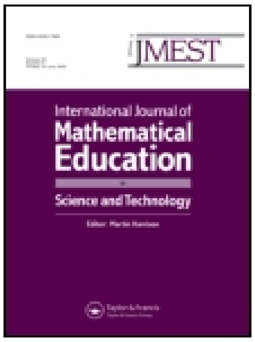

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Geoff Smith , Leigh Wood , Mary Coupland , Brian Stephenson , Kathryn Crawford & Geoff Ball

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Constructing mathematical examinations to assess a range of knowledge and skills

by GEOFF SMITH, LEIGH WOOD, MARY COUPLAND, BRIAN STEPHENSON
School of Mathematical Sciences, University of Technology, Sydney, P.O. Box 123, Broadway, NSW 2007, Australia

KATHRYN CRAWFORD
Faculty of Education, University of Sydney, Australia

and GEOFF BALL
School of Mathematics and Statistics, University of Sydney, Australia

(*Received and reviewed by an AEB member*)

In this paper, we describe aspects of a programme to enhance student learning in undergraduate mathematics. We present ways of constructing formal examinations which assess a range of knowledge and skills and which encourage students to reflect on their learning. To assist with this process, we propose a taxonomy to classify assessment tasks ordered by the nature of the activity required to complete each task successfully, rather than in terms of difficulty. An extensive list of university level examples is given to illustrate descriptors in the taxonomy and to provide ideas for those interested in implementing an alternative approach to writing examination questions.

1. Introduction

This paper describes aspects of a programme to improve student learning in undergraduate mathematics. We demonstrate ways of constructing formal examinations which assess a range of knowledge and skills and which encourage students to reflect on their learning. Analyses of students' responses, marking methods and other types of assessment will be published later. We have compiled a list of questions which illustrate alternative approaches to examination questions. We have found it useful to classify these questions in a taxonomy, but other readers may not wish to follow this approach. These readers can simply treat this paper as a list of novel and, hopefully, useful approaches to examination questions.

Logistics and tradition have meant that assessment in mathematics has relied heavily on formal examinations. There are significant difficulties in changing this in early undergraduate years and little educational reason to do so. However, examinations often test a narrow range of skills and we have been investigating ways to broaden the skills tested. In addition, textbooks are virtual clones of each other, with long sets of repetitive exercises. These factors all encourage a surface approach to learning. To quote from Ramsden [1]

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> It has become clear from numerous investigations that:
>
> Many students are accomplished at complex routine skills in science, mathematics and humanities, including problem solving algorithms.
>
> Many have appropriated enormous amounts of detailed knowledge, including knowledge-specific terminology.
>
> Many are able to reproduce large quantities of factual information on demand.
>
> Many are able to pass examinations.
>
> But many are *unable* to show that they understand what they have learned, when asked simple yet searching questions that test their grasp of the content.
>
> In summary, the research seems to indicate that, at least for a short period, students retain vast quantities of information. On the other hand, many seem to forget much of it and do not appear to make good use of what they do remember [1, pp. 30–31].

This is not new. Over the years, teachers and lecturers have recognized these facts and have attempted to influence student learning by a number of methods, such as changing the style of their teaching, by attempting to give clearer explanations, by giving more examples, by preparing better lecture notes and so on, all on the assumption that mathematics need only be presented logically in order to be learned. However, research has shown that students are often more motivated to learn material or methods that are of direct relevance to passing, and will adapt their learning styles and do what they perceive is necessary to pass assessment tasks. This means that changing teaching methods without due attention to assessment methods is not sufficient [1].

Our assumptions are:

(i) In the interests of higher quality in education, a deep approach to learning mathematics is to be valued over a surface approach, and students entering university with a surface approach should be given every encouragement to progress to a deep approach. It has been shown [2] that students able to adopt a deep approach to study tended to achieve at a higher level after a year of university study.

(ii) The adoption of a surface approach to learning is more widespread than might be expected in university students [2] and is a learned response, gained from previous experience in school [3–5] and often reinforced by university practices.

(iii) Students are capable of changing their approach to learning, from a surface approach towards a deeper approach, and will do so if they see it as necessary in order to succeed.

(iv) Assessment drives what students learn. It controls their approach to learning by directing them to take either a surface approach or a deep approach to learning [1]. The types of questions that we set show students what we value and how we expect them to direct their time. Good questions are those which help to build concepts, alert students to misconceptions and introduce applications and theoretical ideas.

(v) It is possible to use a taxonomy to classify a set of tasks ordered by the nature of the activity required to complete each task successfully, rather than in

Group A	Group B	Group C
Factual knowledge	Information transfer	Justifying and interpreting
Comprehension	Application in new situations	Implications, conjectures and comparisons
Routine use of procedures		Evaluation

Figure 1.

terms of difficulty. Activities which need only a surface approach appear at one end, while those requiring a deeper approach appear at the other end.

There are several taxonomies that one could use, depending on the purpose. One of the best known is Bloom's taxonomy [6], which gives a hierarchy of concepts.

Bloom's taxonomy is quite good for structuring assessment tasks, but does have some limitations in the mathematical context. We propose a modification of Bloom's taxonomy, the MATH taxonomy (mathematical assessment task hierarchy) for the structuring of assessment tasks. The categories in the taxonomy are summarized in Figure 1. A detailed list of descriptors is given in section 2.

We have found it helpful to use a grid (Figure 2) which combines subject topics with the descriptors of the taxonomy. The grid entries are references to particular questions on the examination paper. This enables us to more readily determine the balance of assessment tasks on the examination paper. Most of the mathematics examination papers we have analysed are heavily biased towards group A tasks.

Students enter tertiary institutions with most of their mathematical learning experience in group A tasks, with some experience with group B tasks [2]. Their experience in group C tasks in mathematics is severely limited or non-existent. One of the aims of tertiary education in mathematics should be to develop skills at all three levels.

MATH Taxonomy \ Topic	Topic 1	Topic 2	Topic 3	Topic 4	Topic 5
Factual knowledge					
Comprehension					
Routine use of procedures					
Information transfer					
Applications in new situations					
Justifying and interpreting					
Implications, conjectures, comparisons					

Figure 2.

2. Descriptors

It is easy to have mistaken assumptions about the skills that a particular task assesses. If a student is asked to prove a theorem given in lectures, a correct answer may be given by a student who understands the theorem and its significance and can apply it in relevant situations or prove similar theorems. However, the most we can assume is that the student can reproduce the theorem on demand; this style of assessment cannot discriminate between different types of learning which can lead to the same response. If we are content with this, then the question is satisfactory, but if we wish to be sure that the student understands the theorem and has not merely learned it by rote, then we need to ask more probing questions. It is essential to be clear about the desired outcomes of our assessment and to be able to identify the types of assessment tasks which are reliable indicators of these outcomes.

The list of descriptors given below attempts to force all types of mathematical examination assessment into one of eight categories. There will certainly be borderline cases or cases which do not fit comfortably in any category or cases which fit into more than one category. However, it is not our aim to be able to uniquely characterize every conceivable assessment task. Rather, the aim of the descriptors is to assist with writing examination questions, and to allow the examiner's judgement, objectives and experience to determine the final evaluation of an assessment task.

We have also (section 3) given a list of examples to illustrate these descriptors. Some of these depend on the prior knowledge of the student. A student who succeeds in proving an unseen theorem is demonstrating an ability to apply knowledge to new situations, but may only be demonstrating factual recall when proving it for a second time.

There is a loose hierarchy in this list of descriptors; finding the area of a circle requires factual knowledge (the formula πr^2), comprehension (knowing what each of the three symbols represent, knowing that multiplication and exponentiation are implied) and the use of a routine procedure (evaluation after substitution). Finding the area without previous knowledge of the method (as was the case for the ancient Greeks) would require other skills. The assumptions being made in each group of questions are given.

It is important to understand, however, that no hierarchy of difficulty is implied as we move down the list. It is the *nature* of the activity we are interested in, not the degree of difficulty, which in any case is a very subjective and elusive concept. (Some definitions: *Easy*—I can do it. *Difficult*—I can't do it.)

2.1. *Group A*

Factual knowledge and fact systems. The difficulty and depth of the material may cover a wide range from remembering a specific formula or definition (factual knowledge) [Examples 1, 2 and 3 in section 3] to learning a complex theorem (a fact system) [Examples 4 and 5 in section 3], but the only skill required is to bring to mind *previously learned* information in the form that it was given.

Comprehension. It is quite possible to reproduce knowledge without understanding. To demonstrate comprehension of factual knowledge, students should:

- be able to decide whether or not conditions of a simple definition are satisfied [Examples 6–9]. By a *simple definition*, we mean one which is a matter of terminology, making use of previously acquired knowledge or skills, e.g. *a*

linear first-order differential equation is one which satisfies.... The student has merely learned a new term, but not one which requires a significant conceptual change in their mathematical understanding;
- understand the significance of the symbols in a formula (both implicit and explicit) and show an ability to substitute in a formula;
- be able to recognize examples and counterexamples [Example 10].

Routine procedures. This requires the ability to use material in a way which goes beyond simple factual recall. The essential feature is that when the procedure or algorithm is properly used, all people solve the problem correctly and in the same way. This does not preclude the possibility that there may be more than one routine procedure applicable to a given problem. Examples include pattern recognition, such as evaluating a formula after substitution, or solving a differential equation [Examples 11 and 12]. Students would have been expected to have worked on problems using these procedures in drill exercises. In some cases, there may be several distinct processes underlying a particular procedure and although students may be able to state the general procedure to be followed and understand its principles, they may be unable to carry out the detail. As an example, a student may know that the area under a curve can be obtained by integration and may be able to set up the integral correctly, but be unable to do all but the simplest integrations.

2.2. *Group B*

Information transfer. This may be shown by the ability to perform the following tasks:

- transformation of information from one form to another—verbal to numerical or vice versa [Example 13];
- deciding whether or not conditions of a conceptual definition are satisfied. A *conceptual definition* is one whose understanding requires a significant change in the student's mode of thought or mathematical knowledge, for example, the definition of a limit or of linear independence. Deciding whether or not a definition is simple or conceptual will often be a subjective judgement, but, given our aims, we do not regard this as important;
- recognizing the applicability of a formula or method in different or unusual contexts [Examples 14 and 15];
- recognizing the inapplicability of a generic formula in particular contexts [Example 16];
- summarizing in non-technical terms for a different audience, or paraphrasing;
- framing a mathematical argument from a verbal outline of the method;
- explaining the relationships between component parts of the material;
- explaining processes;
- reassembling the [given] component parts of an argument in their logical order [Example 17].

Application in new situations. Ability to choose and apply appropriate methods or information in new situations, including the following:

- modelling real life settings;
- proving a previously unseen theorem or result which goes beyond the routine use of procedures [Examples 18 and 20];
- extrapolation of known procedures to new situations [Example 19];

- choosing and applying appropriate statistical techniques;
- choosing and applying appropriate algorithms.

2.3. *Group C*

Justifying and interpreting. Ability to justify and/or interpret a given result or a result derived by the student. This includes:

- proving a theorem in order to justify a result, method or model;
- the ability to find errors in reasoning [Examples 21 and 22];
- recognizing the limitations in a model and being able to decide if a model is appropriate [Example 23];
- recognition of computational limitations and sources of error;
- interpreting a regression model;
- discussing the significance of given examples and counterexamples;
- recognition of unstated assumptions.

Implications, conjectures, and comparisons. Given or having found a result/situation, the student has the ability to draw implications and make conjectures and the ability to justify or prove these. The student also has the ability to make comparisons, with justification, in various mathematical contexts. Examples are:

- the ability to make conjectures based, for example, on inductive or heuristic arguments, and then to prove these conjectures by rigorous methods [Examples 24 and 25];
- comparisons between algorithms [Example 27];
- the ability to deduce the implications of a given result;
- the construction of examples and counterexamples.

Evaluation. Evaluation is concerned with the ability to judge the value of material for a given purpose based on definite criteria. The students may be given the criteria or may have to determine them. This includes the following:

- the ability to make judgements [Examples 29 and 30];
- the ability to select for relevance [Example 29];
- the ability to coherently argue the merits of an algorithm [Example 31];
- organizational skills;
- creativity, which includes going beyond what is given, restructuring the information into a new whole and seeing implications of the information which is not apparent to others.

3. Examples

A series of sample questions which illustrates the above list of descriptors is given below. We assume that the questions would be asked of students currently studying the topic under consideration, since, in all these examples, the actual category into which the question fits could depend on the prior experience of the student; what may be factual knowledge to a graduate mathematician could be incomprehensible to a clever high school student. There can also be various levels of difficulty for particular skills. It is more difficult, for example, to memorize a proof of a theorem than to memorize a simple definition. If the exact question (or very close pattern) has been seen before, then the student can answer questions entirely by rote. It is up to us to be clever and creative enought to design questions that discourage rote learning and that test the skills we wish to assess. You will know best what your students' prior experience has been, so it is your classification that matters.

With minor exceptions, the examples are taken from a pure mathematics core which would be common knowledge in areas such as statistics, operations research and applied mathematics. People from these areas should have no difficulty in understanding our ideas and adapting them to their own disciplines.

3.1. *Factual knowledge and fact systems*

The assumption in these questions is that students have met the material in the form required for the answer.

Factual knowledge

Example 1 • *What is the formula for the area of a circle?*
Example 2 • *State Cramer's rule for solving a system of equations.*
Example 3 • *What is meant by the term* ***linear differential equation****?*

Fact systems

Example 4 • *(a) State the comparison test for series of non-negative terms.*
• *(b) State and prove the ratio test for the convergence of a series of positive terms.*
Example 5 • *State and prove the Hahn–Banach theorem.*

3.2. *Comprehension of factual knowledge*

We assume that various routine skills needed to do the problems (such as partial differentiation in the second example) are familiar to the students, and that they have done drill work in similar but not identical problems.

Example 6 • *Decide, given reasons, whether or not the following differential equation is linear*

$$xy' + y = \exp(x)$$

Example 7 • *Show that $x^3 - 3xy^2$ is a harmonic function.*
Example 8 • *Answer True or False* [7].
[ ] *All continuous functions are differentiable*
[ ] *Some continuous functions are not differentiable*
[ ] *All differentiable functions are continuous*
Example 9 • *Indicate whether the following statements are* ***true*** *or* ***false****.*
[ ] *A function may be differentiable at z_0, but not analytic at z_0.*
[ ] *A function may be differentiable at z_0, and also analytic at z_0.*
[ ] *A function may be analytic at z_0, but not differentiable at z_0.*
[ ] *A function may be analytic everywhere in the complex plane.*
[ ] *A function may be analytic nowhere in the complex plane.*

Trials of questions such as examples 8 and 9 above show that students who can correctly quote the relevant definitions may nevertheless be unable to answer these questions correctly. This shows that the ability to quote a definition may be a meaningless skill.

Example 10 • *Identify the surface*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

3.3. *Routine use of procedures*

Here we assume students have done drill in tasks similar to the ones being assessed.

Example 11 • *Solve the initial value problem*

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = kQ \qquad Q(0) = Q_0$$

given that $Q(\tau) = Q_0/2$, *where* $\tau = 5568$.

Example 12 • *Let C be the circle* $|z-1| = 1$. *Evaluate the integral*

$$\int_c \frac{\mathrm{d}z}{z^2-1}.$$

3.4. *Information transfer*

Example 13 • *Here is an attempted proof of a form of L'Hôpital's rule:*

Statement:

$$\textit{If } f(a) = g(a) = 0 \textit{ then } \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof:

$$(a)\quad \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$(b)\quad = \lim_{x\to a}\frac{(f(x)-f(a))/(x-a)}{(g(x)-g(a))/(x-a)}$$

$$(c)\quad = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Explain carefully what is happening in each of the steps, labelled (*a*), (*b*) *and* (*c*). *Explain where there could be difficulties with the proof. What conditions should be added to the statement to make the proof valid?*

Any capable high school calculus student would have the technical skills, namely the theory of maxima and minima, to solve the next two problems, but in fact many university students have difficulty with them.

Example 14 • *Find all the real roots of the equation* $3x^4 + 4x^3 - 12x^2 + 30 = 0$, *or explain why no solutions exist.*

Example 15 • *Show that* $x^3 + cx + d = 0$ *has only one real root if* $c \geq 0$.

Example 16 • *A function is defined by*

$$f(x) = \begin{cases} \sin x & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Find $f'(x)$.

This last question comes from a study by Harel and Kaput [8, p. 85]. A common response was:

$$f'(x) = \begin{cases} \cos x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This shows that the students were blithely applying an algorithm to each point rather than considering the real meaning of differentiation, which is not a pointwise concept. It is a good question because it addresses vividly a common student misconception. The following example appears in [9].

Example 17 • *For any finite set S, a field F of subsets of S and a real-valued function P on F, a* ***probability space*** *is defined by the following three axioms:*

(*i*) $P(A) \geq 0$ for all $A \in F$ (*ii*) $P(S) = 1$

(*iii*) $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$ for all $A_i \in F$ with $A_i \cap A_j = \varnothing \quad i \neq j$

We want to prove the theorem: $P(\varnothing) = 0$. *The steps of the proof are given, but they are not in the correct order. Arrange them to form a logically valid proof.*

(*a*) *so* $P(\varnothing) = 2P(\varnothing)$

(*b*) *now,* $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ (*by axiom iii*)

(*c*) *take* $A_1 = A_2 = \varnothing$

(*d*) *so* $P(\varnothing) = 0$

(*e*) $\varnothing \in F$ (*since F is a field*)

(*f*) *then* $A_1 \cup A_2 = A_1 \cap A_2 = \varnothing$.

3.5. *Application to new situations*

Our assumption in this section is that the students have not met any of the results they are asked to prove. The first example was an examination question given to students who had not previously encountered Liouville's theorem or Cauchy's inequalities.

Example 18 • *Suppose f is analytic within and on the circle* $|z - z_0| = R$, *denoted by C, and let* M_R *denote the maximum value of* $|f(z)|$ *on C. Use Cauchy's integral formula to derive Cauchy's inequalities:*

$$(*) \qquad |f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n} \qquad (n = 1, 2, 3, \ldots).$$

The above result can be used in the proof of the following important theorem:

If a function f is entire and bounded in the complex plane, then $f(z)$ *is constant throughout the plane (Liouville's theorem). Prove this result by the following method:*

Step 1 *Use the fact that f is entire to show that inequality* (*) *above with* $n = 1$ *holds for any choice of* z_0 *and R.*

Step 2 *Use the fact that f is bounded to show that there is a constant M such that*

$$|f'(z_0)| \leq \frac{M}{R},$$

where z_0 *is any fixed point in the plane and R is arbitrarily large.*

Step 3 *Show that the inequality in step 2 can only hold if $f'(z_0)=0$ and hence show that f is a constant function.*

The next example assumes students have only met linear differential equations in their studies.

Example 19 • *Solve the following two equations by showing that the indicated substitution transforms the equation to one which is linear in x and v.*

1. $y' + 2y = x/y^2 \qquad v = y^3$
2. $y' - 5y = -\frac{5}{2}xy^3 \qquad v = y^{-2}$

Generalize the method in 1 and 2 above to solve

$$y' + P(x)y = Q(x)y^n$$

The next example could be given in a class test or as a tutorial problem to students early in a complex variables course before they had met the Cauchy–Riemann equations.

Example 20 • *Use the method outlined below to show that if the function f defined by $f(z) = u(x,y) + iv(x,y)$ has a derivative at $z_0 = x_0 + iy_0$, then the first partial derivatives of u and v exist at z_0 and*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at z_0. These are the ***Cauchy–Riemann equations****.*

Step 1 *Write the derivative of f as a limit.*

Step 2 *Express this limit in terms of u and v.*

Step 3 *Evaluate this limit in two ways and compare the results.*

3.6. *Justifying and interpreting*

Example 21 • *Here is an attempted proof of Cauchy's mean value theorem:*

Statement: *If functions f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there exists a value c in (a, b) such that:*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Proof:

Apply the ordinary mean value theorem to f:

$$\exists c \in (a,b)\colon f(b) - f(a) = f'(c)(b-a) \tag{1}$$

Apply the ordinary mean value theorem to g:

$$\exists c \in (a,b)\colon g(b) - g(a) = g'(c)(b-a) \tag{2}$$

Divide equation (1) by equation (2):

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)(b-a)}{g'(c)(b-a)}$$

Cancel $(b-a)$ on the right and cross multiply to obtain the result. The attempted proof is actually invalid. Examine it carefully and write a short criticism of it [8].

Example 22 • *Here are two arguments, one to show that* $\sin^{-1}x - \cos^{-1}x = \pi/2$ *and the other that* $\sin^{-1}x + \cos^{-1}x = \pi/2$. *They cannot both be correct (and may both be wrong). Find and explain the error(s) in reasoning.*

We know that

$$\cos y = \sin(y + \pi/2)$$

for all y, so suppose

$$x = \cos y = \sin(y + \pi/2).$$

Then

$$y = \cos^{-1}x \tag{1}$$

and

$$y + \pi/2 = \sin^{-1}x \tag{2}$$

Subtraction of equation (1) *from equation* (2) *gives the result*

$$\sin^{-1}x - \cos^{-1}x = \pi/2.$$

On the other hand, we also know that

$$\cos(\pi/2 - y) = \sin y$$

for all y, so suppose

$$x = \cos(\pi/2 - y) = \sin y.$$

Then

$$y = \sin^{-1}x \tag{3}$$

and

$$\pi/2 - y = \cos^{-1}x \tag{4}$$

Addition of equations (3) *and* (4) *gives the result*

$$\sin^{-1}x + \cos^{-1}x = \pi/2.$$

Example 23 • *One way of modelling population growth is to assume that the population increases at a rate proportional to the size of the population, that is* $P' \propto P$. *When would this be an appropriate model for population growth? What assumptions are being made and what are the limitations of the model?*

3.7. *Implications, conjectures, and comparisons*

The following problem required students to use a computer program to multiply given matrices and then to make conjectures based on the results they obtained.

Example 24 • *This problem investigates the similarity of nth powers of matrices. You are given two square matrices A and B and a nonsingular matrix P that satisfy the relationship* $B = P^{-1}AP$.

(*i*) *Check that* $B = P^{-1}AP$ *as claimed.*

(*ii*) *Calculate* B^2 *and* $P^{-1}A^2P$.

(*iii*) *Calculate* B^3 *and* $P^{-1}A^3P$.

(*iv*) *Calculate* B^4 *and* $P^{-1}A^4P$.

(*v*) *Let C and D be any two similar matrices. Make a conjecture about the similarity of* C^n *and* D^n, *for* $n = 1, 2, 3, \ldots$

(*vi*) *Prove your conjecture.*

The next two examples appear in [10].

Example 25 • *Take the expression* $n^2 + n + 17$, *let* $n = 1$, *and evaluate the result. Is it a prime number? Substitute* $n = 2$. *Is the result a prime number? Substitute values of n from 3 to 10. Are the results all prime numbers? Can you come to a general conclusion? Are you using deductive or inductive arguments? Are you certain of your conclusion? Is the conclusion actually true?*

Example 26 • *Prove or disprove the following: The expression* $n^2 + n + 41$ *represents a prime number for any natural number n.*

Example 27 • *Compare the method of undetermined coefficients with variation of parameters for second order linear differential equations.*

3.8. *Evaluation*

Example 28 • *Is it possible to prove the result* $e^{ix} = \cos x + i \sin x$? *Give reasons for your answer.*

Example 29 • *Here are two definitions of a complex number:*

1 *The equation* $x^2 = -1$ *has no real roots, but we may invent an imaginary unit i for which* $i^2 = -1$. *We then define a complex number as a combination* $p + iq$ *formed from two real numbers p and q and the imaginary unit i.*

2 *The complex numbers can be defined as the set* $\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$ *together with certain standard arithmetical operations defined on this set.*

Compare the two definitions. Your answer could include:
The circumstances under which each definition would be appropriate.
The relative merits of each definition from a mathematical point of view.
Historical aspects of these definitions.
A demonstration of the equivalence of the definitions.

Example 30 • *Write a short exposition evaluating the relative merits of Leibniz's and Newton's notation for differentiation.*

Example 31 • *Explain why the method of Laplace transforms works so well for linear differential equations with constant coefficients* ***and*** *integro-differential equations involving a convolution* [11].

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