Contents

Chapter 6

Optimisation Theory

We are often interested in how large or how small a quantity can be. Often the quantity depends on a variable and we need to find which value of the variable gives the largest or smallest value of the quantity of interest. For example the stress in a metal bar may vary along its length and we may want to know the point of maximum stress. An optimum value of a quantity is the maximum or minimum possible value of the quantity. $\square$

6.1 Critical points

Let f be a (real) function.

A number c is called a *local maximising* point of f if and only if there are numbers a and b with $a < c < b$ such that

$$f(x) \leq f(c) \quad \text{whenever } a < x < b,$$

in which case we say that f has a local maximum at c.

A number c is called a *local minimising* point of f if and only if there are numbers a and b with $a < c < b$ such that

$$f(x) \geq f(c) \quad \text{whenever } a < x < b,$$

in which case we say that f has a local minimum at c.

If c is a local maximising or local minimising point of f and f is differentiable at c, then $f'(c) = 0$.

[We can justify the statement above by the following argument. Recall

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Supppose that c is a local minimising point so that $f(c+h) - f(c) \geq 0$ if $a < c+h < b$ (that is $a - c < h < b - c$).

If $f'(c) < 0$ then, from the definition of limits,

$$\frac{f(c+h) - f(c)}{h} < 0 \quad \text{if } h \text{ is sufficiently close (but not equal) to } 0$$

but this is impossible because $\dfrac{f(c+h) - f(c)}{h} \geq 0$ if $0 < h < b - c$.

If $f'(c) > 0$ then, from the definition of limits,

$$\frac{f(c+h) - f(c)}{h} > 0 \quad \text{if } h \text{ is sufficiently close (but not equal) to } 0$$

but this is impossible because $\dfrac{f(c+h) - f(c)}{h} \leq 0$ if $a - c < h < 0$.

The only other possibliity is $f'(c) = 0$. A similar argument holds if c is a local maximising point.] $\square$

We say that f is *stationary* at c and call c a *critical point* of f if $f'(c) = 0$. The corresponding point $(c, f(c))$ of the graph $y = f(x)$ is called a critical point *of the graph.*

Local maximising points and local minimising points are critical points. The critical points of the graph are precisely the points where the slope of the tangent is zero.

To find the local maximising and minimising points, first find the critical points and then decide which (if any) of the critical points are local maximising and which are local minimising.

To find the critical points of a function f one can try to calculate the derivative and then find where the derivative is zero (in other words solve the equation $f'(x) = 0$). Some examples are discussed in lectures. $\square$

In Figure 6.1 the graph of a function is given.

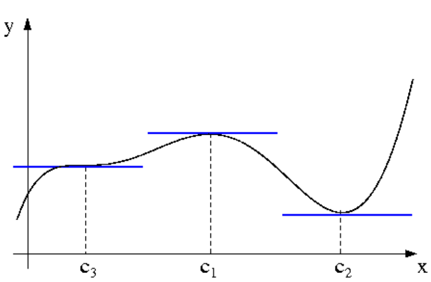

Figure 6.1: Graph of a function

The function has three critical points c_1, c_2 and c_3. It is fairly clear from the graph that c_1 (the middle critical point) is a local maximising point, c_2 is a local minimising point and c_3 is neither local minimising nor local maximising.

A critical point is not necessarily local maximising or local minimising.

6.2 Classifying Critical Points

[The results of subsection 3.1.2 will be used in this section.]

We say that c is an *isolated* critical point of a function f if c is a critical point of f and there are no other critical points of f near c, more precisely there are numbers a and b with $a < c < b$ such that c is the only critical point of f between a and b. $\square$

Consider a function f and let c be an isolated critical point of f. Thus there are numbers a and b with $a < c < b$ such that c is the only critical point of f between a and b. Assume that f is differentiable at all points in the interval from a to b. Thus

$$f'(x) \neq 0 \quad \text{if } a < x < c \text{ or } c < x < b$$

so $f'(x)$ is positive or negative if $a < x < c$ or $c < x < b$. There are three cases to consider.

1. $f'(x)$ is positive whenever $a < x < c$ and $f'(x)$ is negative whenever $c < x < b$. In this case c is a local maximising point.

   [To see this: since $f'(x) > 0$ for $a < x < c$, $f(x)$ must be strictly increasing on the interval from a to c and so $f(x) < f(c)$ for $a < x < c$ and since $f'(x) < 0$ for $c < x < b$, $f(x)$ must be strictly decreasing on the interval from c to b and so $f(x) < f(c)$ for $c < x < b$. Hence $f(x) \leq f(c)$ for all x between a and b, which means that c is a local maximum.]

2. $f'(x)$ is negative whenever $a < x < c$ and $f'(x)$ is positive whenever $c < x < b$. In this case c is a local minimising point.

   [To see this: $f(x)$ must be decreasing on the interval from a to c and and $f(x)$ must be increasing on the interval from c to b. A similar argument to the previous case shows that c is a local minimum.]

3. Either $f'(x)$ is positive for all x in the intervals from a to c and c to b or $f'(x)$ is negative for all x in the intervals from a to c and from c to b. In this case c is neither a local maximising point nor a local minimising point.

   [To see this: the assumptions are sufficient to show that either that f is strictly increasing on the interval from a to b or f is strictly decreasing on the interval from a to b, so in any open interval containing c there will be numbers d and e with $d < c < e$ such that either $f(d) < f(c) < f(e)$ or $f(d) > f(c) > f(e)$ respectively. Hence c is neither local maximising nor local minimising.]

A *point of inflexion* of a function is a point at which the second derivative of the function is zero.

In case (3) either $f'(x) \geq 0$ for $a < x < b$ (or $f'(x) \leq 0$ for $a < x < b$) but $f'(c) = 0$ so c is a local minimising (or local maximising point) of the *derivative* f' and thus, assuming the f is twice differentiable at c, $f''(c) = 0$. Hence, in case (3), c is a *point of infexion* of f. $\square$

It can be shown (under reasonable conditions on f) that the sign of $f'(x)$ is the same for all numbers in the interval from a to c (so that if f' takes, for instance, a positive value at one number in the interval then f' takes positive values at all numbers in the interval) and the same is true for the interval from c to b. It follows that exactly one of the cases above occurs at an isolated critical point. The results are the basis of the *sign test* (sometimes called the *first derivative test*) for determining the nature of an isolated critical point.

Sign Test Let c be an isolated critical point of a function f.

(a) If the sign of $f'(x)$ changes from positive to negative as x increases and passes c then c is a local maximising point of f.

(b) If the sign of $f'(x)$ changes from negative to positive as x increases and passes c then c is a local minimising point of f.

(c) If the sign of $f'(x)$ is the same for x on both sides of c then c is neither a local maximising point nor a local minimising point and is a point of inflexion.

For instance suppose we apply the sign test to the function g, whose *derivative* has the graph shown in Figure 6.2 (that is the graph $dy/dx = g'(x)$).

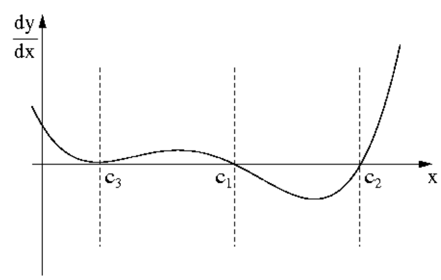

Figure 6.2: Graph of the derivative of a function

The function has three critical points at c_1, c_2 and c_3 (because g' is zero at these points). We see that $g'(x)$ changes from positve to negative as x increases and passes c_1 so we deduce that c_1 is a local maximising point. Also we see that $g'(x)$ changes from negative to positive as x increases and passes c_2 and so c_2 is a local minimising point. Finally we see that $g'(x)$ is positive on both sides of c_3 and so c_3 is neither local maximising nor local minimising. In fact (as you may have guessed) the function g whose derivative is indicated in Figure 6.2, is the function whose graph is given in Figure 6.1 and so the predictions of the sign test are confirmed in this case. Also note that c_3 is a local minimising point of the derivative g' as predicted. $\square$

There is another test for classifying critical points:

Second Derivative Test Let c be an isolated critical point of a function f.

(a) If $f''(c) < 0$ then c is a local maximising point.

(b) If $f''(c) > 0$ then c is a local minimising point.

(c) If $f''(c) = 0$ then no conclusion may be drawn (c may may be local maximising, local minimising or neither)

[Justification. (a) Suppose that $f''(c) < 0$. From the definition of limits we can deduce that

$$\frac{f'(c+h) - f'(c)}{h} < 0 \quad \text{for all } h \text{ 'near' but not equal to 0.}$$

Indeed there must be a number $\delta > 0$ such that

$$\frac{f'(c+h) - f'(c)}{h} < 0 \quad \text{if } -\delta < h < 0 \text{ or } 0 < h < \delta.$$

But c is a critical point of f so $f'(c) = 0$ and hence

$$\frac{f'(c+h))}{h} < 0 \quad \text{if } -\delta < h < 0 \text{ or } 0 < h < \delta.$$

If $-\delta < h < 0$ then $f'(c+h) > 0$ and if $0 < h < \delta$ then $f'(c+h) < 0$.

Thus $f'(x)$ changes from positive to negative as x increases and passes c and so c is local maximising point by the sign test.

(b) is justified by a similar argument.

(c) Consider the functions given by

$$f(x) = x^4, \qquad g(x) = -x^4, \qquad h(x) = x^3 \ .$$

Note first that 0 is an isolated critical point of each of the functions (indeed it is the only critical point of the functions). Since $f(x) \geq 0$ for any value of x, 0 is a local minimising (infact minimising) point. Clearly 0 is a local maximising point of g. It is clear that $h(x) < 0 = h(0)$ if $x < 0$ and $h(x) > 0 = h(0)$ if $x > 0$ and so 0 is neither local maximising or local minimising point of h. Since the value of second derivatives of each function is 0, (c) is justified.] $\square$

For interest the graph of the *second derivative* of the function whose graph was given in Figure 6.1 is given in Figure 6.3. Note that c_3 is a point inflexion. There are another two points of inflexion (one between c_3 and c_1 and one between c_1 and c_2) which are not a critical points.

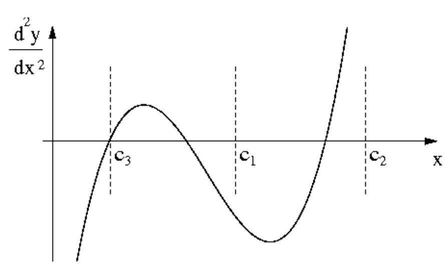

Figure 6.3: Graph of the second derivative of the function

$\square$

The second derivative test is often a lot easier to use than the sign test but it has the disadvatage that sometimes it doesn't come up with an answer (case (c) in second derivative test). However problems that arise from real life applications it is almost certain that case (c) will not occur. To compare the methods consider 6.2.1 Example.

6.2.1 Example. Find and determine the nature of the critical points of the function f given by $f(x) = x^2(x-1)$.

Solution. Differentiating gives

$$f'(x) = 2x(x-1) + x^2 = x(3x-2)$$

$f'(x) = 0$ when $x = 0$ and when $x = 2/3$. Two critical points: 0 and 2/3.

Sign test

The sign of x changes from negative to positive as x increases and passes 0 but $3x-2$ is negative for any x near 0 and so $f'(x) = x(3x-2)$ changes from positive to negative as x increases and passes 0.

The sign of $3x-2$ changes from negative to positive as x increases and passes 0 and x is positive for any x near 2/3 and so $f'(x) = x(3x-2)$ changes from negative to positive as x increases and passes 0.

Thus by the sign test 2/3 is a local minimising point.

Second derivative test

Differentiating again:

$$f''(x) = 6x - 2 \ .$$

$f''(0) = -2$ which is negative so f has a local maximum at 0.

$f''(2/3) = 4$ which is positive so f has a local minimum at 2/3.

[On this occasion the second derivative test was quicker because the second differentiation was easy and because we didn't hit an awkward case. One isn't always so lucky.] □

6.3 Global Maxima and Minima

In practical problems we are more likely to be interested in the maximum and minimum values of a quantity rather than the values which are locally maximum or locally minimum. For these problems it isn't quite enough to find the critical points, but it nearly is. □

Let f be a function and let A and B be numbers with $A < B$. We discuss the problem of finding a number c in the closed interval from A to B for which $f(c)$ is the largest (or smallest) value of f in the interval. (Recall that the closed interval from A to B consists of all the numbers between A and B together with the endpoints A and B.) Thus we want to find a number c with $A \le c \le B$ such that

$$f(x) \le f(c) \quad (\text{or } f(x) \ge f(c)) \qquad \text{for all } x \text{ such that } A \le x \le B.$$

We say that $f(c)$ is the *maximum* value of f on the interval and c is a *maximising point* if $f(x) \le f(c)$ for all x in the interval. We sometimes refer to the 'global' maximum.

We say that $f(c)$ is the *minimum* value of f on the interval and c is a *minimising point* if $f(x) \geq f(c)$ for all x in the interval. We sometimes refer to the 'global' minimum. $\square$

We remark that if c is a maximising point lying strictly between A and B (so $A < c < B$) then c is clearly a local maximising point and so is a critical point of f. Similarly if c is a minimising point.

We restrict to values in an interval because real life problems are like that. In real life situations there will be commonsense restrictions on the variables.

As an example consider the function whose graph is sketched in Figure 6.4

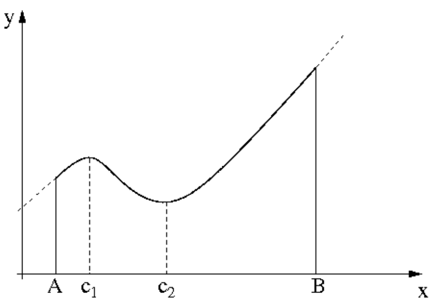

Figure 6.4: Graph of function

In the figure we see that the maximum of the function on the closed interval from A to B is achieved at B (an endpoint of the interval). The minimum is achieved at c_2 (which is not an endpoint). Thus B and c_2 are respectively maximising and minimising points on the interval.

This example reflects the basic result:

6.3.1 Theorem. *Let f be a real function and let A and B be numbers with $A < B$. Suppose that f is differentiable at each point of the closed interval from A to B. The maximum and minimum values of f on the interval are achieved and they are achieved either at a critical point lying strictly between A and B or at one of the endpoints, A or B*

The proof is beyond the scope of this course but the theorem leads to the following procedure:

1. Find all critical points c of f with $A < c < B$.

2. Calculate the value $f(c)$ at each of these critical points and the value of f at the end points A and B. Select the greatest and smallest number from these values of f.

Examples are discussed in lectures. $\square$

Engineering problems are usually phrased in terms of variables. Suppose we want to find the maximum (or minimum) value of a variable y. Our results apply in the case y depends on another variable, x say. In this case y is a function of x, call the function f then

$$y = f(x), \quad \frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{d^2y}{dx^2} = f''(x) \ .$$

The critical points of y are the values of x where $dy/dx = 0$, that is the critical points of f. Our previous results apply directly to f but clearly a maximum (or minimum) value of $f(x)$ is a maximum or minimum of y. We give a couple of examples below.

6.3.2 Example. For a certain belt drive, the power P transmitted is a function of the speed v of the belt, the law being

$$P = Tv - av^3 \ ,$$

where T is the tension in the belt and a some constant. For safety reasons the belt is not allowed to have a speed greater than 12. Find the maximum power if $T = 600$, $a = 2$ and the maximising value. Is the answer different if the maximum speed allowed is 8?

Solution. First find the critical points:

$$P = 600v - 2v^3, \qquad \frac{dP}{dv} = 600 - 6v^2$$

so dP/dv is zero when $v = \pm 10$.

We are interested in values of v between 0 and 12 that is $0 \le v \le 12$ (speed is always non-negative). We can forget about the critical point at -10.

So we have just the one relevant critical point to worry about, the one at $v = 10$. The two endpoints are $v = 0$ and $v = 12$. Next calculate P for each of these values and see which is the largest:

$$\begin{aligned}
&\text{when } v = 0, \ \ P = 0, \\
&\text{when } v = 10, \ \ P = 6000 - 2000 = 4000, \\
&\text{when } v = 12, \ \ P = 7200 - 3456 = 3744.
\end{aligned}$$

(We don't hold out a lot of hope for $v = 0$, since this would indicate that the machine was switched off, but we calculated it anyway.)

Thus the maximum power occurs when $v = 10$ and is 4000.

When the interval is reduced so that $0 \le v \le 8$, neither of the critical points is in the interval. That being the case, we just have the endpoints to worry about. The maximum this time occurs when $v = 8$ and is given by $P = 4800 - 1024 = 3776$. □

6.3.3 Example. A box of maximum volume is to be made from a sheet of card measuring 16 inches by 10 inches . It is an open box and the method of construction is to cut a square from each corner and then fold.

Solution. Let x inches be the length side of the square which is cut from each corner. Then the two sides of the box have length $16 - 2x$ and $10 - 2x$ and the volume V (in cubic inches) of the resulting box is given by

$$\begin{aligned} V &= (16-2x)(10-2x)x \\ &= 4x(8-x)(5-x) \\ &= 4(x^3-13x^2+40x) \end{aligned}$$

and so

$$\frac{dV}{dx} = 4(3x^2-26x+40)$$

The critical points occur when

$$3x^2-26x+40=0\ ,$$

that is when

$$\begin{aligned} x &= \frac{26 \pm \sqrt{676-480}}{6} \\ &= \frac{26 \pm 14}{6} \\ \text{so } x &= 2 \text{ or } 40/6 \end{aligned}$$

The commonsense restrictions are $0 \le x \le 5$. The only critical point in the interval is $x = 2$.

Now calculate of V at the critical point and the two endpoints:

$$\begin{aligned} &\text{when } x=0,\ \ V=0, \\ &\text{when } x=2,\ \ V=144, \\ &\text{when } x=5,\ \ V=0, \end{aligned}$$

so the maximum volume possible is 144, occurring when $x = 2$. □

6.4 Mean Value Theorem

A consequence of 6.3.1 Theorem is the following very useful result.

6.4.1 Theorem. *(Mean Value Theorem) Let f be a real function and suppose that f is differentiable at each point of an interval. Then for any two numbers c and d in the interval with $c < d$ there is a number t with $c < t < d$ such that*

$$f(d) - f(c) = f'(t)\,(d-c)\ .$$

Proof. Consider the function g defined by

$$g(x) = f(x) - f(c) - \frac{f(d)-f(c)}{d-c}\,(x-c)\ .$$

Differentiating gives

$$g'(x) = f'(x) - \frac{f(d)-f(c)}{d-c}$$

and also note

$$g(c) = 0, \qquad g(d) = 0 \ .$$

If g is a constant on the interval from c to d then $g'(x) = 0$ for all x between c and d so the result is trivial in this case.

So we may suppose that g is not constant. By Theorem 6.3.1, g must take a maximum or minimum value at a critical point t with $c < t < d$ (if it took both its maximum and minimum values at endpoints then g would be zero on the interval and thus constant). Since t is a maximising or minimising point $g'(t) = 0$, that is

$$f'(t) - \frac{f(d) - f(c)}{d - c} = 0$$

which gives the required result. □

Figure 6.5 indicates a possible value of t in a particular case (there are two other possible values (can you see them?))

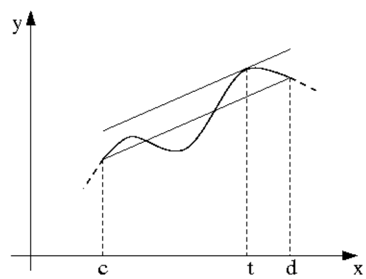

Figure 6.5: Graph of a function

A consequence of the mean value theorem is the following result which we have discussed previously.

6.4.2 Theorem. *Let f be a real function and suppose that $f'(x) = 0$ for all x in some interval. Then f is constant in the interval.*

Proof. Choose any two points c and d in the interval with $c < d$. By the mean value theorem there exits t with $c < t < d$ such that

$$f(d) - f(c) = f'(t)(d - c) \ .$$

But $f'(t) = 0$ so $f(d) = f(c)$. We have shown that f take the same value at any two points in the interval and thus f must be constant. □

When we discussed this result previously, we relied on intuition for a justification. It is good to have a proof so that we are certain it is true in all cases.

Other results introduced in subsection 3.1.2 are also consequences of the mean value theorem.

6.4.3 Theorem. *Let f be a real function and suppose that f is differentiable at each point of an interval. Let a and b be numbers in the interval with $a < b$. Suppose that $f'(x) > 0$ for all x with $a < x < b$. Then $f(x)$ is strictly increasing in the interval $a \leq x \leq b$.*

Proof. If $a \leq c < d \leq b$ then there exists t with $c < t < d$ such that

$$f(d) - f(c) = f'(t)\,(d - c)\ .$$

But $f'(t) > 0$ and $d - c > 0$ and so

$$f(d) - f(c) > 0\ .$$

□

The corresponding result for the case $f'(x) < 0$ is left as an exercise. Again these increasing and decreasing results have been mentioned previously, but now we don't have to rely on intuition for a justification.