UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

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EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 5 (DIFFERENTIATION)

1. Two rockets are launched vertically from pads 100 metres apart, with the second launch two seconds after the first. Both ascend in such a way that t seconds after launch they are at height of $25t^2$ metres. At what rate is the distance between them changing 2 seconds after the launch of the second rocket?

2. Two straight roads lead off from a junction, making an angle of sixty degrees ($\pi/3$ radians) with each other. A walker sets off down one at a speed of 3 mph, and an hour later a runner sets off down the other at a speed of 10 mph. At what rate is the distance between them changing 1 hour after the runner sets off? [You will need to use the cosine formula, which was mentioned briefly in the second lecture. Ask a tutor if you have forgotten it.]

3. Let x and y be variables and suppose that y depends on x. Show that if $xy + \sin y = 2$ then

$$\frac{dy}{dx} = \frac{-y}{x + \cos y}$$

4. A curve has equation $(y + 2)x^2 = y^2(6 - y)$. Find dy/dx at the point $x = 2$, $y = 2$. Write down an equation of the tangent at the point $(2, 2)$.

5. Let P be the point where the curve $x^2 + 3xy + 2y^2 + 4x + y = 10$ meets the positive y-axis. Find P and find an equation of the tangent line to the curve at P.

6. Suppose that $x = 4t^2 + 2t$ and $y = 3t^2 - t$. Find dy/dx in terms of t.

7. Consider the curve given parametrically by $x = t^2$, $y = t^3$. Find an equation of the tangent line to this curve at the point $(1, 1)$.

8. Let $f(x) = 3x^3 + 2x^{-3}$. Calculate $f''(x)$, $f'''(x)$ and $f^{iv}(x)$.

9. If $y = \sin x$ what are the first five derivatives of y with respect to x ? What are $y^{(25)}$ and $y^{(101)}$?

10. Show that $p(x) = \alpha \sin x + \beta \cos x$, where α and β are constants, satisfies the equation $p'' = -p$.

11. A curve is defined by the equation $xy^3 - 2x^2y^2 + x^4 = 1$. Show that the point (1, 2) lies on the curve and find the value of dy/dx and d^2y/dx^2 at this point.

12. Newton's law says that force equals mass times acceleration; Hooke's law says that the tension in a spring is proportional to the extension of the spring beyond its natural length. If you put these two together you get the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where x is the extension, t is time (both in appropriate units) and ω some constant. Show that $x = \sin \omega t$ is a solution to this differential equation (that is it satisfies the equation for all values of t). Show that $x = \cos \omega t$ is also a solution, and then show that, for any constants A and B, $x = A \sin \omega t + B \cos \omega t$ is a solution.

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EG1504 ENGINEERING MATHEMATICS 1
SOLUTIONS TO EXERCISES 5 (DIFFERENTIATION)

1. The second rocket it is at height $25t^2$ metres t seconds after its launch. The first rocket has been travelling for $t+2$ seconds, and so its height is $25(t+2)^2$. So the difference between their heights is $25(t^2+4t+4-t^2) = 100(t+1)$. The horizontal difference between them is 100m. Therefore the distance, s, between them is given by

$$s^2 = 100^2(t+1)^2 + 100^2 \ .$$

Putting $t=2$ into this equation shows that when $t=2$ we have $s = 100\sqrt{10}$. Differentiating through the equation gives

$$2s\frac{ds}{dt} = 100^2 2(t+1) \ .$$

So when $t=2$ we get $2.100\sqrt{10}\frac{ds}{dt} = 100^2.6$, and hence $\frac{ds}{dt} = 30\sqrt{10}$ metres per second.

2. The same routine as in the last question: t hours after the runner sets off he has travelled $10t$ miles. The walker has been travelling for $t+1$ hours and has travelled $3(t+1)$ miles. The cosine formula tells us that if s is the distance between the two people,

$$\begin{aligned} s^2 &= (10t)^2 + (3(t+1))^2 - 2.10t.3(t+1)\cos(\pi/3) \\ &= 100t^2 + 9(t+1)^2 - 30(t^2+t) \end{aligned}$$

When $t=1$: $s = \sqrt{76}$.

Differentiating gives

$$2s\frac{ds}{dt} = 200t + 18(t+1) - 30(2t+1)$$

When $t=1$: $\frac{ds}{dt} = \frac{73}{\sqrt{76}}$.

3. Differentiating through the equation

$$\frac{d}{dx}(xy + \sin y) = 0$$

gives

$$y + x\frac{dy}{dx} + \cos y.\frac{dy}{dx} = 0 \quad \text{hence} \quad \frac{dy}{dx} = \frac{-y}{x+\cos y} \ .$$

4. To get dy/dx you differentiate both sides of the equation (implicit differentiation). The result will be an equation linking x, y and dy/dx. From it you will be able to calculate dy/dx.

The left hand side is the product of $(y+2)$ and x^2. So its derivative is $\frac{dy}{dx}.x^2 + (y+2).2x$. The right hand side is $6y^2 - y^3$. Differentiating this (using the chain rule) we get $(12y - 3y^2)\frac{dy}{dx}$ because $\frac{d}{dx}(6y^2 - y^3) = \frac{d}{dy}(6y^2 - y^3).\frac{dy}{dx}$. So we have

$$\frac{dy}{dx}.x^2 + (y+2)2x = (12y - 3y^2)\frac{dy}{dx}$$

When $x=2$ and $y=2$: $dy/dx = 2$.

So an equation of the tangent is $y - 2 = 2(x-2)$, i.e. $y = 2x - 2$.

5. The curve meets the y-axis when $x = 0$:

$$0 + 0 + 2y^2 + 0 + y = 10 \quad \text{or} \quad 2y^2 + y - 10 = 0$$

Solving this quadratic for y we get $y = 2$ and $y = -5/2$. So the curve cuts the *positive* y-axis at $(0, 2)$.

Now differentiate:

$$2x + 3y + 3x\frac{dy}{dx} + 4y\frac{dy}{dx} + 4 + \frac{dy}{dx} = 0$$

When $x = 0$ and $y = 2$ we get $dy/dx = -10/9$.

So an equation of the tangent line at P is $9y = -10x + 18$.

6. Using the parametric differentation formula:

$$\frac{dx}{dt} = 8t + 2 \qquad \frac{dy}{dt} = 6t - 1$$

So

$$\frac{dy}{dx} = \frac{6t - 1}{8t + 2}$$

7. To use parametric differentiation find the value of t for which $x = 1$ and $y = 1$, this is easy because we want $t^2 = 1$ and $t^3 = 1$, the second of these tells us that $t = 1$ (not -1). We have to find the slope (dy/dx) at $t = 1$.

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2 \qquad \frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$$

So the curve has slope $3/2$ at $(1, 1)$ and the tangent line has equation $y - 1 = (3/2)(x - 1)$.

[In fact you don't need parametric differentiation in this case since $t = x^{\frac{1}{2}}$ so $y = x^{\frac{3}{2}}$ and hence $dy/dx = (3/2)x^{\frac{1}{2}}$.]

8. We have $f(x) = 3x^3 + 2x^{-3}$ and we use $d/dx(x^n) = nx^{n-1}$.

$$f'(x) = 9x^2 - 6x^{-4}; \quad f''(x) = 18x + 24x^{-5}; \quad f'''(x) = 18 - 120x^{-6}; \quad f^{iv}(x) = 720x^{-7}.$$

9. The first few derivatives are, in order:

$$\cos x, -\sin x, -\cos x, \sin x, \cos x, -\sin x, -\cos x, \sin x \ldots$$

It should now be fairly obvious that we have a pattern here. The derivatives run in a cycle of *four*: sin,cos,$-$sin,$-$cos and back to sin. To find any derivative of $\sin x$ we just have to count up in fours. Derivatives number $0, 4, 8, 12, \ldots$ are all $\sin x$, derivatives number $1, 5, 9, 13, \ldots$ are all $\cos x$, derivatives number $2, 6, 10, 14, \ldots$ are all $-\sin x$ and finally, derivatives number $3, 7, 11, 15, \ldots$ are all $-\cos x$.

You can say it in a more condensed way:

$$\frac{d^{4n}}{dx^{4n}}\sin x = \sin x$$
$$\frac{d^{4n+1}}{dx^{4n+1}}\sin x = \cos x$$
$$\frac{d^{4n+2}}{dx^{4n+2}}\sin x = -\sin x$$
$$\frac{d^{4n+3}}{dx^{4n+3}}\sin x = -\cos x$$

So, in particular, the 25$^{\text{th}}$ derivative is $\cos x$ and the 101$^{\text{st}}$ derivative is also $\cos x$.

10. We just have to work out the second derivative of $p(x)$, plug it into the equation and see whether or not it fits:

$$p'(x) = \alpha \cos x - \beta \sin x$$
$$p''(x) = -\alpha \sin x - \beta \cos x = -p(x)$$

(as required).

11. Clearly (1, 2) lies on the curve because $x = 1$, $y = 2$ satisfy the equation. We want to find the derivative of y with respect to x:

$$y^3 + 3xy^2 \frac{dy}{dx} - 4xy^2 - 4x^2 y \frac{dy}{dx} + 4x^3 = 0$$

Plug in $x = 1$ and $y = 2$ (which does satisfy the equation) and get

$$8 + 12\frac{dy}{dx} - 16 - 8\frac{dy}{dx} + 4 = 0 \quad \text{hence} \quad \frac{dy}{dx} = 1.$$

Now differentiate through again

$$3y^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} + 6xy \frac{dy}{dx}\frac{dy}{dx} + 3xy^2 \frac{d^2y}{dx^2} - 4y^2 - 8xy \frac{dy}{dx} - 8x \frac{dy}{dx} - 4x^2 \frac{dy}{dx}\frac{dy}{dx} - 4x^2 y \frac{d^2y}{dx^2} + 12x^2 = 0$$

Into this put $x = 1$, $y = 2$ and $dy/dx = 1$:

$$12 + 12 + 12 + 12\frac{d^2y}{dx^2} - 16 - 16 - 16 - 4 - 8\frac{d^2y}{dx^2} + 12 = 0$$

and so $d^2y/dx^2 = 1$.

12. If $x = \sin \omega t$ we have $dx/dt = \omega t \cos \omega t$ and $d^2/dx^2 = -\omega^2 \sin \omega t$. Substituting shows that the equation is satisfied for all values of t. The argument is similar for $x = \cos \omega t$, and not really any different for the last part. The second derivative of $A \sin \omega t + B \cos \omega t$ is $-\omega^2(A \sin \omega t + B \cos \omega t)$.