{dx^2} + 2k\frac{dp}{dx} + (\omega^2 + k^2)p = 0$$

10. [Put here to give those who have sailed through the rest of the sheet something to get their teeth into.] A heavy weight is attached to the end of a rope 32 metres long. The rope passes over a pulley 15 metres above ground level, and the other end is attached (at ground level) to a tractor. Initially the rope is taut and the weight is on the ground. The tractor then moves away, as shown in Figure 1, at a rate of 4 metres per second. How quickly is the weight rising 3 seconds after the tractor starts to move? How quickly is the angle θ changing at this same instant?

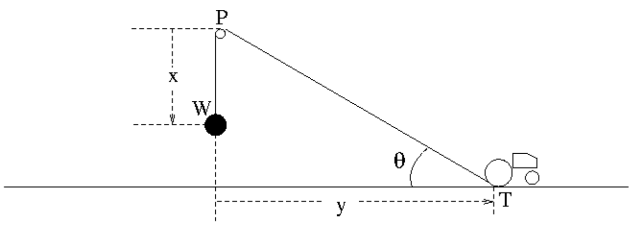

Figure 1:

In the diagrams W is the weight, P the pulley and T the tractor. The pulley is a simple one, and so you can assume that WPT is a triangle and that $|WP|$ plus $|PT|$ between them account for the full length of the rope.

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EG1504 Engineering Mathematics 1
Solutions to Exercises 6 (Differentiation)

1. Differentiate y and x with respect to t and then take the ratio.

$$\frac{dx}{dt} = r(1-\cos t) \qquad \frac{dy}{dt} = r\sin t\ .$$

Therefore

$$\frac{dy}{dx} = \frac{\sin t}{1-\cos t}\ .$$

When $t = \pi/3$:

$$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \qquad y = r\left(1-\frac{1}{2}\right), \qquad \frac{dy}{dx} = \sqrt{3}\ .$$

So the tangent has equation

$$y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)\ .$$

(i) $y = 0$ when $\cos t = 1$, and $\cos t = 1$ whenever t is of the form $2k\pi$ for some integer k. (An *integer* is a 'whole number').

(ii) For these values of t, $\sin t = 0$ and so $x = 2k\pi r$. Therefore the curve touches the x-axis at points that are $2\pi r$ apart. This makes sense because the circle does a full rotation in between each of these points, and $2\pi r$ is the distance round the circle's edge.

(iii) Recall that $\cos t$ lies between 1 and -1. So the largest that $1-\cos t$ could be is $1-(-1) = 2$. Therefore the greatest that y can be is $2r$. This happens when the point whose path is being followed is directly above the point of contact between circle and ground. The point is then the diameter of the circle away from the ground, and the diameter is $2r$.

(iv) See answers to (ii) and (iii).

2. Answers: $\arccos(-0.2) = 1.7722$ so $\beta = 1.7722$ or $2\pi - 1.7722 = 4.511$ (to 4 decmal places).

3. Answers:

$$f'(x) = \frac{2x}{\sqrt{1-(x^2-2)^2}} = \frac{2x}{\sqrt{4x^2-x^4-3}}$$

$$g'(x) = \frac{-\cos x}{\sqrt{1-(1+\sin x)^2}}$$

$$h'(x) = 0 \quad \text{because the function is constant - why?}$$

$$k'(x) = -3(\arctan x)^{-4}\frac{1}{1+x^2}$$

$$l'(x) = \frac{1}{x}\frac{1}{\sqrt{x^2-1}}$$

$f(x)$ is defined for those values of x that make x^2-2 lie between -1 and 1, i.e. $1 \leqslant x^2 \leqslant 3$.

$g(x)$ is defined for those values of x that give $-1 \leqslant 1+\sin x \leqslant 1$, i.e. $-2 \leqslant \sin x \leqslant 0$. This means that we must have $\sin x \leqslant 0$.

$h(x)$ is defined for $-1 \leqslant x \leqslant 1$.

$k(x)$ is defined if $x \neq 0$. ($\arctan(x) = 0$ if and only if $x = 0$.)

$l(x)$ is defined if $x^2 \geqslant 1$. ($\tan^{-1}$ is often used for arctan.)

4. These are simple and important:

$$e^{\ln x+\ln y} = e^{\ln x}e^{\ln y} = xy, \qquad \ln(e^x e^y) = \ln(e^{x+y}) = x+y\ .$$

5. Answers in order:

$$\frac{1}{1+x}, \qquad \frac{1}{x}e^{1+\ln x}, \qquad \frac{e^x}{1+e^x}, \qquad 5e^{5x+2}$$
$$\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}, \qquad \cos(x)e^{1+\sin x}$$
$$\ln\left(\left(\frac{1-x}{1+x}\right)^{\frac{3}{2}}\right) = \frac{3}{2}(\ln(1-x)-\ln(1+x)) \quad \text{so we get} \quad \frac{3}{2}\left(\frac{-1}{1-x} - \frac{1}{1+x}\right),$$
$$e^x\sin 2x + 2e^x\cos 2x, \qquad \frac{\cos x}{\sin x}, \qquad \frac{1}{x}\cdot\frac{1}{\ln x}, \qquad e^x\exp(e^x)$$

6. We just have to differentiate the function twice and then substitute in the equation.

$$\begin{aligned} p &= Ae^{2t} + Be^{3t} \\ \dot{p} &= 2Ae^{2t} + 3Be^{3t} \\ \ddot{p} &= 4Ae^{2t} + 9Be^{3t} \end{aligned}$$

So

$$\ddot{p} - 5\dot{p} + 6p = 4Ae^{2t} + 9Be^{3t} - 10Ae^{2t} - 15Be^{3t} + 6Ae^{2t} + 6Be^{3t} = 0\ .$$

7. (i) Take logs before differentiating:

$$\ln f(x) = \frac{2}{3}(\ln(1+x^2) - \ln(1-x^2))\ .$$

Differentiation is now easy,

$$\frac{f'(x)}{f(x)} = \frac{2}{3}\left(\frac{2x}{1+x^2} + \frac{2x}{1-x^2}\right)\ .$$

Multiply through by $f(x)$ and tidy up to get the answer.

$$f'(x) = \frac{2}{3}\left(\frac{4x}{(1+x^2)(1-x^2)}\right)\left(\frac{1+x^2}{1-x^2}\right)^{\frac{2}{3}}\ .$$

(ii) Again take logs:

$$\ln f(x) = \frac{5}{7}\ln(1+x) + \frac{3}{5}\ln(1+x^2) - \frac{3}{2}\ln(1-x) - \ln(1-x^2)$$

so

$$\frac{f'(x)}{f(x)} = \frac{5}{7(1+x)} + \frac{6x}{5(1+x^2)} + \frac{3}{2(1-x)} + \frac{2x}{1-x^2},$$

now multiply through by $f(x)$ as in the previous question.

8. Taking logs (natural) we get $\ln(f(x)) = x \ln a$. Differentiating this we get

$$\frac{f'(x)}{f(x)} = \ln a \quad \text{hence} \quad f'(x) = \ln(a)a^x$$

[Remark: At the start of the subsection on exponential and logarithms we claimed that

$$\lim_{h\to 0} \frac{a^h - 1}{h} = C(a)$$

where $C(a)$ was a constant depending on a. The claim was indeed correct and you now know that $C(a) = \ln(a)$. Also note that $C(e) = \ln(e) = 1$ as claimed.]

9. We've already done lots of this type of question. A tutor will help if you had problems.

10. Let x and y be the distances in matres indicated in the Figure. In the initial position $|WP| = 15$, and so $|PT| = 32 - 15 = 17$. Pythagoras's theorem now tells you that, in this starting position, $y = 8$.

Pythagoras's theorem gives us the equation $15^2 + y^2 = |PT|^2$, i.e. $15^2 + y^2 = (32 - x)^2$.

Differentiate both sides with respect to t to get

$$y\frac{dy}{dt} = -(32 - x)\frac{dx}{dt}$$

After 3 seconds $y = 8 + 12 = 20$ (We calculated the starting value of y to be 8 earlier, and it has increased by 4 for each of the 3 seconds.) Substituting this value for y into the equation linking y and x we get $15^2 + 20^2 = (32 - x)^2$, from which it follows that $32 - x = 25$. Put these values, and the given one for dy/dt into the equation we got from differentiating, and you find that $20\, dy/dt = -25\, dx/dt$, and so the weight is rising at 3.2 metres/sec. (The minus sign tells you that x is decreasing, which is what you would expect.)

The figure we also have that

$$\tan\theta = \frac{15}{y} \quad \text{and hence that} \quad \sec^2\theta\frac{d\theta}{dt} = -\frac{15}{y^2}\frac{dy}{dt}\ .$$

When $y = 20$, $\tan\theta = 3/4$, and so $\sec^2\theta = 1 + \dfrac{9}{16} = \dfrac{25}{16}$. Substitute these numbers in, and you have your answer. The units are radians per second.