Author	AP/IC/SD/AS
Date	17.1.013
Code	

University of Aberdeen
Department of Mathematical Sciences

EG1504 Engineering Mathematics 1
Exercises 8 (Approximation and Taylor Series)

1. Use the approximation $f(a+h) \approx f(a) + hf'(a)$, with a suitable choice of the function, a and h to compute an approximate value for

$$1.02^7, \qquad \ln(1.03), \qquad \frac{1}{\sqrt{9.4}} .$$

Compare your answers with the "accurate" ones you get from a calculator.

2. Show that if x is small then $\dfrac{1}{1+x} \approx 1 - x$, and that $\sqrt{1+x} \approx 1 + \dfrac{1}{2}x$.

3. Using the third order approximation, show that if x is small then

$$\frac{1+x}{1-x} \approx 1 + 2x + 2x^2 + 2x^3 .$$

By working out some values on your calculator try to estimate the range of values of x for which this approximation is accurate to within 1% (that is the *relative* error is less than 0.01).

4. Write down the first four derivatives of $\sin(x)$. Use these results and Taylor's Theorem to write down the next *non-zero* term in the expansion

$$\sin(x) = x - \frac{x^3}{3!} + \ldots .$$

Explain how you derive the term.

5. Find the Maclaurin series for the functions $\cosh x$ and $\sinh x$.

6. You can get the Maclaurin series for $\sin 2x$ simply by taking the Maclaurin series for $\sin x$ and replacing x by $2x$. So

$$\sin 2x = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \cdots$$

Use the same idea to write down the first few terms of the Maclaurin series for the functions e^{3x}, e^{-x}, $\cos x^2$, $\ln(1-x)$, $\ln(1+3x)$. For what values of x is the Maclaurin series for $\ln(1+3x)$ valid?

7. Use the binomial series to write down the first few terms of the Maclaurin series for the following functions:

$$\sqrt{1-x}, \quad \frac{1}{(1+x)^2}, \quad (1-2x)^{3/4}, \quad (1-x^2)^{-3/5}$$

In each of these cases give the range of values of x for which the full expansion is valid. Use a Maclaurin series to calculate $(0.9)^{3/4}$ to 6 decimal places.

8. Use Newton's method to find an approximation to the unique solution of the equation

$$x - e^{-x} = 0$$

(The solution lies in the interval $[0, 1]$.) Take $x = 0$ as an initial estimate of the solution.

9. The equation $x^4 - x - 2 = 0$ has a solution somewhere between $x = 1$ and $x = 2$. Find this solution accurate to 4 decimal places. Check that your answer is correct.

University of Aberdeen
Department of Mathematical Sciences

EG1504 Engineering Mathematics 1
Solutions to Exercises 8 (Approximation and Taylor Series)

1. Let $f(x) = x^7$, so $f'(x) = 7x^6$. Taking $a = 1$ and $h = 0.02$ leads to the approximation

$$1.02^7 = f(1 + 0.2) \approx 1 + 0.02.7.(1)^6 = 1.14.$$

The calculator gives 1.14869 to 5 decimal places.

Let $f(x) = \ln(x)$ so $f'(x) = 1/x$. Taking $a = 1$ and $h = 0.3$ leads to the approximation

$$\ln(1.03) = f(1 + 0.3) \approx \ln(1) + 0.03.1/1 = 0.03.$$

The calculator gives ≈ 0.02956 to 5 decimal places.

Let $f(x) = x^{-1/2}$ so $f'(x) = \frac{-1}{2}x^{-3/2}$. Taking $a = 9$ and $h = 0.4$ leads to the approximation

$$\frac{1}{\sqrt{9+0.4}} = f(9 + 0.4) \sim \frac{1}{3} - \frac{1}{2}.\frac{1}{27} = 17/54 = 0.3148 \quad \text{to 4 decimal places.}$$

The calculator gives 0.32616 to 5 decimal places.

2. Let $f(x) = 1/(1 + x)$. The linear approximation formula is

$$f(x + h) \approx f(x) + hf'(x)$$

This example talks about x 'being small', i.e. close to zero. So we perform our approximation about 0:

$$f(0 + h) \approx f(0) + hf'(0)$$

We have $f(0) = 1$. All that remains is to work out the derivative.

$$f'(x) = \frac{-1}{(1+x)^2} \qquad \text{so} \qquad f'(0) = -1$$

So the linear approximation becomes $f(h) \approx 1-h$, or changing h to x (it's just a name), $f(x) \approx 1-x$.

The second one uses exactly the same process. $f(x) = \sqrt{1+x}$. $f(0) = 1$. $f'(x) = \frac{1}{2}(1 + x)^{-1/2}$ and $f'(0) = \frac{1}{2}$. So

$$f(x) \approx f(0) + xf'(0) = 1 + \frac{1}{2}x$$

3. The third order approximation is

$$f(x) \approx 1 + 2x + \frac{4}{2}x^2 + \frac{12}{6}x^3 = 1 + 2x + 2x^2 + 2x^3 \ .$$

The (absolute) error in the approximation is

$$\begin{aligned}\frac{1+x}{1-x} - (1 + 2x + 2x^2 + 2x^3) &= \frac{1 + x - (1 + 2x + 2x^2 + 2x^3)(1 - x)}{1-x} \\ &= \frac{1 + x - (1 + 2x + 2x^2 + 2x^3) + x(1 + 2x + 2x^2 + 2x^3)}{1-x} \\ &= \frac{2x^4}{1-x}\end{aligned}$$

Thus the relative error is $\dfrac{2x^4}{1+x}$.

Suppose that $x \geqslant 0$. Then $2x^4/(1+x) \leqslant 2x^4$ thus the relative error will be less than 0.01 if $x^4 < 0.005$.

But $\sqrt[4]{0.005} = 0.266$ to 3 decimal places, so if $0 \leqslant x \leqslant 0.265$ then the relative error is less than 0.01.

Suppose that $-0.5 \leqslant x < 0$. Then $2x^4/(1+x) \leqslant 2x^4/(1-0.5) = 4x^4$.

But $\sqrt[4]{0.0025} = 0.224$ to 3 decimal places, so if $-0.223 < x < 0$ then the relative error is less than 0.01.

Our answer is $-0.223 < x < 0.265$. Did you do better with your calculator?

4. Let $f(x) = \sin x$. then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and $f^{(4)}(x) = \sin x$. The next two terms in the Taylor expansion about zero are

$$+\frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 = 0 + \frac{x^5}{5!}.$$

(so you needed the fifth derivative as well).

5. There are two ways to do this. You can proceed in the usual way, working out derivatives, or you can use the known series for the exponential function.

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots)$$

so

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

6. Answers:

$$\begin{aligned}
e^{3x} &= 1 + 3x + (3x)^2/2 + (3x)^3/6 + \cdots = 1 + 3x + \frac{9}{2}x^2 + \frac{27}{6}x^3 + \cdots \\
e^{-x} &= 1 - x + x^2/2 - x^3/6 + x^4/24 - \cdots \\
\cos x^2 &= 1 - x^4/2 + x^8/24 - x^{12}/720 + \cdots \\
\ln(1-x) &= -x - x^2/2 - x^3/3 - x^4/4 - \cdots \\
\ln(1+3x) &= 3x - (3x)^2/2 + (3x)^3/3 - (3x)^4/4 + \cdots
\end{aligned}$$

The series for $\ln(1+x)$ is valid for $-1 < x \leqslant 1$, so the series for $\ln(1+3x)$ is valid for $-1 < 3x \leqslant 1$, i.e. $-1/3 < x \leqslant 1/3$.

7. Answers:

$$\begin{aligned}
\sqrt{1-x} &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots \\
(1+x)^{-2} &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \\
(1-2x)^{3/4} &= 1 - \frac{3}{2}x - \frac{3}{8}x^2 - \frac{5}{16}x^3 - \cdots \\
(1-x^2)^{-3/5} &= 1 + \frac{3}{5}x^2 + \frac{12}{25}x^4 + \frac{52}{125}x^6 + \cdots
\end{aligned}$$

The third one is valid for $-1/2 < x < 1/2$, the others are valid for $-1 < x < 1$.

$0.9^{3/4} = 0.924021$ to 6 decimal places.

8. Newtons method: starting with x_0

$$x_{n+1} = x_n + h_n, \qquad h_n = -\frac{f(x_n)}{f'(x_n)} .$$

Let $f(x) = x - e^{-x}$ and let $x_0 = 0.0$. Applying Newton's method gives successive values, which are given in the table:

n	x_n	h_n
0	0.0	0.5
1	0.5	0.06631
2	0.56631	0.00083
3	0.56714	0.00000

(tabular figures to 5 decimal places). The root appears to be 0.5671 to 4 decimal places.

9. The answer is $x = 1.3532$.