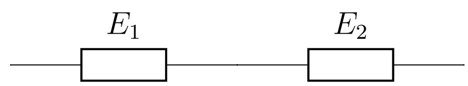
**Solution** **Homework 1** ISyE 6420 September 17, 2019

# 1 Problem 1 (Circuit)

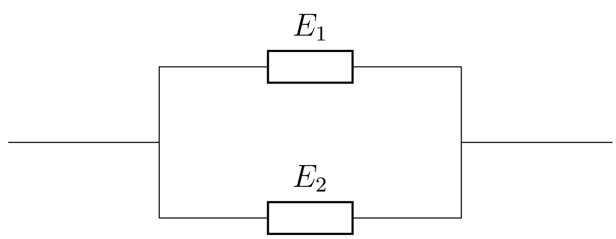
Let the probability of an element $E_i$ of a system $S$ being operational at time $t$ be $p_{E_i}(t)$. Suppose all elements work independently. We consider the following two ways of connection of elements.

- For sequential connection of elements, that is




  Then $p_S(t) = p_{E_1}(t) \cdot p_{E_2}(t)$.

- For parallel connection of elements, that is




  Then $p_S(t) = p_{E_1 \cup E_2}(t) = p_{E_1}(t) + p_{E_2}(t) - p_{E_1}(t) \cdot p_{E_2}(t)$. (This is similar to the union of events)
  Alternatively, We let $q_{E_i}(t) = 1 - p_{E_i}(t)$ be the probability of element $E_i$ failing before time $t$, then

$$\begin{aligned} p_S(t) &= 1 - q_S(t) = 1 - q_{E_1}(t) \cdot q_{E_2}(t) \\ &= 1 - (1 - p_{E_1}(t)) \cdot (1 - p_{E_2}(t)) = p_{E_1}(t) + p_{E_2}(t) - p_{E_1}(t) \cdot p_{E_2}(t) \end{aligned}$$

(a) Based on the problem statement, we have

$$p_1(t) = e^{-t}, p_2(t) = e^{-2t}, p_3(t) = e^{-t/2}, p_4(t) = e^{-t/3}, p_5(t) = e^{-t}.$$

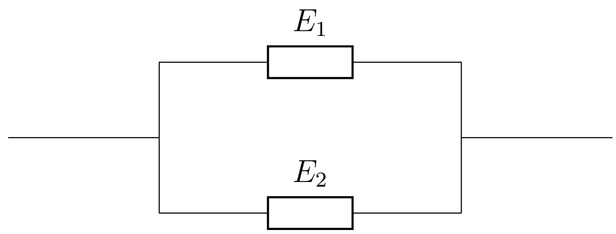
We consider the following two hypotheses:

$$\begin{aligned} &H_1 : E_5 \text{ works at time } t; \\ &H_2 : E_5 \text{ fails up to time } t \text{ (This is equivalent to } H_1^c\text{ )} \end{aligned}$$

We know that $\mathrm{P}(H_1) = e^{-t}$ and $\mathrm{P}(H_2) = 1 - e^{-t}$.
By law of Total Probability, we have

$$\mathrm{P}(S) = \mathrm{P}(S|H_1)\mathrm{P}(H_1) + \mathrm{P}(S|H_2)\mathrm{P}(H_2).$$

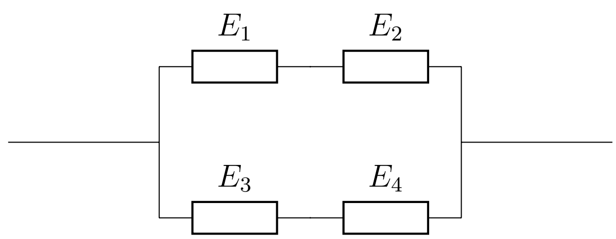
Under hypothesis $H_1$, $S$ is equivalent to




Thus,

$$P(S|H_1) = 1 - (1 - e^{-t/2}) \cdot (1 - e^{-t}) = e^{-t} + e^{-t/2} - e^{-3t/2}$$

Under hypothesis $H_2$, $S$ is equivalent to




Thus,

$$\begin{aligned} P(S|H_2) &= 1 - (1 - (e^{-t} \cdot e^{-2t})) \cdot (1 - (e^{-t/2} \cdot e^{-t/3})) \\ &= 1 - (1 - e^{-3t}) \cdot (1 - e^{-5t/6}) = e^{-3t} + e^{-5t/6} - e^{-23t/6}. \end{aligned}$$

Hence,

$$\begin{aligned} P(S) &= (e^{-t} + e^{-t/2} - e^{-3t/2})(e^{-t}) + (e^{-3t} + e^{-5t/6} - e^{-23t/6})(1 - e^{-t}) \\ &= e^{-29t/6} - e^{-4t} - e^{-23t/6} + e^{-3t} - e^{-5t/2} + e^{-2t} - e^{-11t/6} + e^{-3t/2} + e^{-5t/6} \end{aligned}$$

For $t = 1/2$, $P(S) = 0.843069$.
The plot of the probability that the system $S$ is operational at time $t$ is shown below in Figure 1.

(b) We need to find $P(H_1|S)$. That is

$$\begin{aligned} P(H_1|S) &= \frac{P(S|H_1)P(H_1)}{P(S)} = \frac{(e^{-t} + e^{-t/2} - e^{-3t/2})e^{-t}}{P(S)} \\ &= \frac{(e^{-1/2} + e^{-(1/2)/2} - e^{-(3)(1/2)/2})e^{-1/2}}{0.843069} = \frac{0.553741}{0.843069} = 0.656831 \end{aligned}$$

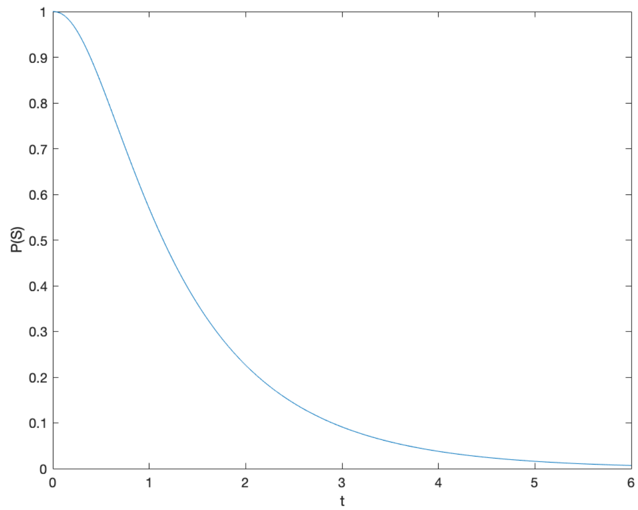



Figure 1: Plot of the probability that the system $S$ is operational at time $t$

## 2 Two Batches.

We let the first batch be the batch with all products conforming, and let the second batch be the batch that contains 20% non-conforming products. We define the following events.

$$H_1 : \text{The first batch is selected;} \quad H_2 : \text{The second batch is selected;}$$
$$A : \text{The first selected product from the selected batch is conforming;}$$
$$B : \text{The second selected product is non-conforming.}$$

We know that $\mathrm{P}(H_1) = \mathrm{P}(H_2) = \frac{1}{2}$ as the batch is selected randomly. We also know that $\mathrm{P}(A|H_1) = 1$ and $\mathrm{P}(A|H_2) = 0.8$ based on the problem setting. We hope to find $\mathrm{P}(B|A)$. By law of Total Probability, we have

$$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}(A|H_1)\mathrm{P}(H_1) + \mathrm{P}(A|H_2)\mathrm{P}(H_2)\\ &= (1)(0.5) + (0.8)(0.5) = 0.9\end{aligned}$$

and

$$\begin{aligned}\mathrm{P}(B|A) &= \mathrm{P}(B, H_1|A) + \mathrm{P}(B, H_2|A)\\ &= \mathrm{P}(B|H_1, A)\mathrm{P}(H_1|A) + \mathrm{P}(B|H_2, A)\mathrm{P}(H_2|A)\end{aligned}$$

By Bayes rule, we have

$$\begin{aligned}\mathrm{P}(H_1|A) &= \frac{\mathrm{P}(A|H_1)\mathrm{P}(H_1)}{\mathrm{P}(A)} = \frac{(1)(0.5)}{0.9} = \frac{5}{9}\\ \mathrm{P}(H_2|A) &= \frac{\mathrm{P}(A|H_2)\mathrm{P}(H_2)}{\mathrm{P}(A)} = \frac{(0.8)(0.5)}{0.9} = \frac{4}{9}\end{aligned}$$

| | | Truth | | |
|---|---|---|---|---|
| | | 1 | 0 | Total |
| Classified | 1 | 52 | 18 | 70 |
| | 0 | 13 | 37 | 50 |
| | Total | 65 | 55 | 120 |

Table 1: Results of the classifier

Hence, we have

$$P(B|A) = (0)(\frac{5}{9}) + (0.2)(\frac{4}{9}) = \frac{4}{45}$$

# 3 Classifier.

Based on the problem statement, we have the following information.

By law of Total Probability and Bayes rule, we have

$$\begin{aligned}
P(\text{item} = 1|\text{cls} = 1) &= \frac{P(\text{cls} = 1|\text{item} = 1)P(\text{item} = 1)}{P(\text{cls} = 1)} \\
&= \frac{P(\text{cls} = 1|\text{item} = 1)P(\text{item} = 1)}{P(\text{cls} = 1|\text{item} = 1)P(\text{item} = 1) + P(\text{cls} = 1|\text{item} = 0)P(\text{item} = 0)} \\
&= \frac{(52/65)(0.01)}{(52/65)(0.01) + (18/55)(0.09)} = 0.0241
\end{aligned}$$

Here *sensitivity* of classifier is $52/65 = 0.8$ and $specificity$ is $37/55 \approx 0.67272$. Since the prevalence of 1's is only 1%, the *positive predicted value*, that is $P(\text{item} = 1|\text{cls} = 1)$, is only 2.5%. This is source of "prosecutor's paradox".