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Chapter 2

Complex Numbers

2.1 Introduction

You are familiar with the following situation with regard to quadratic equations:

> The equation $x^2 = 1$ has two roots $x = 1$ and $x = -1$.
> The equation $x^2 = -1$ has no roots because you cannot take the square root of a negative number.

Long ago mathematicians decided that this was too restrictive. They did not like the idea of an equation having no solutions — so they invented them. They proved to be very useful, even in practical subjects like engineering.

Consider the general quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$. The usual formula. obtained by "completing the square" gives the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 \geq 4ac$ we are happy. If $b^2 < 4ac$ then the number under the square root is negative and you would say that the equation has no solutions. In this case we can write $b^2 - 4ac = (-1)(4ac - b^2)$ and $4ac - b^2 > 0$. So, in an obvious formal sense,

$$x = \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a},$$

and now the only 'meaningless' part of the whole formula is $\sqrt{-1}$.

So we might say that any quadratic equation either has "real" roots in the usual sense or else has roots of the form $p + q\sqrt{-1}$, where p and q belong to the real number system $\mathbb{R}$.

The expressions $p + q\sqrt{-1}$ do not make any sense as real numbers, but there is nothing to stop us from playing around with them *as symbols*. In fact, playing around with them proves to be very useful for applications to problems in differential equations, electrical circuit theory and fluid mechanics.

Although we don't discuss it formally here, there is a number system larger than $\mathbb{R}$ containing a special number j such that $j^2 = -1$, called the complex numbers, and written $\mathbb{C}$. This number system can be put on just as proper or correct a foundation as $\mathbb{R}$, and so, although we introduce it as a device to do calculations, there is no logical objection to its use. Informally, you can think of $j = \sqrt{-1}$ but remember that $(-j)^2 = -1$ too.

We call these numbers *complex numbers*; the special number j is called an *imaginary* number, even though j is just as "real" as the real numbers and complex numbers are probably simpler in many ways than real numbers.

Engineering usage is different from that of mathematicians or physicists. One of the important early uses in engineering is in connection with electrical circuits, and in particular, in calculating current flows. The symbol i is reserved for current, and so j is used for $\sqrt{-1}$, while mathematicians use i for this *imaginary* number.

For manipulations, remember that

$$\boxed{j^2 = -1}$$

2.1 Definition. A **complex number** is any expression of the form $x + jy$, where x and y are ordinary real numbers. The collection of all complex numbers is denoted by $\mathbb{C}$.

Note that all real numbers are complex numbers as well: $x = x + 0j$.

2.2 The Arithmetic of Complex Numbers

Using the 'rule' $j^2 = -1$ we can build up an 'arithmetic' of complex numbers which is very similar to that of ordinary numbers. In this Section we define what we mean by the sum, difference, product and ratio of complex numbers. All the definitions are derived by assuming that the ordinary rules of arithmetic work with the addition that $j^2 = -1$.

In what follows let a, b, c, d are ordinary real number, and let $z = a + jb$ and $w = c + jd$ be two complex numbers.

$$z = w \quad \text{iff} \quad a = c \quad \text{and} \quad b = d \qquad (\textbf{Equality})$$

$$z + w = (a + c) + j(b + d) \qquad (\textbf{Addition})$$

$$z - w = (a - c) + j(b - d) \qquad (\textbf{Subtraction})$$

$$zw = (ac - bd) + j(bc + ad) \qquad (\textbf{Multiplication})$$

$$\frac{z}{w} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \qquad (\textbf{Division})$$

Equality, Addition and Subtraction are fairly obvious. Note that the definition of equality enables us to 'equate real parts' and 'equate imaginary parts' when we have two complex numbers that are equal. The definitions of multiplication and division are more complicated. They come about as follows.

Multiplication Multiply out $(a + jb)(c + jd)$ just as you usually would

$$zw = (a + jb)(c + jd) = ac + jbc + ajd + j^2bd.$$

Now add in the information that $j^2 = -1$ and get

$$zw = ac + jbc + jad - bd = (ac - bd) + j(bc + ad),$$

as given above.

Division This is even more complicated. Start by noting that

$$(c + jd)(c - jd) = c^2 - j^2d^2 = c^2 + d^2.$$

Now use this to rearrange the quotient as follows:

$$\frac{z}{w} = \frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2},$$

again as given above.

This arithmetic works in the same way as ordinary arithmetic. You use the usual rules.

It is not really worth remembering the 'formula' for the product and quotient. It is better to calculate them in the same way that I derived the formulae.

2.2 Example. If $z = 2 + 3j$ and $w = 1 - 2j$ what are $z + w$, $z - w$, zw and z/w?

Solution. We compute using these rules.

$$\begin{aligned}
z + w &= (2 + 3j) + (1 - 2j) = (2 + 1) + (3 - 2)j = 3 + j. \\
z - w &= (2 + 3j) - (1 - 2j) = (2 - 1) + (3 + 2)j = 1 + 5j. \\
zw &= (2 + 3j)(1 - 2j) = 2 - 4j + 3j - 6j^2 = 2 - 4j + 3j + 6 = 8 - j. \\
\frac{z}{w} &= \frac{2 + 3j}{1 - 2j} = \frac{(2 + 3j)(1 + 2j)}{(1 - 2j)(1 + 2j)} = \frac{2 + 3j + 4j - 6}{1 + 4} = \frac{-4 + 7j}{5} = -\frac{4}{5} + \frac{7}{5}j.
\end{aligned}$$

2.3 Example. If $z = 5 - 6j$, $w = 2 + 7j$ and $x = -1 - j$ what are $2z + w - x$, $z^2 - x^2$, xzw, $1/z$, $z(w - 2x)$?[1]

[1] You see here the difficulty of trying to reserve particular symbols for particular meanings. Usually x is a real number, but here we have no need of symbols for real numbers, but need three different ones for complex numbers. So x gets temporarily used as a complex number.

Solution. Again computing, we have:

$$\begin{aligned}
2z + w - x &= 2(5-6j) + (2+7j) - (-1-j) = 10 - 12j + 2 + 7j + 1 + j = 13 - 4j;\\
z^2 - x^2 &= (5-6j)(5-6j) - (-1-j)(-1-j),\\
&= 25 - 60j - 36 - (1 - 1 + 2j) = -11 - 62j;\\
xzw &= (-1-j)(5-6j)(2+7j) = (-1-j)(10 - 12j + 35j + 42),\\
&= (-1-j)(52+23j) = -52 - 52j - 23j + 23 = -29 - 75j;\\
\frac{1}{z} &= \frac{5+6j}{(5-6j)(5+6j)} = \frac{5+6j}{25+36} = \frac{5+6j}{61};\\
z(w-2x) &= (5-6j)(2+7j+2+2j) = (5-6j)(4+9j),\\
&= 20 - 24j + 45j + 54 = 74 + 21j.
\end{aligned}$$

2.4 Example. What are the roots of the quadratics $x^2 + x + 1 = 0$, $x^2 - 2x + 3 = 0$?

Solution. Using the usual formula, we get for roots as follows:

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm j\sqrt{3}}{2}$$

and

$$\frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm j\sqrt{8}}{2} = 1 \pm j\sqrt{2}.$$

Powers of j. What are the powers of j? Starting from $j^2 = -1$ we get

$$j^2 = -1, \quad j^3 = -j, \quad j^4 = 1, \quad j^5 = j, \quad \ldots$$

The powers go in a cycle of length 4.

2.2.1 Square Roots

The square root of a complex number z is any complex number w such that $w^2 = z$. Given a particular z, it is not too hard to calculate the square roots of z.

2.5 Example. Let $z = 1 - 5j$; calculate the square roots of z.

Solution. Write $w = a + jb$ where a and b are real. Then

$$1 - 5j = (a + jb)(a + jb) = (a^2 - b^2) + 2jab.$$

So our problem reduces to that of solving two simultaneous *real* equations;

$$a^2 - b^2 = 1 \quad \text{and} \quad 2ab = -5$$

The second one gives $b = -5/(2a)$. Put this into the first and get

$$a^2 - \frac{25}{4a^2} = 1 \quad \text{or} \quad 4a^4 - 4a^2 - 25 = 0$$

This looks a bit worrying because it is an equation of degree 4. The trick is to notice that it is actually just a quadratic for a^2. So, by the usual formula for quadratics,

$$a^2 = \frac{4 \pm \sqrt{16+400}}{8} = \frac{1 \pm \sqrt{26}}{2}$$

Now a is definitely a real number, so its square cannot be negative. So the only possibility is that

$$a^2 = \frac{1}{2}(1+\sqrt{26}) \quad \text{and} \quad a = \pm\sqrt{\frac{1+\sqrt{26}}{2}}$$

This gives us two possible values for a. The corresponding values of b are then obtained from $b = -5/(2a)$. This gives **two** values for w, differing by a factor of -1.

2.2.2 Complex Conjugates

Let $z = x + jy$ be a complex number. We say that z is *real* if $y = 0$, and *purely imaginary* if $x = 0$. The real number x is called the *real part* of z and written $x = \Re z$. The real number y is called the *imaginary part* of z and written $y = \Im z$.

The *complex conjugate* of the complex number $z = x + jy$ is the complex number $\bar{z} = x - jy$. Thus z and $\bar{z}$ have the same real part, while $z + \bar{z}$ has 0 as its imaginary part. Note that

$$z\bar{z} = x^2 + y^2, \qquad z + \bar{z} = 2x, \qquad z - \bar{z} = 2jy.$$

2.6 Example. Let $z_1 = 2 + 3j$, $z_2 = 4j$ and $z_3 = -j$. Give the real and imaginary parts, and the complex conjugates of z_1, z_2 and z_3.

Solution. If $z_1 = 2 + 3j$ then z_1 has real part 2, imaginary part 3 and complex conjugate $\bar{z}_1 = 2 - 3j$.

If $z_2 = 4j$ then z_2 is purely imaginary and $\bar{z}_2 = -4j$.

If $z_3 = -j$ then z_3 is purely imaginary and $\bar{z}_3 = j$.

2.3 The Argand Diagram

You are familiar with the representation of real numbers as points along a line:

A complex number $z = x + jy$ is specified by *two* real numbers x and y. So it is often useful to think of a complex number as being represented by the point in a plane with Cartesian coordinates (x, y). This representation is called the **Argand diagram** or the **complex plane**.

Figure 2.1: The real line.

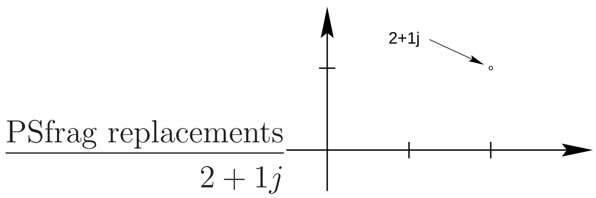


Figure 2.2: The Argand diagram or complex plane.

2.4 Modulus and Argument

Thinking in terms of the Argand diagram we can specify the position of the complex number $z = x + jy$ on the plane by giving the polar coordinates of the point (x, y).

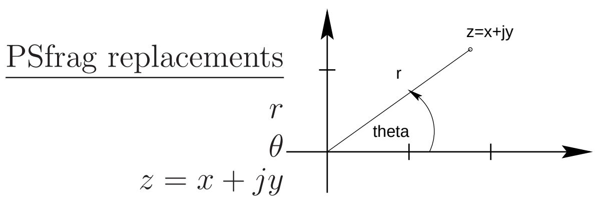


Figure 2.3: The modulus - argument representation of z.

The polar coordinate r is the distance from O to P and is called the *modulus* of the complex number z and written as $|z|$.

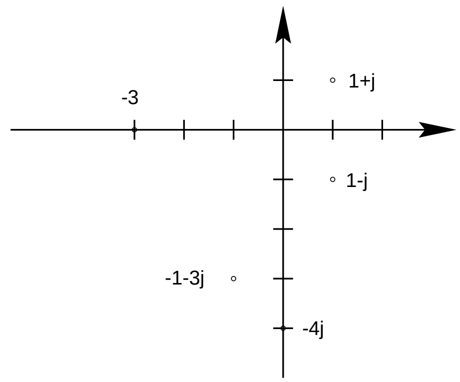
$$r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

The polar coordinate θ is called *an argument* of z. If we take θ in the range $-\pi < \theta \leq \pi$ then we call it *the (principal) argument* of z and we denote it by $\arg(z)$. Note that any argument of z differs from $\arg(z)$ by an integer multiple of 2π (working in radians) or of $360°$ (working in degrees)[2].

Since $x = r\cos\theta$ and $y = r\sin\theta$ we can write z in terms of its modulus and argument as

$$\boxed{z = r(\cos\theta + j\sin\theta)} \qquad r \geq 0, \quad -\pi < \theta \leq \pi.$$

[2]You are reminded that there is a very good reason for working in radians: the derivative of $\sin x$ is $\cos x$ *only* when the angle is measured in radians; if degrees are used there is a constant $\pi/180$ in the formula: there is a similar reason for measuring arguments in radians.



r9cm

Figure 2.4: Plotting points: Example 2.7.

This is called writing z in *polar form* or *modulus - argument form.* Any non-zero complex number can be written in this form. The point 0 is a slightly special case, it has $r = 0$ but the angle θ is not defined.

2.7 Example. Give the real and imaginary parts, complex conjugate and the modulus and argument of each of the complex numbers $z = 1 + j$, $z = 1 - j$, $z = -4j$, $z = -3$, $z = -1 - 3j$.

Solution. The given complex numbers are plotted in the complex plane in Fig 2.4.

$z = 1 + j$ has real part 1, imaginary part 1, complex conjugate $\bar{z} = 1 - j$ and modulus $|z| = \sqrt{1 + 1} = \sqrt{2}$. The argument of z is $\pi/4$.

$$z = \sqrt{2}(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}).$$

$z = 1 - j$ has real part 1, imaginary part -1, complex conjugate $\bar{z} = 1 + j$ and modulus $|z| = \sqrt{2}$. The argument of z is, according to our conventions, $-\pi/4$.

$$z = \sqrt{2}(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}).$$

Clearly $z = -4j$ has real part 0, imaginary part -4, complex conjugate $\bar{z} = 4j$ and modulus $|z| = \sqrt{0 + (-4)^2} = 4$. The argument of z is $-\pi/2$, and

$$z = 4\left(\cos -\frac{\pi}{2} + j\sin -\frac{\pi}{2}\right).$$

$z = -3$ has real part -3, imaginary part 0, complex conjugate $\bar{z} = -3 = z$ and modulus $|z| = 3$. The argument of z is, according to our conventions, π so $z = 3(\cos\pi + j\sin\pi)$.

$z = -1 - 3j$ has real part -1, imaginary part -3, complex conjugate $\bar{z} = -1 + 3j$ and modulus $|z| = \sqrt{1 + 9} = \sqrt{10}$. The argument of z has to be found with the aid of a calculator. It lies in the range $-\pi < \theta < -\pi/2$ (third quadrant) and has value

$$\theta = \arctan\frac{-3}{-1} - \pi = -1.8925.$$

It is perhaps of interest that this problem with arctan is quite common; so common that many computer languages, starting with Fortran, have two version of the function, typically called `atan` and `atan2`. The first one genuinely computes the inverse tangent function, and returns an angle between $-\pi/2$ and $\pi/2$; the second function is the "proper" one in our context andit takes the *two* arguments need to compute the angle to within 2π.

2.5 Products

Let $z = r(\cos\theta + j\sin\theta)$ and $w = s(\cos\varphi + j\sin\varphi)$ be two complex numbers in polar form. Thus $r = |z|$ and $\theta = \arg(z)$, while $s = |w|$ and $\varphi = \arg(w)$.

Consider the product of z and w:

$$\begin{aligned} zw &= rs(\cos\theta + j\sin\theta)(\cos\varphi + j\sin\varphi) \\ &= rs((\cos\theta\cos\varphi - \sin\theta\sin\varphi) + j(\sin\theta\cos\varphi + \cos\theta\sin\varphi)) \\ &= rs(\cos(\theta+\varphi) + j\sin(\theta+\varphi)) \end{aligned}$$

This tells us that the modulus of zw is just the product of the moduli of z and w:

$$|zw| = |z|\,|w|$$

and, provided we adjust the angles to the correct range by adding or subtracting multiples of 2π, the argument of the product is the sum of the arguments:

$$\arg(zw) = \arg(z) + \arg(w) \qquad (\text{modulo} \quad 2\pi).$$

For example, if z has argument $120°$ and w has argument $150°$ then an argument of zw is $120 + 150 = 270$, which is not in the right range, so we subtract $360°$ and get the principal argument, which is $-90°$ (or $-\pi/2$ radians).

Similarly, for $w \neq 0$,

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

and

$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w) \qquad (\text{modulo} \quad 2\pi).$$

2.6 De Moivre's Theorem

If we repeat the process of the above section over and over again we can show that if $z \neq 0$ and n is a positive whole number then $|z^n| = |z|^n$ and $n\arg(z)$ is an argument of z^n.

Since $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ and $\arg(1/z) = -\arg(z)$ we also get the same results if n is a negative integer. So we have the above formulae for all integer values of n ($n = 0$ is easy — check).

The result is often put in the following useful form, which is known as **de Moivre's Theorem**. If n is any whole number then

$$\boxed{z = r(\cos\theta + j\sin\theta) \quad \Rightarrow \quad z^n = r^n(\cos n\theta + j\sin n\theta)}$$

Let me tell you of one other notation at this point, which looks a bit obscure at the moment but which you will meet a lot in later years:

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

So

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta} \quad \text{and} \quad z^n = r^n e^{nj\theta}.$$

2.7 The Roots of Unity

The problem here is to solve the equation $z^n = 1$, where n is usually a positive whole number.

Write *both* sides of the equation in polar form. Let z have polar form

$$z = r(\cos\theta + j\sin\theta) \quad \text{so that} \quad z^n = r^n(\cos n\theta + j\sin n\theta)$$

We know that

$$1 = 1(\cos 0 + j\sin 0)$$

So our equation becomes

$$r^n(\cos n\theta + j\sin n\theta) = 1(\cos 0 + j\sin 0)$$

Now two complex numbers in standard polar form are equal if and only if their moduli and arguments are equal. In the case of the argument this statement has to be handled with care. It means 'are equal if reduced to the proper range'. So, for example, 10° and 370° count as equal from this point of view.

So we can say that $r^n = 1$ and that $n\theta$ and 0 are equal up to the addition of some multiple of 2π radians.

$$r^n = 1 \qquad n\theta = 0 + 2k\pi$$

where k is some whole number.

Since r is real and positive, the only possibility for r is $r = 1$.

The other equation gives us

$$\theta = 0 + 2\pi\frac{k}{n}.$$

This, in principle, gives us infinitely many answers! One for each possible whole number k. But not all the answers are different. Remember that changing the angle by 2π does not change the number z.

The distinct solutions, of which there are n, are given by $r = 1$ and

$$\theta = 2\pi\frac{k}{n} \qquad k = 0, 1, 2, 3, \ldots, n-1$$

and we can write these solutions as

$$z_k = \cos\theta_k + j\sin\theta_k \quad \text{where} \quad \theta_k = 2\pi\frac{k}{n} \qquad k = 0, 1, 2, \ldots n-1$$

That looks rather complicated. It becomes a lot simpler if you think in terms of the Argand diagram. All the solutions have modulus 1 and so lie on the circle of radius 1 centred at the origin. The solution with $k = 1$ is just $z = 1$. The other solutions are just $n - 1$ other points equally spaced round this circle, with angle $2\pi/n$ between one and the next. This is illustrated in Fig 2.5 in fact for the case $n = 17$.

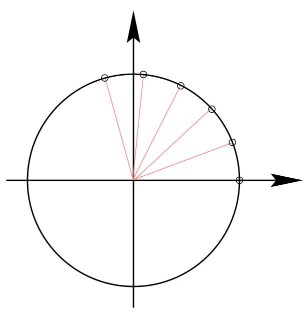


Figure 2.5: The n^{th} roots of 1.

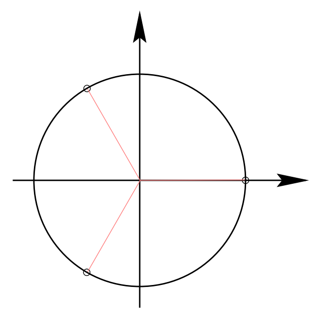


Figure 2.6: The three cube roots of 1.

Let's look at some specific examples. The cube roots of unity are the solutions to $z^3 = 1$. There are three of them and they are

$$z_0 = 1, \qquad z_1 = \cos 2\pi/3 + j\sin 2\pi/3, \qquad z_2 = \cos 4\pi/3 + j\sin 4\pi/3$$

Note that $z_2 = \bar{z}_1$, $z_2 = z_1^2$ and $1 + z_1 + z_2 = 0$. the roots are shown in Fig 2.6.

Similarly the fourth roots of unity are the solutions of $z^4 = 1$ and these are

$$z = 1, \quad z = j, \quad z = -1, \quad z = -j.$$

A picture for $n = 4$ together with those for $n = 5$ and $n = 6$ is given in Fig 2.7

We can do other equations like this in much the same way.

2.8 Example. Find the solutions of the equation $z^4 = j$.

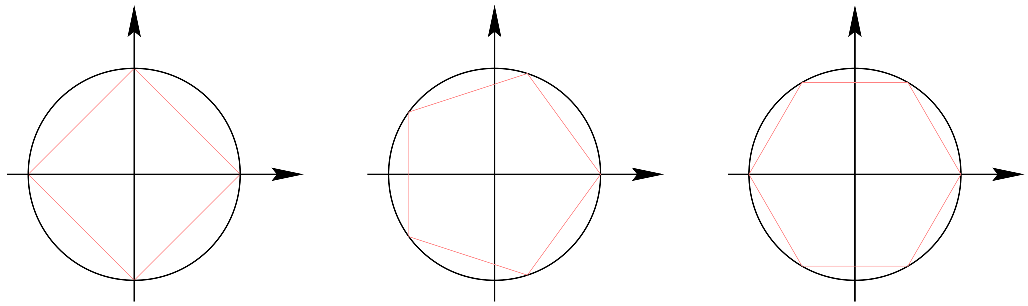


Figure 2.7: The n^{th} roots of 1 for $n = 4, 5, 6$.

Solution.

Put $z = r(\cos\theta + j\sin\theta)$. Then $z^4 = r^4(\cos 4\theta + j\sin 4\theta)$. We know that $j = 1(\cos\pi/2 + j\sin\pi/2$. So our equation becomes

$$r^4(\cos 4\theta + j\sin 4\theta) = 1(\cos\pi/2 + j\sin\pi/2)$$

Therefore

$$r = 1 \quad \text{and} \quad 4\theta = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad \theta = \frac{\pi}{8} + k\frac{\pi}{2}$$

There are 4 distinct solutions, given by $k = 0, 1, 2, 3$. They form a square on the unit circle.

2.8 Polynomials

We have learned how to manipulate complex numbers, and suggested that they will prove valuable in Engineering calculations. The original motivation for introducing them was to give the equation $x^2 = -1$ two roots, namely j and $-j$, rather than it having no roots. It turns out that this is *all* we have to do to ensure that every polynomial has the *right* number of roots. We now discuss this, and a number of other basic results about polynomials, that are quite useful to know.

A *polynomial* in x is a function of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a's are (real or complex) numbers and $a_n \neq 0$. For example

$$p(x) = x^3 - 2x + 4, \quad q(t) = 5t^8 - t^4 + 6t^3 - 1.$$

The highest power in the polynomial is called the *degree* of the polynomial. The above examples have degrees 3 and 8.

A number a (real or complex) is said to be a *root* of the polynomial $p(x)$ if $p(a) = 0$. Thus $x = 1$ is a root of $x^2 - 2x + 1$.

The first important result about polynomials is that a number a (real or complex) is a root of the polynomial $p(x)$ if and only if $(x - a)$ is a *factor* of $p(x)$, in the sense that we can write $p(x)$ as

$$p(x) = (x - a)q(x)$$

where $q(x)$ is another polynomial. This result is often called the *remainder theorem* . For example, $x = 2$ is a root of $p(x) = x^3 + x^2 - 7x + 2$ and it turns out that

$$p(x) = (x - 2)(x^2 + 3x - 1).$$

Note that necessarily the polynomial q has degree one less than the degree of p.

It may be the case that you can pull more than one factor of $(x-a)$ out of the polynomial. For example, 2 is a root of $p(x) = x^3 - x^2 - 8x + 12$ and it turns out that

$$p(x) = (x - 2)(x - 2)(x + 3).$$

In such cases a is said to be a *multiple root* of $p(x)$. The *multiplicity* of the root is the number of factors $(x - a)$ that you can take out. In the above example, 2 is a root of multiplicity 2, or a *double root*. A root is called a *simple root* if it produces only one factor. Multiple roots are a considerable pain in the neck in many applications, but they have the advantage that the Fundamental Theorem of Algebra, Theorem 2.9 takes a simple form.

There is a simple test for multiplicity. Suppose a is a root of $p(x)$, so that $p(a) = 0$. If, in addition, $p'(a) = 0$ (derivative) then a is a multiple root. To take the above example: $p(x) = x^3 - x^2 - 8x + 12$. We have $p'(x) = 3x^2 - 2x - 8$ and $p(2) = 0$ and we have $p'(2) = 0$, so we know that 2 is a multiple root.

Let me prove this result. Since a is a root of $p(x)$, we can write $p(x) = (x - a)q(x)$ where $q(x)$ is another polynomial. By the product rule,

$$p'(x) = (x - a)q'(x) + q(x).$$

So $p'(a) = 0.q'(a) + q(a) = q(a)$. Since $p'(a) = 0$ we have $q(a) = 0$. But this means that $q(x)$ has $(x - a)$ as a factor — and hence that $p(x)$ has $(x - a)$ as a factor more than once.

You should check the converse: if a is a multiple root of $p(x)$ (so that $p(x) = (x-a)^2 q(x)$ for some polynomial $q(x)$) then $p(a) = p'(a) = 0$.

The next result is fundamental. I am not going to attempt to prove it in detail; it requires some rather fancy mathematics!.

2.9 Theorem (Fundamental Theorem of Algebra)**.** *Let p be* any *polynomial of degree n. Then p can be factored into a product of a constant and n factors of the form $(x-a)$, where a may be real or complex.*

Also, the factorisation is unique; you cannot find two essentially different factorisations for the same polynomial. The factors need not all be different because of multiple roots. The fact that there cannot be *more than* n such factors is fairly obvious, since we would have the wrong degree. What is not at all obvious is that we have all the factors that we want. Note that this result does not tell you how to find these factors; just that they must be there!

The result is often stated loosely as: a polynomial of degree n must have exactly n roots. You have to allow complex roots or the theorem is not true. For example $p(x) = x^2 + 1$ has no real roots at all. Its roots are $x = \pm j$ and it factorises as $p(x) = (x - j)(x + j)$.

We have already seen this result in action when solving equations earlier in the Chapter. I told you then that you can take it for granted that an equation like $z^7 = 2 + j$ will have exactly 7 solutions. In fact, if $w \neq 0$ then $p(z) = z^n - w$ $(n \geq 1)$ always has exactly n *distinct* roots because we know that it must have n roots in all and it cannot have any multiple roots because $p'(z) = nz^{n-1}$ has only 0 as a root and 0 is not a root of $p(z)$.

There is one other result about roots of polynomials that is worth knowing. Suppose we have a polynomial with *real*, as opposed to complex, coefficients. Suppose that the complex number z is a root of the polynomial. Then the complex conjugate $\bar{z}$ is also a root. So you get two roots for the price of one. You can see this in the example of the previous paragraph. $x^2 + 1$ has j as a root, so it automatically must have $-j$ as a root as well.

2.10 Example. Let $p(z) = z^4 - 4z^3 + 9z^2 - 16z + 20$. Given that $2 + j$ is a root, express $p(z)$ as a product of real quadratic factors and list all four roots, drawing attention to any conjugate pairs.

Solution. Since p has real coefficients, and complex roots occur in pairs consisting of a root and its complex conjugate. Given that $2 + j$ is a root, it follows that $2 - j$ must also be a root, and so the quadratic

$$(z - (2 + j))(z - (2 - j)) = z^2 - 4z + 5$$

must be a factor. Dividing the given polynomial by this factor gives

$$p(z) = z^4 - 4z^3 + 9z^2 - 16z + 20 = (z^2 - 4z + 5)(z^2 + 4).$$

The roots of $z^2 + 4$ are $2j$ and its complex conjugate, $-2j$. Thus the given polynomial, of degree four, has two pairs of complex conjugate roots.

Having seen how useful the result can be in practice, let me give a proof, because it is really a very simple manipulation with complex conjugates.

2.11 Proposition. *Let P be a polynomial with real coefficients, and assume that $p(z_0) = 0$. Then $p(\bar{z}_0) = 0$.*

Proof. Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

and assume that $a_0,\ a_1, \ldots a_n \in R$. Thus p is a polynomial with real coefficients. Let $p(z_0) = 0$, so that

$$a_0 + a_1 z_0 + a_2 z_0^2 + \cdots + a_n z_0^n = 0.$$

We have

$$p(\bar{z}_0) = a_0 + a_1 \bar{z}_0 + a_2 \bar{z}_0^2 + \cdots + a_n \bar{z}_0^n$$

and since each coefficient is real,

$$\begin{aligned} &= \bar{a}_0 + \bar{a}_1 \bar{z}_0 + \bar{a}_2 \bar{z}_0^2 + \cdots + \bar{a}_n \bar{z}_0^n \\ &= p(\bar{z_0}) = 0 \quad \text{since } z_0 \text{ is a root of } p. \end{aligned}$$

Thus $\bar{z}_0$ is a root of p as claimed.

Of course if $z_0 \in \mathbb{R}$, the result tells us nothing, since in that case $z_0 = \bar{z}_0$. But as we saw in the example, if we have found one complex root, we can immediately get hold of another one; the complex roots come in pairs.

2.12 Example. Express $z^5 - 1$ as a product of real linear and quadratic factors.

Solution. We rely on our knowledge of the n^{th} roots of unity from Section 2.7. Let

$$\alpha = \exp\left(\frac{2\pi j}{5}\right) = \cos\left(\frac{2\pi}{5}\right) + j \sin\left(\frac{2\pi}{5}\right)$$

Then the roots of $z^5 - 1 = 0$ are α, α^2, α^3, α^4 and 1 and

$$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1) = (z-1)(z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4).$$

For convenience, write $\beta = \alpha^2$, and note that $\bar{\beta} = \alpha^3$ while $\bar{\alpha} = \alpha^4$. Our problem is to factorise $z^4 + z^3 + z^2 + z + 1$ as a product of real quadratic factors. We know the roots are α, $\bar{\alpha}$, β and $\bar{\beta}$. Now construct the quadratic with roots α and $\bar{\alpha}$. We have

$$(z-\alpha)(z-\bar{\alpha}) = z^2 - (\alpha + \bar{\alpha}) + \alpha.\bar{\alpha} = z^2 - 2\Re(\alpha) + 1$$

where $\Re(\alpha)$ is the real part of α. Since $(z-\beta)(z-\bar{\beta})$ behaves in the same way, we have

$$\begin{aligned} z^5 - 1 &= (z-1)(z^2 - 2\Re(\alpha) + 1)(z^2 - 2\Re(\beta) + 1), \\ &= (z-1)\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right) + 1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right) + 1\right). \end{aligned}$$

and this is a product of real linear and quadratic factors.