Chapter 2
Mathematical Competencies and PISA

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Abstract The focus of this chapter is on the notion of mathematical competence and its varying role in the PISA mathematics frameworks and reports of PISA results throughout the first five survey administrations, in which mathematical literacy is a key concept. The chapter presents the genesis and development of the competency notion in Denmark, especially in the so-called KOM project, with a view to similar or related notions developed in different environments and contexts, and provides a detailed description of the eight competencies identified in the KOM project. Also the relationship between the mathematical competencies and the fundamental mathematical capabilities of the PISA 2012 Framework is outlined and discussed.

Introduction

The notion of mathematical competence—which will be introduced and discussed in greater detail below—has been present in some way or another in all the PISA mathematics frameworks from the very beginning in the late 1990s. However, the actual role of mathematical competencies in the PISA frameworks and in the reporting of PISA outcomes has been subject to considerable evolution across the five PISA surveys completed so far; that is, until 2013.

These facts provide sufficient reason for including a chapter on the role of mathematical competencies within PISA in this book. The structure of the chapter is as follows. After this introduction comes a section in which the genesis of the notion of mathematical competence is presented and its history briefly outlined. It may be worth noticing that the inception of this notion—in the specific version presented in this chapter—took place more or less at the same time but completely independently of the launching of PISA in 1997. Subsequently, the trajectories of development of mathematical competencies and PISA, respectively, became intertwined in several interesting ways. The section to follow next considers further

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aspects of the notion of mathematical competence in a general setting not specifically focused on PISA. Then comes the core of this chapter, namely an analysis and discussion of the changing role of mathematical competencies within PISA, both in relation to the mathematics frameworks of the different PISA survey administrations, and to the reporting of PISA outcomes. That section also includes a discussion of the transformation of the original competencies into a particular set of competencies that have proved significant in capturing and characterising the intrinsic demands of PISA items.

Brief History of the General Notion of Competencies and a Side View to Its Relatives

Traditionally, in most places mathematics teaching and learning have been defined in terms of a curriculum to be taught by the teacher and achieved by the student. Typically, a curriculum used to be a sequence—organised by conceptual and logical progression—of mathematical concepts, terms, topics, results and methods that people should know, supplemented with a list of procedural and technical skills they should possess. In curriculum documents, the generally formulated requirements are often accompanied by illustrative examples of tasks (including exercises and problems) that students are expected to be able to handle when/if they have achieved the curriculum.

However, there have always been mathematics educators (e.g. Hans Freudenthal (1973, 1991), who kept emphasising that mathematics should be perceived as an activity) who have insisted that coming to grips with what it means to be mathematically competent cannot be adequately captured by way of such lists. There is significantly more to be said, they believe, in the same way as no sensible person would reduce the definition of linguistic competence in a given language to lists of the words, orthography and grammatical rules that people have to know in that language. Already in the first IEA study (Husén 1967), the precursor to and generator of the later TIMSS studies, mathematics is defined by way of two dimensions, mathematical topics and five cognitive behaviour levels:

> (a) knowledge and information: recall of definitions, notations, concepts; (b) techniques and skills: solutions; (c) translation of data into symbols or schemas and vice versa; (d) comprehension: capacity to analyse problems, to follow reasoning; (e) inventiveness: reasoning creatively in mathematics. (Niss et al. 2013, p. 986)

Heinrich Winter (1995) spoke about three fundamental, general experiences that mathematics education should bring about: coming to grips with essential phenomena in nature, society and culture; understanding mathematical objects and relations as represented in languages, symbols, pictures and formulae; fostering the ability to engage in problem solving, including heuristics.

Also, the notions of numeracy, mathematical literacy, and quantitative literacy have been coined so as to point to essential features of mathematical mastery,

geared towards the functional use of mathematics, that go beyond factual knowledge and procedural skills (see also Chap. 1 in this volume). Moreover, newer curriculum documents such as the NCTM Standards of 1989 (National Council of Teachers of Mathematics 1989) also involve components that are not defined in terms of factual knowledge and procedural skills. The *Standards* identify five ability-oriented goals for all K-12 students: (1) that they learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically (NCTM 1989, p. 5).

Let these few examples suffice to indicate that lines of thought do exist that point to (varying) aspects of mathematical mastery that go beyond content knowledge and procedural skills. The notion of mathematical competence and competencies was coined and developed in the same spirit, albeit not restricted to functional aspects as above.

From the very beginning, the graduate and undergraduate mathematics studies at Roskilde University, Denmark, designed and established in 1972–1974, and continuously developed since then, were described partly in terms of the kinds of overarching mathematical insights and competencies (although slightly different words were used at that time) that graduates were supposed to develop and possess upon graduation. Needless to say, the programme documents also included a list of traditional mathematical topics that students should become familiar with. For a brief introduction to the mathematics studies at Roskilde University, see Niss (2001). In the 1970s and 1980s aspects of this way of thinking provided inspiration for curriculum development in lower and upper secondary mathematics education in Denmark.

In the second half of the 1990s executives of the Danish Ministry of Education wanted the Ministry to chart, for each school subject, what was called 'the added value' generated within the subject by moving up through the education levels, from elementary and primary (Grades K-6), over lower secondary (Grades 7–9) through to the upper secondary levels (Grades 10–12 in different streams), with a special emphasis on the latter levels. It was immediately clear to the mathematics inspectors and other key officers in the Ministry that the added value could not be determined in a sensible manner by merely pointing to the new mathematical concepts, topics and results that are put on the agenda in the transition from one level or grade to the next. But what else could be done? The officers in the Ministry turned to me for assistance, and after a couple of meetings I devised a first draft of what eventually became a system of mathematical competencies. The underlying thinking was greatly influenced by the philosophy underpinning the mathematics studies at Roskilde University. The fundamental idea was to try to answer two questions.

The first question springs from noting that any observer of mathematics teaching and learning in the education system, at least in Denmark, will find that what happens in elementary and primary mathematics classrooms, in lower secondary classrooms, in upper secondary classrooms and, even more so, in tertiary classrooms, displays a dramatic variability, not only because the mathematical topics

and concepts dealt with in these classrooms are different, but also, and more importantly, because topics, concepts, questions and claims are dealt with in very different ways at different levels—in particular when it comes to justification of statements—even to the point where mathematics at, say, the primary level and at the tertiary level appears to be completely different subjects. So, given this variability, what is it that makes it reasonable to use the same label, mathematics, for all the different versions of the subject called mathematics across education levels? Differently put, what do all these versions have in common, apart from the label itself? Next, if we can come up with a satisfactory answer to the question of what very different versions of mathematics have *in common*, the second question is then to look into how we can use this answer to account, in a unified and non-superficial manner, for the obvious *differences* encountered in mathematics education across levels.

As we have seen, the commonalities in the different versions of mathematics do not lie in any *specific* content, as this is clearly very different at different levels. Whilst it is true that content introduced at one level remains pertinent and relevant for all subsequent levels, new content is introduced at every level. The general rational numbers of the lower secondary level are not dealt with at the primary level. The trigonometry or the polynomials of the upper secondary level have no presence at the primary or lower secondary levels. The general vector spaces, analytic functions or commutative rings of the tertiary level have no place at the upper secondary level. In other words, in terms of specific content, the only content that is common to all levels are small natural numbers (with place value) and names of well-known geometrical figures. Well, but instead of specific content we might focus on more abstract *generic* content such as numbers and the rules that govern them, geometric figures and their properties, measure and mensuration, all of which are present at any level, albeit in different manifestations. Yes, but the intersection would still be very small, as a lot of post-elementary mathematics cannot be subsumed under those content categories. Of course, we might go further and adopt a *meta-perspective* on content, as is done in PISA, and consider phenomenological content categories such as *Space and shape*, *Change and relationships*, *Quantity*, and *Uncertainty and data*, all of which are present at any level of mathematics education. However, this does not in any way imply that these categories cover all mathematical content beyond the lower secondary level. For example, an unreasonable amount of abstraction and flexibility of interpretation would be required to fit topics such as integration, topological groups or functional analysis into these categories. Finally, one might consider taking several steps up the abstraction ladder and speak, for example, of mathematics as a whole as the science of patterns (Devlin 1994, 2000), a view that does provide food for thought but is also so abstract and general that one may be in doubt of what is actually being said and covered by that statement. If, for instance, people in chemistry, in botany, or in art and design wished to claim—which wouldn't seem unreasonable—that they certainly profess sciences of patterns, would we then consider these sciences part of mathematics? Probably not.

Instead of focusing on content, I chose to focus on mathematical activity by asking what it means to be mathematically competent. What are the characteristics of a person who, on the basis of knowledge and insight, is able to successfully deal with a wide variety of situations that contain explicit or implicit mathematical challenges? *Mathematical competence* is the term chosen to denote this aggregate and complex entity. I wanted the answers to these questions to be specific to mathematics, even if cast in a terminology that may seem generalisable to other subjects, to cover all age groups and education levels, and to make sense across all mathematical topics, without being so general that the substance evaporates. The analogy with linguistic competence touched upon above was carried further as an inspiration to answering these questions. If linguistic competence in a language amounts to being able to understand and interpret others' written texts and oral statements and narratives in that language, as well as to being able to express oneself in writing and in speech, all of this in a variety of contexts, genres and registers, what would be the counterparts with regard to mathematics? Clearly, people listen, read, speak and write about very different things and in very different ways when going to kindergarten and when teaching, say, English history to PhD students. However, the same four components—which we might agree to call linguistic competencies—play key parts at all levels.

Inspired by these considerations, the task was to identify the key components, the *mathematical competencies* analogous to linguistic competencies, in *mathematical competence*. The approach taken was to reflect on and theoretically analyse the mathematical activities involved in dealing with mathematics-laden, challenging situations, taking introspection and observation of students at work as my point of departure.

It is a characteristic of mathematics-laden situations that they contain or can give rise to actual and potential *questions*—which may not yet have been articulated—to which we seek valid answers. So, it seems natural to focus on the competencies involved in posing and answering different sorts of questions pertinent to mathematics in different settings, contexts and situations. The first competency then is to do with key aspects of mathematical thinking, namely the nature and kinds of questions that are typical of mathematics, and the nature and kinds of answers that may typically be obtained. This is closely related to the types, scopes and ranges of the statements found in mathematics, and to the extension of the concepts involved in these statements, e.g. when the term 'number' sometimes refers to natural numbers, sometimes to rational numbers or complex numbers. The ability to relate to and deal with such issues was called the *mathematical thinking competency*. The second competency is to do with identifying, posing and solving mathematical problems. Not surprisingly, this was called the *mathematical problem handling competency*. It is part of the view of mathematics education nurtured in most places in Denmark, and especially at Roskilde University, that the place and role of mathematics in other academic or practical domains are crucial to mathematics education. As the involvement of mathematics in extra-mathematical domains takes place by way of explicit or implicit mathematical models and modelling, individuals' ability to deal with existing models and to engage in model

construction (active modelling) is identified as a third independent competency, the *mathematical modelling competency*. The fourth and last of this group of competencies focuses on the ways in which mathematical claims, answers and solutions are validated and justified by mathematical reasoning. The ability to follow such reasoning as well as to construct chains of arguments so as to justify claims, answers and solutions was called the *mathematical reasoning competency*.

The activation of each of these four competencies requires the ability to deal with and utilise mathematical language and tools. Amongst these, various representations of mathematical entities (i.e. objects, phenomena, relations, processes, and situations) are of key significance. Typical examples of mathematical representations take the form of symbols, graphs, diagrams, charts, tables, and verbal descriptions of entities. The ability to interpret and employ as well as to translate between such representations, whilst being aware of the sort and amount of information contained in each representation, was called the *mathematical representation competency*. One of the most important categories of mathematical representations consists of mathematical symbols, and expressions composed of symbols. The ability to deal with mathematical symbolism—i.e. symbols, symbolic expressions, and the rules that govern the manipulation of them—and related formalisms, i.e. specific rule-based mathematical systems making extensive use of symbolic expressions, e.g. matrix algebra, was called the *mathematical symbols and formalism competency*. Considering the fact that anyone who is learning or practising mathematics has to be engaged, in some way or another, in receptive or constructive communication about matters mathematical, either by attempting to grasp others' written, oral, figurative or gestural mathematical communication or by actively expressing oneself to others through various means, a *mathematical communication competency* is important to include. Finally, mathematics has always, today as in the past, made use of a variety of physical objects, instruments or machinery, to represent mathematical entities or to assist in carrying out mathematical processes. Counting stones (calculi), abaci, rulers, compasses, slide rulers, protractors, drawing instruments, tables, calculators and computers, are just a few examples. The ability to handle such physical aids and tools (mathematical technology in a broad sense) with insight into their properties and limitations is an essential competency of contemporary relevance, which was called the *mathematical aids and tools competency*. In the next section, a figure depicting the competencies as the petals of a flower is presented (Fig. 2.1).

We now have identified eight mathematical competencies, which are claimed to form an exhaustive set of constituents of what has been termed mathematical competence. The first published version of these competencies (in Danish) can be found in Niss (1999) in a journal published then by the Danish Ministry of Education. Each of the competencies can be perceived as the ability to successfully deal with a wide variety of situations in which explicit or implicit mathematical challenges of a certain type manifest themselves. By addressing and playing out in mathematics-laden situations, the competencies do not deal with mathematics as a whole. Therefore, the set of competencies was complemented with three kinds of *overview and judgement concerning mathematics as a discipline*: the actual use of

The competency flower
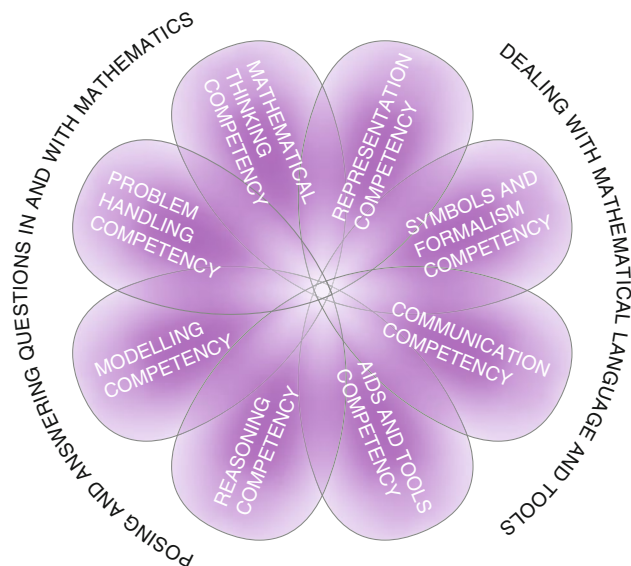
Fig. 2.1 The 'competency flower' from the KOM project

mathematics in society and in extra-mathematical domains, the specific nature and characteristics of mathematics as a discipline compared and contrasted with other scientific and scholarly disciplines, and the historical development of mathematics in society and culture.

Soon after, in 2000, a Danish government agency and the Danish Ministry of Education jointly established a task force to undertake a project to analyse the state of affairs concerning the teaching and learning of mathematics at all levels of the Danish education system, to identify major problems and challenges within this system, especially regarding progression of teaching and learning and the transition between the main sections of the system, and to propose ways to counteract, and possibly solve, the problems thus identified. I was appointed director of the project with Tomas Højgaard Jensen serving as its academic secretary. The project became known as the *KOM project* (KOM = Kompetencer og matematiklæring, in Danish, which means "Competencies and the learning of mathematics"), because the main theoretical tool adopted by the task force to analyse mathematics education in Denmark was the set of eight mathematical competencies, and the three kinds of overview and judgement, introduced above. More specifically, the actual presence and role of the various competencies in mathematics teaching and learning at different levels were analysed. This allowed for the detection of significant differences in the emphases placed on the individual competencies in different sections of the education system. This in turn helped explain some of the observed problems of transition between the sections as well as insufficient progression of teaching and

learning within the entire system. The competencies were also used in a normative manner to propose curriculum designs, modes and instruments of assessment, and competency-oriented teaching and learning activities from school to university, teacher education included. In the next section we shall provide a more detailed account of further aspects of the competencies and their relationship with mathematical content. The formal outcome of the KOM project was the publication, in Danish, of the official report on the project (Niss and Jensen 2002). However, during and after the completion of the project a huge number of meetings, seminars and in-service courses were held throughout Denmark and in other countries to disseminate and discuss the ideas put forward by the project. Also, the project informed—and continues to inform—curriculum design and curriculum documents in mathematics at all levels of the education system in Denmark. An English translation of the most important sections of the KOM report was published in 2011 (Niss and Højgaard 2011).

Concurrently with the KOM project similar ideas emerged elsewhere in the world. To mention just one example, consider the influential *Adding It Up* (National Research Council 2001), produced by the Mathematics Learning Study Committee under the auspices of the National Research Council, edited by Kilpatrick, Swafford and Findell, and published by the National Academies in the USA. In this book we read the following (p. 116):

> Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen *mathematical proficiency* to capture what we believe is necessary for anyone to learn mathematics successfully. Mathematical proficiency, as we see it, has five components, or *strands*:
>
> - *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
> - *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
> - *strategic competence*—ability to formulate, represent, and solve mathematical problems
> - *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
> - *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.
>
> These strands are not independent; they represent different aspects of a complex whole.
> (National Research Council 2001, p. 116)

Although different in the specifics from the conceptualisation put forward by the competency approach, which focuses on what it takes to *do* mathematics, the approach in *Adding It Up* is an attempt to capture what it takes to *learn* mathematics, and hence what is characteristic of an individual who has succeeded in learning it.

A more recent attempt, in some respects closer to that of the competency approach, can be found in the first part of the US *Common Core State Standards Initiative*, which identifies (2010, pp. 1–2) what is called eight "Standards for Mathematical Practice" common to all (school) levels as below.

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

Since the first inception of the competency approach to mathematics, the KOM project and its ramifications have been subject to a lot of further development and follow-up research in various parts of the world. This, together with experiences gained from various sorts of uses of the competency approach in different places and contexts, has given rise to conceptual and terminological development and refinement. This is not the place to elaborate on these developments. Suffice it to mention that one modification of the scheme is essential in the research done by some of the MEG members to capture and characterise item difficulty in PISA, see the next section and in Chap. 4 of this volume.

Further Aspects of the Notion of Competency

It should be underlined that the eight competencies are not mutually disjoint, nor are they meant to be. (Note differences here with the closely related scheme for item rating in Chap. 4 of this volume.) On the contrary the whole set of competencies has a non-empty intersection. In other words, the competencies do not form a partition of the concept of mathematical competence. Yet each competency has an identity, a 'centre of gravity', which distinguishes it from the other competencies. The fact that all competencies overlap can be interpreted such that the activation of each competency involves a secondary activation of the other competencies, details depending on the context. Consider, for example, the modelling competency. Working to construct a model of some situation in an extra-mathematical context presupposes ideas of what sorts of mathematical questions might be posed in such a context and of what sorts of answers can be expected to these questions. In other words, the thinking competency is activated. Since the very purpose of constructing a mathematical model is to mathematise aspects and traits of the extra-mathematical situation, leading to the posing of mathematical problems that then have to be solved, the problem handling competency enters the scene. Carrying out the problem solving processes needed to solve the problems arising from the mathematisation normally requires the use of mathematical representations, as well as manipulating symbolic expressions and invoking some formalism, alongside using mathematical aids and tools, e.g. calculators or computers, including

mathematical software. In other words the representation competency, the symbols and formalism competency, and the aids and tools competency are all activated as part of the process of solving the problem(s) posed. In order to validate, and eventually justify, the solutions and answers obtained as a result of the modelling steps just mentioned, the reasoning competency has to be activated. Finally, beginning and undertaking the modelling task usually requires activation of the receptive side of the communication competency, whereas presenting the modelling process, the model constructed, the model results and their justification, to others activates the constructive side of the communication competency.

In the KOM project we chose to represent the set of competencies as the *competency flower* shown in Fig. 2.1. Each petal represents a competency. They are all distinct petals although they overlap. The density of the shading of each petal is maximal in the middle, at the 'centre of gravity', and fades away towards the boundary. The centre of the flower is the non-empty intersection of all the competencies. Even though a given petal may seem to have a larger intersection with its two neighbours than with the other petals, this is not meant to point to a closer relationship amongst neighbouring petals than amongst other sets of petals.

Possessing a mathematical competency is clearly not an issue of all or nothing. Rather we are faced with a continuum. How, more specifically, can we then describe the extent of an individual's possession of a given competency? The approach taken by the KOM project was to identify three dimensions of the possession of any competency, called *degree of coverage*; *radius of action*; and *technical level*.

A more detailed description of each of the competencies includes a number of aspects employed to characterise that competency. Take, for instance, the representation competency. One of its aspects is to interpret mathematical representations. Another aspect is to bring representations to use, a third is to translate between representations, whereas a fourth aspect is to be aware of the kind and amount of information about mathematical entities that is contained—or left out— in a given representation. Moreover, all of these aspects pertain to any specific mathematical representation under consideration. The *degree of coverage* of a given competency, in this case the representation competency, then refers to the extent to which a person's possession of the competency covers all the *aspects involved in the definition and delineation of that competency*. The more aspects of the competency the person possesses, the higher the degree of coverage of that competency with that person.

Each competency is meant to deal with and play out in challenging mathematics-laden situations that call for the activation of that particular competency. Of course, there is a wide variety of such situations, some more complex and demanding than others. For example, the communication competency can be put to use in situations requiring a person to show and explain how he or she solved a certain task, but it can also be put to use in situations where the person is requested to present and defend his or her view of mathematics as a subject. The *radius of action* of a given

competency refers to the *range of different kinds of contexts and situations* in which a person can successfully activate the competency. The wider the variety of contexts and situations in which the person can activate the competency, the larger the radius of action of that competency with that person.

Different mathematics-laden situations give rise to different levels of mathematical demands on a given competency. The symbols and formalism competency, for instance, can be activated in situations that require dealing with arithmetic operations on concrete rational numbers using the rules that govern the operations. It can also be activated, however, in situations that require finding the roots of third degree polynomials, or the solution of separable first order differential equations. The *technical level* on which an individual possesses a given competency, in this case the symbols and formalism competency, refers to the *range of conceptual and technical mathematical demands* that person can handle when activating the competency at issue. The broader the set of demands the person can handle with respect to the competency, the higher the technical level on which the person possesses that competency.

The three dimensions of the possession of a competency allow us to characterise progression in competency possession by an individual as well as by groups or populations. A person's possession of a given competency *increases* from one point in time to a later point in time, if there is an increase in at least one of the three dimensions, degree of coverage, radius of action or technical level, and no decrease in any of them at the same time. This can be extended to groups or entire populations if some notion of average is introduced. Taking stock of the change of average competency possession for all eight competencies across groups or populations allows us to capture progression (or regression for that matter) in mathematical competence at large for those groups or populations. The three dimensions can also be used to compare the intended or achieved mathematical competency profiles of different segments of the education system, or even of different such systems. It is worth noting that such comparisons over time within one section of the education system, or at the same time between segments or systems, attribute at most a secondary role to mathematical content.

One issue remains to be considered. What is the relationship between mathematical competencies and mathematical content? In the same way as it is true that linguistic competencies are neither developed nor activated in environments without the presence of spoken or written language, mathematical competencies are neither developed nor activated without mathematical content. Since one and the same set of mathematical competencies are relevant from kindergarten to university, and vis-à-vis any kind of mathematical content, we can neither derive the competencies from the content, nor the content from the competencies.

The position adopted in the KOM project is that the eight competencies and any set of mathematical content areas, topics, should be perceived as constituting two independent, orthogonal spaces.

Analysis and Discussion of the Role of Competencies
Within PISA

It should be borne in mind when reading this section that for all official PISA documents published by the OECD the final authorship and the corresponding responsibility for the text lie with the OECD, even though the international contractors under the leadership of the Australian Council for Educational Research, in turn seeking advice from the Mathematics Expert Group, was always, of course, a major contributor to the publications 'behind the curtains'.

In the first PISA survey administration, in 2000, mathematics was a minor assessment domain (reading being the major domain). The initial published version of the Framework (OECD 1999), gives emphasis to a version of the eight mathematical competencies of the KOM project. In the text they actually appear as 'skills' ('mathematical thinking skill', 'mathematical argumentation skill', 'modelling skill', 'problem posing and solving skill', 'representation skill', 'symbolic, formal and technical skill', 'communication skill', and 'aids and tools skill') but under the section headed 'Mathematical competencies' (p. 43), the opening paragraph uses the term 'competency'. This is the first indication of reservations and (later) problems with the OECD concerning the term 'mathematical competency'. In the Framework, 'mathematical competencies' was presented as one of two major aspects (p. 42), the other one being 'mathematical big ideas', along with two minor aspects, 'mathematical curricular strands' and 'situations and contexts'. Together these aspects were used as organisers of the mathematics (literacy) domain in PISA 2000. Based on the point of view that the individual competencies play out collectively rather than individually in real mathematical tasks (p. 43), it was not the intention to assess the eight competencies individually. Instead, it was decided to aggregate them (quite strongly) into what were then called 'competency classes'—Class 1: reproduction, definitions, and computations; Class 2: connections, and integration for problem solving; Class 3: mathematical thinking, generalisation and insight. The Framework emphasises that all the skills are likely to play a role in all competency classes. The degree of aggregation of the competencies into competency classes is very high, so that the competency classes take precedence as an organising idea, while the competencies are recognised to play a component role in all mathematical activity.

Soon after, in a precursor publication to the official report of PISA 2000, (OECD 2000) the terms 'competencies' and 'skills' of the Framework were replaced with the term 'mathematical processes' (p. 50). The headings are unchanged, except that the word 'skill' is omitted in each of them. Similarly, the 'competency classes', including the very term, were preserved but now referred to as 'levels of mathematical competency'.

The first results of PISA 2000 were officially reported in 2001 (OECD 2001). As to the competencies, they almost disappeared in that report. The notion of mathematical processes as composed of different kinds of skills was preserved. The competency classes of the 1999 Framework were changed to 'competency clusters'

simply labelled 'reproduction', 'connections' and 'reflection' (p. 23). Apart from that no traces of the competencies are left in the report, including in Chap. 2 in which the findings concerning mathematical literacy are presented.

Mathematics was the major domain in PISA 2003. In the Framework (OECD 2003), it is interesting to observe that the eight mathematical competencies are back on stage in a slightly modified version. In outlining the main components of the mathematics assessment, the Framework reads:

> The process of mathematics as defined by general mathematical competencies. These include the use of mathematical language, modelling and problem solving skills. Such skills, however, are not separated out in different text [sic, should be test] items, since it is assumed that a range of competencies will be needed to perform any given mathematical task. Rather, questions are organised in terms of 'competency clusters' defining the type of thinking skill needed. (OECD 2003, p. 16)

This short text, six lines in the original, succeeds in interweaving process, competencies and skills, whilst letting questions be organised by way of competency clusters that define thinking skills. However, in the chapter devoted to mathematical literacy (Chap. 1), there is a clearer—and much more detailed—account of the competencies and their role in the Framework. Taking its point of departure in mathematisation, focusing on what is called, there, 'the mathematisation cycle' (p. 38), (and called the modelling cycle in the PISA 2012 Framework (OECD 2013), see also Chap. 1 of this volume) the role of the competencies is to underpin mathematisation. The Framework reads:

> An individual who is to engage successfully in mathematisation in a variety of situations, extra- and intra-mathematical contexts, and overarching ideas, needs to possess a number of mathematical competencies which, taken together, can be seen as constituting comprehensive mathematical competence. Each of these competencies can be possessed at different levels of mastery. To identify and examine these competencies, OECD/PISA has decided to make use of eight characteristic competencies that rely, in their present form, on the work of Niss (1999) and his Danish colleagues. Similar formulations may be found in the work of many others (as indicated in Neubrand et al. 2001). Some of the terms used, however, have different usage among different authors. (OECD 2003, p. 40)

The Framework moves on to list the competencies and their definition. These are 'Thinking and reasoning', 'Argumentation', 'Communication', 'Modelling', 'Problem posing and solving', 'Representation', 'Using symbolic, formal and technical language and operations', and 'Use of aids and tools' (pp. 40–41). The three competency clusters of the PISA 2000 report (reproduction, connections, and reflection) were preserved in the PISA 2003 Framework, but whilst the competencies didn't appear in the description of these clusters in PISA 2000, they were indeed present in PISA 2003. For each of the three clusters, the ways in which the competencies manifest themselves at the respective levels are spelled out in the Framework (OECD 2003, pp. 42–44 and 46–47, respectively).

How then, do the competencies figure in the first report on the PISA 2003 results (OECD 2004)? In the summary on p. 26 the competencies as such are absent; only the competency clusters are mentioned. In Chap. 2, reporting in greater detail on the mathematics results, the competencies are only listed by their headings (p. 40) when

the report briefly states that they help underpin the key process, identified as mathematisation. In the description of the competency clusters (pp. 40–41) there is no mention of the competencies. Even though competencies are referred to in the previous paragraph (p. 40), they do not appear in the competency clusters. The description of the six levels of general proficiency in mathematics (p. 47) employs some elements from the competency terminology. So, the re-introduction of the competencies into the Framework of PISA 2003 was not really maintained in the reporting of the outcomes.

Apart from what seems to be a general reservation within the OECD towards using the notion of competency in relation to a specific subject—they prefer to use the term to denote more general, overarching processes such as cross-curricular competencies (OECD 2004, p. 29)—there is also a more design-specific and technical reason for the relative absence of the competencies in the report. The classification system for PISA items (that which is called the metadata in Chap. 7 of this volume) did not include information on the role of the eight competencies in the individual items. An item was not classified with respect to all the competencies, only assigned to one of the three competency clusters and other characteristics such as overarching idea, response type etc. This means that there were no grounds on which the PISA results could attribute any role to the individual competencies except in more general narratives such as the proficiency level descriptions. In retrospect one may see this as a deficiency in the Framework. If the eight competencies were to play a prominent role in the design of the PISA mathematics assessment, each of the competencies, and not only the competency clusters, would have to be used in the classification of all the items.

In 2009 the OECD published an in-depth study on aspects of PISA 2003 mathematics done by a group of experts from within and outside the MEG in collaboration with the OECD (2009a). In this report, the eight competencies re-emerge under the same headings as in the 2003 Framework, and with the following opening paragraph:

> An individual who is to make effective use of his or her mathematical knowledge within a variety of contexts needs to possess a number of mathematical competencies. Together, these competencies provide a comprehensive foundation for the proficiency scales described further in this chapter. To identify and examine these competencies, PISA has decided to make use of eight characteristic mathematical competencies that are relevant and meaningful across all education levels. (OECD 2009a, p. 31)

On the following pages (pp. 32–33) of the report, each of the competencies is presented as a key contributor to mathematical literacy.

Science was the major domain in PISA 2006, whereas mathematics was a minor domain so the 2006 Framework (OECD 2006) was pretty close to that of 2003 for mathematics. The central mathematical process was still mathematisation, depicted by way of the mathematisation cycle (p. 95). The competencies were introduced as one of the components in the organisation of the domain:

> The competencies that have to be activated in order to connect the real world, in which the problems are generated, with mathematics, and thus to solve the problems. (p. 79)

Otherwise, the role and presentation of the competencies (pp. 96–98) resembled those of 2003, as did the three competency clusters and the description of their competency underpinnings.

The reporting of the mathematics outcomes of PISA 2006 (OECD 2007) is rather terse, focused on displaying and commenting on a set of items and on presenting the six proficiency levels, the same as used in 2003. In the report, there is no explicit reference to the competencies, even though words from the competency descriptions in the Framework are interspersed in the level descriptions. In this context it is interesting to note that the term 'competencies' does in fact appear in the very title of the report, but in the context of science, "PISA 2006. Science Competencies for Tomorrow's World".

As regards the competencies, the PISA 2009 Mathematics Framework (OECD 2009b) is very close to 2003 and 2006, with insignificant changes of wording here and there. It is interesting, though, that the heading of the section presenting the competencies has been changed to "the cognitive mathematical competencies". The overall reporting of the 2009 mathematics outcomes (OECD 2010) does not deviate from that of 2006. The same is true of the role of the competencies.

In PISA 2012, mathematics was going to be the major domain for the second time. In the course of the previous PISA survey administrations certain quarters around the world had aired some dissatisfaction with the focus on mathematical literacy and with the secondary role attributed to classical content areas in the assessment framework. It was thought, in these quarters, that by assessing mathematical literacy rather than 'just mathematics', the domain became more or less misrepresented. With reference to the need to avoid monopolies, there were also parties in OECD PISA who wanted to diversify the management of PISA, which throughout the life of PISA had taken place in a Consortium (slightly changing over time) led by the Australian Council for Education Research (ACER). Several authors of chapters in this book have personally witnessed expressions of dissatisfaction with aspects of the design of PISA mathematics and an increasing ensuing pressure on those involved in PISA mathematics to accommodate the dissatisfaction.

This is not the place to go into details with evidence and reflections concerning the activities that took place behind the public stage of PISA, but one result of these activities was that PISA mathematics 2012 was launched in a somewhat different setting to what was the case in the previous survey administrations. First, a new agency *Achieve*, from the USA, was brought in to oversee, in collaboration with ACER, the creation of a new Mathematics Framework, especially with regard to the place of mathematical content areas. Secondly, a number of new MEG members were appointed to complement the set of members in the previous MEG which was rather small because mathematics was a minor domain in PISA 2006 and 2009. The opening meeting of the new MEG was attended by a chief officer of the OECD who gave clear indications of the desired change of course with respect to PISA mathematics 2012.

The process to produce a Framework for PISA 2012 mathematics became a lengthy and at times a difficult one, in particular because it took a while for the

MEG to come to a common understanding of the boundary conditions and the degrees of freedom present for the construction of the Framework. After several meetings and iterations of draft texts, the MEG eventually arrived at a common document—submitted to the OECD in the northern autumn of 2010—which was to everyone's satisfaction, even though several compromises had of course to be made, but at a scale that was acceptable to all members, as well as to *Achieve*, ACER and eventually the PISA Governing Board.

Some of the compromises were to do with the competencies and their role in the Framework. We shall take a closer look at these issues below. Before doing so, it is worth mentioning that as the very term 'mathematical competencies' was not acceptable to the OECD for PISA 2012, the term chosen to replace it was 'fundamental mathematical capabilities', whilst it was acknowledged that these had been called 'competencies' in previous Frameworks (OECD 2013, pp. 24 and 30). As will be detailed below, the names, definitions, and roles of these capabilities have, in fact, been changed as well.

Technically speaking the definition of mathematical literacy in the 2012 Framework (p. 25) appeared to be rather different from the ones used in previous Frameworks. However, in the view of the MEG the only difference was that the new definition attempted to explicitly bring in some of the other Framework elements in the description so as to specify more clearly, right at the beginning in the definition, what it means and takes to be mathematically literate. So, the change has taken place on the surface rather than in the substance.

In the introduction to the Framework (OECD 2013, p. 18), the mathematical processes are summarised as follows:

> *Processes*: These are defined in terms of three categories (*formulating situations mathematically*; *employing mathematical concepts*, *facts*, *procedures and reasoning*; *and interpreting*, *apply* [sic] *and evaluating mathematical outcomes*—referred to in abbreviated form as *formulate*, *employ* and *interpret*)—and describe what individuals do to connect the context of a problem with the mathematics and thus solve the problem. These three processes each draw on the seven fundamental mathematical capabilities (*communication*; *mathematising*; *representation*; *reasoning and argument*; *devising strategies for solving problems*; *using symbolic*, *formal and technical language and operations*; *using mathematical tools*) which in turn draw on the problem solver's detailed mathematical knowledge about individual topics.

The role of the fundamental mathematical capabilities—a further modification of the eight mathematical competencies of the KOM project and of the previous four Frameworks—in the 2012 Framework is to underpin the new reporting categories of the three processes (*Formulate—Employ—Interpret*) (see Chap. 1 of this volume.) A detailed account of how this is conceptualised is given on pages 30–31 and in Fig. 1.2 in the Framework (OECD 2013). Apart from the change of terminology from 'mathematical competencies' to 'fundamental mathematical capabilities', which is primarily a surface change, what are the substantive changes involved—signalled by the new headings of the fundamental capabilities—and what caused them? (As 'competency' is the generally accepted term in several quarters outside PISA, we continue to use this term rather than fundamental

mathematical capabilities in the remainder of this chapter.) There are three such changes. First, there are some changes in the number and names of the competencies. For example, in the particular context of PISA it was never possible to really disentangle the mathematical thinking competency from the reasoning competency, especially as the former was mainly present indirectly and then closely related to the latter. It was therefore decided to merge them under the heading 'reasoning and argument'. This change is predominantly of a pragmatic nature.

The second, and most significant, change is in the definition and delineation of the fundamental capabilities. In the first place, this change is the result of research done over almost a decade by members of the MEG with the purpose of capturing and characterising the intrinsic mathematical competency demands of PISA items (see Chap. 4 in this book). The idea is to attach a competency vector, the seven components of which are picked from the integers 0,1,2,3, to each item. Over the years, in this research, it became increasingly important to reduce or remove overlap across the competency descriptions, primarily in order to produce clear enough descriptions for experts to be able to rate the items in a consistent and reliable manner. It was also because the scheme was used to predict empirical item difficulty, which imposed certain requirements in order for it to be psychometrically reliable. This means that the fundamental mathematical capabilities are defined and described in such a way that overlap between them is minimal. This is in stark contrast to the original system of competencies, all of which, by design, overlap. Even though there is a clear relationship between the eight competencies and the seven fundamental mathematical capabilities (e.g. 'communication' corresponds to 'communication', 'modelling' corresponds to 'mathematising', 'thinking and reasoning' together with 'argumentation' correspond to 'reasoning and argumentation') the correspondence between the two sets is certainly not one-to-one. In the final formulation of the 2012 Framework it was decided to use the descriptions and delineations from the PISA research project to define the fundamental mathematical capabilities. This implies that the set of mathematical competencies does not make the set of fundamental mathematical capabilities superfluous, nor vice versa. They have different characteristics and serve different purposes, namely providing a general notion of mathematical competence and a scheme to analyse the demands of PISA items, respectively. From that perspective it can be seen as a stroke of luck that the requirement to introduce a new terminology eventually served to avoid confusion of the scheme of the KOM project (and the earlier versions of the PISA Framework) and the 2012 Framework.

The third change was one of order. The fundamental mathematical capabilities of the 2012 Framework occur in a different order than did the mathematical competencies of the previous survey administrations. The reason for this reordering was an attempt to partially (but not completely) emulate the logical order in which a successful problem solver meets and approaches a PISA item. First, the problem solver reads the stimulus and familiarises himself or herself with what the task is all about. This requires the receptive part of 'communication'. Next, the problem solver engages in the process of mathematising the situation (i.e. 'mathematising'), whilst typically making use of mathematical representations

(i.e. 'representation') to come to grips with the situation, its objects and phenomena. Once the situation has been mathematised, the problem solver has to devise a strategy to solve the ensuing mathematical problems (i.e. 'devising strategies for solving problems'). Such a strategy will, more often than not, involve 'using symbolic, formal and technical language and operations', perhaps assisted by 'using mathematical tools'. Then comes an attempt to justify the solutions and mathematical conclusions obtained by adopting the strategy and activating the other capabilities (i.e. 'reasoning and argument'). Finally the problem solver will have to communicate the solution process and its outcome as well as its justification to others. This takes us back to 'communication', now to its expressive side.

At the time of writing this chapter, the official report of PISA 2012 had not yet been published. So, it is not possible to consider the way in which the three processes and the fundamental mathematical capabilities fare in the reporting. This is, of course, even more true of PISA 2015 and subsequent PISA survey administrations, which are in the hands of a completely different management, even though my role as a consultant to the agency in charge of producing the PISA 2015 Framework allows me to say that this Framework is only marginally different from the PISA 2012 Framework.

Concluding Remarks

This chapter has attempted to present the genesis, notion and use of mathematical competencies in Denmark and in other places with a side view to analogous ideas and notions, so as to pave the way for a study of the place and role of mathematical competencies and some of their close relatives, fundamental mathematical capabilities, in the Frameworks and reports of the five PISA survey administrations that at the time of writing have almost been completed (September 2013). The chapter will be concluded by some remarks and reflections concerning a special but significant issue of the relationship between competencies (capabilities) and the entire Framework. In a condensed form this issue can be phrased as a question: 'what underpins what?'

From the very beginning of PISA the approach to the key constituent of the mathematics assessment, i.e. mathematical literacy, was based on mathematical modelling and mathematisation of situations in contexts, although the specific articulation of this in the Framework varied from one survey administration to the next, as did the related terminology. In other words, modelling and mathematisation were always at the centre of PISA. However, the eight mathematical competencies, and most recently the seven fundamental capabilities, were part of the Frameworks as well. Now, do we detect here a potential paradox or some kind of circularity, since modelling (mathematising) is one of the eight competencies (seven capabilities) underpinning the whole approach, above all modelling? It is not exactly surprising that a set of competencies that includes modelling can serve to underpin modelling. If modelling is in centre, why do we need the other competencies then?

Alternatively, would it have been better (if possible) to specify mathematical literacy in terms of the possession of all the mathematical competencies, without focusing especially on the modelling competency, the possession of which would then become a corollary?

Let us consider the first question. When it comes to the eight competencies, it was mentioned in a previous section that the fact that they all overlap means that even when the emphasis is on one of the competences, the others enter the field as 'auxiliary troops' in order for the competency at issue to be unfolded and come to fruition. It is therefore consistent with this interpretation to have the entire system of competencies underpin the modelling competency. One might say, though, that were it only for PISA, in which the emphasis is on the modelling competency, that competency might have been omitted from the list in order to avoid the tiny bit of circularity that is, admittedly, present. However, as the competency scheme is a general one used in a wide variety of contexts, and not only in PISA, it would be unreasonable to remove it from the list solely because of its special use in PISA. What about the seven fundamental mathematical capabilities in the 2012 Framework, then? Here the circularity problem has actually disappeared, at least terminologically speaking, because the seven capabilities do not contain one called modelling, only mathematising (and in a more limited sense than it sometimes has), and because the term mathematising is not used in the modelling cycle in the Framework, as it has been replaced by 'formulating situations mathematically'. So, in the 2012 Framework it is indeed the case that the capabilities underpin this process as well as the other two, 'employing mathematical concepts, facts procedures and reasoning', and 'interpreting, applying and evaluating mathematical outcomes'.

As to the second question, since the eight competencies are meant to constitute mathematical competence and mastery at large, the option mentioned would have amounted to equating mathematical literacy and mathematical competence. This is certainly a possible but not really a desirable option. The perspective adopted in PISA, right from the outset, was not to focus on young people's acquisition of a given subject, such as mathematics, but on their ability to navigate successfully as individuals and citizens in a multifaceted society as a result of their compulsory schooling. This zooms in on putting mathematics to use in a variety of mainly extra-mathematical situations and contexts, in other words the functional aspects of having learnt mathematics. This is what mathematical literacy is all about, being brought about by way of modelling. I, for one, perceive mathematical literacy as a proper subset of mathematical competence, which implies that for someone to be mathematically competent he or she must also be mathematically literate. Even though mathematical literacy does indeed draw upon (aspects of) all the competencies, it does not follow that all the competencies are represented at a full scale and in an exhaustive manner. So, the converse implication, that a mathematically literate person is also necessarily mathematically competent, does not hold. Mathematical competence involves operating within purely mathematical structures, studying intra-mathematical phenomena such as the irrationality of $\sqrt{2}$ and π

even though this is never really required in the physical world, and at a higher level understanding the role of axioms, definitions and proofs.

These remarks are meant to show that what at face value may appear, to some, as a kind of circularity or inconsistency in the PISA Frameworks concerning mathematical literacy, mathematical competence and competencies, fundamental mathematical capabilities, modelling and mathematising are, as a matter of fact, basically logically coherent in a closer analysis.

It will be interesting to follow, in the years to come, how mathematical competencies are going to be developed from research as well as from practice perspectives. At the very least, putting the competencies on the agenda of mathematics education has offered new ways of thinking about what mathematics education is all about.

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