1 Estimating the Precision Parameter of a Rayleigh Distribution.

(a) As the prior on ξ follows $\mathcal{E}(\lambda)$ and the likelihood is $f(r|\xi) = \xi r \exp\Big\{-\frac{\xi r^2}{2}\Big\}$, we find the posterior as

$$\begin{aligned}\pi(\xi|r) &\propto f(r|\xi)\pi(\xi) = \xi r \exp\Big\{-\frac{\xi r^2}{2}\Big\} \cdot \lambda \exp\{-\lambda\xi\} \\ &= r\lambda\xi \exp\Big\{-\xi\Big(\lambda + \frac{r^2}{2}\Big)\Big\} \propto \xi \exp\Big\{-\xi\Big(\lambda + \frac{r^2}{2}\Big)\Big\}.\end{aligned}$$

Thus, the posterior is gamma $\mathcal{G}a\Big(2, \lambda + \frac{r^2}{2}\Big)$.

(b) We first find the posterior distribution when there are n data points. As the prior on ξ follows $\mathcal{E}(\lambda)$ and the likelihood is

$$L(\xi|r_1,\ldots,r_n) = \prod_{i=1}^n f(r_i|\xi) = \Big(\prod_{i=1}^n r_i\Big)\xi^n \exp\Big\{-\frac{\xi}{2}\sum_{i=1}^n r_i^2\Big\},$$

we find the posterior as

$$\begin{aligned}\pi(\xi|r) &\propto \Big(\prod_{i=1}^n r_i\Big)\xi^n \exp\Big\{-\frac{\xi}{2}\sum_{i=1}^n r_i^2\Big\} \cdot \lambda\exp\{-\lambda\xi\} \\ &= \Big(\prod_{i=1}^n r_i\Big)\lambda\xi^n \exp\Big\{-\xi\Big(\lambda + \frac{1}{2}\sum_{i=1}^n r_i^2\Big)\Big\} \propto \xi^n \exp\Big\{-\xi\Big(\lambda + \frac{1}{2}\sum_{i=1}^n r_i^2\Big)\Big\}.\end{aligned}$$

Thus, the posterior is gamma $\mathcal{G}a\Big(n+1, \lambda + \frac{1}{2}\sum_{i=1}^n r_i^2\Big)$. The Bayesian estimator of ξ is

$$\mathbb{E}_{\xi|r_1,\ldots,r_n}\xi = \frac{n+1}{\lambda + \frac{1}{2}\sum_{i=1}^n r_i^2}$$

Based on the given data, we have $n = 4$ and $\lambda + \frac{1}{2}\sum_{i=1}^n r_i^2 = \lambda + (0.5)(3^2+4^2+2^2+5^2) = \lambda + 27$. The Bayesian estimator of ξ is $\frac{5}{\lambda+27}$.

(c) For $\lambda = 1$, we know that $[\xi|r] \sim \mathcal{G}a(5, 28)$. By Rstudio, we obtain that $[0.058, 0.366]$.

2 Estimating Chemotherapy Response Rates.

(a) We let p be the proportion of the cancer patients that respond to the new chemotherapy treatment. We elicit a beta prior, namely $\mathcal{B}eta(\alpha, \beta)$. By setting the mean μ and standard deviation σ of the beta prior using $\mu = 0.9, \mu - 2\sigma = 0.8$, which is $\mu = 0.9, \sigma = 0.05$, we have

$$\mu = \frac{\alpha}{\alpha+\beta} = 0.9 \text{ and } \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.05^2.$$

By solving the above equation, we get $\alpha = 31.5$ and $\beta = 3.5$. For each single patient, the probability for him responding the treatment with be p. Therefore, the likelihood as follows:

$$\mathcal{L}(p;\boldsymbol{x}) = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}.$$

Furthermore, we can obtain that the posterior is

$$f(p\big|\boldsymbol{x}) \propto \mathcal{L}(p;\boldsymbol{x})p^{\alpha-1}(1-p)^{\beta-1} = p^{\alpha-1+\sum_{i=1}^n x_i}(1-p)^{\beta-1+n-\sum_{i=1}^n x_i},$$

which shows that the posterior follows beta distribution $\mathcal{B}eta(\alpha + \sum_{i=1}^n x_i, \beta + n - \sum_{i=1}^n x_i)$.

Thus the Bayes estimate is $\frac{\alpha+\sum_{i=1}^n x_i}{\beta+n-\sum_{i=1}^n x_i+\alpha+\sum_{i=1}^n x_i} = \frac{31.5+22}{3.5+8+31.5+22} = 0.823$.

(b) Note from (a), we have $p|\boldsymbol{x} \sim \mathcal{B}eta(53.5, 11.5)$. Then we have $[0.722, 0.905]$.

(c) First we compute the p_0 and p_1. As $p|\boldsymbol{x} \sim \mathcal{B}eta(53.5, 11.5)$ from (a), p_0 can be computed as 0.7054 using CDF from beta distribution. Similarly, p_1 is computed as 0.2946.

We then compute π_0 and π_1. As $p \sim \mathcal{B}eta(31.5, 3.5)$, π_0 can be computed as 0.9585 using CDF from beta distribution. Similarly, π_1 is computed as 0.0415.

Thus, we find Bayes factor as $B_{10} = \frac{p_1/p_0}{\pi_1/\pi_0} = \frac{0.2946/0.7054}{0.0415/0.9585} = 9.6459$. This follows that $\log_{10} B_{10} = 0.9843$, which shows substantial evidence against H_0.

(d) We use the following OpenBUGS code to perform the computation.

```
#Model
model {
x ~ dbin(p, M);
p ~ dbeta(31.5, 3.5);
pH0 <- step(p-0.8);
}

#Data
list(M=30, x=22)
```

We obtain the result, shown in Figure 1, after running the above code. Based on the result, the Bayes estimator is 0.8231 (shown as the mean value of p), the 95% equal-tailed credible set for p is [0.7216,0.9052], which are both very close the theoretical value computed from (a) and (b).

Node statistics

	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
p	0.8231	0.04709	1.45E-4	0.7216	0.8265	0.9052	1001	100000
pH0	0.7074	0.455	0.001435	0.0	1.0	1.0	1001	100000

Figure 1: Result to OpenBUGS code