Contents

Chapter 4

Applications and Further Differentiation

4.1 More on Derivatives

4.1.1 Rate of change

For engineers it is most important to understand that the derivative of a variable gives the 'instantaneous' rate of change of a variable in the sense described below.

Let x and y be real variables and suppose that y depends on x so there is a function f such that

$$y = f(x).$$

For a fixed value a of x let δy be the change in y if x changes from a to $a + \delta x$ so that

$$\delta y = f(a + \delta x) - f(a)$$

If $\delta x > 0$ then the *average* rate of change of y with respect to x over the interval from a to $a + \delta x$ is

$$\frac{\delta y}{\delta x} = \frac{f(a + \delta x) - f(a)}{\delta x}$$

(if $\delta x < 0$ then this is the average rate of change of y with respect to x over the interval from $a + \delta x$ to a).

If f is differentiable at a then

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \ .$$

But as δx gets smaller and smaller, $\delta y/\delta x$ becomes the average rate of change in y over smaller and smaller intervals starting (or finishing) at a. Thus the limit is interpreted as the *instantaneous* rate of change of y with respect to x at a. We can state the following important result.

If $y = f(x)$ then $\dfrac{dy}{dx} = f'(x)$ gives the *instantaneous* rate of change of y with respect to x at each value of x.

It is common practice to use the word *instantaneous* even when the variable x does not represent time. The word is frequently omitted. In future we say *'rate of change'* rather than 'instantaneous rate of change'. The next example illustrates the terminology in a particular case.

4.1.1 Example. Consider an object moving along a directed line. Let x be the *signed* distance of the object from a fixed point O on the line (as in Figure 4.1).

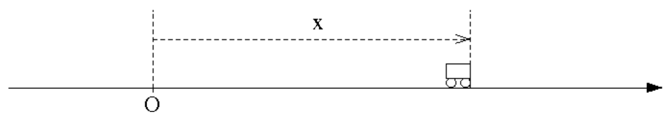

Figure 4.1: Object moving along a directed line.

Let t be the time from some instant, x will depend on t,

$$x = f(t) \text{ say, so that } \frac{dx}{dt} = f'(t).$$

For a fixed value a of t let δx be the change in x in the time interval from $t = a$ to $t = a + \delta t$, so

$$\delta x = f(a + \delta t) - f(a)\ .$$

The average rate of change of x with respect to t in the time interval from a to $a+\delta t$ if $\delta t > 0$ or in the interval from $a + \delta t$ to a if $\delta t < 0$ is

$$\frac{\delta x}{\delta t} = \frac{f(a + \delta t) - f(a)}{\delta t}\ ,$$

which is called the *average velocity* of the object in the time interval.

The *velocity* of the object at time $t = a$ is the (instantaneous) rate of change of x with respect to t when $t = a$ and, therefore, is given by

$$\left.\frac{dx}{dt}\right|_{x=a} = f'(a)\ .$$

Note that dx/dt is a variable depending on time t, at each instant it gives the velocity *at that instant.*

If δt is small then the *distance* covered in the time interval is $|\delta x|$ and so the *average speed* in the interval is $|\delta x/\delta t|$ (recall that the average speed of an object in a time interval is the distance travelled divided by the length of the time interval). It follows that the speed of the object is given by the variable $|dx/dt|$ (the absolute value of dx/dt). (If the 'object' is a car then at each instant the value of $|dx/dt|$ should be a same as the speedometer reading at each instant of time.)

We should say something about units. It has been implicitly assumed that distances are measured in metres and time in seconds and so the dx/dt gives the velocity in metres per second. If we measured distances in miles and time in hours then dx/dt would give the velocity in miles per hour. $\square$

4.1.2 Increasing and decreasing

Let f be a function and suppose that $f'(x) > 0$ whenever x is in some interval of numbers. We will refer to the interval as I. The graph of such a function is given in Figure 4.2(a).

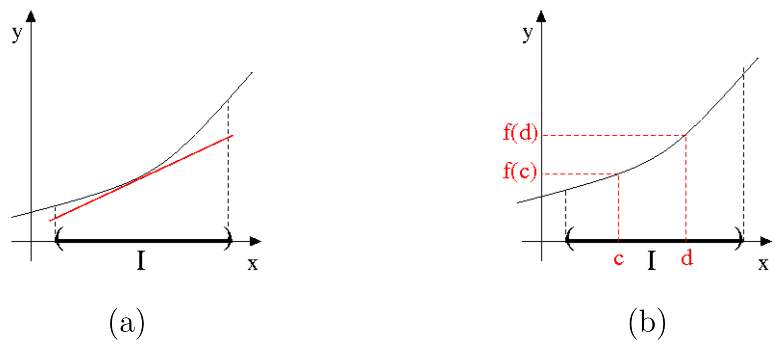

Figure 4.2: Graph of function increasing on an interval I.

By the assumption the slope of the tangent at each point of the graph over I is *positive* and so the part of the graph of f over I must 'slope upwards to the right' (as indicated by the example in Figure 4.2(a)). Thus $f(x)$ *increases* as x *increases* in the interval I.

Indeed f is *strictly increasing* on I meaning that if c and d are numbers in I with $c < d$ then $f(c) < f(d)$ (see Figure 4.2(b)). The following resullt can be proved.

If $f'(x) > 0$ for all values of x in some interval I then f is strictly increasing on I.

Now suppose that $f'(x) < 0$ whenever x is in some interval of numbers, which we again refer to as I. The graph of such a function is given in Figure 4.3(a).

In this case $f(x)$ *decreases* as x *increases* in the interval I.

Indeed f is *strictly decreasing* on I meaning that if c and d are numbers in I with $c < d$ then $f(c) > f(d)$ (see Figure 4.3(b)). The following resullt can be proved.

If $f'(x) < 0$ for all values of x in some interval I then f is strictly decreasing on I.

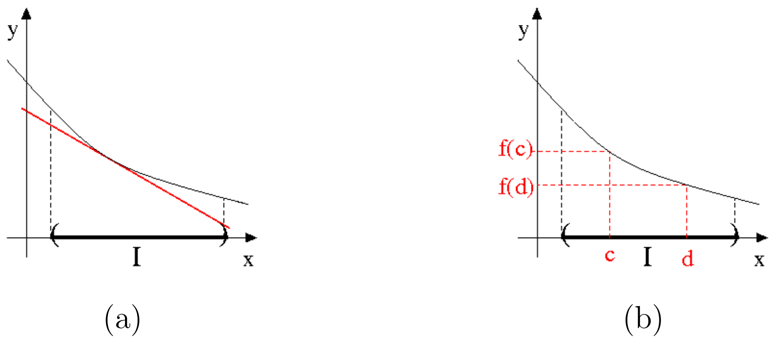

Figure 4.3: Graph of function decreasing on an interval I.

There is another important result which we mention at this stage. We know that the derivative of a constant function is 0. The converse of this result is holds.

If $f'(x) = 0$ for all values of x in some interval then f is *constant* in the interval (which means that there is a fixed number c such that $f(x) = c$ whenever x is in the interval).

An important consequence of this result is the following.

If $f'(x) = c$ (a constant) for all x in some interval then

$$f(x) = cx + d \quad \text{where } d \text{ is a constant}$$

for all values of x in the interval.

[To see this consider $g(x) = f(x) - cx$ where c is the constant value of $f'(x)$. Differentiating gives $g'(x) = f'(x) - c = c - c = 0$ and so $g(x)$ is constant in the interval. $g(x) = d$ say. Thus $f(x) = cx + d$.]

4.1.2 Example. Consider an object moving along a directed line. Let x be the signed distance from a fixed point O on the line (see Example 4.1.1). Let t be the time from some instant (x depends on t). Recall that dx/dt is the *velocity* of the object. Preceeding results show that:

- if $dx/dt > 0$ in some time interval then x increases as t increases in the interval, in other words the object is moving in the direction of the line;
- if $dx/dt < 0$ in some time interval then x decreases as t increases in the interval, in other words the object is moving in the direction opposite to that of the line.

For these reasons we usually say that dx/dt is the velocity *in the direction of the line*, a positive velocity meaning the object is moving in the direction of the line, a negative velocity meaning it is moving in the opposite direction (see Figure 4.4). Recall that

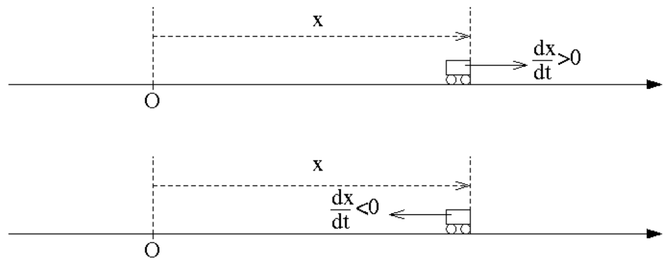

Figure 4.4: Direction of moving object.

the speed of the object is the absolute value of the velocity (so the velocity gives speed as well as direction of motion).

Also we note the following results:

- if $dx/dt = 0$ in some time interval then x is constant, in other words the object is stationary;
- if $dx/dt = c$ (where c is a constant) in some time interval then $x = ct + d$ for some constant d for all values of t in the interval. Moreover if we know the value of x at some instant then we can determine d.

If the velocity is zero then the particle is stationary. □

4.1.3 Exercise. A particle is moving along a directed line. The velocity of the particle in the direction of the line is a constant -5 metres per second. Let x be the signed distance in metres of the particle from a fixed point on the line and let t be the time in seconds from some instant. If $x = 15$ when $t = 2$ find the value of x when $t = 10$. □

4.2 Some Examples

We discuss some examples .

4.2.1 Example. A stone is thrown vertically upwards into the air so that its height in metres is given by $y = 5 + 20t - 5t^2$, where t is the time in seconds elapsed since launch.

(1) How fast is the stone travelling when $t = 1$ and when $t = 3$?

(2) When does the stone reach the maximum height ?

(3) What is its average speed between times $t = 1$ and $t = 3$?

Solution. [Note that dy/dt is the velocity of the stone in the vertical direction. The sign of dy/dt picks out the direction of travel. If dy/dt is positive, y is increasing, and so the stone is rising If dy/dt is negative, the stone is falling.]

(1) Differentiating gives $dy/dt = 20 - 10t$.

When $t = 1$: $\dfrac{dy}{dt} = 10$ and the speed is $\left|\dfrac{dy}{dt}\right| = 10$ metres per second.

When $t = 3$: $\dfrac{dy}{dt} = -10$ and the speed is $\left|\dfrac{dy}{dt}\right| = 10$ metres per second.

['How fast' is asking for the speed (not the velocity).]

(2) We have $dy/dx = 20 - 10t$ so $dy/dt = 0$ when $t = 2$.

If $0 < t < 2$ then $dy/dt > 0$ and hence the stone is rising in the time interval from $t = 0$ to $t = 2$.

If $t > 2$ then $dy/dt < 0$ and hence the stone is falling in the time interval from $t = 2$.

Since the stone is rising before time $t = 2$ and falling afterwards it must reach its greatest height when $t = 2$ (that is when $dy/dt = 0$).

(3) [The average speed is the total distance travelled divided by the time taken.]

When $t = 1$, $y = 5 + 20 - 5 = 20$ feet. When $t = 3$, $y = 5 + 60 - 45 = 20$ (so the change in height is zero). The total distance travelled is the distance travelled on the way up plus the distance travelled on the way down.

Distance travelled up $= y|_{t=2} - y|_{t=1} = (5 + 40 - 20) - 20 = 25 - 20 = 5$ metres,

distance travelled down $= y|_{t=2} - y|_{t=3} = 25 - (5 + 60 - 45) = 25 - 20 = 5$ metres.

Total distance travelled is 10 metres. Average speed is $10/2 = 5$ metres per second.

□

In Example 4.2.1 we were dealing with a situation that we could picture in our heads and the variable that we were interested in, height, was given to us explicitly in terms of the other key variable, time. Normally things aren't quite so simple.

4.2.2 Example. A ball is thrown so that its position t seconds after launch is given by $x = 32t$, $y = 32t - 16t^2$. (Here x and y are the Cartesian coordinates relative to axes with the x-axis horizontal and y-axis vertically upwards. The unit length is one foot. Air resistance is being ignored.)

(1) How far is the ball from the start at the top of its flight?

(2) At what rate is the distance between the ball and its position of launch increasing when the ball is at the top of its flight?

Solution. This time the situation is more complicated, and so we need a picture (see Figure 4.5).

When $t = 0$ the ball is at the point $(0, 0)$ (the axes were chosen so that the ball is launched at the origin of the coordinate system). The picture shows the position of the ball at some unspecified time t (and the path which we expect it to travel along).

Let w be the distance of the ball from its launch point in feet, so

$$w^2 = x^2 + y^2$$

(w is a variable which depends on t).

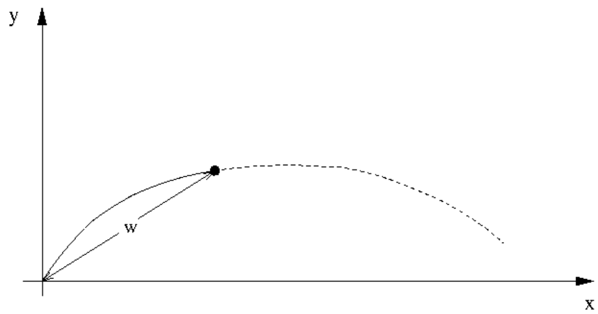

Figure 4.5: Thrown ball

(1) [As in Example 4.2.1, the ball is rising when $dy/dt > 0$, and falling when $dy/dt < 0$. Assuming the ball is rising initially, it will start to fall at the instant when dy/dt changes from positive to negative, that is when $dy/dt = 0$. This will be the instant when the ball is at the top of its flight. Our strategy will be to calculate dy/dt, find the value of t for which it is 0, and then evaluate w at this value of t.] (There is a general point to note here: if you are tackling a problem, inside mathematics or out of it, it helps if you first think about a strategy for solving it.)

Differentiating y with respect to t we get $dy/dt = 32 - 32t$, which is 0 when $t = 1$. Note that $dy/dt > 0$ when $0 < t < 1$ and is thus rising initially (until $t = 1$) and clearly falling afterwards.

When $t = 1$: $x = 32$ and $y = 16$ so $w^2 = x^2 + y^2 = 32^2 + 16^2 = 16^2 \times 5$.

Hence $w = 16\sqrt{5}$ at the top of the flight.

(2) For this we need to calculate dw/dt when $t = 1$. The procedure is to try to get w in terms of t, to differentiate and then to put $t = 1$. [Note that putting in the particular value of t comes last in the process. It is no use fixing t and then trying to differentiate !!!]

$$\begin{aligned} w^2 &= x^2 + y^2 \\ &= (32t)^2 + (32t - 16t^2)^2 \\ &= 1024t^2 + 1024t^2 - 1024t^3 + 256t^4 \\ &= 2048t^2 - 1024t^3 + 256t^4 \\ &= 256(8t^2 - 4t^3 + t^4) \end{aligned}$$

That is

$$w^2 = 256(8t^2 - 4t^3 + t^4) \tag{4.1}$$

[Takng square roots would give w explicitly as a function of t (note that w is positive so we would take the *positive* square root), we could then differentiate directly. It turns out it is a little easier to differentiate both sides of the equation (4.1) so this is what we do here. The equation is satisfied for all values of t, therefore the derivative of the left hand side of (4.1) with respect to t must be identical to the derivative of the right hand side with respect to t.]

By the chain rule,

$$\frac{d}{dt}(w^2) = \frac{d}{dw}(w^2).\frac{dw}{dt} = 2w\,\frac{dw}{dt}$$

and so differentiating (4.1) with respect to t gives:

$$2w\,\frac{dw}{dt} = 256(16t - 12t^2 + 4t^3)$$

When $t = 1$: $w = 16\sqrt{5}$ and so $\dfrac{dw}{dt} = \dfrac{256}{32\sqrt{5}}\,.\,(16 - 12 + 4) = \dfrac{64}{\sqrt{5}}$.

Thus the answer to (2) is $64/\sqrt{5}$ feet per second.

$\square$

Now we draw up a general plan for these type of problems:

- Get clear in your mind what you are trying to find and just what it is you have been given in the way of information.
- If possible, sketch a diagram of the situation in a 'general' state.
- Define the variables of interest carefully.
- Try to express the quantity you are interested in in terms of a single variable. This may involve you in doing some calculations to eliminate surplus variables.

4.2.3 Example. Water is being poured into a (right circular) cone at the rate of 5cc/sec. The cone is 10cm deep, and the angle between the side of the cone and the vertical is thirty degrees ($\pi/6$ radians). How fast is the level rising when the depth of water in the cone is 5cm?

Solution. [First sketch a diagram to see what's going on at general time. Figure 4.6 might suffice (θ in the figure is $\pi/6$).]

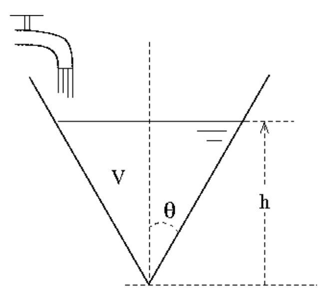

Figure 4.6: Filling the cone

[Next define the variables of interest carefully. We are given the rate of change of volume of water in the cone (with respect to time) and we want the rate of change of depth of water in the cone so we are interested in the variables volume of water and depth of water in the cone.]

Let V be the volume of water in the cone in cubic centimetres (cc), let h be the depth of water in the cone in centimetres (cm) and let t be the time in seconds from a fixed instant. (Units are chosen in line with the information given. In this case there is no obvious instant from which to measure time.)

[Get clear in your mind what you are trying to find and what you have been given. In this case we want $\dfrac{dh}{dt}$ and have been given information about $\dfrac{dV}{dt}$. So a sensible strategy is to look for the connection between h and V.]

The Engineering Mathematics Handbook tells us that $V = \dfrac{1}{3}\pi r^2 h$ (r is the radius of the cone at the top of the water).

[This formula contains three variables, which is one too many for our purposes. So look for a way of getting rid of one of them, and it is clearly r that we would like to see go.]

Trigonometry enables us to express r in terms of h: $r = h.\tan\frac{\pi}{6}$. Thus

$$V = \frac{1}{9}\pi h^3.$$

[This doesn't give h as a function of t, as written it doesn't even give h as a function of V, but none of this matters. Both V and h depend on t and the equation is satisfied for all values of t. Thus the derivative of the left hand side with respect to t is identical to the derivative of the right handside with respect to t.]

Differentiating with respect to t:

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \frac{1}{3}\pi h^2\cdot\frac{dh}{dt}$$

[We know dV/dt, we know the value of h that interests us, and so if we plug in these values, we can get the required value of dh/dt.] □

4.2.4 Example. An observer stands 100 metres from the launch pad of a rocket, which blasts off vertically so that its height at time t is given by $h = 25t^2$. How quickly is the angle of elevation (observer to rocket) increasing two seconds after launch?

Solution. First we sketch a diagram (Figure 4.7, the observer is at O).

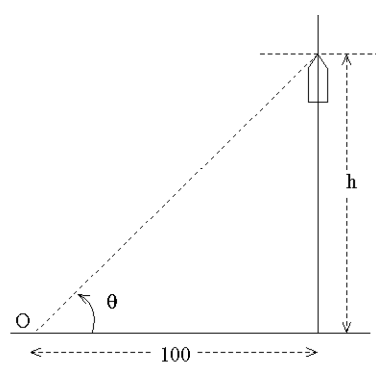

Figure 4.7: Rocket

Let θ be the angle of elevation in radians ($0 \le \theta < \pi/2$). (The other variables are defined in the question.)

We want $\dfrac{d\theta}{dt}$ when $t = 2$. From the diagram $h = 100 \tan\theta$. Hence

$$25t^2 = 100 \tan\theta.$$

[We could solve this equation to get θ as a function of t and hence calculate the derivative we need. However (at this stage) we do not know how to differentiate the resulting function of t.]

The equation is satisfied for all values of t (so the derivative of the left hand side of the equation must be identical to the derivative of the right hand side). Differentiating the equation with respect to t gives

$$50t = 100 \sec^2\theta . \frac{d\theta}{dt} . \quad \text{Hence} \quad \frac{d\theta}{dt} = \frac{1}{2} t . \cos^2\theta .$$

When $t = 2$: $h = 100$, so $\tan\theta = 100/100 = 1$, so $\theta = \pi/4$ and we obtain

$$\frac{d\theta}{dt} = \frac{1}{2} 2 . \cos^2(\tfrac{\pi}{4}) = \tfrac{1}{2} \text{ radians per second.}$$

□

4.3 Further Methods of Differentiation

4.3.1 Implicit differentiation

Let x and y be variables and suppose that y depends on x. Suppose also that x and y satisfy the equation

$$x^2 + y^2 = \sin(x + y) + 2 \tag{4.2}$$

for all values of x.

Can we work out the derivative dy/dx ?

This would be no problem if we could solve the equation to determine y in terms of x and thus write y explicitly as a function of x because we could then differentiate the function directly to get dy/dx. But in this case we cannot solve the equation.

(There is no problem if x satisfied the simpler equation, $3y + 4x = 2$, because in this case we can solve this equation for y in terms of x easily, $y = (2 - 4x)/3$ and differentiate to get $dy/dx = -4/3$.)

Even though we cannot solve equation (4.2) it is still quite easy to find a formula for the derivative.

Equation (4.2) is satisfied for all values of x, and so the the derivative of the left hand side of the equation with respect to x must be identical to the derivative of the right hand side with respect to x.

Differentiating both sides of such an equation with respect to x is called *implicit differentiation* with respect to x.

For equation (4.2) implicit differentiation gives

$$2x + 2y.\frac{dy}{dx} = \cos(x+y).\big(1 + \frac{dy}{dx}\big)\,,$$

collecting terms

$$(2x - \cos(x+y)) + (2y - \cos(x+y)).\frac{dy}{dx} = 0$$

and so

$$\frac{dy}{dx} = -\frac{2x - \cos(x+y)}{2y - \cos(x+y)}.$$

Note there is a snag, implicit differentiation gives the derivative as a function of x and y, rather than a function of just x.

At this stage we emphasise that when we differentiate (4.2) we must remember that we assumed that y depends on x and is thus a *function* of x. Hence, for instance,

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\,\frac{dy}{dx} = 2y\,\frac{dy}{dx}$$

by the chain rule. Some of you may prefer the following approach to the differentiation. Suppose that $y = f(x)$ and so from (4.2)

$$x^2 + (f(x))^2 = \sin(x + f(x)) + 2$$

for all values of x. Differentiate with respect to x to get:

$$2x + 2f(x)\,f'(x) = \cos(x + f(x))\,(1 + f'(x))$$

that is

$$2x + 2y.\frac{dy}{dx} = \cos(x+y)).\big(1 + \frac{dy}{dx}\big)$$

as before.

4.3.1 Example. Suppose that y depends on x and the equation

$$x^3 = 2x^2y^2 + 3(y-1)$$

is satisfied for all values of x. Calculate the value of dy/dx when $x = 2$ and $y = 1$.

Solution. [For the question to make sense the values $x = 2$ and $y = 1$ must satisfy the equation. They do.]

Differentiating with respect to x (implicitly) gives

$$3x^2 = 2(2x.y^2 + x^2.2y.\frac{dy}{dx}) + 3\frac{dy}{dx}$$

by the chain rule and the product rule (remembering that y depends on x). Collecting terms

$$(3x^2 - 4xy^2) + (-4x^2y - 3)\,\frac{dy}{dx} = 0 \tag{4.3}$$

Thus

$$\frac{dy}{dx} = \frac{3x^2 - 4xy^2}{4x^2y + 3}.$$

When $x = 2$ and $y = 1$:

$$\frac{dy}{dx} = (12 - 8)/(16 + 3) = 4/19.$$

□

The technique of implicit differentiation was used in some of the examples in Section 4.2.

4.3.2 Parametric differentiation

Let x and y be variables and suppose that y depends on x. In many applications we have to deal with the situation that x and y are given as functions of another variable. Suppose that

$$x = f(u), \quad y = g(u)$$

where f and g are known functions and u is a variable. In this situation u is often referred to as a *parameter*.

Can we work out the derivative of y with respect to x?

This would be no problem if we can solve the equation $x = f(u)$ to get u as an explicit function of x, $u = h(x)$ say, because then $y = g(h(x)$ which we should be able to differentiate using the chain rule.

(For instance in the (very) simple case

$$x = t + 2, \quad y = \sin(t)$$

we can solve $x = t + 2$ to get $t = x - 2$ and so $y = \sin(x - 2)$ which we can differentiate.)

In the general case it is usually very difficult to solve for u as a function of x (and even if we could then subsequent calculations could be horrendous). Fotunately there is an 'easy' way to calculate dy/dx, by *parametric differentiation.*

First note that

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

so that

$$\frac{dy}{dx} = \frac{dy}{du} \bigg/ \frac{dx}{du} \quad \text{provided} \quad \frac{dx}{du} \neq 0. \tag{4.4}$$

In this case the snag is that (4.4) gives dy/dx as a function of u.

4.3.2 Example. Let x and y be variables and suppose that

$$x = \theta - \cos\theta, \qquad y = 1 + \cos\theta\ .$$

Find $\dfrac{dy}{dx}$ when $\theta = \pi/2$.

Solution. (We assume y depends on x.) Here θ is the parameter.

$$\frac{dx}{d\theta} = 1 + \sin\theta, \qquad \frac{dy}{d\theta} = -\sin\theta\ ,$$

and so

$$\frac{dy}{dx} = \frac{-\sin\theta}{1+\sin\theta} \quad \text{provided } 1 + \sin\theta \neq 0.$$

When $\theta = \pi/2$: $1 + \sin\theta = 2 \neq 0$ and so $\dfrac{dy}{dx} = \dfrac{1}{2}$. □

4.4 Applications to Planar Curves

4.4.1 Introduction

A curve in the (x, y)-plane can be given in different ways, for instance:

- as a graph of a function;
- as an equation involving x and y;
- parametrically.

If a curve is given as the graph $y = f(x)$ then the curve s made up of all the points which can be expressed as $(x, f(x))$ for some value of x. (Sometimes the values of x are restricted, for instance

$$y = f(x), \qquad (a \leq x \leq b)$$

refers to that part of the graph which lies between the lines $x = a$ and $x = b$.) We sketched the graph $y = x^2 + 3x - 2$ in Section 2.2.

In general an equation of the form $F(x, y) = 0$ where F is a function a two variables determines a curve. The curve consists of the all the points (x, y) which satisfy the satisfy the equation. (For instance we know that the equation $x^2 + y^2 - 1 = 0$ determines the unit circle in the (x, y)-plane.) Some care must be taken because there may be no points satisfying such an equation (for example $x^2 + y^2 = -1$), or there may be just one point (for example $x^2 + y^2 = 0$).

Often the simplest way of describing a curve is 'parametrically', this means the curve is given in the form

$$x = f(u), \qquad y = g(u) \tag{4.5}$$

where f and g are functions and u is some variable called a *parameter* of the curve. In general as u increases the point (x, y) given by (4.5) will trace out a curve in the plane. The curve consists of all the points which can be expressed as $(f(u), g(u))$ for some value of u (the values of u may be restricted). An important example is the curve traced out by a particle moving in the plane. In this case the curve can be parameterised by time t,

$$x = f(t), \qquad y = g(t)$$

(the particle is at the point $(f(t), g(t))$ at time t).

4.4.2 Tangents

In the case of a curve given as a graph of the form $y = f(x)$ we know that tangents to the graph have slope given by $dy/dx = f'(x)$. Knowing the slope of the tangent means we can write down an equation of the tangent very easily.

Suppose that a curve is given in the form

$$F(x, y) = 0 \tag{4.6}$$

where F is a function a two variables. If we differentiate the equation with respect to x formally, that is *assuming* y is a function of x (implicit differentiation), then after some rearrangement we will get an equation of the form

$$f(x, y) + g(x, y).\frac{dy}{dx} = 0 \tag{4.7}$$

where f and g are functions of two variables. Thus

$$\frac{dy}{dx} = -\frac{f(x, y)}{g(x, y)} \quad \text{provided that } g(x, y) \neq 0. \tag{4.8}$$

It is not clear that (4.8) gives the slope of a tangent since we have not justified the assumption that the equation determines y as a function of x.

Let a and b be numbers such that $x = a$ and $y = b$ satisfy equation (4.6), so that (a, b) is a point on the curve and suppose that $g(a, b) \neq 0$. It can be proved under very general assumptions (implicit function theorem) that part of the curve near (a, b) can be expressed as a graph of the form $y = h(x)$. In other words the equation (4.6) does determine y as a function of x (at least if we restrict to points close to (a, b)). Moreover $h'(a) = -f(a, b)/g(a, b)$.

Thus (4.8) does indeed give the tangent to the curve determined by (4.6) at any point (a, b) on the curve for which $g(a, b) \neq 0$.

4.4.1 Example. Find the equation of the tangent to the curve

$$x - y + x^2y^2 = 1$$

at the point (1, 1).

Solution. [Note that $x = 1$, $y = 1$ satisfies the equation so the point (1, 1) is a point of the curve]

$$x - y + x^2y^2 = 1.$$

Differentiate with respect to x (implicit differentiation):

$$1 - \frac{dy}{dx} + \frac{d}{dx}(x^2y^2) = 0.$$

[Using the product rule and the chain rule (assuming y is a function of x):

$$\begin{aligned}\frac{d}{dx}(x^2y^2) &= 2xy^2 + x^2\frac{d}{dx}(y^2)\\ &= 2xy^2 + x^2.\frac{d}{dy}(y^2).\frac{dy}{dx}\\ &= 2xy^2 + x^2 2y\,.\,\frac{dy}{dx},\end{aligned}$$

and substituting into the equation....]

$$1 - \frac{dy}{dx} + \left(2xy^2 + x^2 2y\,\frac{dy}{dx}\right) = 0$$

Rearranging and collecting terms gives

$$(1 + 2xy^2) + \left(2x^2y - 1\right) . \frac{dy}{dx} = 0$$

(which is exactly of the form (4.7)). This gives

$$\frac{dy}{dx} = -\frac{1 + 2xy^2}{2x^2y - 1} \quad \text{provided } 2x^2y - 1 \neq 0.$$

When $x = 1$ and $y = 1$: $2x^2y - 1 \neq 0$ and the slope of the tangent at $(1,\, 1)$ is -3 so the tangent line at $(1,\, 1)$ has equation

$$y + 3x = 4.$$

□

Now consider the case of a curve given parametrically in the form

$$x = f(u), \qquad y = g(u)$$

If the curve can be expressed as a graph, $y = h(x)$ say, then by (4.4)

$$\frac{dy}{dx} = \frac{dy}{du} \Big/ \frac{dx}{du} \quad \text{provided } \frac{dx}{du} \neq 0.$$

In fact it may not be possible to express a curve given parametrically as a graph. However suppose that we want the slope of the tangent at the point $(x_0,\, y_0)$ on the curve given by

$$x_0 = f(u_0), \quad y_0 = g(u_0).$$

If the value of dx/du when $u = u_0$ (that is $f'(u_0)$) is not equal to zero, then it can be proved that the *part* of the curve near $(x_0,\, y_0)$ can be expressed as a graph in the form $y = h(x)$ and $dy/dx = h'(x)$ is given by (4.4). Thus the slope of the tangent at $(x_0,\, y_0)$ is

$$\left.\frac{dy}{du}\right|_{u=u_0} \Big/ \left.\frac{dx}{du}\right|_{u=u_0} = g'(u_0)/f'(u_0).$$

In most examples it is very difficult, if not impossible, to actually calculate $h(x)$. Luckily to use the formula above does not require calculating $h(x)$.

4.4.2 Example. A curve is given parametrically as

$$x = \cos^3 t\,, \qquad y = \sin^3 t$$

What is the equation of the tangent at the point where $t = \pi/3$?

Solution. [The option of getting y as a function of x is not attractive. Even if you could do it, the result is likely to be a mess and not the sort of function you would relish differentiating.]

From the results above ('t' replaces 'u')

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} .$$

Finding the derivatives of x and y with respect to t is easy enough:

$$x = \cos^3 t \quad \text{and so} \quad \frac{dx}{dt} = 3\cos^2 t(-\sin t) \quad \text{(chain rule)},$$

$$y = \sin^3 t \quad \text{and so} \quad \frac{dy}{dt} = 3\sin^2 t(\cos t).$$

Therefore

$$\frac{dy}{dx} = \frac{3\sin^2 t \cos t}{3\cos^2 t(-\sin t)} = -\tan t$$

The derivative is given as a function of the parameter t.

When $t = \pi/3$, $\tan t = \sqrt{3}$, and so the slope of the tangent at the point with this parameter value is $-\sqrt{3}$.

Also, when $t = \pi/3$: $\quad y = \frac{3\sqrt{3}}{8}, \quad x = \frac{1}{8}$ so the tangent has equation

$$\left(y - \frac{3\sqrt{3}}{8}\right) = -\sqrt{3}\left(x - \frac{1}{8}\right) .$$

□

4.5 Higher Order Derivatives

The derivative of a real function is another real function. That being so we can often differentiate the derivative, thereby getting the derivative of the derivative. These secondary and later derivatives are called *higher order derivatives* of the original function.

Let f be a function then:

- the derivative of f is denoted by f',
- the derivative of f' is denoted by f'',
- the derivative of f'' is denoted by f''' and so on.

The derivatives f', f'', f''', ... are called the first, second, third, ... order derivatives of f respectively.

To avoid too many dashes the n-th order derivative of f is denoted by $f^{(n)}$ (note the brackets round the n).

Sometimes roman numerals are used in place of (n), that is f^i, f^{ii}, f^{iii}, f^{iv}, f^v, etc.

If y depends on x, in Leibniz's notation,

- the derivative of y with respect to x is $\dfrac{dy}{dx}$,
- the derivative of $\dfrac{dy}{dx}$ with respect to x is $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, shortened to $\dfrac{d^2y}{dx^2}$,
- the derivative of $\dfrac{d^2y}{dx^2}$ with respect to x is $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$, shortened to $\dfrac{d^3y}{dx^3}$ and so on.

If $y = f(x)$ then clearly

$$\frac{dy}{dx} = f'(x), \quad \frac{d^2y}{dx^2} = f''(x), \quad \frac{d^3y}{dx^3} = f'''(x), \quad \ldots$$

The derivatives $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}$, ... are called the first, second, third , ... order derivatives of y with respect to x

The n-th order derivative of y with respect to x is denoted by $\dfrac{d^ny}{dx^n}$.

There is nothing mysterious. You know how to differentiate, and what you can do once, you can do repeatedly.

See lecture notes for examples.

The most common every day use of second order derivative is in the notion of acceleration.

Consider a particle travelling along a directed line. Let x be the (signed) distance of the particle from an origin (that is x is the Cartesian coordinate relative to the origin) and let t be the time from a fixed instant then

$$\frac{d^2x}{dt^2} \text{ is the instantaneous rate of change of } \frac{dx}{dt} \text{ with respect to } t$$

We know that dx/dt is the velocity of the particle, and so d^2x/dt^2 is the instantaneous rate of change of velocity, called the *acceleration* of the particle. (If the acceleration is positive then the velocity is increasing and if it is negative the velocity is decreasing (with respect to time).)

Now consider a particle moving in a plane. If x, y be the (Cartesian) coordinates of the particle rellative to a fixed set of axes in the plane then

$$\begin{aligned}
\frac{dx}{dt} &= \text{velocity in the direction of the } x\text{-axis} \\
\frac{dy}{dt} &= \text{velocity in the direction of the } y\text{-axis} \\
\text{and} \quad \frac{d^2x}{dt^2} &= \text{acceleration in the direction of the } x\text{-axis} \\
\frac{d^2y}{dt^2} &= \text{acceleration in the direction of the } y\text{-axis}
\end{aligned}$$

□

Implicit differentiation can be used to find higher order derivatives.

4.5.1 Example. Suppose that a variable y depends on x and satisfies

$$16x^2 = (x-1)^2(x^2+y^2)$$

Find the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ when $x = 3$ and $y = 3\sqrt{3}$.

Solution. [Note that is $x = 3$ and $y = 3\sqrt{3}$ satisfy the equation.]

Differentiating with respect to x (implicitly) gives

$$32x = 2(x-1)(x^2+y^2) + (x-1)^2\left(2x + 2y\frac{dy}{dx}\right) \tag{4.9}$$

(by the product rule and the chain rule). Cancelling the 2 which appears throughout (4.9) gives

$$16x = (x-1)(x^2+y^2) + (x-1)^2\left(x + y\frac{dy}{dx}\right), \tag{4.10}$$

Differentiating (4.10) with respect to x gives

$$\begin{aligned}
16 = (x^2+y^2) + (x-1)\left(2x + 2y\frac{dy}{dx}\right) + 2(x-1)\left(x + y\frac{dy}{dx}\right) + \\
(x-1)^2\left(1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right)
\end{aligned} \tag{4.11}$$

When $x = 3$ and $y = 3\sqrt{3}$: from (4.10)

$$dy/dx = -\sqrt{3}$$

Substituting the values $x = 3$, $y = 3\sqrt{3}$ and $dy/dx = -\sqrt{3}$ into (4.11) gives the required value of d^2y/dx^2 (this last calculation is left as an exercise). □

We can use parametric differentiation to calculate higher derivatives.

If x and y are *any* variables given as functions of a parameter u

$$x = f(u), \qquad y = g(u)$$

say. Formula (4.4) gives dy/dx as a function of u, so

$$x = f(u), \qquad \frac{dy}{dx} = k(u)$$

say. We can now apply the formula (4.4) with y replaced by dy/dx to get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = \frac{d}{du}\Big(\frac{dy}{dx}\Big) \Big/ \frac{dx}{du} = \frac{k'(u)}{f'(u)} \quad \text{provided} \quad \frac{dx}{du} \neq 0, \tag{4.12}$$

which gives d^2y/dx^2 as a function of u.

4.5.2 Example. A particle is moving round a circle of radius r and centre at the origin in the $(x,\,y)$-plane. The x and y coordinates of the particle are given by

$$x = r\cos\omega t\,, \qquad y = r\sin\omega t$$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

Solution. First get the derivatives of y and x with respect to t.

$$\frac{dy}{dt} = r\omega\cos\omega t\,, \quad \frac{dx}{dt} = -r\omega\sin\omega t\,, \quad \text{and so} \quad \frac{dy}{dx} = -\cot\omega t$$

From the (4.12) with 'u' replaced by 't':

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}$$

and so

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dt}\,(-\cot\omega t)\Big/\,(-r\omega\sin\omega t) \\ &= \big(\omega\,\mathrm{cosec}^2\,\omega t\big)/\,(-r\omega\sin\omega t) \\ &= -\frac{1}{r}\,\mathrm{cosec}^3\,\omega t\ . \end{aligned}$$

$\square$