ISYE 6420: Homework 1

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1. Circuit

Instructions

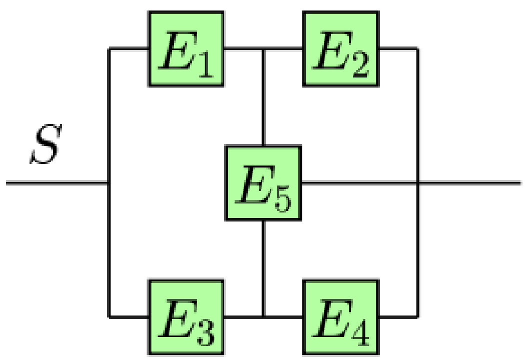


Figure 1: Components $E_1, \ldots E_4$ at operational at time t with probabilities e^{-t}, e^{-2t}, $e^{-t/2}$, $e^{-t/3}$ and e^{-t}, respectively.

The system S consists of five independent elements $E_i, i = 1, \ldots, E_5$, connected as in Figure 1. Probability that the element E_i is operational at the end of time interval $[0, t]$ is given as

$$p_i(t) = e^{-\lambda_i t}, \quad t \geq 0,$$

for $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 1/2, \lambda_4 = 1/3, \lambda_5 = 1$.
(a) Find the probability that the system S will be operational at time t. Plot this probability as a function of time t. What is this probability for $t = 1/2$?
(b) Find the probability that component E_5 was operational at time $t = 1/2$, if the system was operational at that time.

Hint: If you consider (b), it is conditional probability, more precisely, a posterior probability of the hypothesis $H_1 : E_5$ operational at time t, given that system S is operational at t. Thus, solve part (a) as a total probability with $H1$ and $H_2 = H_1^C$ as hypotheses. Under the two hypotheses the system simplifies as in Figure 2 and it is easy to find $P(S|H_1)$ and $P(S|H_2)$. Then (b) is just a Bayes formula.

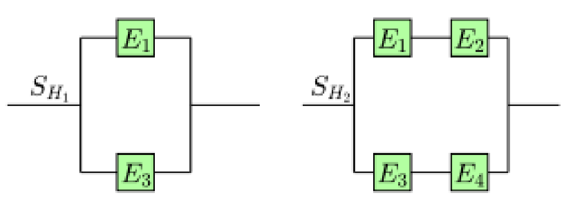


Figure 2: Left: System under hypothesis $H_1 : E_5$ operational; Right: System under hypothesis $H_2 : E_5$ not operational.

Response

(a)

We first formulate $P(S|H_1)$ (using the notation of the hint) as follows.

$$\begin{aligned} P(S|H_1) &= 1 - P((E_1 \cup E_3)') \\ &= 1 - P(E_1' \cap E_3') \\ &= 1 - (1 - P(E_1))(1 - P(E_3)) \\ &= 1 - (1 - e^{-t})(1 - e^{-t/2}) \\ &= 1 - (1 - e^{-t} - e^{-t/2} + e^{t^2/2}) \\ &= e^{-t} + e^{-t/2} - e^{-3t/2}. \end{aligned}$$

Note that $P((E_1 \cup E_3)'$ represents the probability that neither E_1 nor E_3 works, which can also be expressed as $P(E_1' \cap E_3')$, which, in words, represents the probability that both E_1 and E_3 fail. (This is just a consequence of De Morgan's laws (https://en.wikipedia.org/wiki/De_Morgan%27s_laws).)

Also, note that we achieved the same result more directly using the addition law of probability (https://en.wikipedia.org/wiki/Probability_axioms#Further_consequences), which corresponds to the expression $P(S|H_1) = P(E_1) + P(E_3) - P(E_1 \cap E_3)$ here.

Similarly, we can find $P(S|H_2)$ as follows.

$$\begin{aligned} P(S|H_2) &= 1 - P((E_1 \cap E_2) \cup (E_3 \cap E_4)) \\ &= 1 - (1 - P(E_1)P(E_2))(1 - P(E_3)P(E_4)) \\ &= 1 - (1 - e^{-t}e^{-2t})(1 - e^{-t/2}e^{-t/3}) \\ &= 1 - (1 - e^{-3t})(1 - e^{-5t/6}) \\ &= e^{-3t} + e^{-5t/6} - e^{-23t/6}. \end{aligned}$$

Then, by the law of total probability (https://en.wikipedia.org/wiki/Law_of_total_probability), it follows that

$$\begin{aligned} P(S) &= P(S|H_1)P(H_1) + P(S|H_2)P(H_2)P(H_2) \\ &= P(S|H_1)P(H_1) + P(S|H_2)P(H_2)(1 - P(H_1)) \\ &= (e^{-t} + e^{-t/2} - e^{-3t/2})(e^{-t}) + (e^{-3t} + e^{-5t/6} - e^{-23t/6})(1 - e^{-t}). \end{aligned}$$

We can write functions in `R` to codify these formulations.

```
f_sh1 <- function(t) {
  exp(-t) + exp(-t / 2) - exp(-3 * t / 2)
  # Or
  # 1 - (1 - exp(-t)) * (1 - exp(-t / 2))
}
f_sh2 <-  function(t) {
  exp(-3 * t) + exp(-5 * t / 6) - exp(-23 * t / 6)
  # Or
  # 1 - (1 - exp(-t) * exp(-2 * t)) * (1 - exp(-t / 2) * exp(-t / 3))
}
f_h1 <- function(t) {
  exp(-t)
}
f_h2 <- function(t) {
  1 - f_h1(t)
}
f_s <- function(t) {
  f_sh1(t) * f_h1(t) + f_sh2(t) * f_h2(t)
}
```

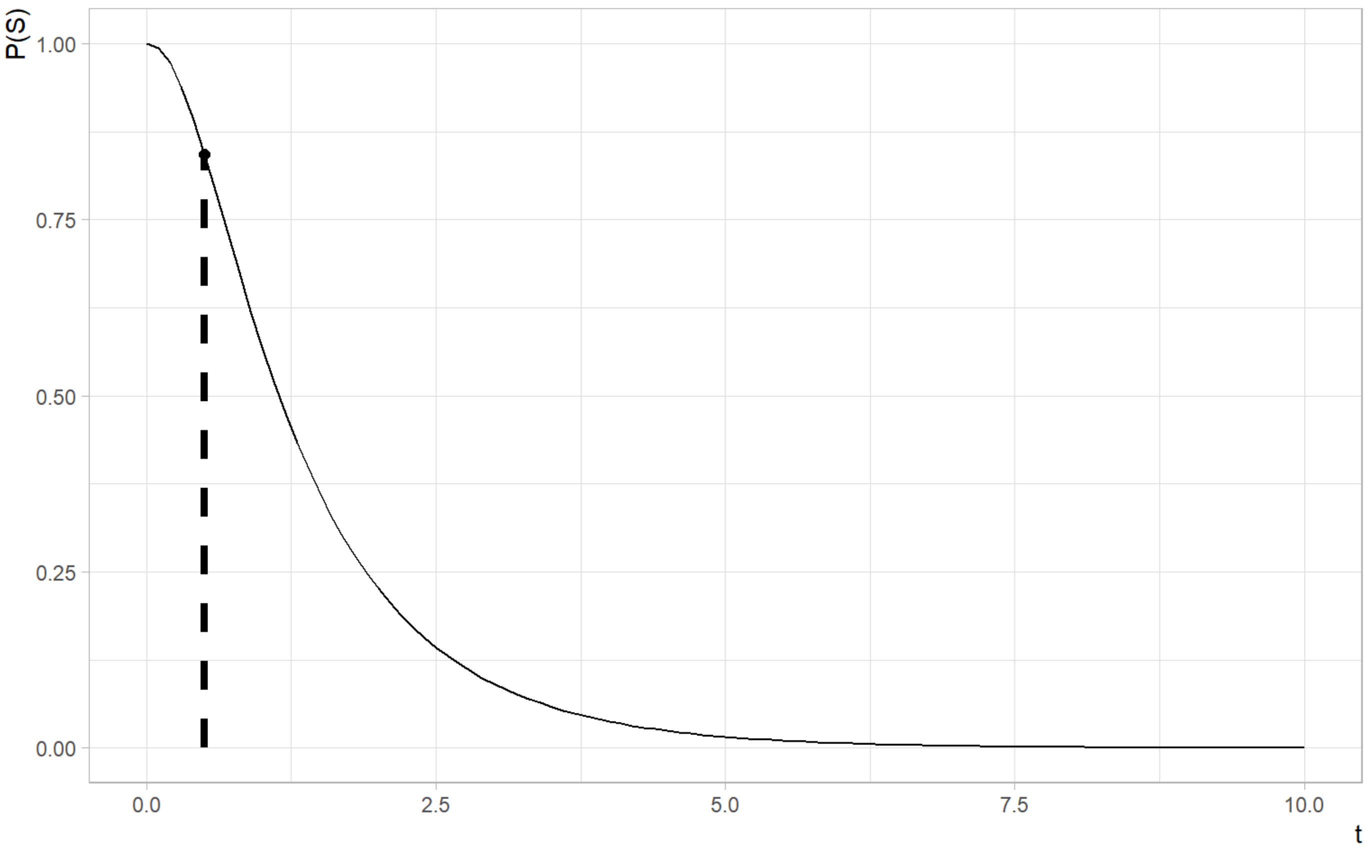
And, finally, we can use these functions (namely, the last one defined above) to calculate the probability that the system S will be operational at time $t = 0.5$.

```
t_x <- 0.5
p_s_x <- f_s(t_x)
p_s_x
```

```
## [1] 0.8430491
```

We find that $P(S) = 0.843$ at $t = 0.5$.

Below is a plot of this probability as a function of time t for $0 \leq t \leq 10$.



(b)

See the formulation for $P(H_1|S)$ below.

$$\begin{aligned} P(H_1|S) &= \frac{P(S|H_1)P(H_1)}{P(S)} \\ &= \frac{(e^{-t}+e^{-t/2}-e^{-3t/2})(e^{-t})}{(e^{-t}+e^{-t/2}-e^{-3t/2})(e^{-t})+(e^{-3t}+e^{-5t/6}-e^{-23t/6})(1-e^{-t})}. \end{aligned}$$

We can implement this in `R` as follows.

```
f_h1s <- function(t) {
  f_sh1(t) * f_h1(t) / f_s(t)
}
p_h1s_x <- f_h1s(t_x)
p_h1s_x
```

```
## [1] 0.6568315
```

We find that $P(H_1|S) = 0.657$ at at $t = 0.5$.

2. Two Batches

Instructions

There are two batches of the same product. In one batch all products are conforming. The other batch contains 20% non-conforming products. A batch is selected at random and one randomly selected product from that batch is inspected. The inspected product was found conforming and was returned back to its batch. What is the probability *that the second product, randomly selected from the same batch, is found non-conforming?*

Hint: This is both Bayes rule and Total Probability. The two hypotheses concern the type of batch. For the first draw they are equally likely (batch is selected at random), but for the second draw, they are updated by the information on first draw via Bayes rule. Updated probabilities of hypotheses are used in the total probability formula.

Response

Let B_1 and B_2 represent the batch with all conforming products and the batch with 20% non-conforming products respectively. Also, let $C = 1$ and $C = 0$ denote conforming and non-conforming samples respectively. The problem description indicates that $P(B_1) = P(B_2) = 0.5, P(C = 1|B_1) = 1, P(C = 1|B_2) = 0.8$, where the $P(B_1), P(B_2)$ terms represent prior probabilities. Then, we formulate the posterior probability $P(B_1|C)$ as follows.

$$\begin{aligned} P(B_1|C) &= \frac{P(C=1|B_1)P(B_1)}{P(C=1|B_1)P(B_1)+P(C=1|B_2)P_0(B_2)} \\ &= \frac{(1)(0.5)}{(1)(0.5)+(0.8)(0.5)} \\ &= \frac{0.5}{0.5+0.4} = \frac{5}{9} \approx 0.555. \end{aligned}$$

Then we can calculate $P(C = 1)$ as follows.

$$\begin{aligned} P_1(C = 1) &= P(C = 1|B_1)P_1(B_1) + P(C = 1|B_2)P_1(B_2) \\ &= P(C = 1|B_1)P_1(B_1) + P(C = 1|B_2)(1 - P_1(B_1)) \\ &= (1)(\tfrac{5}{9}) + (0.8)(1 - (\tfrac{5}{9})) \\ &= \tfrac{41}{45} \approx 0.911. \end{aligned}$$

Finally, we find that $P(C = 0) = 1 - P_1(C = 1) = \frac{4}{45} \approx 0.089$.

3. Classifier

Instructions

In a machine learning classification procedure the items are classified as 1 or 0. Based on a training sample of size 120 in which there are 65 1's and 55 0's, the classifier predicts 70 1's and 50 0's. Out of 70 items predicted by the classifier as 1, 52 are correctly classified.

From the population of items where the proportion of 0-labels is 99% (and 1-labels 1%), an item is selected at random. What is the probability that the item is of label 1, if the classifier says it was.

Response

First, let X represent the actual value and Y represent the predicted value. Next, let's consider a general two-by-two "confusion matrix" (https://en.wikipedia.org/wiki/Confusion_matrix) and it's relationship to counts of true positives (TP), true negatives (TN), false positives (FP), and false negatives (FN).

		Predicted	
		$Y = 0$	$Y = 1$
Actual	$X = 0$	TN	FP
	$X = 1$	FN	TP

Given the problem setup, the cells of the table are as follows. (The instruction-provided numbers are emphasized. The other numbers represent “calculations”.)

		Predicted $Y=0$	$Y=1$	Y_{total}
Actual	$X=0$	37	18	**55**
	$X=1$	13	**52**	**65**
	X_{total}	**50**	**70**	**120**

The problem description indicates

$$P(X=0)=0.99,$$
$$P(X=1)=0.01.$$

(Note that we would define

$$P(X=0)=\frac{55}{120}=\frac{11}{24},$$
$$P(X=1)=\frac{65}{120}=\frac{13}{24}$$

if we were to use the “original” population.)

Additionally, we identify the following.

$$P(Y=1|X=1)=\frac{52}{65}=\frac{4}{5},$$
$$P(Y=1|X=0)=\frac{18}{55}.$$

Next, let’s create “placeholder” expressions A, B (to simplify notation) as follows.

$$A=P(Y=1|X=1)P(X=1)=(\frac{4}{5})(0.01)=0.008.$$

and

$$B=P(Y=1|X=0)P(X=0)=(\frac{18}{55})(0.99)=0.324.$$

Then, using Bayes’ formula, we have the following.

$$\begin{aligned} P(X=1|Y=1) &= \frac{P(Y=1|X=1)P(X=1)}{P(Y=1|X=1)P(X=1)+P(Y=1|X=0)P(Y=1|X=0))} \\ &= \frac{A}{A+B} \\ &= \frac{(0.008)}{(0.008)+(0.324)} \\ &\approx 0.02410. \end{aligned}$$

Thus, we have found that the probability that the item is of label 1 if the classifier indicates that it is 1 is 0.02410.

Note that if we had used the “original” ratio of 0’s and 1’s, we would have found a very different answer.

$$A=P(Y=1|X=1)P(X=1)=(\frac{4}{5})(\frac{13}{24})=\frac{13}{30}.$$

$$B = P(Y = 1|X = 0)P(X = 0) = (\frac{18}{55})(\frac{11}{24}) = \frac{3}{30}.$$

$$\begin{aligned} P(X = 1|Y = 1) &= \tfrac{A}{A+B} \\ &\approx 0.8125. \end{aligned}$$