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UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 9 (INTEGRATION)

1. *Basic integrals using only the simplest of combining rules.*

Find indefinite integrals for each of the following functions of x:

$$x^5, \qquad 5x^4, \qquad 6x^{-3}, \qquad x^{-2}+\sin x, \qquad x+1+x^{-1}, \qquad \cos x+\sec^2 x\ .$$

2. *In this next batch you need to rearrange the integrand a little before you can write down the integral. For the rearrangements you will need to make use of things such as the binomial theorem and some of the trig formulas.*

Find indefinite integrals for each of the following functions of x:

$$\cos^2 x-\sin^2 x, \qquad \sin x\cos x, \qquad (x+x^{-1})^3, \qquad \cos^2 x+\sin^2 x,$$

$$\tan^2 x, \qquad \sin^2 x, \qquad \sinh^2 x, \qquad \frac{1}{e^x}\ .$$

3. *In this one you have to resort to a bit of trial and error. Make a guess on the basis of things you know and which have roughly the same form, test your guess by differentiating and then make any necessary adjustments.*

Find indefinite integrals for each of the following functions of x:

$$e^{-3x}, \qquad \cos 3x+\sin 2x, \qquad \sec^2(-2x), \qquad \frac{1}{25+x^2}, \qquad \frac{1}{\sqrt{1-9x^2}}\ .$$

With the remaining questions you are calculating areas and will probably find that a simple sketch will be helpful.

4. Find the area of the region under the curve $y=x^3$ between $x=0$ and $x=2$.

5. Find the points where the line $y=x$ meets the curve $y=\sqrt{x}$. Then find the area of the region bounded by the line and the curve.

6. Find the area of the region between the curve $y=x^2+1$ and the line $y=5$.

7. Find the area of the region between the curve $y=x^5$ and the line $y=x$.

8. Find the area of the region between the curve $y=\cosh x$ and the horizontal line through that point on the curve which has x-coordinate 1.

9. Find the area of the region between the curve $y=\cos x$, the x-axis and the vertical lines $x=0$ and $x=\dfrac{2\pi}{3}$.

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EG1504 Engineering Mathematics 1
Solutions to Exercises 9 (Integration)

1. $x^6/6, \quad x^5, \quad -3x^{-2}, \quad -x^{-1} - \cos x, \quad x^2/2 + x + \ln x, \quad \sin x + \tan x$

2. The first function can be written as $\cos 2x$ and this has integral $\frac{1}{2}\sin 2x$.
The second equals $\frac{1}{2}\sin 2x$ which has integral $-\frac{1}{4}\cos 2x$.
For the third expand the bracket using the binomial theorem. This gives $x^3 + 3x + 3x^{-1} + x^{-3}$ and an integral for this is $\frac{1}{4}x^4 + \frac{3}{2}x^2 + 3\ln x - \frac{1}{2}x^{-2}$.
The fourth is just a complicated way to write 1, and so the integral here is just x.
$\tan^2 x = \sec^2 x - 1$, the integral of $\sec^2 x$ is $\tan x$, and so an integral here is $\tan(x) - x$.
The next requires a little thought before you spot the trick. Nonetheless, it is a trick worth noting because $\sin^2 x$ is a commonly occurring expression. Look at the formula for $\cos 2x$ in terms of $\sin x$ and rearrange it to get $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$. Now integrate the righthand side getting $\frac{1}{2}x - \frac{1}{4}\sin(2x)$.
For the last the simplest approach is to write $\sinh x$ as $\frac{1}{2}(e^x - e^{-x})$, square and expand. This gives you $\frac{1}{4}(e^{2x} - 2 + e^{-2x})$, which can be integrated without too much difficulty to get $\frac{1}{8}(e^{2x} - 4x - e^{-2x})$.
The function is just a disguised version of e^{-x}, and this has integral $-e^{-x}$.

3. Your first try is probably e^{-3x}. Differentiating this using the chain rule gives you minus three times the answer you want. So divide by minus three and try again, this time with $-e^{-3x}/3$. This works, and you use the same approach with the others. With the second you try $\sin 3x - \cos 2x$, discover that this doesn't quite work and modify it to $\dfrac{1}{3}\sin 3x - \dfrac{1}{2}\cos 2x$. The other answers are

$$-\frac{1}{2}\tan(-2x)\,, \qquad \frac{1}{5}\arctan(\frac{x}{5})\,, \qquad \frac{1}{3}\arcsin 3x$$

4. Since $x^3 \geqslant 0$ for $0 \leqslant x \leqslant 2$, the required area is given by

$$\text{Area} = \int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4\ .$$

5. The points of intersection are easily found to be $(0,0)$ and $(1,1)$. The curve lies over the line between these two points, and so the area between the two is the area under the curve minus the area under the line. The area under the curve is given by

$$\text{Area} = \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}\,x\sqrt{x}\right]_0^1 = \frac{2}{3}\ .$$

That under the line is $1/2$. (Integrate if you must here, but half base times height is easier.) So the area between the line and the curve is $1/6$.

6. The curve and the line cross at the points $x = \pm 2, y = 5$. The area you want is the difference between the area under the line and the area under the curve. The former area is a rectangle with base 4 and height 5. The latter area is

$$\int_{-2}^{2}(x^2+1)\,dx = \cdots = \frac{28}{3}\ .$$

So the answer is $20 - 28/3 = 32/3$.

7. The line and curve meet when $x = -1$, when $x = 0$ and when $x = 1$, and there is an obvious symmetry which enables us to assert that the total area is twice that for the section between $x = 0$ and $x = 1$. So

$$\text{Area} = 2\left(\int_0^1 (x - x^5)\,dx\right) = 2\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{2}{3}\ .$$

8. This is another area under the line minus area under the curve job. The line is $y = \cosh 1 = \dfrac{e + e^{-1}}{2}$, and so the area under the line is rectangular with this height and base length 2 (the other point where the line crosses the curve is when $x = -1$). So the rectangle has area $e + e^{-1}$. The area under the curve is $\sinh(1) - \sinh(-1)$ which is $e - e^{-1}$. So the area required is $2e^{-1}$.

9. You need to be careful with this one because part of the area is above the axis and part below. So split the area up into the section above and the section below, and calculate each separately.

$$\text{Area above} = \int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$$

Below the axis the area is minus the integral and so

$$\text{Area below} = -\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos x\,dx = -\left[\sin x\right]_{\pi/2}^{2\pi/3} = 1 - \frac{\sqrt{3}}{2}$$

$$\text{Total area} = 2 - \frac{\sqrt{3}}{2}\ .$$