Support Vector Machine

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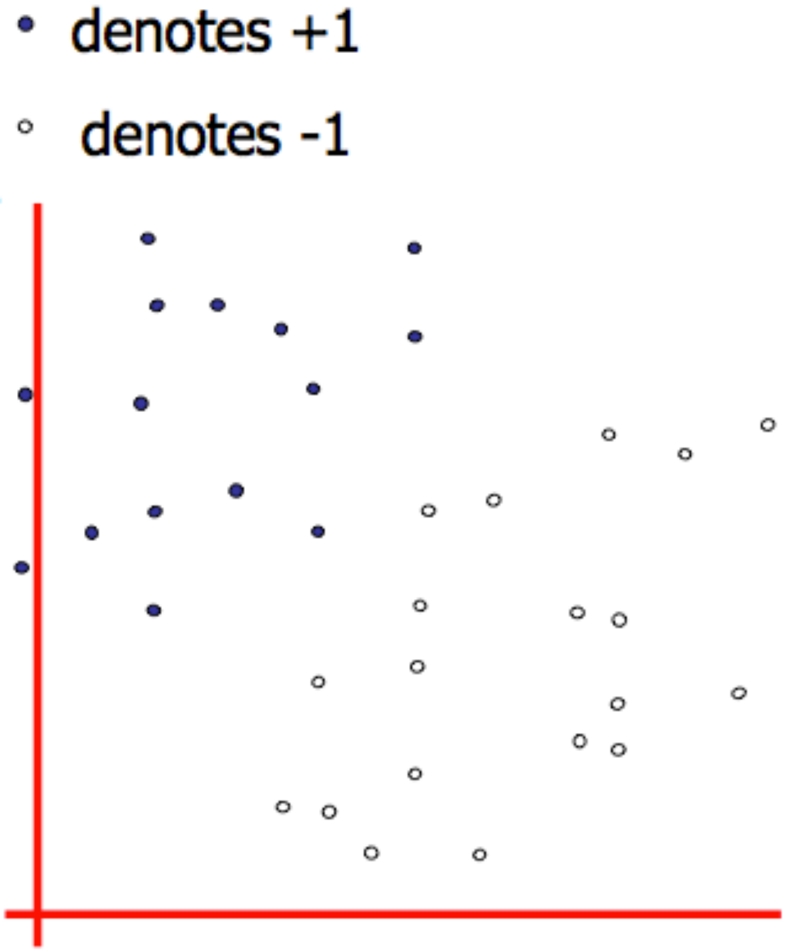
Outline

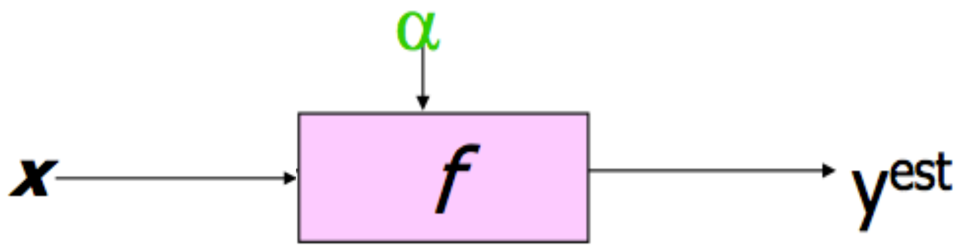
- Probabilistic and Bayesian Analytics
- Classification
  - Naïve Bayes Classifier
  - Support vector machines (SVM)
  - Decision Trees
- Association Rule Mining
- Feature Selection
- Visualization I and II
- Case study
- Data Mining Issues

What You Should Know

- Linear SVMs
- The definition of a maximum margin classifier
- What QP (Quadratic Programming) can do for you
  - for this class, you don't need to know how it does it
- How we deal with noisy data, i.e. misclassified data
- How we permit non-linear boundaries
- How SVM Kernel functions permit us to pretend we're working with ultra-high-dimensional basis- function terms

Linear Classifiers

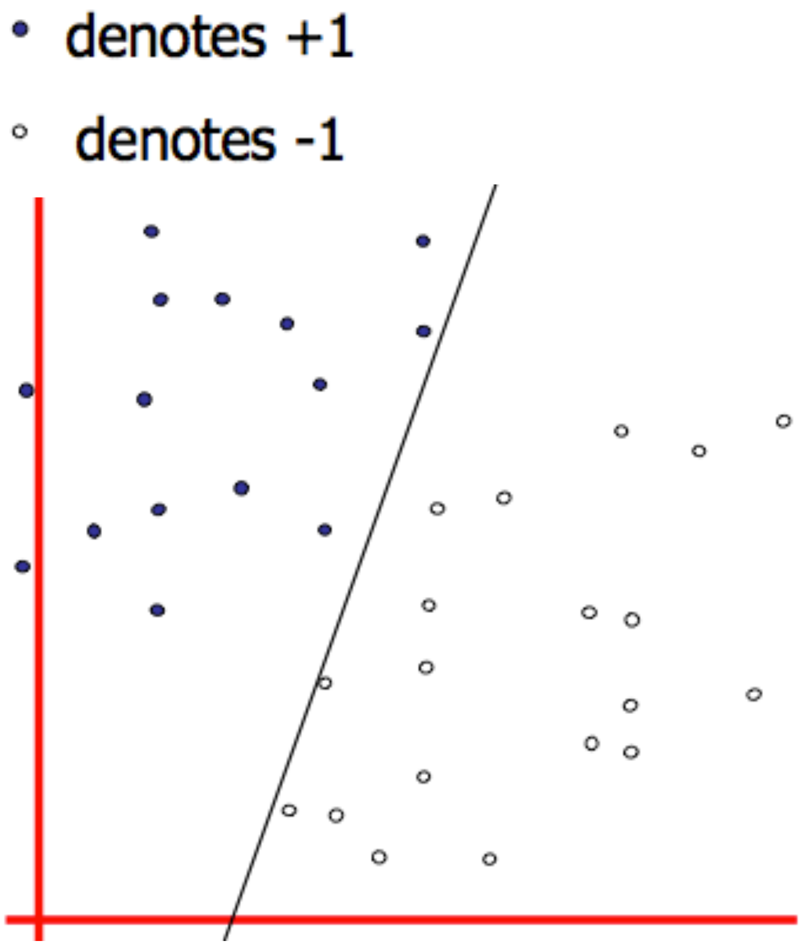


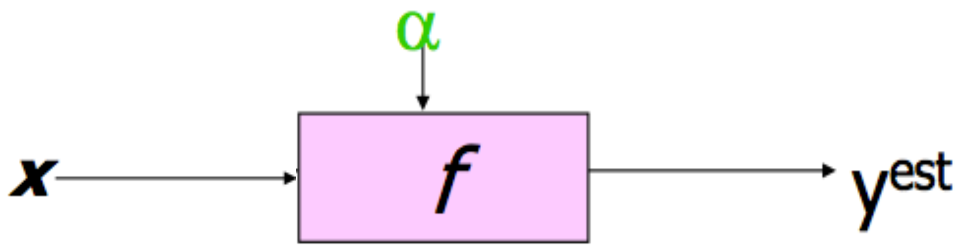


$$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} - b)$$

How would you classify this data?

Linear Classifiers





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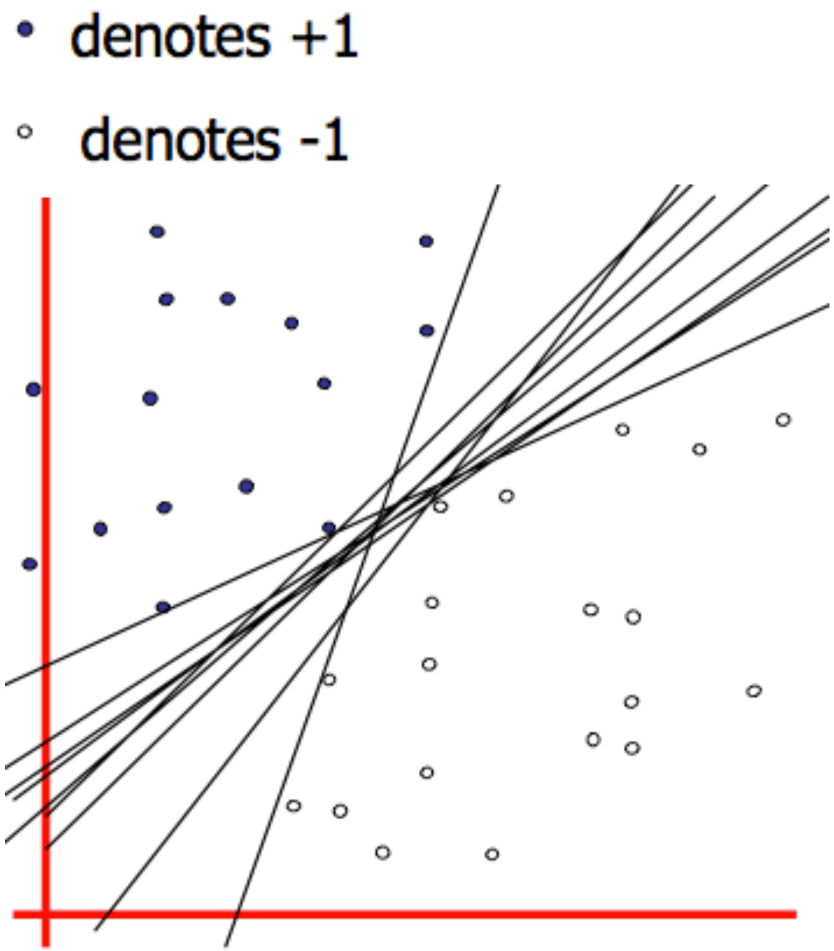
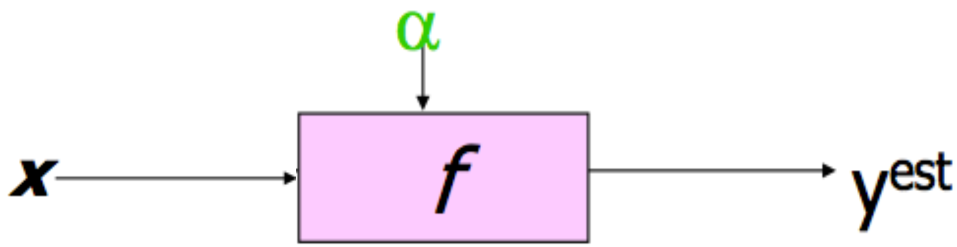


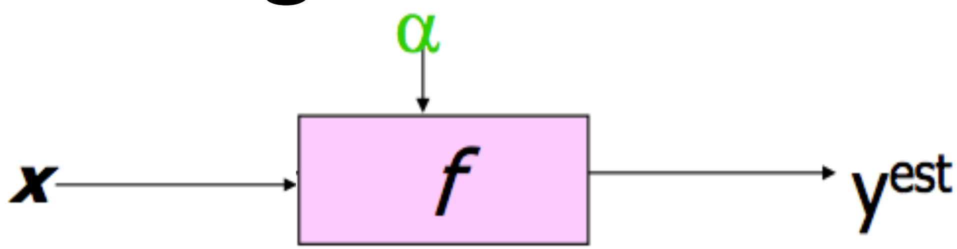


$$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} - b)$$

Any of these would be fine …

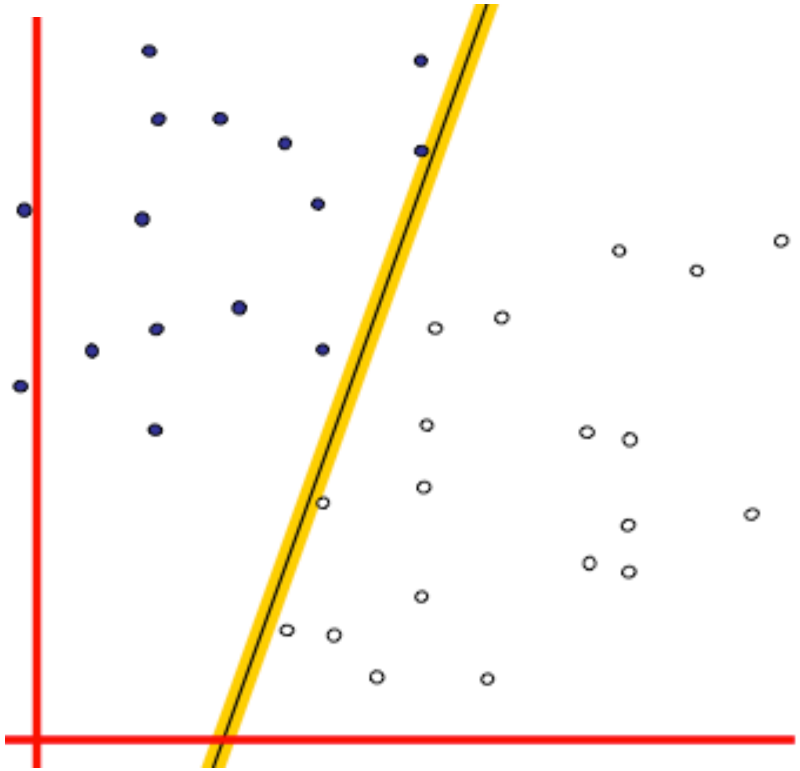
… but which is best?

Classifier Margin



• denotes +1

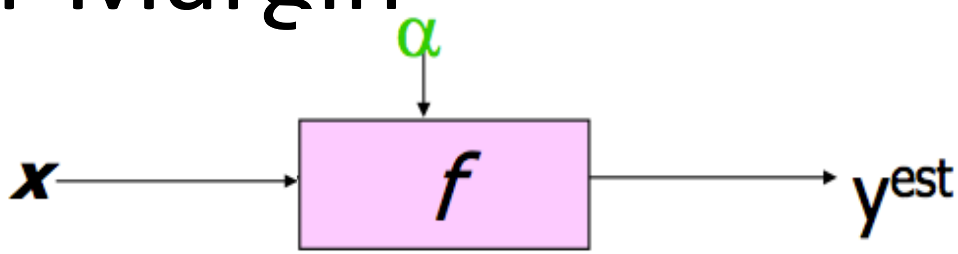
◦ denotes -1



$$f(\mathbf{x},\mathbf{w},b) = \mathrm{sign}(\mathbf{w}\cdot\mathbf{x} - b)$$

Define the margin of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.

Classifier Margin



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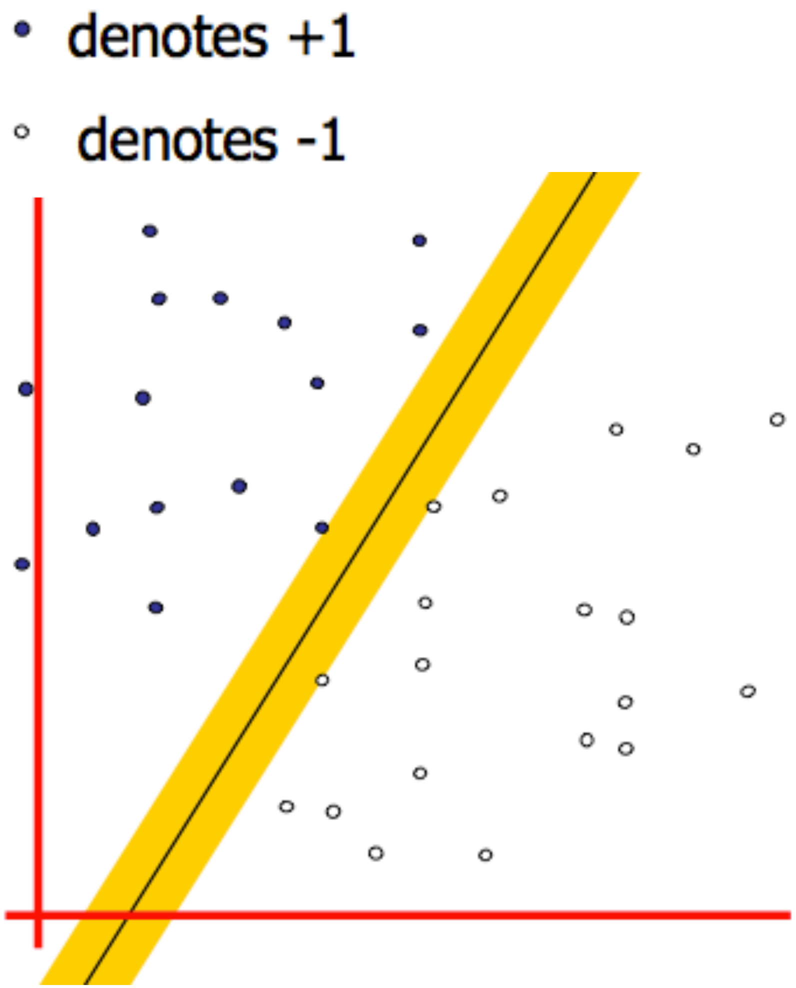
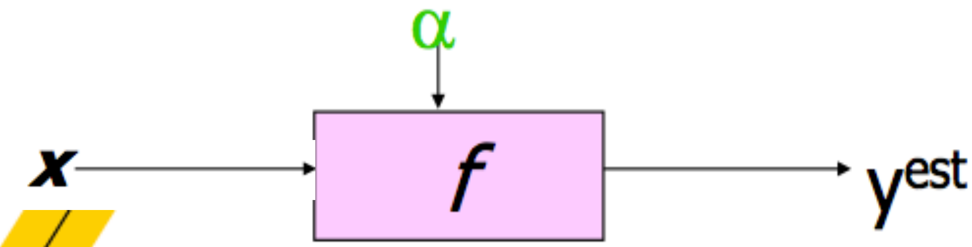


The maximum margin linear classifier is the linear classifier with the, um, maximum margin.

This is the simplest kind of SVM (Called an LSVM, i.e. Linear SVM)

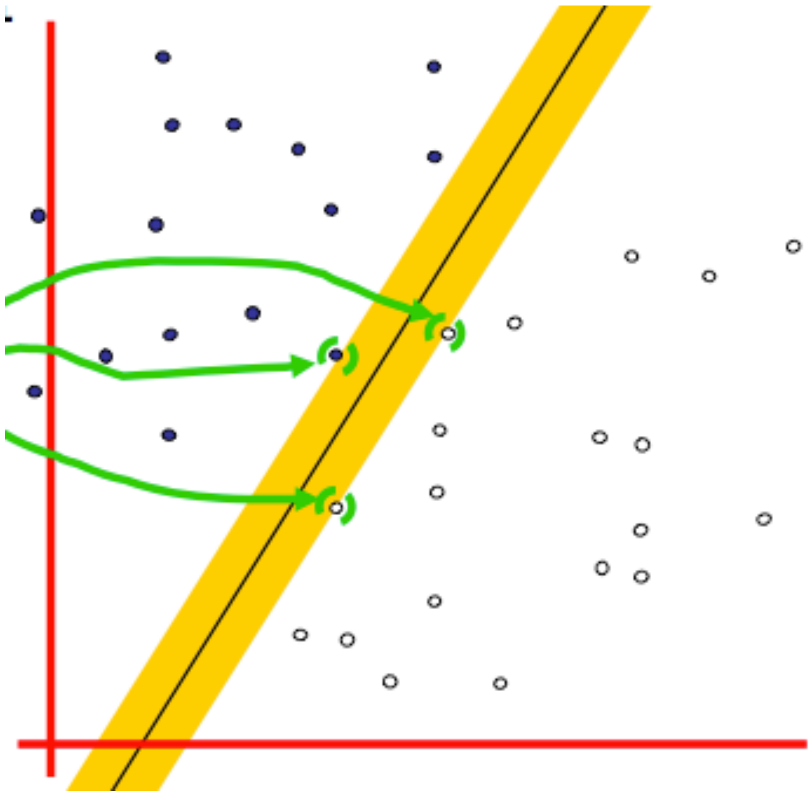
Maximum Margin

- denotes +1
- ∘ denotes -1



$$f(\mathbf{x},\mathbf{w},b) = \mathrm{sign}(\mathbf{w}\cdot\mathbf{x} - b)$$

Support Vectors are those datapoints that the margin pushes up against



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Why Maximum Margin

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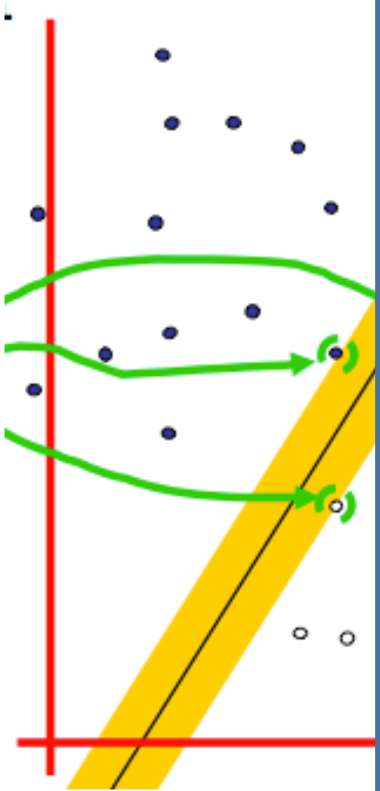
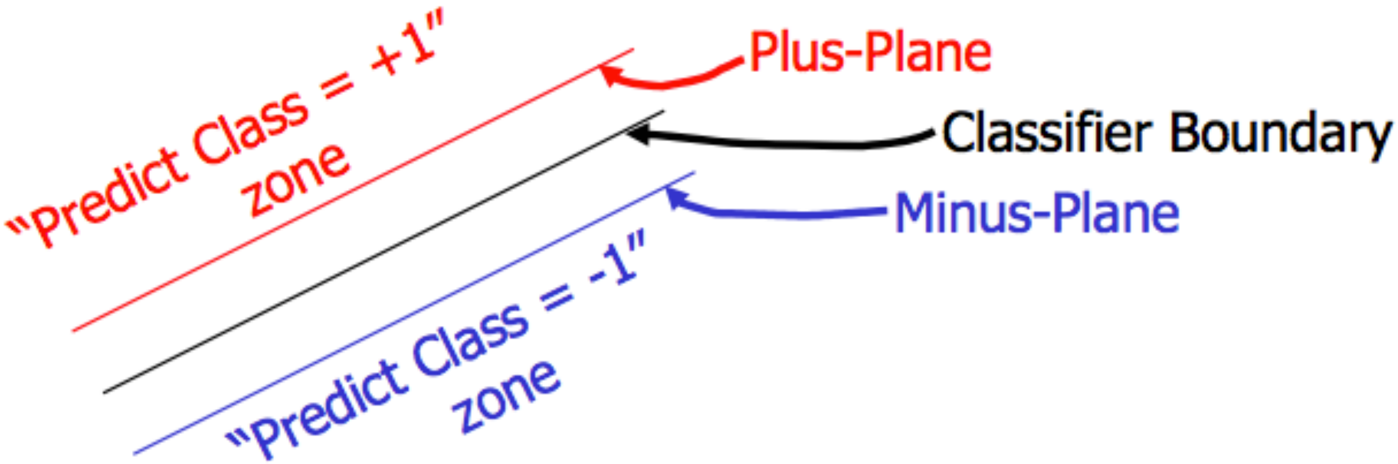
Support Vectors are those datapoints that the margin pushes up against



1. Intuitively this feels safest.
2. If we've made a small error in the location of the boundary (it's been jolted in its perpendicular direction) this gives us least chance of causing a misclassification
3. LOOCV is easy since the model is immune to removal of any non-support vector datapoints.
4. Empirically it works very very well.

Specifying a line and margin



- Plus-plane = {**x**:**w**.**x**+b=+1}
- Minus-plane= {**x**:**w**.**x**+b=-1}

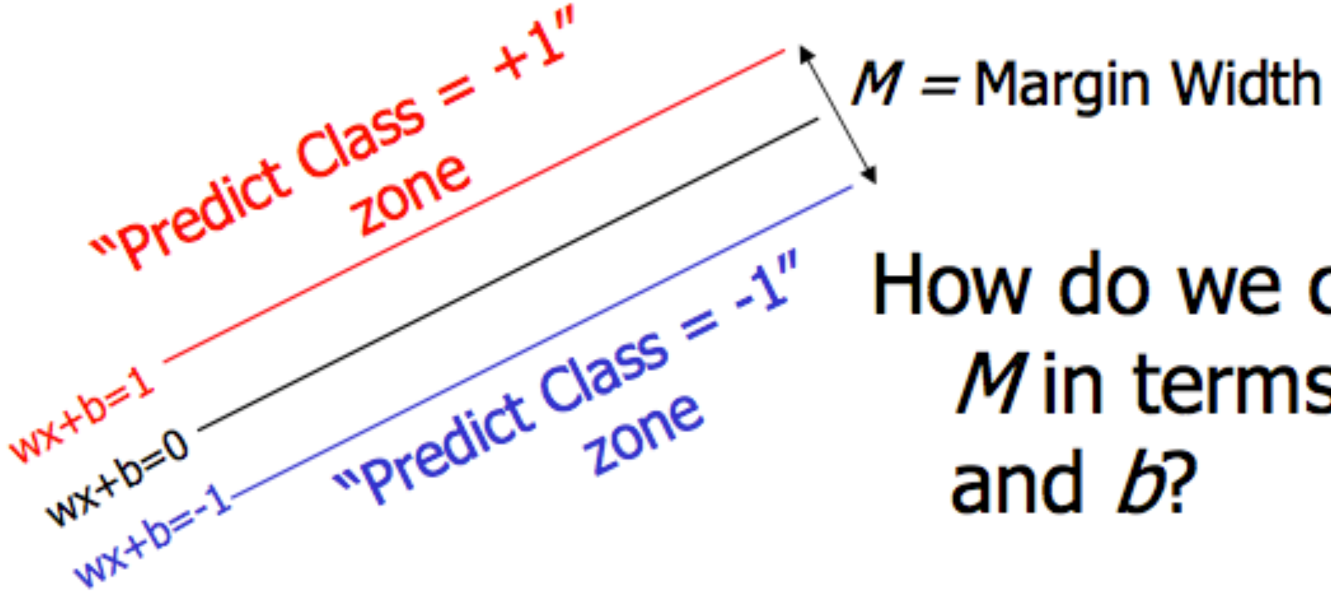
Classify as..

+1 if **w**.**x**+b>=1

-1 if **w**.**x**+b<=-1

Universe Explodes if -1 < **w** . **x** + b < 1

Compute the margin width

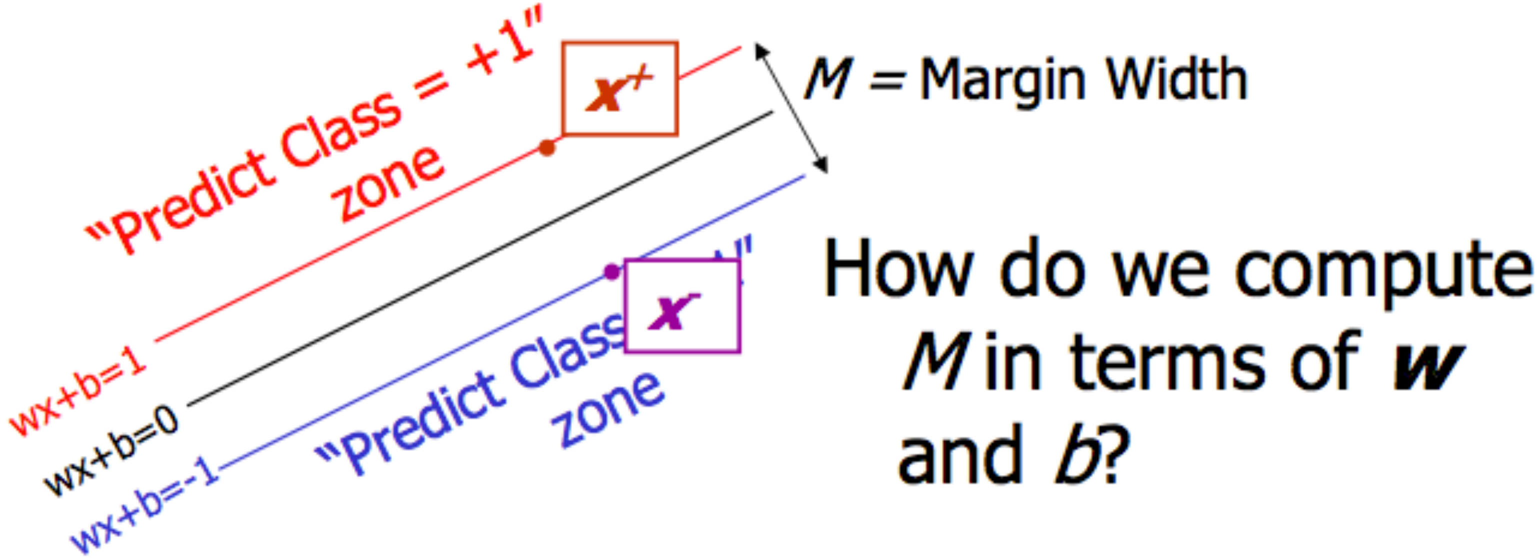


How do we compute *M* in terms of ***w*** and *b*?

- Plus-plane = {**x**:**w**.**x**+b=+1}
- Minus-plane= {**x**:**w**.**x**+b=-1}

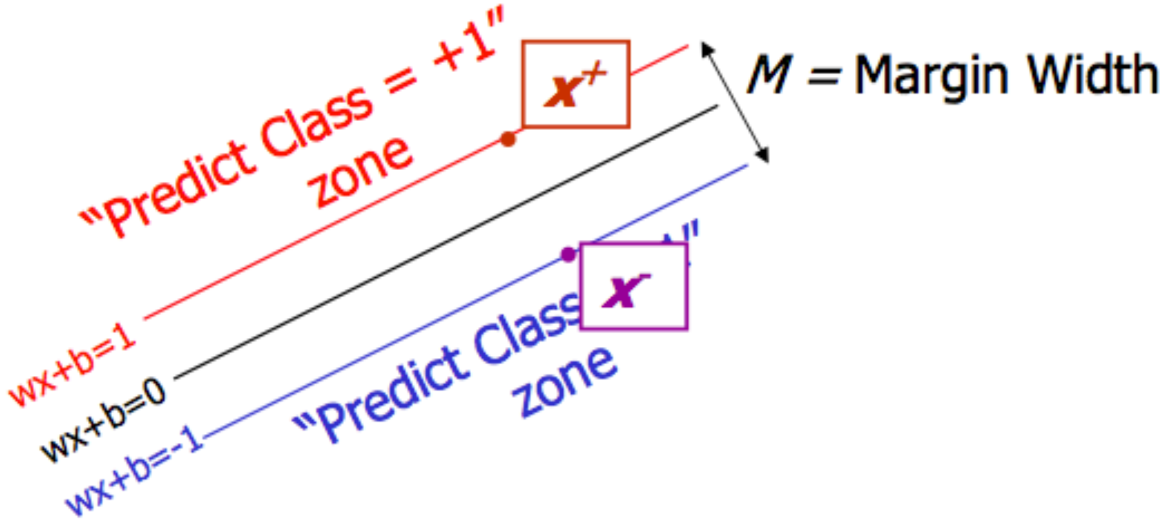
Claim: The vector **w** is perpendicular to the Plus Plane. Why?

Computing the margin width



- Plus-plane = {**x**:**w**.**x**+b=+1}
- Minus-plane= {**x**:**w**.**x**+b=-1}
- The vector w is perpendicular to the Plus Plane
- Let x^- be any point on the minus plane
- Let x^+ be the closest plus-plane-point to x^-.
- Claim:$x^+ = x^- + \lambda w$ for some value of λ. Why?

Computing the margin width

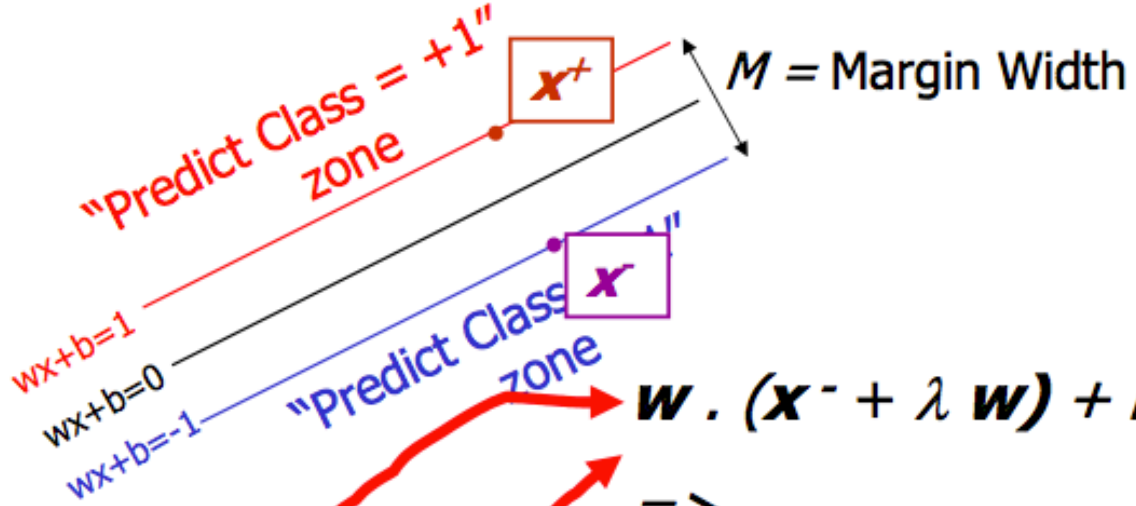


What we know:

- $\boldsymbol{w} \cdot \boldsymbol{x}^+ + b = +1$
- $\boldsymbol{w} \cdot \boldsymbol{x}^- + b = -1$
- $\boldsymbol{x}^+ = \boldsymbol{x}^- + \lambda\, \boldsymbol{w}$
- $|\boldsymbol{x}^+ - \boldsymbol{x}^-| = M$

It's now easy to get M
in terms of $\boldsymbol{w}$ and b

Computing the margin width



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- $|\mathbf{x}^+ - \mathbf{x}^-| = M$

It's now easy to get M in terms of $\mathbf{w}$ and b

$$\mathbf{w} \cdot (\mathbf{x}^- + \lambda \, \mathbf{w}) + b = 1$$

=>

$$\mathbf{w} \cdot \mathbf{x}^- + b + \lambda \, \mathbf{w} \cdot \mathbf{w} = 1$$

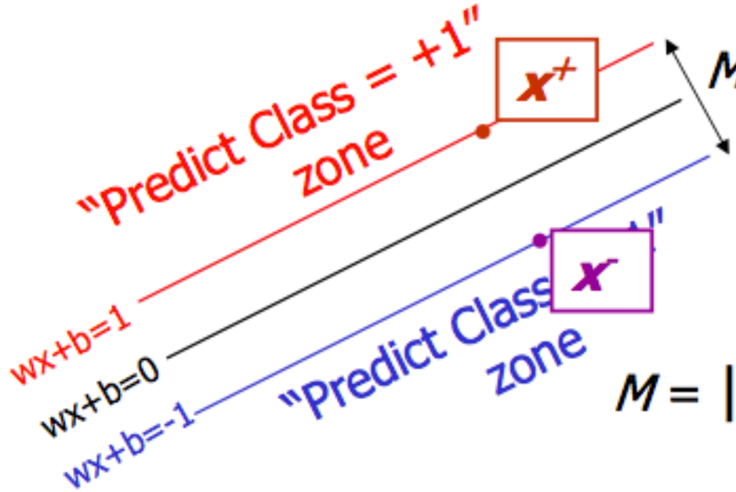
=>

$$-1 + \lambda \, \mathbf{w} \cdot \mathbf{w} = 1$$

=>

$$\lambda = \frac{2}{\mathbf{w} \cdot \mathbf{w}}$$

Computing the margin width



$$M = \text{Margin Width} = \frac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$$

What we know:

- $\boldsymbol{w} \,.\, \boldsymbol{x}^{+} + b = +1$
- $\boldsymbol{w} \,.\, \boldsymbol{x}^{-} + b = -1$
- $\boldsymbol{x}^{+} = \boldsymbol{x}^{-} + \lambda\, \boldsymbol{w}$
- $|\boldsymbol{x}^{+} - \boldsymbol{x}^{-}| = M$
- $$\lambda = \frac{2}{\mathbf{w}.\mathbf{w}}$$

$$M = |\boldsymbol{x}^{+} - \boldsymbol{x}^{-}| = |\lambda\, \boldsymbol{w}| =$$

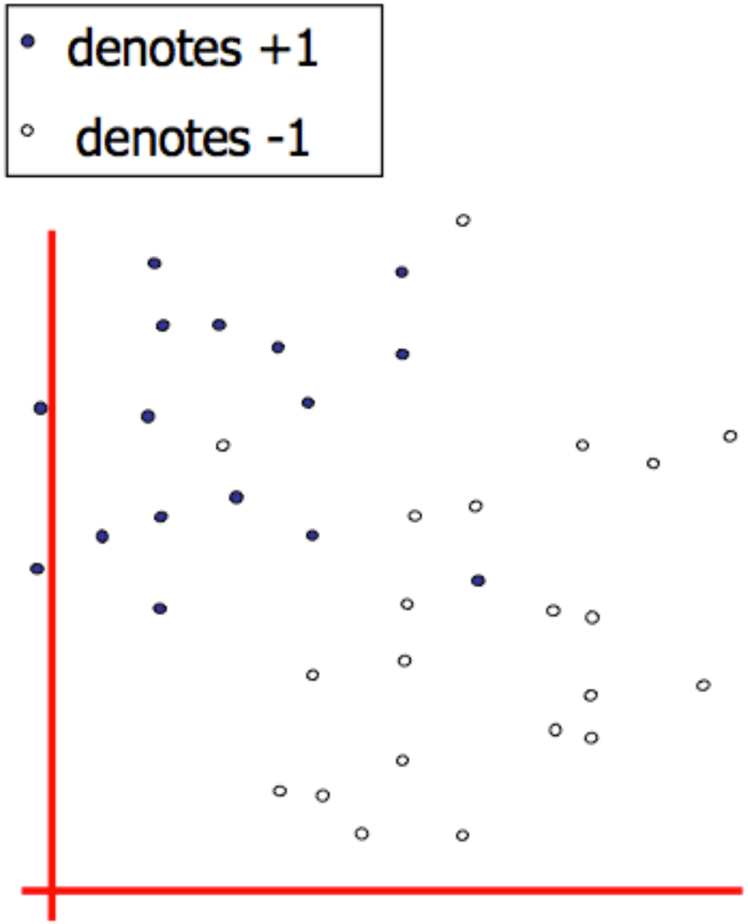
$$= \lambda\, |\mathbf{w}| = \lambda \sqrt{\mathbf{w}.\mathbf{w}}$$

$$= \frac{2\sqrt{\mathbf{w}.\mathbf{w}}}{\mathbf{w}.\mathbf{w}} = \frac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$$

Learning the Maximum Margin Classifier

- Given a guess of **w** and b we can
  - Compute whether all data points in the correct half-planes
  - Compute the width of the margin
- So now we just need to write a program to search the space of **w**’s and b’s to find the widest margin that matches all the datapoints.
- How? Learning via Quadratic Programming
  - Out of the scope of the course

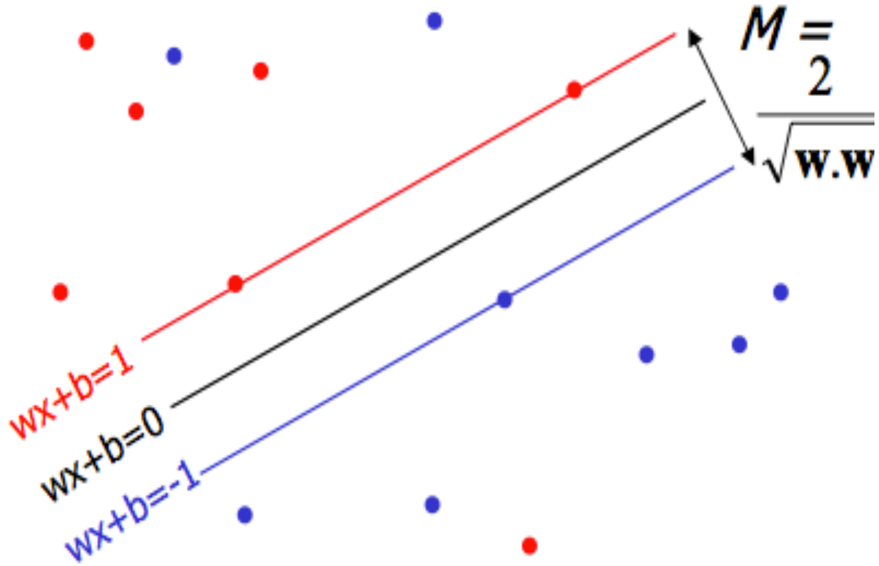
Noise, Uh-oh!



This is going to be a problem!
What should we do?

Minimize: **w.w** + D
where D is the distance of error points to their correct place

Learning Maximum Margin with Noise

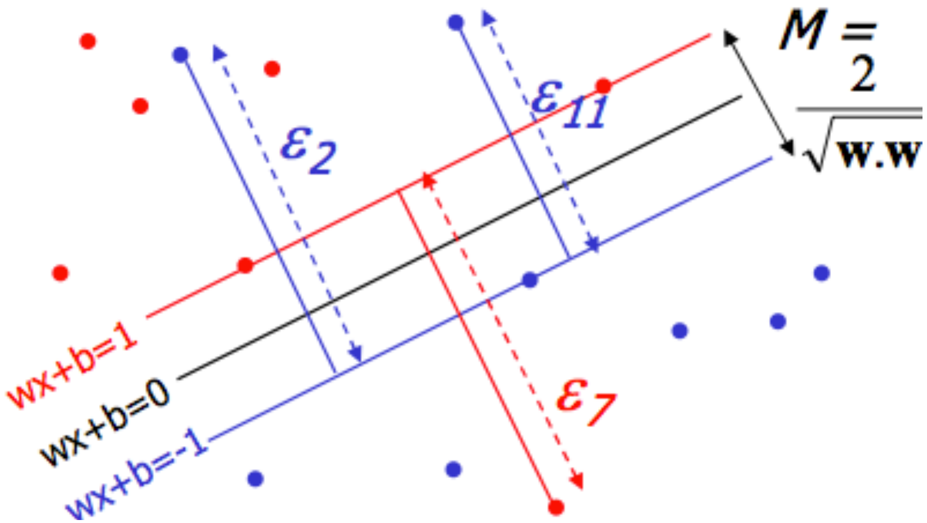


Given guess of W, b, we can

- Compute sum of distances of points to their correct zones
- Compute the margin width. Assume R datapoints, each ($\mathbf{x}_k$,y_k) where y_k = +/- 1

- How many constraints will we have?

Learning Maximum Margin with Noise



What should our quadratic optimization criterion be?

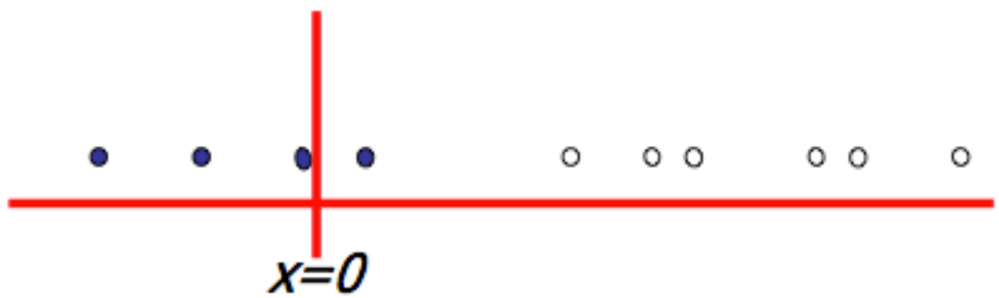
Minimize $\frac{1}{2}\mathbf{w}.\mathbf{w} + C\sum_{k=1}^{R}\varepsilon_k$

Given guess of W, b can

- Compute sum of distances of points to their correct zones
- Compute the margin width. Assume R datapoints, each ($\mathbf{x}_k$,y_k) where y_k = +/- 1
- ε: the slack variable
- C: control the trade-off between the slack variable and the margin.

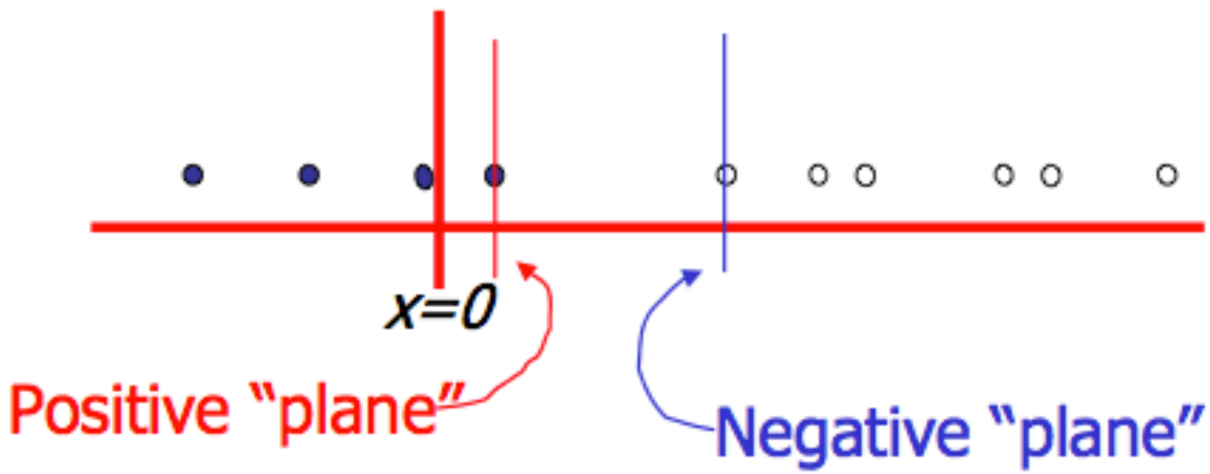
Suppose we’re in 1-dimension

What would SVMs do with this data?



Suppose we're in 1-dimension

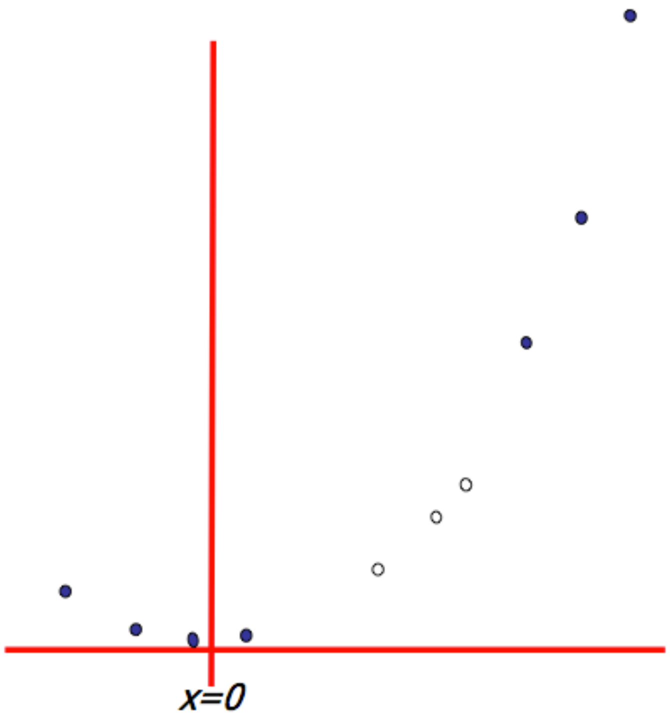
Not a big surprise



Harder 1-dimensional dataset

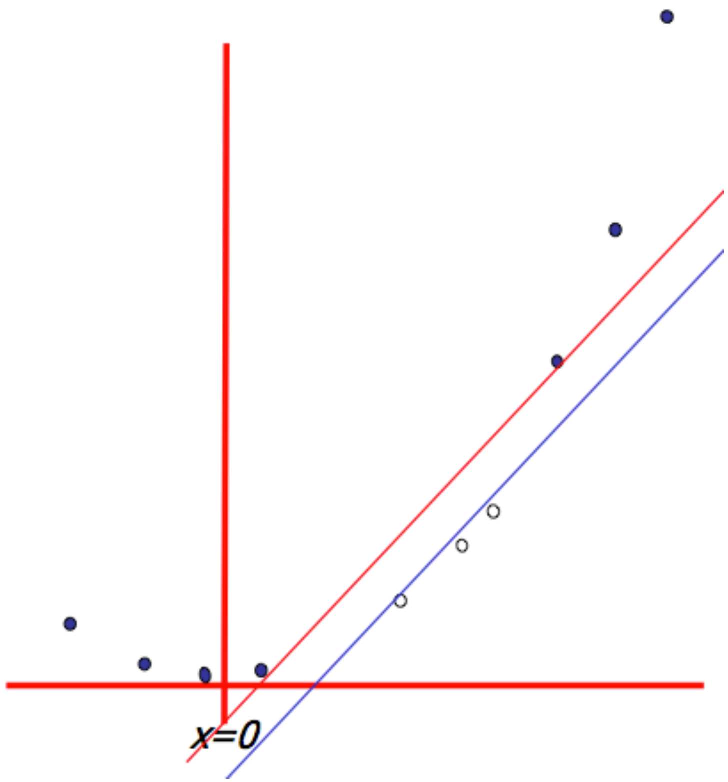
What can be done about this? i.e. linear not separable

Harder 1-dimensional dataset



- Non-linear basis functions to rescue!
- To project original datapoints to higher dimensions within which datapoints are separable
- $\mathbf{z}_k = (x_k, x_k^2)$

Harder 1-dimensional dataset



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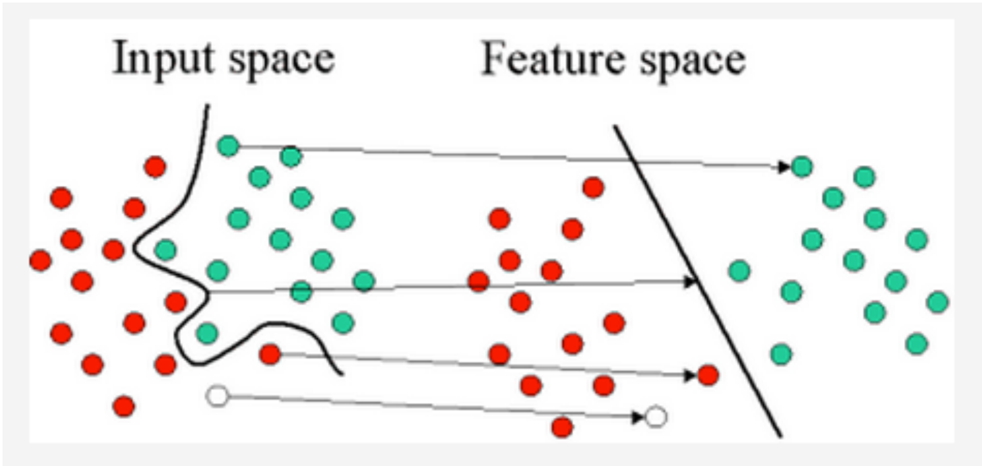
Common SVM Kernel functions

Kernel trick: SVMs can efficiently perform a non-linear classification using what is called the kernel trick, implicitly mapping their inputs into high-dimensional feature spaces.

Kernel functions:

- polynomial functions
- radial basis functions
- sigmoid functions



Summary

- The definition of a maximum margin classifier
- How Maximum Margin can be turned into a QP problem
- How we deal with noisy data, i.e. misclassified data
  - slack variable
- How we permit non-linear boundaries
  - SVM Kernel functions permit us to pretend we're working with ultra-high-dimensional basis-function terms
  - And in the new feature space, datapoints are linearly separable

Readings

- An excellent tutorial on VC-dimension and Support Vector Machines:
  – C.J.C. Burges. A tutorial on support vector machines for pattern recognition. Data Mining and Knowledge Discovery, 2(2):955-974, 1998. http://citeseer .nj.nec.com/burges98tutorial.html
- The VC/SRM/SVM Bible:
  – Statistical Learning Theory by Vladimir Vapnik, Wiley- Interscience; 1998