Author	AP/IC/SD/AS
Date	17.1.013
Code	

University of Aberdeen
Department of Mathematical Sciences

EG1504 Engineering Mathematics 1
Exercises 1 (Revision)

1. Use the Pascal triangle to obtain the expansions of

$$(x+y)^7 \qquad \text{and} \qquad (x+1/x)^8.$$

2. What is the coefficient of x^2 in the expansion of $(x+1/x)^{20}$?

3. As practice in the use of 'Σ' notation, write out expansions of the following sums:

$$\sum_{n=1}^{5} n^2, \qquad \sum_{n=0}^{4} n^3, \qquad \sum_{j=-2}^{2} j(j+1).$$

4. The binomial theorem can be stated with summation notation as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This formula is true for **all** values of x and y. So you would get a true statement if you replaced x and y by particular numbers. By choosing appropriate values for x and y show that:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

5. Show that the line L passing through the points $(3,4)$ and $(6,10)$ has equation $y = 2x-2$. Find an equation of the line through the point where L meets the y-axis which is perpendicular to L. Show that these two lines and the x-axis form a triangle of area 5.

6. Find an equation of the line through the point $(1,6)$ with slope 2. Find the points P, Q where this line meets the x and y axes respectively. Find the coordinates of the point R on the line half way between P and Q. Find the equation of the circle with PQ as diameter. Find the coordinates of the points where the line $x+y=1$ meets this circle.

7. Find an equation for the circle with centre at $(1,2)$ and radius 3. Find the coordinates of the four points in which this circle meets the axes. Show that the quadrilateral formed by these four points has area $4\sqrt{10}$.

8. Use the formulae for $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$ to obtain formulae for $\sin(2x)$, $\cos(2x)$, $\sin(3x)$, $\cos(3x)$, $\sin(4x)$ and $\cos(4x)$, in terms of $\sin x$ and $\cos x$.

If you found the previous questions easy then you should attempt the three questions on the next page. (The rest of you can ignore them.)

9. A water tank has three drainage taps of different sizes. Tap A will drain the tank in twenty minutes, tap B in thirty and tap C in forty. How long will it take to drain the tank if all three are taps are opened? (You may assume that the taps do not interfere with each other's efficient operation.)

10. When you plot experimental data in the hope of discovering a physical law, the thing that you are always half hoping will appear is a straight line. Of course, not all physical laws are that simple, but you can often get round this by changing what you plot. For example, if you suspect that the quantities x, y that you have been measuring are related by a law of the form $y = a + bx^2$ for some numbers a, b, the way both to verify your suspicion and to find the values for a and b is to plot, not y against x, but y against x^2. The following experimental data gives values for the solubility of a certain chemical in water at various temperatures:

$T°C$	20	25	30	35	40	45	50
s mol m^{-3}	32.8	38.7	45.7	54.1	64.0	75.7	89.5

Draw a suitable graph to show that, allowing for small errors in observation, there is a relationship between s and T of the form $s = 16 + aT + bT^3$ and find a and b.

11. This is a calculating device known to bookmakers. It concerns the situation where a client picks three horses, A, B, C and makes a set of bets, all at the same stake. The bets are "A to win", "B to win", "C to win", "A and B both to win", "A and C both to win","B and C both to win", and "A, B and C all to win". So there are seven bets in all. The point of the trick is to enable you to work out the total payout quickly and in your head. I shall pay you the compliment of assuming you know nothing about betting on horses and explain how it works: When a bookmaker offers odds of say 3–1 against a horse, he means that each pound staked by the client will produce winnings of £3 if the horse wins; so the client hands over his pound, and if he is lucky his "return" is £4 (his original stake plus his winnings). When stating odds fractions are avoided; so you say 9–2 rather than 4.5–1. With a bet such as "A and B both to win" what happens is that your stake initially goes on horse A, and if this wins the whole of the return is wagered on horse B. It is easy to work out that the return on a successful "double" is the product of what would have been the returns on the two singles, i.e. if a £1 stake on A would have given a return r_a, and one on B a return r_b, then the return for a £1 stake on "A and B both to win" is $r_a r_b$. With a treble you multiply the three individual returns. The claim is that, if the three horses are at odds of x_1–y_1, x_2–y_2 and x_3–y_3, the return on the "three singles, three doubles and a treble" bet is one less than would be the return on a simple treble at odds of $(x_1 + y_1)$–y_1, $(x_2 + y_2)$–y_2 and $(x_3 + y_3)$–y_3. So, if the three horses are offered at 2–1, 3–1 and 7–2, the bookmaker modifies this to 3–1, 4–1 and 9–2, multiplies the three returns of 4, 5 and 5.5 getting 110, and then subtracts 1 to get the payout to the client of 109. The question is why does this give the right answer?

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EG1504 Engineering Mathematics 1
Solutions to Exercises 1 (Revision)

1. The first few rows of the Pascal triangle are:

$$\begin{array}{r|ccccccccccccccccc}
n=0 & &&&&&&&& 1 &&&&&&&& \\
n=1 & &&&&&&& 1 && 1 &&&&&&& \\
n=2 & &&&&&& 1 && 2 && 1 &&&&&& \\
n=3 & &&&&& 1 && 3 && 3 && 1 &&&&& \\
n=4 & &&&& 1 && 4 && 6 && 4 && 1 &&&& \\
n=5 & &&& 1 && 5 && 10 && 10 && 5 && 1 &&& \\
n=6 & && 1 && 6 && 15 && 20 && 15 && 6 && 1 && \\
n=7 & & 1 && 7 && 21 && 35 && 35 && 21 && 7 && 1 & \\
n=8 & 1 && 8 && 28 && 56 && 70 && 56 && 28 && 8 && 1
\end{array}$$

So $(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.

You do exactly the same thing for $(x+1/x)^8$. If in doubt, expand the expression $(x+y)^8$ and then replace y by $1/x$. The answer is

$$(x+\frac{1}{x})^8 = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56\frac{1}{x^2} + 28\frac{1}{x^4} + 8\frac{1}{x^6} + \frac{1}{x^8}$$

2. [If you feel energetic you can expand $(x+1/x)^{20}$ out in full and pick out the terms which just involve x^2 rather than some other power of x. Alternatively, you can use your head first.]

Any term in the expansion of this expression will have the form:

$$\binom{20}{m} x^{20-m} \frac{1}{x^m}.$$

The power of x in this term is $20-m-m = 20-2m$. If this is to be x^2 then we must have $m=9$. So the required coefficient is:

$$\binom{20}{9} = \frac{20!}{9!\,11!} = 167960.$$

3. The expansions are:

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

$$\sum_{n=0}^{4} n^3 = 0^3 + 1^3 + 2^3 + 3^3 + 4^3 = 100$$

$$\sum_{j=-2}^{2} j(j+1) = -2(-2+1) + (-1)(-1+1) + 0(0+1) + 1(1+1) + 2(2+1) = 10.$$

4. For the first one, put $x=y=1$. For the second one, put $x=1$ and $y=-1$.

5. The slope of L is $\dfrac{10-4}{6-3} = 2$.

Thus L has equation $y = 2x + c$ where $4 = 2 \times 3 + c$ (so $c = -2$) and hence an equation is $y = 2x - 2$.

The line L meets the y-axis when $x = 0$ and, therefore, $y = -2$. Any line perpendicular to L has slope $-\frac{1}{2}$. The one of interest passes through $(0, -2)$ so has equation

$$y + 2 = -\frac{1}{2}(x - 0) \quad \text{or} \quad 2y = -4 - x$$

This line meets the x-axis at $x = -4$. L meets x-axis at (1, 0) The required area of the triangle is $\frac{1}{2}base \times height$, which gives $2 \times 5/2 = 5$.

6. The straight line has equation $y - 6 = 2(x - 1)$. When $y = 0$ we have $x = -2$, so P is the point $(-2, 0)$. When $x = 0$ we have $y = 4$, so Q is the point $(0, 4)$. We get the coordinates of R by adding up corresponding coordinates of P and Q and dividing by 2, so $R = (-1, 2)$. The radius of the circle with PQ as diameter is half of $|PQ|$. By Pythagoras's theorem, $PQ^2 = 2^2 + 4^2$. So the radius is $\sqrt{5}$. We can now write down the equation of the circle as $(x+1)^2 + (y-2)^2 = 5$, which simplifies to $x^2 + y^2 + 2x - 4y = 0$. To find the points of intersection with the line $x + y = 1$, write the line equation as $y = 1 - x$ and substitute this expression for y in the circle equation. The result is the quadratic $2x^2 + 4x - 3 = 0$, which has roots $x = -1 \pm \sqrt{10}/2$. These two values for x are the x-coordinates of the points of intersection; to get the corresponding values for y use the equation $y = 1 - x$, getting $y = 2 \mp \sqrt{10}/2$.

7. The equation of the circle centre (a, b) and radius r is

$$(x - a)^2 + (y - b)^2 = r^2$$

In this case:

$$(x - 1)^2 + (y - 2)^2 = 9 \quad \text{or} \quad x^2 - 2x + y^2 - 4y = 4$$

When $x = 0$ the equation for y is $(y - 2)^2 = 8$ which gives the answers $y = 2 \pm \sqrt{8}$. When $y = 0$ the equation for x is $(x - 1)^2 = 5$ which gives the answers $x = 1 \pm \sqrt{5}$. The vertices of the quadrilateral are $(1 + \sqrt{5}, 0)$, $(0, 2 + \sqrt{8})$, $(1 - \sqrt{5}, 0)$, $(0, 2 - \sqrt{8})$. To get the area, think of this as two triangles each of base $2\sqrt{5}$. One has height $2 + \sqrt{8}$ and the other has height $\sqrt{8} - 2$ (be careful about the sign, the coordinate is negative). So the area of the quadrilateral is

$$\sqrt{5} \times (2 + \sqrt{8} + \sqrt{8} - 2) = 2\sqrt{5 \times 8} = 4\sqrt{10}.$$

8. For the first two, put $\theta = \phi = x$ in the formulae for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.

$$\begin{aligned}\sin(2x) &= \sin(x + x)\\ &= \sin x \cos x + \cos x \sin x\\ &= 2 \sin x \cos x\end{aligned}$$

In the same way,

$$\begin{aligned}\cos(2x) &= \cos x \cos x - \sin x \sin x\\ &= \cos^2 x - \sin^2 x\\ &= \cos^2 x - (1 - \cos^2 x)\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x \quad \text{similarly.}\end{aligned}$$

Continuing in the same vein,

$$\sin 3x = \sin(x + 2x) = \sin x \cos 2x + \cos x \sin 2x = \sin x(2\cos^2 x - 1) + \cos x(2 \sin x \cos x)$$

So

$$\sin 3x = 4\sin x\cos^2 x - \sin x$$

Similarly

$$\cos 3x = \cos x - 4\cos x\sin^2 x$$

For $\sin 4x$ and $\cos 4x$ you can either think of $4x$ as $3x + x$ or as $2x + 2x$.

$$\begin{aligned}\sin(4x) &= \sin(2x + 2x)\\ &= 2\sin 2x\cos 2x\\ &= 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)\\ &= 4\sin x\cos^3 x - 4\sin^3 x\cos x\end{aligned}$$

$$\begin{aligned}\cos(4x) &= 1 - 2\sin^2 2x\\ &= 1 - 2(2\sin x\cos x)^2\\ &= 1 - 8\sin^2 x\cos^2 x\end{aligned}$$

With all these last four you can get alternative forms for the answer by making use of $\sin^2 x+\cos^2 x = 1$. So if your answer differs from mine, try to turn it into mine by making use of this identity.

9. One way to think of this is to work out how much each tank could empty in two hours. Tap A could empty 6 tanks, tap B 4 and tap C 3. Therefore, the three together could empty 13. So, if they can empty 13 in 120 minutes, it takes them $120/13 \approx 9.23$ minutes to empty one.

The alternative approach is to think in terms of "speeds". Speed is inversely proportional to time, and the three emptying speeds add. So if t is the time the tank takes to empty when all the taps are open,

$$\frac{1}{t} = \frac{1}{20} + \frac{1}{30} + \frac{1}{40}\,.$$

10. Rearrange the supposed relationship as $s - 16 = aT + bT^3$ and then divide both sides by T. You are then looking at something of the form $y = mx + c$, i.e. at a "line equation" with y replaced by $(s - 16)/T$ and x replaced by T^2. So the plot to make is $(s - 16)/T$ against T^2. This should give you the values $a = .72$ and $b = .0003$.

11. If the odds on horse A are x_a–y_a, the winnings for a one pound stake are x_a/y_a, and so the return, r_a, is $1 + (x_a/y_a)$ (return=stake+winnings). If all three horses win, the return on the seven bets is $r_a + r_b + r_c + r_ar_b + r_ar_c + r_br_c + r_ar_br_c$. Now look at the effect on the returns of modifying the odds in the way indicated. Changing x_a–y_a to $(x_a + y_a)$–y_a would increase the winnings to $(x_a + y_a)/y_a$ i.e. to $(x_a/y_a) + 1$. So the winnings for a one pound stake have been increased by one; so also has the return. This means that the return on the modified treble is $(r_a + 1)(r_b + 1)(r_c + 1)$. Multiply this expression out and compare it with the earlier one.