UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

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EG1504 ENGINEERING MATHEMATICS 1
EXERCISES 2 (COMPLEX NUMBERS)

1. Express the following complex numbers in the form $x + jy$ where x and y are real.

$$(2+3j)(1-j), \qquad \frac{2+3j}{1-j}.$$

2. Mark the positions of each of the following complex numbers on a diagram of the Complex Plane and calculate each modulus and argument ($-\pi < \theta \leqslant \pi$).

$$j, \qquad 2, \qquad -2, \qquad 1+j, \qquad -1-j, \qquad \frac{1-j\sqrt{3}}{2}, \qquad \frac{-1-j\sqrt{3}}{2}.$$

3. Let z and w be defined by

$$z = \frac{1}{1+2j}, \qquad \frac{1}{w} = \frac{2}{1-3j} + \frac{2}{3+j}.$$

Find the real and imaginary parts of z and w.

4. **(a)** Let $z = 1-2j$ and $w = 3+j$ be complex numbers. Express each of the following complex numbers in the form $x + jy$ where x and y are real numbers.

$$w - 2z, \quad \frac{1}{z}, \quad \left|\frac{w-\bar{w}}{w+\bar{w}}\right|.$$

(b) Express the complex number

$$z = -\sqrt{3} + j$$

exactly in modulus - argument form. Hence find the modulus and principal argument of z^4.

5. Find the square roots of the following complex numbers

$$2j, \qquad -3, \qquad 3-4j, \qquad -5+12j.$$

6. **(a)** Express the following complex numbers in the form $x + yj$ where x and y are real.

$$(-1+2j)(5-4j), \qquad \frac{2+3j}{1-2j}.$$

(b) Let $z = 2-2j$. Find $\bar{z}$, $|z|$, $\arg(z)$, and write z in polar form.

7. Find the real and imaginary parts, the modulus and the argument of the complex number

$$z = \frac{(1+j)^2}{1-j}.$$

By using the polar form of z, or otherwise, find the modulus and argument of z^3.

8. Write the following complex numbers in polar form:

$$z = -1 + \sqrt{3}j, \qquad w = -1 - j.$$

Now use your answers and de Moivre's theorem to find z^4 and w^6. Convert the results back into Cartesian form at the end.

9. Find all solutions to the equation $z^4 = -1$ by writing both z and -1 in polar form and using de Moivre's theorem. Use the result to factorise $z^4 + 1$ into the form

$$(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)$$

where α, β, γ and δ are complex numbers.

10. Use de Moivre's theorem to find all the solutions to each of the following equations:

$$z^3 = j, \qquad z^5 = -1, \qquad z^{20} = 1.$$

In each case the answers all have modulus 1. Mark them on the Complex Plane.

11. Find all solutions w to the equation

$$w^3 = -27j$$

and mark them on an Argand diagram.

12. We know by de Moivre's theorem that for real x

$$(\cos x + j\sin x)^n = \cos nx + j\sin nx.$$

(a) Expand the left hand side of the above expression in the cases that $n = 3, 4, 5$ using the binomial theorem.

(b) Equate the real and imaginary parts of both sides to get expressions for $\sin 3x$, $\cos 4x$, $\sin 5x$ in terms of $\sin x$ and $\cos x$.

13. Let $p(z) = z^4 - 4z^3 + 9z^2 - 16z + 20$. Given that $2 + j$ is a root, express $p(z)$ as a product of real quadratic factors and list all four roots, drawing attention to any conjugate pairs.

14. Let

$$p(z) = z^5 - 5z^4 + 8z^3 - 2z^2 - 8z + 8.$$

Show that $p(2) = 0$. Show also that $z^2 - 2z + 2$ is a factor of $p(z)$. Hence write p as a product of linear factors.

15. Let

$$p(z) = z^3 - 5z^2 + 8z - 6.$$

Given that $p(1 + j) = 0$, write p as a product of linear factors.

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EG1504 Engineering Mathematics 1
Solutions to Exercises 2 (Complex Numbers)

1. Multiplying as usual,

$$(2+3j)(1-j) = 2+3j-2j+3 = 5+j;$$

the quotient needs a real denominator, so as usual,

$$\frac{2+3j}{1-j} = \frac{(2+3j)(1+j)}{(1-j)(1+j)} = \frac{2+5j-3}{1+1} = \frac{-1+5j}{2}.$$

2. The positions are given in Fig 1. Here are the moduli and arguments; j has modulus 1 and

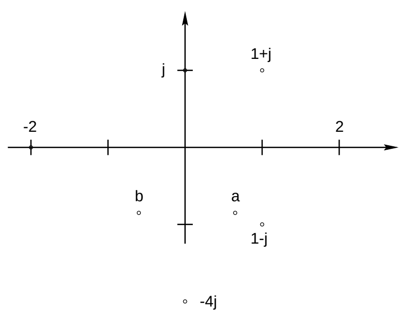

Figure 1: Positions of complex numbers on the Argand diagram; the diagram is roughly to scale, and we write $a = (1-j\sqrt{3})/2$ and $b = (-1-j\sqrt{3})/2$.

argument $\frac{\pi}{2}$; 2 and -2 both have modulus 2; 2 has argument 0 and -2 has argument π; $1+j$ has modulus $\sqrt{(1^2+1^2)} = \sqrt{2}$ and argument $\pi/4$; $-1-j$ also has modulus $\sqrt{2}$ but has argument $-\frac{3}{4}\pi$. The last two both have modulus 1 and arguments $-\frac{1}{3}\pi$,$-\frac{2}{3}\pi$.

3. We have

$$z = \frac{1}{1+2j} = \frac{1-2j}{(1+2j)(1-2j)} = \frac{1-2j}{5}.$$

Thus z has real part 0.2 and imaginary part 0.4.

Again

$$\frac{1}{w} = \frac{2}{1-3j} + \frac{2}{3+j} = \frac{2(3+j)+2(1-3j)}{(1-3j)(3+j)} = \frac{8-4j}{6-8j} = \frac{4-2j}{3-4j}.$$

Thus

$$w = \frac{3-4j}{4-2j} = \frac{(3-4j)(4+2j)}{(4-2j)(4+2j)} = \frac{12+8-16j+6j}{16+4} = 1-\frac{1}{2}j,$$

and so w has real part 1 and imaginary part 0.5.

4. **(a)** Adding, we get $3 + j - 2(1 - 2j) = 1 + 5j$. We have

$$\frac{1}{z} = \frac{1}{1-2j} = \frac{1+2j}{(1+2j)(1-2j)} = \frac{1+2j}{5}.$$

Thus $1/z$ has real part 0.2 and imaginary part 0.4.

Finally

$$\left|\frac{w-\bar{w}}{w+\bar{w}}\right| = \left|\frac{3+j-3+j}{3+j+3-j}\right| = \left|\frac{2j}{6}\right| = \frac{1}{3}.$$

This has real part 1/3 and imaginary part 0.

(b) Recall the "30, 60, 90" triangle, with sides 1, $\sqrt{3}$, and 2. Then

$$z = -\sqrt{3} + j = 2\exp\left(\frac{5\pi}{6}\right)$$

and so

$$z^4 = 16\exp\left(\frac{20\pi}{6}\right) = 16\exp\left(\frac{-2\pi}{3}\right),$$

where the second form is in terms of the modulus (16) and *principal* argument $(-2\pi/3)$ of z^4.

5. I don't mind whether you do these algebraically or by deMoivre. Since a later question uses deMoivre I will be algebraic here.

The idea is that if we want the square root of $a + jb$ then we are looking for complex numbers $x + jy$ such that

$$a + jb = (x + jy)^2 = (x^2 - y^2) + 2jxy.$$

Let's do $-5 + 12j$. Comparing real and imaginary parts we get the equations

$$x^2 - y^2 = -5, \qquad 2xy = 12.$$

From the second equation, since x cannot be zero, we get

$$y = \frac{12}{2x} = \frac{6}{x}.$$

Put this into the first equation and get

$$x^2 - \frac{36}{x^2} = -5 \quad \text{so } x^4 + 5x^2 - 36 = 0.$$

This is a quadratic for x^2 with positive solution $x^2 = 4$. So the two possibilities for x are ± 2. The corresponding values of y are therefore $y = \pm 12/4 = \pm 3$. So the square roots of $-5 + 12j$ are

$$\pm(2 + 3j).$$

Similarly, the square roots of $2j$ are $\pm(1 + j)$.

The square roots of -3 are $\pm j\sqrt{3}$.

The square roots of $3 - 4j$ are $\pm(2 - j)$.

6. **(a)** $3+14j$, $-4/5+7/5j$. **(b)** $\bar{z}=2+2j$, $|z|=\sqrt{2^2+(-2)^2}=\sqrt{8}=2\sqrt{2}$, $\arg(z)=-\pi/4$ so that in polar form we have $z=2\sqrt{2}(\cos(-\pi/4)+j\sin(-\pi/4))$.

7. We first simplify the given number:

$$z=\frac{(1+j)^2)}{1-j}=\frac{1+2j-1}{(1-j)}=\frac{2j(1+j)}{(1-j)(1+j)}=\frac{2j(1+j)}{1+1}=-1+j.$$

Thus $|z|=\sqrt{2}$, while $\arg z=3\pi/4$. We now use de Moivre's theorem, so z^3 has modulus $2\sqrt{2}$ and argument derived from $9\pi/4$. Thus

$$z^3=2\sqrt{2}\left(\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)=2+2j.$$

8. We have $z=2(\cos 2\pi/3+\sin 2\pi/3)$, so $z^4=2^4(\cos 8\pi/3+\sin 8\pi/3)=16(\cos 2\pi/3+\sin 2\pi/3)$. Hence, $z^4=8+8\sqrt{(3)}j$.

Similarly $w=\sqrt{2}(\cos(-3\pi/4)+\sin(-3\pi/4))$, so $w^6=\sqrt{2}^6(\cos(-18\pi/4)+\sin(-18\pi/4))=8(\cos(-\pi/2)+\sin(-\pi/2))$. Hence $w=-8j$.

9. Put $z=r(\cos\theta+j\sin\theta)$ where $r\geq 0$. Then by de Moivre's theorem we have

$$z^4=r^4(\cos 4\theta+j\sin 4\theta)=-1=1(\cos\pi+j\sin\pi).$$

Hence $r^4=1$ (and so $r=1$ since $r>0$) and

$$4\theta=\pi+2\pi k \text{ where } k \text{ is any integer.}$$

So, dividing through by 4,

$$\theta=\frac{\pi}{4}+\frac{k\pi}{2}$$

For answers in the range $-\pi<\theta\leqslant\pi$ take $k=1,0,-1,-2$ (alternatively, just take $k=0,1,2,3$). These give the answers

$$\theta=\frac{3}{4}\pi,\qquad \frac{1}{4}\pi,\qquad -\frac{1}{4}\pi,\qquad -\frac{3}{4}\pi,$$

which give the complex numbers

$$\frac{-1+j}{\sqrt{2}},\qquad \frac{1+j}{\sqrt{2}},\qquad \frac{1-j}{\sqrt{2}},\qquad \frac{-1-j}{\sqrt{2}}.$$

So

$$z^4+1=(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)$$

where $\alpha,\ldots,\delta$ are the above solutions.

Taking them in complex conjugate pairs we get

$$(z-\alpha)(z-\delta)=z^2+\sqrt{2}z+1$$

and

$$(z-\beta)(z-\gamma)=z^2-\sqrt{2}z+1.$$

So

$$z^4+1=(z^2+\sqrt{2}z+1)(z^2-\sqrt{2}z+1).$$

10. The answers are obtained in the same way as in the previous question. They are:

$$z^3 = j: \qquad \theta = \frac{5}{6}\pi, \quad \frac{1}{6}\pi, \quad -\frac{1}{2}\pi,$$

$$z^5 = -1: \qquad \theta = \pi, \quad \frac{3}{5}\pi, \quad \frac{1}{5}\pi, \quad -\frac{1}{5}\pi, \quad -\frac{3}{5}\pi,$$

$$z^{20} = 1: \qquad \theta = \frac{k\pi}{10} \text{ for } k = -9, \ldots, 10$$

11. We have $w = -27j = 3^3 \exp\left(-\frac{1}{2}\right)\pi j = 3^3 \exp\left(2k - \frac{1}{2}\right)\pi j$ for any integer k. Thus by de Moivres theorem, we have $w = 3\exp\left(\frac{2k}{3} - \frac{1}{6}\right)\pi j$, with distinct solutions occurring when $k = 0$, 1 and 2. Thus the three solutions are

$$w = 3\exp\left(-\frac{1}{6}\right)\pi j, \quad 3\exp\left(\frac{2}{3} - \frac{1}{6}\right)\pi j, \quad \text{and} \quad 3\exp\left(\frac{4}{3} - \frac{1}{6}\right)\pi j.$$

These can be written, in simplified form as

$$w = 3\exp\left(-\frac{\pi j}{6}\right), \quad 3j, \quad \text{and} \quad 3\exp\left(-\frac{5\pi j}{6}\right).$$

The three roots are shown in Fig. 2.

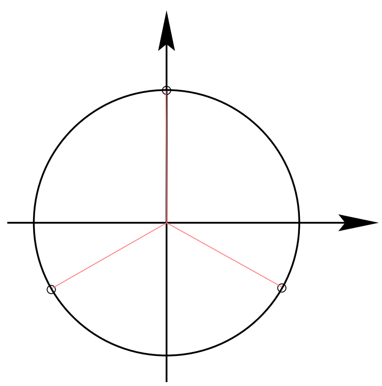

Figure 2: The three solutions of $w^3 = -27j$.

12. Using the binomial theorem on the LHS we get

$$(\cos x + j\sin x)^3 = \cos^3 x + 3j\cos^2 x \sin x - 3\cos x \sin^2 x - j\sin^3 x,$$

which tidies to give

$$\cos 3x + j\sin 3x = (\cos^3 x - 3\cos x \sin^2 x) + j(3\cos^2 x \sin x - sin^3 x).$$

So

$$\cos 3x = \cos^3 x - 3\cos x \sin^2 x \qquad \text{and} \qquad \sin 3x = 3\cos^2 x \sin x - \sin^3 x.$$

Use the fact that $\cos^2 x + \sin^2 x = 1$ to express $\cos 3x$ in terms of $\cos x$ and $\sin 3x$ in terms of $\sin x$.

The same methods yield

$$\cos 4x = \sin^4 x - 6\sin^2 x \cos^2 x + \cos^4 x = 8\cos^4 x - 8\cos^2 x + 1,$$
$$\sin 5x = \sin x(\sin^4 x - 10\sin^2 x \cos^2 x + 5\cos^4 x) = \sin x(16\sin^4 x - 20\sin^2 x + 5).$$

13. Since p has real coefficients, and complex roots occur in pairs consiting of a root and its complex conjugate. Given that $2+j$ is a root, it follows that $2-j$ must also be a root, and so the quadratic

$$(z-(2+j))(z-(2-j)) = z^2 - 4z + 5$$

must be a factor. Dividing the given polynomial by this factor gives

$$p(z) = z^4 - 4z^3 + 9z^2 - 16z + 20 = (z^2 - 4z + 5)(z^2 + 4).$$

The roots of $z^2 + 4$ are $2j$ and its complex conjugate, $-2j$. Thus the given polynomial, of degree four, has two pairs of complex conjugate roots.

14. Doing the long division of polynomials, we see that

$$p(z) = (z^2 - 2z + 2) \cdot (z^3 - 3z^2 + 4) = (z^2 - 2z + 2) \cdot q(z) \quad \text{(say)}.$$

Also $q(2) = 0$, so $p(2) = 0$. The remainder theorem now shows that $(z-2)$ is a factor of $q(z)$. Again doing the division, we have

$$p(z) = (z^2 - 2z + 2) \cdot (z-2) \cdot (z^2 - z - 2).$$

Finally we factor each of the quadratics. Using the quadratic formula on the first, the roots of $z^2 - 2z + 2$ are seen to be $1+j$ and $1-j$. Thus

$$z^2 - 2z + 2 = (z - 1 - j) \cdot (z - 1 - j).$$

The same method works on the second quadratic, or it can be factored directly to give

$$(z^2 - z - 2) = (z-2) \cdot (z+1).$$

Putting this all together shows that

$$p(z) = (z - 1 - j) \cdot (z - 1 - j) \cdot (z-2)^2 \cdot (z+1).$$

15. Since $p(1+j) = 0$ and p is a real polynomial, it follows that $1-j$ is a root and hence that

$$(z-(1+j))(z-(1-j)) = z^2 - 2z - 2$$

is a factor. Dividing, we see that $p(z) = (z^2 - 2z - 2)(z-3)$. Thus we can write p as a product of linear factors as

$$p(z) = (z-(1+j))(z-(1-j))(z-3).$$