1 Cell Clusters in 3D Petri Dishes.

Since the number of cell clusters follows a Poisson distribution with the rate parameter $\lambda = 5$, that is the number of cell clusters X has the pdf as

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

(a) We compute the probability when $X = 0$ as

$$P(X = 0) = \frac{(5)^0 e^{-5}}{0!} = e^{-5} \approx 0.006738.$$

(b) We compute the probability when $X \geq 1$ as

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) = 1 - e^{-5} \\ &= 1 - e^{-5} \approx 0.9933. \end{aligned}$$

(c) We compute the probability when $X > 8$ as

$$P(X > 8) = 1 - \sum_{k=0}^{8} P(X = k) = 1 - \sum_{k=0}^{8} \frac{(5)^k e^{-5}}{k!} \approx 0.06809.$$

(d) We compute the probability when $4 \leq X \leq 6$ as

$$\begin{aligned} P(4 \leq X \leq 6) &= P(X = 4) + P(X = 5) + P(X = 6) \\ &= \frac{(5)^4 e^{-5}}{4!} + \frac{(5)^5 e^{-5}}{5!} + \frac{(5)^6 e^{-5}}{6!} \approx 0.4972. \end{aligned}$$

2 Silver-Coated Nylon Fiber.

Since the time in hours between blockages has the exponential distribution with the rate parameter 1/10, we obtain its pdf $f(x)$ and cdf as follows:

$$f(T = t) = \frac{1}{10}\exp(-\frac{1}{10}t) \quad \text{and} \quad P(T \leq t) = 1 - \exp(-\frac{1}{10}t) \quad \text{for } t > 0.$$

(a) We compute the probability of $T \geq 10$

$$P(T \geq 10) = 1 - P(T < 10) = \exp(-1).$$

(b) We compute the probability of $T \leq 15$

$$P(T \leq 15) = 1 - \exp(-\frac{3}{2}).$$

(c) We compute the probability of $T \geq 20$ conditioning on $T \geq 10$

$$P(T \geq 20 \big| T \geq 10) = \frac{P(T \geq 20)}{P(T \geq 10)} = \frac{\exp(-2)}{\exp(-1)} = \exp(-1).$$

3 2-D Density Tasks.

(a) For $x > 0$, we have

$$f_X(x) = \int_x^\infty f(x,y)dy = \int_x^\infty \lambda^2 e^{-\lambda y} dy = \left[\lambda^2\Big(-\frac{1}{\lambda}\Big)e^{-\lambda y}\right]\Bigg|_x^\infty = \lambda e^{-\lambda x},$$

which matches with the pdf of exponential distribution with rate parameter λ. Thus, we know that marginal distribution $f_X(x)$ is exponential $\mathcal{E}(\lambda)$.

(b) For $y > 0$, we have

$$f_Y(y) = \int_0^y f(x,y)dx = \int_0^y \lambda^2 e^{-\lambda y} dy = \left[\lambda^2 e^{-\lambda y} x\right]\Bigg|_0^y = \lambda^2 y e^{-\lambda y},$$

which matches with the pdf of gamma distribution with shape parameter 2 and rate parameter λ. Thus, we know that marginal distribution $f_Y(y)$ is Gamma $\mathcal{G}a(2,\lambda)$.

(c) For $y \geq x$, we have

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}, y \geq x,$$

which shows that conditional distribution $f_{Y|X}(y|x)$ is shifted exponential.

(d) For $x \leq y$, we have

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}} = \frac{1}{y}, y \geq x,$$

which shows that conditional distribution $f_{X|Y}(x|y)$ is uniform $\mathcal{U}(0,y)$.

4 Nylon Fiber Continued.

(a) Note the mean of the $\mathcal{E}(\lambda)$ is $1/\lambda$. Therefore, the classical estimate of λ is

$$\hat{\lambda} = \frac{1}{\bar{T}} = \frac{3}{3+8+13} = \frac{1}{8}.$$

(b) Denote $\boldsymbol{T} = (T_1, T_2, T_3)$. We first compute the posterior distribution of λ:

$$\begin{aligned} p(\lambda\big|\boldsymbol{T}) \propto p(\boldsymbol{T}|\lambda)\pi(\lambda) &= \lambda^3 \exp(-\lambda(T_1+T_2+T_3)) \cdot \frac{1}{\sqrt{\lambda}} \\ &= \lambda^{5/2} \exp(-\lambda(T_1+T_2+T_3)). \end{aligned}$$

Therefore, $[\lambda\big|\boldsymbol{T}] \sim \mathcal{G}a(7/2, T_1+T_2+T_3)$. Under the minimum mean square error, the Bayes estimator is

$$\mathbb{E}_{\lambda|\boldsymbol{T}}\lambda = \frac{7/2}{T_1+T_2+T_3} = \frac{7}{48}.$$