A New Model for Scaling Response Patterns: An Application of the Quasi-Independence Concept

LEO A. GOODMAN*

To analyze the "scalability" of the observed response patterns for a set of m dichotomous items, we introduce a model in which a given individual in the population is either "intrinsically scalable" or "intrinsically unscalable" (with respect to the m items), and there are d different types of "intrinsically scalable" individuals. With this model, using methods already developed for the study of quasi-independence in contingency tables, we show how to (a) test whether the model fits the observed data, (b) estimate the proportion of intrinsically scalable (and unscalable) individuals, and (c) estimate the distribution of the intrinsically scalable individuals among the d different scale types.

1. INTRODUCTION

Consider first the situation where there are four dichotomous items, say, A, B, C and D, and let (i, j, k, l) denote the response pattern in which the responses on items A, B, C and D are at levels i, j, k and l, respectively (for $i = 1, 2$; $j = 1, 2$; $k = 1, 2$; $l = 1, 2$). In this situation, there are 2^4 possible response patterns; but suppose we consider for the moment the special case in which only five patterns are actually observable,

$$(1, 1, 1, 1),\ (1, 1, 1, 2),\ (1, 1, 2, 2),\ (1, 2, 2, 2),\ (2, 2, 2, 2). \quad (1.1)$$

In this special case, we have a Guttman scale, all respondents are scalable, and there are five different scale types (see, e.g., [8]).

In actual practice, response patterns other than those listed in (1.1) will also occur. To analyze the "scalability" of the observed patterns (when the five patterns in (1.1) correspond to the scale types), we introduce the following model.

> We consider a population in which individuals are classified into one of *six* mutually exclusive and exhaustive categories, viz., a single category for the "intrinsically unscalable" individuals and five other categories for the "intrinsically scalable" individuals in the five scale types corresponding to the response patterns in (1.1). For the "intrinsically unscalable" individuals, the responses on items A, B, C and D are mutually independent; and the response patterns for the "intrinsically scalable" individuals correspond to their scale type.[1] For the sake of simplicity, the category for the intrinsically unscalable will be labeled category zero, and the other five categories will be numbered in order from one to five.

Let π_t denote the probability that an individual will be in the tth category (for $t = 0, 1, 2, 3, 4, 5$); let $\pi_{it}^{\bar A}$ denote the *conditional* probability that his response will be at level i on variable A (for $i = 1, 2$), given that he was in the tth category; and let the conditional probabilities $\pi_{jt}^{\bar B}$, $\pi_{kt}^{\bar C}$, $\pi_{lt}^{\bar D}$ be defined similarly (for $j = 1, 2$; $k = 1, 2$; $l = 1, 2$). Let π_{ijkl}^{ABCD} denote the probability of obtaining response pattern (i, j, k, l); and let $\pi_{ijklt}^{\bar A\bar B\bar C\bar D}$ denote the *conditional* probability of obtaining this pattern, given that the individual was in the tth category. Our model states that

$$\pi_{ijkl}^{ABCD} = \sum_{t=0}^{5} \pi_t \pi_{ijklt}^{\bar A\bar B\bar C\bar D}, \quad (1.2)$$

where

$$\pi_{ijkl0}^{\bar A\bar B\bar C\bar D} = \pi_{i0}^{\bar A}\pi_{j0}^{\bar B}\pi_{k0}^{\bar C}\pi_{l0}^{\bar D}, \quad (1.3)$$

$$\pi_{11111}^{\bar A\bar B\bar C\bar D} = \pi_{11122}^{\bar A\bar B\bar C\bar D} = \pi_{11223}^{\bar A\bar B\bar C\bar D} = \pi_{12224}^{\bar A\bar B\bar C\bar D} = \pi_{22225}^{\bar A\bar B\bar C\bar D} = 1, \quad (1.4)$$

with $\pi_{ijklt}^{\bar A\bar B\bar C\bar D} = 0$ otherwise. Formula (1.2) states that the individuals in the population can be classified into six mutually exclusive and exhaustive categories ($t = 0, 1, 2, \cdots, 5$); (1.3) states that the responses on items A, B, C and D are mutually independent for the individuals in the 0th category; and (1.4) states that the response pattern for an individual in the tth category (for $t = 1, 2, 3, 4, 5$) corresponds to his scale type (with probability one).

From (1.2)–(1.4), we see that

$$\begin{aligned}
\pi_{1111}^{ABCD} &= \pi_1 + \pi_0\pi_{10}^{\bar A}\pi_{10}^{\bar B}\pi_{10}^{\bar C}\pi_{10}^{\bar D}, \\
\pi_{1112}^{ABCD} &= \pi_2 + \pi_0\pi_{10}^{\bar A}\pi_{10}^{\bar B}\pi_{10}^{\bar C}\pi_{20}^{\bar D}, \\
\pi_{1122}^{ABCD} &= \pi_3 + \pi_0\pi_{10}^{\bar A}\pi_{10}^{\bar B}\pi_{20}^{\bar C}\pi_{20}^{\bar D}, \\
\pi_{1222}^{ABCD} &= \pi_4 + \pi_0\pi_{10}^{\bar A}\pi_{20}^{\bar B}\pi_{20}^{\bar C}\pi_{20}^{\bar D}, \\
\pi_{2222}^{ABCD} &= \pi_5 + \pi_0\pi_{20}^{\bar A}\pi_{20}^{\bar B}\pi_{20}^{\bar C}\pi_{20}^{\bar D};
\end{aligned} \quad (1.5)$$

and for any response pattern (i, j, k, l) that does *not* correspond to one of the specified scale types, we obtain

$$\pi_{ijkl}^{ABCD} = \pi_0\pi_{i0}^{\bar A}\pi_{j0}^{\bar B}\pi_{k0}^{\bar C}\pi_{l0}^{\bar D}. \quad (1.6)$$

* Leo A. Goodman is the Charles L. Hutchinson Distinguished Service Professor, Departments of Statistics and Sociology, University of Chicago, Chicago, Ill. 60637, and research associate at the Population Research Center of the University. This research was supported by Research Contract No. SOC 72-05228 A03 from the Division of the Social Sciences of the National Science Foundation. The author is indebted to D. Andrews, M. Burawoy, C. Clogg, O.D. Duncan, S. Haberman and P.F. Lazarsfeld for helpful comments.

[1] The meaning of this model, and the interpretation of results obtained with it, will be discussed later. The methods which will be presented here for analyzing this model are somewhat similar to methods developed earlier (using the quasi-independence concept) to analyze the mover-stayer model (see, e.g., [3])

December 1975, Volume 70, Number 352
Applications Section

Formula (1.6) states that a response pattern (i, j, k, l) that does not correspond to one of the specified scale types can be obtained from an individual in the 0th category (with conditional probability $\pi_{i0}^{\bar{A}}\pi_{j0}^{\bar{B}}\pi_{k0}^{\bar{C}}\pi_{l0}^{\bar{D}}$, given the individual in the 0th category); and (1.5) states that a pattern that corresponds to one of the specified scale types can be obtained *either* from an individual in the corresponding scale-type category (with conditional probability one, given the individual in the scale-type category) or from an individual in the 0th category (with conditional probability $\pi_{i0}^{\bar{A}}\pi_{j0}^{\bar{B}}\pi_{k0}^{\bar{C}}\pi_{l0}^{\bar{D}}$, given the individual in the 0th category).

Let S denote the set of possible response patterns that correspond to scale types, and let U denote the set of possible patterns that do not correspond to scale types. (In the particular situation just considered, there are five patterns in S, and there $2^4 - 5 = 11$ patterns in U.) Let us now delete (i.e., blank out) the patterns in S. Considering only the remaining patterns (i.e., those in U), (1.6) states that the responses on items A, B, C and D are quasi-independent (see, e.g., [2]). This formula can be rewritten as

$$\pi_{ijkl}^{ABCD} = \alpha_i\beta_j\gamma_k\delta_l, \quad \text{for} \quad (i, j, k, l) \in U, \qquad (1.7)$$

with

$$\begin{aligned} \pi_{i0}^{\bar{A}} &= \alpha_i/(\alpha_1 + \alpha_2), \quad \pi_{j0}^{\bar{B}} = \beta_j/(\beta_1 + \beta_2), \\ \pi_{k0}^{\bar{C}} &= \gamma_k/(\gamma_1 + \gamma_2), \quad \pi_{l0}^{\bar{D}} = \delta_l/(\delta_1 + \delta_2), \\ \pi_0 &= (\alpha_1 + \alpha_2)(\beta_1 + \beta_2)(\gamma_1 + \gamma_2)(\delta_1 + \delta_2). \end{aligned} \qquad (1.8)$$

To calculate the maximum-likelihood estimates, $\hat{\pi}_{i0}^{\bar{A}}$, $\hat{\pi}_{j0}^{\bar{B}}$, $\hat{\pi}_{k0}^{\bar{C}}$, $\hat{\pi}_{l0}^{\bar{D}}$ and $\hat{\pi}_0$, for the corresponding parameters in (1.6), we first calculate $\hat{\alpha}_i$, $\hat{\beta}_j$, $\hat{\gamma}_k$, $\hat{\delta}_l$ corresponding to the parameters in the quasi-independence model (1.7), using the kind of iterative procedure presented for this purpose in [2, p. 1118–19].[2] The maximum-likelihood estimates, $\hat{\pi}_{i0}^{\bar{A}}$, $\hat{\pi}_{j0}^{\bar{B}}$, $\hat{\pi}_{k0}^{\bar{C}}$, $\hat{\pi}_{l0}^{\bar{D}}$ and $\hat{\pi}_0$, can then be obtained from (1.8) by replacing each term on the right side of these formulas (the $\alpha_i, \beta_j, \gamma_k, \delta_l$) by the corresponding quantity just calculated (the $\hat{\alpha}_i, \hat{\beta}_j, \hat{\gamma}_k, \hat{\delta}_l$). To calculate the maximum-likelihood estimate $\hat{\pi}_t$ (for $t = 1, 2, 3, 4, 5$), we use (1.5) to obtain the formulas

$$\begin{aligned} \hat{\pi}_1 &= p_{1111}^{ABCD} - \hat{\pi}_0\hat{\pi}_{10}^{\bar{A}}\hat{\pi}_{10}^{\bar{B}}\hat{\pi}_{10}^{\bar{C}}\hat{\pi}_{10}^{\bar{D}}, \\ \hat{\pi}_2 &= p_{1112}^{ABCD} - \hat{\pi}_0\hat{\pi}_{10}^{\bar{A}}\hat{\pi}_{10}^{\bar{B}}\hat{\pi}_{10}^{\bar{C}}\hat{\pi}_{20}^{\bar{D}}, \\ \hat{\pi}_3 &= p_{1122}^{ABCD} - \hat{\pi}_0\hat{\pi}_{10}^{\bar{A}}\hat{\pi}_{10}^{\bar{B}}\hat{\pi}_{20}^{\bar{C}}\hat{\pi}_{20}^{\bar{D}}, \end{aligned} \qquad (1.9)$$

etc., where p_{ijkl}^{ABCD} is the observed proportion of response patterns that are (i, j, k, l). The maximum-likelihood estimates $\hat{\pi}_{ijkl}^{ABCD}$ of the corresponding π_{ijkl}^{ABCD} can then be obtained from (1.5)–(1.6) by replacing each term on the right side of these formulas by the corresponding maximum-likelihood estimate.

[2] Further comments on this iterative procedure, and alternative procedures, will be included in Sections 7 and 8 (see, e.g., (8.5)–(8.7)). To apply the methods proposed here, we assume that the observed frequency f_{ijkl} of response pattern (i, j, k, l) is greater than zero for each pattern (i, j, k, l) in set U; or, more generally, that it is greater than zero for enough of the patterns in U to estimate the parameters in the quasi-independence model. (Some cases where problems arise in the estimation of the parameters will be considered later; see the discussion of Model H_6 in Section 4 and the first model presented in the appendix.)

We assume here that the probability π_t is greater than zero. In this case, the corresponding maximum-likelihood estimate $\hat{\pi}_t$ will also be greater than zero, with probability approaching one, as the sample size n approaches infinity. On the other hand, if any of the $\hat{\pi}_t$ (for $t = 1, 2, 3, 4, 5$) calculated by (1.9) turn out to be negative (which would happen with probability approaching zero), the method just described would need to be modified slightly. We comment later on this situation (see Section 4), but for the time being we will assume that it does not happen.

For the response patterns that are in set S, we see from (1.5) and (1.9) that

$$\hat{\pi}_{ijkl}^{ABCD} = p_{ijkl}^{ABCD}, \quad \text{for} \quad (i, j, k, l) \in S. \qquad (1.10)$$

In other words, for (i, j, k, l) in the set S, the observed proportion p_{ijkl}^{ABCD} is fitted perfectly (except for round-off error) by the corresponding $\hat{\pi}_{ijkl}^{ABCD}$ under the model considered here.

Let f_{ijkl} and F_{ijkl} denote the observed and expected frequency of response pattern (i, j, k, l) in a sample of n individuals, and let $\hat{F}_{ijkl}$ denote the maximum-likelihood estimate of F_{ijkl} under the model. Note that

$$f_{ijkl} = np_{ijkl}^{ABCD}, \quad F_{ijkl} = n\pi_{ijkl}^{ABCD}, \quad \hat{F}_{ijkl} = n\hat{\pi}_{ijkl}^{ABCD}. \qquad (1.11)$$

From (1.10) we see that

$$\hat{F}_{ijkl} = f_{ijkl}, \quad \text{for} \quad (i, j, k, l) \in S. \qquad (1.12)$$

To test whether the model is congruent with the observed data, we compare the observed f_{ijkl} with the corresponding $\hat{F}_{ijkl}$ for (i, j, k, l) in set U, using either the usual goodness-of-fit chi-square statistic

$$\sum_U (f_{ijkl} - \hat{F}_{ijkl})^2/\hat{F}_{ijkl}, \qquad (1.13)$$

or the corresponding chi-square based on the likelihood-ratio statistic

$$2\sum_U f_{ijkl}\log(f_{ijkl}/\hat{F}_{ijkl}), \qquad (1.14)$$

where $\sum_U$ denotes summation over all (i, j, k, l) in set U.[3] The chi-square value obtained from (1.13) or (1.14) can be assessed by comparing its numerical value with the tabulated percentiles of the chi-square distribution. The number of degrees of freedom for testing the model will be

$$2^4 - 5 - 5 = 6; \qquad (1.15)$$

i.e., the number of possible response patterns $(2^4 - 5)$ in set U minus the number of independent parameters estimated under the quasi-independence model for U (i.e., $\pi_{10}^{\bar{A}}, \pi_{10}^{\bar{B}}$, $\pi_{10}^{\bar{C}}$, $\pi_{10}^{\bar{D}}$ and π_0).

In this calculation of the degrees of freedom for testing the model, the number of patterns in set S was taken to be five, the full set of scale types as described earlier. More generally, in the situation where there are m dichotomous items, there will be 2^m possible response

[3] From (1.12) we see that the chi-square values (1.13) and (1.14) would remain unchanged if the summation was over *all* (i, j, k, l) in the 2^4 table. On the other hand, in view of (1.12), there is no need to sum over the (i, j, k, l) in set S.

1. Observed Cross-Classification in Three Different Sets of Data

Response pattern				Data set		
Item						
A	B	C	D	Stouffer-Toby questionnaire (1)	McHugh test (2)	Lazarsfeld-Stouffer questionnaire (3)
1	1	1	1	42	23	75
1	1	1	2	23	5	69
1	1	2	1	6	5	55
1	1	2	2	25	14	96
1	2	1	1	6	8	42
1	2	1	2	24	2	60
1	2	2	1	7	3	45
1	2	2	2	38	8	199
2	1	1	1	1	6	3
2	1	1	2	4	3	16
2	1	2	1	1	2	8
2	1	2	2	6	4	52
2	2	1	1	2	9	10
2	2	1	2	9	3	25
2	2	2	1	2	8	16
2	2	2	2	20	34	229

patterns, there will be $m + 1$ possible patterns corresponding to the full set of scale types, there will be $m + 1$ independent parameters estimated under the quasi-independence model for U, and thus the number of degrees of freedom for testing the model will be

$$2^m - 2(m + 1). \tag{1.16}$$

Formula (1.16) applies to the situation where the full set of $m + 1$ scale types can occur. In situations where this is *not* the case, a slight modification of (1.16) will be required (see (4.11)).

To illustrate the general approach proposed, we apply it to the following three sets of data presented in Table 1:

1. The Stouffer-Toby response patterns for respondents to questionnaire items on role conflict (see [5, 11, 17]). This table cross-classifies 216 respondents with respect to whether they tend toward *universalistic* values (+) or *particularistic* values (−) when confronted by each of four different situations of role conflict.[4]
2. The McHugh test data on creative ability in machine design (see [6, 12, 15]). This table cross-classifies 137 engineers with respect to their dichotomized scores (above the subtest mean (+) or below (−)) obtained on each of four different subtests that were supposed to measure creative ability in machine design.[5]
3. The Lazarsfeld-Stouffer response patterns for noncommissioned officers responding to items on attitude toward the Army (see [10, 16]). This table cross-classifies a sample of 1,000 noncommissioned officers with respect to their dichotomized responses (favorable (+) or unfavorable (−)) obtained on each of four different items on general attitudes toward the Army.[6]

For the sake of simplicity, we focus our attention in this article on the case where there are four dichotomous items (as in each set of data in Table 1), but we also note that our methods can be applied more generally to the case where there are m dichotomous items ($m = 3, 4, \cdots$).[7] The three different sets of data in Table 1 provide an opportunity to illustrate different kinds of considerations that arise in the application of the general approach proposed here. Each set of data will be considered separately (see Sections 2, 3 and 4 for data sets 1, 2 and 3, respectively), as will some additional points that apply more generally (see Sections 5 to 8). We begin with the analysis of the first set of data.

2. THE ANALYSIS OF THE STOUFFER-TOBY DATA

From the Stouffer-Toby data (Column 1 of Table 1), we see that the observed frequencies f_{ijkl}, for the response patterns included in set S described by (1.1), are

$$f_{1111} = 42, \quad f_{1112} = 23, \quad f_{1122} = 25, \quad f_{1222} = 38, \quad f_{2222} = 20. \tag{2.1}$$

Set S consists of five patterns; and the remaining 11 patterns in the 2^4 table form set U. We delete the five patterns in S from the 2^4 table, and consider only the 11 patterns in U. We next consider the quasi-independence model H_1, which states that the responses on items A, B, C and D are quasi-independent with respect to set U; and we apply to set U the method described in Section 1 to obtain the maximum-likelihood estimates of the parameters $\pi_{10}^{\bar{A}}$, $\pi_{10}^{\bar{B}}$, $\pi_{10}^{\bar{C}}$, $\pi_{10}^{\bar{D}}$, and π_0 (see comments following (1.8)).

For the Stouffer-Toby data under H_1, we obtain estimates of the parameters,

$$\hat{\pi}_{10}^{\bar{A}} = .77, \quad \hat{\pi}_{10}^{\bar{B}} = .38, \quad \hat{\pi}_{10}^{\bar{C}} = .44, \quad \hat{\pi}_{10}^{\bar{D}} = .19, \quad \hat{\pi}_0 = .68, \tag{2.2}$$

which can then be used to calculate the corresponding $\hat{\pi}_{ijkl}^{ABCD}$ for the patterns (i, j, k, l) in set U (by replacing the π's on the right side of (1.6) by the corresponding $\hat{\pi}$'s); and we then obtain the corresponding $\hat{F}_{ijkl}$ under H_1 (using the last formula in (1.11)), which we have presented in Table 2. (All calculations in this table and throughout were carried to more significant digits than reported here.) Next we use (1.9) to obtain

$$\hat{\pi}_1 = .18, \ \hat{\pi}_2 = .03, \ \hat{\pi}_3 = .03, \ \hat{\pi}_4 = .03, \ \hat{\pi}_5 = .05; \tag{2.3}$$

and we use (1.13) and (1.14) to obtain a goodness-of-fit chi-square of 1.01 and a likelihood-ratio chi-square of .99, each with six degrees of freedom (see (1.15)). Thus, the

[4] For these data, the letters A, B, C, D in Table 1 denote the dichotomous responses when confronted by the four different role conflicts. In Table 1, level 1 on a given dichotomous response denotes a positive (+) response, and level 2 denotes a negative (−) response.

[5] For these data, the letters A, B, C, D in Table 1 denote the dichotomized scores on the four subtests. For ease of later exposition, we have interchanged in Table 1 the order of the subtests which were considered first and second in the earlier literature.

[6] For these data, the letters A, B, C, D in Table 1 denote the dichotomized responses on the four items. For ease of later exposition, these items appear in Table 1 in the same order as in [10], which is the opposite order from that used in [16].

[7] The case where $m = 3$ and $m = 2$ will be discussed briefly in the appendix. Further insight into the meaning of the model introduced in this article can be obtained from the results presented in the appendix and from some of the other results presented in the following sections.

2. Observed and Estimated Expected Frequencies for the Stouffer-Toby Data Under Two Different Quasi-Independence Models, H_1 and H_2[a]

Response pattern: Item A	B	C	D	Observed frequency	Expected frequency estimated under model H_1	Expected frequency estimated under model H_2
1	1	1	1	42	—	—
1	1	1	2	23	—	20.19
1	1	2	1	6	4.72	5.50
1	1	2	2	25	—	25.96
1	2	1	1	6	5.99	5.92
1	2	1	2	24	24.74	27.93
1	2	2	1	7	7.56	7.61
1	2	2	2	38	—	35.91
2	1	1	1	1	1.14	1.10
2	1	1	2	4	4.73	5.18
2	1	2	1	1	1.44	1.41
2	1	2	2	6	5.97	6.66
2	2	1	1	2	1.83	1.52
2	2	1	2	9	7.57	7.17
2	2	2	1	2	2.31	1.95
2	2	2	2	20	—	—

[a] In H_1, the full set of scale types is deleted. In H_2, only the two extreme scale types are deleted.

model under consideration here fits the data very well indeed.

For purposes of comparison, we consider next the usual model (H_0) in which responses on items A, B, C and D are mutually independent for the individuals in the population. This model can be viewed as stating that all the individuals are intrinsically unscalable and that there are no scale types; i.e., that $\pi_0 = 1$ and $\pi_t = 0$, for $t = 1, 2, 3, 4, 5$ (see (1.2)–(1.3) and (1.5)–(1.6)). To test Model H_0, the usual chi-square methods of testing for mutual independence can be applied, with $2^4 - 5 = 11$ degrees of freedom (see, e.g., [4]).[8] The chi-square values obtained thereby, for the Stouffer-Toby data, are included in Table 3a.[9] Comparing the chi-square values for H_0 with those for H_1, we see the dramatic improvement obtained when H_0 is replaced by H_1![10]

The parameters π_1 and π_5 in Model H_1 pertained to the scale types corresponding to the extreme response patterns (1, 1, 1, 1) and (2, 2, 2, 2), respectively; whereas π_2, π_3 and π_4 pertain to the scale types corresponding to the less extreme response patterns (1, 1, 1, 2), (1, 1, 2, 2) and (1, 2, 2, 2), respectively (see (1.4)–(1.5)). Instead of five different scale types, we next consider the situation

3. Some Estimated Parameters and Chi-Square Values for Various Examples of the New Scaling Model

Model	Number of scale types	Estimated proportion $\hat{\pi}_0$ of intrinsically unscalable individuals	Degrees of freedom	Goodness-of-fit chi-square	Likelihood-ratio chi-square
		a. Stouffer-Toby Data			
H_0	0	1.00	11	104.11	81.08
H_1	5	.68	6	1.01	.99
H_2	2	.78	9	2.28	2.28
		b. McHugh Data			
H_1	5	.49	6	5.63	5.90
H_2	2	.67	9	23.08	22.59
H_3	4	.52	7	7.11	7.42
		c. Lazarsfeld-Stouffer Data			
H_4	4	.67	7	26.09	26.50
H_5	8[a]	.47	3[a]	5.47	5.86
H_6	12	.04[b]	1	.42	.44
H_7	4	.67	7	25.05	25.01

[a] For comments on the results actually obtained for the Lazarsfeld-Stouffer data, see discussion in the text.
[b] Estimate obtained under condition (4.9).

where there are only two different scale types; i.e., the types corresponding to the two extreme response patterns,

$$(1, 1, 1, 1) \quad \text{and} \quad (2, 2, 2, 2). \tag{2.4}$$

Consider now the situation where set S consists of the two response patterns in (2.4); and the remaining 14 patterns in the 2^4 table form the set U. We delete the two patterns in S from the 2^4 table, and consider only the 14 patterns in U. The corresponding quasi-independence model H_2 states that responses on items A, B, C and D are quasi-independent with respect to set U; and we now apply to set U the general approach described in Section 1.

For the Stouffer-Toby data under H_2, we obtain estimates of the parameters,

$$\hat{\pi}_{10}{}^{\bar{A}} = .80, \quad \hat{\pi}_{10}{}^{\bar{B}} = .42, \quad \hat{\pi}_{10}{}^{\bar{C}} = .44, \quad \hat{\pi}_{10}{}^{\bar{D}} = .17, \quad \hat{\pi}_0 = .78; \tag{2.5}$$

and the estimates corresponding to (2.3) are now .18 and .05 under H_2, for the two scale types corresponding to the response patterns in (2.4).[11] The $\hat{F}_{ijkl}$ under H_2 are included in Table 2, and the chi-square values for testing H_2 are included in Table 3a.[12] From Table 3a we see that H_2 also fits the data very well.

[8] Note that there are five more degrees of freedom for testing H_0 than for testing H_1 (see (1.15)); this corresponds to the fact that π_t is set at zero (for $t = 1, 2, 3, 4, 5$) under H_0 but not under H_1. More generally, in the situation where there are m dichotomous items, the number of degrees of freedom for testing H_0 will be $[2^m - (m + 1)]$ (see, e.g., [4]); there are $m + 1$ more degrees of freedom for testing H_0 than for testing H_1 (see (1.16)); and this corresponds to the fact that π_t is set at zero (for $t = 1, 2, \cdots, m + 1$) under H_0 but not under H_1.

[9] Table 3a includes the results for H_0, some of the results for H_1 presented earlier in this section, and some results for Model H_2 which we shall consider next.

[10] This comparison of the chi-square values can be used to test the null hypothesis that $\pi t = 0$ (for $t = 1, 2, 3, 4, 5$), with five degrees of freedom (see Footnote 8). The difference between the chi-squares obtained with statistic (1.14) gives the likelihood-ratio chi-square for testing this null hypothesis, assuming that Model H_1 is true (see, e.g., [2]). Later we show how to test the null hypothesis that $\pi_t = 0$ for a given subset of the five π_t ($t = 1, 2, 3, 4, 5$).

[11] The $\hat{\pi}_1$ and $\hat{\pi}_5$ under H_1 in (2.3) agreed to the order of accuracy reported here with the corresponding estimates just presented under H_2, but this need not be the case in general.

[12] For testing H_2, the number of degrees of freedom will be $2^4 - 2 - 5 = 9$ (as indicated in Table 3a); i.e., the number of possible response patterns $(2^4 - 2)$ in set U (obtained with H_2) minus the number of independent parameters estimated under the quasi-independence model for U. (See related comment following (1.15), and the more general formulas (4.10)–(4.11) later.)

From the results for H_1 and H_2 in Table 3a, we see that the estimated proportion $\hat{\pi}_0$ of "intrinsically unscalable" individuals is somewhat smaller under H_1 than under H_2. On the other hand, because H_1 has three more scale types than H_2, there are three more parameters to be estimated under H_1 than under H_2; and thus, there are three less degrees of freedom for testing H_1 than for testing H_2. By comparing the corresponding likelihood-ratio chi-squares, we obtain a difference of 1.29 (i.e., $2.28 - .99 = 1.29$), which can be assessed in terms of the tabulated percentiles of the chi-square distribution with three degrees of freedom (i.e., $9 - 6 = 3$). The difference in the chi-squares (1.29) can be used to test the null hypothesis that the three additional scale types under H_1 do not contribute in a statistically significant way to the fit of the model (i.e., that the estimated values of π_2, π_3 and π_4 in (2.3) do not differ from zero in a statistically significant way). This null hypothesis is supported by the data.

The method just described (see also Footnote 10) for comparing two different likelihood-ratio chi-squares can be applied more generally to examine any given subset of the $\hat{\pi}$'s in (2.3) to determine whether they differ from zero in a statistically significant way. To save space, we shall not give the numerical details here, but summarize briefly: Assuming H_1 to be true, we can show that $\hat{\pi}_2$, $\hat{\pi}_3$ and $\hat{\pi}_4$ in (2.3) are not significantly different from zero, but $\hat{\pi}_1$ and $\hat{\pi}_5$ are.[13] Similarly, assuming H_2 to be true, we can also show that the latter two $\hat{\pi}$'s continue to be significantly different from zero.[14]

We note that $\hat{\pi}_0$ under Model H_1 will be larger than the *observed* proportion of individuals whose response patterns do not correspond to the scale types specified by the model (i.e., $68/216 = .31$); and, similarly, $\hat{\pi}_0$ under H_2 will be larger than the corresponding *observed* proportion (i.e., $154/216 = .71$). This is because the individuals whose observed response patterns correspond to scale types will include both the individuals who are in scale-type categories *and* some individuals who are not.[15] Thus, the proportion of individuals whose observed response patterns correspond to scale types will tend to overestimate the proportion of individuals who are "intrinsically scalable."[16] The latter proportion is estimated here by $1 - \hat{\pi}_0$.

3. THE ANALYSIS OF THE McHUGH DATA

We consider next the data in Column 2 of Table 1. For the response patterns in set S described in (1.1), we apply the same methods used in Section 2 to study Model H_1, and we obtain estimates of the parameter for the McHugh data,

$$\begin{gathered} \hat{\pi}_{10}{}^{\bar{A}} = .43, \quad \hat{\pi}_{10}{}^{\bar{B}} = .43, \quad \hat{\pi}_{10}{}^{\bar{C}} = .57, \\ \hat{\pi}_{10}{}^{\bar{D}} = .68, \quad \hat{\pi}_0 = .49. \end{gathered} \tag{3.1}$$

The estimates corresponding to (2.3) are now

$$\hat{\pi}_1 = .13, \; \hat{\pi}_2 = .02, \; \hat{\pi}_3 = .09, \; \hat{\pi}_4 = .04, \; \hat{\pi}_5 = .23; \tag{3.2}$$

and the chi-square values for testing H_1 are included in Table 3b for these data. Here too the model under consideration fits the data well.

In Section 2 on the analysis of the Stouffer-Toby data, we considered both H_1 corresponding to the five scale types and H_2 corresponding to the two extreme scale types. We have also applied these two models to the McHugh data, and the results thus obtained are summarized in Table 3b. From the chi-square values for H_2 in Table 3b, we see that this particular model does not fit the McHugh data.

Instead of having five different scale types as in H_1, or two different scale types as in H_2, consider the situation where there are four different scale types; viz., the types corresponding to the response patterns

$$(1,1,1,1), \quad (1,1,2,2), \quad (1,2,2,2), \quad \text{and} \quad (2,2,2,2). \tag{3.3}$$

In this situation, set S consists of these four patterns, and set U consists of the remaining 12 patterns in the 2^4 table. The corresponding quasi-independence model H_3 states that responses on items A, B, C and D are quasi-independent with respect to set U. Applying the general approach described earlier, we now obtain estimates for the parameters under H_3,

$$\begin{gathered} \hat{\pi}_{10}{}^{\bar{A}} = .47, \quad \hat{\pi}_{10}{}^{\bar{B}} = .46, \quad \hat{\pi}_{10}{}^{\bar{C}} = .59, \\ \hat{\pi}_{10}{}^{\bar{D}} = .66, \quad \hat{\pi}_0 = .52; \end{gathered} \tag{3.4}$$

and the estimates corresponding to (3.2) are .12, .09, .04, and .23 under H_3, for the scale types corresponding to the four patterns listed in (3.3). From the chi-square values for H_3 in Table 3b, we see that this model fits the data well.

From the results for H_1 and H_3 in Table 3b, we see that the estimated proportion $\hat{\pi}_0$ of "intrinsically unscaled" individuals is somewhat smaller under H_1 than under H_3. On the other hand, because H_1 has one more scale type than H_3, there is one more parameter to be

[13] The difference in the chi-squares just presented tested the null hypothesis that $\pi_2 = \pi_3 = \pi_4 = 0$ (with three degrees of freedom), and the difference in the chi-squares considered in Footnote 10 tested the null hypothesis that $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 0$ (with five degrees of freedom), assuming H_1 to be true. The general method can also be applied to test separately each null hypothesis stating that $\pi_t = 0$ for a specified value of t (with one degree of freedom for each test), or each null hypothesis stating that $\pi_t = 0$ for two specified values of t (with two degrees of freedom for each test), etc.

[14] The difference in the chi-squares for H_0 and H_2 (see Table 3a) can be used to test the null hypothesis that $\pi_1 = \pi_5 = 0$ (corresponding to the two $\hat{\pi}$'s that are relevant here), with two degrees of freedom. We can also test separately the null hypothesis that $\pi_1 = 0$ and the null hypothesis that $\pi_5 = 0$ (with one degree of freedom for each test), using the general method mentioned at the end of Footnote 13.

[15] See (1.15) and its interpretation (included in the paragraph after (1.6)). In the model introduced here, the "intrinsically scalable" individual's observed response pattern is in set S (in accordance with the usual "perfect scale" concept) and it cannot be in set U; whereas the "intrinsically unscalable" individual's responses on items A, B, C and D are mutually independent, and so his observed pattern can be in set S or in set U. For comments on some models in which the "intrinsically scalable" individual's observed pattern can be in either set S or U, see Section 6.

[16] The magnitude of this overestimation is equal to $\hat{\pi}_0 \hat{\pi}_{i0}{}^{\bar{A}} \hat{\pi}_{j0}{}^{\bar{B}} \hat{\pi}_{k0}{}^{\bar{C}} \hat{\pi}_{l0}{}^{\bar{D}}$ summed over all patterns (i, j, k, l) in set S (see (1.9)). This sum will depend on the proportion of individuals whose observed response patterns are in set U and the observed one-way marginal distributions (with respect to items A, B, C and D) obtained for those individuals with patterns in U. (For some related results, see (8.5)–(8.7) and the appendix.)

estimated under H_1 than under H_3; and thus, there is one less degree of freedom for testing H_1 than for testing H_3. By comparing the corresponding likelihood-ratio chi-squares, we obtain a difference of 1.52 (i.e., $7.42 - 5.90 = 1.52$), which can be assessed in terms of the tabulated percentiles of the chi-square distribution with one degree of freedom (i.e., $7 - 6 = 1$). Here we find that the additional scale type under H_1 does not contribute in a statistically significant way to the fit of the model (i.e., that the estimated value of π_2 in (3.2) does not differ from zero in a statisically significant way).

By applying the same general method used in the preceding paragraph and also near the end of Section 2, we find that, assuming H_1 to be true, the estimate $\hat{\pi}_2$ in (3.2) is not significantly different from zero, but $\hat{\pi}_1$, $\hat{\pi}_3$, $\hat{\pi}_4$ and $\hat{\pi}_5$ are significantly different from zero. Similarly, assuming H_3 to be true, we can also show that the latter $\hat{\pi}$'s continue to be significantly different from zero.

4. THE ANALYSIS OF THE LAZARSFELD-STOUFFER DATA

In Section 3, analyzing the McHugh data, we started with a model (H_1) having five scale types and ended with a model (H_3) having four scale types. We now analyze the Lazarsfeld-Stouffer data, and for reasons which will become clear later, we start with a model that has four scale types; viz., the types corresponding to the scale patterns

$$(1, 1, 1, 1),\quad (1, 1, 1, 2),\quad (1, 2, 2, 2),\quad (2, 2, 2, 2). \tag{4.1}$$

Note that (4.1) deletes from (1.1) the middle pattern (1, 1, 2, 2), but it includes the two most extreme patterns (as in (2.4) for H_2) and also the two next most extreme patterns. Set S now consists of the four patterns in (4.1), and set U consists of the remaining 12 patterns in the 2^4 table. The corresponding quasi-independence model H_4 states that responses on items A, B, C and D are quasi-independent with respect to set U. Applying the general approach described earlier, we obtain estimates for the parameters under H_4,

$$\begin{gathered}\hat{\pi}_{10}{}^{\bar{A}} = .75, \quad \hat{\pi}_{10}{}^{\bar{B}} = .47, \quad \hat{\pi}_{10}{}^{\bar{C}} = .36, \\ \hat{\pi}_{10}{}^{\bar{D}} = .30, \quad \hat{\pi}_0 = .67;\end{gathered} \tag{4.2}$$

and the estimates corresponding to (3.2) are .05, .01, .08, .19 under H_4, for the scale types corresponding to the four patterns listed in (4.1).

Since the four patterns in (4.1) are the first, second, fourth and fifth patterns listed in (1.1) (with the third in (1.1) deleted), the preceding corresponding four estimated parameters can be expressed as

$$\hat{\pi}_1 = .05, \quad \hat{\pi}_2 = .01, \quad \pi_3 = 0, \quad \hat{\pi}_4 = .08, \quad \hat{\pi}_5 = .19. \tag{4.3}$$

For Model H_4, the parameter π_3 in (4.3), which pertains to the scale type corresponding to pattern (1, 1, 2, 2), is assumed to be equal to zero, since this pattern was not included in (4.1); whereas for H_1, all five π's (including π_3) are to be estimated from the data, and all of them are assumed to be positive, or at least nonnegative (see, e.g., (2.3) and (3.2)).

When Model H_1 is applied to the Lazarsfeld-Stouffer data using the general approach described earlier, we find that the estimate of π_3 obtained from (1.8) turns out negative, which indicates that the maximum-likelihood estimate of π_3 must be zero (since $\pi_3 \geqq 0$). In this case, to obtain the maximum-likelihood estimates of the parameters under H_1, we will first need to make π_3 equal to zero. Because of this, the maximum-likelihood estimates of the parameters under H_1 will be equal to the corresponding quantities obtained for H_4.

These comments can be applied more generally whenever the estimate obtained from (1.9) is negative for a subset of the π's. In this case, to obtain the maximum-likelihood estimates of the parameters under the model, we will first need to make one or more of π's in that particular subset equal to zero.

Let us return now to Model H_4. From the chi-square values for this model presented in Table 3c, we see that this particular model does not fit the Lazarsfeld-Stouffer data. From previous remarks, we also see that the same chi-square values would be obtained for H_1 as for H_4, when these values are based on the maximum-likelihood estimates. Since neither model H_4 (with its four scale-type categories) nor H_1 (with its five scale-type categories) fits these data, we consider next a model in which there are still more categories corresponding both to "scale types" and "demi-scale types."

If we consider the set of five patterns listed in (1.1) as corresponding to the "scale types," then three additional patterns could be viewed as corresponding to "demi-scale types,"

$$(1, 1, 2, 1),\ (1, 2, 1, 2),\ (2, 1, 2, 2). \tag{4.4}$$

Pattern 1 in (4.4) would be a scale type if items C and D are interchanged; pattern 2 would be a scale type if items B and C are interchanged; and pattern 3 would be a scale type if items A and B are interchanged. The scale-type categories corresponding to the five patterns in (1.1) were numbered earlier in order from one to five; and now we continue this by using 6, 7 and 8 to denote the demi-scale-type categories corresponding to the three patterns in (4.4).

Now set S consists of the eight patterns in (1.1) and (4.4), and set U consists of the remaining eight patterns in the 2^4 table. The corresponding quasi-independence model H_5 states that responses on items A, B, C, D are quasi-independent with respect to set U. Applying the general approach described here, we obtain estimates of the parameters under H_5,

$$\begin{gathered}\hat{\pi}_{10}{}^{\bar{A}} = .74, \quad \hat{\pi}_{10}{}^{\bar{B}} = .35, \quad \hat{\pi}_{10}{}^{\bar{C}} = .44, \\ \hat{\pi}_{10}{}^{\bar{D}} = .35, \quad \hat{\pi}_0 = .47;\end{gathered} \tag{4.5}$$

and the estimates corresponding to (3.2) are

$$\begin{gathered}\hat{\pi}_1 = .06, \quad \hat{\pi}_2 = .03, \quad \hat{\pi}_3 = .05, \quad \hat{\pi}_4 = .12, \\ \hat{\pi}_5 = .20, \quad \hat{\pi}_6 = .03, \quad \pi_7 = .00, \quad \hat{\pi}_8 = .04.\end{gathered} \tag{4.6}$$

To test H_5 with its eight categories for scale and demi-scale types, there will be three degrees of freedom (i.e., $2^4 - 8 - 5 = 3$);[17] and from the chi-square values for H_5 in Table 3c, we find that this model fits the data.

Let us return now for a moment to Models H_1 and H_4. We noted earlier that to test H_1 with its five scale types there will usually be six degrees of freedom (see, e.g., H_1 in Tables 3a and 3b); but in the special case considered earlier, where we needed to make the parameter π_3 equal to zero (see (4.3)), we actually replaced H_1 by H_4, and so there were seven degrees of freedom (rather than six) for testing the model (see H_4 in Table 3c). Similarly, to test H_5 with its eight categories for scale and demi-scale types, there will usually be three degrees of freedom, as noted previously. But when this model was actually applied to the Lazarsfeld-Stouffer data, we found that the parameter π_7 needed to be equated to zero (see (4.6)), and so there were four degrees of freedom (rather than three) for testing the model.[18]

By applying the same general methods used at the end of Section 3, we can show that, assuming H_5 to be true, the $\hat{\pi}$'s in (4.6) are significantly different from zero (except for π_7, which was set at zero).

Since H_5 fits the data, there is no need to consider models in which there are still more categories. However, in situations where H_5 does not fit, we may wish to consider models that include both the scale and demi-scale types listed in (1.1) and (4.4), and also the demi-demi-scale types

$$(1, 2, 1, 1), \quad (2, 1, 1, 2), \quad (1, 2, 2, 1), \quad (2, 2, 1, 2). \qquad (4.7)$$

Pattern 1 in (4.7) would be a demi-scale type if items B and C are interchanged, and it would be a scale type if items B and D are interchanged; pattern 2 would be a demi-scale type if items A and B are interchanged, and it would be a scale type if items A and C are interchanged; pattern 3 would be a demi-scale type if items C and D are interchanged, and it would be a scale type if items B and D are interchanged; pattern 4 would be a demi-scale type if items B and C are interchanged, and it would be a scale type if items A and C are interchanged.

There are 12 patterns in (1.1), (4.4) and (4.7). Consider now the case where set S consists of these 12 patterns, and set U consists of the remaining four patterns in the 2^4 table,

$$(2, 1, 1, 1), \quad (2, 2, 1, 1), \quad (2, 1, 2, 1), \quad (2, 2, 2, 1). \qquad (4.8)$$

The corresponding quasi-independence model H_6 with respect to this particular set U is actually equivalent to the model that states that responses on items B and C are conditionally independent of each other, given that the responses on items A and D are at levels 2 and 1, respectively.[19] Thus, to test H_6, the usual chi-square test of independence in the 2×2 contingency table (formed by the four response patterns in (4.8)) can be applied. From the chi-square values for H_6 in Table 3c, we see that this model fits the data very well indeed.

As we noted earlier, for the quasi-independence models considered here we first estimated the parameters $\pi_{10}^{\bar{A}}$, $\pi_{10}^{\bar{B}}$, $\pi_{10}^{\bar{C}}$, $\pi_{10}^{\bar{D}}$, π_0 (see, e.g., (4.5)) using the data in set U. However, for the particular set U in (4.8), the parameters $\pi_{10}^{\bar{A}}$, $\pi_{10}^{\bar{D}}$ and π_0 are not identifiable (unless some restrictions are imposed on these parameters), since set U includes only response patterns in which the responses to items A and D are at levels 2 and 1, respectively.[20] In view of the preceding restrictions on the responses to items A and D in set U, we might accordingly introduce restrictions on $\pi_{10}^{\bar{A}}$ and $\pi_{10}^{\bar{D}}$,

$$\pi_{10}^{\bar{A}} = 0, \quad \pi_{10}^{\bar{D}} = 1; \qquad (4.9)$$

and in that particular case, the parameter π_0 can be estimated by the observed proportion p_U of individuals in the 2^4 table whose response patterns are in set U.[21]

We now use the symbol $\#S$ to denote the number of different response patterns in set S. For the 2^4 table, the corresponding number of patterns in set U will be $2^4 - \#S$. From the data in set U, we need to estimate the five parameters $\pi_{10}^{\bar{A}}$, $\pi_{10}^{\bar{B}}$, $\pi_{10}^{\bar{C}}$, $\pi_{10}^{\bar{D}}$ and π_0; and when these are identifiable (which is the case for all the models considered so far, except for H_6), the number of degrees of freedom for testing the corresponding quasi-independence model will be

$$2^4 - \#S - 5. \qquad (4.10)$$

(Compare (4.10) with (1.15).) More generally, in the situation where there are m dichotomous items, the corresponding number of degrees of freedom will be

$$2^m - \#S - (m + 1). \qquad (4.11)$$

(Compare (4.11) with (1.16).) Formulas (4.10) and (4.11) can be applied to all the models considered so far, except for H_6 (see Table 3). Even in the case of H_6, these formulas can be applied if a simple adjustment is made to take account of the fact that *two* additional restrictions have been imposed on the parameters (see (4.9)). In this case, the number of degrees of freedom for testing the model will be $2^4 - 12 - 5 + 2 = 1$, which agrees with

[17] See related comments following (1.15), and the more general formulas (4.10)–(4.11) later.

[18] In order not to confuse the reader who might want to apply H_5 with its eight categories for scale and demi-scale types, Table 3c gives the number of scale (and demi-scale) types, and the number of degrees of freedom, usually appropriate for this model; but in the actual application to the Lazarsfeld-Stouffer data, there really were seven scale (and demi-scale) types (rather than eight), and four degrees of freedom (rather than three).

[19] In the notation used in [4], this model states that $[B \otimes C | A = 2, D = 1]$.

[20] Another example similar to H_6 in this respect will be included as the first model presented in the appendix. For H_6 and for this model in the appendix, the corresponding set U does not provide sufficient data for the estimation of some of the relevant parameters; there are too few patterns in U in both cases. In the rest of this article, this problem does not arise.

[21] The probability that an "intrinsically unscalable" individual's response pattern will be in set U defined by (4.8) is $\pi_{20}^{\bar{A}}\pi_{10}^{\bar{D}}$, which will be equal to one under condition (4.9). In this case, the observed proportion p_U is the maximum-likelihood estimate of π_0. On the other hand, if the particular restriction (4.9) is not imposed, but instead the parameters $\pi_{10}^{\bar{A}}$ and $\pi_{10}^{\bar{D}}$ are equated to two specified numerical values which are such that the product $\pi_{20}^{\bar{A}}\pi_{10}^{\bar{D}}$ is not equal to one (nor equal to zero), then the corresponding maximum-likelihood estimate of π_0 would be greater than p_U. (In passing, we note that $E\{p_U\} = \pi_0\pi_{20}^{\bar{A}}\pi_{10}^{\bar{D}}$). For related comments, see the appendix.

the result obtained from our discussion of Model H_6 earlier in this section (see Table 3c).[22]

We now comment briefly on Model H_7 in Table 3c. As noted earlier, if Model H_1 (with its five scale types) is applied to the Lazarsfeld-Stouffer data, we obtain H_4 (with its four scale types). Similarly, if H_1 (with its five scale types) is applied to the Lazarsfeld-Stouffer data with the order of items C and D interchanged in the data, we obtain another model having four scale types which we shall call H_7.[23] Comparing the chi-squares for H_7 and H_4 in Table 3c, we see that H_7 fits the data slightly better than does H_4. Comparing the estimated proportions $\hat{\pi}_0$ in Table 3c, we see that the same numerical value (to the order of accuracy reported here) is obtained for H_7 and H_4.[24]

Since H_7 fits the data slightly better than does H_4, and the estimated proportion $\hat{\pi}_0$ is not greater for H_7 than for H_4, we pursue further the analysis of the Lazarsfeld-Stouffer data with the order of items C and D interchanged (as was done in H_7). Since H_7 does not really fit the data (see Table 3c), we consider next Model H_5 (with its eight categories for scale and demi-scale types) and H_6 (with its 12 categories for scale, demi-scale, and demi-demi-scale types) applied to the data with the order of the items interchanged (as in H_7). The results obtained in these applications did not actually improve the original results presented in Table 3c for Models H_5 and H_6 (when the order of the items was not interchanged), and to save space we do not go into these details here. If the results obtained in these applications would have improved the original results (e.g., if H_5 would have fit the data better when the order of the items was interchanged, and if the estimated proportion $\hat{\pi}_0$ were not greater in that case), then we would have suggested that the order of the items should be interchanged when considering scaling models for this particular set of data.[25]

5. UNIFORM, BIFORM AND MULTIFORM SCALES

The scaling model in which set S consists of the five response patterns in (1.1) describes the case (H_1) where the four items (A, B, C, D) are ordered $ABCD$; and the ordering $ABDC$ is obtained when pattern (1, 1, 2, 1) replaces pattern 2 in (1.1). In the former case, all "intrinsically scalable" individuals in the population conform to the ordering $ABCD$; and in the latter case, they conform to the ordering $ABDC$. Consider now the case where some of the "intrinsically scalable" individuals in the population conform to the ordering $ABCD$ and the other "intrinsically scalable" individuals conform to the ordering $ABCD$. In this case, the "scale-type categories" include both the scale types under the ordering $ABCD$ *and* the scale types under the ordering $ABDC$. Set S now consists of six patterns: (1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2), (2, 2, 2, 2), (1, 1, 2, 1); viz., the five patterns in (1.1) *and* also pattern (1, 1, 2, 1). The general methods applied in Section 4 (see, e.g., discussion of H_5) can be applied directly to the case considered here. (To save space, we shall not go into these details.)

The model obtained when set S consists of the six patterns just listed is an example of a "biform" scale, in which the two forms correspond to the $ABCD$ and $ABDC$ orderings; whereas the usual scaling Model (H_1) is an example of a "uniform" scale corresponding to the single ordering $ABCD$. For any two given orderings of the items, the corresponding "biform" scale can be analyzed using the general methods described here. For the "biform" scales, set S will consist of no fewer than six patterns (as in the case just considered) and no more than eight patterns (as would be the case if the two given orderings were, say, $ABCD$ and $DABC$). More generally, in the situation where there are m dichotomous items, set S would consist of $m + 1$ response patterns in the "unifrom" scales, and between $m + 2$ and $2m$ response patterns in the "biform" scales.

These comments about "biform" scales can be directly extended to "multiform" scales. These kinds of models will be of substantive interest in various applied contexts, and in addition they will provide us with another way to view the demi-scale and demi-demi-scale models introduced in Section 4: For the demi-scale model H_5, with its eight categories for scale and demi-scale types (see (1.1) and (4.4)), using comments of the kind immediately following (4.4), we find that set S consists of the scale types under the ordering[26]

$$ABCD,\ ABDC,\ ACBD,\ BACD,\ BADC.$$

Thus, the demi-scale model H_5 is an example of a "quinqueform" scale, with the five forms corresponding to the orderings just listed.

For the demi-demi-scale model H_6, with its 12 categories for scale, demi-scale, and demi-demi-scale types (see (1.1), (4.4) and (4.7)), using comments of the kind immediately following (4.7), we find that set S consists of the scale types under the following orderings in addition to the orderings just listed[27]

$$ADCB,\ CBAD,\ ACDB,\ BCAD,\ ADBC,\ CABD,\ CADB.$$

[22] For further comments on identifiability in models of this general kind, see [5].

[23] The fact that a four scale-type model was obtained when H_1 was applied to a given set of data does not necessarily mean that a four scale-type model would also be obtained when H_1 is applied to the same set of data with the order of two of the items interchanged; but it just happened to turn out that way for the Lazarsfeld-Stouffer data.

[24] The numerical value of $\hat{\pi}_0$ obtained for H_7 does not necessarily have to be the same as that obtained for H_4, but it just happened to turn out that way for these data.

[25] More generally, for any given scaling model of the kind introduced in this article, the two criteria just noted (i.e., goodness-of-fit and magnitude of $\hat{\pi}_0$) can be used in attempting to determine the appropriate order of the items, when this order is not clearly established otherwise.

[26] For each of the orderings listed here, the corresponding five scale-type response patterns are all included within the eight patterns in (1.1) and (4.4). Letting d_i denote the difference between the letter in the ith position ($i = 1, 2, 3, 4$) in a given ordering and the corresponding letter in the ordering $ABCD$, we see that $|d_i| \leq 1$ (for $i = 1, 2, 3, 4$) for each ordering listed here. (In calculating the difference between letters, we replace the letters A, B, C, D by the numbers 1, 2, 3, 4, respectively.) Each ordering listed here (except for $ABCD$) is obtained by interchanging the items within a pair of adjacent items, for one or more such disjoint pairs, in the ordering $ABCD$.

[27] For each of the orderings listed here, the corresponding five scale-type response patterns are all included within the twelve patterns in (1.1), (4.4) and (4.7). For d_i defined as in Footnote 26, we see that $|d_i| \leq 2$ (for $i = 1, 2, 3, 4$), *and* $|\sum_{i=1}^{j} d_i| \leq 2$ (for $j = 1, 2, 3, 4$), for each ordering listed here.

Thus, the demi-demi-scale model H_6 is an example of a "duodecaform" scale, with the 12 forms corresponding to the orderings listed in total in this paragraph.

To illustrate the application of the multiform scales, we return for a moment to the Stouffer-Toby data. We shall now let $\mathcal{F}_1$ denote the uniform scale model (i.e., the scale model H_1 introduced in Section 2) using the ordering *ABCD*. Let $\mathcal{F}_2$ denote the biform scale model using the orderings *ABCD* and *ACBD*; let $\mathcal{F}_5$ denote the quinqueform scale model discussed earlier in this section (i.e., the demi-scale model H_5 introduced in Section 4); let $\mathcal{F}_8$ denote the octaform scale model using the orderings *ACDB, CADB, CABD* and the five orderings in $\mathcal{F}_5$; and let $\mathcal{F}_{12}$ denote the duodecaform scale model discussed earlier in this section (i.e., the demi-demi-scale model H_6 introduced in Section 4). The scale-type response patterns for $\mathcal{F}_1$, $\mathcal{F}_5$, and $\mathcal{F}_{12}$ were listed earlier, and we display in the figure the corresponding patterns for the other two models just described ($\mathcal{F}_2$ and $\mathcal{F}_8$).[28] Table 4 summarizes some of the results obtained

4. Some Estimated Parameters and Chi-Square Values for the Uniform Scale Model and Some Multiform Scale Models Applied to the Stouffer-Toby Data

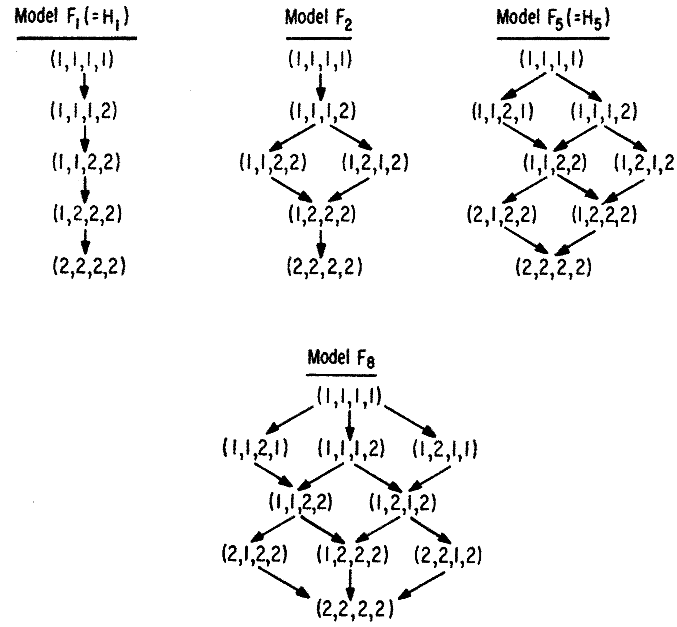
Model	Number of scale types	Estimated proportion $\hat{\pi}_0$ of intrinsically unscalable individuals	Degrees of freedom	Goodness-of-fit chi-square	Likelihood-ratio chi-square
$\mathcal{F}_1(=H_1)$	5	.68	6	1.01	.99
$\mathcal{F}_2$	6[a]	.68	5[a]	1.01	.99
$\mathcal{F}_5(=H_5)$	8[a]	.59	3[a]	.11	.11
$\mathcal{F}_8$	10[a]	.59	1[a]	.03	.03
$\mathcal{F}_{12}(=H_6)$	12	.03[b]	1	.00	.00

[a] For comments on the results actually obtained for the Stouffer-Toby data, see Footnote 29.

[b] Estimate obtained under condition (4.9).

with these models.[29] Since Model $\mathcal{F}_1$ fits the data well (see also H_1 in Table 3a), there was no need to introduce additional scale-type patterns into $\mathcal{F}_1$ (to obtain $\mathcal{F}_2$, $\mathcal{F}_5$, $\cdots$); but we did so here for illustrative purposes, and

The Scale-Type Response Patterns in the Uniform Scale and in Some Multiform Scales



to show how $\hat{\pi}_0$ changes with the increase in scale-type patterns.[30]

6. THE NEW SCALING MODEL, THE USUAL SCALING MODEL AND THE USUAL LATENT CLASS MODELS

The scaling model was described at the beginning of this article by formulas (1.5)–(1.6) for the case (H_1), where the response patterns in (1.1) formed set S; and we then saw how to extend this model to the case where a proper subset of the patterns in (1.1) formed set S and to the case where set S consisted of the patterns in (1.1) and some other specified patterns as well. We shall now comment on the relationship between these models, the more usual Guttman scaling model, and the more usual latent class models. For the sake of simplicity, we shall focus our attention on the relationship between H_1 and the other more usual approaches; but these comments can be extended to cover the other scaling models that can be obtained with the general approach introduced here.

We noted earlier that the usual Guttman scaling model is a special case of Model H_1 in which all individuals in the population under consideration are assumed to be "intrinsically scalable." In terms of the parameters used in (1.5)–(1.6) to describe H_1, the usual Guttman scaling model assumes that $\pi_0 = 0$. This assumption is not supported by the data considered here, and the estimates of π_0 in Tables 3 and 4 shed some light on the magnitude of the discrepancy between the actual value of π_0 and the assumed value of zero.

[28] For purposes of comparison, Models $\mathcal{F}_1$ and $\mathcal{F}_5$ are also included in the figure; the corresponding display of $\mathcal{F}_{12}$ is somewhat more complicated than those included there. Each of these models could be displayed in various ways; e.g., the patterns in $\mathcal{F}_5$ could also be displayed as in $\mathcal{F}_2$ with patterns (1, 1, 2, 1) and (2, 1, 2, 2) inserted to the left of (1, 1, 1, 2) and (1, 2, 2, 2), respectively; or, equivalently, as in $\mathcal{F}_8$ with patterns (1, 2, 1, 1) and (2, 2, 1, 2) deleted. In the displays for both $\mathcal{F}_2$ and $\mathcal{F}_8$, the response patterns that are at levels (1, 2) and (2, 1) on the joint variable (*B, C*) are on the left and right sides, respectively, of the display; and the patterns that are at levels (1, 1) and (2, 2) on variable (*B, C*) are on a vertical center line. Each of the models considered in this section can be described in terms of its scale-type patterns (as in Section 4), or in terms of the orderings that yield those patterns (as in this section), or in terms of various properties of the corresponding displays.

[29] With respect to Models $\mathcal{F}_2$, $\mathcal{F}_5$ and $\mathcal{F}_8$ in Table 4, a comment of the same kind as in Footnote 18 should be included here. In the actual application to the Stouffer-Toby data, for Model $\mathcal{F}_2$, there really were five scale types (rather than six), and six degrees of freedom (rather than five); for $\mathcal{F}_5$, there really were seven scale types (rather than eight), and four degrees of freedom (rather than three); for $\mathcal{F}_8$, there really were nine scale types (rather than ten), and two degrees of freedom (rather than one). For $\mathcal{F}_2$ and $\mathcal{F}_5$, the scale-type category corresponding to response pattern (1, 2, 1, 2) needed to be void; and for $\mathcal{F}_8$, the scale-type category corresponding to (1, 2, 1, 1) needed to be void.

[30] The estimate $\hat{\pi}_0$ will depend upon the proportion of individuals whose observed response patterns are in set U *and* the observed one-way marginal distributions (with respect to items *A, B, C* and *D*) obtained for those individuals with patterns in U. For some further insight into $\hat{\pi}_0$, see related comments in the appendix.

Consider next the usual latent class model in which there are six latent classes, which we shall number 0, 1, 2, $\cdots$, 5. We shall use the letter X to denote the corresponding latent variable, with its six latent classes (or levels). We let π_t^X denote the probability that an individual in the population will be at level t on variable X (for $t = 0, 1, 2, \cdots, 5$); we also let $\pi_{it}^{\bar{A}X}$ denote the *conditional* probability that the individual will be at level i on variable A (for $i = 1, 2$), given that he was at level t on variable X; and we let the *conditional* probabilities $\pi_{jt}^{\bar{B}X}$, $\pi_{kt}^{\bar{C}X}$, $\pi_{lt}^{\bar{D}X}$ be defined similarly (for $j = 1, 2$; $k = 1, 2$; $l = 1, 2$). As earlier, we let π_{ijkl}^{ABCD} denote the probability that an individual will be at level (i, j, k, l) on the joint variable (A, B, C, D); we also let π_{ijklt}^{ABCDX} denote the probability that an individual will be at level (i, j, k, l, t) on the joint variable (A, B, C, D, X); and we let $\pi_{ijklt}^{\bar{A}\bar{B}\bar{C}\bar{D}X}$ denote the *conditional* probability that the individual will be at level (i, j, k, l) on the joint variable (A, B, C, D), given that he was at level t on variable X. The latent class model can be expressed by

$$\pi_{ijkl}^{ABCD} = \sum_{t=0}^{5} \pi_{ijklt}^{ABCDX}, \tag{6.1}$$

$$\pi_{ijklt}^{ABCDX} = \pi_t^X \pi_{ijklt}^{\bar{A}\bar{B}\bar{C}\bar{D}X}, \tag{6.2}$$

and

$$\pi_{ijklt}^{\bar{A}\bar{B}\bar{C}\bar{D}X} = \pi_{it}^{\bar{A}X} \pi_{jt}^{\bar{B}X} \pi_{kt}^{\bar{C}X} \pi_{lt}^{\bar{D}X}. \tag{6.3}$$

Formula (6.1) states that the individuals can be classified into six mutually exclusive and exhaustive latent classes ($t = 0, 1, 2, \cdots, 5$) with respect to the latent variable X; (6.2) follows directly from the definition of the π's appearing in (6.2); and (6.3) states that within the tth latent class the levels on variables A, B, C and D are mutually independent (for $t = 0, 1, 2, \cdots, 5$).

Since this latent class model has four manifest variables (A, B, C, D) and six latent classes, the parameters in the model will not be identifiable unless some restrictions are imposed (see, e.g., [5]). Various kinds of restrictions could be imposed. Consider now the following set of restrictions imposed on the parameters

$$\begin{aligned}
\pi_{11}^{\bar{A}X} &= \pi_{11}^{\bar{B}X} = \pi_{11}^{\bar{C}X} = \pi_{11}^{\bar{D}X} = 1, \\
\pi_{12}^{\bar{A}X} &= \pi_{12}^{\bar{B}X} = \pi_{12}^{\bar{C}X} = \pi_{22}^{\bar{D}X} = 1, \\
\pi_{13}^{\bar{A}X} &= \pi_{13}^{\bar{B}X} = \pi_{23}^{\bar{C}X} = \pi_{23}^{\bar{D}X} = 1, \\
\pi_{14}^{\bar{A}X} &= \pi_{24}^{\bar{B}X} = \pi_{24}^{\bar{C}X} = \pi_{24}^{\bar{D}X} = 1, \\
\pi_{25}^{\bar{A}X} &= \pi_{25}^{\bar{B}X} = \pi_{25}^{\bar{C}X} = \pi_{25}^{\bar{D}X} = 1.
\end{aligned} \tag{6.4}$$

From (6.1)–(6.3) and (6.4), we see that

$$\begin{aligned}
\pi_{1111}^{ABCD} &= \pi_0^X \pi_{10}^{\bar{A}X} \pi_{10}^{\bar{B}X} \pi_{10}^{\bar{C}X} \pi_{10}^{\bar{D}X} + \pi_1^X, \\
\pi_{1112}^{ABCD} &= \pi_0^X \pi_{10}^{\bar{A}X} \pi_{10}^{\bar{B}X} \pi_{10}^{\bar{C}X} \pi_{20}^{\bar{D}X} + \pi_2^X, \\
\pi_{1122}^{ABCD} &= \pi_0^X \pi_{10}^{\bar{A}X} \pi_{10}^{\bar{B}X} \pi_{20}^{\bar{C}X} \pi_{20}^{\bar{D}X} + \pi_3^X, \\
\pi_{1222}^{ABCD} &= \pi_0^X \pi_{10}^{\bar{A}X} \pi_{20}^{\bar{B}X} \pi_{20}^{\bar{C}X} \pi_{20}^{\bar{D}X} + \pi_4^X, \\
\pi_{2222}^{ABCD} &= \pi_0^X \pi_{20}^{\bar{A}X} \pi_{20}^{\bar{B}X} \pi_{20}^{\bar{C}X} \pi_{20}^{\bar{D}X} + \pi_5^X;
\end{aligned} \tag{6.5}$$

and for any other level (i, j, k, l) on the joint variable (A, B, C, D), we obtain

$$\pi_{ijkl}^{ABCD} = \pi_0^X \pi_{i0}^{\bar{A}X} \pi_{j0}^{\bar{B}X} \pi_{k0}^{\bar{C}X} \pi_{l0}^{\bar{D}X}. \tag{6.6}$$

Comparing (6.5)–(6.6) with (1.5)–(1.6), we see that the scaling model H_1 is a special case of a latent class model; i.e., it is the latent class model (6.1)–(6.3) with the additional restrictions in (6.4) imposed on the parameters.

For purposes of comparison, we consider next a latent class model that closely resembles the usual Guttman scaling model; viz., Proctor's model [14].[31] It differs from the model introduced here in that it has five latent classes[32] rather than six (i.e., it has no "intrinsically unscalable" class); and under this model an "intrinsically scalable" individual's response on a given item will correspond to his scale type with probability π (rather than one). Thus, the conditional probabilities in (6.4) are equated to π here (rather than to one), and the "response error rate" is $1 - \pi$. This model is a special case of the *restricted* latent structures considered in [5, 6], and the statistical methods developed in those articles can be applied to this special case to obtain the results reported in Table 5.[33] For the three sets of data considered, by comparing Table 5 with Tables 3 and 4, we find that the general approach introduced earlier led to models that fit the data better than Proctor's model.[34]

5. Some Estimated Parameters and Chi-Square Values for Proctor's Model of Guttman Scaling, Applied to the Data in Table 1

Data	Number of scale types	Response error rate	Degrees of freedom	Goodness-of-fit chi-square	Likelihood-ratio chi-square
Stouffer-Toby	5	15	10	28.03	27.16
McHugh	5[a]	.18	10[a]	25.20	20.31
Lazarsfeld-Stouffer	5	.16	10	71.54	75.60

[a] For comments on the results actually obtained for the McHugh data, see Footnote 33.

In addition to Proctor's model, other latent class and latent distance models could be applied to these data; but to save space we shall not go into this here. The interested reader can compare the results presented here

[31] This model was also discussed briefly in [11]. It is a special case of a latent distance model discussed in [9, 10] and of the more general latent distance model discussed in [11]. For this special case, Proctor [14] provided maximum-likelihood estimates of the parameters in the model, and he showed how the special case could be used to analyze various sets of data.

[32] More generally, when there are m dichotomous items, there will be $m + 1$ latent classes in Proctor's model.

[33] With respect to the results for the McHugh data in Table 5, a comment of the same kind as in Footnote 18 should be included here. In the actual application to the McHugh data, there really were three scale types (rather than five), and twelve degrees of freedom (rather than ten); the scale-type categories corresponding to the response patterns (1, 1, 1, 2) and (1, 2, 2, 2) needed to be void.

[34] The appropriate models in Tables 3 and 4 fit the data better than Proctor's model, but there were fewer degrees of freedom associated with the former models. The scaling models introduced earlier can be modified, by methods described later in Section 8, to obtain new scaling models that have more degrees of freedom associated with them.

using the new scaling models with results obtained using various latent class and latent distance models.[35]

7. SOME FURTHER COMMENTS ON ESTIMATION METHODS

We began this article by showing how the methods which had been developed earlier for the analysis of the quasi-independence model could be applied to calculate the maximum-likelihood estimates of the parameters in the new scaling models introduced here. On the other hand, in view of the equivalence noted in Section 6, between these new scaling models and the corresponding latent class model described there (see (6.1)–(6.4)), an alternative method for estimating these parameters is available; viz., the general procedure presented in [5, 6] for calculating maximum-likelihood estimates of the parameters in a wide variety of latent class models.[36] However, the method described in Section 1 is somewhat easier to apply.

In calculating the maximum-likelihood estimates using the method described in Section 1 (applying the quasi-independence model to set U), we *first* calculated the estimates $\hat{\pi}_{10}^{\bar{A}}$, $\hat{\pi}_{10}^{\bar{B}}$, $\hat{\pi}_{10}^{\bar{C}}$, $\hat{\pi}_{10}^{\bar{D}}$, $\hat{\pi}_0$ using the iterative procedure mentioned earlier, and *then* calculated $\hat{\pi}_t$ (for $t = 1, 2, \cdots, 5$) and $\hat{\pi}_{ijkl}$ (see (1.9) and comments following it). Another alternative to this method would be to *first* calculate the estimates $\hat{\pi}_{ijkl}$, under the quasi-independence model applied to set U, using an appropriate kind of iterative procedure (see, e.g., [1; 2, p. 1119–20]);[37] and *then* calculate the estimates $\hat{\pi}_{10}^{\bar{A}}$, $\hat{\pi}_{10}^{\bar{B}}$, $\hat{\pi}_{10}^{\bar{C}}$, $\hat{\pi}_{10}^{\bar{D}}$, and $\hat{\pi}_t$ (for $t = 0, 1, 2, \cdots, 5$).

Let $\Omega_{.0}^{\bar{A}}$ denote the odds in favor of a level-1 (rather than a level-2) response on item A for the individuals in the population who are in the 0th category, i.e.,

$$\Omega_{.0}^{\bar{A}} = \pi_{10}^{\bar{A}}/\pi_{20}^{\bar{A}}; \tag{7.1}$$

and let the odds $\Omega_{.0}^{\bar{B}}$, $\Omega_{.0}^{\bar{C}}$, $\Omega_{.0}^{\bar{D}}$ be defined similarly. From (7.1) and (1.6), we see that

$$\Omega_{.0}^{\bar{A}} = \pi_{1jkl}^{ABCD}/\pi_{2jkl}^{ABCD}, \tag{7.2}$$

for any pair of response patterns $(1, j, k, l)$ and $(2, j, k, l)$ that are included in set U; and similar formulas can be obtained for the $\Omega_{.0}^{\bar{B}}$, $\Omega_{.0}^{\bar{C}}$, $\Omega_{.0}^{\bar{D}}$. From (7.1) we see that

$$\pi_{10}^{\bar{A}} = \Omega_{.0}^{\bar{A}}/(1 + \Omega_{.0}^{\bar{A}}). \tag{7.3}$$

To obtain the maximum-likelihood estimate for $\pi_{10}^{\bar{A}}$, we insert the corresponding estimates $\hat{\pi}_{1jkl}^{ABCD}$ and $\hat{\pi}_{2jkl}^{ABCD}$ on the right side of (7.2) to obtain $\hat{\Omega}_{.0}^{\bar{A}}$, and we then insert $\hat{\Omega}_{.0}^{\bar{A}}$ on the right side of (7.3) to obtain $\hat{\pi}_{10}^{\bar{A}}$. A similar kind of calculation will also yield $\hat{\pi}_{10}^{\bar{B}}$, $\hat{\pi}_{10}^{\bar{C}}$, $\hat{\pi}_{10}^{\bar{D}}$; and from (1.6) we see that $\hat{\pi}_0$ can be obtained as follows:

$$\hat{\pi}_0 = \hat{\pi}_{ijkl}^{ABCD}/(\hat{\pi}_{i0}^{\bar{A}}\hat{\pi}_{j0}^{\bar{B}}\hat{\pi}_{k0}^{\bar{C}}\hat{\pi}_{l0}^{\bar{D}}), \tag{7.4}$$

for any pattern (i, j, k, l) that is included in set U. Having thus obtained $\hat{\pi}_{10}^{\bar{A}}$, $\hat{\pi}_{10}^{\bar{B}}$, $\hat{\pi}_{10}^{\bar{C}}$, $\hat{\pi}_{10}^{\bar{D}}$ and $\hat{\pi}_0$, we can then use (1.9) to obtain estimates for the remaining parameters π_t, for $t = 1, 2, \cdots, 5$.[38]

We have just noted two alternatives to the estimation method presented in Section 1: first, a method based on the more general procedure appropriate for latent class models; and second, a method based on the quasi-independence model (as in Section 1), but differing in detail from the procedure suggested earlier. Each of these methods yields the same results. The researcher can choose whichever technique he finds most convenient.

8. SOME NEW SCALING MODELS THAT ARE MORE PARSIMONIOUS

In our discussion of the quasi-independence model for set U, we noted in Section 1 that there were five independent parameters estimated under the model, $\pi_{10}^{\bar{A}}$, $\pi_{10}^{\bar{B}}$, $\pi_{10}^{\bar{C}}$, $\pi_{10}^{\bar{D}}$ and π_0. The first four parameters are the *conditional* probabilities of being at level 1 on variables A, B, C and D, respectively, for the individuals in the 0th category (viz., the "intrinsically unscalable" category). In some contexts, there may be good reason to consider models in which it is assumed, e.g., that the following condition holds true:

$$\pi_{10}^{\bar{A}} = \pi_{10}^{\bar{B}} = \pi_{10}^{\bar{C}} = \pi_{10}^{\bar{D}}; \tag{8.1}$$

or, more generally, that the parameters in some specified subset of the four parameters in (8.1) are equal to each other. With these kinds of restrictions imposed on the parameters in the model, the number of independent parameters estimated under the model is reduced. (For example, under (8.1), there will be two rather than five independent parameters estimated under the model.) With models of this kind, we can assess whether a particular model is congruent with the observed data, and whether the particular restrictions (e.g., (8.1)) are supported by the data.

In Section 7 we noted that the general estimation procedure presented in [5, 6] can be applied to a wide variety of latent class models including the scaling models considered earlier, where restrictions of the kind described in (6.4) are imposed. In addition, this general estimation procedure can also be applied to scaling models of the kind just described where restrictions of the kind described in (8.1) are imposed (see, e.g., [6]).

---

[35] See, e.g., [5, 6, 7, 10, 11, 13]. The methods for analyzing the new scaling models, presented earlier, are somewhat easier to apply than the corresponding methods for the latent class and latent distance models. Compare, e.g., the analysis of the Stouffer-Toby data in [5], or the McHugh data in [6], with the analysis of these data presented here. Considering the various models that may fit a given set of data, the choice among them would depend, in part, on which models are more meaningful in the particular substantive context.

[36] This general procedure can be applied to latent class models in which restrictions of the kind described by (6.4) are imposed, or in which various other kinds of restrictions are imposed. As noted in Section 6, we used this general procedure to obtain the results reported in Table 5 for Proctor's model, and it can also be used to obtain the results reported in Tables 2, 3 and 4. Other latent class and latent distance models can also be analyzed using this general procedure.

[37] The $\hat{\pi}_{ijkl}$ can be calculated using, e.g., the computer program ECTA (Everyman's Contingency Table Analyzer), prepared by R. Fay and the author, and available from the author.

[38] As noted in Section 4, whenever the estimate of the π_t obtained from (1.9) turns out to be negative for a subset of the π_t's (for $t = 1, 2, 3, 4, 5$), to obtain the maximum-likelihood estimates of the parameters under the model, we first need to make one or more of the π_t's in that particular subset equal to zero, thus deleting the corresponding response pattern from set S.

The kind of restriction described in (8.1) pertained to the same level (level 1) on each of the variables A, B, C and D. On the other hand, in some contexts, we might want to consider the more general kind of restriction

$$\pi_{i0}{}^{\bar{A}} = \pi_{j0}{}^{\bar{B}} = \pi_{k0}{}^{\bar{C}} = \pi_{l0}{}^{\bar{D}}, \qquad (8.2)$$

for a specified level i, j, k, l on variables A, B, C, D, respectively (e.g., level 1, 2, 1, 2, rather than level 1, 1, 1, 1 as in (8.1)). Even more generally, we might want to consider the restriction that the parameters in some specified subset of the four parameters in (8.2) are equal to each other. To analyze scaling models in which these more general kinds of restrictions are imposed, we can also apply the general estimation procedure just mentioned. The results in Table 6 were obtained in this way, for various *restricted* scaling models applied to the data in Table 1.

Model H_2' in Table 6 states that H_2 is true *and* that the parameters in the model satisfy the condition

$$\pi_{10}{}^{\bar{B}} = \pi_{10}{}^{\bar{C}}. \qquad (8.3)$$

Because of the restriction imposed in (8.3), model H_2' has one less parameter to estimate than H_2, and there is one more degree of freedom for testing H_2' than for testing H_2. Model H_2'' in Table 6 states that H_2 is true *and* that the parameters in the model satisfy the two conditions

$$\pi_{10}{}^{\bar{B}} = \pi_{10}{}^{\bar{C}} \quad \text{and} \quad \pi_{10}{}^{\bar{A}} = \pi_{20}{}^{\bar{D}}. \qquad (8.4)$$

Because of the two restrictions imposed in (8.4), model H_2'' has two less parameters to estimate than H_2, and there are two more degrees of freedom for testing H_2'' than for testing H_2. By comparing the chi-square values for H_2 and H_2', and then the chi-square values for H_2' and H_2'', we see that the restrictions imposed in (8.3) and in (8.4) are supported by the data to which those particular models were applied (i.e., the Stouffer-Toby data). These three models (i.e., H_2, H_2', H_2'') fit the data very well indeed, but H_2' is slightly more parsimonious than H_2, and H_2'' is slightly more parsimonious than H_2'.

The restricted scaling models in Table 6 for the McHugh data and the Lazarsfeld-Stouffer data can be analyzed by the same methods just used for the analysis of the Stouffer-Toby data. We leave these details for the interested reader.

In Section 1 we presented a method for calculating the maximum-likelihood estimates of the parameters in the scaling model introduced there (i.e., the unrestricted scaling model), which was based on the procedure used for the quasi-independence model; and in Section 7 we noted that an alternative method could be based on the more general procedure appropriate for latent class models. In this section we used the general procedure appropriate for latent class models to calculate the maximum-likelihood estimates of the parameters in the *restricted* scaling models introduced here. For those who wish to analyze *restricted* scaling models but prefer to use procedures similar to those used for the quasi-independence model, we now show how this can be done.

For response pattern (i, j, k, l), let $\delta_{ijkl} = 1$ if this pattern is in set U, and let $\delta_{ijkl} = 0$ otherwise. Under the usual (i.e., unrestricted) quasi-independence model for set U (see (1.7)), the maximum-likelihood estimate $\hat{\pi}_{ijkl}^{ABCD}$ of the corresponding probability π_{ijkl}^{ABCD} (i.e., the probability of obtaining pattern (i, j, k, l)) can be calculated from

$$\hat{\pi}_{ijkl}^{ABCD} = \hat{\alpha}_i\hat{\beta}_j\hat{\gamma}_k\hat{\delta}_l, \quad \text{for} \quad (i, j, k, l) \in U, \qquad (8.5)$$

where the $\hat{\alpha}_i$, $\hat{\beta}_j$, $\hat{\gamma}_k$, $\hat{\delta}_l$ satisfy

$$\begin{aligned} \overset{A}{p}_{i\ldots} &= \hat{\alpha}_i \sum_{j,k,l} \delta_{ijkl}\hat{\beta}_j\hat{\gamma}_k\hat{\delta}_l, & \overset{B}{p}_{.j..} &= \hat{\beta}_j \sum_{i,k,l} \delta_{ijkl}\hat{\alpha}_i\hat{\gamma}_k\hat{\delta}_l, \\ \overset{C}{p}_{..k.} &= \hat{\gamma}_k \sum_{i,j,l} \delta_{ijkl}\hat{\alpha}_i\hat{\beta}_j\hat{\delta}_l, & \overset{D}{p}_{\ldots l} &= \hat{\delta}_l \sum_{i,j,k} \delta_{ijkl}\hat{\alpha}_i\hat{\beta}_j\hat{\gamma}_k, \end{aligned} \qquad (8.6)$$

6. Some Estimated Parameters and Chi-Square Values for Various Restricted Scaling Models, Applied to the Data in Table 1

Model	*Number of scale types*	*Restrictions imposed upon the parameters*	$\hat{\pi}_0$	*Degrees of freedom*	*Goodness-of-fit chi-square*	*Likelihood-ratio chi-square*
			a. Stouffer-Toby Data			
H_2'	2	$\pi_{10}^{\bar{B}} = \pi_{10}^{\bar{B}}$	.78	10	2.42	2.39
H_2''	2	$\pi_{10}^{\bar{B}} = \pi_{10}^{\bar{C}}$, $\pi_{10}^{\bar{A}} = \pi_{20}^{\bar{D}}$	.77	11	2.85	2.72
			b. McHugh Data			
H_3'	4	$\pi_{10}^{\bar{A}} = \pi_{10}^{\bar{B}}$	.52	8	7.11	7.43
H_3''	4	$\pi_{10}^{\bar{A}} = \pi_{10}^{\bar{B}}$, $\pi_{10}^{\bar{C}} = \pi_{10}^{\bar{D}}$	.52	9	8.00	8.17
H_3'''	4	$\pi_{10}^{\bar{A}} = \pi_{10}^{\bar{B}} = \pi_{10}^{\bar{C}} = \pi_{10}^{\bar{D}}$	.56	10	13.45	12.91
			c. Lazarsfeld-Stouffer Data			
H_5'	7	$\pi_{10}^{\bar{B}} = \pi_{10}^{\bar{D}}$	.48	5	5.46	5.87
H_5''	7	$\pi_{10}^{\bar{B}} = \pi_{10}^{\bar{D}} = \pi_{20}^{\bar{A}}$	.49	6	10.06	9.65

where $p^A_{i\cdots}$ is calculated from the observed proportion p^{ABCD}_{ijkl},

$$p^{A}_{i\cdots} = \sum_{j,k,l} \delta_{ijkl} p^{ABCD}_{ijkl} , \qquad (8.7)$$

and where $p^B_{\cdot j\cdot\cdot}$, $p^C_{\cdot\cdot k\cdot}$ and $p^D_{\cdot\cdot\cdot l}$ are calculated similarly. On the other hand, under the *restricted* quasi-independence model for set U, where, e.g., the restriction in (8.8) is imposed on the parameters,

$$\pi_{10}^{\bar{A}} = \pi_{10}^{\bar{B}}, \qquad (8.8)$$

(1.7) would be replaced by

$$\pi^{ABCD}_{ijkl} = \alpha_i\alpha_j\gamma_k\delta_l, \quad \text{for} \quad (i, j, k, l) \in U, \qquad (8.9)$$

since $\beta_j = \alpha_j$ (for $j = 1, 2$) under the restriction in (8.8), (8.5) would be replaced by

$$\hat{\pi}^{ABCD}_{ijkl} = \hat{\alpha}_i\hat{\alpha}_j\hat{\gamma}_k\hat{\delta}_l, \qquad (8.10)$$

and (8.6) would be replaced by[39]

$$\begin{aligned} &p^{A}_{i\cdots} + p^{B}_{\cdot i\cdot\cdot} \\ &\quad = \hat{\alpha}_i\Big[\sum_{j,k,l} \delta_{ijkl}\hat{\alpha}_j\hat{\gamma}_k\hat{\delta}_l + \sum_{t,k,l} \delta_{tikl}\hat{\alpha}_t\hat{\gamma}_k\hat{\delta}_l\Big], \\ &p^{C}_{\cdot\cdot k\cdot} = \hat{\gamma}_k \sum_{i,j,l} \delta_{ijkl}\hat{\alpha}_i\hat{\alpha}_j\hat{\delta}_l, \quad p^{D}_{\cdot\cdot\cdot l} = \hat{\delta}_l \sum_{i,j,k} \delta_{ijkl}\hat{\alpha}_i\hat{\alpha}_j\hat{\gamma}_k. \end{aligned} \qquad (8.11)$$

The kind of iterative procedure presented in [2, p. 1118–9] for calculating the $\hat{\alpha}_i$, $\hat{\beta}_j$, $\hat{\gamma}_k$, $\hat{\delta}_l$ in (8.6) can be modified in a straightforward way in order to calculate the corresponding quantities in (8.11). Similar modifications can be made for the other kinds of *restricted* quasi-independence models considered earlier in this section (see, e.g., (8.1)–(8.4)).

APPENDIX

We noted in Section 1 that the model and methods introduced can be applied directly to the case where there are m dichotomous variables, for $m = 3, 4, \cdots$. Further insight into this approach can be obtained by considering in more detail the special case where $m = 2$ and $m = 3$. When $m = 2$, a problem of identifiability arises, and the discussion of this problem here will shed further light on our analysis of H_6 in Section 4 and on related matters. When $m = 3$, we can present explicit formulas for the estimated parameters, and these formulas will shed further light on the meaning of $\hat{\pi}_0$ and related quantities.

Consider first the situation where there are two dichotomous items, say, A and B, and let (i, j) denote the response pattern in which responses on items A and B are at levels i and j, respectively (for $i = 1, 2; j = 1, 2$). Corresponding to the five scale-type patterns in (1), we now have three such patterns:

$$(1, 1),\ (1, 2),\ (2, 2). \qquad (A.1)$$

Here set S consists of the three patterns in (A.1), and set U consists of the remaining pattern (2, 1) in the 2×2 table.[40]

Corresponding to the six categories in the model (H_1) introduced earlier, we now have a total of four categories: the "intrinsically unscalable" category and the three "intrinsically scalable" categories pertaining to the patterns in (A.1). The parameters in the scaling model are now π_t (for $t = 0, 1, 2, 3$) and the conditional probabilities $\pi_{i0}^{\bar{A}}$ and $\pi_{j0}^{\bar{B}}$ pertaining to the 0th category (i.e., the intrinsically unscalable category).

Since set U consists only of one response pattern, we face an identifiability problem here similar to the one faced with H_6 in Section 4. To deal with this problem, we need to impose some restrictions on the relevant conditional probabilities (see, e.g., (4.9)). Accordingly, we now introduce restrictions on $\pi_{10}^{\bar{A}}$ and $\pi_{10}^{\bar{B}}$,

$$\pi_{10}^{\bar{A}} = 0\ , \quad \pi_{10}^{\bar{B}} = 1\ ; \qquad (A.2)$$

and in that particular case, the parameter π_0 can be estimated by the observed proportion $p_U = p_{21}$.[41]

More generally, if we set $\pi_{10}^{\bar{A}}$ and $\pi_{10}^{\bar{B}}$ equal to two specified numerical values, then $\hat{\pi}_0 = p_{21}/\pi_{20}^{\bar{A}}\pi_{10}^{\bar{B}}$, and we obtain the explicit formulas for the maximum likelihood estimate of π_t, for $t = 1, 2, 3$,[42]

$$\begin{aligned} \hat{\pi}_1 &= p_{11} - \hat{\pi}_0\pi_{10}^{\bar{A}}\pi_{10}^{\bar{B}} = p_{11} - p_{21}\Omega^{\bar{A}}\ , \\ \hat{\pi}_2 &= p_{12} - \hat{\pi}_0\pi_{10}^{\bar{A}}\pi_{20}^{\bar{B}} = p_{12} - p_{21}\Omega^{\bar{A}}/\Omega^{\bar{B}}\ , \\ \hat{\pi}_3 &= p_{22} - \hat{\pi}_0\pi_{20}^{\bar{A}}\pi_{20}^{\bar{B}} = p_{22} - p_{21}/\Omega^{\bar{B}}\ , \end{aligned} \qquad (A.3)$$

where $\Omega^{\bar{A}}$ denotes the odds defined as in (7.1) and $\Omega^{\bar{B}}$ is defined similarly. (Compare (A.3) with (1.9).)

In this particular case, since set U consists only of one response pattern, the corresponding model of quasi-independence cannot be tested.

Consider next the case where there are three dichotomous items, say, A, B and C, and let (i, j, k) denote the response pattern in which items A, B and C are at levels i, j and k, respectively (for $i = 1, 2; j = 1, 2; k = 1, 2$). Corresponding to the three scale-type patterns in (A.1), we now have four such patterns,

$$(1, 1, 1),\ (1, 1, 2),\ (1, 2, 2),\ (2, 2, 2)\ . \qquad (A.4)$$

Here set S consists of the four patterns in (A.4), and set U consists of the remaining four patterns in the $2 \times 2 \times 2$ table,

$$(1, 2, 1),\ (2, 1, 1),\ (2, 2, 1),\ (2, 1, 2)\ . \qquad (A.5)$$

In this case, we obtain the explicit formulas for the maximum-likelihood estimates of the odds defined as in (7.1),[43]

$$\hat{\Omega}_{\cdot 0}^{\bar{A}} = p_{121}/p_{221}\ , \quad \hat{\Omega}_{\cdot 0}^{\bar{B}} = p_{211}/p_{221}\ , \quad \hat{\Omega}_{\cdot 0}^{\bar{C}} = p_{211}/p_{212}\ ; \qquad (A.6)$$

and the corresponding estimated probabilities $\hat{\pi}_{10}^{\bar{A}}$, $\hat{\pi}_{10}^{\bar{B}}$, $\hat{\pi}_{10}^{\bar{C}}$ can be calculated directly from the estimated odds (see (7.3)). Similarly, we obtain the explicit formula for $\hat{\pi}_0$,[44]

$$\hat{\pi}_0 = (p_{121} + p_{221})(p_{211} + p_{221})(p_{211} + p_{212})/p_{211}p_{221}\ . \qquad (A.7)$$

Letting π^{ABC}_{ijk0} denote the probability that an individual will be in the 0th category *and* will be at level (i, j, k) on the joint variable (A, B, C), we see that

$$\hat{\pi}^{ABC}_{ijk0} = \hat{\pi}_0\hat{\pi}_{i0}^{\bar{A}}\hat{\pi}_{j0}^{\bar{B}}\hat{\pi}_{k0}^{\bar{C}}\ , \qquad (A.8)$$

[39] Note that the formula on the first line in (8.11) is obtained from the two formulas on the first line in (8.6), by setting $\hat{\beta}_j = \hat{\alpha}_j$ in the two formulas in (8.6), then adding the corresponding terms on the left side of the two formulas, and equating the sum thus obtained on the left with the corresponding sum obtained on the right side of the two formulas.

[40] Letting p_U denote the observed proportion of individuals whose response patterns are in set U, we see that, in this particular case, this proportion is equal to the observed proportion p_{21} of individuals whose response pattern is at level (2, 1) on the joint variable (A, B).

[41] See related comments in Footnote 21.

[42] To avoid unnecessary complications, the specified numerical values of $\pi_{10}^{\bar{A}}$ and $\pi_{10}^{\bar{B}}$ should be such that none of the $\hat{\pi}_t$ in (A.3) are negative, and $\hat{\pi}_0 \leq 1$. This can be accomplished, e.g., with (A.2).

[43] For the sake of simplicity, we assume here that $p_{ijk} > 0$, for $(i, j, k) \in U$; or, more generally, that $p_{ijk} > 0$ for enough of the patterns (i, j, k) in set U so that the relevant parameters can be estimated. In this particular case, we need to assume that $p_{211} > 0$ and $p_{221} > 0$ (see (A.7) and (A.10)). For a related comment see Footnote 3.

[44] The $\hat{\pi}_0$ defined by (A.7) will satisfy the inequality $0 < \hat{\pi}_0 \leq 1$, if none of the $\hat{\pi}_t$ in (A.9) are negative. Similarly, the estimate $\hat{\pi}_0$ obtained from (1.8) will satisfy this inequality if none of the $\hat{\pi}_t$ in (1.9) are negative. This result for $\hat{\pi}_0$ holds true more generally when there are m dichotomous items ($m = 3, 4, \cdots$).

and the formulas corresponding to (1.9) and (A.3) can be expressed[45]

$$\begin{aligned} \hat{\pi}_1 &= p_{111} - \hat{\pi}^{ABC}_{1110} \;, \quad \hat{\pi}_2 = p_{112} - \hat{\pi}^{ABC}_{1120} \;, \\ \hat{\pi}_3 &= p_{122} - \hat{\pi}^{ABC}_{1220} \;, \quad \hat{\pi}_4 = p_{222} - \hat{\pi}^{ABC}_{2220} \;. \end{aligned} \tag{A.9}$$

For the $\hat{\pi}^{ABC}_{ijk0}$ in (A.8), we obtain the explicit formulas[46]

$$\begin{aligned} \hat{\pi}^{ABC}_{ijk0} &= p_{ijk} \;, \quad \text{for} \quad (i, j, k) \in U \;, \\ \hat{\pi}^{ABC}_{1110} &= p_{121}p_{211}/p_{221} \;, \quad \hat{\pi}^{ABC}_{1120} = p_{121}p_{212}/p_{221} \;, \\ \hat{\pi}^{ABC}_{1220} &= p_{121}p_{212}/p_{211} \;, \quad \hat{\pi}^{ABC}_{2220} = p_{221}p_{212}/p_{211} \;. \end{aligned} \tag{A.10}$$

For each pattern (i, j, k) in set S, the preceding formulas show how the corresponding $\hat{\pi}^{ABC}_{ijk0}$ is determined in terms of the p_{ijk} for some patterns (i, j, k) in set U.

In this particular case, since there are zero degrees of freedom for testing the corresponding quasi-independence model (see (1.16) with $m = 3$), this model cannot be tested.[47] Thus, both in the case where $m = 3$ and $m = 2$ this model could not be tested; but all the parameters in the model are identifiable when $m = 3$, and they are not when $m = 2$ (unless additional restrictions are imposed).

[*Received August 1974. Revised June 1975.*]

REFERENCES

[1] Bishop, Y.M.M. and Fienberg, S.E., "Incomplete Two-Dimensional Contingency Tables," *Biometrics*, 25 (March 1969), 119–28.

[2] Goodman, Leo A., "The Analysis of Cross-Classified Data: Independence, Quasi-Independence, and Interactions in Contingency Tables With or Without Missing Entries," *Journal of the American Statistical Association*, 63 (December 1968), 1091–131.

[3] ———, "On the Measurement of Social Mobility: An Index of Status Persistence," *American Sociological Review*, 34 (December 1969), 831–50.

[4] ———, "The Multivariate Analysis of Qualitative Data: Interactions Among Multiple Classifications," *Journal of the American Statistical Association*, 65 (March 1970), 225–56.

[5] ———, "Exploratory Latent Structure Analysis Using Both Identifiable and Unidentifiable Models," *Biometrika*, 61 Part 2, (1974), 215–31.

[6] ———, "The Analysis of Systems of Qualitative Variables When Some of the Variables Are Unobservable. Part I—A Modified Latent Structure Approach," *Americal Journal of Sociology*, 79 (March 1974), 1179–259.

[7] ———, "The Analysis of Systems of Qualitative Variables When Some of the Variables Are Unobservable. Part II—The Use of Modified Latent Distance Models," Unpublished manuscript.

[8] Guttman, Louis, "The Basis for Scalogram Analysis," in Samuel A. Stouffer *et al.*, eds., *Measurement and Prediction, Studies in Social Psychology in World War II, Vol. IV*, Princeton: University Press, 1950, 60–90.

[9] Hays, David G. and Borgatta, Edgar F., "An Empirical Comparison of Restricted and General Latent Distance Analysis," *Psychometrika*, 19 (December 1954), 271–9.

[10] Lazarsfeld, Paul F., "The Interpretation and Computation of Some Latent Structures," in Samuel A. Stouffer *et al.*, eds., *Measurement and Prediction, Studies in Social Psychology in World War II, Vol. IV*, Princeton: University Press, 1950, 413–72.

[11] ——— and Henry, Neil W., *Latent Structure Analysis*, Boston: Houghton-Mifflin, Co., 1968.

[12] McHugh, R.B., "Efficient Estimation and Local Identification in Latent Class Analysis," *Psychometrika*, 21 (December 1956), 331–47.

[13] Murray, J.R., "Statistical Models for Qualitative Data with Classification Errors," Ph.D. thesis, University of Chicago, 1971.

[14] Proctor, C.H., "A Probabilistic Formulation and Statistical Analysis of Guttman Scaling," *Psychometrika*, 35 (March 1970), 73–8.

[15] Schumacher, C.F., Maxson, G.R. and Martinek, H., "Tests for Creative Ability in Machine Design," Project ONR 458, Armed Services Technical Information Agency 21 284, 1953.

[16] Stouffer, Samuel A., "An Overview of the Contributions to Scaling and Scale Theory," in Samuel A. Stouffer *et al.*, eds., *Measurement and Prediction, Studies in Social Psychology in World War II, Vol. IV*, Princeton: Univeristy Press, 1950, 3–45.

[17] ——— and Toby, Jackson, "Role Conflict and Personality," *American Journal of Sociology*, 56 (March 1951), 395–406. Reprinted in Samuel A. Stouffer, *Social Research to Test Ideas*, New York: The Free Press, 1962. Reprinted in part in Matilda White Riley, *Sociological Research: A Case Approach*, New York: Harcourt, Brace and World, Inc., 1963.

[45] As with (1.9) and (A.3), we shall assume that none of the $\hat{\pi}_t$ in (A.9) are negative. If this is not the case, then a modification of the kind noted in Footnote 38 will be required here.

[46] The explicit formulas for the corresponding $\hat{\pi}^{ABC}_{ijk0}$ can be inserted in (A.9) to obtain explicit formulas for the $\hat{\pi}_t$ (for $t = 1, 2, 3, 4$).

[47] The corresponding chi-square values will be equal to zero, as can be seen from the first line of (A.10).